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Honorable Mention

Dividing a Circle with the Least Curve

Team Members:	Chung Yin Chan, Rennie Lee
Teacher:	Mr. Ka Wo Leung
School:	Hong Kong True Light College

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DIVIDING A CIRCLE WITH THE LEAST CURVE

TEAM MEMBERS
Chung Yin Chan, Rennie Lee

TEACHER
Mr. Ka Wo Leung

SCHOOL
Hong Kong True Light College

Abstract. In this project we planned to study the division of a circle with the shortest curve. In a party, we often divide a circular cake into equal and unequal parts. Suppose that bacteria grow on the exposed surface area of a cake. In order to keep the cake hygienic, we should divide the cake with the shortest cut. We investigated this problem by using a simple mathematical model: dividing a circle into equal or unequal areas with the shortest curve.

The first possible solution was the radius method. It meant that we used radii to divide a circle into parts. But, were there any ways to divide a circle with a curve shorter than that of the radius method?

The results included:

1. Radius method is the solution of the problem for $n = 2, 3$ and equal division.
2. Radius method is not a solution of the problem for $n = 4$ and equal division.
3. Orthogonal circular arc is the solution of the problem for $n = 2$ and unequal division.
4. We found a necessary condition of the problem for $n = 3$ and unequal division by a "Y-shaped" curve.

[See reviewer's comment (2) and (3)]

Introduction

After reading the book written by Polya [10], we were interested in the problem which is actually the problem in our project $n = 2$ and equal division. We thought that we could have our ways to solve the problem by geometry and calculus.

By isoperimetric inequality and its variants, we would like to claim the curve must be a circle, a straight line or a circular arc. Actually, it was not very easy to prove that the region, enclosed by the shortest curve, with a non-major segment, had to be convex because bad operations might move parts of the curve outside the circle. However, when we overcame this difficulty, we had the claim and could use simple geometry theorems to compare the length of the radius method and that of a circular cut.

Besides geometrical method, we could also use calculus to solve the problem after we had the claim in the former paragraph. An interesting result was that the circular arcs or the straight line had to be orthogonal to the circle no matter it was an equal division or not. Therefore, we solved the problem for $n = 2$ and unequal division.

Then we came to the idea of this project. We planned to study the division of a circle with the shortest curve. The first possible solution was the radius method. It meant that we used radii to divide a circle into equal or unequal areas. But, were there any ways to divide a circle with a curve less than that of the radius method? If so, what was the method with the shortest curve? We planned to study this problem in details. More precisely, we would start our investigation with two directions:

1. equal division or unequal division,
2. $n = 2, 3, 4, \ldots$ [See reviewer's comment (4)]

For the problem for $n = 3$ and equal division, radius method consisted of three radii with 120°. That meant O was the many different centres of the three points on the circumference. Obviously, shortest curve related to the Fermat point. Therefore, we tried to use the Fermat point in our proof. The four cases:

Case 1: Two closed curves inside the circle,

Case 2: Two closed curves nested together,

Case 3: A closed curve and a curve ending on the circle, and

Case 4: Two curves ending on the circle,

could be proven not a solution to this problem. The remaining task was to cope with the

Case 5: A "Y-shaped" curve.

We had two directions of thought here. First, we tried to find some necessary condition of this problem to justify a curve not to be the shortest. Second, we tried to use the Fermat point to solve this problem. As O was the Fermat point of any three points divides the circumference equally, we hoped to find some operations to satisfy the following two conditions:

1. the curve was shorter after operation; and
2. the end-points of the curve divide the circumference equally after operation.

Later, we solved this problem.

Some of the necessary conditions for the problem of $n = 3$ and equal division could be generalized to that of unequal division.

When surfing on the web for the minimum spanning tree problem for another competition about System Modeling & Optimization, we accidentally found the minimum Steiner tree problem and the Euclidean Steiner tree. We used this idea to disprove that the radius method is a solution of the problem for $n = 4$ and equal division, although we could not find the shortest cut. Actually, the key idea of disproof was only the Fermat point.

Symbols used [See reviewer's comment (5)]

- We define the circle to be divided as the unit circle, named S. Therefore, the area is π.
- A generalized circular arc means a circular arc or a line segment.
- d_n = the minimum length of a curve ending on S to enclose an area of $\frac{\pi}{n}$.
- $l(\text{curve})$ = length of the curve. For example, $l(\text{the unit circle}) = 2\pi$

With reference to Fig 1.1,

- $A(\theta)$ = the area enclosed by a minor segment of S and an orthogonal generalized circular arc corresponding to an angle θ.
- $p(\theta)$ = the length of an orthogonal generalized circular arc corresponding to an angle θ.
- $p(A)$ = the minimum length of a curve ending on S to enclose an area of A

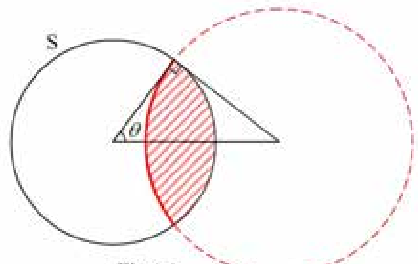


Fig 1.1

Literature Review and Lemmas [See reviewer's comment (6a)]

Isoperimetric Inequality and its variants

From the books written by Polya [10], Choi [1], and the web pages [3-5], we had the followings:

The Isoperimetric Inequality: Closed curves, circle is the best

- Among all planar shapes with the same perimeter, the circle has the largest area .... (1.1a)
- Among all planar shapes with the same area, the circle has the shortest perimeter .... (1.1b)

Curves with end-points on a straight line, semi-circle is the best

- Among all planar curves with two end-points lying on a straight line and the same length, the semi-circle encloses the largest area with the straight line .... (1.2a)
- Among all planar curves with two end-points lying on a straight line and the same enclosed area with the straight line, the semi-circle has the shortest length .... (1.2b)

Curves with two fixed end-points, circular arc is the best

- Among all planar curves with two fixed end-points and the same length, the circular arc encloses the largest area with the line segment joining the two fixed points .... (1.3a)
- Among all planar curves with two fixed end-points and the same enclosed area with the line segment joining the two fixed points, the circular arc has the shortest length .... (1.3b)

Theorems (1.2b) and (1.3b) are not actually from the books. However, starting from theorems (1.2a) and (1.3a), using the spirit of the proof of the equivalence of Isoperimetric Inequality (1.1a) and (1.1b) in [6], we can prove them.

Fermat Point

From the papers written by Cheung [2], Cheng [3], and the web pages [8], we had the followings:

- Fermat point of a triangle is a point such that the sum of its distance from the three vertices of the triangle is a minimum. The Fermat point is unique.

If $\triangle ABC$ is a triangle with all angles less than $120°$, then

- P is the Fermat point $\iff$ angles among AP, BP and CP are $120°$. ...(2)

Shortest curve between two points

From web page [12], the triangle inequality can be extended by mathematical induction to the following:

- Any polygonal path is longer than the line segment joining its end-points.

By definition, the arc-length of a curve is the least upper bound of the lengths of all polygonal approximations of the curve. The above statement can be further extended to the following:

- The shortest curve between two points is a straight line. .... (3)

Lemma 4. *The shortest path (curve) between a fixed point inside a circle and a point on the circle is the shorter line segment lying on a diameter of the circle (OA in Fig 2.1 and Fig 2.2). .... (4)*

By theorem (3), it suffices to prove that OA is shorter than or any line segment joining O and a point on the circle. [See reviewer's comment (6b)]

Proof by Calculus:

Equation of the circle:

$$(x-(p-r))^2+y^2=r^2 \text{ for } x\in[p-2r,p]$$

$$\begin{aligned} l &= \text{the distance between } O \text{ and a point } (x,y) \\ &\quad \text{on the circumference} \\ &= \sqrt{x^2+y^2} \\ &= \sqrt{x^2+r^2-x^2+2(p-r)x-(p-r)^2} \\ &= \sqrt{r^2+2(p-r)x-(p-r)^2} \\ \frac{dl}{dx} &= \frac{p-r}{\sqrt{r^2+2(p-r)x-(p-r)^2}} < 0 \quad \text{as } p<r \end{aligned}$$

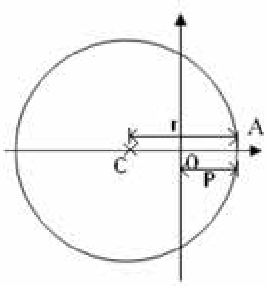


Fig 2.1

Therefore, the minimum l is p at $x=p$.

□

Proof by Geometry (Fig 2.2).

Let C be the centre of the fixed circle C_1 and O be the fixed point. Extend CO to meet the circle at A. Draw a circle C_2 with centre O and radius OA. Draw a straight line T passing A and $\perp CA$.

T is the common tangent of the two circles. O lies inside $C_1 \implies C_1$ and C_2 touch each other internally.

[See reviewer's comment (6c)]

Therefore, C_2 lies inside C_1
Hence, any line segment joining O and a point on C_1 is not shorter than OA.

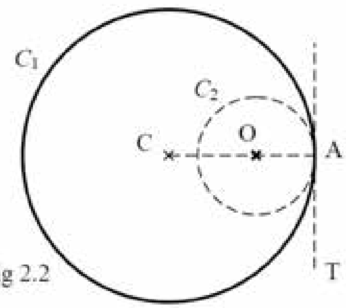


Fig 2.2

$\square$

Lemma 5. *In a triangle, longer side opposite larger angle, and vice versa.* . . . *(5)*

From the web pages [10-11], or by the sine formula, we can have this lemma.

Lemma 6. *If* $A \in \{(x_1, y_1) : x_1^2 + y_1^2 = 1, x_1 \leq 0\}$ *and* $B \in \{(x_2, 0) : 0 \leq x_2 \leq 1\}$, *then the length of any curve* $\widehat{AB} \geq 1$. . . . . *(6)*

Again, by theorem (3), it suffices to prove that $AB \geq 1$.

