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WHAT CAN YOU SPLIT AND DRAW?

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Abstract. Building on 'What Can You Draw (2023)' by Florian Frick and Fei Peng, this paper generalises 'drawability', a topological framework combining unions and exclusions of regions, to arbitrary brush shapes and infinite drawing sequences, enabling the study of more complex figures. 'n-layer drawability' is introduced as a new metric to quantify the complexity of undrawable sets. Our central theoretical result establishes that a figure is undrawable if and only if its boundary contains 'containment loops' ($\in$-loops). To analyse these loops, we developed the layer drawability graph and prove its chromatic number provides a lower bound on n-layer drawability. Finally, we prove this complexity metric is unbounded. Case studies of practical drawing are also discussed.

Contents

1. Introduction

In their 2023 paper, 'What Can You Draw?'[1], Florian Frick and Fei Peng laid the groundwork for the topological study of drawability. They established the core problem of which sets in $\mathbb{R}^2$ can be constructed with a unit disk 'pencil' and 'eraser' and introduced the key concepts of strokes and finite drawing sequences. Their paper is foundational to the topic, therefore gaining familiarity with it is highly recommended before proceeding.
Building on their work, this paper introduces three principal extensions to their framework and uses these new tools to investigate a novel question about drawability. Our main contributions are:

- Generalization of the drawing tool: We move beyond the unit-disk brush to allow for a 'brush' $b \subset \mathbb{R}^2$ of any arbitrary shape. This requires replacing the original neighbourhood notation with the more general language of Minkowski sums.
- Extension to infinite steps: We expand the drawing process from a finite sequence of strokes to a countably infinite sequence, defining the final image as the set-theoretic limit of the process. This generalization allows for the construction of a richer class of sets, thereby deepening the theoretical scope of drawability within topology.
- Introduction to layer drawability: Using this generalised framework, we introduce and analyse the metric of 'n-layer drawability'- the minimum number of drawable sets whose union can form a given target image. This creates a new measure of complexity for undrawable figures.

Proofs involving technical details will be attached in the Appendix (Section 8.2). Also, AI is used throughout the paper but limited to assisting

with the polishing of language and ensuring clarity. Core research questions, mathematical insights, and proof strategies are entirely our own work.

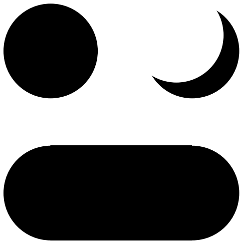


FIGURE 1A Examples of drawable sets ($b = B_1(\mathbf{0})$) provided in Frick and Peng's *'What can you draw?'*

FIGURE 1B Examples of undrawable sets ($b = B_1(\mathbf{0})$) provided in Frick and Peng's *'What can you draw?'*

1.1. Preliminaries and Definitions.

Preliminary 1.1.1 (r-ball, Boundary, Set-theoretic limit).

1. $B_r(x) = \left\{p \in \mathbb{R}^2 : |p - x| < r\right\}$ is the $\mathbb{R}^2$-ball centred at x of radius r.
2. The boundary of a set $P \subseteq \mathbb{R}^2$ follows the conventional way:
$$\partial P = \left\{x \in \mathbb{R}^2 : \forall \varepsilon > 0, \quad B_\varepsilon(x) \cap P \neq \varnothing \wedge B_\varepsilon(x) \cap P^{\complement} \neq \varnothing\right\}$$
3. A sequence of subsets $(P_n)_{n\in\mathbb{N}}$ converges to a set Q, written as $\lim_{n\to\infty} P_n := Q$, if for every point $x \in \mathbb{R}^2$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $(x \in P_n \iff x \in Q)$.

Definition 1.1.2 (Brush, Step and Stroke).

1. A brush $b \subset \mathbb{R}^2$ represents the shape of the drawing tool currently in use, and must be open and bounded.
2. The k-th step, denoted as $A_k \subseteq \mathbb{R}^2$ for $k \in \mathbb{N}$, is the set where the centre of the brush is applied during the k-th action. A sequence of complete steps to draw a set is denoted as $A = (A_n)_{n\in\mathbb{N}_1}$.
3. The k-th *stroke*, denoted by σ_k, is the region on the canvas marked by the k-th step. σ_k is chosen to satisfy the inclusion condition:
$$A_k \oplus b \subseteq \sigma_k \subseteq \overline{A_k \oplus b}$$
Here, the Minkowski sum $(A_k \oplus b)$ geometrically represents the total area covered by the brush b as its centre traces the path A_k. This formulation provides a general way to model the act of drawing with a tool of any arbitrary shape. A sequence of chosen strokes is denoted by $\sigma = (\sigma_n)_{n\in\mathbb{N}_1}$.

Remark. The freedom in defining a stroke is an intentional choice to elevate the analysis beyond the topological technicalities of open versus closed sets. This abstraction allows our framework to focus on the more fundamental geometric conflicts between the *shape of the brush* and the *boundary of the figure*. It sidesteps pathological cases where drawability hinges merely on the inclusion or exclusion of a boundary line, a scenario illustrated in Figure 3. While an analysis restricted to purely open or closed strokes is a valid mathematical exercise, this paper adopts a more general definition to prioritise the study of these shape-based constraints.

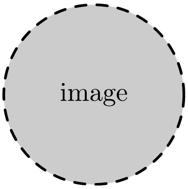


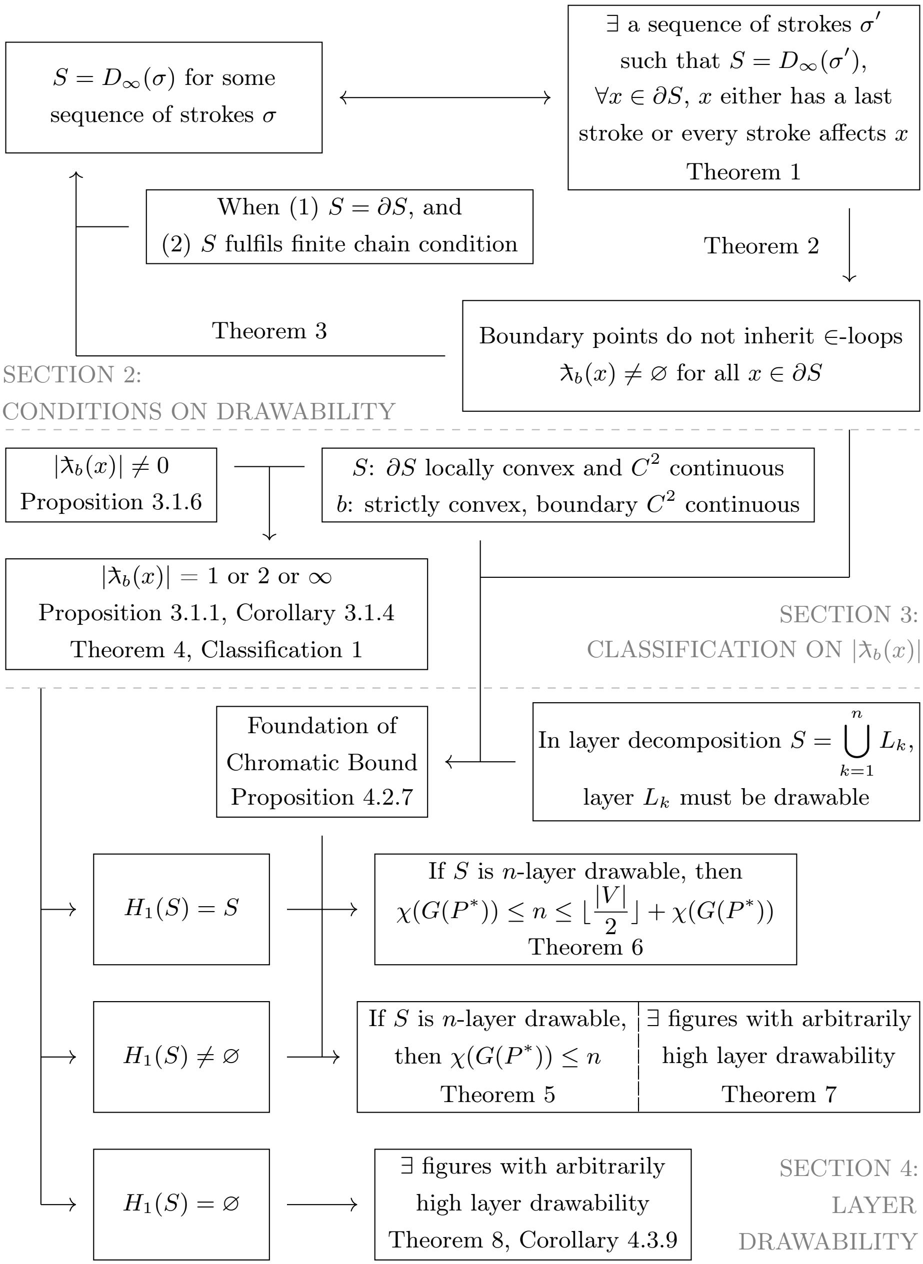
FIGURE 2. The image is $B_{0.99}(\mathbf{0})$. This is undrawable with an open unit disk, but drawable with a closed unit disk.

Definition 1.1.3 (Drawable sets)**.** Given a brush b and a sequence of strokes $\sigma = (\sigma_k)_{k\in\mathbb{N}}$. The k-stage image $D_k(\sigma)$ is defined as:

$$D_k(\sigma) := \begin{cases} \varnothing & \text{if } k = \ 0 \\ D_{k-1}(\sigma) \cup \sigma_k & \text{if } k \text{ is odd (a drawing stroke)} \\ D_{k-1}(\sigma) \setminus \sigma_k & \text{if } k \text{ is even and } k \neq 0 \text{ (an eraser stroke)} \end{cases}$$

A set $S \subseteq \mathbb{R}^2$ is said to be b-drawable if there exists a sequence of strokes σ such that $\lim_{n\to\infty} D_n(\sigma) = S$. The collection of all b-drawable sets is denoted by $\mathcal{D}_b$.
$D_\infty(\sigma) = S$ will be used as shorthand for $\lim_{n\to\infty} D_n(\sigma) = S$.

1.2. **Overview of the Theoretical Framework.** To provide a clear overview of the structure of this paper, the following flowchart illustrates the logical flow of our theoretical framework. It connects the theorems from each section, serving as a roadmap for the technical arguments that follow. Brush $b \subset \mathbb{R}^2$ is initially assumed to be open and bounded.



2. Equivalence Forms and Practical Conditions on Drawability

This section translates the abstract definitions of drawability into a rigorous topological framework, with the ultimate goal of establishing a practical test for determining whether a figure can be drawn. Our investigation focuses on the sequence of strokes at the figure's boundary, where we will show that undrawability arises from a fundamental structure: *'containment loop' ($\in$-loop).* The analysis culminates in Theorem 3, providing a set of necessary and sufficient conditions to assess a figure's drawability.

2.1. **Strokes and Their Properties.** This subsection builds the necessary vocabulary, starting with a rigorous definition of a stroke 'affecting' a point.

Definition 2.1.1 (Notion of Affecting)**.** In a sequence of strokes σ, a stroke σ_k *affects* the point $x \in \mathbb{R}^2$, denoted as the predicate $\Delta_k(\sigma, x)$, if the image changes in every neighbourhood of x. Symbolically,

$$\forall \varepsilon > 0 \quad D_{k-1}(\sigma) \cap B_\varepsilon(x) \neq D_k(\sigma) \cap B_\varepsilon(x).$$

Below is a useful fact when proofs involve the notion of affecting.

Proposition 2.1.2. In a sequence of strokes σ, for any $x \in \mathbb{R}^2$, $\Delta_k(\sigma, x)$ holds $\implies x \in \overline{\sigma_k}$.

Proof. If $x \notin \overline{\sigma_k}$, $\exists \delta > 0, B_\delta(x) \cap \sigma_k = \varnothing$. Then, $D_{k-1}(\sigma) \cap B_\delta(x) = D_k(\sigma) \cap B_\delta(x)$. In other words, $\Delta_k(\sigma, x)$ does not hold. ■

Definition 2.1.3 (Last Stroke and Persisting Limit of Strokes)**.** Given $S \subseteq \mathbb{R}^2$, a brush b, and a sequence of strokes σ such that $S = \lim_{n\to\infty} D_n(\sigma)$, then,

- The *last stroke* of $x \in \partial S$, denoted as the predicate $\lambda_k(\sigma, x)$, holds if and only if
$$\begin{cases} \Delta_k(\sigma, x) \text{ holds} \\ \forall l > k, \Delta_l(\sigma, x) \text{ is false} \\ \exists \delta > 0 \text{ such that } D_k(\sigma) \cap B_\delta(x) = D_\infty(\sigma) \cap B_\delta(x) \end{cases}$$
The last stroke of any $x \in \partial S$ is unique if it exists.
-
$$R^+ = \bigcap_{n=1}^{\infty} \sigma_{2n-1} \quad \text{and} \quad R^- = \bigcap_{n=1}^{\infty} \sigma_{2n}$$
are said to be the drawing and erasing *persisting limits* of σ respectively.

Definition 2.1.4 (One-sidedness)**.** Let $S \subseteq \mathbb{R}^2$. A set $C \subseteq \mathbb{R}^2$ is *one-sided* on some $x \in \partial S$ when $\exists \delta > 0$ s.t. $(\text{int}(C) \cap B_\delta(x) \cap S = \varnothing)$ or $(\text{int}(C) \cap B_\delta(x) \cap S^{\complement} = \varnothing)$.

Proposition 2.1.5. Let $S \subseteq \mathbb{R}^2$. If $C \subset \mathbb{R}^2$ is one-sided on some $x \in \partial S$, then $C' \subseteq C$ is also one-sided on x.

Proof. The statement is straight-forward by that $\mathrm{int}(C') \cap B_\delta(x) \subseteq \mathrm{int}(C) \cap B_\delta(x)$. ∎

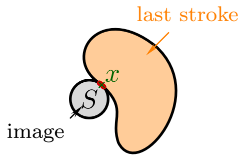


FIGURE 3A Visualization of the last stroke of x. The neighbourhood of x is also stabilised after the last stroke.

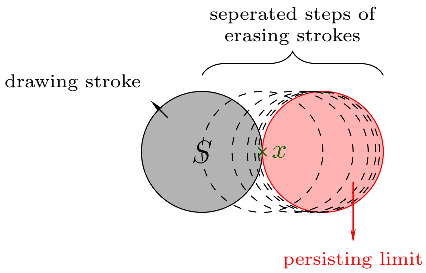


FIGURE 3B Visualization of one of the persisting limits (R^-). The erasing strokes secure the convergence of the final image of S by asymptotically approaching the 'external tangent brush' at x. The tangency nature of R^+, R^- plays a central role in proofs.

To build a universal test for drawability, we need to abstract away from any particular sequence of drawing and consider the static geometry of the figure itself. A set of all geometrically plausible 'candidate' strokes that could, in principle, serve as a final stroke for a boundary point.

Definition 2.1.6 (Hypothetical Last Strokes)**.** Let $S \subseteq \mathbb{R}^2$, $x \in \partial S$ and $b \subseteq \mathbb{R}^2$ be the brush. The set of hypothetical last strokes, denoted as $\lambdabar_b(x)$, is a set whenever $\alpha \in \lambdabar_b(x)$,

- α is a translation of the brush: $\alpha = b \oplus v$ for some $v \in \mathbb{R}^2$,
- $x \in \partial\alpha$,
- α is one-sided on x.

Below are some examples of possible $\lambdabar_b(x)$ for $b = B_1(0)$:

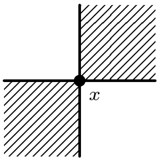


FIGURE 4A Point x with an empty $\lambdabar_b(x)$.

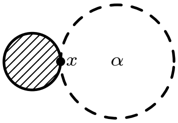


FIGURE 4B Point x with 1 element in $\lambdabar_b(x)$.

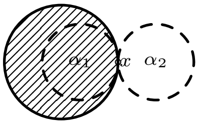


FIGURE 4C Point x with 2 elements in $\lambdabar_b(x)$.

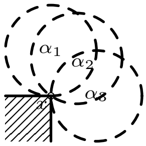


FIGURE 4D Point x with infinite elements in $\lambdabar_b(x)$.

To connect the abstract drawing process to the final image's geometry, we must first reveal the structural properties that govern any valid sequence of strokes. *Decreasing form* is a method for restructuring an infinite sequence of strokes to simplify analysis. Then we will establish the fundamental geometric constraints that any last stroke or persisting limits must satisfy, laying the groundwork for our main theorems on drawability.
Extensive use of properties of Minkowski sum is present in the following proof, please see Section 8.3.

Definition 2.1.7 (Decreasing Form)**.** Let $\sigma = (\sigma_k)_{k\in\mathbb{N}}$ be a sequence of strokes. The *decreasing form* of σ, denoted $\hat{\sigma}$, is the sequence of strokes $(\hat{\sigma}_n)_{n\in\mathbb{N}}$ where the n-th stroke is defined as:

$$\hat{\sigma}_n := \bigcup_{k=0}^{\infty} \sigma_{n+2k}$$

Remark.

1. The term 'decreasing' refers to the nested nature of this form: the total area covered by the n-th drawing stroke, $\hat{\sigma}_n$, contains all subsequent drawing strokes, i.e., $\hat{\sigma}_{n+2} \subseteq \hat{\sigma}_n$. $\hat{\sigma}_1$ can be thought of as the *'widest' possible pencil stroke*, encompassing every region that will ever be drawn upon, while $\hat{\sigma}_2$ represents the *total area the eraser will ever touch.*
2. Strokes in decreasing form are valid stroke forms (see Section 8.2). Let A be the corresponding steps of σ. For any $n \in \mathbb{N}$, $A_n \oplus b \subseteq \sigma_n \subseteq \overline{A_n \oplus b}$ by definition. It follows that $b \oplus \bigcup_{k=0}^{\infty} \overline{A_{n+2k}} \subseteq \hat{\sigma}_n \subseteq b \oplus \bigcup_{k=0}^{\infty} \overline{A_{n+2k}}$ (See Section 8.2). Hence, for each $n \in \mathbb{N}$, $\hat{A}_n \oplus b \subseteq \sigma_n \subseteq \overline{\hat{A}_n \oplus b}$ with $\hat{A}_n = \bigcup_{k=0}^{\infty} \overline{A_{n+2k}}$. Hence, each stroke in $\hat{\sigma}$ is of stroke form.

Lemma 2.1.8 (Properties of the Decreasing Form)**.** Let σ be a sequence of strokes such that $S = D_\infty(\sigma)$, and let $\hat{\sigma}$ be its decreasing form. The following properties hold:

1. $D_\infty(\hat{\sigma}) = S$.
2. For any $n \geq 2$, if the stroke $\hat{\sigma}_n$ does not affect some $x \in \partial S$, then no subsequent stroke $\hat{\sigma}_m$ with $m \geq n$ affects x.
3. If there is $k \in \mathbb{N}$ such that $\Delta_j(\hat{\sigma}, x)$ does not hold for all $j > k$, then

$$\exists \delta > 0 \text{ such that } D_k(\hat{\sigma}) \cap B_\delta(x) = D_\infty(\hat{\sigma}) \cap B_\delta(x)$$

4. The first stroke of the decreasing form, $\hat{\sigma}_1$, must affect every point $x \in \partial S$.
5. Any $x \in \partial S$ either has a last stroke or is affected by every stroke in its decreasing form $\hat{\sigma}$.

Proof. Due to the technicality of the proof, please see it in Appendix (Section 8.2). ■

Lemma 2.1.9 (Last Stroke Properties)**.**
Let $S = D_\infty(\sigma)$ be a drawable set with strokes σ, with $\lambda_k(\sigma, x)$ holding for some $x \in \partial S$.
1. For any $j \in \mathbb{N}$, if $x \in int(\sigma_j)$, then $k > j$.
2. $x \in \partial\sigma_k$ and σ_k is one-sided on x.

Proof. Due to the high technicality of the proof, please see it in Appendix (Section 8.2). ■

Lemma 2.1.10 (Persisting Limit Properties)**.**
Let $S = D_\infty(\hat{\sigma})$ be a drawable set with strokes in decreasing form $\hat{\sigma}$, with R^+ and R^- as the persisting limits of $\hat{\sigma}$.
1. $R^+ \subseteq S$ and $R^- \subseteq S^{\complement}$.
2. R^+ and R^- are of stroke forms. More specifically, $r \oplus b \subseteq R \subseteq \overline{r \oplus b}$ for $R = R^+, R^-$ and $r = \bigcap_{n=1}^{\infty} \overline{\hat{A}_{2n-1}}, \bigcap_{n=1}^{\infty} \overline{\hat{A}_{2n}}$ respectively, where $\hat{A}$ are the corresponding steps of $\hat{\sigma}$.
3. $\partial S \cap \mathrm{int}(R^+) = \varnothing$ and $\partial S \cap \mathrm{int}(R^-) = \varnothing$.
4. For any $x \in \partial S$, if every stroke of $\hat{\sigma}$ affects x, then $x \in \partial R$, and R^+, R^- are one-sided on x.

