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A GENERALIZATION OF THE GAUSS SUM

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Abstract. This essay will analyze a function that is a generalization of the Gauss sum. The function happens to be closely related to the cycle index of the symmetric group, which will also be analyzed. Some properties of the Gauss sum will be generalized. A number-theoretic inequality is also obtained.

1. Introduction

By Theorem 33, j_χ, the main function in this thesis, is a generalization of the Ramanujan's sum, the Gauss sum, and the coefficient of the cyclotomic polynomial. [See reviewer's comment (1)] It is hoped that the properties of them can be understood better through the investigation of j_χ.

Some notation that is used in the report is introduced at the beginning. Then, the main theorems, especially the equation connecting j_χ and the cycle index of the symmetric group, are presented. After that, properties of the cycle index of the symmetric group and its relationship with the Gaussian binomial coefficient are displayed. Some properties of the Gauss sum related to its separability and primitive Dirichlet characters are generalized. Finally, some special values of j_χ are evaluated. An inequality version of a well-known equation connecting the mobius function and the Euler totient function will also be demonstrated.

2. Notation

This essay will use the following notation.

Notation 1. $\mathbb{N}$ *is the set of positive integers,* $\mathbb{C}$ *is the set of complex numbers, and* $\emptyset$ *is the empty set.*

Notation 2. *$\phi(n)$ is the Euler totient function.*

Notation 3. *$\mu(n)$ is the Mobius function.*

Notation 4. *$\chi(n)$ is a Dirichlet character and $\chi_1(n)$ is the principal Dirichlet character. χ^m represents the Dirichlet character for which $\chi^m(n) = (\chi(n))^m$ for all $n \in \mathbb{Z}$. [See reviewer's comment (2)]*

Notation 5. *ζ_k is $e^{2\pi i/k}$.*

Notation 6. *For a Dirichlet character $\chi \mod k$, denote $G(n,\chi) = \sum_{a=1}^{k} \chi(a)\zeta_k^{an}$ as the Gauss sum associated with χ. The Ramanujan's sum $c_k(n)$ denotes $G(n,\chi_1)$.*

Notation 7. *$A_n = \{1,2,\ldots,n\}$, $A_{n,m} = \{X \subset A_n \mid X \text{ has } m \text{ elements.}\}$*

Notation 8. *For a set S, define $\alpha(S) = \sum_{x\in A} x$ and $\beta(S) = \prod_{x\in A} x$. Also, define $\alpha(\emptyset) = 0$ and $\beta(\emptyset) = 1$.*

Notation 9. *Denote the Gaussian binomial coefficient by $\begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_{r=1}^{m} \frac{q^{n-m+r}-1}{q^r-1}$.*

It is well-known that $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is a polynomial in q.

3. Relationships between Different Values of j_χ

The following function is the main function that will be investigated in this paper.

Definition 10. *Let $z \in \mathbb{C}$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, and χ be a Dirichlet character $\mod n$. Define*

$$j_\chi(z,n,m) = \sum_{S \in A_{n,m}} \chi(\beta(S)) z^{\alpha(S)}$$

[See reviewer's comment (3)]

We will also define $j(z,n,m) = j_{\chi_1}(z,n,m)$.

Remark 11. *In the following parts, unless otherwise stated, whenever $j_\chi(z,n,m)$ appears, it is understood that $z \in \mathbb{C}, n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}$, and χ is a Dirichlet character $\mod n$.*

Remark 12. *Note that we have*

$$j_\chi(z,n,m) = \begin{cases} 1 & \text{if } m = 0 \\ \displaystyle\sum_{1 \le a_1 < a_2 < \cdots < a_m \le n} \prod_{r=1}^{m} (\chi(a_r) z^{a_r}) & \text{if } 1 \le m \le \phi(n) \\ 0 & \text{if } m > \phi(n) \end{cases}$$

In the second case, the sum is taken over all $a_1, a_2, \ldots, a_m$ satisfying $1 \le a_1 < a_2 < \cdots < a_m \le n$. Note that $j_\chi(z, n, m)$ is also the coefficient of x^{n-m} in $\prod_{k=1}^{n}(x + \chi(k)z^k)$. That is, we have

$$\prod_{k=1}^{n}(x + \chi(k)z^k) = \sum_{m=0}^{n} j_\chi(z, n, m)x^{n-m}$$

Theorem 13. *For all $m \ge 1$, we have*

$$j_\chi(z, n, m) = \frac{1}{m}\sum_{l=1}^{m}(-1)^{l+1} j_{\chi^l}(z^l, n, 1) j_\chi(z, n, m-l) \tag{1}$$

