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ON THE GENERALIZATION OF THE TRIANGLE PEG PROBLEM TO HIGHER DIMENSIONS

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Abstract. A lot of work has been put into solving special cases of the famous Square Peg Problem, which focuses on the two dimensional space. The aim of this project is to investigate the generalization of the Triangle Peg Problem into manifolds of higher dimensions.

By considering the Triangle Peg Problem, we have successfully proven the Tetrahedron Peg Problem, i.e. for every smooth compact connected surface, there exists four distinct points on the surface which can form a regular tetrahedron. Finally, by induction, we have generalized a variant of the Triangle Peg Problem to even higher dimensions.

Keywords. Triangle Peg Problem, Tetrahedron Peg Problem, Jordan Curve, Manifolds

Contents

1. Introduction

1.1. **Motivation.** The famous unsolved problem in Mathematics, the Square Peg Problem (or Toeplitz' Conjecture) proposed by Otto Toeplitz in 1911, discusses if every simple closed curve in the two dimensional plane inscribes a square. More preciously,

Conjecture 1.1 (Square Peg Problem)**.** *For all simple closed curve $\gamma \subset \mathbb{R}^2$, there exist four distinct points $x, y, z, w \in \gamma$ such that they form a square, i.e.*

$$\|x-y\| = \|y-z\| = \|z-w\| = \|w-x\|.$$

This problem has remained unsolved for over a century.

In the early stage of our research, we were inspired by the paper 'Equilateral triangles and continuous curves' by Mark D. Meyerson [1]. It proves several results related to the triangle peg problem and even the stronger, fixed peg variant of it, which states that each point on every simple closed curve in the two dimensional plane is a vertex of an inscribed equilateral triangle. Formally,

Theorem 1.2 (Fixed Equilateral Triangle Peg Problem)**.** *Given any simple closed curve $\gamma \subset \mathbb{R}^2$ and a point $x \in \gamma$, there exist two distinct points $y, z \in \gamma$ such that x, y, z is an equilateral triangle, i.e.*

$$\|x-y\| = \|y-z\| = \|z-x\|.$$

Based on the results, we decided to make a proof on the Regular Tetrahedron Peg Problem, and it aided us to deduce some strategies to tackle it in the early phase.

We have also considered other possible shapes such as cuboids and pyramids. However, considering the simple properties of a tetrahedron, we believe that the tetrahedron is the most suitable for our generalized topic.

1.2. **Outline of the Paper.** In our research, we firstly revisit the Equilateral Triangle Peg Problem and examine the crucial properties of simple closed curves and equilateral triangles to understand the possible methods to generalize and solve the problem.

Then, we attempt to extend our vision from two to three dimensional space to investigate the Regular Tetrahedron Peg Problem: the possibility that all four vertices of a regular tetrahedron lie on a three dimensional compact surface, mainly focusing on its fixed peg variant.

Problem 1.3. *Under what condition for a compact surface $S \subset \mathbb{R}^3$ and being given a point $x \in S$ do there exist three distinct points $y, z, w \in S$ which form a regular tetrahedron with x?*

Finally, manifolds in higher dimensions would also be a part of our investigations. By considering the outline of our proofs in lower dimensions, we have generalized the Regular Tetrahedron Peg Problem to higher dimensional manifolds, which will be further discussed below.

1.3. **Notations and Definitions.** Before we begin the investigation, let us define some common terminologies and symbols used in this document.

Definition 1.4. $\mathbb{R}$ *denotes the set of all real numbers.*

In this document, an n-dimensional point $P \in \mathbb{R}^n$ may sometimes refer to the position vector $\overrightarrow{OP}$ where O is the origin.

Definition 1.5. *An n-dimensional unit sphere* $\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} : \|x\| = 1\right\}$.

Definition 1.6. *A curve $\gamma \subset \mathbb{R}^n$ is the image of a continuous function $f : [0,1] \to \mathbb{R}^n$. f is called the parametrization of γ.*

Definition 1.7. *Define a function $R : (\mathbb{R}^2, \mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^2$ with*

$$R(y, x, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} (y - x) + x,$$

where x and y are the column vector representations of the points, which returns the image of y under a rotation about x by an angle θ anti-clockwise.

Definition 1.8. *∂S denotes the boundary of the set S.*

Definition 1.9. $\mathrm{cl}(S)$ *denotes the closure of the set S, i.e. $S \cup \partial S$.*

2. Revisiting the Jordan Curve

2.1. **Introduction to the Jordan Curve.** As aforementioned, the Equilateral Triangle Peg Problem investigates simple closed curves, i.e. Jordan curves. We define Jordan curves as follows:

Definition 2.1. *A curve is closed if $f(0) = f(1)$ where $f : [0,1] \to \mathbb{R}^2$ is its parametrization.*

Definition 2.2. *A Jordan curve is a closed curve whose parametrization $f : [0,1] \to \mathbb{R}^2$ is injective on $[0,1)$.*

In addition, the definition of the Jordan arc is useful in later proofs.

Definition 2.3. *A Jordan arc is a proper, path-connected subset of a Jordan curve.*

Let us recall an important fact about continuity:

Theorem 2.4 (Intermediate value theorem)**.** *Given a continuous function $f : [0,1] \to \mathbb{R}$, if $f(0) \leq f(1)$, then $\forall y \in [f(0), f(1)], \exists x \in [0,1]$ such that $f(x) = y$. Similarly, if $f(0) \geq f(1)$, then $\forall y \in [f(1), f(0)], \exists x \in [0,1]$ such that $f(x) = y$.*

In addition, Jordan curves have a simple property, which is stated in the Jordan curve theorem.

Theorem 2.5 (Jordan curve theorem). *For a Jordan curve γ, $\mathbb{R}^2 \setminus \gamma$ consists of exactly two connected components, one bounded and one unbounded, and γ is the boundary of each component.* [3]

The following definition is adopted for easier reference to the bounded and unbounded commponents (which are, in fact, the "interior" and the "exterior") of a Jordan curve in this paper:

Definition 2.6. *For every Jordan curve γ, define γ_{int} and γ_{ext} to be its bounded (i.e. the interior) and unbounded (i.e. the exterior) connected component of $\mathbb{R}^2 \setminus \gamma$ respectively.*

In other words, a Jordan curve γ divides the 2D plane $\mathbb{R}^2$ into three connected components, the curve γ, the interior γ_{int} and the exterior γ_{ext}.

When the Jordan curve theorem and the intermediate value theorem are put together, we get the following fact:

Lemma 2.7. *Given a Jordan curve γ and a curve with parametrization $f : [0, 1] \to \mathbb{R}^2$, if $f(0) \in \gamma \cup \gamma_{int}$ and $f(1) \in \gamma \cup \gamma_{ext}$, or $f(0) \in \gamma \cup \gamma_{ext}$ and $f(1) \in \gamma \cup \gamma_{int}$, then $\exists x \in [0, 1]$ where $f(x) \in \gamma$.*

Proof. By the Jordan curve theorem, define a sign function over $\mathbb{R}^2$ of

$$\epsilon(x) = \begin{cases} -1, & x \in \gamma_{int} \\ 0, & x \in \gamma \\ 1, & x \in \gamma_{ext} \end{cases} .$$

And since γ is closed, there exists a well-defined and continuous signed distance function to γ, $d : \mathbb{R}^2 \to \mathbb{R}$ with

$$d(x) = \epsilon(x) \min \{\|y - x\| : y \in \gamma\},$$

which returns the closest distance of x to γ with the sign indicating whether x is inside or outside γ. Note that $d \circ f$ is a continuous function.