Proof. Due to the high technicality of the proof, please see it in Appendix (Section 8.2). ■

Lemma 2.1.9 confirms that any actual last stroke or persisting limit from a drawing sequence is geometrically well-behaved at the boundary point it finalises. However, such strokes can be large and complex sets. The following proposition provides a crucial simplification. It demonstrates that within any such stroke, we can always find a *minimal, idealised component*—a simple translation of the brush—that inherits all the necessary local properties at x.

Proposition 2.1.11. For any image $S \subseteq \mathbb{R}^2$, if there exists $T \subseteq \mathbb{R}^2$ of stroke form (that is, $\exists t \subseteq \mathbb{R}^2$, such that $t \oplus b \subseteq T \subseteq \overline{t \oplus b}$) such that T is one-sided on x and $x \in \partial T$, then $\exists \alpha \in \lambdabar_b(x)$ such that $\alpha \subseteq \operatorname{int}(T)$.

Proof. By Proposition 8.3.1, we have that $\partial T \subseteq \partial(t \oplus b)$. Because of the property of Minkowski sum (8.3.4), $x \in \partial t \oplus \partial b$ holds as b is bounded, so there must be $v \in \partial t$ such that $x \in v \oplus \partial b$, which implies that $x \in v \oplus \partial b$ and so $x \in \partial(v \oplus b)$.
We now prove that $v \oplus b \subseteq \operatorname{int}(T)$. Recall that $v \in \partial t \subseteq \bar{t}$, so $v \oplus t \subseteq \bar{t} \oplus b$. As b is open, by Proposition 8.3.7, $\bar{t} \oplus b = t \oplus b$, and by Proposition 8.3.1 (2), $t \oplus b \subseteq \operatorname{int}(T)$. So, $v \oplus b \subseteq \operatorname{int}(T)$.
By Proposition 2.1.5, $v \oplus b$ is also on-sided on x. Hence, $v \oplus b \in \lambdabar_b(x)$. ■

Corollary 2.1.12. Let $S = D_\infty(\sigma)$ be a drawable set with strokes σ.

1. For some $x \in \partial S$, if $\lambda_k(\sigma, x)$ holds for some $k \in \mathbb{N}$, then $\exists \alpha \in \lambdabar_b(x)$ such that $\alpha \subseteq \operatorname{int}(\sigma_k)$.
2. Let $\hat{\sigma}$ be the decreasing form of σ with persisting limits R^+ and R^-. If some $x \in \partial S$ is affected by every stroke in $\hat{\sigma}$, then $\exists \alpha^+, \alpha^- \in \lambdabar_b(x)$ such that $\alpha^+ \subseteq \operatorname{int}(R^+)$ and $\alpha^- \subseteq \operatorname{int}(R^-)$.

Proof.

1. From Lemma 2.1.9 (2), σ_k is one-sided and $x \in \partial \sigma_k$. By Proposition 2.1.11, the statement follows.
2. From Lemma 2.1.10 (4), R^+ and R^- are one-sided and $x \in \partial R$. By Proposition 2.1.11, the statement follows.

■

2.2. **Equivalence Theorems for Drawability.** Section 2.1 established the necessary local properties of strokes at the boundary. However, drawability is ultimately a global question of whether these locally-valid strokes interfere with one another across the figure. This section resolves this question by formalizing this conflict as the presence of *'containment loop' ($\in$-loop)*. Our two main equivalence theorems make this concept rigorous: *Theorem 1* provides a crucial structural simplification of the drawing process itself, which in turn allows us to prove our primary result, *Theorem 3*, establishing that the absence of these loops is the necessary condition for a figure to be drawable.

Theorem 1 (Last Stroke or All Affecting Existence)**.** *S is b-drawable if and only if there exists a sequence of strokes σ' such that $S = D_\infty(\sigma')$ and that each $x \in \partial S$ either has a last stroke or is affected by every stroke in σ'. Moreover, The decreasing form $\hat{\sigma}$ of σ is a valid choice of σ'.*

Proof. ($\Longleftarrow$): If there exists strokes σ' such that $S = D_\infty(\sigma')$, then S is b-drawable.

($\Longrightarrow$): $S = D_\infty(\sigma)$ for some sequences of strokes by assumption. Let $\hat{\sigma}$ be the decreasing form of σ and $S = D_\infty(\hat{\sigma})$ by Lemma 2.1.8 (1). By Lemma 2.1.8 (5), each $x \in \partial S$ either has a last stroke or is affected by every stroke in $\hat{\sigma}$. ■

The following definitions formalize this obstruction as a cyclical relationship of containment between boundary points and their finalising strokes.

Definition 2.2.1 (Terminologies in Containment)**.** Let $S \subseteq \mathbb{R}^2$, and $b \subseteq \mathbb{R}^2$ be the brush. Denote $[n] = \{1, 2, ..., n\}$.

1. Let $x, y \in \partial S$ and $\alpha \in \lambdabar_b(x)$. x is said to *contain* y with α if $y \in \alpha$.
2. Let $x_i \in \partial S$ and $\alpha_i \in \lambdabar_b(x_i)$, where $i \in [n]$. $(x_j, \alpha_j)_{j=1}^{n}$ *forms* a containment-loop when x_j contains x_{j+1} with α_j for $j \in [n-1]$ and x_n contains x_1 with α_n.
3. $x_1, x_2, ..., x_n \in \partial S$, where $n \geq 2$, is said to *inherit* a containment-loop when
   (a) $\forall \alpha_i \in \lambdabar_b(x_i)$ where $i \in [n]$, there exists a subsequence $(l_k)_{k=1}^{m}$ of first n positive integers such that $(x_{l_j}, \alpha_{l_j})_{j=1}^{m}$ forms a containment-loop.
   (b) X does not satisfy (a) when $X \subset \{x_1, ..., x_n\}$.
4. Let $x_i \in \partial S$ such that $x_1, x_2, ..., x_n$ does not inherit $\in$-loop, and $\alpha_i \in \lambdabar_b(x_i)$, where $i \in [n]$. $(x_j, \alpha_j)_{j=1}^{n}$ forms a *containment-chain* ($\in$-chain) with length n when x_j contains x_{j+1} with α_j for $j \in [n-1]$.

For simplicity, $\in$-loop is used as a short form of containment-loop.

The visualization below helps the reader in digesting the non-trivial terminologies regarding concepts of containment.

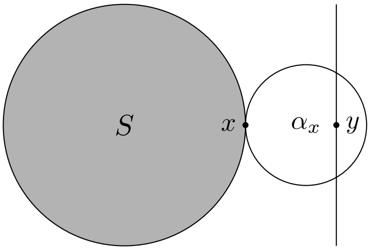


Figure 5a α_x is selected from $\lambdabar_b(x)$. point x *contains* y with α_x.

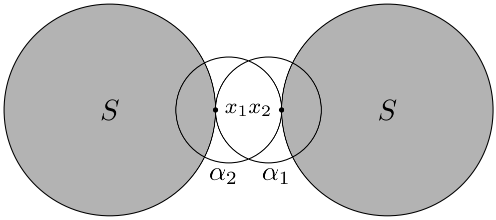


Figure 5b The most trivial case of *forming* $\in$-loop. Assume α_1 and α_2 are the only hypothetical last strokes of x_1 and x_2 respectively. x_1 contains x_2 with α_1 and x_2 contains x_1 with α_2.

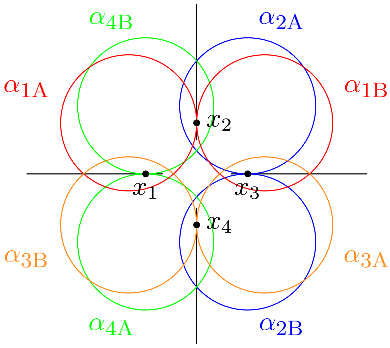


Figure 5c The case where points x_1, x_2, x_3, x_4 *inherit* a $\in$-loop, assuming each point has two hypothetical last strokes. The choices among $\lambdabar_b(x_i)$ are equivalent to consider four hypothetical last strokes with distinct colours. In any case, either two or four of the hypothetical last strokes form a $\in$-loop.

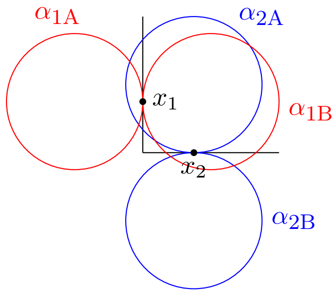


Figure 5d The case where points x_1, x_2 *do not* inherit a $\in$-loop. x_1 *does not* contain x_2 with α_{1B} and this combination denies x_1 and x_2 from inheriting a $\in$-loop.

Theorem 2 (Necessity of Drawability)**.** If S is drawable with brush b, then $\forall x \in \partial S$, x satisfies $\lambdabar_b(x) = \varnothing$, and any finite $X \subseteq \partial S$ does not inherit $\in$-loop.

Proof. We will prove this lemma by proving its contrapositive, that is, if there is $x \in \partial S$ such that $\lambdabar_b(x) = \varnothing$, or if there are points $\{x_1, x_2, ...x_n\} \subseteq \partial S$ that inherit containment loop, then S is undrawable. In both cases, we first assume that S is drawable with a sequence of strokes σ and consider the decreasing form $\hat{\sigma}$ of σ. By Theorem 1, either $x \in \partial S$ has a last stroke or every stroke affects x.

Case 1: $\exists x \in \partial S$ such that $\lambdabar_b(x) = \varnothing$.

By Corollary 2.1.12, there exists some $\alpha \in \lambdabar_b(x)$, and hence $\lambdabar_b(x) \neq \varnothing$. But it is given that $\lambdabar_b(x) = \varnothing$. Contradiction! Hence, S must be undrawable in such case.

Case 2: there are points $x_1, x_2, ..., x_n \subseteq \partial S$ that inherit containment loop.

Assume that some $x \in \{x_1, x_2, ...x_n\}$ is affected by every stroke in $\hat{\sigma}$. Then by Corollary 2.1.12, $\exists \alpha \in \lambdabar_b(x)$ such that $\alpha \subseteq \operatorname{int}(R)$, where R is one of the persisting limits of $\hat{\sigma}$, but as it is given that $\{x_1, x_2, ...x_n\}$ inherits $\in$-loop, so we know that there is $y \in \{x_1, x_2, ...x_n\}$ such that $y \in \alpha$ (otherwise, $\{x_1, x_2, ...x_n\} \setminus \{x\}$ should also inherit $\in$-loop, which violates that $\{x_1, x_2, ...x_n\}$ inherits $\in$-loop), so $y \in \operatorname{int}(R)$, but by Lemma 2.1.10 (3), $\operatorname{int}(R) \cap \partial S = \varnothing$. Contradiction! As no $x \in \{x_1, x_2, ...x_n\}$ can have persisting stroke, they must all have last strokes.
Let the last strokes of $x_1, x_2, ..., x_n$ be $\sigma_{k_1}, \sigma_{k_2}, ..., \sigma_{k_n}$. There must be $\alpha_i \in \lambdabar_b(x_i)$ such that $\alpha_i \subseteq \operatorname{int}(\sigma_{k_i})$ by Corollary 2.1.12, where $i \in [n]$. Recall that, as $x_1, x_2, ..., x_n$ inherits a $\in$-loop, so there is a subsequence $\{l_j\}_{j=1}^m$ of first n positive integers such that $(x_{l_j}, \alpha_{l_j})_{j=1}^m$ forms a $\in$-loop. That is, $x_{l_j+1} \in \alpha_{l_j}$ for $j \in [n]$ and $x_{l_1} \in \alpha_{l_m}$. Notice that $\alpha_{l_j} \subseteq \operatorname{int}(\sigma_{k_i})$ for each $i \in \lfloor m \rfloor$ and hence $x_{l_j+1} \in \operatorname{int}(\sigma_{k_{l_j}})$ for $j \in \lfloor m-1 \rfloor$ and $x_{l_1} \in \operatorname{int}(\sigma_{k_{l_m}})$. By Lemma 2.1.9 (1), $k_{l_1} < k_{l_2} < ... < k_{l_m} < k_{l_1}$. Contradiction! Hence, S is not drawable in such case.

Hence, if S is drawable with brush b, then $\forall x \in \partial S$, $\lambdabar_b(x) = \varnothing$, and any finite $X \subseteq \partial S$ does not inherit $\in$-loop. ■

With Theorem 2 established, drawability can be practically tested. In this section, we will show how the property of last strokes (Lemma 2.1.9) contributes to the hypothetical last strokes, introduce the idea of containment and the containment loop ($\in$-loop), and ultimately, a theorem with an if and only if condition on images that are undrawable.

Definition 2.2.2 (Finite chain condition)**.** Suppose that we have a class of figures that contains $S \subseteq \mathbb{R}^2$ such that,
if
1. every $x \in \partial S$, $\lambdabar_b(x) \neq \varnothing$, and
2. there are no $\in$-loops inherited for any finite set of points in ∂S,

then there is a choice $(\alpha_x)_{x \in \partial S}$ with each $\alpha_x \in \lambdabar_b(x)$ such that:
1. there is no finite $X \subseteq \partial S$ that forms $\in$-loop with corresponding $(\alpha_x)_{x \in X}$, and
2. there is a finite upper bound on the length of $\in$-chains that form with $(\alpha_x)_{x \in \partial S}$.

Then, this class of figures satisfies the finite chain condition.

Theorem 3 (Equivalence on Drawability on Boundary Figures)**.** For any $S \subseteq \mathbb{R}^2$ such that $S = \partial S$, and that S satisfies the finite chain condition, S is drawable if and only if every $x \in \partial S$ satisfies $\lambdabar_b(x) \neq \varnothing$ and that there are no $\in$-loops inherited.

Proof. ($\Longrightarrow$): If S is drawable, then by Theorem 2, $\forall x \in \partial S, \lambda_b(x) \neq \varnothing$ and that there are no $\in$-loops inherited.

($\Longleftarrow$): It is given that every $x \in \partial S$ satisfies $\lambda_b(x) \neq \varnothing$ and that there are no $\in$-loops inherited. By Definition 2.2.2, there is a choice $(\alpha_x)_{x\in\partial S}$ with each $\alpha_x \in \lambda_b(x)$ such that there is no finite $X \subseteq \partial S$ that forms $\in$-loop with corresponding $(\alpha_x)_{x\in X}$, and there is a finite upper bound N on the length of $\in$-chains that form with $(\alpha_x)_{x\in\partial S}$.

Constructing a finite order on ∂S: We then want to construct a finite partition $(P_n)_{n=0}^N$ of ∂S such that for each $n \in \{0, 1, ..., N\}$, $\forall y \in P_n, \forall x \in \bigcup_{k=0}^{n} P_k, x \notin \alpha_y$.
First, for each $x \in \partial S$ we define

$$\rho(x) = \begin{cases} 0, & \text{if there is no } y \in \partial S \text{ such that } x \in \alpha_y \\ \sup\{\rho(y) \mid y \in \partial S, x \in \alpha_y\}, & \text{otherwise} \end{cases}$$

that is, one less than the length of the longest $\in$-chain that ends at x. Note that for every $x \in \partial S$, $\rho(x)$ is well-defined and finite, as there are no $\in$-loops formed, and that there is a finite upper bound on the length of $\in$-chains formed. With this, we can take $P_n = \{x \in \partial S \mid \rho(x) = n\}$ for each $n \in \{0, 1, ..., N\}$. Thus, we have constructed the desired partition $(P_n)_{n=0}^N$.
We will then construct a sequence of strokes $(\sigma_n)_{n\in\mathbb{N}}$ that aims to draw S.
Constructing the strokes: For each $n \in \{0, 1, ..., N\}$, we construct σ_{2n+1} and σ_{2n+2} with the following:

$$\sigma_{2n+1} = P_n \cup \bigcup_{y\in P_n} \alpha_y$$

$$\sigma_{2n+2} = \bigcup_{y\in P_n} \alpha_y$$

Recall that each $\alpha_x = v_x \oplus b$. We show that σ_{2n+1} is a valid stroke form with the corresponding step as $A_{2n+1} = \bigcup_{y\in P_n} \overline{v_y}$.

$$\begin{aligned} A_{2n+1} \oplus b = b \oplus \bigcup_{y\in P_n} \overline{v_y} & \\ = \bigcup_{y\in P_n} (b \oplus \overline{v_y}) & \qquad \text{(by Proposition 8.3.6)} \\ = \bigcup_{y\in P_n} (b \oplus v_y) & \qquad \text{(by Proposition 8.3.7, as } b \text{ is open)} \\ = \bigcup_{y\in P_n} \alpha_y & \\ \subseteq P_n \cup \bigcup_{y\in P_n} \alpha_y & \\ = \sigma_{2n+1} & \end{aligned}$$

Recall that for any $x \in P_n$, $x \in \partial\alpha_x \subseteq \overline{\alpha_x}$.

$$
\begin{aligned}
\sigma_{2n+1} &= P_n \cup \bigcup_{y\in P_n} \alpha_y \\
&= \bigcup_{y\in P_n} (y \cup \alpha_y) \\
&\subseteq \bigcup_{y\in P_n} (y \cup \overline{\alpha_y}) \\
&= \bigcup_{y\in P_n} \overline{\alpha_y} \\
&= \bigcup_{y\in P_n} \overline{v_y \oplus b}
\end{aligned}
$$

$$
\begin{aligned}
(\text{cont.}) &= \bigcup_{y\in P_n} (\overline{v_y} \oplus \overline{b}) \text{ (by Proposition 8.3.3, as } b \text{ is bounded)} \\
&= \overline{b} \oplus \bigcup_{y\in P_n} \overline{v_y} \text{ (by Proposition 8.3.6)} \\
&\subseteq \overline{b} \oplus \overline{\bigcup_{y\in P_n} \overline{v_y}} \\
&= \overline{b \oplus \bigcup_{y\in P_n} \overline{v_y}} \\
&= \overline{A_{2n+1} \oplus b}
\end{aligned}
$$

Hence we have shown that $A_{2n+1} \oplus b \subseteq \sigma_{2n+1} \subseteq \overline{A_{2n+1} \oplus b}$ with $A_{2n+1} = \bigcup_{y\in P_n} \overline{v_y}$, so σ_{2n+1} is of stroke form. The proof for σ_{2n+2} follows an even simpler logic, so we will not show it.

At last, for any $m > 2N+2$, we define $\sigma_m = \varnothing$.

Proving that the strokes σ draw S: For any $n \in \{0, 1, ..., N\}$, we show that $D_{2n+2}(\sigma) = \bigcup_{k=0}^{n} P_k$. We will use induction. For $n = 0$, we write $D_2(\sigma) = \sigma_1 \setminus \sigma_2 = (P_0 \cup \bigcup_{y\in P_0} \alpha_y) \setminus \bigcup_{y\in P_0} \alpha_y$. Recall that we have defined P_0 in step 1 to be disjoint from $\bigcup_{y\in P_0} \alpha_y$, so this is equal to C_0. Then, fix some $n \in \{0, 1, ..., N-1\}$ and assume that $D_{2n+2}(\sigma) = \bigcup_{k=0}^{n}$. We will show that $D_{2n+4}(\sigma) = \bigcup_{k=0}^{n+1} P_k$. We write $D_{2n+4}(\sigma) = (D_{2n+2}(\sigma) \cup \sigma_{2n+3}) \setminus \sigma_{2n+4}$. By assumption, this is equal to

$$
\begin{aligned}
\left(\bigcup_{k=0}^{n} P_k \cup \sigma_{2n+3}\right) \setminus \sigma_{2n+4} &= \left(\bigcup_{k=0}^{n} P_k \cup P_{n+1} \cup \bigcup_{y\in P_{n+1}} \alpha_y\right) \setminus \bigcup_{y\in P_{n+1}} \alpha_y \\
&= \left(\bigcup_{k=0}^{n+1} P_k \cup \bigcup_{y\in P_{n+1}} \alpha_y\right) \setminus \bigcup_{y\in P_{n+1}} \alpha_y
\end{aligned}
$$

Recall that we have defined P_{n+1} in step 1 to be disjoint from $\bigcup_{y\in P_{n+1}} \alpha_y$, so this is equal to $\bigcup_{k=0}^{n+1} P_k$. Hence, $D_{2N+2}(\alpha) = \bigcup_{k=0}^{N} P_k = \partial S = S$. Recall that all strokes of σ after $m > 2N+2$ are empty, so $D_\infty(\sigma) = D_{2N+2}(\sigma)$ holds, and so, $D_\infty(\sigma) = S$. Hence, S is drawable. ∎

3. Classifications on $|\lambdabar_b(x)|$ for Strictly Convex Brush b

Section 2 establishes that a figure is undrawable if a finite set of its boundary points inherits $\in$-loops. As these loops are constructed from sequences of $\alpha \in \lambdabar_b(x)$, the potential to form such a loop at a point $x \in \partial S$ is determined by its set of available $\lambdabar_b(x)$. This section therefore provides a necessary foundation by classifying $|\lambdabar_b(x)|$ for a convex brush b, connecting the size of $\lambdabar_b(x)$ to the local geometry of ∂S. Necessary preliminaries in convex geometry are attached in Preliminary 8.1.1.