Proof by Algebra:

$$\begin{aligned} AB &= \sqrt{(x_1 - x_2)^2 + y_1^2} \\ &= \sqrt{x_1^2 - 2x_1x_2 + x_2^2 + y_1^2} \\ &\geq \sqrt{x_1^2 + y_1^2} = 1 \end{aligned}$$

as $x_1, x_2 \leq 0$, $x_2^2 \geq 0$. $\square$

Proof by Geometry (Fig 2.3):

$$\angle AOB \leq 90^\circ \quad \text{or} \quad B = O$$

$$\Longrightarrow \angle AOB \leq 90^\circ \quad \text{or} \quad B = O$$

$$\Longrightarrow \quad AB > AO \quad \text{or } AB = AO \quad \text{(lemma 4)}$$

$$\Longrightarrow \quad AB \geq AO = 1$$

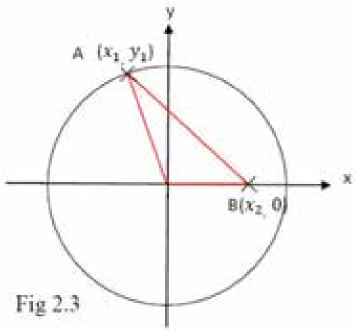


Fig 2.3

Therefore, we proved lemma (6). $\square$

Lemma 7 (a). *If l is the shortest curve with end-points on the circle dividing a circle into 2 EQUAL parts, then*

1. *l is a generalized circular arc; and*
2. *The region enclosed by l and a non-major arc must be convex* . . . . *(7a)*

Proof.
In Fig 2.4, A_1 is the region enclosed by any curve (blue) and the minor arc $\widehat{AB}$, with

$$\text{area of } A_1 = \text{area of } A_2$$

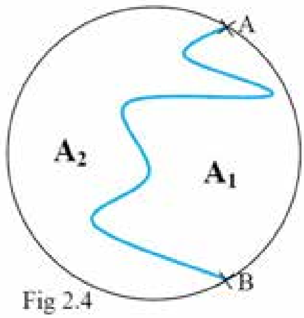


Fig 2.4

In Fig 2.5, we use an elastic band (red curve) to trap the A_1. We noted that

$$\text{area of } A_1' \geq \text{area of } A_1,$$

$$A_1' \text{ is convex, and}$$

$$l(\text{red curve}) \leq l(\text{blue curve}) \quad (\text{theorem 3})$$

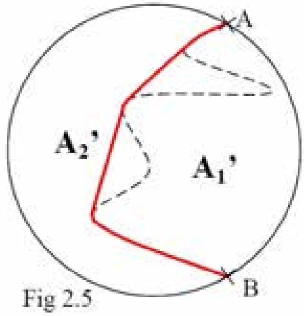


Fig 2.5

In Fig 2.6, we move the red curve in the direction $\perp A'B'$ until

$$\text{area of } A_1'' = \text{area of } A_1$$

Let A'' and B'' be the points of intersection between the circle and the red curve. We coloured the red curve within A'' and B'' green. We noted that

$$\text{area of } A_1'' = \text{area of } A_1,$$
$$A_1' \text{is convex} \implies A_1'' \text{ is convex, and}$$
$$l(\text{green curve}) \leq l(\text{red curve})$$

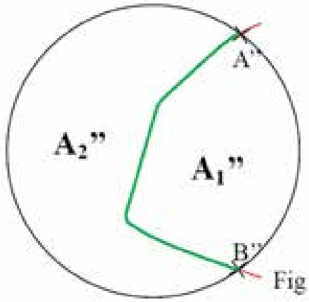


Fig 2.6

In Fig 2.7, the orange curve is the circular arc with end-points A'' and B'' and an enclosed area equal to area of A_1''.

By variant of isoperimetric inequality (1.3b),

$$l(\text{orange curve}) \leq l(\text{green curve}),$$

equality holds iff the green curve is a generalized circular arc.
As a result, if the blue curve is not a generalized circular arc, we can find an orange curve so that it also divides the circle into 2 equal parts and

$$l(\text{orange curve}) < l(\text{blue curve}).$$

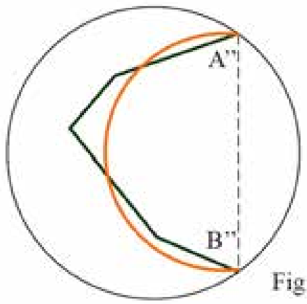


Fig 2.7

□

We proved the lemma 7(a) with two remarks:

1. End-points of orange and blue curves may not be the same.
2. Orange and blue curves divide the circle into equal areas.

Lemma 7 (b). *If l is the shortest curve with FIXED end-points on the circle dividing a circle into 2 parts of GIVEN AREAS, then*

1. *l is a generalized circular arc; and*
2. *The region with area not less than the segment with the common arc on S must be convex. . . . . (7b)*

[See reviewer's comment (6d)]

Proof.

In Fig 2.8, A_1 is the region enclosed by any curve (blue) and the minor arc $\widehat{AB}$, with

$$\text{area of } A_1 > \text{area of minor segment } AB$$

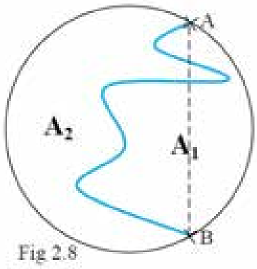


Fig 2.8

In Fig 2.9, we use an elastic band (red curve) to trap the A_1. We noted that

$$\text{area of } A_1' \geq \text{area of } A_1,$$
$$A_1' \text{ is convex, and}$$
$$l(\text{red curve}) \leq l(\text{blue curve}) \quad (\text{theorem } 3)$$

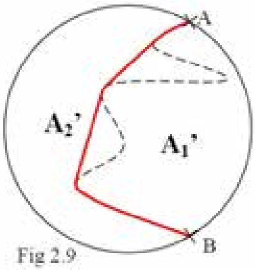


Fig 2.9

In Fig 2.10, we draw the line AB and move the line to left until area of $A_1'' =$ area of A_1.
We noted that

$$A_1' \text{is convex} \implies A_1'' \text{ is convex, and}$$
$$l(\text{green curve}) \leq l(\text{red curve}) \quad (\text{theorem } 3)$$

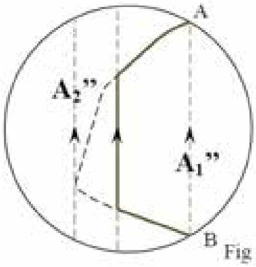


Fig 2.10

In Fig 2.11, the orange curve is the circular arc with end-points A and B and an enclosed area equal to area of A_1''.

By variant of isoperimetric inequality (1.3b),

$$l(\text{orange curve}) \leq l(\text{green curve})$$

equality holds iff the green curve is a generalized circular arc.

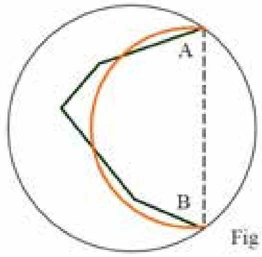


Fig 2.11

As a result, if the blue curve is not a generalized circular arc, we can find an orange curve so that it also divides the circle into 2 parts of given areas and

$$l(\text{orange curve}) < l(\text{blue curve})$$

For the case of area of A_1 < area of minor segment AB, equivalently,

$$\text{area of } A_2 > \text{area of major segment } AB,$$

do similar operations on A_2, we can obtain the same result.

For the case of area of A_1 = area of minor segment AB, the solution is trivial, minimum curve is the line segment AB. □

We proved the lemma 7(b) with two remarks:

1. End-points of orange and blue curves are the same.
2. Orange and blue curves divide the circle into 2 parts of given areas.

Lemma 7 (c)**.** *If l is the shortest curve with end-points on the circle dividing a circle into 2 parts of GIVEN AREAS, then*

1. *l is a generalized circular arc;*
2. *The region enclosed by l and a non-major arc must be convex. ....(7c)*

Proof.

By lemma 7(b), l is a generalized circular arc. Let A_1 be the region enclosed by l and a non-major arc. By suitable rotation, end-points of l, A and B, are on the same vertical line. Let A_0B_0 be a vertical line with end-points on the circle such that the area minor segment enclosed by A_0B_0 = area of A_1.

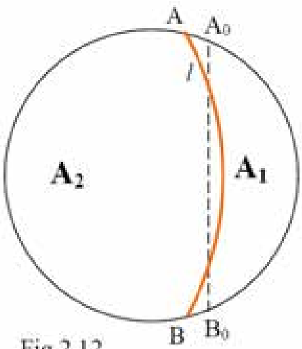


Fig 2.12

Suppose, on the contrary, A_1 is concave (Fig 2.12). Then we have

$$\text{area of } A_1 < \frac{\pi}{2} \text{ and } \widehat{A_0B_0} < \widehat{AB} \leq \pi.$$

$l > AB > A_0B_0$ which is a contradiction.

By contradiction, A_1 is convex. □

We proved the lemma 7(c) with two remarks:

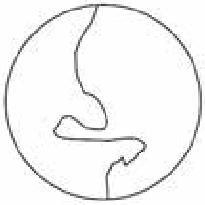
1. End-points of curves are not fixed.
2. The curve divides the circle into 2 parts of given areas.

Solving the problem for n = 2 and equal division

To divide a circle into 2 equal areas, there are two possible ways:

Case 1:
A closed curve which is embedded fully inside the circle

Fig 3.1

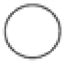


Case 2:
A curve which ends on the circumference of the circle

Fig 3.2

Case 1: A closed curve which is embedded fully inside the circle

According to the isoperimetric inequality (1.1b), a circle has the shortest perimeter. Therefore, perimeter of circle must be smaller than that of other closed curves.

$l($  $) \leq l($  $)$

Let r be the radius of a circle in Case 1, then

$$\pi r^2 = \frac{\pi}{2} \implies r = \frac{\sqrt{2}}{2}$$

$$\implies \text{perimeter of a circle in Case 1} = 2\pi r = \sqrt{2}\pi > 4.44 > 2$$

However, the length of a diameter of S is 2. The perimeter of a circle in Case 1 is larger than the length of a diameter of S. A diameter of S also divides S into 2 equal areas. Therefore, Case 1 must not be the solution.

