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$3n+1$ CONJECTURE (COLLATZ CONJECTURE)

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Abstract. The aim of our project is to investigate the $3n+1$ conjecture. It is very hard to give a general path for each natural number to arrive at 1. So we investigate its negation i.e. there exists a natural number k with no path to 1. There are two possibilities: either k takes a path which becomes a cycle to after n steps, or its path is increasing indefinitely. These two possibilities lead us to study pre-numbers of any odd natural number and the number of peaks of paths. In the project, several interesting results were obtained by studying backward paths, number of peaks and cycles or forward paths.

Firstly, we tried our best to trace back the path by studying all the pre-numbers of any natural number. We successfully showed that every odd number not divisible by 3 has infinitely many pre-numbers by finding them explicitly, and hence obtained a beautiful known result: the odd numbers A and $4A+1$ fall into the same number. Moreover, we also found that two third of these pre-numbers has infinitely many pre-numbers. Continuing in this way, we constructed a decreasing path of any length to arrive at any given number. This leads to a beautiful corollary: there are infinitely many distinct decreasing paths of different lengths to 1.

Secondly, we studied the peaks of the path of any number and obtained a theorem that for any T-number k and natural number r, there exists a path to k of length $2r$ with exactly r peaks.

Thirdly, we assumed, on the contrary, that there is a path with the same beginning and ending, and obtained a constraint on both the length and the sum of powers of 2.

Fourthly, we found infinitely many pairs which meet before 1 and fall into the same path afterward.

Finally, after investigating the possibilities of general (a,b,c)-Conjectures, we concluded that the only possible conjectures are $3n+b$ Conjectures.

[1]This work is done under the supervision of the author's teacher, Mr. Chi-Keung Lai

1. Introduction

In this essay, all unknowns are natural numbers unless otherwise specified. An odd number which is not divisible by 3 is called a **T-number**.

Define a function $f : \mathbb{N} \to \mathbb{N}$ by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even,} \\ 3x+1 & \text{if } x \text{ is odd.} \end{cases}$$

Let $f^{(1)}(x) = f(x)$, $f^{(2)}(x) = (f \circ f)(x)$ and $f^{(n)}(x) = (\underbrace{f \circ \cdots \circ f}_{n \text{ times}})(x)$, for any natural number n.

The $3n+1$ Conjecture (or Collatz Conjecture)

For any natural number k, there exists a natural number n such that $f^{(n)}(k) = 1$. In other words, every natural number k will eventually become 1 under a finite number of operations of f, or 1 is the black hole attracting all natural numbers to it under f. Since all even numbers can be reduced to odd number by f, thus the conjecture is true if it is true for all odd natural numbers. So we focused on studying all odd natural numbers.

For example, $f(3) = 10, f(f(3)) = 5, f(f(f(3))) = 16, f(f(f(f(3)) = 8, f^{(5)}(3) = 4, f^{(6)}(3) = 2, f^{(7)}(3) = 1$. For simplicity, the process can be represented by the path $3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$. Actually, putting any even number into the function f repeatedly until an odd number is obtained is equivalent to removing all factors of 2 of the even number. It is very clumsy to write $5 \to 16 \to 8 \to 4 \to 2 \to 1$. In this essay, only odd numbers in the path will be shown for simplicity e.g. $3 \to 5 \to 1$. The number 3 takes two steps *instead of five steps* to 1 while the number 7 ($7 \to 11 \to 17 \to 13 \to 5 \to 1$) takes 5 steps. For convenience, a new function $h : D \to D$, where D is the set of all odd natural numbers, is introduced and defined as $h(x) = \frac{3x+1}{2^r}$, where r is the natural number making $\frac{3x+1}{2^r}$ an odd natural number e.g. $h(5) = \frac{3\times5+1}{2^4} = 1$.

If $3n+1$ Conjecture is not correct, then there are two possibilities:

1. There exists an odd natural number r other than 1 such that $h^{(n)}(r) = r$ for some n.

2. There exists a natural number q such that $h^{(n)}(q)$ will never be 1 no matter how large n is. In other words, the path starting from q must have a rising trend as a whole, because there are only finite integers below q.

