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GENERALISING ORTHOCENTRES OF TRIANGLES TO SIMPLICES AS THE ISOGONAL CONJUGATES OF THE CIRCUMCENTRES

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Abstract. In this paper, we have generalised the orthocentre of a triangle as the isogonal conjugate of the circumcentre of a simplex. Along this generalisation, we have also carried two intriguing properties of the orthocentre of a triangle over to higher dimensions, which says that the isogonal conjugate of the circumcentre of a simplex is either the incentre or an excentre of its pedal simplex, and is also the radical centre of the facetal circumhyperspheres of the simplex.

To this end, we have extended the scope of isogonal conjugation with respect to simplices to non-interior points through developing new algebraic and geometric characterisations for it. We have also obtained a higher-dimensional analogue of a curious property of isogonal conjugates with respect to triangles, which says that when both a point and its isogonal conjugate with respect to a simplex are projected onto the facets, the projections formed are co-hyperspherical.

Keywords. orthocentre, simplex, barycentric coordinates, isogonal conjugate, circumcentre, pedal simplex, inexcentre, incentre, facetal circumhypersphere, radical centre

Contents

1. Introduction

1.1. **Background.** The challenge of generalising notions and results in Euclidean geometry in the plane to higher dimensions has attracted numerous mathematics lovers to work on. One primary direction of generalisation is of triangles to simplices, and investigating how much geometry can or cannot be carried over. See [9] for an excellent survey paper on this field. See also [2], [3], [10], [16], [19], [23], [27] and [28] from recent years for simplex versions of famous triangle theorems such as Ceva's, Menelaus', Pompeiu's, Wallace's, Miquel's, Thébault's, van Aubel's and the Steiner–Routh theorems.

A comparatively young but already quite fruitful subfield is that of constructing simplex centres that can bring along the most important properties of the original triangle centres to higher dimensions. Successful ones include incentre, excentre, centroid, circumcentre, orthocentre, nine-point centre, symmedian/Lemoine/Grebe point, Gergonne point, Nagel point, Spieker/cleavance centre, Feuerbach point and Fermat–Torricelli point, which are only the first eight, the tenth, the eleventh and the thirteenth members of Kimberling centres under the framework of triangle centres in [11] and [12]. See [1], [4], [5], [8], [13], [20], [22], [24] and [26] for these works mostly from recent years.

While seeking possibilities to generalise other triangle centres to simplices is a definite research direction, we have worked on the equally natural direction of creating equivalents or alternatives to the existing generalisations. Here we describe what have motivated our research and what have inspired the core ideas behind the present work.

1.2. **Motivations and Inspirations.** Our research began with having learned the oddity that the orthocentre of a simplex may not exist because the altitudes may not be concurrent. The so-called Monge point, antimedial circumcentre and orthic inexcentre have generalised the orthocentres of triangles to tetrahedra/simplices from quite distinct perspectives. They were invented through focusing on alternative definitions of the orthocentre of a triangle other than using altitudes, so as to perform novel geometric constructions on tetrahedra/simplices and/or to utilise known tetrahedron/simplex centres. See [5], [6] and [7] for those properties that have been preserved to higher dimensions by these three generalisations.

How about the many more unpreserved properties? In this paper, we will establish higher-dimensional analogues (Theorems 25 and 27) for the following two properties through a fourth perspective which considers the orthocentre of a triangle as the isogonal conjugate of the circumcentre (see [12] for instance): *The orthocentre H of ΔABC is*

(i) *either the incentre or an excentre of its pedal triangle ΔDEF.* (see Figures 1 and 2 resp.)

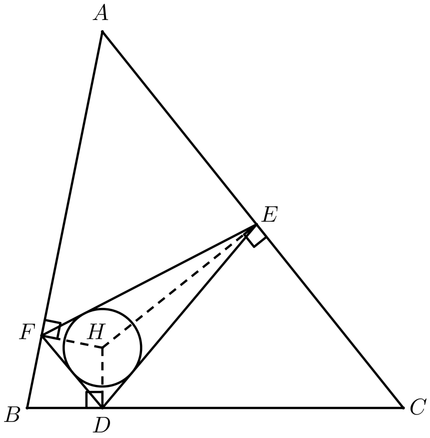

FIGURE 1.

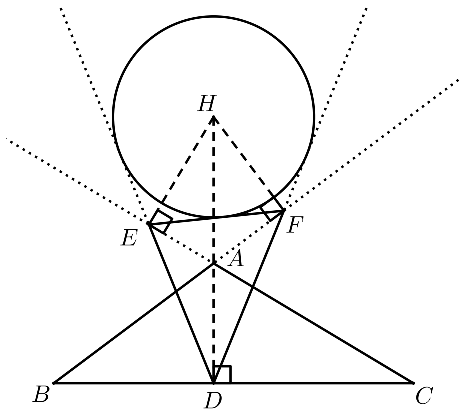

FIGURE 2.