Case 2: A curve which ends on the circumference of the circle

Applying Lemmas 7(a) or 7(b) or 7(c), we have the following:

1. The shortest curve must be a generalized circular arc.
2. Region enclosed by a curve and a non-major arc must be convex.

Method 1: To prove that the minimum cut is the radius method by geometry

Suppose AB is not a diameter and $\widehat{AMB}$ the shortest cut.

By suitable rotation, x-coordinates of A and B are the same and positive.

Suppose, on the contrary, that $\widehat{AMB}$ does not cut PQ.

$$\frac{\pi}{2} = \text{area of AMBE} < \text{area of PQE} = \frac{\pi}{2}.$$

By contradiction, $\widehat{AMB}$ cuts PQ.

By symmetry about x-axis, let F and G be the points of intersection between $\widehat{AMB}$ and PQ such that F and G lies on the positive and negative y-axis, respectively.

$$\widehat{AF} > PF \qquad \text{(Lemma 4)}$$

$$\widehat{FMG} > FG \qquad \text{(Theorem 3)}$$

$$\widehat{GB} > GQ \qquad \text{(Lemma 4)}$$

$$\begin{aligned} \widehat{AB} &= \widehat{AF} + \widehat{FMG} + \widehat{GB} \\ &> PF + FG + GQ = PQ \end{aligned}$$

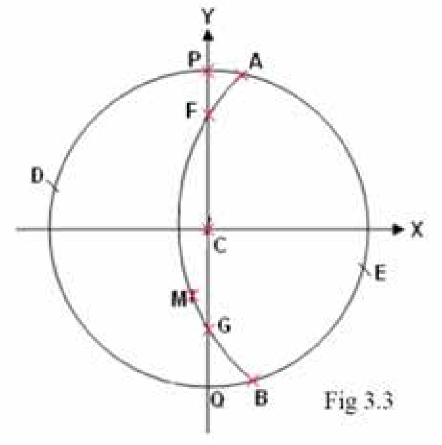


Fig 3.3

Therefore, the minimum cut is the radius method and the length of cut is 2.

Method 2: To prove that the minimum cut is the radius method by calculus

$$\begin{aligned} A &= \text{area} = \frac{\pi}{2} \\ &= \theta - \frac{1}{2}\sin 2\theta + r^2\phi - \frac{1}{2}r^2 \sin 2\phi \qquad (8) \\ p &= \text{length of curve} = 2r\phi \qquad (9) \\ r &= \frac{\sin\theta}{\sin\phi} \qquad (10) \end{aligned}$$

Sub (10) into (8),

$$\begin{aligned} A &= \frac{\pi}{2} \\ &= \theta - \frac{1}{2}\sin 2\theta + \frac{\phi \sin^2\theta}{\sin^2\phi} - \frac{\sin^2\theta\cos\phi}{\sin\phi} \qquad (11) \end{aligned}$$

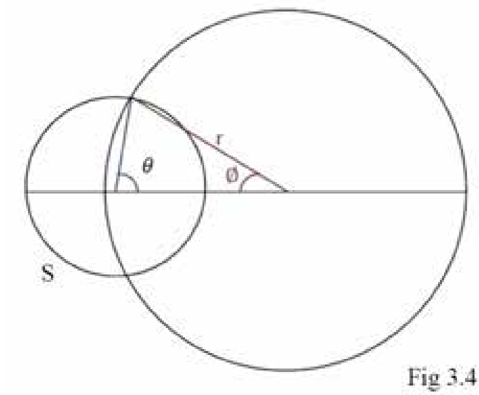


Fig 3.4

Differentiate both sides with respect to θ,

$$\begin{aligned}
0 &= 1 - \cos 2\theta + \frac{2\phi \sin\theta\cos\theta}{\sin^2\phi} - \frac{2\sin\theta\cos\theta\cos\phi}{\sin\phi} \\
&\quad + \frac{d\phi}{d\theta}\left(\frac{\sin^2\theta}{\sin^2\phi} - \frac{2\phi\sin^2\theta\cos\phi}{\sin^3\phi} + \frac{\sin^2\theta\sin\phi}{\sin\phi} + \frac{\sin^2\theta\cos^2\phi}{\sin^2\phi}\right) \\
0 &= 2\sin^2\theta + \frac{2\sin\theta\cos\theta(\phi - \sin\phi\cos\phi)}{\sin^2\phi} + \frac{d\phi}{d\theta}\left[\frac{2\sin^2\theta}{\sin^3\phi}(\sin\phi - \phi\cos\phi)\right] \\
\frac{d\phi}{d\theta} &= -\frac{[\sin\theta\sin^2\phi + \cos\theta(\phi - \sin\phi\cos\phi)]\sin\phi}{\sin\theta(\sin\phi - \phi\cos\theta)} \qquad (12)
\end{aligned}$$

Sub (10) into (9),

$$p = 2\phi\frac{\sin\theta}{\sin\phi} \qquad (13)$$

Differentiate both sides with respect to θ,

$$\begin{aligned}
\frac{dp}{d\theta} &= 2\left[\frac{d\phi}{d\theta}\left(\frac{\sin\theta}{\sin\phi}\right) + \frac{\cos\theta\sin\phi - \frac{d\phi}{d\theta}\cos\phi\sin\theta}{\sin^2\phi}\phi\right] \\
\frac{1}{2}\frac{dp}{d\theta} &= \frac{d\phi}{d\theta}\left(\frac{\sin\theta(\sin\phi - \phi\cos\phi)}{\sin^2\phi}\right) + \frac{\phi\cos\theta}{\sin\phi} \qquad (14)
\end{aligned}$$

Sub (12) into (14)

$$\begin{aligned}
&\frac{1}{2}\frac{dp}{d\theta} \\
&= -\frac{[\sin\theta\sin^2\phi + \cos\theta(\phi - \sin\phi\cos\phi)]\sin\phi}{\sin\theta(\sin\phi - \phi\cos\theta)}\left(\frac{\sin\theta(\sin\phi - \phi\cos\theta)}{\sin^2\phi}\right) + \frac{\phi\cos\theta}{\sin\phi} \\
&= -\frac{[\sin\theta\sin^2\phi + \cos\theta(\phi - \sin\phi\cos\phi)]}{\sin\phi} + \frac{\phi\cos\theta}{\sin\phi} \\
&= -\sin\theta\sin\phi + \cos\theta\cos\phi \\
&= \cos(\theta + \phi) \qquad (15)
\end{aligned}$$

Differentiate both sides with respect to θ,

$$
\begin{aligned}
&\frac{1}{2}\frac{d^2p}{d\theta^2} \\
&= -\left(1+\frac{d\phi}{d\theta}\right)\sin(\theta+\phi) \\
&= -\left(1-\frac{[\sin\theta\sin^2\phi+\cos\theta(\phi-\sin\phi\cos\phi)]\sin\phi}{\sin\theta(\sin\phi-\phi\cos\phi)}\right)\sin(\theta+\phi) \\
&= -\left(\frac{\sin\theta(\sin\phi-\phi\cos\phi)-[\sin\theta\sin^2\phi+\cos\theta(\phi-\sin\phi\cos\phi)]\sin\phi}{\sin\theta(\sin\phi-\phi\cos\phi)}\right) \\
&\quad\times\sin(\theta+\phi) \\
&= -\left(\frac{\sin\theta\sin\phi(1-\sin^2\phi)-\phi(\sin\theta\cos\phi+\cos\theta\sin\phi)+\cos\theta\sin^2\phi\cos\phi}{\sin\theta(\sin\phi-\phi\cos\phi)}\right) \\
&\quad\times\sin(\theta+\phi) \\
&= -\left(\frac{\sin\phi\cos\phi(\sin\theta\cos\phi+\cos\theta\sin\phi)-\phi(\sin\theta\cos\phi+\cos\theta\sin\phi)}{\sin\theta(\sin\phi-\phi\cos\phi)}\right) \\
&\quad\times\sin(\theta+\phi) \\
&= \left(\frac{\phi-\sin\phi\cos\phi}{\sin\theta(\sin\phi-\phi\cos\theta)}\right)\sin^2(\theta+\phi) \\
&= \left(\frac{2\phi-\sin 2\phi}{2\sin\theta\cos\phi(\tan\phi-\phi))}\right)\sin^2(\theta+\phi) \qquad (16)
\end{aligned}
$$

To obtain the extreme value of p, $\dfrac{dp}{d\theta}=0$. By equation (15), $\theta+\phi=\dfrac{\pi}{2}$.

As $2\phi>\sin 2\phi$ and $\tan\phi>\phi$ for $0<\theta,\phi<\dfrac{\pi}{2}$, by equation (16), $\left.\dfrac{d^2p}{d\theta^2}\right|_{\theta+\phi=\frac{\pi}{2}}>0$.

Therefore, the minimum cut must be an orthogonal general circular arc. .... (17)

Sub $\phi=\dfrac{\pi}{2}-\theta$ into (11) and (13)

$$
\begin{aligned}
A(\theta) &= \theta-\frac{1}{2}\sin 2\theta+\frac{\left(\frac{\pi}{2}-\theta\right)\sin^2\theta}{\sin^2\left(\frac{\pi}{2}-\theta\right)}-\frac{\sin^2\theta\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} \\
&= \theta-\frac{1}{2}\sin 2\theta+\frac{\left(\frac{\pi}{2}-\theta\right)\sin^2\theta}{\cos^2\theta}-\frac{\sin^3\theta}{\cos\theta} \\
p(\theta) &= 2\left(\frac{\pi}{2}-\theta\right)\frac{\sin\theta}{\sin\left(\frac{\pi}{2}-\theta\right)} \\
&= (\pi-2\theta)\tan\theta
\end{aligned}
$$

We can solve the problem for $A = \dfrac{\pi}{2}$ by plotting the graph with the software WINPLOT.