Definition 3.0.1. For some S, for some $x \in \partial S$, ∂S is *locally convex* at x when $\exists \delta > 0$ such that $B_\delta(x) \cap S$ is a convex curve.

To ensure a tractable analysis, we impose the following geometric constraints:

- The brush b must be a strictly convex set with a C^2 continuous boundary.
- The image S must have a ∂S that is piecewise differentiable and a locally convex curve at the point of analysis, $x \in \partial S$.

While the proofs will focus on cases where S is locally convex at x, the principles apply symmetrically when its complement, S^{C}, is locally convex at x.
The key to classifying the available $\lambdabar_b(x)$ at a point $x \in S$ lies in the local geometric constraints imposed by the boundary itself. The locally supporting (and opposite) half-plane is the fundamental construct for this analysis.

Proposition 3.0.2. For some $x \in \partial S$, if ∂S is not differentiable at x, then there are infinitely many locally supporting half-planes for S at x.

Proposition 3.0.3. For some $x \in \partial S$, if ∂S is differentiable at x, then there exists a unique tangent line at x and hence a unique locally opposite half-plane $\mathcal{H}_S^+(x)$.

Proof. Please see Section 8.2 for both proofs. ∎

The uniqueness of the tangent line at differentiable points, established above, is a powerful constraint. It allows us to describe the geometric relationship between the image S and a potential $\lambdabar_b(x)$ that is tangent to it. We provide the necessary vocabulary below.

Definition 3.0.4 (Tangency Conditions and Notations)**.** Let S be a locally convex region with a differentiable boundary, and let b be the brush shape. We define the following properties related to tangency at a point $x \in \partial S$.

1. Two figures are *tangent* at a point x if they share the same tangent line at x.
2. We denote $\mathbf{N}_S^+(x)$ as the unit outward-pointing normal vector (the vector that points to the locally opposite half-plane) of ∂S at x, and $\mathbf{N}_S^-(x)$ as the unit inward-pointing normal vector. (If ∂S is locally a straight line at x, then either vector can be $\mathbf{N}_S^+(x)$ while the other one is $\mathbf{N}_S^-(x)$.)
3. Classifications of Tangency: Two locally convex figures, X and Y, that are tangent at x are classified as:

- *Oppositely Tangent* if they locally lie on opposite sides of their common tangent line. This condition is equivalent to their outward normal vectors being opposite: $\mathbf{N}^+_X(x) = -\mathbf{N}^+_Y(x)$.
- *Agreeingly Tangent* if they locally lie on the same side of their common tangent line. This is equivalent to their outward normal vectors being identical: $\mathbf{N}^+_X(x) = \mathbf{N}^+_Y(x)$.

3.1. **Conditions when $\lambdabar_b(x) = \infty$ and $|\lambdabar_b(x)| \neq \varnothing$.** This subsection addresses two important questions regarding $|\lambdabar_b(x)|$: under what conditions is it infinite, and can it equal to 0? We will demonstrate that the differentiability of the boundary is the critical factor that leads to $|\lambdabar_b(x)| = \infty$, as well as $|\lambdabar_b(x)| = 0$ is impossible under our setting.

Proposition 3.1.1. For some $x \in \partial S$, if ∂S is not differentiable at x, then $|\lambdabar_b(x)| = \infty$.

Proof. For some $x \in \partial S$ at which ∂S is not differentiable, then by Proposition 3.0.2, there are infinitely many locally supporting half-planes for S at x.
For some $\alpha = b \oplus v$, since α is strictly convex, for each $x \in \partial\alpha$, $\exists$ a unique $\mathcal{H}^+_\alpha(x)$ (given by Proposition 3.0.3), and therefore a unique $\mathcal{H}^-_\alpha(x)$ with ℓ as its boundary. Conversely, given a half-plane and an x on the boundary of which, $\exists$ a unique $\alpha' = b \oplus v'$ for some v' such that $x \in \partial\alpha'$ and the half-plane is the locally opposite half-plane of α' at x. Therefore, for each locally supporting half-plane $\mathcal{H}^-_S(x)$ for S at x, it is also an locally opposite half-plane $\mathcal{H}^+_\alpha(x)$ for some α at x.
Then, $\mathrm{int}(\mathcal{H}^-_\alpha(x)) \cap \alpha = \mathrm{int}(\mathcal{H}^+_S(x)) \cap \alpha = \varnothing$. By a fundamental result in topology, $\mathrm{int}(\mathcal{H}^+_S(x)) \cap \alpha = \varnothing \iff \mathcal{H}^+_S(x) \cap \mathrm{int}(\alpha) = \varnothing$. Hence, $x \in \partial\alpha$ and $\mathrm{int}(\alpha) \cap S = \varnothing$, which means $\alpha \in |\lambdabar_b(x)|$. Therefore, there are infinite such αs, hence $|\lambdabar_b(x)|$ is infinite. ∎

Proposition 3.1.1 effectively resolves the case for non-differentiable points, showing they lead to an unrestricted, infinite $|\lambdabar_b(x)|$. We now turn our attention to the more common case of points where the boundary ∂S is differentiable.

Lemma 3.1.2. $\forall x \in \partial S$, if $\lambdabar_b(x)$ is non-empty, then $\forall \alpha \in \lambdabar_b(x)$, α is tangent to S at x.

Proposition 3.1.3. For every differentiable $x \in \partial S$, there are 2 translations of brush $b \oplus v_1$ and $b \oplus v_2$ such that they are tangent to S at x for some $v_1, v_2 \in \mathbb{R}^2$.

Proof. Please see Section 8.2 for the proof of both. ∎

Corollary 3.1.4. For differentiable points $x \in \partial S$, $|\lambdabar_b(x)| \leq 2$.

Proof. A direct consequence of Lemma 3.1.2, and Proposition 3.1.3. ∎

The preceding results lead to a significant conclusion: for any differentiable point x, $|\lambdabar_b(x)|$ can contain at most two elements. These two candidates correspond to the two distinct geometric ways the brush can be tangent to the boundary. In one case, the brush and the image lie on opposite sides of their common tangent line; in the other, they lie on the same side. We will define both cases explicitly.

Definition 3.1.5. For all $x \in \partial S$, define α_x^+ as the translation of brush $\alpha = b \oplus v$ in Proposition 3.1.3 satisfying $\mathbf{N}_S^+(x) = -\mathbf{N}_{\alpha_x^+}^+(x)$. ($\alpha_x^+$ is oppositely tangent to S). Similarly, define α_x^- as the remaining α satisfying $\mathbf{N}_S^+(x) = \mathbf{N}_{\alpha_x^-}^+(x)$. ($\alpha_x^-$ is agreeingly tangent to S).
The existence of such is guaranteed, as any α_x^+ oppositely tangent to S a x satisfies $\mathbf{N}_S^+(x) = -\mathbf{N}_{\alpha_x^+}^+(x)$, while any α_x^- agreeingly tangent to S a x satisfies $\mathbf{N}_S^+(x) = \mathbf{N}_{\alpha_x^-}^+(x)$.

Proposition 3.1.6. $\forall x \in \partial S, \alpha_x^+ \in \lambdabar_b(x)$. Consequently, $\lambdabar_b(x) \neq \varnothing$.

Proof. α_x^+ is oppositely tangent to S at x and satisfies $\mathbf{N}_S^+(x) = -\mathbf{N}_A^+(x)$. $\alpha_x^+ \in \lambdabar_b(x)$ if and only if α_x^+ fulfils the One-sidedness property.
By definition of oppositely tangent, S and α_x^+ lie on locally opposite half-planes. Therefore,
$\operatorname{int}(\alpha_x^+) \cap S \cap B_\delta(x) = \varnothing$ for some $\delta > 0$. Hence, α_x^+ is one-sided and $\alpha_x^+ \in \lambdabar_b(x)$. Thus, $\lambdabar_b(x) \neq \varnothing$. ■

Therefore, we have demonstrated $|\lambdabar_b(x)| = 0$ is impossible under our setting.
up

3.2. **Conditions on $|\lambdabar_b(x)| = 1, 2$.** Recognizing that at a differentiable point x, there can be no more than two possible candidates for $\lambdabar_b(x)$, our goal now is to distinguish between the scenarios where $|\lambdabar_b(x)|$ equals 1 and when it equals 2. This section will focus on addressing this question.

Lemma 3.2.1. Let x be a differentiable point on ∂S. Then

$$|\lambdabar_b(x)| = 1 \iff \alpha_x^- \notin \lambdabar_b(x) \iff \forall \varepsilon > 0, S^{\complement} \cap \operatorname{int}(\alpha_x^-) \cap B_\varepsilon(x) \neq \varnothing.$$

Proof. The first equivalence follows from Corollary 3.1.4 and Proposition 3.1.3 and that α_x^+ and α_x^- are the only possible candidates of $\lambdabar_b(x)$.
The second equivalence is a direct quote of the definition of one-sided. By definition, $\alpha_x^- \notin \lambdabar_b(x) \iff \forall \varepsilon_2 > 0, B_{\varepsilon_2}(x) \cap S \cap \operatorname{int}(\alpha_x^-) \neq \varnothing \wedge B_{\varepsilon_2}(x) \cap S^{\complement} \cap \operatorname{int}(\alpha_x^-) \neq \varnothing$.
Since α_x^- and S lie on the same locally supporting halfplane, $\forall \varepsilon_3 > 0, B_{\varepsilon_3}(x) \cap S \cap \operatorname{int}(\alpha_x^-) \neq \varnothing$. Then, the equivalence follows. ■

Then, we will proceed to show how to differentiate when $|\lambdabar_b(x)| = 1$ or 2 based on certain characteristics of S at x. Specifically, we will utilise the curvature of S at x (Please find the notation and formula at Appendix 8.1.1). For the curvature to exist and be manageable, we require S to be a locally convex figure and that ∂S is *at least C^2 continuous everywhere* (its second derivative is continuous). We will prove the iff condition for $|\lambdabar_b(x)| = 1$ first.

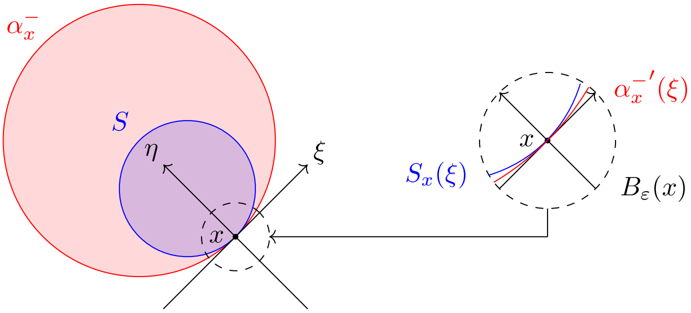


FIGURE 6. Some S and α_x^- that satisfy the theorem.

Lemma 3.2.2. For all $x \in \partial S$, if $|K_S(x)| \neq |K_{\alpha_x^-}(x)|$, then x satisfies $|K_S(x)| > |K_{\alpha_x^-}(x)| \iff \forall \varepsilon > 0, S^{\complement} \cap \operatorname{int}(\alpha_x^-) \cap B_\varepsilon(x) \neq \varnothing$.

Proof. Create a coordinate plane with axis ξ and η for some x, such that the ξ-axis is parallel to the tangent line at x, while the η-axis is perpendicular to the ξ-axis and passes through x. α_x^- and S are fully contained in the half-plane $\eta \geq 0$. Also, x has coordinates (0,0) in the $\xi\eta$ plane. Around some small neighbourhood around x, ∂S and $\partial\alpha_x^-$ can be represented as a function $S_x(\xi)$ and $\alpha_x^-(\xi)$. By construction, $S_x(0) = \alpha_x^-(0) = 0$, $S_x'(0) = \alpha_x^{-\prime}(0) = 0$, $S_x'(\xi) \geq 0$ and $\alpha_x^{-\prime}(\xi) \geq 0$.

Then, for the L.H.S. of the theorem, by formula for curvature for some function $f(x)$ at some point $p = (\xi(p), f(\xi(p)))$ ($\xi(p)$ denotes the ξ-coordinate of a point p), $K(p) = \dfrac{f'(\xi(p))}{(1 + f'(\xi(p))^2)^{\frac{3}{2}}}$. At $p = x$, we get $K_{S_x}(x) = \dfrac{S_x'(\xi(x))}{(1 + S_x'(\xi(x))^2)^{\frac{3}{2}}}$ and $K_{A_x^-}(x) = \dfrac{\alpha_x^{-\prime}(\xi(x))}{(1 + \alpha_x^{-\prime}(\xi(x))^2)^{\frac{3}{2}}}$. Note that $S_x'(\xi(x))$ and $\alpha_x^{-\prime}(\xi(x))$ equal to 0 by construction. Therefore, $K_{S_x}(x) = S_x'(\xi(x))$ and $K_{A_x^-}(x) = \alpha_x^{-\prime}(\xi(x))$. Therefore, L.H.S. can be rewritten as $S_x'(\xi(x)) > \alpha_x^{-\prime}(\xi(x))$. This is equivalent to saying $S_x'(0) > \alpha_x^{-\prime}(0)$.

Note that the R.H.S. of the theorem, $\forall \varepsilon > 0, S^{\complement} \cap \operatorname{int}(\alpha_x^-) \cap B_\varepsilon(x) \neq \varnothing$, means that locally, some part of the stroke α_x^- lies in the exterior of S. In our coordinate system, this is equivalent to the condition that for any small neighbourhood of the origin, there exists a point k where the boundary of the stroke lies 'below' the boundary of the image, i.e., $\alpha_x^-(k) < S_x(k)$. Therefore, R.H.S. can be rewritten as $\forall \varepsilon' > 0, \exists k \in [-\varepsilon', \varepsilon'], S_x(k) > \alpha_x^-(k)$.

Then, we can rewrite the theorem in an equivalent function form:

For all $x \in \partial S$, if $S_x'(0) \neq \alpha_x^{-\prime}(0)$, then x satisfies $S_x'(0) > \alpha_x^{-\prime}(0) \iff \forall \varepsilon' > 0, \exists k \in [-\varepsilon', \varepsilon'], S_x(k) > \alpha_x^-(k)$.

To prove the iff condition, let P be the L.H.S. of the statement while Q is the R.H.S. We will first prove $P \implies Q$:

When $S'_x(0) > {\alpha_x^-}'(0)$, then $S'_x(\xi)$ is curving more than ${\alpha_x^-}'(\xi)$ at $\xi = 0$. For positive ξ, since $S'_x(0) = {\alpha_x^-}'(0) = 0$ by construction, that implies $S'_x(\xi) > {\alpha_x^-}'(\xi)$ for some positive ξ. Then, as $S_x(0) = \alpha_x^-(0) = 0$ by construction, that implies $S_x(\xi) > \alpha_x^-(\xi)$ for some positive ξ close to x. The same can be said for negative ξ by symmetry. Hence, $P \implies Q$ is proved.

For $Q \implies P$, define $g(\xi)$ as $S_x(\xi) - \alpha_x^-(\xi)$. Note that $g(\xi)$ has a local minimum at $\xi = 0$ since $g(0) = 0$ and $g(\varepsilon') > 0$ for some ε' in an arbitrarily small punctured neighbourhood (neighbourhood without x) of x. Also, $g'(0) = 0$. Using a linear approximation for $g(\xi)$ around $\xi = 0$ through Taylor's Theorem and Lagrange's Remainder, we can say that $g(\xi) = g(0) + g'(0)\xi + \dfrac{g'(k')}{2}\xi^2$ for some $k' \in [0, \xi]$. Substituting in the values, we get $g(\xi) = \dfrac{g'(k')}{2}\xi^2$ for some $k' \in [0, \xi]$. Since for some $\xi \neq 0$ but close to 0, $g(\xi) > 0$ by assumption, therefore there exists some k such that $g'(k') > 0$. Note that k' can be arbitrarily close to 0. Therefore, by continuity, $g'(0) \geq 0$. implying $S'_x(0) \geq {\alpha_x^-}'(0)$. However, by assumption, $S'_x(0) \neq \alpha_x^-(0)$, implying $S'_x(0) > {\alpha_x^-}'(0)$. This is precisely condition P. Hence, $Q \implies P$ is true, and the theorem is proved. ■

Lemma 3.2.2 reveals that the validity of the agreeingly tangent stroke α_x^- hinges on its curvature relative to S. α_x^- is disqualified if the boundary of S is more sharply curved than the brush's ($K_S(x) > K_{\alpha_x^-}(x)$), as this causes an overlap with the exterior $S^{\complement}$, violating one-sidedness. The following theorem formalises this insight, creating a test for $|\lambda_b(x)|$ by comparing curvatures at points with matching normal vectors.

Theorem 4. $\forall x \in \partial S$, Given that $|K_S(x)| \neq |K_b(x')|$ whenever $\mathbf{N}_S^+(x) = \mathbf{N}_b^+(x')$ for some $x' \in \partial b$ (initial condition). Then x satisfies

$$|\lambda_b(x)| = 1 \iff (\forall x' \in \partial b, \mathbf{N}_S^+(x) = \mathbf{N}_b^+(x') \implies |K_S(x)| > |K_b(x')|).$$

Proof. By Lemma 3.2.2, if $|K_S(x)| \neq |K_{\alpha_x^-}(x)|$, then $\forall \varepsilon > 0, S^{\complement} \cap \operatorname{int}(\alpha_x^-) \cap B_\varepsilon(x) \neq \varnothing \iff |K_S(x)| > |K_{A_x^-}(x)|$. Then, by Lemma 3.2.1, $|\lambda_b(x)| = 1 \iff |K_S(x)| > |K_{A_x^-}(x)|$. Note that since A_x^- is a translation of b, $K_{A_x^-}(x) = K_b(x')$ when $\mathbf{N}_{A_x^-}^+(x) = \mathbf{N}_b^+(x')$ for some $x' \in \partial b$. Also, note that $\mathbf{N}_{A_x^-}^+(x) = \mathbf{N}_S^+(x)$. Hence, the theorem is proved. ■

Remark. By that $|\lambda_b(x)| = 1$ or 2, the if and only if condition of $|\lambda_b(x)| = 2$ is the negation of Corollary 3.2.2.

Below is a summary of classification on $|\lambda_b(x)|$.

Classification 1 (Size of $\lambdabar_b(X)$)**.**

1. Confirm that the brush b must be a strictly convex set with a C^2 continuous boundary, and the image S must have a piecewise differentiable boundary that is also a locally convex curve at the point of analysis, $x \in \partial S$.
2. If a point is non-differentiable, then $|\lambdabar_b(x)| = \infty$. (Proposition 3.1.1)
3. Under stricter conditions on S, i.e.
   - ∂S is C^2 continuous everywhere, and
   - For the unique point $x' \in \partial b$ such that $\mathbf{N}_S^+(x) = \mathbf{N}_b^+(x')$, $|K_S(x)| \neq |K_b(x')|$.

   then we can differentiate when $|\lambdabar_b(x)| = 1$ or 2 based on the curvature condition in Theorem 4.

4. Layer Drawability

The fundamental philosophy of Layer Drawability is that *undrawable figures can often be decomposed into a collection of drawable sets, or 'layers'.* Finding the minimum number of layers required is a highly non-trivial problem, because the most efficient partitioning is not determined by the number of curves, but by the distribution of $\in$-loop across them. The core challenge is that $\in$-loops often span separate components. A frequent used definition hence is introduced:

Definition 4.0.1 (Embodiment)**.** Disjoint curves, $C_1, \ldots, C_n$, are said to *embody* a $\in$-loop when there exists points $p_1 \in C_1, \ldots, p_n \in C_n$ such that $p_1, \ldots p_n$ inherit a $\in$-loop.

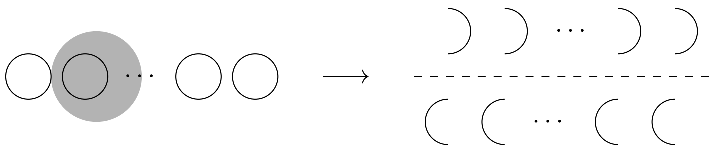


Figure 7. A family of circles with collinear centres, which is undrawable by the embodiment of $\in$-loops between adjacent pairs of circles. The family can be partitioned into *two* drawable subsets.

This section develops a practical and universal method, rooted in graph theory, to establish a lower bound for n-layer drawability by analysing the structure of these embodied loops.

4.1. Definition and Figure Restrictions.

The images S within the discussion of layer drawability follow the restriction, point 1 and 3, stated in Classification 1 on ∂S to ensure that $H_1(S) \neq \varnothing$.

The brush, b, is chosen to be a strictly convex region for the same purpose to become a necessary assumption to classify $|\lambdabar_b(x)|$.