The aim of this essay is to study how a number itself affects its own path, and whether the effect can be generalized to help prove the conjecture.

2. Pre-numbers

By trying some small odd numbers, they all fall into one eventually.

$$3 \to 5 \to 1$$
$$9 \to 7 \to 11 \to 17 \to 13 \to 5 \to 1$$
$$23 \to 35 \to 53 \to 5 \to 1$$

If we call an odd number x a pre-number of an odd number y if and only if $h(x) = y$, then 3, 13 and 53 are pre-numbers of 5. Therefore it is very natural to investigate the relationship between the number and its pre-numbers. We obtained the following lemmas and theorem.

Lemma 1. *If p is odd, then $3p$ has no pre-numbers.*

Proof. Assume, on the contrary, that $3p$ has a pre-number n i.e. $h(n) = 3p$. Then $3p = \frac{3n+1}{2^n}$ for some b. $3p \times 2^b = 3n + 1$ leads to a contradiction that $0 \equiv 1 \pmod 3$. Thus $3p$ has no pre-numbers. □

Corollary 2. *Every path contains at most one multiple of three as its beginning.*

Lemma 3. *Every odd number $k = 3p + 1$ has infinitely many pre-numbers in the form*

$$a_n = \frac{k \cdot 4^n - 1}{3}, \quad \text{where } n \geqslant 1.$$

Proof. If $k = 3p + 1$, then $k \cdot 4^n - 1 \equiv 1 \cdot 1^n - 1 \equiv 0 \pmod 3$. Since $\frac{k \cdot 4^n - 1}{3}$ is an odd number, therefore $\frac{k \cdot 4^n - 1}{3} \to 3\left(\frac{k \cdot 4^n - 1}{3}\right) + 1 = k \cdot 4^n \to k$. Hence $h\left(\frac{k \cdot 4^n - 1}{3}\right) = k$ and $\frac{k \cdot 4^n - 1}{3}$ are pre-numbers of k for all n.

On the other hand, if m is a pre-number of k, then $h(m) = k$ i.e. $k = \frac{3m+1}{2^b}$ for some b. So $k2^b = 3m + 1 \Rightarrow (-1)^b \equiv 1(\mod 3) \Rightarrow b = 2n$ for some n. Therefore $m = \frac{k \cdot 4^n - 1}{3}$ and the lemma is proved. $\square$

Lemma 4. *Every odd number $k = 3p + 2$ has infinitely many pre-numbers in the form*

$$b_n = \frac{2k \cdot 4^n - 1}{3}, \quad \textit{where } n \geqslant 0.$$

Proof. If $k = 3p + 2$, then $2k \cdot 4^n - 1 \equiv 2 \cdot 2 \cdot 1^n - 1 \equiv 0 \pmod 3$. Since $\frac{2k \cdot 4^n - 1}{3}$ is an odd integer, therefore $\frac{2k \cdot 4^n - 1}{3} \to 3\left(\frac{2k \cdot 4^n - 1}{3}\right) + 1 = 2k \cdot 4^n \to k$. So $h\left(\frac{2k \cdot 4^n - 1}{3}\right) = k$ and hence $\frac{2k \cdot 4^n - 1}{3}$ are pre-numbers of k for all n.

On the other hand, if m is a pre-number of k, then $h(m) = k$ i.e. $k = \frac{3m+1}{2^b}$ for some b. So $k2^b = 3m+1 \Rightarrow 2(-1)^b \equiv 1 \pmod 3 \Rightarrow b = 2n+1$ for some $n \geqslant 0$. Therefore $m = \frac{k \cdot 2^{2n+1} - 1}{3} = \frac{2k \cdot 4^n - 1}{3}$ and the lemma is proved. $\square$

Theorem 5. *(cf.* ***Reviewer's Comment*** *1) Let k be a T-number other than* 1.

