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PRODUCING ANNOYING SOUNDS WITH CHALK BY VARYING ANGLE OF CONTACT

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Abstract. This paper investigates how varying the angle of contact α between a chalk and blackboard affects how annoying the sound produced is after drawing some distance d. Annoyance is quantified through Zwicker's psychoacoustic annoyance model [1]. Chalk is modeled as a cylinder and its crumbling is modeled by a downward translation "into" a chalkboard rotated at an angle α. This model is used to calculate the chalk's area of contact with the board at any d and α. Next, the relationship between contact area and psychoacoustic annoyance is determined experimentally. The set of angles that minimize and maximize annoyance are then found, measured using two annoyance metrics: instantaneous and accumulated annoyance.

Contents

1. Introduction

Hagoromo chalk has famously been dubbed the "Rolls Royce of chalk". Drawing with the smoothest lines and softest sounds in the world, it has famously been used by the greatest mathematicians. It was popularized by the Big Think YouTube video, which has now amassed tens of millions of views. In fact, this video brought our attention to the idea of writing on a chalkboard.

Since we are students, we do not have access to Hagoromo chalk. Yet, as aspiring scientists and mathematicians, we became very interested in investigating the sound produced by the regular chalk we have. Our goal was to find another way to produce as "good" (or as "bad") a sound as possible using the chalk that we had. This led to our research question: When drawing with chalk, drawing at which angle relative to the chalkboard will produce the most/least annoying sound?

"Annoyance" is a seemingly qualitative and subjective measure, however many in the field of psychophysics have developed models to quantify it. Psychoacoustics is the branch of psychophysics that studies how our brains perceive sound through the human auditory system. This field has developed mathematical models for psychoacoustic annoyance that have been experimentally verified, which are what we will be using in this paper.

To determine the relationship between angle and psychoacoustic annoyance, we conduct an experiment and approximate a function using polynomial regression on

the data collected. We then analytically calculate the set of angles that produce the most and least annoying sounds.
Section 2 will define the psychoacoustic terminology used in the rest of the paper. Section 3 determines how the chalk's orientation affects the area of contact in two and three dimensions, respectively. Further, Section 4 details the experiment we conducted and approximates the relationship between contact area and psychoacoustic annoyance. The results can be seen in Appendix B. Then, Section 5 calculates the angles that produce the maximum and minimum annoyance. Finally, Section 6 discusses evaluating the experiment and discusses further work. Note that Appendix D contains links to the interactive Desmos's that can help to easier interpret the paper.

2. Definitions

Our definition of an "annoying" sound is characterized by 4 psychoacoustic metrics: perceived loudness, sharpness, roughness, and fluctuation strength.

2.1. **Perceived Loudness.** First, perceived loudness quantifies how loud a sound appears to the human ear, measured with the unit phons. It can be calculated by the sound pressure level (dB) and the frequency (Hz) of a particular sound as seen in equation 1.

$$L_p = \frac{100}{3}\log\left(\frac{10^{\alpha_f\cdot(L+L_U)\cdot 0.1\text{ dB}^{-1}} - 10^{\alpha_f(T_F+L_U)\cdot 0.1\text{ dB}^{-1}}}{(4\cdot 10^{-10})^{0.3-\alpha_f}} + 10^{0.072}\right) \tag{1}$$

L_p is perceived loudness measured in phons. L is the sound-pressure-level in dB. T_f is threshold for hearing in dB. α_f is exponent for loudness perception. L_U is magnitude of the linear transfer function, where the latter three parameters can be found in a look-up table provided by ISO 226:2023 [2].

2.2. **Sharpness.** Second, the sharpness of a sound, measured in the unit acum, is determined by the presence of higher frequencies in a sound spectrum. High frequencies are often perceived as "more sharp". The equation 2 shows how to calculate sharpness.

$$\sigma = 0.11\frac{\int_0^{24\text{ Bark}} N'(z)\cdot g(z)\,dz}{\int_0^{24\text{ Bark}} N'(z)\,dz}\text{ acum} \tag{2}$$

where z is the critical band and N' is the specific loudness at every individual critical band of hearing (the human hearing range is split into 24 critical bands). The denominator essentially uses an integral to calculate the total loudness over all critical bands (0 to 24) [3]. The numerator introduces a weighing factor $g(z)$, which $g(z)$ is defined by von Bismarck [4] in equation 3.

$$g(z) = \begin{cases} z & 0\text{ Bark} \le z \le 15\text{ Bark} \\ (0.2e^{0.308(z-15)}+0.8)z & 15\text{ Bark} < z \le 24\text{ Bark} \end{cases} \tag{3}$$

The weighing factor accounts for the presence of higher frequency sounds by weighing them more.

2.3. **Roughness.** Third, roughness, measured in the unit asper, quantifies how unpleasant a sound is when perceived by the human ear, caused by the modulations in the sound pressure levels and frequencies that are too high to distinguish separately. Roughness can be calculated as seen in equation 4.

$$R = 0.3 \frac{f_{\text{mod}}}{1000\text{ Hz}} \int_0^{24\text{ Bark}} \frac{\Delta L}{\text{dB}\,\text{Bark}^{-1}}\, dz \text{ asper} \tag{4}$$

f_{mod} is the modulation frequency, which is the rate (in Hz) at which the sound's amplitude or frequency fluctuates. ΔL quantifies how strongly these modulations are perceived, which can be found through the loudness envelope of the sound [5].

2.4. **Fluctuation Strength.** Fourth, the fluctuation strength of a sound, measured in the unit vacil, considers how unpleasant a sound is due to the modulations in the sound pressure levels and frequencies that the human ear can discern separately. Fluctuation strength can be calculated through equation 5 [6]:

$$F = \frac{0.032 f_{\text{mod}} \int_0^{24\text{ Bark}} \Delta L\, dz}{f_{\text{mod}}^2 + 16} \text{ vacil} \tag{5}$$

in which f_{mod} and ΔL represent the same concept as in the formula for roughness.

2.5. **Psychoacoustic Annoyance.** Finally, from the 4 psychoacoustic metrics, Zwicker quantified the concept of psychoacoustic annoyance with the following formula [1]:

$$P = N_5(1 + \sqrt{w_\sigma^2 + w_{FR}^2}) \tag{6}$$

where w_σ describes the effects of sharpness σ and w_{FR} describes the influence of fluctuation strength F and roughness R. They are defined in equations 7 and 8. N_5 is the loudness which is reached or exceeded in 5% of the measurement time. Note that annoyance has no unit.

$$w_\sigma = \left(\frac{\sigma}{\text{acum}} - 1.75\right) \cdot 0.25 \log\left(\frac{N_5}{\text{sone}} + 10\right) \text{ for } S > 1.75 \tag{7}$$

$$w_{FR} = \frac{2.18}{(N_5/\text{sone})^{0.4}} \cdot \left(\frac{0.4F}{\text{vacil}} + \frac{0.6R}{\text{asper}}\right) \tag{8}$$

3. Orientation

In the following section, the chalk is modeled as a cylinder with radius r and height h, oriented at an arbitrary angle α between 0 and $\frac{\pi}{2}$ relative to the chalkboard (see figure 1). We aim to find a function $S(d, \alpha)$ that describes how contact area varies with drawing distance d and angle α.

The regions where the chalk intersects the chalkboard represent the contact points responsible for marking and will be left as the "drawing". In our analysis, we

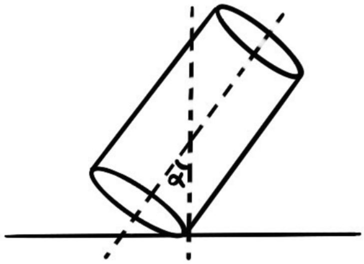


FIGURE 1. Definition of α

treat the chalkboard as a fixed reference plane, and the orientation of the chalk is parameterized by α.