For $f(0) \in \gamma \cup \gamma_{int}$ and $f(1) \in \gamma \cup \gamma_{ext}$, $d \circ f(0) \leq 0$ and $d \circ f(1) \geq 0$. Similarly, for $f(0) \in \gamma \cup \gamma_{ext}$ and $f(1) \in \gamma \cup \gamma_{int}$, $d \circ f(0) \geq 0$ and $d \circ f(1) \leq 0$.

By the intermediate value theorem, $\exists x \in [0, 1]$ where $d \circ f(x) = 0$ and thus $f(x) \in \gamma$. □

This lemma is essential in later proofs as it provides a way to show that there exists a point lying on a Jordan curve.

2.2. **The Equilateral Triangle Peg Problem.** Before we tackle the Equilateral Triangle Peg Problem, let us prove two lemmas related to the closest and furthest points to a Jordan curve from a given point.

Lemma 2.8. *Given a Jordan curve γ and a point $x \in \gamma \cup \gamma_{int}$, let $y \in \gamma$ be a point where $\|y - x\|$ is maximum, i.e. a furthest point from x. Then $\forall z \in \mathbb{R}^2$ where $\|z - x\| = \|y - x\|$, $z \in \gamma \cup \gamma_{ext}$.*

Proof (by contradiction). Assume $z \in \gamma_{int}$. Since γ_{int} is bounded, $\exists r > 0$ where $\forall w \in \gamma_{int}$, $\|w - x\| < r$, and so $\|z - x\| < r$. Then pick a point z_1 on the ray xz where $\|z_1 - x\| \geq r$. Hence $z_1 \notin \gamma_{int}$.

Let a point z_2 continuously move along the line segment zz_1 from $z \in \gamma_{int}$ to $z_1 \in \gamma \cup \gamma_{ext}$. By Lemma 2.7, there exists a z_2 on zz_1 such that $z_2 \in \gamma$. Since xzz_2z_1 is a straight line,

$$\|z_2 - x\| = \|z_2 - z\| + \|z - x\| \geq \|z - x\| = \|y - x\|,$$

which contradicts with $\|y - x\|$ being maximum. Therefore, $z \notin \gamma_{int}$. □

Lemma 2.9. *Given a Jordan curve γ and a point $x \in \gamma \cup \gamma_{int}$, let $y \in \gamma$ be a point where $\|y - x\|$ is minimum, i.e. a closest point to x. Then for every point $z \in \mathbb{R}^2$ where $\|z - x\| = \|y - x\|$, $z \in \gamma \cup \gamma_{int}$.*

Proof (by contradiction). Assume $z \in \gamma_{ext}$. Let a point w continuously move along the line segment xz from x to z. By Lemma 2.7, $\exists w \in \gamma$ on xz. Since $z \in \gamma_{ext}$, $z \neq w$. And since xwz is a straight line,

$$\|w - x\| < \|z - x\| = \|y - x\|,$$

which contradicts with $\|y - x\|$ being minimum. Therefore, $z \notin \gamma_{ext}$. □

With these two lemmas, we can prove the Equilateral Triangle Peg Problem, which, to recall, is:

Theorem 2.10 (Equilateral Triangle Peg Problem)**.** *Given a Jordan curve γ, there exists three distinct points $x, y, z \in \gamma$ such that x, y, z form an equilateral triangle.*

The idea is to consider the third point, z of the equilateral triangle when two of its other points x, y are on γ. Then we will construct a path to move z from the interior to the exterior of γ, hence achieving all three points being on γ in the process. The following is a more concrete proof, mainly focusing on the construction to make z lie in the interior of γ.

Proof. Let $w \in \gamma_{int}$. Let a point $x \in \gamma$ where $\|x - w\|$ is minimum, i.e. x is a closest point on γ to w. Let C be the circle centered at w which passes through x. There exist two points $y_1, z_1 \in C$ such that x, y_1, z_1 form an equilateral triangle. Since $\|y_1 - w\| = \|z_1 - w\| = \|x - w\|$, which is minimum, by Lemma 2.9, $y_1, z_1 \in \gamma \cup \gamma_{int}$. Now let y_3 be the intersection point of γ and ray xy_1 which minimizes $\|y_3 - x\|$.

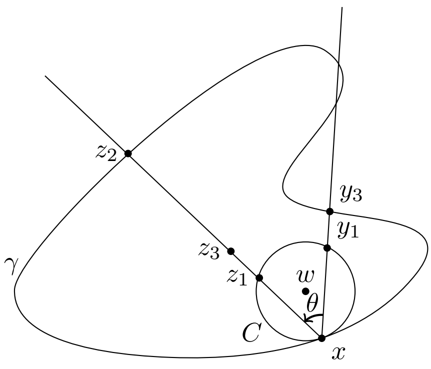

FIGURE 1. Construction of y_3 and z_3.

Similarly, let z_2 be the intersection point of γ and ray xz_1 which minimizes $\|z_2 - x\|$. Without loss of generality, assume $\|y_3 - x\| \leq \|z_2 - x\|$. There exists a point z_3 on the line segment xz_2 which forms an equilateral triangle with x and y_3.

If $z_3 \in \gamma_{ext}$, there exists a point $p \in \gamma$ on the line segment xz_3. Since $z_2 \in \gamma$ and $z_3 \notin \gamma$, $z_3 \neq z_2$. Since xpz_3z_2 is a straight line, $\|p - x\| < \|z_2 - x\|$, which contradicts with $\|z_2 - x\|$ being minimum. So $z_3 \in \gamma \cup \gamma_{int}$. If $z_3 \in \gamma$, the proof is done, so now suppose $z_3 \in \gamma_{int}$. Let $\theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$ be the angle where $z_3 = R\,(y_3, x, \theta)$, i.e. z_3 is the image of y_3 under the rotation about x by θ.

Let a point y_4 where $\|y_4 - x\|$ is maximum, i.e. y_4 is a furthest point from x. Let a point $z_4 = R(y_4, x, \theta)$, which forms equilateral triangle with y_4 and x. Since $\|z_4 - x\| = \|y_4 - x\|$, by Lemma 2.8, $z_4 \in \gamma \cup \gamma_{ext}$.

There exists two distinct Jordan arcs $\gamma_1, \gamma_2 \subset \gamma$ whose end-points are y_3 and y_4 as $y_3 \neq y_4$. Since $y_3, y_4 \neq x$, either γ_1 or γ_2 contains x. Without loss of generality, assume $x \notin \gamma_1$. Now let a point y continuously move along γ_1 from y_3 to y_4. Let a point $z = R(y, x, \theta)$, which forms an equilateral triangle with x and y. Then z will move from $z_3 \in \gamma_{int}$ to $z_4 \in \gamma \cup \gamma_{ext}$. By Lemma 2.7, $\exists y \in \gamma_1$ where $z \in \gamma$ in the process. Since $y \in \gamma_1$ and $x_1 \notin \gamma_1$, $x \neq y$. Therefore, x, y, z are distinct, all on γ and form an equilateral triangle. □

2.3. **A Fixed Peg.** A stronger version of the equilateral triangle peg problem is that for every point x on a Jordan curve γ, there exist two points $y, z \in \gamma$ where x, y, z form an equilateral triangle. In other words, one of the vertices of the equilateral triangle is fixed. This, in fact, has been proven by D. Meyerson [1]. In this paper, we attempt to extend the problem to isosceles triangles with any vertex angle, whose result is useful in later proofs.

First, let us consider a variation where the fixed peg is in the interior of γ, rather than on γ.