(ii) *the radical centre of the three circles with diameters BC, CA and AB.* (see Figure 3)

To this end, we will utilise the geometric transformation of isogonal conjugation with respect to simplices, which was coined in [22] just less than a decade ago (in attempt to generalise symmedian point using the traditional definition). Our first step will be to establish higher-dimensional analogues (Theorem 14) for the following well-known characterisations of isogonal conjugation with respect to triangles: *If X^* is the isogonal conjugate of a point X with respect to ΔABC, then*

(iii) *X^* is the point of concurrence of the reflections of the lines XA, XB and XC across the angle bisectors of $\angle A$, $\angle B$ and $\angle C$ respectively. When*

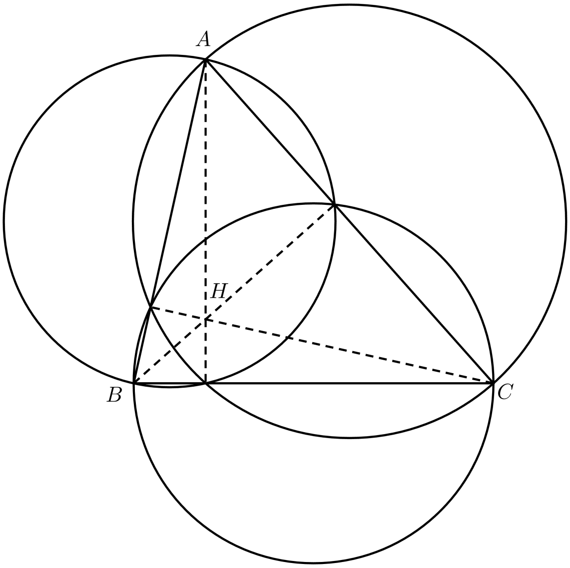

FIGURE 3.

X and X^* are inside ΔABC, we can say that AX and AX^* are equally inclined from AB and AC respectively (and its cyclic variations). (see Figure 4)

(iv) $XX_A \cdot X^*X_A^* = XX_B \cdot X^*X_B^* = XX_C \cdot X^*X_C^*$, *where X_A, X_B and X_C are the projections of X onto the (extended) sides, and X_A^*, X_B^* and X_C^* are those of X^*.* (see Figure 5)

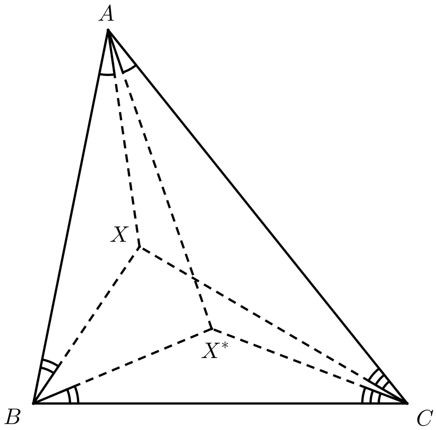

FIGURE 4.

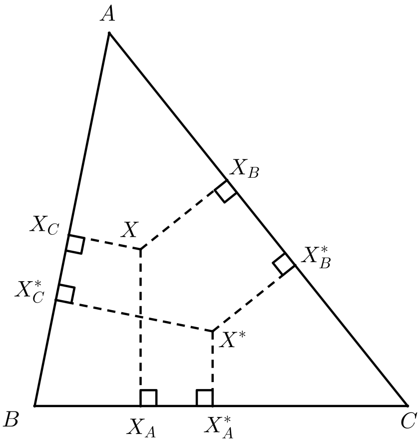

FIGURE 5.

(v) *X^* is the circumcentre of $\Delta X'_A X'_B X'_C$, where X'_A, X'_B and X'_C are the reflections of X across the sides.* (see Figure 6)

(vi) *If the barycentric coordinates of X with respect to ΔABC are $(x_A : x_B : x_C)$, then those of X^* will be*

$$\left(\frac{1}{{h_A}^2 x_A} : \frac{1}{{h_B}^2 x_B} : \frac{1}{{h_C}^2 x_C}\right),$$

where h_A, h_B and h_C are the lengths of the altitudes from A, B and C respectively.