Proof. We have

$$\begin{aligned}
&j_{\chi^l}(z^l, n, 1) j_\chi(z, n, m-l)\\
&= \left(\sum_{a=1}^{n} \chi^l(a) z^{l_a}\right)\left(\sum_{S \in A_{n,m-1}} \chi(\beta(S)) z^{\alpha(S)}\right)\\
&= \sum_{S \in A_{n,m-l+1}} \sum_{k \in S} z^{(l-1)k}\chi(k^{l-1})\chi(\beta(S))z^{\alpha(S)}\\
&\quad + \sum_{S \in A_{n,m-l}} \sum_{k \in S} z^{lk}\chi(k^l)\chi(\beta(S))z^{\alpha(S)}
\end{aligned}$$

This shows that the right hand side of equation 1 is a telescoping sum so we have

$$\begin{aligned}
&\frac{1}{m}\sum_{l=1}^{m}(-1)^{l+1} j_{\chi^l}(z^l, n, 1) j_\chi(z, n, m-l)\\
= &\frac{1}{m}\sum_{S \in A_{n,m}} \sum_{k \in S} \chi(\beta(S))z^{\alpha(S)}\\
&+ \frac{1}{m}\sum_{l=2}^{m} \sum_{S \in A_{n,m-l+1}} \sum_{k \in S}(-1)^{l+1} z^{(l-1)k}\chi(k^{l-1})\chi(\beta(S))z^{\alpha(S)}\\
&+ \frac{1}{m}\sum_{l=1}^{m-1} \sum_{S \in A_{n,m-l}} \sum_{k \in S}(-1)^{l+1} z^{lk}\chi(k^l)\chi(\beta(S))z^{\alpha(S)}\\
&+ \frac{1}{m}\sum_{S \in A_{n,0}} \sum_{k \in S} z^{mk}\chi(k^m)\chi(\beta(S))z^{\alpha(S)}\\
= &\sum_{S \in A_{n,m}} \chi(\beta(S))z^{\alpha(S)}
\end{aligned}$$

since the middle two terms cancel each other out and the last term is zero. [See reviewer's comment (4)] $\square$

Remark 14. *By a similar proof, we have*

$$\sum_{S\in A_{n,m}} z^{\alpha(S)} = \frac{1}{m}\sum_{l=1}^{m}(-1)^{l+1}\left(\sum_{a=1}^{n} z^{la}\right)\left(\sum_{S\in A_{n,m-l}} z^{\alpha(S)}\right)$$

for $m \geq 1$.

Let $Z_{S_m}(a_1, a_2, \ldots, a_m)$ be the cycle index of the symmetric group S_m. [See reviewer's comment (5)] By [4], we have

$$Z_{S_m}(a_1, a_2, \ldots, a_m) = \sum_{j_1+2j_2+\cdots+mj_m=m}\prod_{k=1}^{m}\frac{a_k^{j_k}}{k^{j_k} j_k!} \tag{2}$$

for $m \geq 1$ and

$$Z_{S_m}(a_1, a_2, \ldots, a_m) = \frac{1}{m}\sum_{l=1}^{m} a_l Z_{S_{m-l}}(a_1, a_2, \ldots, a_{m-l}) \tag{3}$$

where Z_{S_0} is defined as 1. Note that this recurrence relation is quite similar to equation 1. In fact, we have the following important theorem, which gives a recurrence relation of the j function, and shows that the j function is closely related to the cycle index of the symmetric group.

Theorem 15.

$$j_\chi(z, n, m) = (-1)^m Z_{S_m}(-j_\chi(z, n, 1), -j_{\chi^2}(z^2, n, 1), \ldots, -j_{\chi^m}(z^m, n, 1)) \tag{4}$$

In other words, we have

$$j_\chi(z, n, m) = \sum_{j_1+2j_2+\cdots+mj_m=m}(-1)^{m+\sum_{k=1}^{m} j_k}\prod_{k=1}^{m}\frac{j_{\chi^k}(z^k, n, 1)^{j_k}}{k^{j_k} j_k!}$$

for $m \geq 1$.

Proof. We will use induction on m. The cases for $m = 0$ and $m = 1$ are trivial.

Assume Equation 4 holds for all $m \leq k-1$ where $k \geq 2$. Then

$$\begin{aligned}
&j_\chi(z,n,k)\\
&= \frac{1}{k}\sum_{l=1}^{k}(-1)^{l+1} j_{\chi^l}(z^l,n,1) j_\chi(z,n,k-l)\\
&= \frac{1}{k}\sum_{l=1}^{k}(-1)^{l+1} j_{\chi^l}(z^l,n,1)\\
&\quad (-1)^{k-l} Z_{S_{k-l}}(-j_\chi(z,n,1), -j_{\chi^2}(z^2,n,1),\ldots,-j_{\chi^{k-l}}(z^{k-l},n,1))\\
&= (-1)^k \frac{1}{k}\sum_{l=1}^{k}(-j_{\chi^l}(z^l,n,1))\\
&\quad Z_{S_{k-l}}(-j_\chi(z,n,1), -j_{\chi^2}(z^2,n,1),\ldots,-j_{\chi^{k-l}}(z^{k-l},n,1))\\
&= (-1)^k Z_{S_k}(-j_\chi(z,n,1), -j_{\chi^2}(z^2,n,1),\ldots,-j_{\chi^{k}}(z^{k},n,1))
\end{aligned}$$