Theorem 2.11. *Given a Jordan curve γ and a point $x \in \gamma_{int}$, for all angle θ, there exists a point $y \in \gamma$ where $R(y, x, \theta) \in \gamma$.*

Proof. Pick two points $y_1, y_2 \in \gamma$ where $\|y_1 - x\|$ is maximum and $\|y_2 - x\|$ is minimum. By Lemma 2.8, $R(y_1, x, \theta) \in \gamma \cup \gamma_{ext}$, and by Lemma 2.9, $R(y_2, x, \theta) \in \gamma \cup \gamma_{int}$. Move a point y along γ from y_1 to y_2. Then $R(y, x, \theta)$ moves from $\gamma \cup \gamma_{ext}$ to $\gamma \cup \gamma_{int}$. And so by Lemma 2.7, $\exists y \in \gamma$ where $R(y, x, \theta) \in \gamma$. □

Corollary 2.12. *Given a Jordan curve γ and a point $x \in \gamma_{int}$, there exists two points $y, z \in \gamma$ such that x, y, z form an equilateral triangle.*

Proof. Notice that, without loss of generality, $z = R(y, x, \frac{\pi}{3})$. Therefore, simply set $\theta = \frac{\pi}{3}$ in the previous theorem. □

Moreover, let us define the smoothness of functions and curves.

Definition 2.13. *A function is smooth at a point x in its domain if it is continuously differentiable at x, i.e. of differentiability class C^1.*

Definition 2.14. *A curve with parametrization f is smooth at a point $f(t)$ if f is smooth at t.*

Now let us consider the original problem with the fixed peg being on the Jordan curve, with the added condition of the curve being smooth at the peg.

Theorem 2.15. *Given a Jordan curve γ and a point $x \in \gamma$ where γ is smooth at x, for all angle $\theta \in [0, \pi)$, $\exists y \in \gamma$ such that $R(y, x, \theta) \in \gamma$ and $y \neq x$.*

The reason why $\theta \neq \pi$ is that there is no such y when γ is strictly convex at x, meaning there exists a straight line passing through x which $\gamma \setminus \{x\}$ lies on one of the open half-planes of the line. An example of such curve is a circle.

Besides, the theorem requires γ to be smooth at x to prevent sharp points at x. A counterexample is a convex polygon.

A way to visualize the theorem is to consider the intersections between γ and the image of γ under the rotation about x by the angle θ anticlockwise.

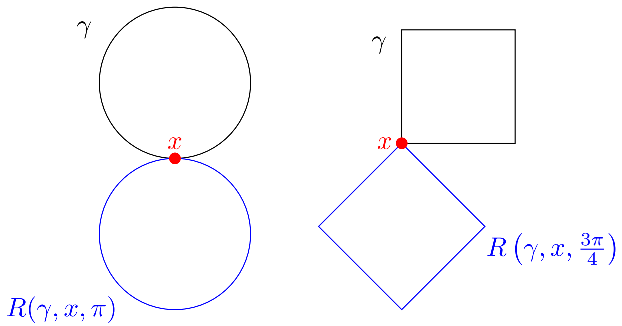

FIGURE 2. Example of a strictly convex curve (left) and a convex polygon (right) where γ and its rotation about x (denoted as $R(\gamma, x, \theta) = \{R(y, x, \theta) : y \in \gamma\}$) does not have a intersection other than x.

Once again, the focus of the proof is on the construction such that the third point is inside the curve.

Proof. Let $y_1 \in \gamma$ where $\|y_1 - x\|$ is maximum. By Lemma 2.8, $R(y_1, x, \theta) \in \gamma \cup \gamma_{ext}$.

Let $D(r) = \{p \in \mathbb{R}^2 : \|p - x\| < r\}$ be the disc around x of radius r. By differentiability and continuity, there exists a tangent line T at x and some $\delta > 0$ where $D(\delta) \cap \gamma$ is a connected Jordan arc, and for all points $z \in \mathrm{cl}(D(\delta)) \cap \gamma$ and $z \neq x$, the angle between the secant line xz and $T < \frac{\pi-\theta}{2}$. Then let $\{y_2, y_3\} = \partial D(\delta) \cap \gamma$ be the two end-points of the Jordan arc in $D(\delta) \cap \gamma$. Let α be the angle between the secant line xy_2 and T, and β be that between xy_3 and T. Then the angle at center formed by y_2 and y_3

$$\angle y_2 x y_3 \geq \pi - \alpha - \beta > \pi - \frac{\pi - \theta}{2} - \frac{\pi - \theta}{2} = \theta.$$

$C(\delta) \setminus \gamma$ has exactly two connected components, C_1, C_2.

Claim. *One of C_1 and C_2 is a proper subset of γ_{int}, and the other one is a proper subset of γ_{ext}.*

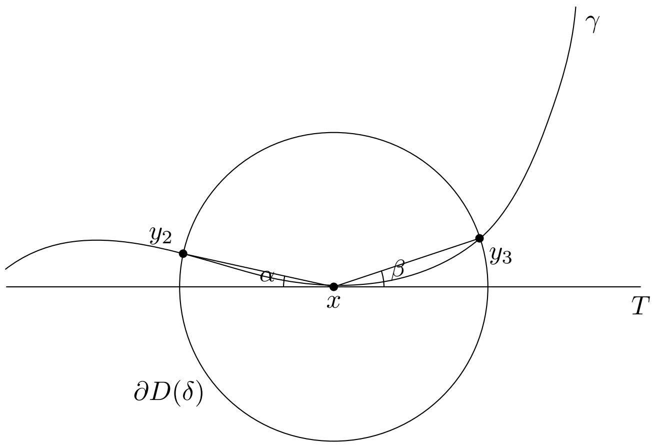

FIGURE 3. Construction of Theorem 2.15.

If there exist $z_1 \in C_1 \cap \gamma_{int}$ and $z_2 \in C_1 \cap \gamma_{ext}$, move a point z from z_1 to z_2 on C_1. Then there exists a $z \in C_1 \cap \gamma$, which contradicts with $C_1 \subset C(\delta) \setminus \gamma$. So C_1 is a proper subset of either γ_{int} or γ_{ext}. The same applies to C_2.

If both $C_1, C_2 \subset \gamma_{int}$, then $D(\delta) \subset \gamma \cup \gamma_{int}$, i.e. γ encloses the whole $D(\delta)$, which contradicts with $D(\delta) \cap \gamma$ being a connected Jordan arc $\subset \gamma$.

If both $C_1, C_2 \subset \gamma_{ext}$, then $\gamma \subset \mathrm{cl}(D(\delta))$, i.e. $\mathrm{cl}(D(\delta))$ encloses the whole γ. Construct a line L perpendicular to T passing through x. Then there exists a point $z \in \gamma \cap L$ and $z \neq x$. Then the angle between line xz and $L = \frac{\pi}{2}$, which contradicts with the angle between secant lines and $T < \frac{\pi-\theta}{2} \leq \frac{\pi}{2}$ for $\theta \geq 0$. So at least one of $C_1, C_2 \subset \gamma_{int}$. Thus, one of C_1 and C_2 is a proper subset of γ_{int}, and the other one is a proper subset of γ_{ext}.

Since $\angle y_2 x y_3 > \theta$, one of $R(y_2, x, \theta)$ and $R(y_3, x, \theta)$ is in C_1 and the other one is in C_2. Therefore, since one of C_1 and C_2 is a proper subset of γ_{int}, without loss of generality, assume $R(y_2, x, \theta) \in \gamma_{int}$.