These results will also enable extending the isogonal conjugation in [22] to non-interior points. A higher-dimensional analogue (Theorem 17) for the following interesting property of isogonal conjugate will also follow as a by-product:

(vii) *The six points X_A, X_B, X_C, X^*_A, X^*_B and X^*_C in* (iv) *are concyclic, with circumcentre being the midpoint M of X and X^*.* (see Figures 5 and 7)

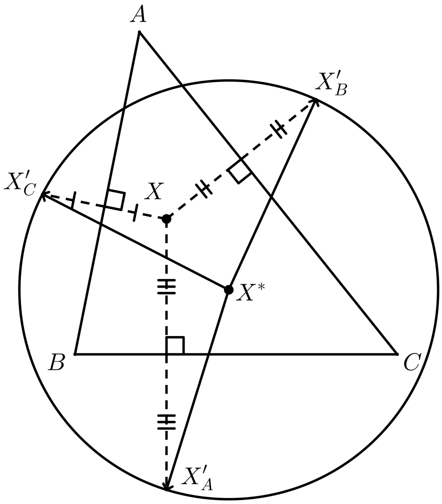

FIGURE 6.

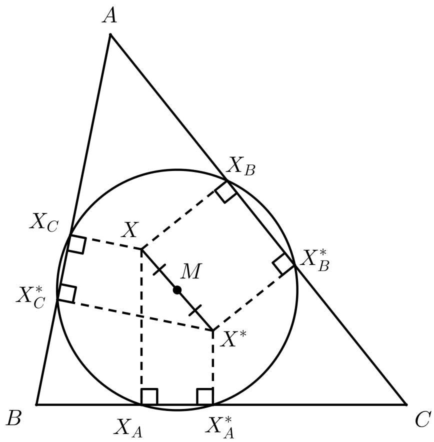

FIGURE 7.

Now we outline the flow of our paper.

1.3. **Outline of Paper.** The aforementioned missions will be accomplished as follows. In Section 1.4, we will define the geometric objects and tools which will be utilised in the upcoming sections, including a simplex and the corresponding barycentric coordinate system, projection and reflection, both absolute and signed distances from a point to a hyperplane, as well as angles between hyperplanes from Definition 1 to Definition 8.

In Section 2.1, we will develop new characterisations of isogonal conjugation for interior points in [22] in algebraic and geometric terms from Definition 9 to Theorem 14. They will be the higher-dimensional analogues of (iii)–(vi) above. We will use these new characterisations to define isogonal conjugation for non-interior points too, and discuss the complications of such a definition from Definition 15 to Theorem 16 by noting the distinction between points which form an isogonal conjugate pair and points which are the isogonal conjugates of each other.

In Section 2.2, we will prove the higher-dimensional analogue of the interesting property (vii) as a $(2n+2)$-point hypersphere in Theorem 17, which contains all the projections of an isogonal conjugate pair onto the extended facets of a simplex.

In Section 3, we will prove one more new characterisation of isogonal conjugate pairs in Lemma 19, which will be useful in proving the upcoming higher-dimensional analogues of (i)–(ii) above. In Section 3.1, we will prove that the isogonal conjugate of the circumcentre of a simplex is an inexcentre of its own pedal simplex from Definition 18 to Theorem 25. In Section 3.2, we will prove that the isogonal conjugate of the circumcentre of a simplex is the radical centre of the facetal circumhyperspheres from Definition 26 to Theorem 27.

In Section 4, we will list some interesting observations made during the production of the paper.

1.4. **Foundations.** In this paper, we will work with vectors, points and various geometric objects in $\mathbb{R}^n$ with $n \geq 2$. Unless otherwise specified, the position vector of a point V (denoted by an italic uppercase letter) will be denoted by $\mathbf{v}$ (the corresponding bold lowercase letter). A set of points and the set of their position vectors will be mentioned interchangeably.

Throughout, we will be speaking of an n-dimensional simplex Δ with the following default notations:

Definition 1. (Simplex, extended facet, extended $(n-2)$-dimensional face) Let $V = \{V_0, V_1, \ldots, V_n\}$ be a set of $(n+1)$ affinely independent points (i.e. the n vectors $\overrightarrow{V_0V_i}$ $(i = 1, \ldots, n)$ are linearly independent). Its convex hull

$$\operatorname{conv}(\{V_0, V_1, \ldots, V_n\}) = \left\{ \sum_{i=0}^{n} a_i \mathbf{v}_i \;\middle|\; \sum_{i=0}^{n} a_i = 1 \text{ and all } a_i \geq 0 \right\}$$

forms an *n-dimensional simplex*, which will denoted by Δ throughout. The *extended facet* opposite to a vertex V_i is the affine hull $\operatorname{aff}(V \setminus \{V_i\})$ of the remaining vertices, which is a hyperplane and will be denoted by F_i throughout. Two extended facets F_i and F_j intersect at the *extended* $(n-2)$*-dimensional face* opposite to V_i and V_j, which is the affine hull $\operatorname{aff}(V \setminus \{V_i, V_j\})$ of the remaining vertices, which is an $(n-2)$-dimensional affine subspace and will be denoted by $F_{i,j}$ throughout.