It is important to note that any interaction between the chalk and the chalkboard will only depend on the orientation of the chalk with respect to the chalkboard and not on absolute spatial coordinates. In other words, the absolute position of the chalk doesn't affect the locus of intersection. Consequently, instead of rotating the chalk, it is much more convenient tilt the chalkboard instead, that is, expressing the chalkboard with the equation $z = (y - r)\tan\alpha$ (figure 2).

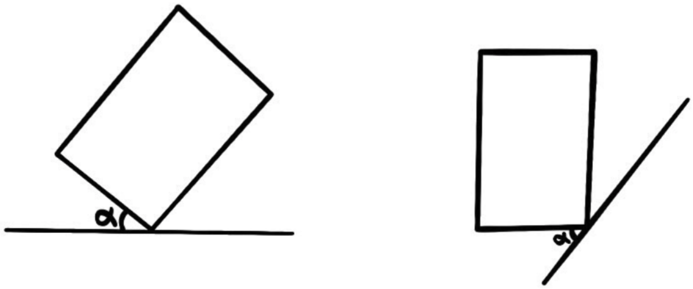


FIGURE 2. Change in reference frame

To analyze the system, we consider an infinitely extended chalkboard. We observe that when a chalk draws by a distance d and some part of the chalk "crumbles". We model this crumble as a downward translation of the chalk by $s = d\sin\alpha$ units along the z-axis (figure 3). d does not have a physical interpretation. Every chalk crumbles differently because they are made of different materials. The physical drawing distance t can be expressed as $t = kd$, where k is a constant experimentally derived.

We will investigate how we can express the area of contact between the chalk and the chalkboard after the chalk has drawn for any given distance d. The set of all possible intersections between the chalk and the chalkboard is E (equation 9).

$$E = \{(x, y, z) : x^2 + y^2 < r^2\} \cap \{(x, y, z) : z = (y - r)\tan\alpha\} \tag{9}$$

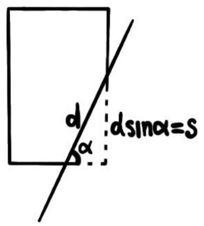


FIGURE 3. Relationship between d and s

We categorize the shape of the chalk into two cases: $2r\tan\alpha \geq h$, and $2r\tan\alpha < h$. The trivial cases $\alpha = 0$ and $\alpha = \dfrac{\pi}{2}$ will be independently discussed as their contact area functions are somewhat different.

3.1. **Trivial Cases.** At $\alpha = 0$, the chalk is upright, and hence the contact area at any drawing distance will remain constant at πr^2. At $\alpha = \dfrac{\pi}{2}$, the contact area will be rectangular, where the length and width of the rectangle are h and $2\sqrt{2dr - d^2}$. In other words, at $\alpha = 0$, $S(d, 0) = \pi r^2$, and at $\alpha = \dfrac{\pi}{2}$, $S(d, \dfrac{\pi}{2}) = 2h\sqrt{2dr - d^2}$.

3.2. **Case 1:** $2r\tan\alpha \geq h$**.** In the first case, as the chalk interacts with the surface of the chalkboard during the drawing process, we observe that there are three distinct phases of contact area. We can see how we have categorized the three subcases in figure 4, where case 1.1 indicates the start of the drawing process and the last indicates the end. Note that $s = s_1 + s_2 + s_3$.

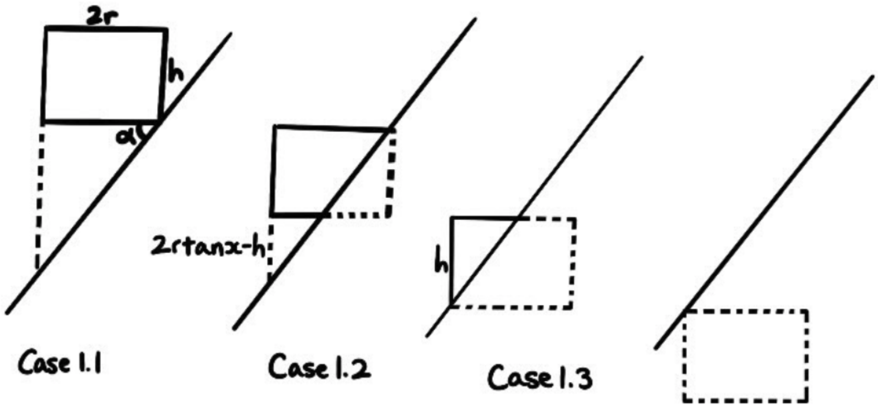


FIGURE 4. Breakdown of case 1

Case 1.1: $0 \leq s \leq h$

Notice that figure 5 is only possible when $2r \tan \alpha \geq h$. The downward translation distance in case 1.1, s_1, is in the range $0 \leq s_1 \leq h$, corresponding to the range $0 \leq d \leq \dfrac{h}{\sin \alpha}$.

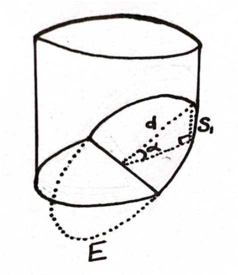


FIGURE 5. Area of intersection for case 1.1

Figure 6 shows the dimensions of the ellipse E. The length of the major radius is $M = \dfrac{r}{\cos \alpha}$. Looking at figure 6 from a bird's eye view, it is apparent that the minor radius is the same as r.

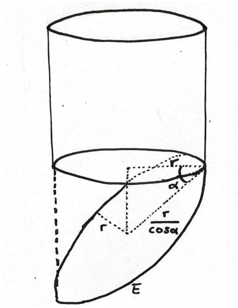


FIGURE 6. Dimensions of the ellipse E

Hence, we obtain an ellipse with the equation:

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{M}\right)^2 = 1 \tag{10}$$

The ellipse can be written as a function of y where $x > 0$:

$$x = \sqrt{M^2 - y^2} \cos \alpha \tag{11}$$

The ellipse is symmetric with respect to the yz plane; therefore, doubling the area of the upper half of the ellipse portion yields the full area. As the chalk travels d_1 units, the area can be evaluated by considering figure 7.

In this case, the contact area will only be portions of this ellipse. We observe that the segment $d_1 = \dfrac{s_1}{\sin\alpha}$ corresponds to the width of the portion of the ellipse.

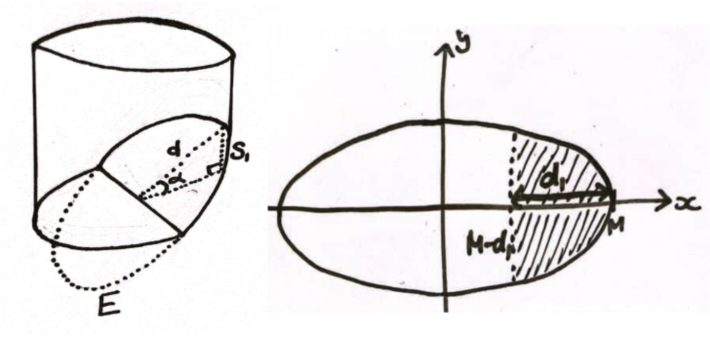


FIGURE 7. E projected on the xy plane for case 1.1

Therefore, we can express this area as a function of the drawing distance d_1:

$$A_1(d_1, \alpha) = 2\cos\alpha \int_{M-d_1}^{M} \sqrt{M^2 - y^2}\, dy \tag{12}$$

Case 1.2 $h < s \leq 2r\tan\alpha$
After s_1 reaches a maximum at $s_1 = h$, we consider the second subcase, where the additional distance is denoted as s_2. s_2 has a range $0 \leq s_2 \leq 2r\tan\alpha - h$. In this case, the total translation $s = s_1 + s_2 = h + s_2$.
We observe the length $\dfrac{h}{\sin\alpha}$ remains constant the entire case, "sliding" along the major axis of E, as seen in figure 8.