(a) *If $k \equiv 1 \pmod 3$, then k has infinitely many pre-numbers $\{a_n\}_1^\infty$ satisfying $a_{n+1} = 4a_n + 1$ and $a_n > k$ for any $n \geqslant 1$.*
(b) *If $k \equiv 2 \pmod 3$, then k has infinitely many pre-numbers $\{b_n\}_1^\infty$ satisfying $bn + 1 = 4b_n + 1$, $b_n > k$ for any $n \geqslant 1$ and $b_0 < k$.*

Proof.

(a) By Lemma 3, the odd number $k = 3p + 1$ has infinitely many pre-numbers $a_n = \frac{k \cdot 4^n - 1}{3}$, thus

$$4a_n + 1 = 4\left(\frac{k \cdot 4^n - 1}{3}\right) + 1 = \frac{k \cdot 4^{n+1} - 1}{3} = a_{n+1}.$$

For $n \geqslant 1$, $a_n \geqslant a_1 = \dfrac{4k - 1}{3} > k$.

(b) By Lemma 4, the odd number $k = 3p + 2$ has infinitely many pre-numbers $b_n = \frac{2k \cdot 4^n - 1}{3}$, thus

$$4b_n + 1 = 4\left(\frac{2k \cdot 4^n - 1}{3}\right) + 1 = \frac{2k \cdot 4^{n+1} - 1}{3} = b_{n+1}.$$

$$\text{For } n \geqslant 1,\ b_n \geqslant b_1 = \frac{8k-1}{3} > k.\ b_0 = \frac{2k-1}{3} = 2p+1 < k. \qquad \square$$

Corollary 6. *The odd numbers A and $4A + 1$ fall into the same path.*

Proof. A is a pre-number of $h(A)$. By Theorem 5, $4A+1$ is also a pre-number of $h(A)$ and therefore A and $4A + 1$ fall into the same path as $h(A)$. $\square$

Theorem 7. *When three consecutive pre-numbers x, y and z of a T-number k are divided by 3, their remainders form the set $\{0, 1, 2\}$.*

Proof. By Theorem 5, $z \equiv y + 1 \equiv x + 2 (\text{mod } 3)$ and hence the set of the remainders is $\{0, 1, 2\}$. $\square$

Corollary 8. *Exactly two of the pre-numbers a_s, a_{s+1} and a_{s+2} of any T-number k have also infinitely many pre-numbers.*

Proof. A path $a_1 \to a_2 \to a_3 \to \cdots \to a_n$ is said to be strictly monotone increasing (decreasing) iff $a_1 < a_2 < a_3 < \cdots < a_n$ ($a_1 > a_2 > a_3 > \cdots > a_n$). By Corollary 8, two third of the pre-numbers of any T-number has also infinitely many pre-numbers. By the above lemmas and theorems, we can construct a decreasing path with any length to any given T-number in Theorem 9. $\square$

Theorem 9. *For any given T-number k and natural number n, there exists a strictly monotone decreasing path of length n to k.*

Proof. We construct a backward path starting with k by first choosing the pre-number a_m (or b_m, where $m \geqslant 1$) of k which is a T-number again so that a_m (or b_m) also has pre-numbers by Corollary 8. By Theorem 5, a_m (or b_m) $> k$. Repeating the process again and again until totally n pre-numbers are chosen in this way, a strictly monotone decreasing path of length n to k will be obtained. $\square$

By Theorem 9, we obtained Corollary 10 which is closer to $3n+1$ Conjecture.

Corollary 10. *There are infinitely many distinct decreasing paths of different lengths to* 1*.*

3. Number of peaks

If $3n + 1$ Conjecture is incorrect, then there exists a path which either has no upper bounds or is a cycle. Therefore we tried to investigate paths with increasing trend. We studied the path of $2^n - 1$ and obtained the following Lemma 11.

Lemma 11. *The path starting from* $(2^n - 1)$ *of length* $n-1$*, (where* $n > 1$*) is*

$$(2^n - 1) \to (3 \cdot 2^{n-1} - 1) \to (3^2 \cdot 2^{n-2} - 1) \to \cdots \to (3^{n-1} \cdot 2 - 1)$$

which is strictly monotone increasing.