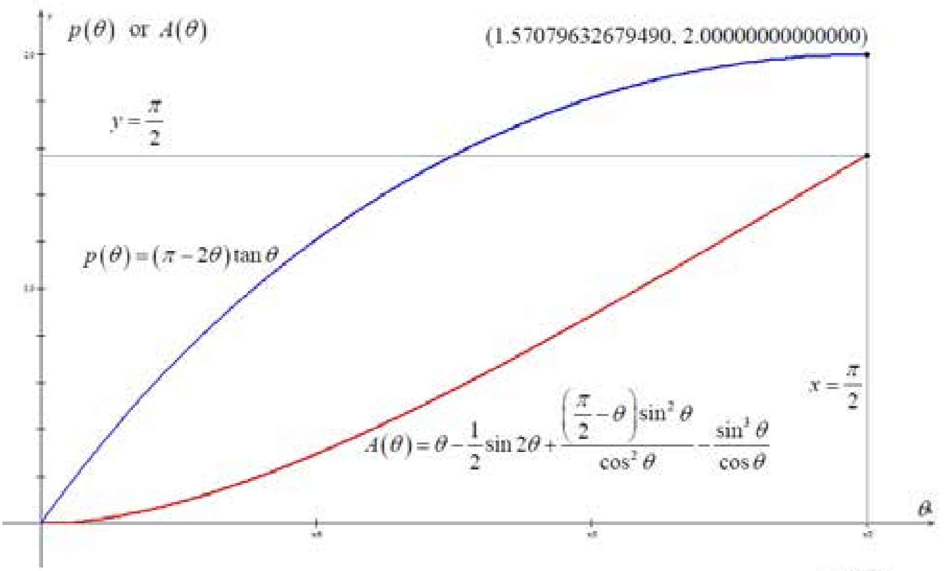


Fig 3.5

From Fig 3.5, $\theta = \dfrac{\pi}{2}, d_2 = p\left(\dfrac{\pi}{2}\right) = 2$ which is the length of a diameter.

Therefore, the minimum cut is the radius method and the length of cut is 2.

Solving the problem for n = 2 and unequal division

To prove that the minimum cut is the orthogonal generalized circular arc

If we use the method of Case 2 in Fig 3.2 on page 10 to divide a circle into 2 unequal areas, by applying Lemmas 7 (c), the shortest curve l satisfies

1. l is a generalized circular arc;
2. The region enclosed by l and a non-major arc must be convex.

By replacing $\dfrac{\pi}{2}$ to any constant in equations (8) and (11), we can still get the conclusion (17).

That is, the minimum cut of this method must be an orthogonal generalized circular arc.

If we use the method of Case 1 in Fig 3.1 on page 10 to divide a circle into 2 unequal areas, by the isoperimetric inequality (1.1b), a circle has the shortest perimeter.

That is, the minimum cut of this method must be a circle.

Next, we need to compare which one is better? An orthogonal generalized circular arc? A circle?

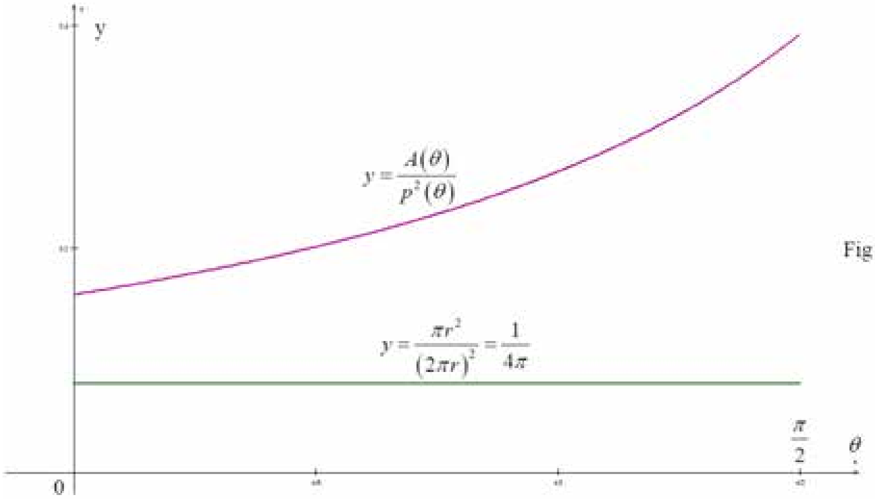
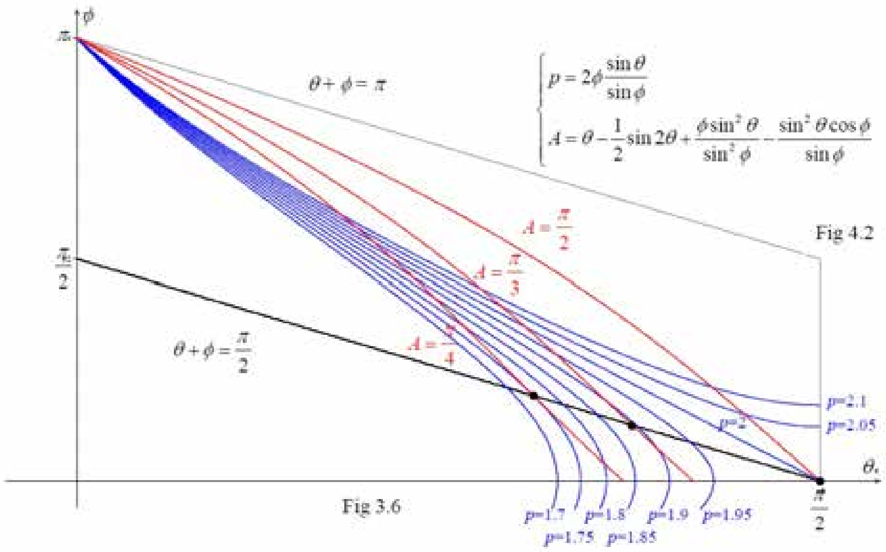


Fig 4.1

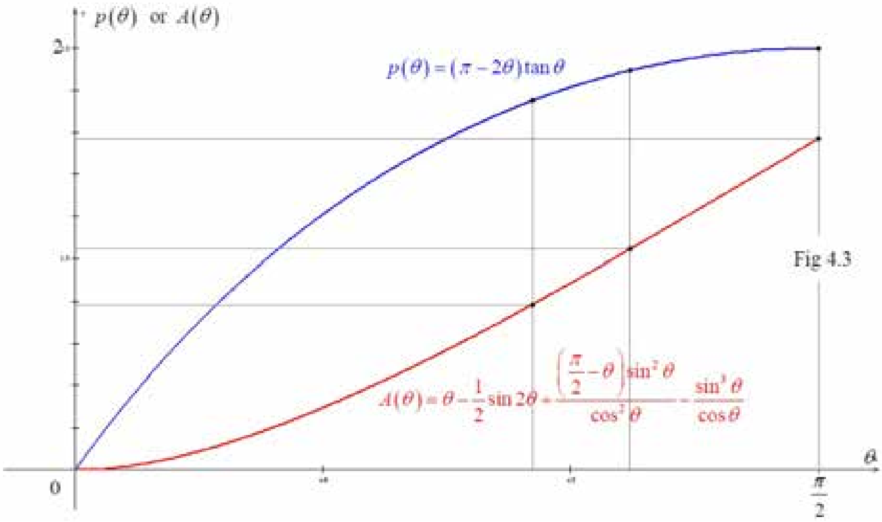
In Fig 4.1, we are sure that purple curve is over green curve. Therefore, we have the following:

Orthogonal generalized circular arc is the solution of the problem for $n = 2$ and unequal division. .... (18)



In Fig 4.2, we can see that at the black dots, the red curves is approximately tangent to a blue curves. So, with the restriction of moving on the same red curves, values of p reach their minimum at the black dots. Equivalently, the necessary and sufficient condition for the problem is $\theta + \phi = 90°$.

To find the minimum length of a curve ending on S to enclose an area of $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{4}$



By reading the values in WINPLOT, we have

θ	$p(\theta)$	$A(\theta)$	Remark
$\frac{\pi}{2} \approx 1.57079632679490$	2	$\frac{\pi}{2}$	This case can be proven geometrically
1.17383106252199	1.89382525073059	$\frac{\pi}{3}$	
0.96620602437632	1.75016083782380	$\frac{\pi}{4}$	

Using Method of Bisection in Excel to check the answer, we get

n	θ	$A(\theta) - \frac{\pi}{n}$	$p(\theta)$
3	1.173	-0.001071732	1.893375
3	1.174	0.00021788	1.893917
4	0.966	-0.000252734	1.749986
4	0.967	0.000974139	1.750833
4	0.9661	-0.000130064	1.750071

Therefore, $d_2 = 2, 1.893 \le d_3 \le 1.894$ and $1.750 \le d_4 \le 1.751$ ....(19)

Solving the problem for $n = 3$ and equal division

To divide a circle to 3 equal areas, there are five possible cases:

Case 1: Two separated closed curves inside the circle (Fig 5.1)

Case 2: Two closed curves nested together (Fig 5.2)

Case 3: A closed curve and a curve ending on the circle (Fig 5.3)

Case 4: Two curves ending on the circle (Fig 5.4)

Case 5: A “Y-shaped” curve (Fig 5.5)

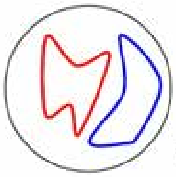


Fig. 5.1

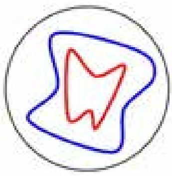


Fig. 5.2

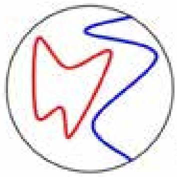


Fig. 5.3

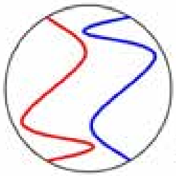


Fig. 5.4


Fig. 5.5

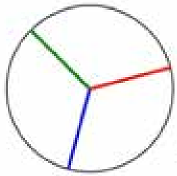


Fig. 5.6

The radius method

Fig 5.6 shows the simplest method to divide a circle into 3 equal parts is the radius method, cutting 3 radii from center. Therefore, the length of curve is 3. Now, we compare the minimum length of the five cases with 3, to find out whether the cases consist of the best solution.