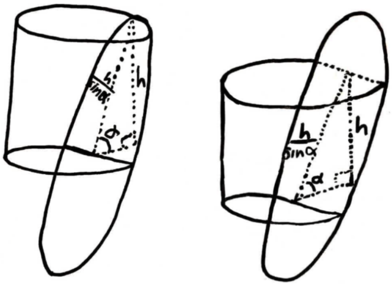


FIGURE 8. Chalk sliding down major axis of E

Recall that $d = \dfrac{s}{\sin\alpha}$ and $\dfrac{h}{\sin\alpha}$ is the maximum of s_1. Hence, we shall denote $D_1 = \dfrac{h}{\sin\alpha}$ since it is the maximum of d_1.

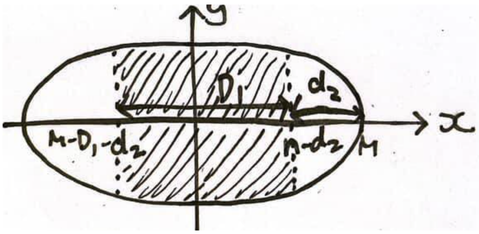


FIGURE 9. E projected on xy plane for case 1.2

Therefore, after the chalk has drawn D_1 units, the area of intersection with the chalkboard after drawing by an additional d_2 units can be expressed as:

$$B_1(d_2, \alpha) = 2\cos\alpha \int_{M-(D_1+d_2)}^{M-d_2} \sqrt{M^2 - y^2}\, dy \tag{13}$$

Case 1.3 $2r\tan\alpha < s \leq 2r\tan\alpha + h$
After d_2 reaches a maximum at $D_2 = \dfrac{2r\tan\alpha - h}{\sin\alpha} = 2M - D_1$ (in other words, $s = 2r\tan\alpha$), we consider the third subcase. The additional distance is denoted as s_3, where $0 \leq s_3 \leq h$.

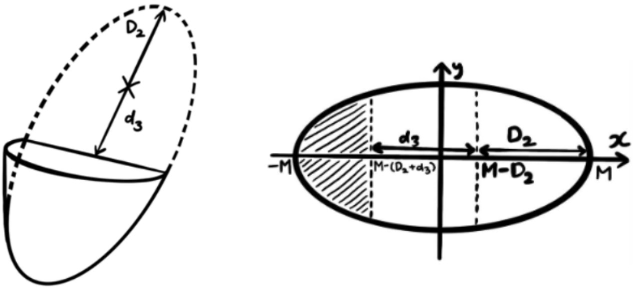


FIGURE 10. E projected on xy plane for case 1.3

Thus, we can express the area of intersection between the chalk and chalkboard after drawing $2r\tan\alpha + d_3$ units as:

$$C_1(d_3, \alpha) = 2\cos\alpha \int_{-M}^{M-(D_2+d_3)} \sqrt{M^2 - y^2}\, dy \tag{14}$$

Finally, when $s > 2r\tan\alpha + h$, the chalk is whittled down to non-existence and no chalk remains. All in all, the area of contact can be expressed by combining equations 12. 13 and 14 as seen in equation 15 (see figure 11 for a plot of case 1):

$$S(d,\alpha)=\begin{cases}2\cos\alpha\displaystyle\int_{M-d}^{M}\sqrt{M^2-y^2}\,dy & 0\le d\le D_1\\ 2\cos\alpha\displaystyle\int_{M-d}^{M+D_1-d}\sqrt{M^2-y^2}\,dy & D_1<d\le 2M\\ 2\cos\alpha\displaystyle\int_{-M}^{M+D_1-d}\sqrt{M^2-y^2}\,dy & 2M<d\le 2M+D_1\end{cases}\tag{15}$$

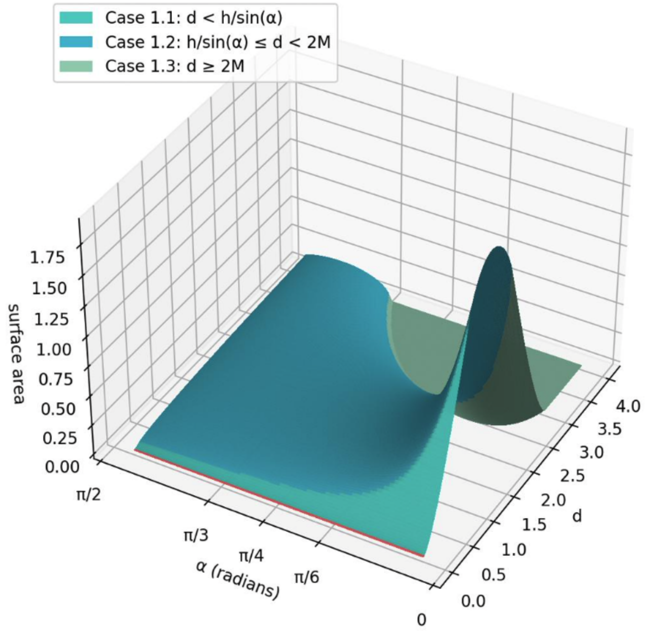


FIGURE 11. Plot of Case 1 for $r=1, h=0.1$

3.3. **Case 2:** $2r\tan\alpha<h$**.** Case 2 is similar to case 1, the major difference being how the subcases are split. The only difference between cases 1 and 2 is that in case 2, there is a subcase where the intersection is a complete ellipse.

Case 2.1: $0\le s\le 2r\tan\alpha$

Similarly to case 1.1, this case is when the chalk's "base" is still there, meaning that it covers only a portion of the ellipse (see figures 7 and 12). In this subcase, s_1 ranges from 0 to $2r\tan\alpha$. We can see the critical case 2.1.1 where the "base" fully disappears (when $s_1=2r\tan\alpha$) as seen in figure 13:

Therefore, for all values $0<s<2r\tan\alpha$, the area of contact can be found using the same method in case 1.1:

$$A_2(d_1,\alpha)=2\cos\alpha\int_{M-d_1}^{M}\sqrt{M^2-y^2}\,dy\tag{16}$$

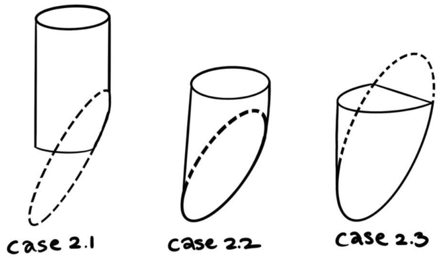


FIGURE 12. Breakdown of Case 2

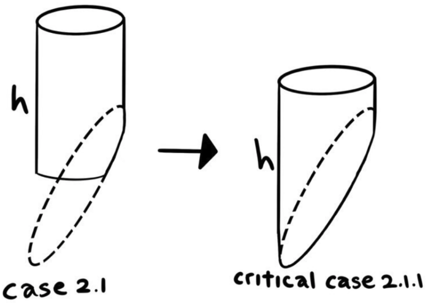


FIGURE 13. Critical Case 2.1.1

Case 2.2: $2r \tan \alpha < s \leq h$
This subcase is what makes case 1 differ from case 2. Case 2.2 is when the contact area is a complete ellipse.
We define s_2 to range from 0 to $h - 2r \tan \alpha$ so that when s_2 reaches its maximum (as seen in critical case 2.2.1, figure 13), this will be the threshold where the cross section is no longer a complete ellipse.