Thus, Equation 4 is true for $m = k$ and this completes the induction. □

Remark 16. *By a similar proof, we have*

$$\begin{aligned}
&\sum_{S\in A_{n,m}} z_0^{\alpha(S)}\\
&= (-1)^m Z_{S_m}\left(-\sum_{a=1}^{n} z_0^a, -\sum_{a=1}^{n} z_0^{2a},\ldots,-\sum_{a=1}^{n} z_0^{m_a}\right)\\
&= (-1)^m \lim_{z\to z_0} Z_{S_m}\left(-\frac{z(z^n-1)}{z-1}, -\frac{z^2(z^{2n}-1)}{z^2-1},\ldots,-\frac{z^m(z^{mn}-1)}{z^m-1}\right)
\end{aligned}$$

Corollary 17. $j_\chi(\zeta_n^r,n,m) = (-1)^m Z_{S_m}(-G(r,\chi), -G(2r,\chi^2),\ldots,-G(mr,\chi^m))$. *In particular,* $\chi(\zeta_n,n,m) = (-1)^m Z_{S_m}(-c_n(1), -c_n(2),\ldots,-c_n(m))$. *Note that this expresses the coefficient of cyclotomic polynomial in terms of the Ramanujan's sum.*

[See reviewer's comment (6)]

The theorem below shows that the j function satisfies a reflection formula.

Theorem 18. *For $z \neq 0$, we have $j_\chi(z,n,m) = j_\chi(z,n,\phi(n)) j_{\overline{\chi}}(z^{-1}, n, \phi(n)-m)$.*

Proof. The case for $m > \phi(n)$ is trivial. Assume $m \leq \phi(n)$. Let $\{a_1, \dots, a_{\phi(n)}\}$ be the reduced residue system modulo n. Then

$$\begin{aligned} j_\chi(z,n,m) &= \sum_{S \in A_{n,\phi(n)-m}} \chi(a_1 a_2 \dots a_{\phi(n)}) z^{a_1+\cdots+a_{\phi(n)}} \overline{\chi(\beta(S))} z^{-\alpha(S)} \\ &= \chi(a_1 a_2 \dots a_{\phi(n)}) z^{a_1+\cdots+a_{\phi(n)}} \sum_{S \in A_{n,\phi(n)-m}} \overline{\chi(\beta(S))} z^{-\alpha(S)} \\ &= j_\chi(z,n,\phi(n)) j_{\overline{\chi}}(z^{-1}, n, \phi(n)-m) \end{aligned}$$

□

The value of $j_\chi(z,n,\phi(n))$ will be determined in Theorem **??**.

We also have the following trivial result.

Theorem 19. *If n is even, then $j_\chi(-z,n,m) = (-1)^m j_\chi(z,n,m)$.*

4. Cycle index and the Gaussian binomial coefficient

Lemma 20. *Let $g(q,n,m) = \sum_{S \in A_{n,m}} q^{\alpha(S)}$. Then we have*

$$g(q,n,m) = \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m+1)}{2}}$$

Proof. A proof is provided in [3]. A slightly different proof will be provided here. Define

$$f(q,n,m) = \sum_{0 \leq a_1 \leq \cdots \leq a_m \leq n} q^{a_1+\cdots+a_m}$$

Then, we have

$$\begin{aligned} f(q,n,m) &= \sum_{r=0}^{n} \sum_{0 \leq a_1 \leq \cdots \leq a_{m-1} \leq r} q^{a_1+\cdots+a_{m-1}} q^r \\ &= \sum_{r=0}^{n} f(q,r,m-1) q^r \end{aligned}$$

We shall use induction to prove that $f(q,n,m) = \begin{bmatrix} m+n \\ m \end{bmatrix}_q$. We have

$$f(q,n,1) = 1 + q + \cdots + q^n = \begin{bmatrix} 1+n \\ 1 \end{bmatrix}_q$$

Assume $f(q,n,k) = \begin{bmatrix} k+n \\ k \end{bmatrix}_q$ where $k \in \mathbb{N}$. We need to prove that

$$f(q,n,k+1) = \begin{bmatrix} k+1+n \\ k+1 \end{bmatrix}_q$$

By the recurrence of f proved above and the induction hypothesis, it suffices to prove that

$$\begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q = \sum_{r=0}^{n} \begin{bmatrix} r+k \\ k \end{bmatrix}_q q^r$$

which is true by induction on n.