Case 1: Two separated closed curves inside the circle

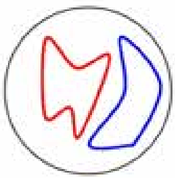


Fig. 5.7

According to the isoperimetric inequality (1.1b), with fixed area, circle has the shortest perimeter. For the minimum length of cut, the two curves are circles.

$$\frac{\pi}{3} = \pi r^2 \implies r = \frac{1}{\sqrt{3}}$$

$$\begin{aligned} \implies l(\text{two such circles}) &\geq 2\left(2\pi \cdot \frac{1}{\sqrt{3}}\right) \\ &> 7.255 \\ &> 3 \end{aligned}$$

As $4r > 2$, we cannot put two such circles inside S. However, we still have

$$l(\text{curve}) > 3.$$

Therefore, Case 1 must not be our solution.

Case 2: Two closed curves nested together

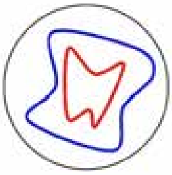


Fig. 5.8

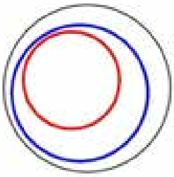


Fig. 5.9

According to the isoperimetric inequality (1.1b), with fixed area, circle has the shortest perimeter. For the minimum length of cut, the two curves are circles.

$$l(\text{two such circles}) \geq 2\pi\left(\frac{1}{\sqrt{3}}+\sqrt{\frac{2}{3}}\right) > 8.757 > 3$$

Therefore, Case 2 must not be our solution.

Case 3: A closed curve and a curve ending on the circle

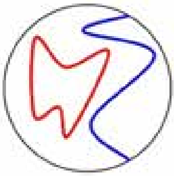


Fig. 5.10

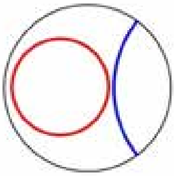


Fig. 5.11

According to the isoperimetric inequality (1.1b) and Theorem (18), with fixed areas, a circle and an orthogonal circular arc has the shortest total length.

$$l(\text{such circles}) \geq \frac{2\pi}{\sqrt{3}} + d_3 > 5.520 > 3$$

Therefore, Case 3 must not be our solution.

Case 4: Two curves ending on the circle

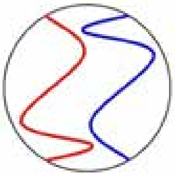


Fig. 5.12

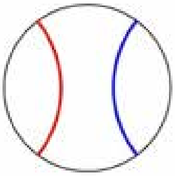


Fig. 5.13

According to the Theorem (18), with fixed areas, orthogonal circular arcs give the shortest total length.

$$l(\text{the curve}) \geq 2d_3 > 3.786 > 3$$

Therefore, Case 4 must not be our solution.

Case 5: A "Y-shaped" curve

Symbols used

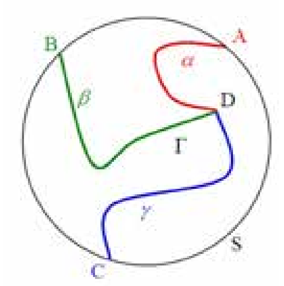


Fig. 5.14

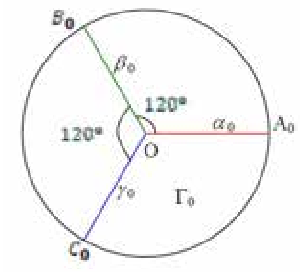


Fig. 5.15

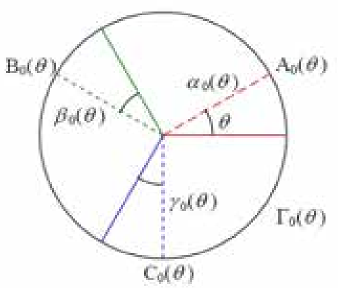


Fig. 5.16

First, we define the three segments (which are coloured in Fig. 5.14) to be α, β, γ such that $l(\alpha) \leq l(\beta) \leq l(\gamma)$. By necessary reflection, α, β, γ are named in anticlockwise direction.

Second, we define Γ to be the "Y-shaped" curve that consists of all the three segments. Equivalently, $\Gamma = \alpha \cup \beta \cup \gamma$.

Third, the region enclosed by α and β means the region enclosed by α, β and the arc $\widehat{AB}$ which is directed in anticlockwise direction. $\widehat{AB}$ may be a major or minor arc but the region must not contain γ.

With reference to the Fig. 5.14, some points are named.

As we are going to compare the minimum length of Γ with that of the radius method, we name the configuration of radius method as show in Fig. 5.15.

It is noted that $A_0 = (1, 0)$.

$X(\theta)$ stands for the result of rotating X by θ anticlockwise through O. Fig. 5.16 shows the result of rotating Fig. 5.15 by θ anticlockwise through O.

We called Γ cuts $\Gamma_0(\theta)$ if and only if α cuts $\alpha_0(\theta)$ and β cuts $\beta_0(\theta)$ and γ cuts $\gamma_0(\theta)$.

We called Γ is minimum if and only if $l(\Gamma) \leq l(\Gamma')$ for all Y-shaped Γ' dividing S as required.

To prove that

if $D \neq O$ and there exists ϕ such that Γ cuts $\Gamma_0(\phi)$, then $l(\Gamma) > l(\Gamma_0)$ .... (20)

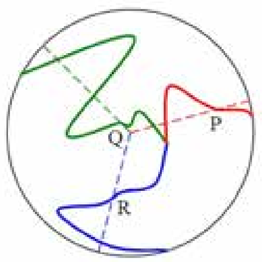


Fig. 5.17

In Fig. 5.17, Γ and $\Gamma_0(\phi)$ are shown in solid and dotted curves respectively.

It is noted that $D \neq O$.

Let P, Q, R be points of intersection between α and $\alpha_0(\phi)$, β and $\beta_0(\phi)$, γ and $\gamma_0(\phi)$ respectively.

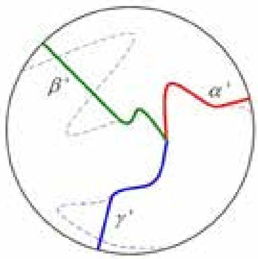


Fig. 5.18

In Fig. 5.18, α, β, γ are shown in dotted curve.

α' is the curve walking from D to P along α and from P to $A_0(\phi)$ along $\alpha_0(\phi)$.

β' and γ' are obtained in similar way.

By Lemma (4),

$$l(\alpha') + l(\beta') + l(\gamma') \leq l(\alpha) + l(\beta) + l(\gamma)$$

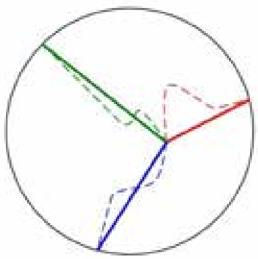


Fig. 5.19

In Fig. 5.19, α', β', γ' are shown in dotted curve.

α'' is the solid line joining D and $A_0(\phi)$.

β'' and γ'' are obtained in similar way.

By Theorem (3),

$$l(\alpha'') + l(\beta'') + l(\gamma'') \leq l(\alpha') + l(\beta') + l(\gamma')$$

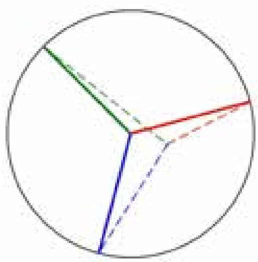


Fig. 5.20

In Fig. 5.20, $\alpha'', \beta'', \gamma''$ are shown in dotted line. Three radii joining from O to $A_0(\phi), B_0(\phi), C_0(\phi)$ are shown in solid line.

By theorem about Fermat point (2), as $D \neq O$,

$$l(\alpha'') + l(\beta'') + l(\gamma'') > 3$$

Therefore,

$$\begin{aligned} l(\Gamma) &\geq l(\alpha') + l(\beta') + l(\gamma') \\ &\geq l(\alpha'') + l(\beta'') + l(\gamma'') \\ &> 3 = l(\Gamma_0) \end{aligned}$$

To prove that if Γ is minimum, then

(a) α, β, γ are genearlized circular arcs,
(b) $l(\Gamma) \leq 3$
(c) $l(\alpha) \leq 1$
(d) $0.786 < 2d_3 - 3 \leq l(\alpha), l(\beta), l(\gamma) \leq 3 - d_3 < 1.107$

Proof of (a). For the curve α,

Step 1) color A and D red

Step 2) color the segments which are not generalized circular arcs blue (including end-points)

Step 3) color the intersections of two different generalized circular arcs blue

Step 4) color the remaining uncolored segments red (which are generalized circular arcs)

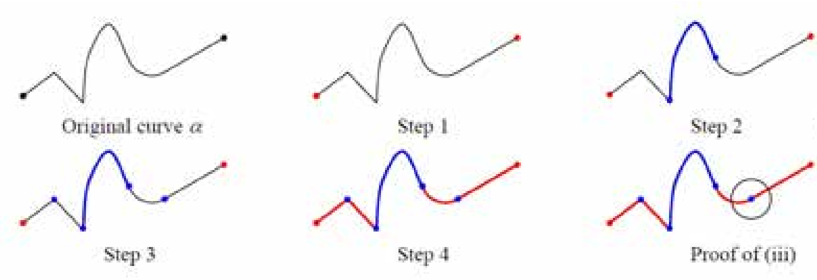


Fig. 5.21

Then,

Case 1) if α does not consist of blue points, then α is a generalized circular arc.

Case 2) if α consists of blue points, then we are going to prove that we can draw a very small circle centred at a blue point so that (i) α divides the circle with one segment, (ii) β and γ does not intersect the circle, and (iii) the segment inside the circle is not a generalized circular arc.

Proof of (i) and (ii): Let P be a blue point. With reference to the Fig. 5.22 on next page,

Step 5) take $0 < r_1 < PD$,

Step 6) take r_2 such that $r_2 \leq r_1$ and $0 < r_2 < \min(\{PQ : Q \text{ lies on } \beta \text{ or } \gamma\})$,

Step 7) take r_3 such that $r_3 \leq r_2$ and $0 < r_3 < \min(\{PQ : Q \text{ lies on S}\})$.