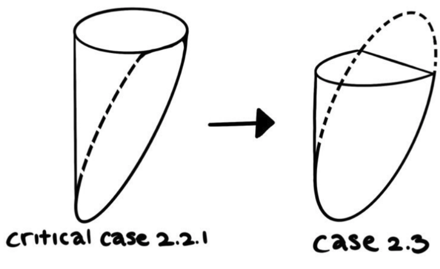


FIGURE 14. Critical Case 2.2.1

As established in figure 6, the minor and major radii of the ellipse are r and M. Using the formula for the area of the ellipse, $B_2(d_2, \alpha)$ is simply defined as:

$$B_2(d_2, \alpha) = \pi M r \tag{17}$$

Case 2.3: $h < s \leq 2r \tan\alpha + h$
Notice that case 2.3 is exactly the same situation as case 1.3 (see figures 10 and 12). Hence, $C_2(d_3, \alpha)$ is:

$$C_2(d_3, \alpha) = 2\cos\alpha \int_{-M}^{M-(D_2+d_3)} \sqrt{M^2 - y^2}\, dy \tag{18}$$

Therefore, when $2r\tan\alpha < h$, we can express the contact area as a function of drawing distance as by combining equations 16, 17 and 18 to get:

$$S(d, \alpha) = \begin{cases} 2\cos\alpha \displaystyle\int_{M-d}^{M} \sqrt{M^2 - y^2}\, dy & 0 \leq d \leq 2M \\ \pi M r & 2M < d \leq D_2 \\ 2\cos\alpha \displaystyle\int_{-M}^{M+D_2-d} \sqrt{M^2 - y^2}\, dy & D_2 < d \leq 2M + D_2 \end{cases} \tag{19}$$

where $D_2 = \dfrac{h}{\sin\alpha}$, since $\dfrac{h}{\sin\alpha}$ is the maximum of d_2. Figure 15 shows a plot for case 2.

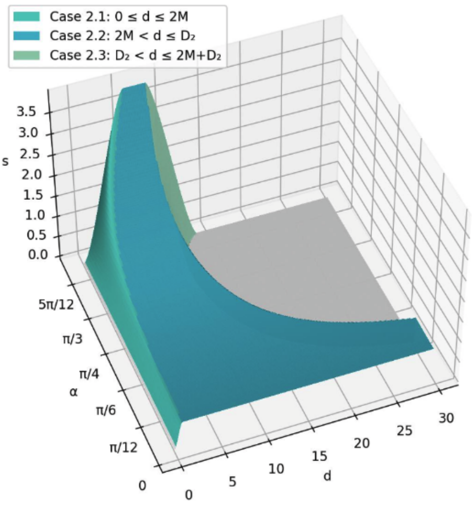


FIGURE 15. Plot of Case 2 for $r = 0.45, h = 8$

3.4. **Three Dimensional Orientation.** It is worth noting that the premise of this problem is three-dimensional. One might reason that "angle" must be described with two angle parameters α and β, similar to spherical coordinates. Consider a chalk rotated at an arbitrary angle α on a chalkboard that is the xy plane. Varying the angle β would be the same as rotating our frame of reference around the z axis. This would not change S, meaning β does not need to be considered at all. Hence, S has been reduced from a function with three parameters to a binary function.

4. Experiment

4.1. **Methodology.**

4.1.1. *Materials and Apparatus.* Chalks with a radius of 0.45 $\pm$ 0.05 cm and a height of 8.0 $\pm$ 0.05 cm were selected to carry out this experiment. The chalks came from a Primo (Morocolor) "100 colored chalk" box [7], meaning that each chalk used had roughly the same physical properties. The chalkboard used had a length of 41.0 $\pm$ 0.05 cm, and it was placed on a table 64.0 $\pm$ 0.05 cm above the ground. Further, a fishing string, a pulley, and a slotted mass set (40 $\pm$ 0.4 g) were used to ensure the net force applied to the apparatus was controlled. Additionally, a CMF Phone 1 was used to record the sounds produced by the chalk.
To ensure that the chalk was held at a fixed angle during each trial, a custom 3D-printed cart was made (figure 16). This cart also regulated normal force via gravitational loading. The cart (7.30 cm $\times$ 9.45 cm $\times$ 5.56 cm) was modeled in TinkerCAD and printed with PLA (0.15 mm layer height, 100% infill).
Post-processing included sanding the clamp protrusions with sandpaper to achieve a tolerance of $\pm$ 0.05 cm, minimizing friction. The clamp is free to move vertically within the chassis, hence applying a constant normal force due to gravity. The wheels reduce sliding friction and noise against the writing surface. The two coaxial holes on opposing sides of the chassis allow for a string to be threaded through, which can be pulled, constraining the cart's motion to be parallel to the writing plane.
During the experiment, we noticed that the cart would tip forward when pulled at a high speed, as the point we pulled was above the cart's center of gravity. To counteract this, we stuck four 10 g masses (total 40 g) on the back of the cart. The link to the CAD file can be found here.

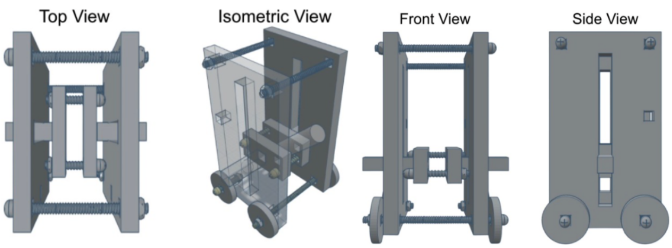


FIGURE 16. Diagram of 3D-printed cart

4.1.2. *Method of Measurement.* During the experiment, we wanted the contact area to remain constant over the course of the recording. Hence, we cut part of the chalk at a certain angle such that the base would be an ellipse with the minor axis to be the radius. For the range of independent variables, we changed the major radius of the ellipse from 0.9 to 1.3 cm with 0.1 cm intervals, such that the contact areas corresponded to 2.54, 2.83, 3.11, 3.39, 3.68 cm^2 respectively. To change the major radius, the chalk was shaved (by repeatedly drawing at an angle) until the major radius reached the desired length.
We then place the processed chalk at the angle at which the base of the ellipse is parallel to the drawing surface in contact. The slotted mass set is tied to the cart with a fishing string and is suspended in the air initially. The mass is then released, applying a force to the cart, allowing it to travel the full length of the board. The sound produced is then recorded and processed.
We controlled the initial distance from the recording device to the cart. To control this, we placed a phone on the left side of the chalkboard, with the microphone facing the right, where the cart would travel. We then placed the cart along the width of the chalkboard, such that its travel path (parallel to the board) would not interfere with the recording device. The exact placement of the recording device and cart can be seen in figure 17.

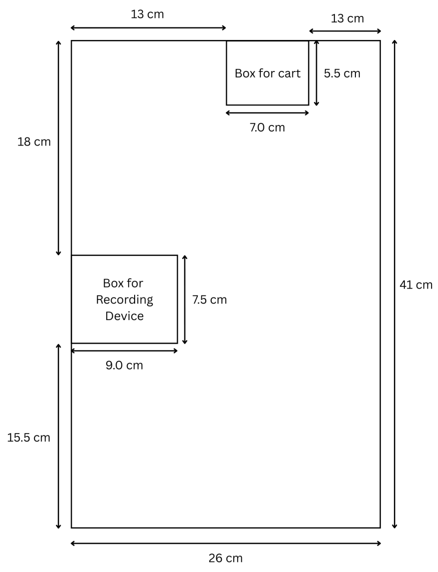


FIGURE 17. Chalkboard Dimensions

Moreover, we noticed that the wheels of the apparatus would also make a low frequency (comparatively to the chalk) sound, the recording was a superposition of the sound that the chalk produced and the sound the wheels produced. To ensure that the latter does not influence our data, we used Audacity's spectral editing function to reduce the lower frequencies of the audio (see figure 18 below).