Let $b_i = a_i + i$ for $1 \leq i \leq m$. Then $0 < a_1 < \cdots < a_m \leq n$ if and only if $0 \leq b_1 \leq \cdots \leq b_m \leq n - m$. Thus,

$$\begin{aligned} g(q,n,m) &= \sum_{0<a_1<\cdots<a_m\leq n} q^{a_1+\cdots+a_m} \\ &= \sum_{0<b_1\leq\cdots\leq b_m\leq n-m} q^{b_1+\cdots+b_m+1+2+\cdots+m} \\ &= q^{\frac{m(m+1)}{2}} f(q,n-m,m) \\ &= \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m+1)}{2}} \end{aligned}$$

□

By comparing Remark 16 and Lemma 20, we can obtain the following theorem.

Theorem 21. *For all $z_0 \in \mathbb{C}$, we have*

$$\begin{bmatrix} n \\ m \end{bmatrix}_{z_0} z_0^{\frac{m(m+1)}{2}} = \lim_{z\to z_0} (-1)^m Z_{S_m}\left(-\frac{z(z^n-1)}{z-1}, -\frac{z^2(z^{2n}-1)}{z^2-1}, \ldots, -\frac{z^m(z^{mn}-1)}{z^m-1}\right)$$

Corollary 22. *For all $n \in \mathbb{C}$ and non-negative integers k, we have*

$$Z_{S_k}\underbrace{(-n,-n,\ldots,-n)}_{k \text{ copies of } -n} = (-1)^k \binom{n}{k} \tag{5}$$

Proof. This follows directly by taking $z_0 = 1$ in Theorem 21. We will present another proof below.

When $k = 0$, both sides are 1 so the equation is true. When $k \geq 1$ and $n = 0$, both sides are 0 so the equation is true. Assume $n \neq 0$. When $k = 1$, both sides are $-n$ so the equation is true. Assume the equation is true for all $k \leq m$ where $m \in \mathbb{N}$. We need to prove that Equation 5 is true for $k = m+1$. By Equation 3, it suffices to prove that

$$(-1)^{m+1}\binom{n}{m+1} = -\frac{n}{m+1}\sum_{l=1}^{m+1}(-1)^{m+1-l}\binom{n}{m+1-l}$$

which is equivalent to

$$\binom{n-1}{m} = \sum_{l=1}^{m+1}(-1)^{l+1}\binom{n}{m+1-l}$$

This is well known and can be proved easily by induction on m. $\square$

We will use the following lemma to prove that $j_\chi(z,n,m)$ is zero for some special values of z,n,m in the subsequent sections.

Lemma 23. *Let $m \in \mathbb{N}$ and $i_1,\dots,i_s$ be positive integers such that*

$$\gcd(i_1,\dots,i_s) \nmid m$$

Let $a_1,\dots,a_m \in \mathbb{C}$. Assume $a_i = 0$ for all $i \in A_m \backslash \{i_1,\dots,i_s\}$. Then $Z_{S_m}(a_1,\dots,a_m)$.

Proof. Whenever $j_1 + 2j_2 + \cdots + mj_m = m$, there exists $i \in A_m \backslash \{i_1,\dots,i_s\}$ for which $j_i \neq 0$ so $\prod_{k=1}^m \frac{a_k^{j_k}}{k^{j_k} j_k!} = 0$. Thus,

$$Z_{S_m}(a_1,a_2,\dots,a_m) = \sum_{j_1+2j_2+\cdots+mj_m=m}\prod_{k=1}^m \frac{a_k^{j_k}}{k^{j_k} j_k!} = 0$$

$\square$

5. Inequalities for j_χ

Theorem 24. *For any positive real number z and integer $n \geq 2$, we have*

$$j(z,n,m) \geq \binom{\phi(n)}{m} z^{\frac{nm}{2}}$$

Equality holds if and only if $z=1, m=0$, or $m \geq \phi(n)$.

Proof. By the arithmetic mean-geometric mean inequality,

$$j(z,n,m) \geq \binom{\phi(n)}{m}\left(z^{\binom{\phi(n)-1}{m-1}\frac{n\phi(n)}{2}}\right)^{\frac{1}{\binom{\phi(n)}{m}}} = \binom{\phi(n)}{m} z^{\frac{nm}{2}}$$

$\square$

[See reviewer's comment (7)]

By [1], we have the following theorem:

let χ be any nonprincipal character modulo n, and let f be a nonnegative function which has a continuous negative derivative $f(x)$ for all $x \geq x_0$. Then if $y \geq x \geq x_0$, we have

$$\sum_{x<a\leq y} \chi(a)f(a) = O(f(x))$$

We will slightly modify the proof of it to prove a result without the big O notation.