Let S_1 be the circle with centre P and radius r_3.

Step 8a) if α divides S_1 with one segment, take $r_4 = r_3$,

Step 8b) otherwise, if more than one segment lies inside S_1, color the segment containing P green and others brown.

Take r_4 such that $r_4 \leq r_3$ and $0 < r_4 < \min(\{PQ : Q \text{ lies on brown segments}\})$

Now, let S_2 be the circle with centre P and radius r_4. By the above method, S_2 satisfies (i) and (ii). □

Proof of (iii). If the circle can be drawn with condition (i), by the coloring method, (iii) holds.

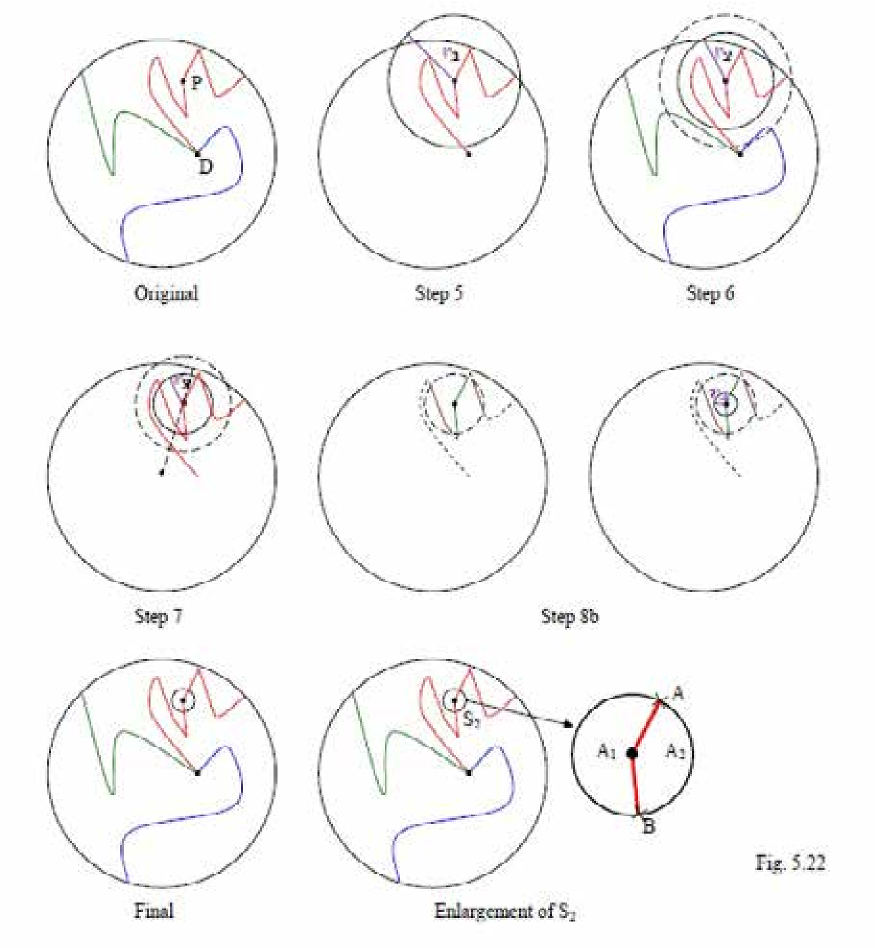


Fig. 5.22

It is noted that the part of α inside S_2 has the properties (i) and (iii). Apply Lemma 7 (b) inside S_2, we can replace that part of α by a generalized circular arc so that $l(\alpha)$ is less than before.

Case 2 (α consists of blue points) contradicts with that Γ is minimum.

Hence, α is of Case 1 and a generalized circular arc. $\square$

By similar argument for β and γ, we proved that (a) α, β, γ are genearlized circular arcs. ....(21)

$\square$

Proof of (b). Γ is minimum $\implies l(\Gamma) \leq l(\Gamma_0) = 3$ ....(22)

$\square$

Proof of (c). If $l(\alpha) > 1$, then $1 < l(\beta) \leq l(\gamma)$, $l(\alpha) + l(\beta) + l(\gamma) > 3$, which contradicts with (b)

By contradiction, $l(\alpha) \leq 1$ ....(23)

$\square$

Proof of (d).

By (22), $l(\alpha) + l(\beta) + l(\gamma) \leq 3$ ....(i)

By (18), $l(\alpha) + l(\beta) \geq d_3$ ....(ii)

$l(\beta) + l(\gamma) \geq d_3$ ....(iii)

$l(\gamma) + l(\alpha) \geq d_3$ ....(iv)

By (19), $d_3 > 1.893$. (i) - (ii) gives $l(\gamma) \leq 3 - d_3 < 1.107$,

(iii) + (iv) - (i) gives $l(\gamma) \geq 2d_3 - 3 > 0.786$.

By similar argument for β and γ,

(d) $0.786 < 2d_3 - 3 \leq l(\alpha), l(\beta), l(\gamma) \leq 3 - d_3 < 1.107$ ....(24)

$\square$

To prove that if Γ is minimum, then α is inscribed in a sector of S with angle θ less than $90°$ ....(25)

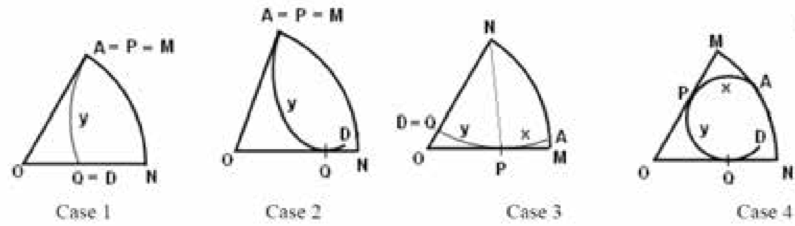


Fig. 5.23

Let P, Q be the points of intersection of the radii and α such that P and Q are near A and D respectively.

Name the sector OMN such that OPM and OQN are straight lines.

Let $x = \widehat{AP}$ and $y = \widehat{PQ}$.

As $PQ \overset{(3)}{\leq} y \leq l(\alpha) - x \overset{(23)}{\leq} 1 - x \overset{(4)}{\leq} 1 - PM = OP$, by Lemma (5), $\angle POQ \leq \angle PQO$.

So, if we call $\theta = \angle POQ$, θ is not the largest angle in $\triangle QOP$. Therefore, (25) is proved.

To prove that if Γ is minimum, then, with θ defined by (25), β must cut the sector $OB_0B_0(\theta)$ .... (26)

Assume, on the contrary, that β does not cut the sector $OB_0B_0(\theta)$.

It is noted that this assumption implies $D \neq O$. We had seven cases:

Case 1) β does not cut OA_0.

Case 2a) β cuts OA_0 once and cuts $OA_0(\theta)$ none.

Case 3) β cuts OA_0 once and cuts $OA_0(\theta)$ twice.

Case 2b) β cuts OA_0 twice and does not cut $OA_0(\theta)$.

Case 4a) β cuts OA_0 twice, cuts $OA_0(\theta)$ and cuts $OA_0(90°)$ none.

Case 5) β cuts OA_0 twice, cuts $OA_0(\theta)$ and cuts $OA_0(90°)$ once.

Case 4b) β cuts OA_0 twice, cuts $OA_0(\theta)$ and cuts $OA_0(90°)$ twice.

We are going to prove that all the above cases are impossible. Then, by contradiction, (26) holds.

Case 1) β does not cut OA_0 (Fig. 5.24).

The region enclosed by α and β is inside the sector OA_0B_0.

The area enclosed by α and $\beta < \frac{\pi}{3}$.

It is impossible for this case.

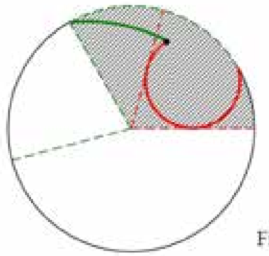


Fig. 5.24

Case 2a) β cuts OA_0 once and cuts $OA_0(\theta)$ none (Fig. 5.25); and

Case 2b) β cuts OA_0 twice and cuts $OA_0(\theta)$ none (Fig. 5.26).

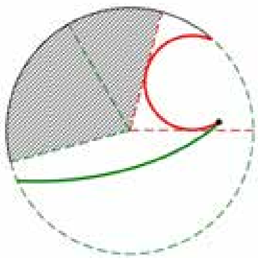


Fig 5.25

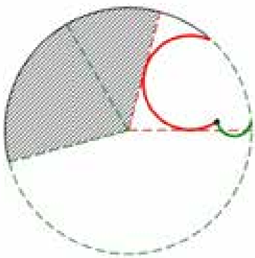


Fig 5.26

The sector $OA_0(\theta)B_0(\theta)$ is inside the region enclosed by α and β.

The area enclosed by α and $\beta > \frac{\pi}{3}$.

It is impossible for this case.

Case 3) β cuts OA_0 once and cuts $OA_0(\theta)$ twice (Fig. 5.27).

The shaded semi-circle is inside the region enclosed by α and β.

The area enclosed by α and $\beta > \frac{\pi}{2}$.

It is impossible for this case.

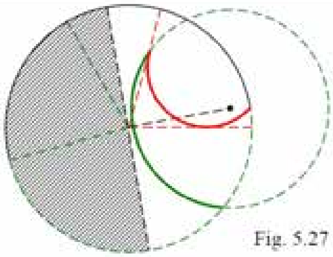


Fig. 5.27

Case 4a) β cuts OA_0 twice, cuts $OA_0(\theta)$ and cuts $OA_0(90°)$ none; and

Case 4b) β cuts OA_0 twice, cuts $OA_0(\theta)$ and cuts $OA_0(90°)$ twice (Fig. 5.28).