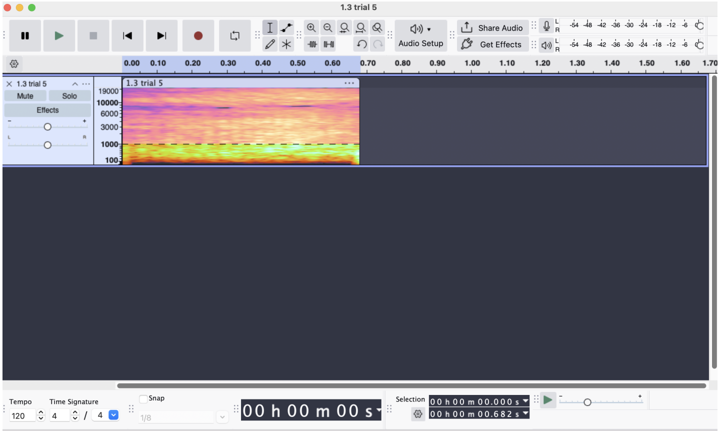


FIGURE 18. Sound Processing in Audacity

After filtering the audio, we processed it through MATLAB for sound analysis and obtain the 4 psychoacoustic metrics for each trial. We then used equation 6 to calculate the annoyance of each trial. All this data can be found in Appendix B.

4.2. **Psychoacoustic Annoyance Function.**

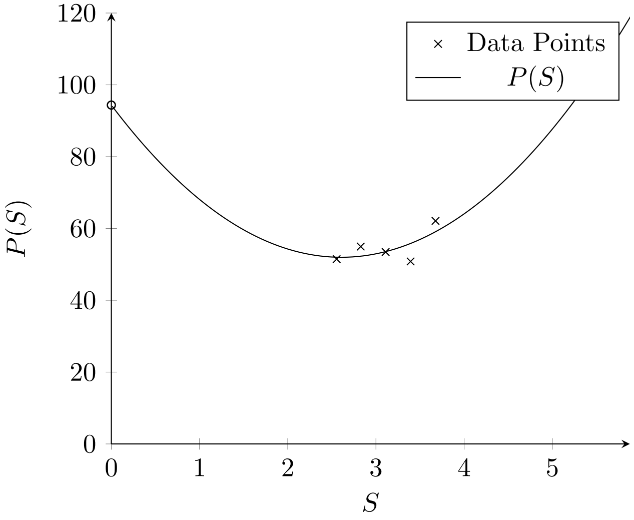
4.2.1. *Instantaneous Annoyance.* The dimensions of the chalk and the rate at which it crumbled made collecting data for small values of S very difficult (that is, it was difficult to control the contact area). A challenge that arose when creating the function was what type of function should be used to create a regression that made the most sense physically.

We decided to use second-order polynomial regression. Polynomial functions are approximations that are much easier to use than other functions, such as exponential or trigonometric functions. This makes it easier to analyze the function in section 5. The polynomial cannot be even-ordered, since the minimum would tend towards negative infinity. Upon observing the formula for annoyance, it is obvious that annoyance cannot be lower than zero. Therefore, the polynomial must have a global minimum greater than zero. Further, using a higher degree polynomial, such as a fourth or sixth-degree, would lead to absurdly large and uninterpretable coefficients, ultimately overfitting the data.

However, at $S = 0$, the chalk is not in contact with the board, contrary to the model's non-zero P-intercept. Therefore, the model has a domain of $0 < S \leq \frac{\pi r}{2}\sqrt{4r^2 + h^2}$ (See section 5.2). The following is the regression function:

$$P(S) = aS^2 + bS + c,\ \ 0 < S \leq \frac{\pi r}{2}\sqrt{4r^2 + h^2} \tag{20}$$

where $a \approx 6.23452, b \approx -32.51125, c \approx 94.3618$. The plot below shows $P(S)$ along with the data points. The raw data can be seen in Appendix B.



4.2.2. *Accumulated Annoyance.* Alternatively, given the radius and height of the chalk, one could also seek to find which α minimizes or maximizes overall annoyance. Specifically, we can plot a function of annoyance to the drawing distance d, and integrate to calculate overall annoyance. We do this because it can be reasoned that an annoying sound that lasts over a period of time is more annoying than a sound that is instantaneously annoying. Since d can be thought of as "time", we can sum the annoyance over d.
We note that the function of annoyance, $T_{A_1}(d,\alpha)$, can be expressed as at any given drawing distance d as follows:

$$T_{A_1}(d,\alpha) = a[A(d,\alpha)]^2 + b[A(d,\alpha)] + c \tag{21}$$

where $A(d,\alpha)$ could apply to both $A_1(d,\alpha)$ or $A_2(d,\alpha)$. The same idea as equation 21 applies to for $T_{B_1}(d)$ and $T_{C_1}(d)$. Now, we can define new functions T_{A_2}, T_{B_2}, and T_{C_2}, which represents the accumulated annoyance up to a drawing distance d.

$$T_{A_2}(d,\alpha) = \int_0^d T_{A_1}(x.\alpha)\ dx \tag{22}$$

$$T_{B_2}(d,\alpha) = T_{A_2}(D_1,\alpha) + \int_{D_1}^d T_{B_1}(x,\alpha)\ dx \tag{23}$$

$$T_{C_2}(d) = T_{B_2}(2M,\alpha) + \int_{2M}^d T_{C_1}(x,\alpha)\ dx \tag{24}$$

Note that if T_{B_2} were to be undefined because $h = 2r\tan\alpha$, $T_{C_2}(d,\alpha) = A_2(D_1,\alpha) + \int_{2M}^d T_{C_1}(x,\alpha)\ dx$.

5. Analysis

5.1. Minimizing P. Instantaneous Annoyance for Case 1:

In case 1, to find the angles α that allow the chalk to achieve the minimum psychoacoustic annoyance, $P_{\min}$, we must identify when the maximum possible contact area $S_{\max}(d, \alpha)$ intersects the value $S_0 = P_{\min} = -\frac{b}{2a}$. At any given α, $S_{\max}(d, \alpha)$ is given by the function $B_1(d, \alpha)$ (equation 13) evaluated at the d that maximizes B_1. This critical distance is found at $d = M + \frac{h}{2\sin\alpha}$. This value represents the midpoint of the "sliding phase" (case 1.2), where the elliptical contact area is largest. Therefore, the maximum contact area for a given α is:

$$S_{\max}(d, \alpha) = B_1\left(M + \frac{h}{2\sin\alpha}, \alpha\right) \tag{25}$$

We define α_{upper} as the solution to the equation:

$$B_1\left(M + \frac{h}{2\sin(\alpha_{\text{upper}})}, \alpha\right) = S_0 \tag{26}$$

The angle α_{upper} serves as a threshold. For any $\alpha > \alpha_{\text{upper}}$, $S_{\max}(d, \alpha) < S_0$. Since $S(d, \alpha)$ begins and ends at 0 and is continuous, its maximum never reaches S_0. Hence, the minimum annoyance $P_{\min}$ is unattainable for these steeper angles. Conversely, as α decreases, $S_{\max}(d, \alpha)$ increases. For any $\alpha < \alpha_{\text{upper}}$, we have $S_{\max}(d, \alpha) > S_0$ and $S(d, \alpha_{\text{upper}}) = S_0$.

The intermediate value theorem guarantees that the horizontal line $S = S_0$ will be intersected at least once (at most twice due to symmetry), meaning the chalk achieves the minimum annoyance at least once during the drawing process. The lower bound α_{lower} is imposed by the geometric condition of case 1 itself, which requires $h \leq 2r\tan\alpha$. Solving for α gives the smallest angle for which the case 1 model is valid:

$$\alpha_{\text{lower}} = \tan^{-1}\left(\frac{h}{2r}\right) \tag{27}$$

Therefore, the set of angles α for which the chalk can produce the minimum annoyance at least once is the interval bounded by these geometric and analytic constraints:

$$Q = \{\alpha : \alpha_{\text{lower}} \leq \alpha \leq \alpha_{\text{upper}}\} \tag{28}$$

We begin by finding all the pairs of h and r where there exists an α_{upper}, we will be ignoring all pairs of h and r that don't meet this requirement, as they will never produce the minimum annoyance, no matter how we adjust α. We can then derive the set of pairs of r and h where the maximum contact area exceeds S_0. In short, r and h are bounded by inequality 29 (see Appendix C for calculations and definitions for U and V).

$$\cos^2\alpha \leq \frac{-V + \sqrt{V^2 + 4U}}{2U} \tag{29}$$

This would mean α_{upper} is the principal solution to equation 30.

$$\cos^2 \alpha_{\text{upper}} = \frac{-V + \sqrt{V^2 + 4U}}{2U} \tag{30}$$

Inequality 29 is plotted in figure 19.