Move a point y from y_1 to y_2 along the Jordan arc of γ not containing x. Then since $R(y_1, x, \theta) \in \gamma \cup \gamma_{ext}$ and $R(y_2, x, \theta) \in \gamma_{int}$, by Lemma 2.7, there exists some $y \neq x$ where $R(y, x, \theta) \in \gamma$. □

Corollary 2.16. *Given a Jordan curve γ and a point $x \in \gamma$, if γ is smooth at x, then there exists two points $y, z \in \gamma$ such that x, y, z form an equilateral triangle.*

Proof. Notice that, without loss of generality, $z = R(y, x, \frac{\pi}{3})$. Therefore, simply set $\theta = \frac{\pi}{3}$ in the previous theorem. □

3. Extending to the 3-Dimensional Space

Here, we investigate the Regular Tetrahedron Peg Problem, i.e. the possibility of forming a regular tetrahedron with its 4 vertices on an arbitrary compact connected surface, which is the 3-dimensional case for the equilateral triangle peg problem.

We define a surface as follows:

Definition 3.1. *A n-dimensional hypersurface (n-hypersurface) is a $(n-1)$-dimensional manifold (a topological space where each point has a neighborhood homeomorphic to the $(n-1)$-dimensional Euclidean space), embedded in n-dimensional Euclidean space.*

Definition 3.2. *A surface is a 3-hypersurface.*

In fact, by the classification of closed surface, a compact connected surface is homeomorphic to either the sphere or the connected sum of g tori where $g \geq 1$.

A compact connected hypersurface is similar to a Jordan curve in the way that both divides the two dimensional space into the interior and the exterior. In fact, a compact connected 2-hypersurface is a Jordan curve. This property is stated in the Jordan-Brouwer separation theorem, a higher dimensional generalization of the Jordan curve theorem.

Theorem 3.3 (Jordan-Brouwer separation theorem)**.** *For a compact connected hypersurface S in $\mathbb{R}^n$, $\mathbb{R}^n \backslash S$ consists of exactly two connected components, one bounded and one unbounded, and S is the boundary of each component.* [2]

Similar to the Jordan curves, we define S_{int} and S_{ext} for easier reference.

Definition 3.4. *For a compact connected n-hypersurface S, define S_{int} and S_{ext} be its bounded (i.e. the interior) and unbounded (i.e. the exterior) connected component of $\mathbb{R}^n \setminus S$ respectively.*

Due to the similarity in properties between Jordan curves and compact connected hypersurfaces, Lemma 3.7 to 3.9 are also applicable to the latter.

Finally, as we will use the locus of points forming an equilateral triangle with two given points extensively in this section, let us define a function for the locus.

Definition 3.5. *Define a function*

$$L(x, y) = \left\{ z \in \mathbb{R}^3 : \|z - x\| = \|z - y\| = \|y - x\| \right\}$$

where $x, y \in \mathbb{R}^3$, returning the locus of points forming an equilateral triangle with x and y.

3.1. **The Regular Tetrahedron.** First, let us take a look at some properties of a regular tetrahedron which will be useful in later proofs.

Lemma 3.6. *Given two points $x, y \in \mathbb{R}^3$, $L(x, y)$ is a circle on the plane normal to the line xy passing through the mid-point of xy.*

Proof. Let z be a movable point such that x,y,z form an equilateral triangle, where $\angle xyz = \angle yzx = \angle zxy = \frac{\pi}{3}$. Given $x, y, z \in \mathbb{R}^3$, the locus of z is the circle rotated about the midpoint of the axis xy. □

Lemma 3.7. *Given three points $x, y \in \mathbb{R}^3$ and $z \in L(x, y)$, a point w forms a regular tetrahedron with x,y,z if and only if $w \in L(x, y)$ and its angle at centre with z is $\cos^{-1} \frac{1}{3}$ on the circle $L(x, y)$.*

Proof. Let $\|y - x\| = l$ and let c be the mid-point of the line segment xy.

Suppose x, y, z and w forms a regular tetrahedron, we have

$$\|x-y\| = \|x-z\| = \|y-z\| = \|x-w\| = \|y-w\| = \|z-w\| = l,$$

and so w is on the locus. Therefore,

$$\|z-c\| = \|w-c\| = l\sin\frac{\pi}{3} = \frac{\sqrt{3}l}{2}.$$

Hence, the angle at centre of w with z is given by

$$\angle zcw = \cos^{-1}\frac{\left(\frac{\sqrt{3}l}{2}\right)^2 + \left(\frac{\sqrt{3}l}{2}\right)^2 - l^2}{2\left(\frac{\sqrt{3}l}{2}\right)\left(\frac{\sqrt{3}l}{2}\right)} = \cos^{-1}\frac{1}{3}.$$

Now suppose w is on the locus and the angle at centre is $\cos^{-1}\frac{1}{3}$. Then, $\|x-y\| = \|x-z\| = \|y-z\| = \|x-w\| = \|y-w\| = l$ and $\|z-c\| = \|w-c\| = l\sin\frac{\pi}{3} = \frac{\sqrt{3}l}{2}$. Therefore,

$$\|z-w\| = \sqrt{\left(\frac{\sqrt{3}l}{2}\right)^2 + \left(\frac{\sqrt{3}l}{2}\right)^2 - 2\left(\frac{\sqrt{3}l}{2}\right)\left(\frac{\sqrt{3}l}{2}\right)\cos\angle zcw} = l$$

and therefore, x, y, z and w forms a regular tetrahedron. □

3.2. **An Extension to the Intermediate Value Theorem.** Just as a continuous function $f : [a,b] \to \mathbb{R}$ has roots if $f(a) \leq 0$ and $f(b) \geq 0$ by the intermediate value theorem, a naturally raised question is whether the extension to a continuous function defined over two dimensions or even n dimensions is correct. More preciously:

Problem 3.8. *Define $D = \{x \in \mathbb{R}^n : r_1 \leq \|x\| \leq r_2\}$ be the domain of the function for some $r_1, r_2 > 0$. Let $C_1 = \{x \in \mathbb{R}^n : \|x\| = r_1\}$ and $C_2 = \{x \in \mathbb{R}^n : \|x\| = r_2\}$. For any continuous function $f : D \to \mathbb{R}$, if*

- $\forall x \in C_1, f(x) \leq 0$; *and*
- $\forall x \in C_2, f(x) \geq 0$,

then does there always exist a compact connected n-hypersurface S where $f(p) = 0$ for all $p \in S$, and the origin $\in S_{int}$?

However, a counterexample can be found by considering the distance function to a variant of the topologist's sine curve,

$$\begin{aligned} & \{(r(t)\cos t, -r(t)\sin t) : t \in (-\pi, 0)\} \\ \cup \quad & \left\{\left(\frac{3}{2}, 0\right)\right\} \\ \cup \quad & \{(r(t)\cos t, r(t)\sin t) : t \in (0, \pi]\} \\ \text{where} \quad & r(t) = \frac{1}{3}\sin\frac{1}{t} + \frac{3}{2}. \end{aligned}$$

Therefore, an additional condition has to be added to f.

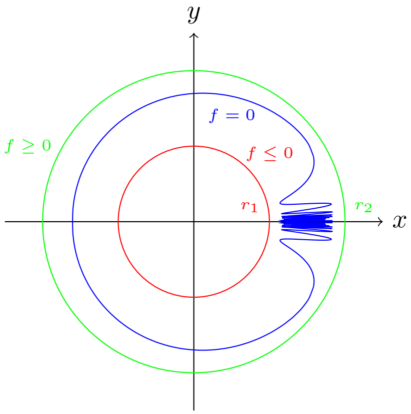

FIGURE 4. An illustration of the proposed extension and its counterexample.