Definition 4.1.1 (Layer decomposition)**.** A *layer decomposition* of an image $S \subseteq \mathbb{R}^2$ with respect to a brush b is a finite collection of sets $L = \{L_1, \ldots, L_n\}$, called *layers*, that satisfies two properties:

1. The image is the union of layers, i.e. $\displaystyle\bigcup_{k=1}^{n} L_k = S$.
2. Each individual layer is drawable with brush b, i.e. $L_k \in \mathcal{D}_b$ for all $k \in \{1, \ldots, n\}$.

Definition 4.1.2 (n-layer drawability)**.** An image $S \subseteq \mathbb{R}^2$ is *n-layer drawable* if the minimum number of layers in a layer decomposition of S is exactly n. In formal notations,

$$n = \min(\{|L| : L \text{ is a layer decomposition of } S\})$$

Definition 4.1.3. Given an image S, define $H_1(S) := \{x \in \partial S : |\lambdabar_b(x)| = 1\}$ and $H_{>1}(S) := \{x \in \partial S : |\lambdabar_b(x)| > 1\}$.

4.2. **Bounds for Layer Drawability on S with $H_1(S) \neq \varnothing$.** $H_1(S)$ plays the key role in securing bounds of layer drawability. With the preset range of S and brush b, the identification of $H_1(S)$ can almost be finished through the use of Classification 1. For the special case where $K_S(x) = K_b(x')$ for some $x \in \partial S$, a local analysis on $x \in \partial S$ is needed to determine whether x satisfies $|\lambdabar_b(x)| = 1$ or 2 by checking if α_x^- is one-sided.

4.2.1. *Primitive Partition.* Our goal is to construct a graph to analyse the $\in$-loops within $H_1(S)$. To do this, we must first partition the curves of $H_1(S)$ into a finite set of arcs, which is intrinsically tied to the geometry of the figure and its potential for forming $\in$-loops. For notational clarity in the upcoming proofs, we denote finite-length curves with endpoints p and q as $\widehat{pq}$.

Definition 4.2.1 (Primitive Points, Arcs, and Partition)**.** Let S be an image under the restriction of the section. We establish the following concepts related to $H_1(S)$:

1. The set of *primitive points*, $X^* \subset H_1(S)$, consists of all points $p \in H_1(S)$ for which there exists a distinct point $q \in \partial S$ ($q \neq p$) such that they share the same hypothetical last stroke. Since $|\lambdabar_b(x)| = 1$ for all $x \in H_1(S)$, let α_x be the unique hypothetical last stroke for a point x. X^* is given by
$$X^* = \{p \in H_1(S) \mid \exists q \in H_1(S), q \neq p, \text{ such that } \alpha_p = \alpha_q\}$$
2. The *primitive partition*, denoted P^*, is the partition of ∂S into a set of open arcs whose endpoints are the primitive points in X^*. Such open arcs are defined as *primitive arcs.* The layer drawability graph inherited from P^* is denoted as $G(P^*)$.

Proposition 4.2.2. Any finite set of points on a primitive arc does not inherit a $\in$-loop.

Proof. Please see Section 8.2 for the proof. ■

The existence of primitive points is equivalent to locating bitangency points of translations of brush $b \oplus v$ and points on ∂S. However, bitangency points are not guaranteed to exist. We must be able to prove that these points actually exist in situations where $\in$-loops are present. Lemma 4.2.5 is an important nature of primitive points we will utilise, that will lead to crucial insights in Graph Theory for the following subsection.

Proposition 4.2.3. Let A be a simple C^2 continuous curve, and let B be a simple closed, strictly convex, C^2 continuous curve. Let $f : [t_0, t_1] \to \mathbb{R}^2$ be a continuous path. If $(A \oplus f(t_0)) \cap B = \varnothing$ and $(A \oplus f(t_1)) \cap B \neq \varnothing$, then there exists $t' \in [t_0, t_1]$ such that $(A \oplus f(t'))$ is externally tangent to B.

Proof. Please see Section 8.2 for the proof. ■

Definition 4.2.4 (Normal-Angle Parameterization)**.** Let C be a simple, regular and C^2 continuous curve. Let the function $\nu : C \to [0, 2\pi)$ map each point $p \in C$ to the directed angle of its outward normal vector. ν is a continuous function secured by C being C^2.

Lemma 4.2.5 (Bitangency)**.** Let C_i and C_j be two disjoint, regular and C^2 continuous curves where C_i and C_j embody a $\in$-loop at points $p_i \in C_i$ and $p_j \in C_j$. If there exists points $q_i \in C_i$ and $q_j \in C_j$ such that p_j and q_j do not inherit a containment loop, then there exists a translation of brush $\alpha = b \oplus v$ that is tangent to both curves at points on $\widehat{p_i q_i}$ and $\widehat{p_j q_j}$.

Proof of Proposition 4.2.5. Without loss of generality, assume p_j does not contain q_j with α_{p_j}. Apply Normal-Angle Parametrization ν on $\widehat{p_i q_i}$ and let the image of ν be $[t_{p_i}, t_{q_i}]$.

Let $f : [t_{q_i}, t_{q_i}] \to \mathbb{R}^2$ be a path such that $b \oplus f(t_x)$ is tangent to C_i at $x \in \widehat{p_i q_i}$, where t_x is the directed angle of the outward normal vector of x. By assumption, $(b \oplus f(t_{p_i})) \cap C_j \neq \varnothing$ and $(b \oplus f(t_{q_i})) \cap C_j = \varnothing$. By Proposition 4.2.3, there exists $t' \in [t_{q_i}, t_{q_i}]$ such that $(b \oplus f(t'))$ is externally tangent to C_j. The Lemma then follows. ■

Remark. The tangency point on both curves are exactly the primitive points in X^*.

By proving a condition that primitive points are guaranteed to exist between conflicting curves, this section has thus established the primitive partition, a precise method for deconstructing $H_1(S)$ into a finite set of meaningful arcs.

4.2.2. *Layer Drawability Graph and Chromatic Bound.* The next step is to build a structure that maps out the conflicts ($\in$-loops) that exist between these components. A graph is naturally the mathematical tool for this purpose. The arcs of our partition will become the vertices, and the relationships between them will be represented by edges. To proceed, see Preliminary 8.1.2 for necessary preliminaries in Graph Theory.

Definition 4.2.6 (Layer Drawability Graph)**.** Let $S \subset \mathbb{R}^2$ be an image and $H_1(S)$ be an non-empty subset of S. Let $Q = \{s_1, \ldots, s_m\}$ be a partition of $H_1(S)$. The *layer drawability graph* for Q is the graph $G(Q) = (Q, E)$, where an edge $\{s_i, s_j\} \in E$ exists if and only if s_i and s_j embody a $\in$-loop.

The Layer Drawability Graph translates our problem into a classic graph colouring task. The rule that arcs embodying an $\in$-loop require separate layers corresponds directly to the rule that adjacent vertices in a graph must be coloured differently. The following proposition formalises this crucial equivalence between the minimum number of layers required and the graph's chromatic number.

Proposition 4.2.7 (Foundation of Chromatic Bound for Layer Drawability)**.**
For any image $S \subseteq \mathbb{R}^2$ where $H_1(S)$ is non-empty, its layer drawability is bounded below by the minimum chromatic number, $\hat{\chi}$, of its layer drawability graphs. That is, whenever $H_1(S)$ can be partitioned into n drawable layers, then

$$n \geq \hat{\chi} \quad \text{where} \quad \hat{\chi} = \min_{Q} \{\chi(G(Q)) \mid Q \text{ is a partition of } H_1(S)\}.$$

Proof. Consider a layer decomposition $L = \{L_1, \ldots, L_n\}$ of S. Define $Q' = \{Q_1, \ldots, Q_n\}$ where $Q_k := \{x \in \partial L_k : |\lambdabar_b(x)| = 1\} \setminus \bigcup_{i=1}^{k-1} Q_i$. Notice that Q' is also a layer decomposition of $H_1(S)$ and layers in Q' are disjoint. If any of the layer Q_k is empty, the inequality $n \geq \hat{\chi}$ holds automatically by the layers used in the partition $H_1(S)$ is less than that of S. Hence, assume each Q_k is non-empty. By Theorem 3, any finite set of points, and inclusively any two points on the boundaries of L_k do not inherit $\in$-loop, $k = 1, 2, ..., n$. Q_k is a subset of the boundaries of L_k, and hence any two points on Q_k do not inherit $\in$-loop. Hence, $\{Q_k\}$ is an independent set of vertices in $G(Q')$. The number of independent sets in $G(Q')$ is then n and Q' is also a partition of $H_1(S)$. By the definition of chromatic number, $n \geq \chi(G(Q) \geq \hat{\chi}$. ■

One may question whether $\chi(G(Q')) = n$ imply that $\hat{\chi}$ will always be n. However, it is possible to further partition Q' into a more complicated layer drawability graph with lower chromatic number. The relevant concept is the key motivation and the below argument proves that this optimal choice is the primitive partition (P^*), which is derived from the figure's intrinsic bitangency geometry.

Definition 4.2.8 (Expanded and reduced-coloured Graphs and Descendants)**.**
Let $G = (V, E)$ be an n-colourable graph derived from a partition of arcs in $H_1(S)$, and let $G' = (V', E')$ be a graph derived from a finer partition.

The set of *descendants* of a vertex $v \in V$ is the set of vertices $S_v \subseteq V'$ created by subdividing the component v. As vertices represent geometric arcs, this is formally expressed as $S_v = \{w \in V' : w \subseteq v\}$.

The graph G' is an *expanded and reduced-coloured graph* of G if it satisfies three properties:

1. Every vertex in V' is coloured from a palette of $n-1$ colours (e.g., $\{c_2, \ldots, c_n\}$).
2. Each descendant set S_v is an independent set in G'.
3. An edge $\{w, z\} \in E'$ exists if and only if $\{u, v\} \in E$ for any $w \in S_u$ and $z \in S_v$.

Finally, for a given colouring c' of G', the *colour set* of the descendants of v is denoted $C(S_v) = \{c'(w) : w \in S_v\}$.

Lemma 4.2.9. Let $G = (V, E)$ be a graph with $\chi(G) = n \geq 2$. For any expanded and reduced-colour graph G' of G, $C(S_u) \cap C(S_v) \neq \varnothing$ for at least one edge $\{u, v\} \in E$.

Proof. Please see Section 8.2 for the proof. ■

This lemma provides a powerful constraint on how colours must be shared when a graph is refined. We are now ready to apply this abstract principle to the geometric problem at hand.

Lemma 4.2.10. Let $S \subseteq \mathbb{R}^2$ be an image and $H_1(S) \neq \varnothing$. Let P^* be the primitive partition of $H_1(S)$. Then

$$\chi(G(P^*)) = \min_Q \{\chi(G(Q)) \mid Q \text{ is a partition of } H_1(S)\}$$

Proof. It is clear that $\chi(G(P^*)) \geq \min_Q \{\chi(G(Q)) \mid Q \text{ is a partition of } H_1(S)\}$ by P^* is also a partition of $H_1(S)$. Assume on the contrary that

$$\chi(G(P^*)) > \min_Q \{\chi(G(Q)) \mid Q \text{ is a partition of } H_1(S)\}.$$

Then there exists a partition Q of $H_1(S)$ such that $\chi(G(Q)) \leq \chi(G(P^*)) - 1$. Further partition Q to Q' such that each arc p in P^* is partitioned into a set of arcs S_p in Q'. Hence arcs in S_p in Q' must not embody any $\in$-loop and the corresponding vertices in $G(Q')$ are mutually independent. Besides, $\chi(G(Q')) \leq \chi(G(Q) \leq \chi(G(P^*)) - 1$.

$G' = (Q', E')$ is then an expanded and colour-reduced graph of $G(P^*)$. Then, by Lemma 4.2.9, $\exists\{u, v\} \in E'$ such that $C(S_u) \cap C(S_v) \neq \varnothing$. $\{u, v\} \in E'$ implies that there exists points $p \in u$ and $q \in v$ such that p and q inherit a $\in$-loop. $C(S_u) \cap C(S_v) \neq \varnothing$ implies that there exists pair points $p_1 \in u$, $q_1 \in v$ such that they must belong to the same layer. Thus p_1 and q_1 must not inherit a $\in$-loop.

By Lemma 4.2.5, there exists a tangent brush $b \oplus v$ at points on arcs $\widehat{pq_1}$ and $\widehat{qp_1}$. But these arcs both are the sub-arcs of a primitive arc. Then, a new primitive point exists on a primitive arc, but P^* should consist of the primitive partition from all primitive points. Contradiction! The lemma is then followed.

■

Theorem 5 (Chromatic Bound for Layer Drawability)**.** Let $S \subseteq \mathbb{R}^2$ be an image and $H_1(S) = \{x \in \partial S : |\lambdabar_b(x)| = 1\}$ is a non-empty subset of S. Let P^* be the primitive partition of $H_1(S)$. Then the n-layer drawability of S, is bounded below by

$$n \geq \chi(G(P^*)).$$

Proof. A direct consequence of Proposition 4.2.7 and Lemma 4.2.10. ■

Ultimately, this section has successfully translated the geometric problem of $\in$-loops into a formal graph-theoretic framework. Applications of practical layer decomposition can be found in Section 5 and Appendix 9.4.

4.2.3. *Upper Bound of Layer Drawability when $S = H_1(S)$.* The chromatic number $\chi(G(P^*))$ provides a firm *lower bound* on n-layer drawability. However, this bound is not always tight. Vertices, with an equivalent meaning of arcs, might still embody a higher-order $\in$-loop and prevent them from being drawn in the same layer.

This gap between the chromatic number and the true n-layer drawability can be arbitrarily large, making a tight, general upper bound elusive. An imagined case is that a colour class with 9 vertices could be structured such that any 3 arcs chosen from it embody an $\in$-loop. Then, any 3 arcs cannot be in the same layer, implying at most 2 arcs can be in the same layer. Then, at least 5 layers is required to draw S. Since the geometric possibilities for S are vast, the existence of such complex graphs cannot be ruled out.

Without deeper geometric constraints on the structure of $G(P^*)$, we can only propose a weak upper bound that depends on both the chromatic number and the total number of vertices.

Theorem 6. Let $S \subset \mathbb{R}^2$ be an image satisfying $S = H_1(S)$, and any two primitive arcs satisfy the finite chain condition in Definition 2.2.2. The the n-layer drawability of S is bounded by:

$$\chi(G(P^*)) \leq n \leq \lfloor \frac{|V|}{2} \rfloor + \chi(G(P^*)).$$

Proof. $\chi(G(P^*)) \leq n$ is given by Theorem 5. Then, to show $n \leq \lfloor \frac{|V|}{2} \rfloor + \chi(G(P^*))$, we maximize n by assuming: In a $\chi(G(P^*))$-colouring of $G(P^*)$, (1) the vertices fulfil any two vertices with different colour classes share an edge, and (2) in every single colour class, any set of primitive arcs of size *three* (which is equivalent to a set of vertices in $G(P^*)$) embody a $\in$-loop.

Whenever three distinct primitive arcs are present in a layer, there exists an embodiment of $\in$-loop, causing the undrawability of the layer by Theorem 2. Then, any two primitive arcs that are represented by vertices in the same colour class are drawable, since they don't embody a $\in$-loop by property of $G(P^*)$ and satisfy the requirements in Theorem 3. Hence, at most two primitive arcs can present in a layer, given that their representing vertices have the same colour. A colour class

can either be completely distributed when it has an even number of vertices, or leave out one vertex, when it has an odd number of vertices.
Then, the number of pairs of distributed arcs, where each pair requires a new layer, is $\leq \lfloor \frac{|V|}{2} \rfloor$, and the number of unpaired arcs, again each of them requiring a new layer, is $\leq \chi(G(P^*))$. Hence, the total (and minimal) number of layers required to draw $S = H_1(S)$ is less than $\lfloor \frac{|V|}{2} \rfloor + \chi(G(P^*))$, hence proving the theorem. ■

Remark. By Definition 2.2.1, the universal existence of the embodiment of set of primitive arcs of size *three* denies the existence of the embodiment of set of primitive arcs of size greater than three. Hence, the assumption of (2) under $\chi(G(P^*))$-colouring is sufficient to maximize the layer drawbility.

The existence of such graph requesting the upper bound in Theorem 6 sounds absurd, however, with infinite possibilities of S, we cannot prove that such graphs do not exist, especially considering the generalization of drawability theories into complex metric spaces. A follow-up conjecture is posted in Section 6.

4.3. **Unboundedness of the Layer Drawability Metric.** The Chromatic Bound establishes a lower limit on layer drawability, but leaves open the question of a universal maximum. This section proves that no such maximum exists: the n-layer drawability metric is *unbounded.* We demonstrate this by constructing families of figures that require an arbitrarily large number of layers. The argument is extended to the challenging case of $H_1(S) = \varnothing$, where we use rigid geometric arrangements of line segments to show that $\in$-loops can be forced even when every point offers multiple last stroke options.

4.3.1. *Unboundedness of Layer Drawability when $H_1(S) \neq \varnothing$.* This subsection constructs a case that the layer drawability of S when $H_1(S) \neq \varnothing$ is unbounded. Please find the detailed proof in Appendix (Section 8.2).

Let $S_m = \displaystyle\bigcup_{k=0}^{m-1} \{x_k : \ |x_k - p_k| \leq 2^{-k}, \text{ where } p_k = (2^{-k}+1)(\cos(2^{-k}\pi), \sin(2^{-k}\pi))\}$.

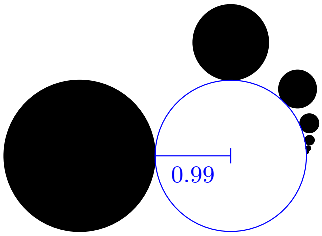


FIGURE 8. A S_7 with a scaling factor of 0.99.

Theorem 7. For any integer $m \geq 2$ and unit disk as the brush shape, there exists an image S_m such that it is m-layer drawable.

Proof. Please see Section 8.2 for the proof. ■

4.3.2. *Unboundedness of Layer Drawability when $H_1(S) = \varnothing$.* The case where $H_1(S)$ is empty presents a great challenge, as our chromatic bound method is inapplicable and the abundance of last stroke choices at every point suggests $\in$-loops can be avoided. We first start from choosing a representing range of images S that reduce the complexity of arguing the unboundedness layer drawability metric.

Definition 4.3.1. Let L denote the union of a finite set of non-intersecting closed line segments $\ell \subset \mathbb{R}^2$, of arbitrary length and orientation.

Fact: Using unit disk as the brush shape b, $|\lambdabar_b(x)| = 2$ or ∞ for every point on line segments, implying $H_1(L) = \varnothing$. Hence, L is a suitable candidate to study the unboundedness layer drawability metric when $H_1(L) = \varnothing$.

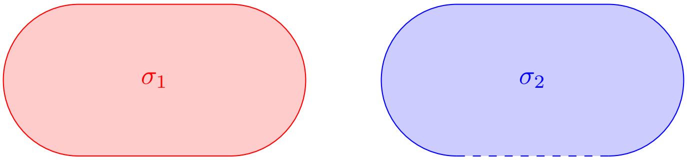


FIGURE 9. σ_1, σ_2 are drawing and erasing strokes respectively. Dashed boundary of σ_2 represents that open part of σ_2. Drawing with σ_1 and erase with σ_2 results in a line segment.

Theorem 8. Let the unit disk be the brush shape b. For all $k \in \mathbb{N}$, there exists an image of L that is at least k-layer drawable.

The layer drawability graph is a powerful tool, but its predictive power relies on the geometrically constrained points in $H_1(S)$. When $H_1(S) = \varnothing$, this method is no longer applicable. Every point on the boundary now offers multiple hypothetical last strokes, allowing $\in$-loops to be potentially avoided by judiciously selecting non-conflicting options.

To prove unboundedness in this scenario, a new technique is required. We must design a geometric construction so rigid that it forces $\in$ loops to form, regardless of which last strokes are chosen. We begin by defining the elemental building block for such a structure.

Definition 4.3.2 (Cross)**.** A cross is a set of line segments $L = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ satisfying the properties below:

1. (a) ℓ_1 and ℓ_3 lies on an auxiliary line l (l is not an element of the cross),
   (b) ℓ_2 and ℓ_4 are both totally inside 2 opposite open half planes which l divided $\mathbb{R}^2$ into, and
   (c) ℓ_2 and ℓ_4, when extended indefinitely, both intersects an auxiliary line segment l_0 which covers the gap between ℓ_1 and ℓ_3.
2. $\ell_1, \ell_2, \ell_3, \ell_4$ embody a $\in$-loop.

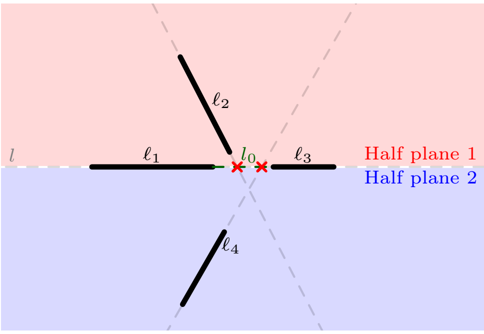


FIGURE 10. A cross.