Lemma 25. *Let χ be any nonprincipal character modulo n, and let f be a nonnegative function which has a continuous nonnegative derivative $f(x)$ for all $x \geq x_0$. Then if $y \geq x \geq x_0$, we have*

$$\left|\sum_{x<a\leq y} \chi(a)f(a)\right| \leq \phi(n)f(x)$$

Proof. Let $\sum_{a\leq x} \chi(a) = A(x)$. By Exercise 6.15 in [1], we have

$$|A(x)| \leq \frac{\phi(k)}{2} = M$$

for all x. By Abel's summation formula, we have

$$\begin{aligned}
\sum_{x<a\leq y} |\chi(a)f(a)| &= \left|f(x)A(x) - f(y)A(y) - \int_x^y A(t)f'(t)dt\right| \\
&\leq |f(x)A(x)| + |f(y)A(y)| + \left|\int_x^y A(t)f'(t)dt\right| \\
&\leq Mf(x) + Mf(y) + M(f(x) - f(y)) \\
&= 2Mf(x) \\
&= \phi(n)f(x)
\end{aligned}$$

□

Lemma 26. *Let χ be any nonprincipal character modulo n. For any real number $z \in (0,1)$, we have $|j_\chi(z,n,1)| \leq \phi(n)z$.*

Proof. Let $r = z^{-1}$ and let ϵ be any real number less than 1. Putting

$$f(t) = r^{-t}, x_0 = 0, x = \epsilon, \text{ and } y = n$$

in Lemma 25, we have

$$|j_\chi(r^{-1},n,1)| = \left|\sum_{\epsilon<a\leq n} \frac{\chi(a)}{r^a}\right| \leq \frac{\phi(n)}{r^\epsilon}$$

Hence, $|j_\chi(r^{-1},n,1)| \leq \frac{\phi(n)}{r}$ and the result follows. □

Theorem 27. *Let $m \in \mathbb{N}$ and $z \in (0,1)$. If $\chi, \chi^2, \dots, \chi^m$ are all nonprincipal characters modulo n, then*

$$|j_\chi(z,n,m)| \le \binom{\phi(n)+m-1}{m} z^m$$

Proof. We have

$$\begin{aligned}
|j_\chi(z,n,m)| &= \left| \sum_{j_1+2j_2+\cdots+mj_m=m} (-1)^{m+\sum_{k=1}^m j_k} \prod_{k=1}^m \frac{j_{\chi^k}(z^k,n,1)^{j_k}}{k^{j_k} j_k!} \right| \\
&\le \sum_{j_1+2j_2+\cdots+mj_m=m} \prod_{k=1}^m \frac{(\phi(n)z^k)^{j_k}}{k^{j_k} j_k!} \\
&= Z_{S_m}(\phi(n),\phi(n),\dots,\phi(n)) z^m \\
&= (-1)^m \binom{-\phi(n)}{m} z^m \\
&= \binom{\phi(n)+m-1}{m} z^m
\end{aligned}$$

by Corollary 22 and Lemma 26. [See reviewer's comment (8)] □

6. Generalization of some properties of the Gauss sum

By [1], if $n \in \mathbb{Z}, k \in \mathbb{N}, \chi$ is a Dirichlet character mod k and $\gcd(n,k)=1$, then $G(n,\chi) = \overline{\chi(n)} G(1,\chi)$. We will generalize this result in Theorem 28 and 29.

Theorem 28. *Let $n \in \mathbb{Z}, k, r \in \mathbb{N}$ and let χ be a Dirichlet character mod k. If $\gcd(n,k)=1$, then $G(nr,\chi^r) = \overline{\chi^r(n)} G(r,\chi^r)$.*

Proof.

$$G(nr,\chi^r) = \sum_{a=1}^k \chi^r(a)\zeta_k^{nra} = \sum_{a=1}^k \overline{\chi^r(n)}\chi^r(na)\zeta_k^{nra}$$

As $\gcd(n,k)=1$, in $\mathbb{Z}/k\mathbb{Z}, n, 2n, \dots, kn$ is a permutation of $1,2,\dots,k$. Thus,

$$G(nr,\chi^r) = \sum_{b=1}^k \overline{\chi^r(n)}\chi^r(b)\zeta_k^{br} = \overline{\chi^r(n)} G(r,\chi^r)$$

□

Theorem 29. *Let $n \in \mathbb{Z}, k \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}$. If $\gcd(n,k)=1$, then*

$$j_\chi(\zeta_k^n, k, m) = \overline{\chi^m(n)} j_\chi(\zeta_k, k, m)$$

Proof. The equation is clearly true for $m = 0$. Assume $m \geq 1$. Then

$$\begin{aligned} & j_\chi(\zeta_k^n, k, m) \\ & = (-1)^m Z_{S_m}(-G(n,\chi), -G(2n,\chi^2), \ldots, -G(mn,\chi^m)) \\ & = (-1)^m Z_{S_m}(-\overline{\chi(n)}G(1,\chi), -\overline{\chi^2(n)}G(2,\chi^2), \ldots, -\overline{\chi^m(n)}G(m,\chi^m)) \\ & = (-1)^m \overline{\chi^m(n)} Z_{S_m}(-G(1,\chi), -G(2,\chi^2), \ldots, -G(m,\chi^m)) \\ & = \overline{\chi^m(n)} j_\chi(\zeta_k, k, m) \end{aligned}$$

□

By [1], if χ is primitive, then $G(n,\chi) = \overline{\chi(n)}G(1,\chi)$ for all $n \in \mathbb{Z}$. [See reviewer's comment (9)] We will generalize this result in Theorem 31.