Theorem 3.9. *Define* $D = \{x \in \mathbb{R}^n : r_1 \leq \|x\| \leq r_2\}$ *be the domain for some* $r_1, r_2 > 0$. *Let* $C_1 = \{x \in \mathbb{R}^n : \|x\| = r_1\}$ *and* $C_2 = \{x \in \mathbb{R}^n : \|x\| = r_2\}$. *For any continuous function* $f : D \to \mathbb{R}$, *if*

- $\forall x \in C_1, f(x) \leq 0$; *and*
- $\forall x \in C_2, f(x) \geq 0$,

and f ***is smooth at*** x ***when*** $f(x) = 0$, *then there exists a compact connected* n*-hypersurface* S *where* $f(p) = 0$ *for all* $p \in S$, *and the origin* $\in S_{int}$.

Proof. Consider the strict subzero set $S_1 = \{x \in D : f(x) < 0\}$. Note that $f(x) = 0$ for all x on the boundary ∂S_1. $C_1 \subseteq \mathrm{cl}(S_1)$, and $C_2 \subseteq D \setminus S_1$.

Let S_2 be the connected component of $\mathrm{cl}(S_1)$ which contains C_1. Let $S_3 = D \setminus (S_2)$. Pick S_4 be the connected component of $\mathrm{cl}(S_3)$ which contains C_2.

Let $S_5 = D \setminus S_4$, which is connected. By the unicoherence of $\mathbb{R}^n$, the intersection $\mathrm{cl}(S_5) \cap S_4 = \partial S_5$ is connected. By the inverse function theorem, since f is smooth at every point in ∂S_5, there exists a compact connected n-hypersurface $S \subset \partial S_5$ where the origin $\in S$. Since $\partial S_5 \subseteq \partial S_1$, $\forall x \in S, f(x) = 0$. $\square$

Theorem 3.10. *Define* $D = \{x \in \mathbb{R}^n : r_1 \leq \|x\| \leq r_2\}$ *be the domain for some* $r_1, r_2 > 0$. *Let* $C_1 = \{x \in \mathbb{R}^n : \|x\| = r_1\}, C_2 = \{x \in \mathbb{R}^n : \|x\| = r_2\}$. *For any continuous function* $f : D \to \mathbb{R}$, *if there exists an open connected set* $C_0 \subset C_1$ *where* $\mathrm{cl}(C_0) \neq C_1$ *such that*

- $\forall x \in C_0, f(x) < 0$;
- $\forall x \in \partial C_0, f(x) = 0$;
- $\forall x \in C_1 \setminus \mathrm{cl}(C_0), f(x) > 0$; *and*
- $\forall x \in C_2, f(x) \geq 0$,

and f *is smooth at* x *when* $f(x) = 0$, *then there exists a compact connected* n*-hypersurface* S *where* $f(p) = 0$ *for all* $p \in S$ *and* $\partial S = \partial C_0$.

The proof is similar to Theorem 3.9.

Proof. Consider the strict subzero set $S_1 = \{x \in D : f(x) < 0\}$. $\forall x \in \partial S_1$, $f(x) = 0$. Then $C_0 \subseteq S_1, \partial C_0 \subseteq \partial S_1$ and $C_1 \setminus \mathrm{cl}(C_0) \subseteq D \setminus \mathrm{cl}(S_1)$.

Pick S_2 be the connected component of S_1 which contains C_0. Let $S_3 = D \setminus S_2$. Pick S_4 be the connected component of S_3 which contains C_2.

Let $S_5 = D \setminus S_4$, which is connected. By the unicoherence of $\mathbb{R}^n$, the intersection $\mathrm{cl}(S_5) \cap S_4 = \partial S_5$ is connected. By the inverse function theorem, since f is smooth at every point in ∂S_5, there exists a connected n-hypersurface $S \subset \partial S_5$ which $\partial C_0 = \partial S$. Since $\partial S_5 \subseteq \partial S_1$, $\forall x \in S, f(x) = 0$. □

We can combine Theorem 3.9 and 3.10 with Theorem 2.11 to achieve the following results, which can be thought of as the intermediate value theorem on a cylinder:

Corollary 3.11. *Given a continuous function* $f : (\mathbb{S}^1, [0,1]) \to \mathbb{R}$ *where* $\forall \theta \in \mathbb{S}^1$, $f(\theta, 0) \leq 0$ *and* $f(\theta, 1) \geq 0$, *if* f *is smooth whenever* $f(\theta, t) = 0$, *then for all angle* ϕ, *there exists some* $\theta \in \mathbb{S}^1$ *and* $t \in [0,1]$ *such that* $f(\theta, t) = f\big(R(\theta, \vec{0}, \phi), t\big) = 0$.

Proof. Let g be a function $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\} \to \mathbb{R}$ with

$$g(\vec{x}) = f(\hat{x}, \|\vec{x}\| - 1).$$

Note that for all $x \in \mathbb{R}^2$, for $\|x\| = 1$, $g(x) = f(\hat{x}, 0) \leq 0$, and for $\|x\| = 2$, $g(x) = f(\hat{x}, 1) \geq 0$.

By Theorem 3.9, there exists a Jordan curve $\gamma \subset \{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$ such that the origin is inside γ and for all points $x \in \gamma$, $g(x) = 0$. Then by Theorem 2.11, there exist a point $x \in \gamma$ where $g(R(x, \vec{0}, \phi)) \in \gamma$.

Therefore, $g(x) = f(\hat{x}, \|\vec{x}\| - 1) = 0$ and $g(R(x, \vec{0}, \phi)) = f(R(\hat{x}, 0, \phi), \|\vec{x}\| - 1) = 0$. And so, for $t = \|\vec{x}\| - 1$ and $\theta = \hat{x}$, $f(\theta, t) = f(R(\theta, \vec{0}, \phi), t) = 0$. □

Corollary 3.12. *Given a continuous function* $f : (\mathbb{S}^1, [0,1]) \to \mathbb{R}$, $\theta_0 \in \mathbb{S}^1$ *and an angle* $\phi_0 \in [0, 2\pi)$ *where*

- $\forall \phi \in (0, \phi_0), f\big(R(\theta_0, \vec{0}, \phi), 0\big) < 0$;
- $f(\theta_0, 0) = f\big(R(\theta_0, \vec{0}, \phi_0), 0\big) = 0$; *and*
- $\forall \theta \in \mathbb{S}^1, f(\theta, 1) \geq 0$,

if f *is smooth whenever* $f(\theta, t) = 0$, *then for all angle* $\phi \in [0, \phi_0]$, $\exists \theta \in \mathbb{S}^1$ *and* $t \in [0,1]$ *such that* $f(\theta, t) = f(R(\theta, \vec{0}, \phi), t) = 0$.

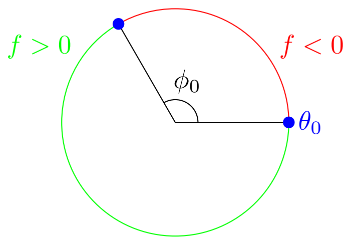

FIGURE 5. Illustration of the condition of $f(\theta, 0)$ in Corollary 3.12.

Proof. Trivially, if $\phi = 0$ or ϕ_0, $f(\theta_0, 0) = f(R(\theta_0, \vec{0}, \phi), 0) = 0$.

Now assume $\phi \in (0, \phi_0)$. Let g be a function $\{x \in \mathbb{R}^2 : 1 \le \|x\| \le 2\} \to \mathbb{R}$ with

$$g(\vec{x}) = f(\hat{x}, \|\vec{x}\| - 1).$$

g satisfies the condition in Theorem 3.10, and so there exists a Jordan arc $\gamma \subset \{x \in \mathbb{R}^2 : 1 \le \|x\| \le 2\}$ whose end-points are θ_0 and $R(\theta_0, \vec{0}, \phi_0)$ such that $\forall x \in \gamma, g(x) = 0$.