Proof. For any $k = 1, 2, 3, \ldots, (n-1)$, $(2^k - 1)$ is odd, therefore

$$f(3^r \cdot 2^k - 1) = 3 \cdot (3^r \cdot 2^k - 1) + 1 = 3^{r+1} \cdot 2^k - 2 = 2(3^{r+1} \cdot 2^{k-1} - 1)$$

and hence $(3^r \cdot 2^k - 1) \to (3^{r+1} \cdot 2^{k-1} - 1)$
Inductively $(2^n - 1) \to (3 \cdot 2^{n-1} - 1) \to (3^2 \cdot 2^{n-2} - 1) \to \cdots \to (3^{n-1} \cdot 2 - 1)$.

Since $\frac{3^r \cdot 2^k}{3^{r+1} 2^{k-1}} = \frac{2}{3} < 1$, therefore $3^r \cdot 2^k - 1 < 3^{r+1} \cdot 2^{k-1} - 1$ and the path is strictly monotone increasing. $\square$

However, the story is quite different if we carry one more step, then

$$f(3^{n-1} \cdot 2 - 1) = 3 \cdot (3^{n-1} \cdot 2 - 1) + 1 = 3^n \cdot 2 - 2 = 2(3^n - 1).$$

In particular, if n is even and positive, then $(3^n - 1)$ is divisible by 4. But $(3^{n-1} \cdot 2 - 1) \to \frac{3^n - 1}{2^b}$ for some $b \geqslant 2$.

$$3^{n-1} \cdot 2 - 1 - \frac{3^n - 1}{4} = \frac{5 \cdot 3^{n-1} - 3}{4} > 0$$
$$\Rightarrow 3^{n-1} \cdot 2 - 1 > \frac{3^n - 1}{4} \geqslant \frac{3^n - 1}{2^b}.$$

Fortunately, the path is decreasing in the n^{th} step and a peak $(3^{n-1}\cdot 2-1)$ is said to be formed. There is only one peak in the path with length n for 2^n-1. Along a path, there can be very few peaks compared with its length and the 'peak-density' can be very low as n is large. Although it is quite difficult to prove that every path starting from an odd number can only has finite number of peaks, we are happy to see that it turns downward at least once after increasing for the first n steps when n is even. We further studied the number of peaks of the path to a given T-number.

By Theorem 5, every T-number in the form $3p+1$ has only pre-numbers larger than itself and that in form $3p+2$ has one pre-number b_0 less than itself in addition. We then studied whether b_0 is still a T-number.

Lemma 12. *If k be a T-number satisfying $k=3p+2$, then k has a pre-number b_r above k such that b_r has a pre-number $b_{r,0}$ below b_r and $b_{r,0}$ is still a T-number.*

Proof. Consider the nine consecutive pre-numbers $b_1, b_2, \ldots, b_9$ of k. By Theorem 5, b_i's are above k and $b_1 \equiv b_4 \equiv b_7 \pmod 3$, $b_2 \equiv b_5 \equiv b_8 \pmod 3$ and $b_3 \equiv b_6 \equiv b_9 \pmod 3$ and $b_3 \equiv b_2+1 \equiv b_1+2 \pmod 3$. Therefore there exists $s \in \{1,2,3\}$ such that $b_s \equiv b_{s+3} \equiv b_{s+6} \pmod 3$. Let $b_s = 3q+2$, then $b_{s,0} = 2q+1 < b_s$.

$b_{s+3} = 64b_s + 21 = 64(3q+2)+21 = 3(64q+49)+2$ and

$b_{s+3,0} = 128q+99 < b_{s+3}$.

$b_{s+6} = 64b_{s+3}+21 = 64(3(64q+49)+2)+21 = 3(64^2q+3185)+2$ and

$b_{s+6,0} = 2(64^2q)+6371 < b_{s+6}$.

So $b_{s,0} = -q+1 \pmod 3$, $b_{s+3,0} = -q \pmod 3$ and $b_{s+6,0} = -q+2 \pmod 3$.