Lemma 30. *Let χ be a Dirichlet character mod k. If χ^m is primitive and d mod m, then χ^d is primitive.*

Proof. Assume χ^d is imprimitive. Then there exists $t \mid k$ with $0 < t < k$ such that $\chi^d(b) = 1$ for all $b \equiv 1 \pmod t$ with $\gcd(b,k) = 1$. Thus,

$$\chi^m(b) = (\chi^d(b))^{\frac{m}{d}} = 1$$

for all $b \equiv 1 \pmod t$ with $\gcd(b,k) = 1$, which means χ^m is imprimitive, a contradiction. □

Theorem 31. *Let $m \in \mathbb{N}$ and $i_1, \ldots, i_s$ be positive integers such that*

$$\gcd(i_1, \ldots, i_s) \nmid m$$

If χ^i is primitive for all $i \in A_m \backslash \{i_1, \ldots, i_s\}$, then $j_\chi(\zeta_k^n, k, m) = \overline{\chi^m(n)} j_\chi(\zeta_k, k, m)$ for all $n \in \mathbb{Z}$.

Proof. If $\gcd(n,k) = 1$, then $j_\chi(\zeta_k^n, k, m) = \overline{\chi^m(n)} j_\chi(\zeta_k, k, m)$ by Theorem 29.

If $\gcd(n,k) \geq 2$, then

$$j_\chi(\zeta_k^n, k, m) = (-1)^m Z_{S_m}(a_1, a_2, \ldots, a_m)$$

where $a_i = -G(ni, \chi^i)$, which equals zero for all $i \in \underline{A_m}\{i_1, \ldots, i_s\}$. By Lemma 23, $Z_{S_m}(a_1, a_2, \ldots, a_m) = 0$. Thus, $j_\chi(\zeta_k^n, k, m) = 0 = \overline{\chi^m(n)} j_\chi(\zeta_k, k, m)$. □

By Lemma 30 and Theorem 31, we have the following corollary.

Corollary 32. *If χ^r is primitive for all $r > \frac{m}{2}$, then*

$$j_\chi(\zeta_k^n, k, m) = \overline{\chi^m(n)} j_\chi(\zeta_k, k, m)$$

for all $n \in \mathbb{Z}$.

7. Special values of j

Theorem 33. *We have*

1. $j(\zeta_k^n, k, 1) = c_k(n)$,
2. $j_\chi(\zeta_k^n, k, 1) = G(n, \chi)$,
3. $(-1)^m j(\zeta_n, n, m) =$ *coefficient of* $x^{\phi(n)-m}$ *in the nth cyclotomic polynomial* $\Phi_n(x)$.

Proof. Trivial. [See reviewer's comment (10)] □

By [2], Gauss has proved that

$$\prod_{r=1,\gcd(r,n)=1}^{n} r$$
$$\equiv \begin{cases} -1 \pmod n & \text{if } n = 4, p^a \text{ or } 2p^a \text{ where } p \text{ is an odd prime and } a \geq 1 \\ 1 \pmod n & \text{otherwise} \end{cases}$$

Thus, we have the following theorem.

Theorem 34.

$$j_\chi(z, n, \phi(n))$$
$$\equiv \begin{cases} \chi(-1) z^{\frac{n\phi(n)}{2}} & \textit{if } n = 4, p^a \textit{ or } 2p^a \textit{ where } p \textit{ is an odd prime and } a \geq 1 \\ z & \textit{if } n = 1 \\ z^{\frac{n\phi(n)}{2}} & \textit{otherwise} \end{cases}$$

Theorem 35. *Let* χ *be a Dirichlet character mod* n*. Let* h *be the order of* χ*, that is, the smallest* $k \in \mathbb{N}$ *such that* $\chi^k = \chi_1$*. Then,*

$$j_\chi(1, n, m) = \begin{cases} (-1)^{\frac{m(h+1)}{h}} \binom{\frac{\phi(n)}{h}}{\frac{m}{h}} & \textit{if } h \mid m \\ 0 & \textit{if } h \nmid m \end{cases}$$

Proof. We have

$$j_\chi(1, n, m) = (-1)^m Z_{S_m}(a_1, \ldots, a_m)$$

where $a_i = -\sum_{a=1}^{n} \chi^i(a)$, which equals $-\phi(n)$ if $h \mid i$ and equals 0 if $h \nmid i$. If $h \nmid m$, then $Z_{S_m}(a_1, \ldots, a_m) = 0$ by Lemma 23.