Construct a open Jordan arc C by constructing a line segment from the origin to θ_0 and then another one to $R(\theta_0, \vec{0}, \phi_0)$. Note that $\forall p \in C$, $\|p\| < 1$. Let γ' be the Jordan curve $C \cup \gamma$. $R(\theta_0, \vec{0}, \phi) \in \gamma'_{int}$.

Pick a point $y_0 \in \gamma'$ such that $\|y_0\|$ is maximum, i.e. a furthest point from the origin, and by Lemma 2.8, $R(y_0, \vec{0}, \phi) \in \gamma' \cup \gamma'_{ext}$. Since $y_0 \in \gamma$, continuously move a point $y \in \gamma$ from θ_0 to y_0. Then by Lemma 2.7, $\exists y \in \gamma$ such that $R(y, \vec{0}, \phi) \in \gamma'$. Since $\|R(y, \vec{0}, \phi)\| = \|y\| \ge 1$, $R(y, \vec{0}, \phi) \notin C$ and so $R(y, \vec{0}, \phi) \in \gamma$. Therefore,

$$y \in \gamma \implies g(y) = 0 \implies f(\hat{y}, \|\vec{y}\| - 1) = 0$$

and

$$R(y, \vec{0}, \phi) \in \gamma \implies g\big(R(y, \vec{0}, \phi)\big) = 0 \implies f\big(R(\hat{y}, \vec{0}, \phi), \|\vec{y}\| - 1\big) = 0$$

□

3.3. **Putting All Together.** Now, we will tackle the tetrahedron peg problem with one fixed peg. As with before, let us first look into the variation in which the fixed peg is inside the surface.

Theorem 3.13. *Given a smooth compact connected surface S and a point $x_0 \in S_{int}$, then there exist three points $w, y, z \in S$ forming a regular tetrahedron with x_0.*

Proof. There exists a function $\mu : (\mathbb{S}^1, \mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}^3$ where for some fixed x and y, $\mu(\theta, x, y)$ maps θ to the corresponding point on $L(x, y)$, and is continuous when x,y are changing.

By the Jordan-Brouwer separation theorem, define a sign function over $\mathbb{R}^3$ of

$$\epsilon(x) = \begin{cases} -1, & x \in S_{int} \\ 0, & x \in S \\ 1, & x \in S_{ext} \end{cases}.$$

In addition, define a signed distance function $d : \mathbb{R}^3 \to \mathbb{R}$ with

$$d(x) = \epsilon(x) \min \{\|y - x\| : y \in S\},$$

returning the minimum distance from a point x to a point on S, negative when inside and positive when outside. Note that d is a continuous function, and is smooth at every $x \in S$.

Pick a point $y_0 \in S$ which minimizes $\|y_0 - x_0\|$, i.e. a closest point to x_0 on S. Consider any point $z \in L(x_0, y_0)$ which forms an equilateral triangle with x_0 and y_0. By Lemma 2.9, since $\|z - x_0\| = \|y_0 - x_0\|$, which is minimal, $z \in S \cup S_{int}$. Therefore, $L(x_0, y_0) \subset S \cup S_{int}$.

Now pick a point $y_1 \in S$ which maximizes $\|y_1 - x_0\|$. For every $z \in L(x_0, y_1)$, $x_0 y_1 z$ form an equilateral triangle. Since $||y_1 - x_0|| = ||z - x_0||$, by Lemma 2.8, $z \in S \cup S_{ext}$. Thus, $L(x_0, y_1) \subset S \cup S_{ext}$.

Move a point y along a path $\gamma : [0,1] \to S$ from $\gamma(0) = y_0$ to $\gamma(1) = y_1$ not passing through x_0. Then $L(x_0, y)$ will continuously transform from $S \cup S_{int}$ to $S \cup S_{ext}$. Since S is a smooth manifold, γ is also smooth. Now define a function $f : (\mathbb{S}^1, [0,1]) \to \mathbb{R}$ with

$$f(\theta, t) = d(\mu(\theta, x_0, \gamma(t))),$$

returning the d value of the point at angle θ on the circle at time t. Since f is a composition of continuous functions, f itself is continuous too. Moreover, as d is smooth whenever $\mu\big(\theta, x_0, \gamma(t)\big) \in S$, f is smooth whenever $f = 0$. Note that for all $\theta \in \mathbb{S}^1$,

$$\mu\left(\theta, x_0, \gamma\left(0\right)\right) \in L\left(x_0, \gamma\left(0\right)\right) \subset S \cup S_{int} \implies f\left(\theta, 0\right) \leq 0$$

and

$$\mu\left(\theta, x_0, \gamma\left(1\right)\right) \in L\left(x_0, \gamma\left(1\right)\right) \subset S \cup S_{ext} \implies f\left(\theta, 1\right) \geq 0.$$

Then by Corollary 3.11, there exists some $\theta \in \mathbb{S}^1$ and $t \in [0,1]$ where $f(\theta, t) = f(R\left(\theta, 0, \cos^{-1}\frac{1}{3}\right), t) = 0$. Let $y = \gamma(t)$. Therefore,

$$f(\theta, t) = d(\mu(\theta, x_0, \gamma(t))) = 0 \implies \mu(\theta, x_0, y) \in S$$

and

$$f\Big(R\big(\theta, 0, \cos^{-1}\frac{1}{3}\big), t\Big) = d\Big(\mu\big(R(\theta, 0, \cos^{-1}\frac{1}{3}), x_0, \gamma(t)\big)\Big) = 0$$
$$\implies \mu\Big(R\big(\theta, 0, \cos^{-1}\frac{1}{3}\big), x_0, y\Big) \in S.$$

Let $z = \mu\left(\theta, x_0, y\right)$ and $w = \mu\left(R\left(\theta, 0, \cos^{-1}\frac{1}{3}\right), x_0, y\right)$. The angle at center formed by z and w in $L(x_0, y)$ is $\cos^{-1}\frac{1}{3}$. By Lemma 3.7, $w, x_0, y, z \in S$ form a regular tetrahedron. □

When $x_0 \in S_{int}$, we can easily find a point $y_0 \in S$ such that any point forming an equilateral triangle with x_0 and y_0 lies on or inside S. For $x_0 \in S$, however, there does not always exist such y_0. An example of such surface is a sphere.

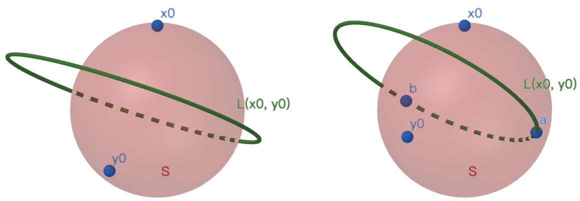

FIGURE 6. The locus of points forming an equilateral triangle with two points $x_0, y_0 \in S$ where S is a sphere. The points a and b represent the intersections between the locus and S.

Therefore, we will make use of the fact that S is smooth at x_0 to help construct a locus which proves the case where $x_0 \in S$.

Theorem 3.14. *Given a smooth compact connected surface S and a point $x_0 \in S$, there exists three distinct points $w, y, z \in S$ forming a regular tetrahedron with x_0.*

Proof. There exists a function $\mu : (\mathbb{S}^1, \mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}^3$ where for some fixed x and y, $\mu(\theta, x, y)$ maps θ to the corresponding point on $L(x, y)$, and is continuous when x,y are changing.