Therefore at least one, say $b_{r,0}$, of $b_{s,0}, b_{s+3,0}, b_{s+6,0}$ is not $0 \pmod 3$ and $b_{r,0}$ is still a T-number. $k < b_{r,0}$ and $b_{r,0} < b_r$. (cf. **Reviewer's Comment** 2) □

By Lemma 12, a peak b_r is formed before k in two steps in the path. Inductively, we can form infinitely many paths of length $2r$ to any T-number with r peaks.

Theorem 13. *For any T-number k and natural number r, there exists a path to k of length $2r$ with exactly r peaks.*

4. Existence of a cycle

In section 3, we had studied the number of peaks to see whether there are any upper bounds. Now we are going to study what happens if some paths form cycles other than $1 \to 1 \to 1$. But up to now, no cycle has been found. If such cycle exists, there is a constraint on both the length and the sum of powers of 2.

Theorem 14. *If a path of length n forms a cycle $x_1 \to x_2 \to x_3 \to \cdots \to x_n \to x_1$ satisfying*

$$x_2 = \frac{3x_1+1}{2^{y_1}}, x_3 = \frac{3x_2+1}{2^{y_2}}, \ldots, x_n = \frac{3x_{n-1}+1}{2^{y_{n-1}}} \text{ and } x_1 = \frac{3x_n+1}{2^{y_n}},$$

then $\dfrac{m}{n} > \dfrac{\log 3}{\log 2}$, where $m = y_1 + y_2 + \cdots + y_n$.

Proof. Assume that there is a path of length n forming a cycle $x_1 \to x_2 \to x_3 \to \cdots \to x_n \to x_1$, i.e. There exists n odd numbers $x_1, x_2, x_3, \ldots, x_n, x_1$ and natural numbers $y_1, y_2, y_3, \ldots, y_n, y_1$ such that

$$\begin{aligned} x_2 &= \frac{3x_1+1}{2^{y_1}} > \frac{3x_1}{2^{y_1}}, x_3 = \frac{3x_2+1}{2^{y_2}} > \frac{3x_2}{2^{y_2}}, \ldots, \\ x_n &= \frac{3x_{n-1}+1}{2^{y_{n-1}}} > \frac{3x_{n-1}}{2^{y_{n-1}}}, \text{ and } x_1 = \frac{3x_n+1}{2^{y_n}} > \frac{3x_n}{2^{y_n}} \end{aligned}$$

$$\therefore\ x_2 x_3 \cdots x_n x_1 > \frac{3^n x_1 x_2 \cdots x_n}{2^{y_1+y_2+\cdots+y_n}}.$$

So $2^m > 3^n$, where $m = y_1 + y_2 + \cdots + y_n$. $m \log 2 > n \log 3 \ \Rightarrow\ \dfrac{m}{n} > \dfrac{\log 3}{\log 2}$. □

As stated in Wikipedia, a number smaller than 10×2^{58} will not be in a cycle and the cycle, if exists, contains no fewer than 35400 numbers (including even terms). Although $2^m > 3^n$, but we believed that 2^m is very close to 3^n as x_i's are at least 10×2^{58}. If $2^m - 3^n = 1$, it will lead to another well-known conjecture stating that $a^b - c^d = 1$ has no integral solution other than $3^2 - 2^3 = 1$.

5. Numbers which will fall into the same path before reaching 1

In the section 2, we concerned the pre-numbers of a T-number and only the relationship between the pre-numbers of the same number is concerned. However, numbers falling into the same path are not necessarily pre-numbers of the same number. For example, consider paths

$$9 \to 7 \to 11 \to 17 \to 13 \to 5 \to 1 \text{ and } 15 \to 23 \to 35 \to 53 \to 5 \to 1,$$

9 and 15 meet at 5 before falling into 1.
Although we couldn't obtain a necessary condition for numbers to fall into the same number (hence the same path), we wanted to find some concrete examples. After investigating and comparing a lot of paths for different numbers, (using http://www.numbertheory.org/php/collatz.html) , we study pairs of odd numbers in approximate ratio of $2:1$ such as $2^{2n}(4B+1)-1$ and $2^{2n-1}(4B+1)-1$. We found that their paths meet before 1 in Theorem 15.