Assume $h \mid m$ and let $m = ht$. Then,

$$
\begin{aligned}
Z_{S_{ht}}(a_1, \ldots, a_{ht}) &= \sum_{j_1+2j_2+\cdots+htj_{ht}=ht} \prod_{k=1}^{ht} \frac{a_k^{j_k}}{k^{j_k} j_k!} \\
&= \sum_{r_1+2r_2+\cdots+tr_t=t} \prod_{s=1}^{t} \frac{(-\phi(n))^{r_s}}{(hs)^{r_s} r_s!} \\
&= Z_{S_t}\left(-\frac{\phi(n)}{h}, -\frac{\phi(n)}{h}, \ldots, -\frac{\phi(n)}{h}\right)
\end{aligned}
$$

The last expression has t copies of $-\frac{\phi(n)}{h}$ and equals $(-1)^t \binom{\frac{\phi(n)}{h}}{t}$ by Corollary 22.

Thus, $j_\chi(1, n, ht) = (-1)^{t(h+1)} \binom{\frac{\phi(n)}{h}}{t}$. □

From Lemma 20, we have the following theorem.

Theorem 36. *For any prime p and z, we have $j(z, p, m) = \begin{bmatrix} p-1 \\ m \end{bmatrix}_z z^{\frac{m(m+1)}{2}}$.*

Theorem 37. *For all $z_0 \in \mathbb{C}$, we have*

$$
j(z_0, n, 1) = \lim_{z \to z_0} \sum_{d \mid n} \frac{\mu(d) z^d (z^n - 1)}{z^d - 1}
$$

Proof. For all z for which $z^n \neq 1$, we have

$$
\begin{aligned}
j(z, n, 1) &= \sum_{a=1, \gcd(a,n)=1} z^a \\
&= \sum_{a=1} \sum_{d \mid \gcd(a,n)} \mu(d) z^a \\
&= \sum_{a=1} \sum_{d \mid a, d \mid n} \mu(d) z^a \\
&= \sum_{d \mid n} \sum_{m=1}^{n/d} \mu(d) z^{md} \\
&= \sum_{d \mid n} \frac{\mu(d) z^d (z^n - 1)}{z^d - 1}
\end{aligned}
$$

As $j(z, n, 1)$ is a polynomial in z, we have

$$
j(z_0, n, 1) = \lim_{z \to z_0} \sum_{d \mid n} \frac{\mu(d) z^d (z^n - 1)}{z^d - 1}
$$

□

Remark 38. *For $n \geq 2$, we have*

$$
\begin{aligned}
j(z_0, n, 1) &= \lim_{z \to z_0} \sum_{d|n} \frac{\mu(d) z^d (z^n - 1)}{z^d - 1} \\
&= \lim_{z \to z_0} \sum_{d|n} \mu(d) \left(1 + \frac{1}{z^d - 1}\right)(z^n - 1) \\
&= \lim_{z \to z_0} \sum_{d|n} \frac{\mu(d)(z^n - 1)}{z^d - 1}
\end{aligned}
$$

Corollary 39. *We have*

1. $\dfrac{\mu(n) z^n}{z^n - 1} = \sum_{d|n} \mu\left(\dfrac{n}{d}\right) \dfrac{j(z, d, 1)}{z^d - 1}$ *for* $z^n \neq 1$,
2. $\dfrac{z(z^n - 1)}{z - 1} = \sum_{d|n} j(z^{\frac{n}{d}}, d, 1)$ *for* $z \neq 1$.

Proof. For $z^n \neq 1$, we have

$$\frac{j(z, n, 1)}{z^n - 1} = \sum_{d|n} \frac{\mu(d) z^d}{z^d - 1}$$

By Mobius inversion formula, we have

$$\frac{\mu(n) z^n}{z^n - 1} = \sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{j(z, d, 1)}{z^d - 1}$$

[See reviewer's comment (11)]

For $z \neq 1$, we have

$$j(z^{\frac{1}{n}}, n, 1) = \sum_{d|n} \frac{\mu(d) z^{\frac{d}{n}} (z - 1)}{z^{\frac{d}{n}} - 1}$$

By Mobius inversion formula, we have

$$\frac{z^{\frac{1}{n}}(z - 1)}{z^{\frac{1}{n}} - 1} = \sum_{d|n} j(z^{\frac{1}{d}}, d, 1)$$

so

$$\frac{z(z^n - 1)}{z - 1} = \sum_{d|n} j(z^{\frac{n}{d}}, d, 1)$$

□

By Theorem 24 and Remark 38, we have the following theorem.