Remark. Unless specified, for any cross mentioned in the following proofs, ℓ_1 and ℓ_3 refers only to the collinear line segments, while ℓ_2 and ℓ_4 refers to the other two.

Proposition 4.3.3. Any cross is 2-layer drawable.

Proof. A cross, by definition, contains points that inherit a $\in$-loop, therefore it can never be 1-layer drawable. However, $L_1 = \ell_1 \cup \ell_3$ and $L_2 = \ell_2 \cup \ell_4$ is always a valid layer decomposition $\{L_1, L_2\}$ of a cross. Therefore, it is 2-layer drawable. ■

The *'cross'* is the basic building block for our argument. While satisfying its positional criteria is straightforward, guaranteeing the crucial loop criterion is not, as the existence of an $\in$-loop depends critically on the brush's size relative to the figure. The following lemma provides the key.

Lemma 4.3.4. For any given set of line segments $L = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ satisfying the positional criteria in Definition 4.3.2, there exists a proportionally scaled figure $L' = \{\ell_1', \ell_2', \ell_3', \ell_4'\}$ of L such that L' forms a cross.

Proof. Please see Section 8.2 for the proof. ■

Corollary 4.3.5. A scaled-down image L' of a cross L is also a cross.

A single cross is sufficient to prove 2-layer drawability. Forcing a higher layer drawability requires a method of escalating complexity through a recursive construction. The core idea is to use the cross structure as a 'forcing gadget' embedded within a larger system. We now introduce a tool $Ł_n$, a family of L that utilises small crosses, together with Lemma 4.3.4, that pushes k to arbitrarily large for some k-layer drawable L.

Definition 4.3.6. ($Ł_n$) Let $l \subset \mathbb{R}^2$ be a vertical line. $Ł_n = L_{left} \cup L_{right} \cup ł^+ \cup ł^-$ is defined by the following positional restrictions and constructions:

1. L_{left} and L_{right} are n-layer drawable and are on the opposite sides of l.

2. Extend the line segments in L_{left} and L_{right} to intersect l. The line segment l_0 connects the two most distant intersection points on l. $ƚ^+$ and $ƚ^-$ are arbitrary short segments extended from the endpoints of l_0 respectively.
3. L_{left} and L_{right} are vertically bounded by $ƚ^+$ and $ƚ^-$, i.e. for all p in L_{left} and L_{right}, we can always find some $p^+ \in ƚ^+$ 'higher' than p and some $p^- \in ƚ^-$ 'lower' than p.

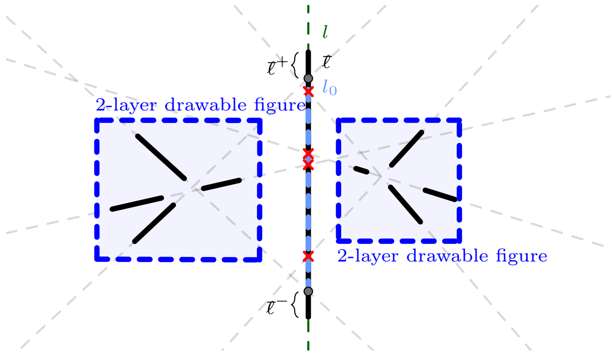


FIGURE 11. An example of $Ⱡ_2$.

Lemma 4.3.7. Assume $Ⱡ_n = L_{\text{left}} \cup L_{\text{right}} \cup ƚ^+ \cup ƚ^-$ exists and $Ⱡ_n$ is n-layer drawable for some $n \geq 2$. Then there exists a scale-down figure $Ⱡ'_n = L'_{\text{left}} \cup L'_{\text{right}} \cup ƚ^{+'} \cup ƚ^{-'}$ of $Ⱡ_n$ such that for any layer decomposition $Ⱡ'_n = \bigcup_{k=1}^{n} S'_k$ and for any $p^{+'} \in ƚ^{+'}$ and $p^{-'} \in ƚ^{-'}$, $\{p^{+'}, p^{-'}\} \not\subset S'_k$ for all $k \in \{1, 2, \dots, n\}$. That is, the set of layers contributed to drawing $ƚ^{+'}$ and $ƚ^{-'}$ must be distinct in a minimal layer decomposition.

Remark. Please see Section 8.2 for the proof.

Lemma 4.3.7 establishes that for a suitably scaled $Ⱡ'_n$ structure, $ƚ^{+'}$ and $ƚ^{-'}$ are forced into distinct drawable layers. However, $Ⱡ_n$ is only capable of forcing the use of distinct layers in drawing. A more powerful construction based on $Ⱡ_n$ is needed. We can proceed with the last geometric structure that is designed to chain the property of $Ⱡ_n$, the final building block in proving Theorem 8.

Definition 4.3.8 ($Ⱡ_{n+1}$). Assume $Ⱡ_n = L_{\text{left}} \cup L_{\text{right}} \cup ƚ^+ \cup ƚ^-$ exists and $Ⱡ_n$ is also n-layer drawable for some $n \geq 2$. $Ⱡ_{n+1}$ is defined by stacking n copies of $Ⱡ_n$ (with an order of$(Ⱡ_n)_1, (Ⱡ_n)_2, \dots, (Ⱡ_n)_n$), such that $ƚ^-$ and $ƚ^+$ from each adjacent pair $(Ⱡ_n)_i$, $(Ⱡ_n)_{i+1}$, $1 \leq i \leq n-1$, are colinear and connected.

Denote the components of $Ⱡ_{n+1}$, $(Ⱡ_n)_j$, by $(Ⱡ_n)_j = (L_{\text{left}})_j \cup (L_{\text{right}})_j \cup ƚ_j^+ \cup ƚ_j^-$, where $1 \leq j \leq n$. $Ⱡ_{n+1}$ is also non self-intersecting by that $(L_{\text{left}})_j$ and $(L_{\text{right}})_j$ are vertically bounded by $ƚ_j^+$ and $ƚ_j^-$, as given by the definition of $Ⱡ_n$.

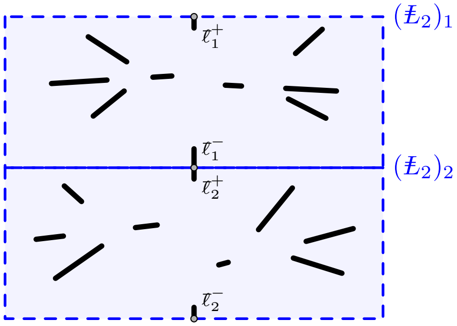


FIGURE 12A An example of $Ł_3$.

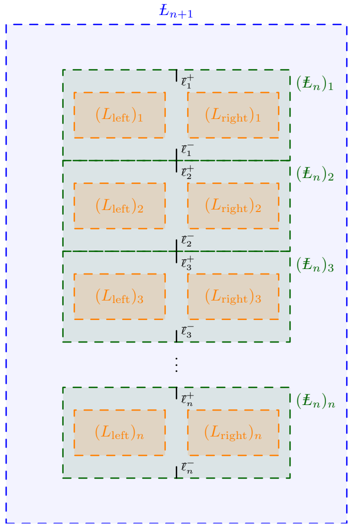


FIGURE 12B $Ł_n$ in general.

Theorem 8. Let the unit disk be the brush shape b. For all $k \in \mathbb{N}$, there exists an image of L that is at least k-layer drawable.

Proof of Theorem 8. The proof is based on induction on n.
When $n = 1$, a line segment is 1-layer drawable. When $n = 2$, a cross is 2-layer drawable. Both are valid candidates of L.

For induction assumption, assume there exists a n-layer drawable L. Then $Ł_n$ is well-defined and is at least n-layer drawable. If $Ł_n$ is $(n+1)$-layer drawable, then the inductive step is completed. Therefore, we will only consider the case where $Ł_n$ is n-layer drawable among all possible constructions under the induction assumption.

Now, construct $Ł_{n+1}$ by stacking n copies of $Ł_n$. We want to prove that $Ł_n$ is $(n+1)$-layer drawable. We let $Ł_{n+1}$ to be n-layer drawable and let $Ł_{n+1} = \bigcup_{k=1}^{n} S_k$, where S_k are the drawable layers of $Ł_{n+1}$ (otherwise the inductive step will already be completed).
Consider $\bigcup_{k=r}^{s} (Ł_n)_k$, where $1 \leq r < s \leq n$. If $\bigcup_{k=r}^{s} (L_{\text{left}})_k$ or $\bigcup_{k=r}^{s} (L_{\text{right}})_k$ are at least $(n+1)$-layer drawable, then the inductive step is completed. Therefore, we will only consider when $\bigcup_{k=r}^{s} (L_{\text{left}})_k$ and $\bigcup_{k=r}^{s} (L_{\text{right}})_k$ are both n-layer drawable. Then, observe that $(\bigcup_{k=r+1}^{s} (L_{\text{left}})_k) \cup (\bigcup_{k=r+1}^{s} (L_{\text{right}})_k) \cup \ell_r^- \cup \ell_s^-$ is a valid construction of

$\text{\L}_n$. We only consider when $(\bigcup_{k=r+1}^{s}(L_{\text{left}})_k) \cup (\bigcup_{k=r+1}^{s}(L_{\text{right}})_k) \cup \ell_r^- \cup \ell_s^-$ is n-layer drawable (otherwise, the induction step is again completed).
We can pick any $p_r^- \in \ell_r^-, p_s^- \in \ell_s^-$, where ℓ_r^-, ℓ_s^- is a subset of some $\text{\L}_n = (\bigcup_{k=r}^{s}(L_{\text{left}})_k) \cup (\bigcup_{k=r}^{s}(L_{\text{right}})_k) \cup \ell_r^- \cup \ell_s^-$ for $1 \le r < s \le n$. By Lemma 4.3.7, there exists a down-scaled figure $\text{\L}'_{n+1}$ (by down-scaling copies of $\text{\L}_n$ simultaneously) such that for all ${p_r^-}', {p_s^-}'$, where $1 \le r < s \le n$, must belong to distinct layers of ${\text{\L}_{n+1}}'$. Hence, ${p_1^-}', {p_2^-}', \ldots, {p_n^-}'$ each belong to a unique layer.

Similar to before, consider $\bigcup_{k=1}^{i}(\text{\L}_n)_k$, where $1 \le i \le n$. However, this time, we will focus on ℓ_1^+. Observe that $(\bigcup_{k=1}^{i}(L_{\text{left}})_k) \cup (\bigcup_{k=1}^{i}(L_{\text{right}})_k) \cup \ell_1^+ \cup \ell_i^-$ is a valid construction of $\text{\L}_n$. Again, we only consider when $(\bigcup_{k=r+1}^{s}(L_{\text{left}})_k) \cup (\bigcup_{k=r+1}^{s}(L_{\text{right}})_k) \cup \ell_r^- \cup \ell_s^-$ is n-layer drawable (otherwise, the induction step is again completed). Then, similar to the above argument, we can deduce that for all $\forall i$, where $1 \le i \le n$, $\forall {p_1^+}' \in {\ell_1^+}', {p_i^-}' \in {\ell_i^-}'$, ${p_1^+}'$ and ${p_i^-}'$ each belong to a unique layer.

Therefore, ${p_1^+}', {p_1^-}', {p_2^-}', \ldots {p_n^-}'$ all belong to distinct layers, and at least $n+1$ distinct layers is needed to draw L' Hence, $\text{\L}'_{n+1}$ is at least $(n+1)$-layer drawable. The inductive step is completed and the statement holds by mathematical induction. ■

Corollary 4.3.9. For all $k \in \mathbb{N}$, there exists an image of L that is exactly k-layer drawable.

Proof. A L which is at least $(k+1)$-layer drawable exists for all $k \in \mathbb{N}$, as given by Theorem 8. By iteratively removing line segments in L, a k-layer drawable L' can be obtained. This is because since each line segment itself can be drawn in one layer, adding a line segment to a figure can at most increase the figure's minimum number of layers needed by 1. Therefore conversely, removing a line segment can at most decrease layers needed by 1. Therefore, a $(k+n)$-drawable L can be reduced to a k-layer drawable $\widetilde{L}$, proving the corollary. ■

From above, we can conclude that there is no upper bound on the number of layers required to draw the image of L in general.

4.4. **Challenges on Layer Drawability when** $\text{int}(S) \neq \varnothing$**.** Extending layer drawability analysis, for example, proposing a effective bound, from boundaries to figures with non-empty interiors introduces significant complexity. The core challenge is that any partition of $\text{int}(S)$ into drawable layers necessarily creates *new, artificial boundaries* that did not exist in the original figure. Unlike the original boundary ∂S, for which the primitive partition P^* provides a canonical set of arcs, there are infinitely many ways to subdivide the interior, making it difficult to define an optimal strategy.

A natural approach is to use the boundary's minimal layering, as determined by the graph $G(P^*)$, to guide the interior partition. One could form closed drawable regions by connecting the *endpoints* of primitive arcs that share the same colour in a minimal colouring.

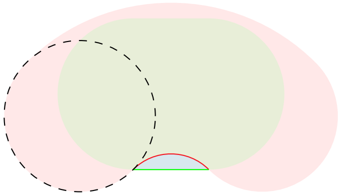


FIGURE 13. The simplest drawing sequence in producing the blue image segment. The red arc belong to $H_1(S)$ but the line segment belongs $H_{>1}(S)$.

However, this intuitive "linking" method fails for two main reasons:

1. Some regions left unassigned. This strategy can leave central portions of the interior unassigned to any layer, requiring further complex partitioning (Figure 14a).
2. New conflict arises. Specifically, the new line-segment boundaries can introduce new $\in$-loops. Primitive arcs that were previously in the same layer may form an undrawable configuration when combined with their new boundaries, as the strokes required to draw them now interfere with each other (Figure 14b).

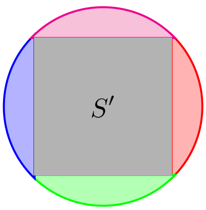


FIGURE 14A Partitioning S' by connecting endpoints on arcs will form new boundaries, where the gray region is unassigned to existing layers.

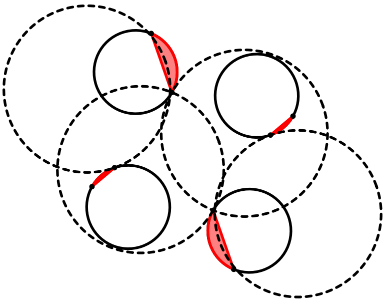


FIGURE 14B Dashed circles represent the last stroke of some primitive points. The red arcs by themselves do not embody any $\in$-loop. However, the arcs together with the line segments together, as the boundaries of red regions, embody at least one $\in$-loop.

The failure of this method reveals a deeper problem: a minimal colouring of $G(P^*)$ guarantees a conflict-free layering for the boundary *in isolation*, but it provides no guarantee of geometric compatibility for a corresponding partition of the interior. The true challenge is not just finding any minimal colouring, but finding a *specific* minimal colouring that also permits a conflict-free division of the interior space. Resolving these intertwined local and global constraints requires a more sophisticated approach and remains a significant open question.

5. Case Study: Layer Drawability of the HKUGA College Emblem

This section applies our framework to determine the n-layer drawability of a concrete example: an outline of the HKUGA College school emblem (Figure 15b), using the brush of shape $b = B_{0.16}(0)$. The analysis follows the methodology detailed in the Appendix 9.4, which is also applicable to a more general class of curves and reach the n-layer drawability of curves.

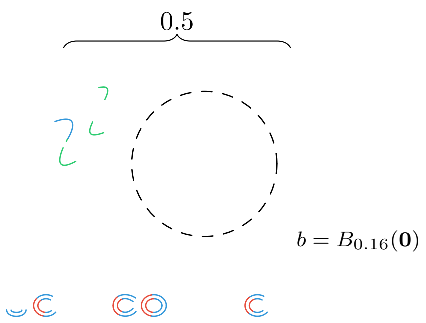


Figure 15a Coloured $H_1(S)$ of the School emblem.

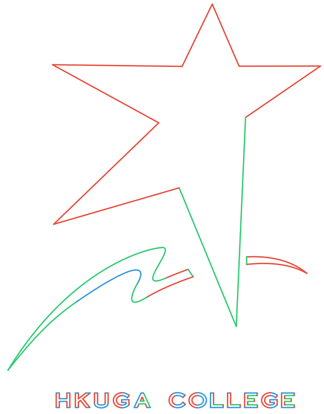


Figure 15b School emblem of HKUGA College with its layer drawability outlined.

Analysis. *Boundary Classification:* We first apply the curvature condition from Theorem 5. Boundary points with curvature $|\kappa(x)| > \dfrac{1}{0.16} = 6.25$ form the set $H_1(S)$, while all other smooth points constitute the set $H_{>1}(S)$. $H_1(S)$ is shown coloured in Figure 15a.

Graph Construction and Chromatic Bound: Next, the primitive partition P^* for $H_1(S)$ is computed by identifying all points of bitangency (through numerical approximations). This partition is used to construct the layer drawability graph

$G(P^*)$, shown in Figure 16. The graph contains several 3-cliques, hinting its chromatic number is at least 3. $\chi(G(P^*)) = 3$ is verified by a valid 3-colouring of $G(P^*)$ presented in Figure 16. From the Chromatic Bound (Theorem 5), we conclude that S requires *at least 3 layers* to be drawn.

Result: The 3-colouring of $G(P^*)$ provides a valid 3-layer partition of $H_1(S)$, as illustrated by the colours in Figure 15bb. The remaining components from $H_{>1}(S)$ (primarily line segments) were successfully distributed among these three layers without creating new $\in$-loops. The animated demonstration justifying the successful drawing of each layer is available at `https://drive.google.com/file/d/14KMz-CwaPaHSM5E2g4nyVCdYZ-83u52k/view?usp=sharing`.

Since a valid 3-layer decomposition was found and the theoretical lower bound is 3, we conclude that the HKUGA College school emblem is *3-layer drawable.*

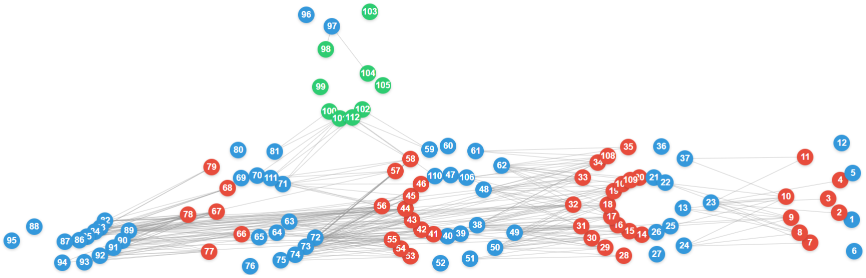


FIGURE 16. The layer drawability graph $G(P^*)$ for the arcs in $H_1(S)$. The spatial layout of the vertices approximates the geometric midpoints of their corresponding arcs.

Further Exploration.

- Two simpler samples on layer drawability are discussed in Appendix (Section 8.3), featuring the existence of odd cycles in $G(P^*)$ and an irregular type of brush shape and figure S.
- An AI application using PoE is attached in Appendix (Section 8.5) can be used by readers to explore the layer drawability for translations of circles with unit circle brush. A link is also provided here: `https://poe.com/Primitive_Partition`.

6. FUTURE RESEARCH DIRECTIONS

This paper builds a foundation for analysing the compositional complexity of geometric figures. The framework can be extended to the theory on higher dimensions and abstract metric spaces. Extending brush shapes b to allow sharp corner also post a new challenge to the existing theory. The immediate challenges below, however, follow directly from the results established in this paper and invite deeper investigation into the nature of $\in$-loops and their graph-theoretic representations.

1. **On the Tightness of the Chromatic Bound**

   A consistent observation across case study in Section 5 and our internal experiments is that the n-layer drawability of a figure precisely matched the chromatic number of its layer drawability graph. This empirical result motivates a strong conjecture:

   Conjecture 6.0.1. Let $S \subseteq \mathbb{R}^2$ be an image such that $\partial S = S = H_1(S)$. Then the n-layer drawability of S is

   $$n = \chi(G(P^*))$$

   While this conjecture is ambitious, its verification for all examined figures suggests a deep connection that is not yet fully understood. Another challenge is that the prerequisite on $H_1(S)$ satisfying the finite chain condition is crucial to apply Theorem 3 and hence verify the drawability of abstract figures. A primary task for future research is to either find a counterexample or identify the specific geometric conditions under which this equality holds. Readers can use the application in Appendix (Section 8.5) for exploration.

2. **Layer Drawability of Interiors and Boundaries**

   This paper excels at images that are equivalent to their boundaries. A natural and challenging extension is to consider figures with non-empty interiors, as noted in Section 4.4. This raises a key exploratory question:

   > *Let $S \subseteq \mathbb{R}^2$ be an image with a non-empty interior. If the layer drawability of the full set S is n and the layer drawability of its boundary ∂S is n^*, what is the relationship between n and n^*? Under what conditions does $n = n^*$ hold?*

   We strongly believe that a generalised version of Theorem 3, proving that the *interior* of figures can also be drawn if its boundary does not embody any $\in$-loops, is essential to answering the above question.