Theorem 15. *(cf.* ***Reviewer's Comment*** *3) For any natural numbers n and non-negative integer B,*

(a) *The paths from $2^{2n}(4B+1)-1$ and $2^{2n-1}(4B+1)-1$ meet at $\frac{3^{2n}(4B+1)-1}{2}$ before 1.*
(b) *The paths from $2^{2n}(4B-1)-1$ and $2^{2n-1}(4B-1)-1$ meet at $\frac{3^{2n}(4B-1)-1}{2}$ before 1.*

Proof.

(a) Since $2^{a+1}3^b(4B+1)-1 \to \frac{3(2^{a+1}3^b(4B+1)-1)+1}{2} = 2^a3^{b+1}(4B+1)-1$, therefore

$$\begin{aligned}2^{2n-1}(4B+1)-1 &\to 2^{2n-2}3(4B+1)-1 \to 2^{2n-3}3^2(4B+1)-1\\ &\to \cdots \to \frac{3^{2n-1}(4B+1)-1}{2}.\end{aligned}$$

As $\frac{3^{2n-1}(4B+1)-1}{2}$ is odd, thus

$$\frac{3^{2n-1}(4B+1)-1}{2} \to 3\left(\frac{3^{2n-1}(4B+1)-1}{2}\right)+1 = \frac{3^{2n}(4B+1)-1}{2}.$$

On the other hand,

$$\begin{aligned}2^{2n}(4B+1)-1 &\to 2^{2n-1}3(4B+1)-1 \to 2^{2n-2}3^2(4B+1)-1\\ &\to \cdots \to \frac{3^{2n}(4B+1)-1}{2} > 1.\end{aligned}$$

So the paths meet at $\frac{3^{2n}(4B+1)-1}{2}$ before 1.

(b) can be proved similarly. □

Corollary 16. $2^{2n}-1$ *and* $2^{2n-1}-1$ *meet at* $\frac{3^{2n}-1}{2}$ *and then fall into the same path. (cf.* ***Reviewer's Comment*** *4)*

Theorem 15 suggests that each pair of $2^{2n}(4B+1)-1$ and $2^{2n-1}(4B+1)-1$ fall into the same path after a finite number of steps. We believed that odd numbers in approximate ratio of $2:1$ can be further studied.

In studying numbers in form of 2^n-1, we surprisingly found many of them fall into the same path early in their paths such as all the paths starting from $2^{43}-1, 2^{45}-1, 2^{47}-1, 2^{49}-1, 2^{51}-1$ and $2^{55}-1$ contain the same number 3133507921263587 and fall into the same path afterward. So we believed that Theorem 15 can be extended to find some consecutive odd integers which meet each other and fall into the same path afterward.

6. Why $3n+1$?

In this section, we studied whether there are similar conjectures other than $3n+1$ Conjecture which is also named as Collatz Conjecture.

Let a, b, c be relatively prime to each other and define a function $g : \mathbb{N} \to \mathbb{N}$ by

$$g(n) = \begin{cases} \frac{n}{c} & \text{iff } n \text{ is divisible by } c, \\ an + b & \text{iff } n \text{ is not divisible by } c. \end{cases}$$

(a, b, c)-Conjecture states that for any natural number n, there exists a natural number m such that $g^{(m)}(n) = 1$. For simplicity, we named $3n + 1$ conjecture as $(3, 1, 2)$-Conjecture. Why did Collatz choose a function with $a = 3$, but not other odd numbers? Firstly, $(1, 1, 2)$-Conjecture is obviously true as an odd input always gives a smaller output after being divided by 2 and reaches 1 after finite number of steps. For $(1, b, 2)$-Conjecture, where $b > 1$, $3b \to b \to b$ implies that all odd multiples of b will end with multiples *of* b rather than one. (cf. **Reviewer's Comment** 5)

Back to $3n + 1$ Conjecture, among all the even numbers, half of them are divisible by 2 but not 4, one fourth of them are divisible by 4 but not 8, one eighth of them are divisible by 8 but not 16, one sixteenth of them are divisible by 16 but not 8 and so on.... The expected value of the powers of 2 of all even numbers can be expressed as $S = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{16} \cdot 4 + \cdots = 2$. Hence we can say that an even number has a factor of 4 on average. In other words, when an odd number takes once the function $f(x)$, it will, on average, be expected to become a little bit over three quarters of its original value, which is generally smaller than itself. So $3n + 1$ conjecture is possibly true. (Note that if so many positive numbers are considered, adding one to $3n$ does not sufficiently affect the ratio of inputs and outputs as a whole result.) When several steps are taken into consideration, we consider all T-numbers and the expected trend is still decreasing.