Pick a point $y_1 \in S$ where $||y_1 - x_0||$ is maximum. Then $L(x_0, y_1) \subset S \cup S_{ext}$.

By differentiability and continuity, there exists a point y_0 in a neighbourhood of x such that $L(x_0, y_0) \cap S = \{z_1, z_2\}$ for some z_1, z_2 with angle at center $\geq \cos^{-1}\frac{1}{3}$ with one of the connected components of $L(x_0, y_0) \setminus \{z_1, z_2\}$ is a subset of S_{int} and the other one is a subset of S_{out}.

Move a point y along a continuous path $\gamma : [0, 1] \to S$ from $\gamma(0) = y_0$ to $\gamma(1) = y_1$ not passing through x. Then $L(x_0, y)$ will continuously transform from "about halfly" inside S and "about halfly" outside S to on or outside S.

Define a function $f : \mathbb{S}^1, (0, 1) \to \mathbb{R}$ with

$$f(\theta, t) = d(\mu(\theta, x_0, \gamma(t))).$$

By Theorem 3.12, $\exists\theta \in \mathbb{S}^1, t \in [0, 1]$ such that $f(\theta, t) = f\big(R(\theta, \vec{0}, \cos^{-1}\frac{1}{3}\big) = 0$. Similar to 3.13, $y = \gamma\left(t\right), z = \mu(\theta, x_0, y)$ and $w = \mu(R(\theta, 0, \cos^{-1}\frac{1}{3}), x_0, y)$ are all on S form a regular tetrahedron with x_0. □

4. Generalizing to Higher Dimensions

After we have proven the regular tetrahedron peg problem for arbitrary smooth compact connected surface, we are going to generalize the results for cases in even higher dimensions, e.g. 4-dimensional space.

4.1. The Regular n-simplex.

First, let us generalize the tetrahedron to higher dimensions.

Definition 4.1. *A regular n-simplex is a convex hull of a set of $(n + 1)$ points $\{x_0, x_1, \ldots, x_n\}$ where for every i, j where $0 \leq i, j \leq n$ and $i \neq j$, $\|x_i - x_j\|$ is equal, i.e. any two points are equidistant from each other.*

Just as before, we are interested in the locus of points forming n-simplexes. Let us consider the cases in $\mathbb{R}^2$ and $\mathbb{R}^3$.

- In $\mathbb{R}^2$, given two points x_1, x_2, the locus of points forming an equilateral triangle with x_1, x_2 is $\left\{R\left(x_2, x_1, \frac{\pi}{3}\right), R\left(x_2, x_1, -\frac{\pi}{3}\right)\right\}$, which is a 0-sphere.
- In $\mathbb{R}^3$, given two points x_1, x_2, the locus of points forming a regular tetrahedron with x_1, x_2 is a circle (1-sphere), shown in Lemma 3.6. Then, after picking a point x_3 on the locus, the locus of forming a regular tetrahedron with the three points is two points, a 0-sphere.

The pattern emerged is formulated and proven in the following theorem, which generalizes it to higher dimensions:

Theorem 4.2. *Given m distinct points $X_1, X_2, \ldots, X_m \in \mathbb{R}^n$ where $2 \leq m \leq n$, let L be the locus of points forming a regular m-simplex with all X_i, where $1 \leq i \leq m$. If all X_i form a regular $(m - 1)$-simplex, then L is an $(n - m)$-sphere.*

Proof (by induction on m). Let R be the side length of the $(m - 1)$-simplex.

Claim. *The locus L is an $(n-m)$-sphere with radius r_m, where*

$$r_m = \begin{cases} R, & m = 1 \\ R\sqrt{1 - \frac{R^2}{4r_{m-1}^2}}, & m > 1 \end{cases}.$$

For $m = 2$, under isometry, assume X_1 is the origin and $X_2 = (R, 0, 0, \cdots, 0)$. A point $P = (p_1, p_2, \cdots, p_n) \in L$ if and only if

$$\begin{cases} p_1^2 + p_2^2 + \cdots + p_n^2 & = R^2, \\ (p_1 - d)^2 + p_2^2 + \cdots + p_n^2 & = R^2 \end{cases}.$$

Subtracting the two equations yields

$$\begin{aligned} (p_1 - R)^2 - p_1^2 &= 0 \\ p_1^2 - 2p_1 R + R^2 - p_1^2 &= 0 \\ p_1 &= \frac{R}{2}. \end{aligned}$$

Then

$$\sum_{i=2}^{n} p_i^2 = \frac{3R^2}{4}.$$

The locus of P satisfying this equation is an $(n-2)$-sphere with radius $r_2 = \sqrt{1 - \frac{R^2}{4R^2}} R = \sqrt{\frac{3}{4}} R = \frac{\sqrt{3}}{2} R$.

When $m = k$ where $2 \le k < n$, let L_0 be the locus. Assume L_0 is an $(n-k)$-sphere with radius r_k.

For $m = k + 1$, let $\widetilde{X} = X_{k+1}$. Since $\widetilde{X}$ forms an k-simplex with all X_i where $1 \le i \le k$, $\widetilde{X} \in L_0$. For a point $P \in L$, i.e. P forms an $(k+1)$-simplex with all X_i, P forms an k-simplex with all X_i for $1 \le i \le k$ and $P\widetilde{X} = R$. Hence, the locus L is all points in L_0 where $P\widetilde{X} = R$.

Let Y be the point in L_0 such that $\widetilde{X}Y$ is a diameter of L_0. Recall that the radius of L_0 is r_k. Then the diameter $\widetilde{X}Y$ is $2r_k$. Notice that L can be in three forms:

- If $R > 2r_k$, then $L = \emptyset$.
- If $R = 2r_k$, then $L = \{Y\}$.
- If $R < 2r_k$, then L is a $(n-k-1)$-sphere.

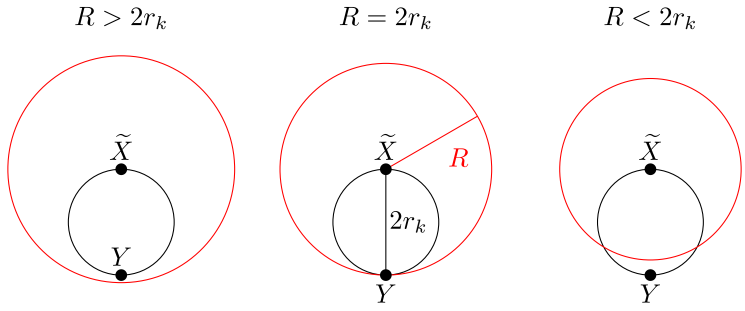

FIGURE 7. An example of the three cases of L when L_0 is a 1-sphere. The black circle shows L_0, and the red circle is centered at $\widetilde{X}$ with radius R. L is the intersection between the two circles. L is, from left to right, $\emptyset$, $\{Y\}$ or a 0-sphere (which is two points).

Notice that $r_m > 0$ for all $m \geq 1$. If $r_m \leq \frac{1}{\sqrt{2}}R$, $m \neq 1$, and

$$\begin{aligned} r_m = R\sqrt{1 - \frac{R^2}{4r_{m-1}^2}} &\leq \frac{1}{\sqrt{2}}R \\ 1 - \frac{R^2}{4r_{m-1}^2} &\leq \frac{1}{2} \\ \frac{R^2}{4r_{m-1}^2} &\geq \frac{1}{2} \\ r_{m-1}^2 &\leq \frac{1}{2}R^2 \\ r_{m-1} &\leq \frac{1}{\sqrt{2}}R, \end{aligned}$$

which, by induction, yields a contradiction. Therefore, $r_m > \frac{1}{\sqrt{2}}R > \frac{1}{2}R$, and so $R < 2r_k$. L is a $(n-k-1)$-sphere.