Definition 2. (Barycentric coordinates) Let $\Delta = \operatorname{conv}(\{V_0, V_1, \ldots, V_n\})$ be a simplex. Each point X admits a unique representation in the form $\mathbf{x} = x_0\mathbf{v}_0 + \cdots + x_n\mathbf{v}_n$, where $x_0, \ldots, x_n \in \mathbb{R}$ with $x_0 + \cdots + x_n = 1$. The *barycentric coordinates* of X with respect to Δ is defined as the ordered tuple $(x_0, \ldots, x_n)$. Given any $a_0, \ldots, a_n \in \mathbb{R}$ with $a_0 + \cdots + a_n \neq 0$, the ordered tuple

$$\left(\frac{a_0}{a_0 + \cdots + a_n}, \ldots, \frac{a_n}{a_0 + \cdots + a_n}\right)$$

is the barycentric coordinates of a point, which we will abbreviate as $(a_0 : \cdots : a_n)$.

Endow $\mathbb{R}^n$ with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Recall that given any subset $S \subseteq \mathbb{R}^n$, the orthogonal complement of S is defined as

$$S^\perp = \{\mathbf{v} \in \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{s} \rangle = 0 \text{ for all } \mathbf{s} \in S\}.$$

In particular, the orthogonal complement $W^\perp$ of an $(n-1)$-dimensional vector subspace W is a 1-dimensional vector subspace. The vectors in $W^\perp$ will be called the normal vectors of W. The normal vectors of unit norm will be called the unit normal vectors of W. For convenience, a (unit) normal vector of the $(n-1)$-dimensional vector subspace parallel to a hyperplane H will also be called a (unit) normal vector of H.

(Orthogonal) projection and reflection will now be defined using the inner product in the usual way:

Definition 3. (Projection of vector onto $(n-1)$-dimensional vector subspace) Let W be an $(n-1)$-dimensional vector subspace with unit normal vector $\mathbf{n}$. The *projection of a vector* $\mathbf{v}$ *onto* W is defined as

$$\operatorname{proj}_W(\mathbf{v}) = \mathbf{v} - \underbrace{\langle \mathbf{v}, \mathbf{n} \rangle\, \mathbf{n}}_{\in W^\perp}.$$

Note that the particular choice of $\mathbf{n}$ does not matter, and proj_W satisfies the linear algebraic definition of projection — a linear transformation T on a vector space such that $T \circ T = T$.

Definition 4. (Projection of point onto hyperplane) Let H be a hyperplane with unit normal vector $\mathbf{n}$, W be the $(n-1)$-dimensional vector subspace parallel to H and A be a point on H. The *projection of a point* P *onto* H is defined as

$$\operatorname{proj}_H(P) = \mathbf{a} + \operatorname{proj}_W\left(\overrightarrow{AP}\right) = \mathbf{a} + \overrightarrow{AP} - \left\langle \overrightarrow{AP}, \mathbf{n} \right\rangle \mathbf{n} = \mathbf{p} - \left\langle \overrightarrow{AP}, \mathbf{n} \right\rangle \mathbf{n}.$$

Note that the particular choice of A and $\mathbf{n}$ do not matter. We will be frequently mentioning the projection of a point X onto F_i, which will be denoted by X_i throughout.

Definition 5. (Reflection of vector across $(n-1)$-dimensional vector subspace) Let W be an $(n-1)$-dimensional vector subspace with unit normal vector $\mathbf{n}$. The *reflection of a vector* $\mathbf{v}$ *across* W is defined as

$$\mathrm{refl}_W(\mathbf{v}) = \mathbf{v} - 2\langle \mathbf{v}, \mathbf{n}\rangle \mathbf{n}.$$

Note that the particular choice of $\mathbf{n}$ does not matter, and refl_W preserves inner product and satisfies the linear algebraic definition of reflection — a linear transformation $T \neq \mathrm{id}$ on a vector space such that $T \circ T = \mathrm{id}$.