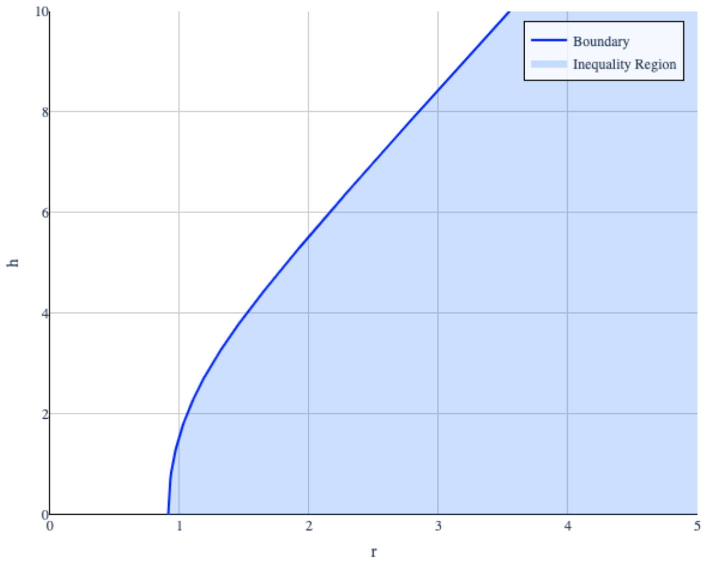


FIGURE 19. $\cos^2 \alpha \leq \dfrac{-V + \sqrt{V^2 + 4U}}{2U}$ for $\alpha = \dfrac{\pi}{4}$

Instantaneous Annoyance for Case 2

On the other hand, if the chalk were to be classified as case 2, we also use the fact that the vertex of $P(S)$ is found at $x = S_0$. Again, we notice the maximum contact area at a given angle is given by $B_2(d, \alpha) = \pi M r$. Therefore, we solve the angle such that:

$$\frac{\pi r^2}{\cos \alpha} = -\frac{b}{2a} \tag{31}$$

$$\alpha = \cos^{-1}\left(-\frac{2a\pi r^2}{b}\right) \tag{32}$$

For any given α, $B_2(d, \alpha)$ is constant, meaning the chalk is the least annoying for the longest period of "time". We define the α solved in equation 32 to be the least annoying angle (for this case), as opposed to if an angle produced a sound minimally annoying for merely an instant.

Accumulated Annoyance

Since annoyance is always nonnegative, the accumulated annoyance must also be a strictly increasing function. Therefore, we only need to consider the end point of the function, which we need to evaluate $T_{C_2}(2M + \frac{h}{\sin\alpha}, \alpha)$. Given the dimensions of the chalk r and h, we can calculate the α that produces the least accumulated annoyance by solving equation 33.

$$\frac{d}{d\alpha}\left[T_{C_2}\left(2M + \frac{h}{\sin\alpha}\right)\right] = 0 \tag{33}$$

However, due to the piecewise and integral nature of the functions $A(d,\alpha)$, $B(d,\alpha)$, and $C(d,\alpha)$, it is unreasonable to calculate an analytical closed-form solution for α in terms of a general r and h. Consequently, the accumulated annoyance problem must be solved numerically. For any given set of parameters of the chalk, the function $T_{C_2}(d,\alpha)$ can be evaluated through numerical integration. We can then apply numerical optimization algorithms, such as a gradient descent over the range of α to find the value that minimizes $T_{C_2}(d,\alpha)$.

5.2. **Maximizing** $\boldsymbol{P}$**.** The chalk has its maximum contact area with the board when the contact area is a full ellipse and its major axis is the diagonal of the chalk (figure 20). Notice that the sides of the right-angled triangle are r and $\frac{h}{2}$, meaning the major radius is simply:

$$M = \sqrt{r^2 + \left(\frac{h}{2}\right)^2} = \sqrt{r^2 + \frac{h^2}{4}} \tag{34}$$

Further, the minor radius is r, meaning the maximum area of the ellipse is:

$$S_{\max} = \pi M r = \frac{\pi r}{2}\sqrt{4r^2 + h^2} \tag{35}$$

Since P is quadratic, $\lim\limits_{S\to 0^+} P = c$. The reason a right-handed limit is taken is because $P(S)$ is only defined over the set $\{S : 0 < S \le \frac{\pi r}{2}\sqrt{4r^2 + h^2}\}$.

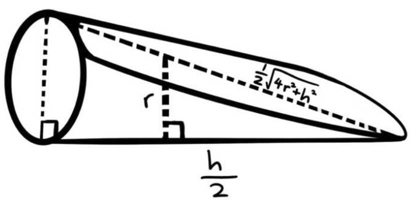


FIGURE 20. Maximum Possible Contact Area

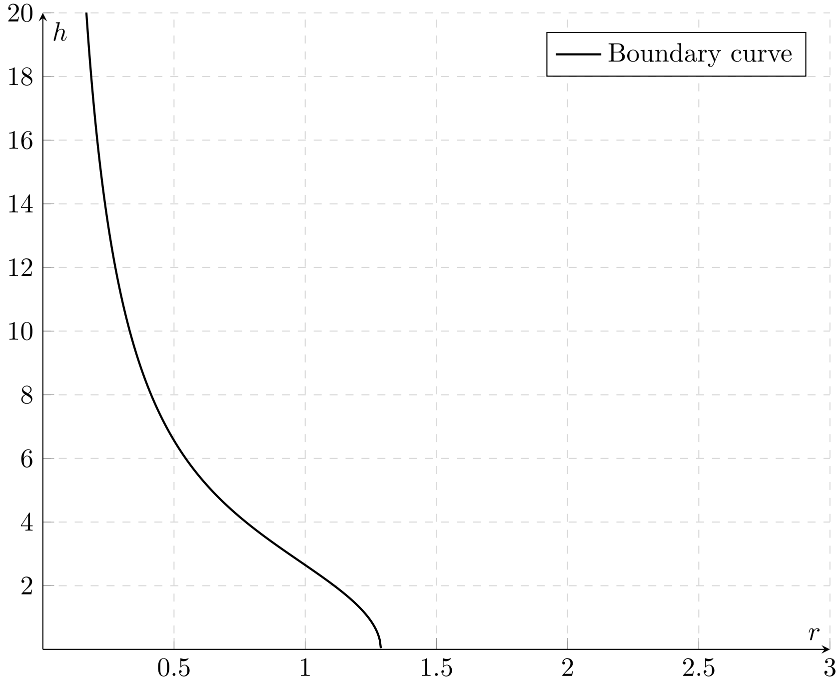
Notice that for $S_{\max} \le -\frac{b}{a}$, $P_{\max}$ is the P-intercept of $P(S)$. Since $S_{\max} = \frac{\pi r}{2}\sqrt{4r^2 + h^2}$, we can solve:

$$\frac{\pi r}{2}\sqrt{4r^2+h^2} \leq -\frac{b}{a} \tag{36}$$

$$h^2 \leq \frac{4b^2 - 4a^2\pi^2 r^4}{a^2\pi^2 r^2} \tag{37}$$

If the chalk in question had dimensions that satisfied inequality 37, $P_{\max} = c$; if else, then $P_{\max} = P(S_{\max})$. Visually, this fact can be interpreted as any point in the region to the left of the curve (seen below) will satisfy the condition $P_{\max} = c$. A plot of this inequality can be found in Appendix D (titled "Case 1: Minimising P").