Consider a hyperplane containing $P, \widetilde{X}, Y$. Recall that $P\widetilde{X} = R$ and $\widetilde{X}Y = 2r_k$. Since $\widetilde{X}Y$ is a diameter of L_0 and $P \in L_0$, $\widetilde{X}P \perp PY$. Using Pythagoras' theorem, $PY = \sqrt{4r_k^2 - R^2}$.

Let O be the center of L. By symmetry, O is on $\widetilde{X}Y$ and $\widetilde{X}Y \perp OP$. $\Delta\widetilde{X}PY \sim \Delta POY$.

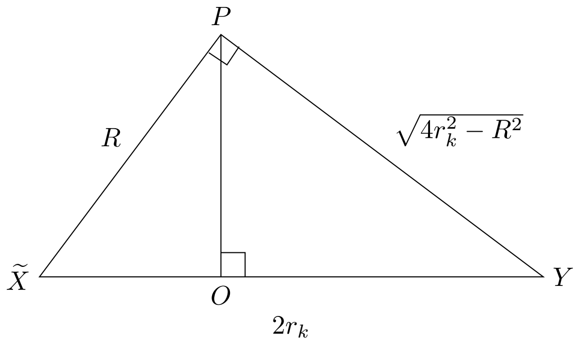

FIGURE 8. A geometric figure of the plane containing P, $\widetilde{X}$ and Y.

Then

$$\begin{aligned}
\frac{P\widetilde{X}}{\widetilde{X}Y} &= \frac{OP}{PY} \\
\frac{R}{2r_k} &= \frac{OP}{\sqrt{4r_k^2 - R^2}} \\
OP &= \frac{R\sqrt{4r_k^2 - R^2}}{2r_k} \\
&= R\sqrt{1 - \frac{R^2}{4r_k^2}} \\
&= r_{k+1}.
\end{aligned}$$

Therefore, L is a $(n-k-1)$-sphere with radius r_{k+1}. □

4.2. **The Regular n-simplex Peg Problem.** Now we have formulate a method of picking points and reducing the number of dimensions of the locus. We can now prove the Regular n-Simplex Peg Problem with a fixed interior peg by induction.

Theorem 4.3. *Given a compact connected n-hypersurface S with $n \geq 2$ and two points $x_0 \in S_{int}$, then there exists n points $x_1, x_2, \dots, x_n \in S$ which form a n-simplex with x_0.*

Proof (by induction on n). **Base case.** For $n = 2$, S is a Jordan curve. This has already been proven in Corollary 2.12.

Induction step. Assume when $n = k$ for a $k \geq 2$, given a smooth compact connected k-hypersurface S and a point $x_0 \in S_{int}$, then there exists k points $x_1, x_2, \dots, x_k \in S$ which form a k-simplex with x_0.

For $n = k+1$, we will define L, μ, d, y_1, f similarly to Theorem 3.13:

Define a function $L(x, y) = \left\{z \in \mathbb{R}^{k+1} : \|z - x\| = \|z - y\| = \|y - x\|\right\}$ where $x, y \in \mathbb{R}^{k+1}$ and $x \neq y$. By Theorem 4.2, the locus is a $(k-1)$-sphere.

There exists a function $\mu : \left(\mathbb{S}^{k-1}, \mathbb{R}^{k+1}, \mathbb{R}^{k+1}\right) \to \mathbb{R}^{k+1}$ where for some fixed x and y, θ, x, y maps θ to the corresponding point on the $(k-1)$-sphere $L(x, y)$, and is continuous when x and y are changing.

By assumption, $L(x_0, y_0) \subset S_{int} \cup S$. Pick a point $y_1 \in S$ which maximizes $\|y_1 - x_0\|$. By Lemma 2.8, $L(x_0, y_1) \subset S \cup S_{ext}$.

Move a point y along a path $\gamma : [0,1] \to S$ from $\gamma(0) = y_0$ to $\gamma(0) = y_1$. Then $L(x_0, y)$ will continuously transform from $S \cup S_{int}$ to $S \cup S_{ext}$. Now define a function $f : \left(\mathbb{S}^{k-1}, [0,1]\right) \to \mathbb{R}$ with

$$f(\theta, t) = d\left(\mu\left(\theta, x_0, \gamma(t)\right)\right).$$

Note that f is continuous. In addition, $f(\theta, 0) \leq 0$ and $f(\theta, 1) \geq 0$ for all $\theta \in \mathbb{S}^{k-1}$.

Now define another function $g : \left\{x \in \mathbb{R}^k : 1 \leq \|x\| \leq 2\right\}$ with

$$g(\vec{x}) = f(\hat{x}, \|x\| - 1).$$

By Theorem 3.9, since $g(x) \leq 0$ for all $x \in \mathbb{R}^k$ where $\|x\| = 1$, and $g(x) \geq 0$ for all $x \in \mathbb{R}^k$ where $\|x\| = 2$, there exists a compact connected k-hypersurface S' such that $\forall p \in S_0$, $g(p) = 0$, and the origin $\in S'_{int}$.

Pick a closest point $y' \in S'$ which minimizes $\|y'\|$. All the points forming an equilateral triangle with the origin and y' is, by Lemma 2.9, on or inside S'. Then by the induction assumption, there exists k distinct points $x'_1, x'_2, \ldots, x'_k$ which forms a k-simplex with the origin.

Let $t = \|x'_1\| - 1 = \|x'_2\| - 1 = \ldots = \|x'_k\| - 1$ and $y = \gamma(t)$. Moreover, for all $1 \leq i \leq k$, let $x_i = \mu\left(\hat{x'_1}, x_0, y\right) \in L(x_0, y)$. Notice that

$$\begin{aligned}
& & x'_i &\in S' \\
\implies & & 0 &= g(x'_i) \\
& & &= f\left(\hat{x'_i}, \|x'_i\| - 1\right) \\
& & &= d\left(\mu\left(\hat{x'_i}, x_0, \gamma(\|x'_i\| - 1)\right)\right) \\
& & &= d\left(\mu\left(\hat{x'_i}, x_0, y\right)\right) \\
& & &= d(x_i) \\
\implies & & x_i &\in S.
\end{aligned}$$

Since all x'_i forms a $(k-1)$-simplex, all x_i also forms a k-simplex. Combined with $x_i \in L(x_0, y)$, the $(k+1)$ points of y and all x_i are on S and form a $(k+1)$-simplex with x_0. $\square$

5. Conclusion

In conclusion, by making use of the fundamental principle in the triangle peg problem, we have generalized it to even higher dimensions, such as the tetrahedron peg problem in the 3-dimensional case. Based on the generalized triangle peg problem, we believe that the case for disphenoids (a class which regular tetrahedra also belong to) and other shapes such as cuboids are some possible directions of future investigations. In addition, as equilateral triangles are ideal shapes in the aesthetics of architecture, the theorems of generalized triangle peg problem may be applicable to fields of architectural studies.

References

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REVIEWERS' COMMENTS

This paper considered the generalization of the classical Triangle Peg problem which says that each point on every simple closed curve in the two dimensional plane is a vertex of an inscribed equilateral triangle. The author extended the idea from plane curves to higher dimensional hypersurfaces cases in Euclidean spaces, proving that given a smooth compact connected surface, there always exist four points on the surface which form a regular tetrahedron. Reviewers think that the authors, being high school students, have good knowledge on college-level mathematical analysis and differential geometry, and suggest that some careful justification on the use of, for instance, implicit/inverse function theorem is needed in the paper.