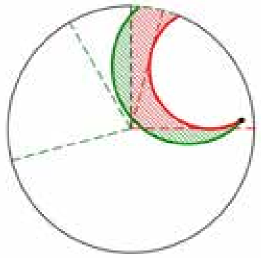


Fig 5.28

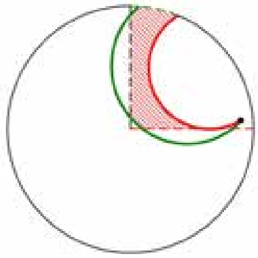


Fig 5.29

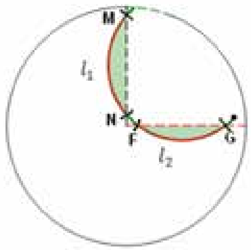


Fig 5.30

Firstly, in Fig. 5.29, we have the area of red shaded part $\leq \dfrac{\pi}{4}$.

In Fig. 5.30, let $\widehat{MN}$ and $\widehat{FG}$ be l_1 and l_2 respectively. (Denote $l_1 = 0$ for Case 4a)

Denote the area enclosed by l_1 with $\widehat{MN}$ and l_2 with $\widehat{FG}$ be A_1 and A_2, respectively.

(Remark: we denote $l_1 = A_1 = 0$ for Case 4a). By (24),

$$l_1 + l_2 \leq l(\beta) < 1.107$$

$$\begin{aligned}\text{The area of green shaded part} &= A_1 + A_2 \\ &\leq \frac{\pi}{2}\left(\frac{l_1}{\pi}\right)^2 + \frac{\pi}{2}\left(\frac{l_2}{\pi}\right)^2 \qquad (1.2\text{b}) \\ &\leq \frac{1}{2\pi}(l_1 + l_2)^2 \\ &< 0.196 \\ &< \frac{\pi}{12}\end{aligned}$$

$$\begin{aligned}\text{The area enclosed by } \alpha \text{ and } \beta &= \text{the area of red and green shaded parts} \\ &< \frac{\pi}{4} + \frac{\pi}{12} = \frac{\pi}{3}\end{aligned}$$

It is impossible for this case.

Case 5) β cuts OA_0 twice, cuts $OA_0(\theta)$ and cuts $OA_0(90°)$ once (Fig. 5.31).

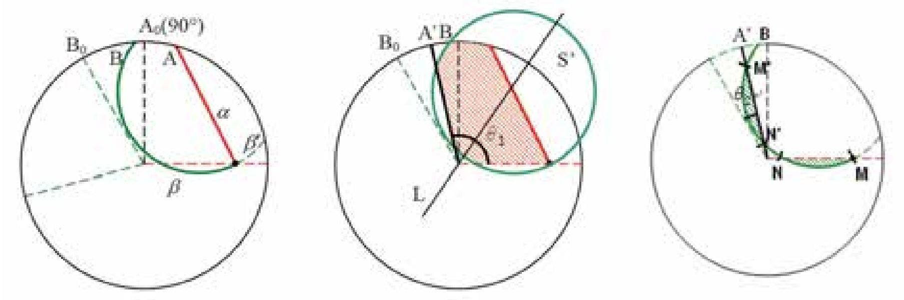


Fig 5.31 Fig 5.32 Fig 5.33

In Fig. 5.32, let S' be the circle containing β. Draw a line L which passes through the centres of S and S'. Under reflection about L, S and S' will not change but A_0, M, N are reflected to A', M', N' respectively.

In Fig. 5.32 and 5.33, let the angle between A_0 and A' be θ_1, and the angle between A' and B_0 be θ_2.

Firstly, in Fig. 5.32, we have the area of red shaded part $\leq \dfrac{1}{2}\theta_1$.

In Fig. 5.33, let $\widehat{MN}$ and $\widehat{M'N'}$ be l_1 and l_2 respectively.

Denote the area enclosed by l_1 with $\widehat{MN}$ and l_2 with $\widehat{M'N'}$ be A_1 and A_2, respectively.

Consider the sector $A'OB_0$ (Fig. 5.34 or 5.35)

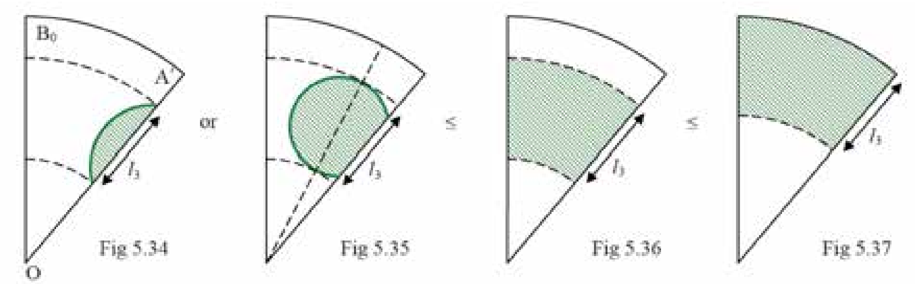


Fig 5.34 Fig 5.35 Fig 5.36 Fig 5.37

$$l_3 \overset{(3)}{\leq} l_2 = l_1 \leq l(\beta) - \widehat{BN} \overset{(24)}{<} 1.107 - \widehat{BN} \overset{(6)}{\leq} 1.107 - 1 = 0.107$$

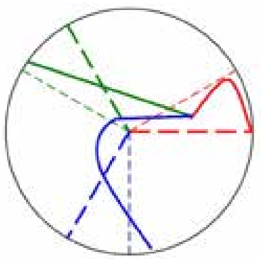
$$A_2 \le \text{green area in Fig. 5.36} \le \text{green area in Fig. 5.37}$$
$$= \frac{1}{2}\theta_2(1-(1-l_3)^2) < \frac{1}{2}\theta_2(1-(0.893)^2) < \frac{1}{4}\theta_2$$

$$\text{The area enclosed by } \alpha \text{ and } \beta = \text{the area of red and green shaded parts}$$
$$< \frac{1}{2}\theta_1 + 2\times\frac{1}{4}\theta_2 = \frac{1}{2}(\theta_1+\theta_2) = \frac{\pi}{3}$$

It is impossible for this case.

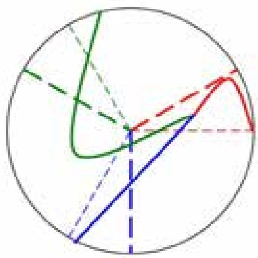
To prove that if Γ is minimum and $D \neq O$, then there exists ϕ such that Γ cuts $\Gamma_0(\phi)$ .... (27)

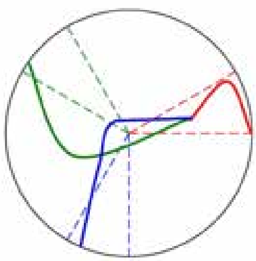
Let ϕ be the angle defined in (25) on page 24.



Case 1) β cuts β_0 first and γ cuts γ_0 first.
Then Γ cuts $\Gamma_0(0)$.

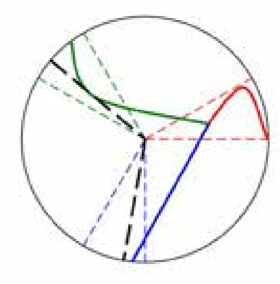
Case 2) β cuts $\beta_0(\theta)$ first and γ cuts $\gamma_0(\theta)$ first.
Then Γ cuts $\Gamma_0(\theta)$.





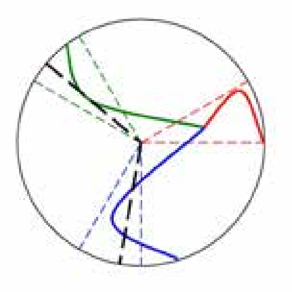
Case 3) β cuts $\beta_0(\theta)$ first and γ cuts γ_0 first.
This case is impossible.

Case 4) β cuts β_0 first and γ cuts $\gamma_0(\theta)$ first.



Let θ' be the largest angle satisfying that $0 \leq \theta' \leq \theta$ and β cuts $\beta_0(\theta')$.

It is impossible that γ does not cut $\gamma_0(\theta')$. Otherwise, as $D \neq O$, the area enclosed by β and $\gamma >$ the area of sector $OB_0(\theta')C_0(\theta') = \dfrac{\pi}{3}$.



Therefore, Γ cuts $\Gamma_0(\theta')$.

To prove that if $D = O$, then $l(\Gamma) \geq l(\Gamma_0)$ and equality holds iff Γ differs Γ_0 by a rotation . . . .(28)

By Theorem (3), $l(\alpha) \geq l(DA) = l(\alpha_0)$ and equality holds iff α is a radius. By similar argument on β and γ, we have $l(\Gamma) \geq l(\Gamma_0)$ and equality holds iff Γ differs Γ_0 by a rotation.

To prove that the minimum cut is the radius method . . . .(29)

Recall some statements proven before,

(20): If $D \neq O$ and there exists ϕ such that Γ cuts $\Gamma_0(\phi)$, then $l(\Gamma) > l(\Gamma_0)$.

(27): If Γ is minimum and $D \neq O$, then there exists ϕ such that Γ cuts $\Gamma_0(\phi)$.

(28): if $D = O$, then $l(\Gamma) \geq l(\Gamma_0)$ and equality holds iff Γ differs Γ_0 by a rotation.

Suppose that Γ is minimum.

$$\begin{aligned} & D \neq O \\ \Longrightarrow\ & D \neq O \text{ and there exists } \phi \text{ such that } \Gamma \text{ cuts } \Gamma_0(\phi) && \text{by Statement (27)} \\ \Longrightarrow\ & l(\Gamma) > l(\Gamma_0) && \text{by Statement (20)} \\ \Longrightarrow\ & \Gamma \text{ is not minimum, which causes a contradiction.} \end{aligned}$$

By contradiction, Γ is minimum $\Longrightarrow$ $D = O$.

By Statement (28), Γ is minimum and $D = O \implies \Gamma$ differs Γ_0 by a rotation.

Therefore, the minimum cut is the radius method.

A necessary condition of the problem for $n = 3$ and unequal division by a "Y-shaped" curve

Statement: If Γ is minimum, then

(a) $\alpha_1, \alpha_2, \alpha_3$ are genearlized circular arcs,

(b) $l(\Gamma) \leq 3$,

(c) $l(\min(\alpha_1, \alpha_2, \alpha_3)) \leq 1$,

(d) $p(A_1) + p(A_2) - 3 \leq l(\alpha_1) \leq 3 - p(A_1)$ ....(30)

By similar argument of (21), (22), (23) on pages 22 - 24, statements (a), (b), (c) hold.