   Answering this question is essential to extending the theory from line drawings to solid shapes. A breakthrough would represent a major step toward a complete theory of drawability for general sets, with potential applications in fields including computational geometry and robotic path planning for tasks such as painting or manufacturing.

7. Conclusion

Classical topology is primarily concerned with the study of sets under operations such as union and intersection. This paper, in contrast, establishes a framework based on the ordered sequence of unions and complements, demonstrating that this is a viable and fruitful research direction. By integrating techniques from several mathematical fields, we have generalised the drawing tool, extended the framework to infinite sequences, and introduced the governing theory of n-layer drawability.

Our research also highlights key open problems for future investigation, such as the generalisation of Theorem 3 for a sufficient condition for drawability for the interior of regions as well, refining the upper bound presented in Theorem 5, and developing efficient methods for partitioning the interior of figures while preserving a minimal layer decomposition. It is our hope that the challenges detailed in Section 6 will inspire subsequent research, further enriching this field in the near future.

8. Appendix

8.1. **Preliminary Knowledge in Convex Geometry and Graph Theory.**

Preliminary 8.1.1 (Common notations in convex geometry)**.**

1. Define the *locally supporting half-plane* for S at x, denoted as $\mathcal{H}_S^-(x)$, as a closed half-plane where its boundary passes through x and $\exists\delta > 0, (B_\delta(x) \cap S) \subset \mathcal{H}_S^-(x)$, and define the *locally opposite half-plane* for S at x, denoted as $\mathcal{H}_S^+(x)$, as $\overline{(\mathcal{H}_S^-(x))^{\complement}}$.

   The existence of $\mathcal{H}_S^-(x)$ is guaranteed by the Supporting Hyperplane Theorem [2], and trivially, $\exists\delta > 0, \mathcal{H}_S^+(x) \cap B_\delta(x) \cap \operatorname{int}(S) = \varnothing$.

2. A line $L \subset \mathbb{R}^2$ is a *locally supporting line* for a set S at a point $x \in \partial S$ if $x \in L$ and S is locally contained in one of the closed half-planes defined by L. A locally supporting line is the boundary of a locally supporting half-plane.

3. For a region R and a differential point x, the *tangent line* of x is defined to be the line passing through x and parallel to the velocity vector of any parametrization of ∂R at x.

4. For a curve with a well-defined tangent line L at a point x, the *normal line* is the unique line passing through x that is perpendicular to L. Any vector parallel to this line is a *normal vector* to the curve at x.

5. The *curvature* of a differentiable curve at a point measures how quickly the curve changes direction at that point. For a curve given by a twice-differentiable function $y = f(x)$, the curvature K at a point $p = (k, f(k))$ is given by the formula:
$$K(p) = \frac{|f'(k)|}{(1 + (f'(k))^2)^{3/2}}$$
   We denote the curvature of a curve R at a point p as $K_R(p)$.

Preliminary 8.1.2 (Notations in graph theory)**.** Let $G = (V, E)$ be a graph.

1. **Neighbourhood**: The *neighbourhood* of a vertex $v \in V$, denoted $N(v)$, is the set of vertices u that is adjacent to v, i.e. $\{u, v\} \in E$.

2. **Independent Set**: A subset of vertices $I \subseteq V$ is an *independent set* if no two vertices in I are adjacent.

3. **Graph Colouring**:

   (a) A *k-colouring* of G is a function $c : V \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every edge $\{u, v\} \in E$.

(b) For a given colouring c, a *colour class* for a colour i is the set of all vertices assigned that colour, i.e., $C_i = \{v \in V \mid c(v) = i\}$.

(c) The *chromatic number* of G, denoted $\chi(G)$, is the minimum integer k for which a k-colouring exists.

(d) A colouring is *minimal* if it uses exactly $\chi(G)$ colours.

8.2. Proofs of Statements Involving Technical Arguments.

Proof of Point 2 from Definition 2.1.7.

$$\bigcup_{k=0}^{\infty}(A_{n+2k} \oplus b) \subseteq \bigcup_{k=0}^{\infty} \sigma_{n+2k} \subseteq \bigcup_{k=0}^{\infty} \overline{A_{n+2k} \oplus b}$$

$$\begin{aligned}\bigcup_{k=0}^{\infty}(A_{n+2k} \oplus b) &= \bigcup_{k=0}^{\infty}(\overline{A_{n+2k}} \oplus b) && \text{(by Proposition 8.3.7, as } b \text{ is open)}\\ &= b \oplus \bigcup_{k=0}^{\infty} \overline{A_{n+2k}} && \text{(by Proposition 8.3.5)}\end{aligned}$$

$$\begin{aligned}\bigcup_{k=0}^{\infty} \overline{A_{n+2k} \oplus b} &= \bigcup_{k=0}^{\infty}(\overline{A_{n+2k}} \oplus \overline{b}) && \text{(by Proposition 8.3.3, as } b \text{ is bounded)}\\ &= \overline{b} \oplus \bigcup_{k=0}^{\infty} \overline{A_{n+2k}} && \text{(by Proposition 8.3.5)}\\ &\subseteq \overline{b} \oplus \overline{\bigcup_{k=0}^{\infty} \overline{A_{n+2k}}} && \\ &= \overline{b \oplus \bigcup_{k=0}^{\infty} \overline{A_{n+2k}}} && \text{(by Proposition 8.3.2)}\end{aligned}$$

$$\bigcup_{k=0}^{\infty} \sigma_{n+2k} = \hat{\sigma}_n \qquad \text{(by definition)}$$

Hence, $b \oplus \bigcup_{k=0}^{\infty} \overline{A_{n+2k}} \subseteq \hat{\sigma}_n \subseteq \overline{b \oplus \bigcup_{k=0}^{\infty} \overline{A_{n+2k}}}$.

■

Proof of Lemma 2.1.8.

1. We prove the equality by showing that for any arbitrary point $y \in \mathbb{R}^2$, $y \in S \iff y \in D_\infty(\hat{\sigma})$.

   As $S = D_\infty(\sigma)$, and by definition of set theoretic limit, for any $y \in \mathbb{R}^2$, there is $N \in \mathbb{N}$ such that $\forall n > N$, $y \in S \iff y \in D_n(\sigma)$. So the set $I(y) := \{k \in \mathbb{N} \mid y \in D_{k-1}(\sigma) \iff y \in D_k(\sigma)\}$ must be finite.

   Note that the final state of y is determined by the parity of its last interaction. If $I(y) = \varnothing$, then $y \notin S$. Otherwise, $y \in S \iff \max I(y)$ is odd.

Let $\hat{I}(y) := \{k \in \mathbb{N} \mid y \in D_{k-1}(\hat{\sigma}) \iff y \in D_k(\hat{\sigma})\}$. The proof reduces to showing that $\hat{I}(y)$ is finite, and $\max I(y) = \max \hat{I}(y)$.

We know that $I(y)$ is finite. If $I(y) = \varnothing$, then there is no σ_n that includes y, so there is no $\hat{\sigma}_n$ that includes y either, then $\hat{I}(y) = \varnothing$ trivially. If $I(y) \neq \varnothing$, let $J = \max I(y)$. We know that $y \in \sigma_J$. Recall that $\hat{\sigma}_J = \bigcup_{k=0}^{\infty} \sigma_{J+2k}$. Since $y \in \sigma_J$, it follows that $y \in \hat{\sigma}_J$, which implies $J \in \hat{I}(y)$. For any integer $m > J$ of the opposite parity, $m \in \hat{I}(y)$ would require $y \in \hat{\sigma}_m$, which would require $y \in \sigma_{m+2k}$ for some $k \geq 0$. But $m + 2k > J$, which contradicts J being the maximum index in $I(y)$. Thus, $m \notin \hat{I}(y)$ for all $m > J$ of the opposite parity. Note that for any $l > J$ of the same parity, $\hat{\sigma}_l$ then cannot change the state of y (regardless of whether $y \in \hat{\sigma}_l$.

This establishes that $\hat{I}(y)$ is finite and $\max \hat{I}(y) = J$, so we have $y \in S \iff y \in D_\infty(\hat{\sigma})$. This holds for all $y \in \mathbb{R}^2$, hence $S = D_\infty(\hat{\sigma})$.

2. We prove by induction that, if $\neg\Delta_n(\hat{\sigma}, x)$ for $n \geq 2$, then $\neg\Delta_m(\hat{\sigma}, x)$ for all $m \geq n$. The inductive step is to show $\neg\Delta_k(\hat{\sigma}, x) \implies \neg\Delta_{k+1}(\hat{\sigma}, x)$ for any $k \geq n$.

   Assume $\neg\Delta_k(\hat{\sigma}, x)$ holds for some $k \geq n$. This means $\exists\delta > 0$ such that $D_k(\hat{\sigma}) \cap B_\delta(x) = D_{k-1}(\hat{\sigma}) \cap B_\delta(x)$. We show that this implies $D_{k+1}(\hat{\sigma}) \cap B_\delta(x) = D_k(\hat{\sigma}) \cap B_\delta(x)$.

   Case 1: $k + 1$ is odd.

$$\begin{aligned}
D_{k+1}(\hat{\sigma}) &= D_k(\hat{\sigma}) \cup \hat{\sigma}_{k+1} \\
D_{k+1}(\hat{\sigma}) \cap B_\delta(x) &= (D_k(\hat{\sigma}) \cap B_\delta(x)) \cup (\hat{\sigma}_{k+1} \cap B_\delta(x)) \\
&= (D_{k-1}(\hat{\sigma}) \cap B_\delta(x)) \cup (\hat{\sigma}_{k+1} \cap B_\delta(x)) \quad \text{(by assumption)} \\
&= ((D_{k-2}(\hat{\sigma}) \cup \hat{\sigma}_{k-1}) \cap B_\delta(x)) \cup (\hat{\sigma}_{k+1} \cap B_\delta(x)) \\
&= (D_{k-2}(\hat{\sigma}) \cap B_\delta(x)) \cup (\hat{\sigma}_{k-1} \cap B_\delta(x)) \cup (\hat{\sigma}_{k+1} \cap B_\delta(x))
\end{aligned}$$

   Since the strokes are of decreasing form, we can write $\hat{\sigma}_{k+1} \subseteq \hat{\sigma}_{k-1}$. The expression then equates to $(D_{k-2}(\hat{\sigma}) \cup \hat{\sigma}_{k-1}) \cap B_\delta(x) = D_{k-1}(\hat{\sigma}) \cap B_\delta(x)$.

   Case 2: $k + 1$ is even.

$$\begin{aligned}
D_{k+1}(\hat{\sigma}) &= D_k(\hat{\sigma}) \setminus \hat{\sigma}_{k+1} \\
D_{k+1}(\hat{\sigma}) \cap B_\delta(x) &= (D_k(\hat{\sigma}) \cap B_\delta(x)) \setminus (\hat{\sigma}_{k+1} \cap B_\delta(x)) \\
&= (D_{k-1}(\hat{\sigma}) \cap B_\delta(x)) \setminus (\hat{\sigma}_{k+1} \cap B_\delta(x)) \quad \text{(by assumption)} \\
&= ((D_{k-2}(\hat{\sigma}) \setminus \hat{\sigma}_{k-1}) \cap B_\delta(x)) \setminus (\hat{\sigma}_{k+1} \cap B_\delta(x)) \\
&= (D_{k-2}(\hat{\sigma}) \cap B_\delta(x)) \setminus ((\hat{\sigma}_{k-1} \cup \hat{\sigma}_{k+1}) \cap B_\delta(x))
\end{aligned}$$

   Since the strokes are of decreasing form, we can write $\hat{\sigma}_{k+1} \subseteq \hat{\sigma}_{k-1}$. The expression then equates to $(D_{k-2}(\hat{\sigma}) \setminus \hat{\sigma}_{k-1}) \cap B_\delta(x) = D_{k-1}(\hat{\sigma}) \cap B_\delta(x)$.

In both cases, we have shown that $D_{k+1}(\hat{\sigma}) \cap B_\delta(x) = D_{k-1}(\hat{\sigma}) \cap B_\delta(x)$. Combined with the assumption, this yields $D_{k+1}(\hat{\sigma}) \cap B_\delta(x) = D_k(\hat{\sigma}) \cap B_\delta(x)$, in other words, $\neg\Delta_{k+1}(\hat{\sigma}, x)$. Hence, by induction, $\neg\Delta_m(\hat{\sigma}, x)$ for all $m \geq n$.

3. Assume that $\forall \varepsilon > 0$ such that $D_k(\hat{\sigma}) \cap B_\varepsilon(x) \neq D_\infty(\hat{\sigma}) \cap B_\varepsilon(x)$. There must be a subsequence q of strokes of $\hat{\sigma}$, with strokes that come after k, of which all have parity opposite to k, and such that $x \in \overline{\bigcup_{k=1}^{\infty} q_k}$, as, if such q does not exist, then $\exists \delta > 0 \quad D_k(\hat{\sigma}) \cap B_\delta(x) = D_\infty(\hat{\sigma}) \cap B_\delta(x)$, which contradicts the assumption. But note that $q_1 = \bigcup_{k=1}^{\infty} q_k$, so $x \in \overline{q_1 \oplus b}$, and q_1 is some stroke in $\hat{\sigma}$ that comes after k with opposite parity to k, and q_1 would affect x. Contradiction! (as $\forall j > k \quad \neg\Delta_j(\hat{\sigma}, x)$) Hence, $\exists \delta > 0$ such that $D_k(\hat{\sigma}) \cap B_\delta(x) = D_\infty(\hat{\sigma}) \cap B_\delta(x)$.
4. Assume there exists a point $x \in \partial S$ such that $\hat{\sigma}_1$ does not affect it, i.e., $\neg\Delta_1(\hat{\sigma}, x)$, that is that $\exists r > 0$ such that $D_0(\hat{\sigma}) \cap B_r(x) = D_1(\hat{\sigma}) \cap B_r(x)$. In other words, $\hat{\sigma}_1 \cap B_\delta(x) = \varnothing$, from which follows that $\hat{\sigma}_1 \cap \hat{\sigma}_2^{\complement} \cap B_r(x) = \varnothing = \hat{\sigma}_1 \cap B_r(x)$, so $\neg\Delta_2(\hat{\sigma}, x)$.

   By property (2) of this lemma, we have $\neg\Delta_k(\hat{\sigma}, x)$ for all $k \geq 1$. And by property (3), we have a $\delta > 0$ such that

$$D_\infty(\hat{\sigma}) \cap B_\delta(x) = D_0(\hat{\sigma}) \cap B_\delta(x)$$
$$S \cap B_\delta(x) = \varnothing \text{ by property (1)}$$

   This contradicts that $x \in \partial S$, which requires that for any $\varepsilon > 0$, $B_\varepsilon(x) \cap S \neq \varnothing$. Therefore, $\Delta_1(\hat{\sigma}, x)$ must hold for any $x \in \partial S$.
5. By property (4) of this lemma, $\hat{\sigma}_1$ affects every point $x \in \partial S$. Then consider two cases for an arbitrary $x \in \partial S$:

   Case 1: $\exists k \geq 2$ such that $\hat{\sigma}_k$ does not affect x. Then by property (2) of this lemma, for all $j > k$, stroke $\hat{\sigma}_j$ does not affect x. By property (3) of this lemma, $\exists \delta > 0$ such that $D_k(\hat{\sigma}) \cap B_\delta(x) = D_\infty(\hat{\sigma}) \cap B_\delta(x)$. Hence, $\hat{\sigma}_k$ is the last stroke of x.

   Case 2: Every stroke in $\hat{\sigma}$ affects x.

   Hence, x either has a last stroke $\hat{\sigma}_k$, or every stroke in $\hat{\sigma}$ affects x.

■

Proof of Lemma 2.1.9. 1. Assume that $x \in \operatorname{int}(\sigma_j \oplus b)$ but $j \geq k$. Note that $\exists \delta_1 > 0$ such that $D_j(\sigma) \cap B_{\delta_1}(x) = S \cap B_{\delta_1}(x)$ as $\lambda_k(\sigma, x)$ holds. But $x \in \operatorname{int}(\sigma_j \oplus b)$ implies that there is $\delta_2 > 0$ such that $D_j(\sigma) \cap B_{\delta_2}(x) = \varnothing$ or $D_j(\sigma)^{\complement} \cap B_{\delta_2}(x) = \varnothing$. Take $\delta_3 = \min(\delta_1, \delta_2)$. If $D_j(\sigma) \cap B_{\delta_2}(x) = \varnothing$, then $D_j(\sigma) \cap B_{\delta_3}(x) = \varnothing$, and as $D_j(\sigma) \cap B_{\delta_3}(x) = S \cap B_{\delta_3}(x)$, $S \cap B_{\delta_3}(x) = \varnothing$. If $D_j(\sigma)^{\complement} \cap B_{\delta_2}(x) = \varnothing$, then $D_j(\sigma)^{\complement} \cap B_{\delta_3}(x) = \varnothing$, and as

$$D_j(\sigma) \cap B_{\delta_3}(x) = S \cap B_{\delta_3}(x)$$

$$\begin{aligned}
(D_j(\sigma) \cap B_{\delta_3}(x))^{\complement} \cap B_{\delta_3}(x) &= (S \cap B_{\delta_3}(x))^{\complement} \cap B_{\delta_3}(x) \\
(D_j(\sigma)^{\complement} \cup B_{\delta_3}(x)^{\complement}) \cap B_{\delta_3}(x) &= (S^{\complement} \cup B_{\delta_3}(x)^{\complement}) \cap B_{\delta_3}(x) \\
D_j(\sigma)^{\complement} \cap B_{\delta_3}(x) &= S^{\complement} \cap B_{\delta_3}(x) \\
S^{\complement} \cap B_{\delta_3}(x) &= \varnothing
\end{aligned}$$

$S \cap B_{\delta_3}(x) = \varnothing$ or $S^{\complement} \cap B_{\delta_3}(x) = \varnothing$ directly contradicts the fact that $x \in \partial S$ (i.e. $\forall \varepsilon > 0$, $B_\varepsilon(x) \cap S \neq \varnothing$ and $B_\varepsilon(x) \cap S^{\complement} \neq \varnothing$). Hence, $k > j$.

2. *Proving $x \in \partial\sigma_k$:* As $\Delta_k(\sigma, x)$ holds, we know that $x \in \overline{\sigma_k}$ by Proposition 2.1.2. Assume that $x \in \mathrm{int}(\sigma_k)$, but by (1) of this Lemma, this results in $k > k$. Contradiction! So $x \notin \mathrm{int}(\sigma_k)$. Hence, $x \in \partial\sigma_k$.

   Proving that σ_k is one-sided on x: Assume σ_k is not one-sided on x. Then, by definition of one-sidedness, this means that for all $\varepsilon > 0$, $\sigma_k \cap B_\varepsilon(x) \cap S \neq \varnothing$ and $\sigma_k \cap B_\varepsilon(x) \cap S^{\complement} \neq \varnothing$.

   From the definition of a last stroke $\lambda_k(\sigma, x)$, there exists a $\delta > 0$ such that $D_k(\sigma) \cap B_\delta(x) = S \cap B_\delta(x)$. Consider two cases based on the parity of k.

   When k is odd, by assumption, there is $p \in \sigma_k \cap B_\delta(x) \cap S^{\complement}$. Since $p \in \sigma_k$, $p \in D_k(\sigma)$, then we have $p \in D_k(\sigma) \cap B_\delta(x) = S \cap B_\delta(x)$. But $p \in S^{\complement}$. Contradiction! The case when k is even follows a symmetric proof.

   Therefore, σ_k is one-sided on x.

Hence, the lemma is proved. ■

Proof of Lemma 2.1.10.

1. We know that $\forall y \in R^+, \forall m \in \mathbb{N}, \quad y \in \sigma_{2m-1}$, and by definition of set theoretic limit, there is $N \in \mathbb{N}$ s.t. $\forall n > N \quad y \in S \iff y \in D_n(\sigma)$, so $y \in S$. A symmetric proof goes for R^-.

2. Let $\hat{\sigma}^+$ be all the odd strokes of $\hat{\sigma}$. We want to prove that $R = \bigcap_{n=1}^{\infty} \sigma_n^+$ is in the form of a stroke, that is, there is non-empty $r \subseteq \mathbb{R}^2$ such that $r \oplus b \subseteq R \subseteq \overline{r \oplus R}$.

   Recall that for each n, $\hat{A}_n^+ \oplus b \subseteq \hat{\sigma}_n^+ \subseteq \overline{\hat{A}_n^+ \oplus b}$, where $(\hat{A}_n^+)$ are the corresponding steps of $\hat{\sigma}^+$.This implies that $\bigcap_{n=1}^{\infty} (\hat{A}_n^+ \oplus b) \subseteq \bigcap_{n=1}^{\infty} \hat{\sigma}_n^+ \subseteq \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+ \oplus b}$ and so $\bigcap_{n=1}^{\infty} (\hat{A}_n^+ \oplus b) \subseteq R \subseteq \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+ \oplus b}$. We want prove that $r \oplus b \subseteq R \subseteq \overline{r \oplus b}$ with $r = \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}$.