How about $(a, b, 2)$-Conjecture when a is odd and greater than 3? On average, we expect that an odd number undergoing the function $h(x)$ once will become $\frac{a}{4}$ of its original value, which is larger than itself. Thus the overall trend is increasing and we would not expect all numbers to reach one eventually. So a must be 3. How about when c is not equal to 2?

For $g(x)$ to make the numbers up and down, it is necessary to find a common factor c of all the numbers in form of $ax + b$ satisfying that a, b and c are relatively prime. Assume that there exists a common factor c (c is relatively prime to a and b) for all numbers in the form of $ax + b$. Since $a(c - 1) + b$ and $a(c + 1) + b$ are all divisible by c, therefore their difference $2a$ is also divisible by c. And their sum $2[ac + b]$ is also divisible by c, so $2b$ is divisible by c. c divides both $2a$ and $2b$. However a and b are relatively prime, so $c = 2$. By the above results, we obtained the last theorem in this paper.

Theorem 17. *Among (a, b, c)-Conjecture, the only possible conjecture is $3n + 1$ Conjecture.*

REFERENCES

[1] http://en.wikipedia.org/wiki/Collatz_conjecture
[2] http://www.numbertheory.org/php/collatz.html

Reviewer's Comments

1. Theorem 5 should be changed to:

   Theorem 5 *Let k be a T-number.*
   (a) If $k \equiv 1 \pmod 3$, then the pre-numbers of k are a_n, $n \geqslant 1$ given by $a_{n+1} = 4a_n + 1$, $a_1 = \frac{4k-1}{3}$. If $k > 1$, we have $a_n > k$ for any $n \geqslant 1$. If $k = 1$, we have $a_1 = k$ and $a_n > k$ for all $n \geqslant 2$.
   (b) If $k \equiv 2 \pmod 3$, then the pre-numbers of k are b_n, $n \geqslant 0$, given by $b_{n+1} = 4b_n + 1$, $b_0 = \frac{2k-1}{3}$. We have: $b_n > k$ for any $n \geqslant 1$ and $b_0 < k$.

2. We need to add the following paragraph in the proof to prepare for Theorem 13:

   Secondly, suppose $k \equiv 1 \pmod 3$ is an odd number. Then we consider the nine consecutive pre-numbers $a_2, a_3, \ldots, a_{10}$ of k. We apply the same argument as above, and obtain the desired conclusion.

3. For $n = 1$, $B = 0$ or $B = 14$,
   $$h\left(\frac{3^{2n-1}(4B+1)-1}{2}\right) = 1;$$
   so is the case for $n = 2$, $B = 414252$. The use of f instead of h is therefore suggested. For part (b), note that with $n = 2$, $B = 1$,
   $$\frac{3^{2n}(4B-1)-1}{2} = 121$$
   does not appear in the sequence (path) from $2^{2n-1}(4B-1)-1 = 23$ at all. Therefore, part (b) in Theorem 15 should be deleted.

4. A passing remark should be added after Corollary 16:

   We note in passing that $h\left(\frac{3^{2n-1}-1}{2}\right) > 1$ unless $n = 1$. Then $2^{2n}-1 \to h(2^{2n}-1) \to h \circ h(2^{2n}-1) \to \cdots$ and $2^{2n-1}-1 \to h(2^{2n-1}-1) \to h \circ h(2^{2n-1}-1) \to \cdots$ meet at $h\left(\frac{3^{2n-1}-1}{2}\right) > 1$, and then fall into the same path (using h throughout).

5. There is the $3n + 371$ conjecture, due to Keith Matthews. Moreover, the author's argument does not say much about b.