Boundary curve for $h^2 \leq \dfrac{4b^2 - 4a^2\pi^2 r^4}{a^2\pi^2 r^2}$



When $P_{\max} = c$, the calculation for the corresponding α is the same as the one done to get to equation 30, except the right-hand-side of inequality 39 is $-\frac{b}{a}$ and instead of being an inequality, it is an equality. This calculation is trivial, as it follows the same method as outlined in Appendix C.

For our experiment, inequality 37 does not hold, meaning $S > -\frac{b}{a}$. In that case, we simply have $\alpha = \frac{\pi}{2} - \tan^{-1}\left(\frac{2r}{h}\right)$, which is the condition such that this maximum contact area exists at some d. This method can be applied to both case 1 chalks and case 2 chalks.

6. Discussion

6.1. **Evaluation of Experiment.** Several methodological limitations must be taken into account when interpreting the results. Our regression analysis relied on 5 data points. A more extensive data set would have helped produce a more accurate model. Ideally, a minimum of 10 spread-out data points would be used to cover a range of values. Further, it is possible that the model we use is not correct, i.e., the function might not be a polynomial. Using different types of models would provide a more accurate representation of the relationship between angle and psychoacoustic annoyance.

Another significant limitation is the failure to isolate the sound produced by the chalk. Our methodology of using a cart to hold the chalk resulted in a lot of sound being produced by the wheels. We attempted to filter out this sound by identifying its frequency, and using spectral filtering to dampen the lower frequencies. However, this process lowers the reliability of our data, as we may not have fully eliminated the wheel noise and dampened some of the lower-end noise produced by the chalk. This contamination affects the psychoacoustic metrics, especially those linked to lower frequencies, such as fluctuation strength and roughness. This systematic error means the sound might not reflect the true value, resulting in inaccuracy. To improve, our current wheel assembly could be replaced with vibration-damping bearings and rubber mounts to minimize the noise generated by the apparatus. We could also attach two microphones, one next to the chalk and another next to the wheels. We could then use destructive interference or spectral subtraction to remove or reduce the wheel sound from the data, resulting in a purer chalk sound, which would yield more accurate results.

Additionally, we had assumed the chalk's material to be perfectly homogeneous and would crumble at a constant rate, making perfectly flat areas of contact. However, in real life, chalks are quite brittle and contain small inconsistencies in their structure. This can lead to non-uniform crumbling, causing the area of contact to differ from the one derived from our formula. This discrepancy may be significant in smaller areas of contact, as the model is more sensitive in those areas. As a result, our model may not be as accurate for small values of S.

Finally, although the experiment was conducted under laboratory conditions, ambient noise was not necessarily fully negated. This could have increased systematic error, reducing the accuracy. Preferably, this experiment should be redone in an anechoic environment to minimize ambient noise. Despite the many limitations of the accuracy of the experiment, the data collected and model created were sufficient "ballpark" estimates for around where the α values of interest might be.

6.2. **Further Work.** Throughout the experiment, we assumed that the chalk crumbles at a uniform rate and the contact area remains elliptical. In fact, the real-life chalk behavior should involve irregular breakage as well as non-elliptical contact patches that should certainly be considered in order to obtain a more precise relationship function.

In the experiment, we mainly focused on how the angle of contact affects the annoyance of the sound it makes. However, another primary factor that could affect the annoyance produced by chalk is its resonant frequency. Typically, the harmonics of the sound produced by the chalk are around the multiples of the lowest resonant frequency. Hence, changing the type of chalk will result in the pitch of the sound produced changing and consequently the annoyance [8]. A piece of chalk's resonant frequency is determined by several physical quantities such as mass, stiffness, and elasticity. Further research could be conducted on how resonant frequency might affect annoyance.

Further, our model breaks down at values very close to 0, since when the contact area gets too small, a sound will not be produced at all. Therefore, our model must have some non-zero minimum boundary that we do not have the capacity to calculate. Perhaps further work can be done to determine such a minimum boundary.

Finally, a purely theoretical framework is much more favorable than a derived equation from the experimentation and regression process, hence further work may focus on connecting the experiment and theory to form a full picture.

7. Conclusion

Findings. In this paper, we discussed the problem of relating the annoyance of the sound produced when changing the angle at which we draw with a chalk. We calculate a function $S(d, \alpha)$, split into two cases depending on the dimensions of the chalk, that relates the distance the chalk has drawn and its angle to the chalkboard to the area of contact. We conduct an experiment to relate S to the annoyance quantity P, defined in Zwicker's psychoacoustic model [1]. We then analytically derive the set of α values that produce the minimum and maximum P and specify the set of (r, h) values for which a minimum exists. We also define a new metric to quantify annoyance: the accumulated annoyance $T_{C_2}(d, \alpha)$.

Limitations. Our experiment has much room for improvement. More trials could have been conducted and better methods could have been used to isolate the sound of the chalk. Our model also assumed that a the chalk is a perfect cylinder that all the possible intersections indeed form an ellipse E. The regression function P that we calculated may not be perfect, but we believe this paper has given a ballpark estimation to work with.

Acknowledgements

We would like to thank our teacher supervisor (and former math teacher) Mr. David Bian for his continued support and invaluable advice. We would also like to thank our physics teacher Mr. Christopher Guy and our lab assistants Ms. Rachel Tang and Ms. Carrie Chow for their supervision of our experiment.

We would like to acknowledge Kyan Cheung for letting us use his MATLAB subscription to process our experiment data. Finally, we would like to thank Elise Chow for drawing all the diagrams. Thank you for bearing with us.

Your help has greatly benefited us and without you, this paper could not have been written.

Appendix A. Perceived Loudness Look-up Table

Table 1. Look-up table for Perceived Loudness [2]

Frequency (Hz)	α_f	L_U (dB)	T_f (dB)
20	0.635	−31.5	78.1
25	0.602	−27.2	68.7
31.5	0.569	−23.1	59.5
40	0.537	−19.3	51.1
50	0.509	−16.1	44.0
63	0.482	−13.1	37.5
80	0.456	−10.4	31.5
100	0.433	−8.2	26.5
125	0.412	−6.3	22.1
160	0.391	−4.6	17.9
200	0.373	−3.2	14.4
250	0.357	−2.1	11.4
315	0.343	−1.2	8.6
400	0.330	−0.5	6.2
500	0.320	0.0	4.4
630	0.311	0.4	3.0
800	0.303	0.5	2.2
1000	0.300	0.0	2.4
1250	0.295	−2.7	3.5
1600	0.292	−4.2	1.7
2000	0.290	−1.2	−1.3
2500	0.290	1.4	−4.2
3150	0.289	2.3	−6.0
4000	0.289	1.0	−5.4
5000	0.289	−2.3	−1.5
6300	0.293	−7.2	6.0
8000	0.303	−11.2	12.6
10 000	0.323	−10.9	13.9
12 500	0.354	−3.5	12.3