We first show that $b \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+} \subseteq \bigcap_{n=1}^{\infty} (\hat{A}_n^+ \oplus b)$. Fix any $y \in b \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}$. There is $a \in \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}$ such that $y \in a \oplus b$. For any $n \in \mathbb{N}$, $a \in \overline{\hat{A}_n^+}$, so $a \oplus b \subseteq \overline{\hat{A}_n^+} \oplus b$ and so $y \in \bigcap_{n=1}^{\infty} (\overline{\hat{A}_n^+} \oplus b)$, which is equal to $\bigcap_{n=1}^{\infty} (\hat{A}_n^+ \oplus b)$. Hence $b \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+} \subseteq \bigcap_{n=1}^{\infty} (\hat{A}_n^+ \oplus b)$.

Then we show that $\bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+ \oplus b} \subseteq \overline{b \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}}$. Recall that each $\overline{\hat{A}_n^+ \oplus b} = \overline{\hat{A}_n^+} \oplus \overline{b}$ (Proposition 8.3.3), so $\bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+ \oplus b} = \bigcap_{n=1}^{\infty} (\overline{\hat{A}_n^+} \oplus \overline{b})$. As $(\overline{\hat{A}_n^+})$ is a sequence of closed sets, and $\overline{b}$ is compact, we have $\bigcap_{n=1}^{\infty} (\overline{\hat{A}_n^+} \oplus \overline{b}) = \overline{b} \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}$ (Proposition 8.3.5). As $\bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+} \subseteq \overline{\bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}}$, we have $\overline{b} \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+} \subseteq \overline{b} \oplus \overline{\bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}}$, which is equal to $\overline{b \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}}$ (Proposition 8.3.3). Hence $\bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+ \oplus b} \subseteq \overline{b \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}}$.

Thus we have shown that $b \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+} \subseteq \bigcap_{n=1}^{\infty} (\hat{A}_n^+ \oplus b) \subseteq R \subseteq \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+ \oplus b} \subseteq \overline{b \oplus \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}}$. In other words, $r \oplus b \subseteq R \subseteq \overline{r \oplus b}$, where $r = \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}$. A symmetric proof goes for R^-.

3. By property 1, $R^+ \subseteq S$, and $\operatorname{int}(S) \cap \partial S = \varnothing$, so $\operatorname{int}(R^+) \cap \partial S = \varnothing$. Same goes for R^-.
4. We show the two properties for $R = R^+, R^-$.

   Firstly, we are proving $x \in \partial R$. We will show that $x \notin \operatorname{int}(R)$ and $x \in \overline{R}$.

   By property 1, $R^+ \subseteq S$ and $R^- \subseteq S^{\complement}$. Since $x \notin \operatorname{int}(S)$ and $x \notin \operatorname{int}(S^{\complement})$, we have $x \notin \operatorname{int}(R^+)$ and $x \notin \operatorname{int}(R^-)$.

   Then we show $x \in \overline{R}$ for $R = R^+$. Let $\hat{\sigma}^+$ be the odd strokes of $\hat{\sigma}$, and $\hat{A}^+$ be the corresponding odd steps.

   Since each stroke $\hat{\sigma}_n^+$ affects x, $x \in \overline{\hat{\sigma}_n^+}$ must hold for each $n \in \mathbb{N}$ (by Proposition 2.1.2). Therefore, $x \in \bigcap_{n=1}^{\infty} \overline{\hat{\sigma}_n^+} = \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+ \oplus b} = \bigcap_{n=1}^{\infty} (\overline{\hat{A}_n^+} \oplus \overline{b})$ (Proposition 8.3.3).

Because $(\overline{\hat{A}_n^+})$ is a decreasing sequence of closed sets and $\overline{b}$ is compact, we can write

$$x \in \bigcap_{n=1}^{\infty} (\overline{\hat{A}_n^+} \oplus \overline{b}) = \left(\bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+} \right) \oplus \overline{b}. \qquad \text{(Proposition 8.3.5)}$$

Let $r^+ = \bigcap_{n=1}^{\infty} \overline{\hat{A}_n^+}$ By property 2, $r^+ \oplus b \subseteq R^+ \subseteq \overline{r^+ \oplus b}$, so $\overline{R^+} = \overline{r^+ \oplus b} = \overline{r^+} \oplus \overline{b}$ (Proposition 8.3.1 and 8.3.3). Since we have shown $x \in r^+ \oplus \overline{b}$, it follows that $x \in \overline{r^+} \oplus \overline{b} \subseteq \overline{R^+}$. A symmetric proof goes for $R = R^-$. Combining both, we conclude that $x \in \partial R$.

To prove R is one-sided on x, for $R = R^+$, as $R^+ \subseteq S$ (by property 1), we can write $R^+ \cap S^{\complement} \neq \varnothing$, so we can simply pick some $\delta > 0$ and conclude that $B_\delta(x) \cap \operatorname{int}(R^+) \cap S^{\complement} = \varnothing$. A symmetric proof goes for $R = R^-$.

Thus we have shown that $x \in \partial R$, and R is one-sided on x for $R = R^+, R^-$. ■

Proof of Proposition 3.0.2. Parameterize ∂S with $\mathbf{r}(t)$, and let $\mathbf{r}(t_0) = x$. When ∂S is not differentiable at x, $\mathbf{v}_1 := \lim_{t \to t_0^-} \mathbf{r}'(t) \neq \lim_{t \to t_0^+} \mathbf{r}'(t) =: \mathbf{v}_2$. Then, take two infinite sequences of points on ∂S, $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$, that converges to x from both sides respectively. Note that $\lim_{n \to \infty} \mathcal{H}_S^+(a_n) = (\mathcal{H}_S^+(x))_1$ have its boundary parallel to $\mathbf{v}_1$ and $\lim_{n \to \infty} \mathcal{H}_S^+(b_n) = (\mathcal{H}_S^+(x))_2$ as such for $\mathbf{v}_2$. Let L_1 and L_2 be the distinct locally supporting lines for the boundary of $(\mathcal{H}_S^+(x))_1$ and $(\mathcal{H}_S^+(x))_2$ of S respectively.

Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward-pointing unit normal vectors to the lines L_1 and L_2, respectively. Since the lines are distinct, $\mathbf{n}_1 \neq \mathbf{n}_2$. For any point $y \in S$, the definition of a locally supporting line implies:

$$(y - x) \cdot \mathbf{n}_1 \leq 0 \quad \text{and} \quad (y - x) \cdot \mathbf{n}_2 \leq 0$$

Now, consider any vector $\mathbf{n}$ in the cone formed by $\mathbf{n}_1$ and $\mathbf{n}_2$. Specifically, let $\mathbf{n}(t)$ be any convex combination of the two normal vectors:

$$\mathbf{n}(t) = (1 - t)\mathbf{n}_1 + t\mathbf{n}_2, \quad \text{for } t \in (0, 1)$$

Since $\mathbf{n}_1 \neq \mathbf{n}_2$, each choice of $t \in (0, 1)$ produces a unique direction.
Let L_t be the line passing through x with normal vector $\mathbf{n}(t)$. We can check if L_t is also a locally supporting line for S. For any point $y \in S$, we have:

$$\begin{aligned}(y - x) \cdot \mathbf{n}(t) &= (y - x) \cdot ((1 - t)\mathbf{n}_1 + t\mathbf{n}_2) \\ &= (1 - t)((y - x) \cdot \mathbf{n}_1) + t((y - x) \cdot \mathbf{n}_2)\end{aligned}$$

Since $(1 - t) > 0$, $t > 0$, and both dot products are non-positive, their weighted sum is also non-positive:

$$(y - x) \cdot \mathbf{n}(t) \leq 0$$

This inequality holds for all $y \in S$. Therefore, the entire set S lies on one side of the line L_t. This means that for every $t \in (0, 1)$, L_t is a valid locally supporting line for S at x. ■

Proof of Proposition 3.0.3. Similar to Proposition 3.0.2, we let a sequence of points converging to x from both sides. However, since x is differentiable, the set limit of the locally opposite half-planes of the points in both sequences converges to the one and only locally opposite half-plane at x. The boundary of $\mathcal{H}_S^+(x)$, defined as ℓ, is the tangent line of S at x by this construction.

Assume there exists some locally convex region R such that ∂R is differentiable at x and $\exists \mathcal{H}_S^+(x)'$, where $\mathcal{H}_S^+(x) \neq \mathcal{H}_S^+(x)'$. Since ℓ is the tangent line of R at x, the boundary of $\mathcal{H}_S^+(x)'$, defined as ℓ', is not tangent to R.

Then, ℓ intersects ℓ' at an angle. Since ∂R is differentiable at x, and ℓ' is not tangent to R, ℓ' will cut the interior of R. Then, $\mathcal{H}_S^+(x)'$ will also cut the interior of R, violating the property of an locally opposite half-plane. Contradiction! Hence, the proposition follows.

■

Proof of Lemma 3.1.2. Assume there exists $x \in \partial S, \exists \alpha \in \lambda_b(x)$, where α is not tangent to S at x. Then, the tangent line of α at x is not equal to the tangent line of S at x.

These two tangent lines divide a small disk $B_\delta(x)$ into four open angular regions.

- Region 1: $\operatorname{int}(H_S^+(x)) \cap \operatorname{int}(H_\alpha^+(x))$. Points here are outside both S and α.
- Region 2: $\operatorname{int}(H_S^-(x)) \cap \operatorname{int}(H_\alpha^-(x))$. There exists points here that are inside of S and inside of α.
- Region 3: $\operatorname{int}(H_S^+(x)) \cap \operatorname{int}(H_\alpha^-(x))$. There exists points here that are outside of S but inside of α.
- Region 4: $\operatorname{int}(H_S^-(x)) \cap \operatorname{int}(H_\alpha^+(x))$. There exists points here that are inside of S but outside of α.

The existence of points from α in both Region 2 (inside S) and Region 3 (outside S) means that $\forall \varepsilon > 0, B_\varepsilon(x) \cap \alpha \cap \operatorname{int}(S) \neq \varnothing$ and $B_\varepsilon(x) \cap \alpha \cap \operatorname{int}(S^{\complement}) \neq \varnothing$. This violates the one-sidedness condition. Therefore, $\alpha \notin \lambda_b(x)$. This contradicts with the initial assumption, hence proving the lemma. ■

Proof of Proposition 3.1.3. x has a unique tangent line. Since b is strictly convex, the set of tangent vectors of all x' in ∂b forms a bijection with its global angle $[0, 2\pi)$, given by a fundamental property of strictly convex figures. Therefore, only two points, p_1 and p_2 in ∂b have their tangent vector parallel to the tangent line of S at x.

To construct an α that touches S at the point x, it is necessary to shift b so that a point ends up on x. This results in two unique translation vectors: $v_1 = x - p_1$ and $v_2 = x - p_2$. These translations, $b \oplus v_1$ and $b \oplus v_2$, make b contact S solely at x locally. The tangent line for both at x aligns with their tangent vector, which parallels the tangent line of S at x. Thus, they coincide with the same tangent line at x, rendering them tangential to S at x. These are the only translations that can achieve this configuration. ■

Proof of Proposition 4.2.2. For all points in a primitive arc, its curvature is non-zero. (if curvature is 0, Theorem 4 implies they satisfy $|\lambda_b(x)| = 2$ and therefore not in $H_1(S)$) Since S is C^2 continuous guarantees every point exists a curvature, the curvature at all points in the primitive arc must be strictly positive or negative. Then, by definition, a primitive arc is a strictly convex curve. By the Supporting Hyperplane Theorem, for any point in the arc x, we can always find a supporting half-plane, and therefore an opposite half-plane of x that intersects the arc only at x. Since the only hypothetical last stroke of any x that lies on the arc, α_x^+ (see Proposition 3.1.6), also lies completely in the opposite half-plane of x, α_x^+ must also only intersect the arc at x, and therefore it is impossible that any α_x^+ contains any other point in the arc, hence any finite set of points on a primitive arc does not inherit a $\in$-loop. ■

Proof of Proposition 4.2.3. Let $A(t) = A \oplus f(t)$. We define the distance function $d(t) = \min_{x \in A(t)} SDF_B(x)$, where $SDF_B(x)$ is the signed distance to B. The continuity of $d(t)$ follows from the 1-Lipschitz property of the signed distance function and the continuity of the path $f(t)$, since it can be shown that $|d(t+\delta) - d(t)| \leq \|f(t+\delta) - f(t)\|$.

The initial conditions state that $d(t_0) > 0$ and $d(t_1) \leq 0$. By the Intermediate Value Theorem, there exists $t' \in [t_0, t_1]$ where $d(t') = 0$, implying that for every point x on the curve $A(t')$, its signed distance from B is non-negative. Then entire curve $A(t')$ must lie in the closed region exterior to B.

Let p be any point of contact between $A(t')$ and B. Because all of $A(t')$ lies outside or on the boundary of B, the contact at p is necessarily external. As both curves are C^2 and B is strictly convex, they must share a common tangent line at p, and hence fulfilling the definition of externally tangent. ■

Proof of Lemma 4.2.9. Assume for contradiction, $\forall \{u, v\} \in E, C(S_u) \cap C(S_v) = \varnothing$. Define a new colouring c^* for G such that $c^*(v) = k_v$, where $k_v \in C(S_v)$.

To show that c^* is a proper colouring, let $\{u, v\}$ be an arbitrary edge in E. By construction, $c^*(u) \in C(S_u)$ and $c^*(v) \in C(S_v)$. From the assumption, since $\{u, v\} \in E$, the sets $C(S_u)$ and $C(S_v)$ must be disjoint, hence they use distinct colours, and therefore, $c^*(u) \neq c^*(v)$.

Then, by definition, c' uses the colours $\{c_2, \ldots, c_n\}$. Clearly, for every vertex $v \in V'$, $C(S_v)$ use only the colours within $\{c_2, \ldots, c_n\}$. By construction, $c^*(v) = k_v \in C(S_v)$, so every colour used in c^* must be an element of $\{c_2, \ldots, c_n\}$. Hence, we showed that c^* is a proper colouring of G, implying $\chi(G) = n - 1$. Contradiction! Hence, the lemma is proved. ■

Proof of Theorem 7. Let the brush b be the open unit disk, $B_1(0)$. Consider the family of figures S_m constructed as the union of m disks, $C_0, C_1, \ldots, C_{m-1}$, where the k-th disk C_k has radius $r_k = 2^{-k}$ and is centred at $p_k = (r_k+1)(\cos(2^{-k}\pi), \sin(2^{-k}\pi))$.

First, we establish that $S_m = H_1(S_m)$. Note that ∂S_m is locally convex and C^2 continuous everywhere. The curvature of the k-th disk is $K_{C_k}(x) = 1/r_k = 2^k$. Since the brush is a unit disk, its curvature is $K_b(x') = 1$ for any $x' \in \partial b$. As $k \geq 0$, the curvature of every disk is $K_{C_k}(x) = 2^k \geq 1$. By Theorem 4, $|\lambda_b(x)| = 1$. This condition holds for all points on all disks except for C_0, where the curvatures on ∂C_0 are equal. For this special case $(K_S(x) = K_b(x'))$, a local analysis shows that the agreeingly tangent stroke α_x^- is not one-sided, thus $|\lambda_b(x)| = 1$ for all points on ∂S_m.

Consider any pair of distinct disks, C_i and C_j. By design, the unique hypothetical last stroke at the point on C_i and C_j closest to the origin is the open unit disk centred at the origin. The unit open disk intersects every disk. Therefore, for any pair of disks C_i, C_j with $i \neq j$, they *embody an* $\in$*-loop.*

Hence, in the layer drawability graph $G(P^*)$ for S_m. every pair of vertices (C_i, C_j) for $i \neq j$ is connected by an edge, implying $G(P^*)$ is the complete graph on m vertices, K_m. The chromatic number of a complete graph K_m is notoriously m. By Theorem 5, the layer drawability of $S_m$is bounded below by

$$n \geq \chi(G(P^*)) = \chi(K_m) = m$$

Thus, S_m is at least *m-layer drawable.* For the upper bound, a valid layer decomposition is to simply assign each disk C_k to its own layer. Since each individual disk is drawable by that each disk is a convex set (from [1]), this is a valid m-layer decomposition. Therefore, the n-layer drawability is at most m. Hence, the theorem follows. ■

Proof of Lemma 4.3.4. Let $C(x, Y, z)$ be a circle tangent to the line segment Y at x and passes through the point z. $C(p_k, \ell_k, p_j)$ for any $p_j \in \ell_j$ and any $p_k \in \ell_k$ where $\{k, j\} \notin \{1, 3\} \cup \{2, 4\}$ is well-defined (1). This is because p_j can never be collinear to ℓ_k, and since all line segments are closed, the shortest distance between p_j and the line extended indefinitely from ℓ_k must be greater than 0.

Note that a circle tangent to ℓ_k at p_k with a radius longer than that of $C(p_k, \ell_k, p_j)$ covers the point p_j (2). Consider $X_{kj} = \max(\{\max(\{|C(p_k, \ell_k, p_j)| : p_j \in \ell_j\}) : p_k \in \ell_k\})$, and let $k_0 = \max(X_{12}, X_{21}, X_{23}, X_{32}, X_{34}, X_{43}, X_{41}, X_{14})$. From (1), since the circles are well-defined, a maximum among them must exist, and k is well-defined. Scale L in $L' = \{\ell_1', \ell_2', \ell_3', \ell_4'\}$ by the factor $k > \dfrac{1}{k_0}$. Since $k \cdot k_0 < 1$, the unit circle tangent to ℓ_k' at p_k' must cover all p_j' on ℓ_j'. Therefore, any 4 points p_1', p_2', p_3', p_4' where $p_k' \in \ell_k'$ must inherit a $\in$-loop, satisfying the second condition of a cross. Hence, such cross exists, and the lemma is proved. ■

Proof of Proposition 4.3.7. Suppose the statement is false. Then, for all scale-down figure $Ł_n'$ of $Ł_n$, there exists a layer decomposition $Ł_n' = \bigcup_{k=1}^{n} S_k'$, that there exists $k \in \{1, 2, \ldots, n\}$ such that we can pick $p^{+\prime} \in ł^{+\prime}$ and $p^{-\prime} \in ł^{-\prime}$, $\{p^{+\prime}, p^{-\prime}\} \subset S_k'$.

Pick $p'_{\text{left}} \in L'_{\text{left}}$ and $p'_{\text{right}} \in L'_{\text{right}}$ such that $p'_{\text{left}} \in S'_k$ and $p'_{\text{right}} \in S'_k$. If such pick is impossible, then the drawable partitions of L'_{left} or L'_{right} do not involve S'_k, implying L'_{left} or L'_{right} are $(n-1)$-layer drawable, which violates the assumptions that both L'_{left} and L'_{right} are n-layer drawable.

Now, $\{p'_{\text{left}}, p^{+'}, p'_{\text{right}}, p^{-'}\} \subset S'_k$. We know $p^{+'} \in \ell^{+'}$ and $p^{-'} \in \ell^{-'}$. Let the line segment where p'_{left} and p'_{right} lie on be ℓ'_{left} and ℓ'_{right} respectively. By definition of Ł_n, $\{\ell^{+'}, \ell^{-'}, \ell'_{\text{left}}, \ell'_{\text{right}}\}$ satisfies the positional criteria in Definition 4.3.2. By Lemma 4.3.4, we can find an $\text{Ł}'_n$ scaled down even more such that $\{\ell^{+'}, \ell^{-'}, \ell'_{\text{left}}, \ell'_{\text{right}}\}$ forms a cross. Then, by definition, $p'_{\text{left}}, p^{+'}, p'_{\text{right}}, p^{-'}$ inherits a $\in$-loop. However, by assumption, $\{p^+, p^-\} \subset S_k$ holds for all scaled down versions of $\{p^+, p^-\}$, and $\{p'_{\text{left}}, p'_{\text{right}}\} \subset S'_k$. Then, we have 4 points in the same supposedly drawable layer that inherit a $\in$-loop, resulting in a contradiction.

Therefore, a desired scale-down figure $\text{Ł}'_n$ exists under the assumption. ■

8.3. **Extra Properties of Minkowski Sum.**

Proposition 8.3.1. If for some $t \in \mathbb{R}^2$, $T \in \mathbb{R}^2$, and brush b, $t \oplus b \subseteq T \subseteq \overline{t \oplus b}$, then the following hold:

1. $\overline{t \oplus b} = \overline{T}$
2. $t \oplus b \subseteq \operatorname{int} T$
3. $\partial(t \oplus b) \supseteq \partial T$

Proof. 1. From $t \oplus b \subseteq T$, it follows that $\overline{t \oplus b} \subseteq \overline{T}$, and with $T \subseteq \overline{t \oplus b}$, we have $\overline{T} = \overline{t \oplus b}$.