Proof. Proof of (d):
By (22),

$$l(\alpha_i) + l(\alpha_j) + l(\alpha_k) \leq 3 \;\ldots\ldots\; \text{(i)}$$

By (18), (ii),

$$l(\alpha_i) + l(\alpha_j) \leq p(A_k) \;\ldots\ldots\ldots\; \text{(ii)}$$

$$l(\alpha_j) + l(\alpha_k) \leq p(A_i) \;\ldots\ldots\ldots\; \text{(iii)}$$

$$l(\alpha_k) + l(\alpha_i) \leq p(A_j) \;\ldots\ldots\ldots\; \text{(iv)}$$

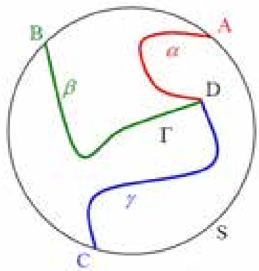


Fig. 5.14

(i) - (iii) gives

$$l(\alpha_i) \leq 3 - p(A_i),$$

(ii) + (iv) - (i) gives

$$l(\alpha_i) \geq p(A_j) + p(A_k) - 3.$$

Therefore we proved (d):

$$p(A_j)+p(A_k)-3 \leq l(\alpha_i) \leq 3-p(A_i)$$

for distinct i, j, k.

□

Disproving the radius method is a solution of the problem for n = 4 and equal division

By [9], the Euclidean Steiner tree problem is about how to connect points together with the shortest length. According to the Euclidean Steiner tree, if the connection has some "Y-shape", the angle between any segments in a Y-shape is $120°$. We use this idea to form our disproving statement.

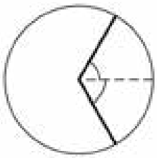


Fig. 6.1

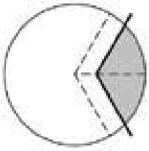


Fig. 6.2

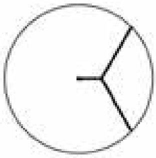


Fig. 6.3

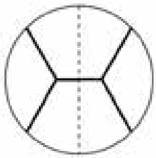


Fig. 6.4

1. We draw 2 segments with $60°$ to horizontal. (Fig 6.1)
2. Move the 2 segments to left until the shaded area equals to $\frac{\pi}{4}$. (Fig. 6.2)
3. Add a line from the centre of the circle to the corner point of the V-Shape. (Fig. 6.3)
4. Reflect the Y-shape by the vertical y-axis.

By Fermat point (2), in Fig. 6.5,

$$OQ + AQ + BQ < OA + OB.$$

Hence,

$$AQ+BQ+CQ+DQ+PQ < OA+OB+OC+OD = 4.$$

[See reviewer's comment (7)]

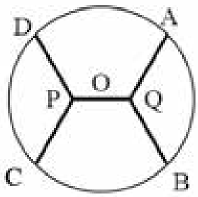


Fig. 6.5

Therefore, the radius method is not a solution of the problem for $n = 4$ and equal division. .. (31)

Calculate the length of cut

$$a = \cos\theta - \frac{\sin\theta}{\sqrt{3}} \text{ and } b = \frac{\sin\theta}{\sin 120°}$$

$$\frac{1}{2}\theta - \frac{1}{2}a\sin\theta = \frac{\pi}{8}$$

$$\frac{1}{2}\theta - \frac{1}{2}\left(\cos\theta - \frac{\sin\theta}{\sqrt{3}}\right)\sin\theta = \frac{\pi}{8}$$

$$\theta - \left(\cos\theta - \frac{\sin\theta}{\sqrt{3}}\right)\sin\theta = \frac{\pi}{4}$$

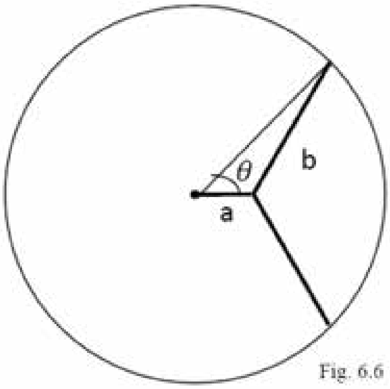


Fig. 6.6

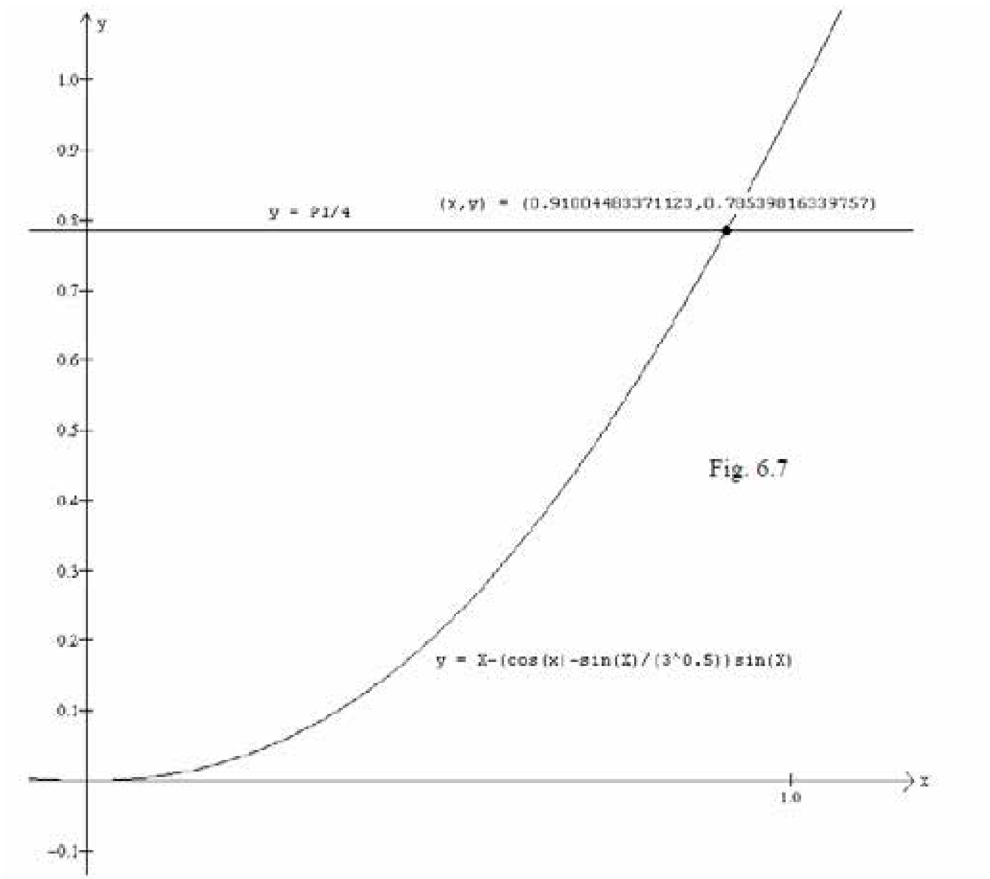


Fig. 6.7

From Fig. 6.7, $\theta = 0.9100448$.

Therefore,

$$\begin{aligned}\text{length of cut of this method} &\approx 4\left(\frac{\sin\theta}{\sin 120^\circ}\right) + 2\left(\cos\theta - \frac{\sin\theta}{\sqrt{3}}\right)\\ &\approx 3.9624372.\end{aligned}$$

Conclusion

When we are cutting a cake or other circular things with the requirement of reducing the cut, we may think that radius method is the best. However, it may not be true.

After we were finishing the project, we discovered that when we divided a circle into 2 or 3 equal parts, the radius method gave the shortest cut. But for the case of 4 equal parts, radius method is not the best. We used the idea of Euclidean Steiner tree to find another method which gave a shorter cut than the radius method. When we divided a circle into 2 unequal parts, the only solution is the orthogonal circular arc. But for the case of 3 unequal parts, we could only found some necessary conditions for the shape and length of cut.

Due to the limitation of time, we could only finish some cases on this problem. We hoped that we could finish the whole problem in the future.

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Reviewer's Comments

The reviewer has some comments about the organization and presentation of this paper, as well as several grammatical mistakes and typos.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. In the title, it may be better to replace "least" by "shortest", for consistency in this paper.
3. In Abstract, the reviewer suggests moving the motivation of this paper: the story of cake dividing, to the introduction, and stating clearly the two directions of the problem: equal and unequal divisions, the definition of n: number of divisions, and roughly describing the main results.
4. In Introduction,
   (a) more precisely, "unequal division" should be "unequal division with given areas", otherwise this case is trivial and confusing.
   (b) the discussion for the problem $n = 3$ may be placed in the part "Solving the problem for $n = 3$ and equal division", and the main results for $n = 2, 3, 4$ should be all discussed here. Moreover, it is better to mention and emphasize the isoperimetric inequality, which is the most useful tool throughout the proof.
5. In Symbols used, definitions of $A(\theta), p(\theta)$ are not so clear, especially for $\theta > \dfrac{\pi}{2}$. It may be better to label them in Fig 1.1. and specify the domain for θ.
6. In Literature Review and Lemmas,
   (a) the first three subsections should be written as three lemmas, just like Lemma (4),
   (b) "theorem (3)" should be "Lemma (3)", "shorter than or" should be "shorter than";
   (c) "O lies inside $C_1 \Rightarrow C_1$ and C_2 touch each other internally" is actually equivalent to the original problem. The reviewer would suggest using "triangle inequality" to prove it directly;
   (d) conclusion 2 is very hard to understand, the reviewer suggests firstly defining the segment associated with the divided region, and then stating the conclusion.
7. The inequality should be
$$AQ + BQ + CP + DP + PQ < OA + OB + OC + OD = 4.$$
8. There are lots of pictures in the paper, and it may be more readable if the symbols for points, curves, regions are marked in the pictures.