Appendix B. Experiment Results

Table 2. Audio Perception Metrics Across Folders and Trials

M (cm)	Trial	Loudness (dB)	Fluctuation (vacil)	Roughness (asper)	Sharpness (acum)	Annoyance
0.9	1	39.470442	0.000001	0.216896	1.933707	43.477883
0.9	2	43.878028	0.000003	0.241736	2.027222	49.966365
0.9	3	45.080413	0.000035	0.232779	1.830992	48.468418
0.9	4	59.461671	0.000003	0.197912	1.599216	64.566874
0.9	5	45.859890	0.000002	0.215406	1.955933	50.843819
1.0	1	44.793084	0.000193	0.209182	2.119052	52.462276
1.0	2	47.180290	0.000796	0.159729	2.299296	58.760585
1.0	3	52.891036	0.000718	0.187890	2.262741	65.372949
1.0	4	41.419929	0.000000	0.191945	1.984357	46.188738
1.0	5	45.942264	0.000475	0.148293	2.036701	52.013032
1.1	1	50.028126	0.001288	0.179946	2.148273	59.225396
1.1	2	45.655283	0.009597	0.149774	2.075526	52.448773
1.1	3	47.815622	0.000002	0.139799	2.113181	55.688646
1.1	4	45.719747	0.000279	0.196858	1.989436	51.137875
1.1	5	43.721950	0.004398	0.135072	1.998940	48.741530
1.2	1	45.788235	0.000000	0.153051	2.199118	54.984278
1.2	2	38.110566	0.000111	0.235852	2.117352	44.605429
1.2	3	42.646510	0.002945	0.188548	2.078983	49.132122
1.2	4	47.459204	0.000000	0.110927	2.185958	56.677614
1.2	5	41.205110	0.000048	0.186279	2.154463	48.679626
1.3	1	52.006985	0.001236	0.222855	2.194903	62.838206
1.3	2	51.265342	0.000009	0.155230	2.181556	61.382593
1.3	3	49.697646	0.000000	0.195500	2.304404	62.217433
1.3	4	50.233933	0.000094	0.153441	2.247474	61.551081
1.3	5	50.884543	0.003481	0.204135	2.248372	62.552385

Table 3. Mean Annoyance

M (cm)	S (cm^2)	Mean Annoyance
0.9	2.5	51.4647
1.0	2.8	54.9595
1.1	3.1	53.4484
1.2	3.4	50.8158
1.3	3.7	62.1083

Appendix C. Calculation For Section 5.1

We begin by plugging in $d = M + \frac{1}{2}D_1$ into $B_1(d, \alpha)$ (for $\alpha = \alpha_{\text{upper}}$, $D_1 = \frac{h}{\sin \alpha_{\text{upper}}}$), and we set it to be an inequality because we are solving for all pairs of h and r where the maximum contact area can be equal to or above the contact area required to produce the minimum annoyance.

$$B_1\left(M + \frac{1}{2}D_1, \alpha\right) = 2\cos\alpha \int_{M-\left(M+\frac{1}{2}D_1\right)}^{M+D_1-\left(M+\frac{1}{2}D_1\right)} \sqrt{M^2 - y^2}\, dy \geq -\frac{b}{a} \tag{38}$$

Since $B_1(d, \alpha)$ is an even function, after simplifying, we get:

$$4\cos\alpha \int_0^{\frac{1}{2}D_1} \sqrt{M^2 - y^2}\, dy \geq -\frac{b}{a} \tag{39}$$

The indefinite integral gives:

$$\int \sqrt{M^2 - x^2}\, dx = \int \sqrt{M^2 - M^2 \sin^2\theta}\; M\cos\theta\, d\theta \tag{40}$$

$$= \int M^2 \cos^2\theta\, d\theta = \frac{M^2}{4}\sin 2\theta + \frac{M^2}{2}\sin\theta \tag{41}$$

$$= \frac{M^2}{4}\sin\left(2\sin^{-1}\frac{x}{M}\right) + \frac{M^2}{2}\sin^{-1}\frac{x}{M} + C = K(x) \tag{42}$$

Hence,

$$4\cos\alpha \int_0^{\frac{1}{2}D_1} \sqrt{M^2 - y^2}\, dy = 4\cos\alpha \cdot K(x)\Big|_{y=0}^{\frac{1}{2}D_1} \geq -\frac{b}{a} \tag{43}$$

$$4\cos\alpha\left[K\left(\frac{1}{2}D_1\right) - K(0)\right] \geq -\frac{b}{a} \tag{44}$$

Since $K(0) = 0$, this simplifies to:

$$4\cos(\alpha)\cdot K\left(\frac{h}{2\sin\alpha}\right) \geq -\frac{b}{a} \tag{45}$$

We then substitute $\alpha = \sin^{-1}\frac{h}{2M}$ into the equation to get:

$$4\cos\left(\sin^{-1}\frac{h}{2M}\right)K(M) \geq -\frac{b}{a} \tag{46}$$

Note that $K(M)$ is like so:

$$K(M) = \frac{M^2}{4}\cdot\sin\left(\frac{\pi}{2}\cdot 2\right) + \frac{M^2}{2}\cdot\frac{\pi}{2} = \frac{M^2\pi}{4} \tag{47}$$

Substituting into the previous inequality yields:

$$M^2\pi\cos\left(\sin^{-1}\frac{h}{2M}\right) \geq -\frac{b}{a} \tag{48}$$

We then use the identity $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$ to simplify the expression (given that $2r\tan\alpha > h \iff \frac{h}{2M} \leq 1$):

$$M^2\pi\sqrt{1-\left(\frac{h}{2M}\right)^2} \geq -\frac{b}{a} \tag{49}$$

After substituting $M = \frac{r}{\cos\alpha}$, we can now express the inequality in terms of h, r, and α:

$$\frac{r^2\pi}{\cos^2\alpha}\sqrt{1-\frac{h^2\cos^2\alpha}{4r^2}} \geq -\frac{b}{a} \tag{50}$$

We then move $\cos^2\alpha$ to the other side and square both sides to get rid of the square root:

$$r^4\pi^2\left(1-\frac{h^2\cos^2\alpha}{4r^2}\right) \geq \frac{b^2\cos^4\alpha}{a^2} \tag{51}$$

We then want to move everything to one side to get a quadratic equation:

$$\frac{b^2\cos^4\alpha}{a^2r^4\pi^2} + \frac{h^2\cos^2\alpha}{4r^2} - 1 \leq 0 \tag{52}$$

To simplify, let $U = \frac{b^2}{a^2r^4\pi^2}$ and $V = \frac{h^2}{4r^2}$:

$$U\cos^4\alpha + V\cos^2\alpha - 1 \leq 0 \tag{53}$$

Since $U > 0$, the inequality will only hold when the value of $\cos^2\alpha$ are in between the two roots. We then apply the quadratic equation to get the following:

$$\frac{-V-\sqrt{V^2+4U}}{2U} \leq \cos^2\alpha \tag{54}$$

$$\frac{-V+\sqrt{V^2+4U}}{2U} \geq \cos^2\alpha \tag{55}$$

Note that the first inequality always holds, as the left hand side is always negative and $\cos^2\alpha$ is positive.

Appendix D. Links To Desmos

Click on these links to view the Desmos.

Case 1 3D desmos

Case 2 3D desmos

Case 1 Minimizing P

References

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EDITOR'S COMMENTS

This is an applied modeling paper that asks at what contact angle α a piece of chalk produces the most or least "annoying" sound, measuring annoyance with Zwicker's psychoacoustic model. The chalk is modeled as a cylinder pressed into a tilted board; the team derives the contact-area function $S(d, \alpha)$ geometrically, relates it to annoyance using an experiment and polynomial regression, and then finds the optimal angles. The premise is playful and original—applying a serious psychoacoustic model to such a down-to-earth question is a creative combination—and the distinction between instantaneous and accumulated annoyance is a thoughtful modeling choice.

The geometric derivation of the contact area across the two cases $2r \tan \alpha \geq h$ and $2r \tan \alpha < h$, in both two and three dimensions, is sound calculus and geometry, and combining an experimentally-fitted function with analytic optimization gives a complete modeling pipeline. Working directly with ISO 226:2023, von Bismarck's sharpness weighting, and Zwicker's formulae shows real diligence. The main things to keep in mind are methodological rather than mathematical: the central relationship between annoyance and contact area rests on a polynomial fit to experimental data, so the conclusions carry some experimental and fitting uncertainty, and the work leans more toward modeling and optimization than toward proving theorems. Overall, it is an insightful paper on mathematical modeling that demonstrated both practical and theoretical aspects of mathematics.