2. From $t \oplus b \subseteq T$, it follows that $\operatorname{int}(t \oplus b) \subseteq \operatorname{int}(T)$. Since $t \oplus b$ is open, it is equal to $\operatorname{int}(t \oplus b)$, so we have $t \oplus b \subseteq \operatorname{int} T$.
3. From (1) of this proposition, we have $\overline{t \oplus b} = \overline{T}$, and from (2), we have $t \oplus b \subseteq \operatorname{int} T$. Combining the two, we get $\overline{t \oplus b} \setminus (t \oplus b) \supseteq \overline{t \oplus b} \setminus (t \oplus b)$. In other words, $\partial(t \oplus b) \supseteq \partial T$.

■

Proposition 8.3.2. For any $P, Q \subseteq \mathbb{R}^2$, $\overline{P} \oplus \overline{Q} \subseteq \overline{P \oplus Q}$.

Proof. Fix any $z \in \overline{P} \oplus \overline{Q}$, we can then rewrite z as $z = p + q$, where there are sequences $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ such that each $p_n \in P$ and $q_n \in Q$, and that $p = \lim_{n\to\infty} p_n$ and $q = \lim_{n\to\infty} q_n$. Note that for each $n \in \mathbb{N}$, $p_n + q_n \in P \oplus Q$, so $\lim_{n\to\infty} (p_n + q_n) \in \overline{P \oplus Q}$. As $\lim_{n\to\infty} (p_n + q_n) = \lim_{n\to\infty} p_n + \lim_{n\to\infty} q_n = p + q = z$. We can conclude that $z \in \overline{P \oplus Q}$. Hence, $\overline{P} \oplus \overline{Q} \subseteq \overline{P \oplus Q}$. ■

Proposition 8.3.3. If either $P \subseteq \mathbb{R}^2$ or $Q \subseteq \mathbb{R}^2$ is bounded, $\overline{P} \oplus \overline{Q} = \overline{P \oplus Q}$ holds.

Proof. ($\supseteq$): WLOG, Q is bounded. Fix any $z \in \overline{P \oplus Q}$. So, there is a sequence $(a_n)_{n\in\mathbb{N}}$ such that each $a_n \in P \oplus Q$ and that $z = \lim_{n\to\infty} a_n$. Then for each $n \in \mathbb{N}$, $a_n = p_n + q_n$, where $p_n \in P$ and $q_n \in Q$. Note that as Q is bounded, the sequence $(q_n)_{n\in\mathbb{N}}$ is also bounded. So by the Bolzano-Weierstrass Theorem, there is a subsequence $(q_{n_k})_{k\in\mathbb{N}}$ such that (q_{n_k}) converges to some q. Note that $q \in \overline{Q}$. Recall that $(a)_{n_k}$ converges to z, and as $a_{n_k} = p_{n_k} + q_{n_k}$, p_{n_k} converges to some $p = \lim_{n\to\infty}(a_{n_k} - q_{n_k}) = z - q$. Note that $p \in \overline{P}$. With any $z = p + q$ where $p \in \overline{P}$ and $q \in \overline{Q}$, we can conclude that $\overline{P \oplus Q} \subseteq \overline{P} \oplus \overline{Q}$.

($\subseteq$): $\overline{P} \oplus \overline{Q} \subseteq \overline{P \oplus Q}$ holds in general.
Hence, $\overline{P} \oplus \overline{Q} = \overline{P \oplus Q}$. ∎

Proposition 8.3.4. If either $P \subseteq \mathbb{R}^2$ or $Q \subseteq \mathbb{R}^2$ is bounded, $\partial(P \oplus Q) \subseteq \partial P \oplus \partial Q$ holds.

Proof. Fix any $z \in \partial(P \oplus Q)$. Note that $z \in \overline{P \oplus Q} = \overline{P} \oplus \overline{Q}$ (as either P or Q is bounded), so $\exists p \in \overline{P} \quad \exists q \in \overline{Q}$ such that $z = p + q$. Assume that $z \notin \partial P \oplus \partial Q$, that is, $\forall x \in \partial P \quad \forall y \in \partial Q \quad z \neq x + y$. Then $p \notin \partial P$ or $q \notin \partial Q$, and so $p \in \operatorname{int}(P)$ or $q \in \operatorname{int}(Q)$. WLOG, $p \in \operatorname{int}(P)$ holds, then $\exists \delta > 0 \quad B_\delta(p) \subset P$. As $q \in \overline{Q}$, $\exists q' \in Q$ such that $|q - q'| < \delta$. Since $\delta < |q - q'| = |z - p + q'| = |z - (p + q')|$, $z \in B_\delta(p+q') = B_\delta(p) \oplus q' \subset P \oplus q' \subseteq P \oplus Q$. Hence $z \in \operatorname{int}(P \oplus Q)$. Contradiction! Hence, $\partial(P \oplus Q) \subseteq \partial P \oplus \partial Q$ holds. ∎

Proposition 8.3.5. Let $P \subseteq \mathbb{R}^2$ be compact, and $(Q_n)_{n\in\mathbb{N}}$ be a decreasing sequence such that for each $n \in \mathbb{N}$, $Q_n \subseteq \mathbb{R}^2$ is closed. $P \oplus \bigcap_{n=1}^{\infty} Q_n = \bigcap_{n=1}^{\infty}(P \oplus Q_n)$ holds.

Proof. ($\subseteq$): Fix any $z \in P \oplus \bigcap_{n=1}^{\infty} Q_n$. So, $z = p + q$ for some $p \in P$ and $q \in Q_n$ for any $n \in \mathbb{N}$. In other words, $p + q \in P \oplus Q_n$ for each $n \in \mathbb{N}$, which means that $z = p + q \in \bigcap_{n=1}^{\infty}(P \oplus Q_n)$. Hence, $P \oplus \bigcap_{n=1}^{\infty} Q_n \subseteq \bigcap_{n=1}^{\infty}(P \oplus Q_n)$ holds.

($\supseteq$): Fix any $z \in \bigcap_{n=1}^{\infty}(P \oplus Q_n)$. So, for any $n \in \mathbb{N}$, $z \in P \oplus Q_n$. In other words, $z = p_n + q_n$ with $p_n \in P$ and $q_n \in Q_n$. Recall that P is compact, so there is subsequence $(p_{n_k})_{k\in\mathbb{N}}$ that converges to some $p \in P$. This implies that (q_{n_k}) converges to some $q = x - p$. Now we need to prove that $q \in \bigcap_{m=1}^{\infty} Q_m$, that is, that for any $m \in \mathbb{N}$, $q \in Q_m$. Fix any $m \in \mathbb{N}$ and any k such that $n_k > m$. Since (Q_n) is decreasing, $q_{n_j} \in Q_{n_j} \subseteq Q_m$ for any $j > k$. As Q_m is closed, it contains all its limit points, so $q \in Q_m$. So $q \in \bigcap_{m=1}^{\infty} Q_m$. Recall that $z = p + q \subseteq P \oplus \bigcap_{n=1}^{\infty} Q_n$, so $P \oplus \bigcap_{n=1}^{\infty} Q_n \supseteq \bigcap_{n=1}^{\infty} (P \oplus Q_n)$.

Hence, $P \oplus \bigcap_{n=1}^{\infty} Q_n = \bigcap_{n=1}^{\infty} (P \oplus Q_n)$. ■

Proposition 8.3.6. For any $P \subseteq \mathbb{R}^2$ and $Q \subseteq 2^{\mathbb{R}^2}$ (that is, for each $M \in Q$, $M \subseteq \mathbb{R}^2$), $P \oplus \bigcup_{M \in Q} M = \bigcup_{M \in Q} (P \oplus M)$ holds.

Proof. For any $z \in \mathbb{R}^2$,

$$\begin{aligned} & z \in P \oplus \bigcup_{M \in Q} M \\ \iff & \exists p \in P \exists M \in Q \quad z \in p \oplus M \\ \iff & \exists M \in Q \exists p \in P \quad z \in p \oplus M \\ \iff & \exists M \in Q \quad z \in P \oplus M \\ \iff & z \in \bigcup_{M \in Q} (P \oplus M) \end{aligned}$$

Hence, $P \oplus \bigcup_{M \in Q} M = \bigcup_{M \in Q} (P \oplus M)$ holds.

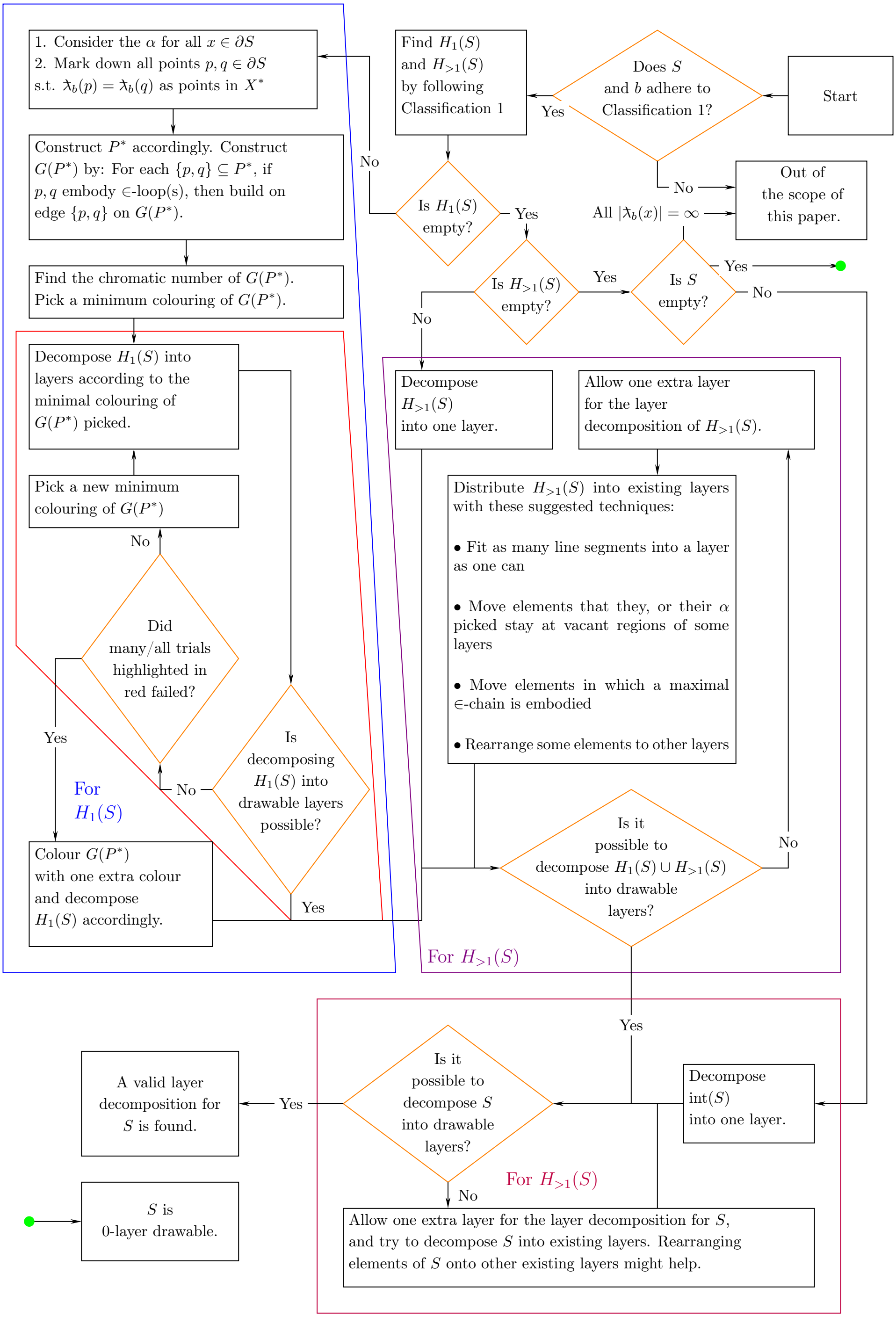
■

Proposition 8.3.7. For any $P \subseteq \mathbb{R}^2$ and open $Q \subseteq \mathbb{R}^2$, $P \oplus Q = \overline{P} \oplus Q$ holds.

Proof. ($\subseteq$): Since $P \subseteq \overline{P}$, this is trivial.
($\supseteq$): Fix any $z \in \overline{P} \oplus Q$, then there is $q \in Q$ and $p \in \overline{P}$ such that $z = p + q$. Note that as Q is open, there is $\delta > 0$ such that $B_\delta(q) \subset Q$. Since $p \in \overline{P}$, we can choose $p' \in B_\delta(p) \cap P$, which implies that $|p' - p| < \delta$. Let $u = p - p' + q$, and note that $|u - q| = |p - p'| < \delta$, so $u \in B_\delta(q) \subset Q$. Then $p - p' + q \in Q$, which implies that $p + q \in p' \oplus Q$, from which follows that $p + q \in P \oplus Q$, so $z \in P \oplus Q$. Hence, $\overline{P} \oplus Q \subseteq P \oplus Q$.

Hence, we have $P \oplus Q = \overline{P} \oplus Q$. ■

8.4. **Methodology in Practical Layer Decomposition and Extra Examples.** The following flowchart illustrates how the theoretical bounds from Section 4 and the layer drawability graph are used to perform a practical layer partition and determine the exact n-layer drawability of a figure.



Extra example 1: The image below is an image S which consists of 4 circles of radius 0,99 at (0,0), $(2.15\cos\left(\frac{j\pi}{3}\right), 2.15\sin\left(\frac{j\pi}{3}\right))$ for $j \in \{0,1,2\}$. The brush b here is the unit closed disk.

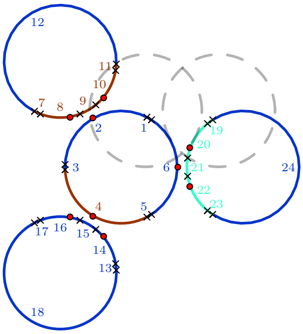


FIGURE 17A The image of S. Black crosses mark all $x \in X^*$. A point on Arc 6 forms a $\in$-loop with a point on Arc 20.

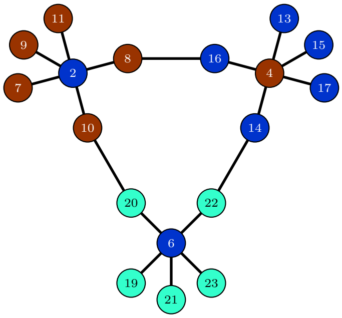


FIGURE 17B $G(P^*)$ of the image on the left. Only non-isolated vertices on the undrawability graph is shown.

We first apply the curvature condition from Theorem 4. Boundary points with curvature $|\kappa(x)| > 1/0.99 \approx 1.0101$ form the set $H_1(S)$, while $H_1(S) = S$.
Next, the primitive partition P^* for $H_1(S)$ is computed by identifying all points of bitangency. This partition is used to construct the layer drawability graph $G(P^*)$, shown in Figure 17b. The graph contains a 9-cycle, hinting its chromatic number is at least 3. Since a valid 3-layer decomposition was found and the theoretical lower bound is 3, we conclude that the S is 3-layer drawable.

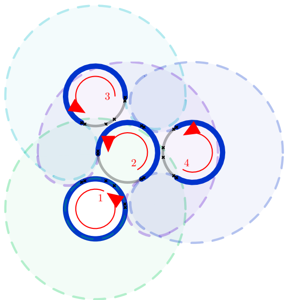


FIGURE 18. An example on how to draw the blue layer of the image above. Order of strokes are represented in red arrows, the strokes are in shifted hues of blue.

Blue layer shown above is a valid subfigure of $H_1(S)$ by construction. The two other layers are drawable in similar orders. Hence, $\chi(G(P^*)) = 3$ is verified by a

valid 3-colouring of $G(P^*)$ presented in Figure 17a. From the Chromatic Bound (Theorem 5), we conclude that S requires *at least 3 layers* to be drawn.
Extra example 2: The image below is an image S which consists of 4 copies of the parametric function $(\cos(t) - 0.3\sin(t), \sin(t) - 0.5\cos^2(t))$. The brush b here is the offset locus (outwards, 1 unit) of said parametric function.

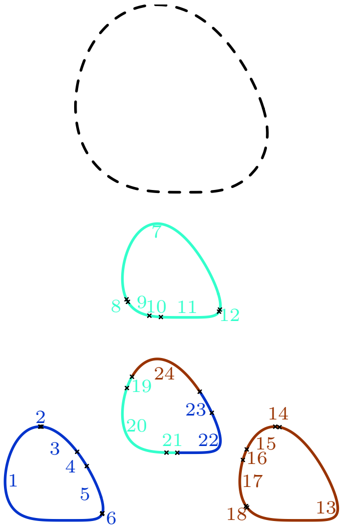


FIGURE 19A S consists of 4 copies of the parametric function $(\cos(t) - 0.3\sin(t), \sin(t) - 0.5\cos^2(t))$. The brush is shown in dotted lines, which is the offset locus (1 unit) of the parametric function.

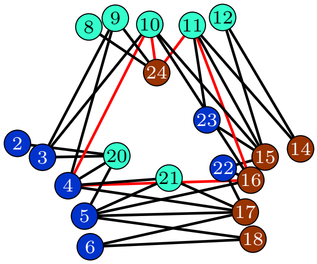


FIGURE 19B $G(P^*)$ of the image on the left. The red edges connecting the vertices form a 5-cycle. Only non-isolated vertices on the undrawability graph is shown.

We first apply the curvature condition from Theorem 4. Since b is '1 unit extruded' from the curves which construct S, which has an effect somewhat like enlarging from the curves, all boundary points x must satisfy Theorem 4 with the brush. Hence, $S = H_1(S)$. The resulting set $H_1(S)$ is shown coloured in Figure 19a.

Next, the primitive partition P^* for $H_1(S)$ is computed by identifying all points of bitangency. This partition is used to construct the layer drawability graph $G(P^*)$, shown in Figure 19b. The graph contains several 3-cliques and odd-cycles, hinting that its chromatic number is at least 3. $\chi(G(P^*)) = 3$ is verified by a valid 3-colouring of $G(P^*)$ presented in Figure 19b. From the Chromatic Bound (Theorem 5), we conclude that S requires *at least 3 layers* to be drawn.

Since a valid 3-layer decomposition was found and the theoretical lower bound is 3, we conclude that the S is 3-layer drawable.

8.5. **PoE Application to Aid Further Research.** To get a better grasp of how $G(P^*)$ of a given S, a AI-assisted tool is developed using PoE via click here or the website: `https://poe.com/Primitive_Partition` to build $G(P^*)$ restricted to the case of S being translation of circles with the same size. The brush shape is default to be the unit disk. A user manual could be found in the application.

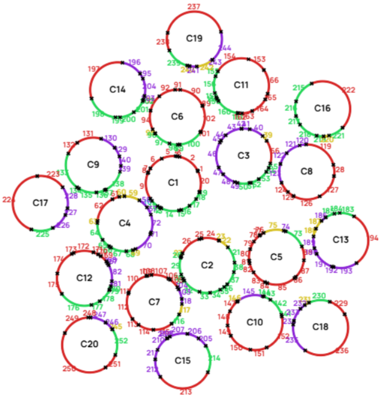


FIGURE 20A Screenshot of the partition of primitive arcs. A minimal colouring is shown automatically.

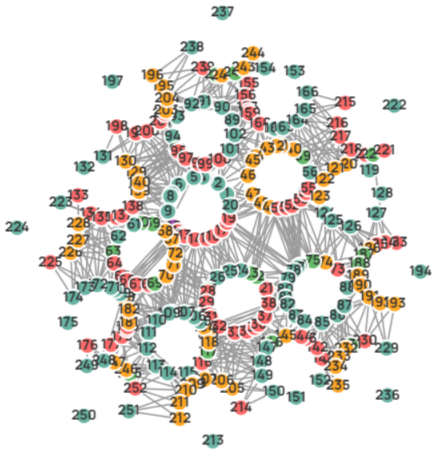


FIGURE 20B Screenshot of $G(P^*)$. A minimal colouring is shown automatically.

References

[1] Florian Frick and Fei Peng. What can you draw? *The American Mathematical Monthly*, 130(1):5–19, 2023.

[2] Kuan Yang. Lecture 4: Supporting and separating hyperplane theorem. MATH3806 lecture notes; relevant material on pp. 5–6.

EDITOR'S COMMENTS

Extending a 2023 paper by Frick and Peng, this work broadens the idea of "drawability" to allow brushes of any shape and infinitely many drawing steps, and then introduces "n-layer drawability" as a way to measure how complex an undrawable figure is. The three generalizations and the new measure are clearly laid out. The central result—that a figure is undrawable exactly when its boundary contains "containment loops"—is a clean and appealing way to capture the idea, and the connection between layer drawability and graph coloring (through the chromatic number) is a creative link between two different areas.

Overall, the paper is well-organized, the flowcharts showing how the theorems fit together. It is an interesting work that brings together convex geometry, topology, and graph theory.