Theorem 40. *For any positive real number z_0 and integer $n \geq 2$, we have*

$$\lim_{z \to z_0} \sum_{d|n} \frac{\mu(d)(z^n - 1)}{z^d - 1} \geq \phi(n) z_0^{\frac{n}{2}}$$

Equality holds if and only if $z_0 = 1$ or $n = 2$.

Remark 41. *Note that this is a generalization of the well-known equation*

$$\sum_{d|n} \mu(d) \frac{n}{d} = \phi(n)$$

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Reviewer's Comments

In this paper under review, the author defines and studies the properties of the following function

$$j_\chi(z,n,m) = \sum_{1\le j_1<j_2<\cdots<j_m\le n} \chi\left(\prod_{i=1}^{m} j_i\right) z^{\sum_{i=1}^{m} j_i}$$

where χ is a Dirichlet character mod n, $z \in \mathbb{C}$, $n, m \in \mathbb{N}$. To justify his claim that $j_\chi(z,n,m)$ is a generalization of various objects in number theory, namely Gauss sums, Ramanujan's sums and coefficients of cyclotomic polynomials, the author derives an assortment of identities and inequalities involving j_χ which specialize to those involving the Gauss sum, etc. For instance, in §6, it is shown that j_χ satisfies

$$j_\chi(\zeta_k^n, k, m) = \overline{\chi^m(n)} j_\chi(\zeta_k, k, m)$$

where ζ_k is a primitive k-th root of unity. This identity is a generalization of the identity involving Gauss sums

$$G(nr,\chi) = \overline{\chi^r(n)} G(r, \chi^r),$$

where $G(n,\chi) := \displaystyle\sum_{a=1}^{k} \chi(a)\zeta^{an}$ for χ being mod k. Moreover, the function j_χ allows the author to deduce the 'quantum' analogue of known identities in combinatorics and number theory. For instance, Theorem 27 shows that j_χ for χ being principal is the quantum analogue of a binomial coefficient. Corollary 30 can be interpreted as the quantum analogue of the fact that the summatory function of the Euler totient function ϕ is the identity function, whereas the inequality in Theorem 31 is the quantum analogue of the Möbius inversion formula

$$\sum_{d|n} \mu(d)\frac{n}{d} = \phi(n).$$

The main tool the author utilizes in proving his results is the expression of j_χ in terms of the cycle index of the symmetric group S_m (Theorem 6). Throughout the paper, the author demonstrates his understanding of, and ability in applying, some notions and results from advanced undergraduate mathematics, notably arithmetic functions and estimate using Abel's summation formula.

While his results are interesting in its own right, the exposition of the paper under review leaves much to be desired. The introduction is poorly written, and context and motivation for the notions he wants to generalize and his main results are blatantly lacking. Rather than mentioning straight right away one of his main results (Theorem 24) in the first sentence of the introduction, which certainly puts off the non-expert reader, he should have taken a step-by-step approach in writing his introduction, beginning with the definitions of Gauss sums, etc. and their significance and applications in number theory, and then justifying his study of j_χ as a generalization of the above notions by saying a few words about Theorem 24. Very often in proofs of his results explanations are inadequate, and more remarks and connecting paragraphs should have been put in place to inform the reader his lines of thought, what the results are for and how one should best interpret his results. For instance, the reviewer is surprised that the author does not mention that Theorem 27, Corollary 30 and Theorem 31 are actually quantum analogues of known identities, even though he cites *Quantum Calculus* as one of the references which the reviewer believes was used in his proofs. Also perplexing is that while the cycle index is one of the main tools he uses in this paper, he does not mention where the cycle index arises. The following are specific mistakes the reviewer found and suggested improvements.

1. As mentioned above, give motivations and say a few words about Theorem 33.
2. Define (principal) Dirichlet characters and say that $\chi(n)$ is a Dirichlet character *modulo k*.
3. There is an inconsistent use of notation. While in Notation 4, χ is (implicitly) assumed to be modulo k, here χ is assumed to be modulo n.
4. More explanations should have been given. For example, the two summands in lines 4 and 5 of the proof actually correspond to two cases, namely $k \in S$ or $k \notin S$ for $S \in A_{n,m-\ell}$.
5. Give context and motivation for cycle index of the symmetric group.
6. Explain why $j(\zeta_n, n, m)$ is a coefficient of a cyclotomic polynomial.
7. The proof only deals with the case $m \leq \phi(n)$. How about the case $m > \phi(n)$?
8. In fact Theorem 6 is used as well.
9. Explain what 'primitive' Dirichlet character means.
10. Refrain from saying 'trivial'. The author should have at least said a few more words in the proof, even though it is trivial to him.
11. It is not at all clear to the reviewer why he can use Möbius inversion formula here, as the function $\dfrac{\mu(d)z^d}{z^d-1}$ is not multiplicative.