Definition 6. (Reflection of point across hyperplane) Let H be a hyperplane with unit normal vector $\mathbf{n}$, W be the $(n-1)$-dimensional vector subspace parallel to H and A be a point on H. The *reflection of a point* P *across* H is defined as

$$\mathrm{refl}_H(P) = \mathbf{a} + \mathrm{refl}_W\left(\overrightarrow{AP}\right) = \mathbf{a} + \overrightarrow{AP} - 2\left\langle \overrightarrow{AP}, \mathbf{n}\right\rangle \mathbf{n} = \mathbf{p} - 2\left\langle \overrightarrow{AP}, \mathbf{n}\right\rangle \mathbf{n}.$$

Note that the particular choice of A and $\mathbf{n}$ do not matter, and refl_H preserves distances between points and satisfies $\mathrm{refl}_H \circ \mathrm{refl}_H = \mathrm{id}$. We will be frequently mentioning the reflection of a point X across F_i, which will be denoted by X_i' throughout.

Finally, we define the notions of distance, direction and angle pertaining to hyperplanes:

Definition 7. (Absolute and signed distances from point to hyperplane, unit normal vector directed towards point, inward normal vector of extended facet) Let H be a hyperplane with unit normal vector $\mathbf{n}$, and A be a point on H. Then, the *distance from a point* P *to* H is defined as

$$d(P,H) = \|\mathbf{p} - \mathrm{proj}_H(P)\| = \left\|\mathbf{p} - \left(\mathbf{p} - \left\langle \overrightarrow{AP}, \mathbf{n}\right\rangle \mathbf{n}\right)\right\| = \left\|\left\langle \overrightarrow{AP}, \mathbf{n}\right\rangle \mathbf{n}\right\| = \left|\left\langle \overrightarrow{AP}, \mathbf{n}\right\rangle\right|.$$

Note that the particular choice of A and $\mathbf{n}$ do not matter. Also note that this is the shortest distance from P to any point on H (see [21], Theorems 3.32 and 3.34).

It will also be useful to define *signed distance* by attaching a sign to $d(P,H)$ as

$$d_\pm(P,H) = \left\langle \overrightarrow{AP}, \mathbf{n}\right\rangle = \begin{cases} d(P,H) & \text{if } \left\langle \overrightarrow{AP}, \mathbf{n}\right\rangle \geq 0 \\ -d(P,H) & \text{if } \left\langle \overrightarrow{AP}, \mathbf{n}\right\rangle < 0. \end{cases}$$

Note that the particular choice of $\mathbf{n}$ matters here. Note that $d_\pm(P,H) > 0$ indicates that $\mathbf{n}$ is *directed towards* (the side of H that contains) P, while $d_\pm(P,H) < 0$ indicates that $\mathbf{n}$ is *directed towards* the opposite side, and $d_\pm(P,H) = 0$ indicates that $P \in H$.

Moreover, in the context of the simplex Δ, we define the *inward unit normal vector* of an extended facet F_i to be the one which is directed towards the opposite vertex V_i, which will be denoted by $\mathbf{n}_i$ throughout. It will be the default unit normal vector whenever $d_\pm(P,F_i)$ is considered.

As far as an angle between two hyperplanes is concerned, their (unit) normal vectors have to be specified in order to give a sense of which region in the space is "containing" the angle:

Definition 8. (Directed hyperplane, angle between directed hyperplanes) Let H be a hyperplane with unit normal vector $\mathbf{n}$, and P be a point not on H. We say that H is *directed towards* P if $\mathbf{n}$ is directed towards P.

Let H_1 and H_2 be two hyperplanes, and P_1 and P_2 be two points not on H_1 and H_2 respectively. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the unit normal vectors of H_1 and H_2 respectively such that H_1 and H_2 are directed towards P_1 and P_2 respectively. Then, we say that

$$\angle(H_1, H_2) = \pi - \arccos\langle \mathbf{n}_1, \mathbf{n}_2\rangle$$

is the *angle between H_1 and H_2 in which H_1 and H_2 are directed towards P_1 and P_2 respectively.*

Throughout, whenever an angle between an extended facet F_i and a hyperplane H is considered, F_i will always be directed towards V_i (i.e. the inward unit normal vector $\mathbf{n}_i$ will be used in $\angle(F_i, H)$).

2. Isogonal Conjugation in Simplices

2.1. **Old and New Characterisations of Isogonal Conjugates.** We begin with the definition of isogonal conjugate pairs for interior points in [22]:

Definition 9. (Isogonal conjugate pair for interior points) Let X and Y be interior points of Δ. Then, X and Y form an *isogonal conjugate pair* if for all $i \neq j$, the hyperplanes $A_X = \operatorname{aff}(F_{i,j} \cup \{X\})$ and $A_Y = \operatorname{aff}(F_{i,j} \cup \{Y\})$ are *equally inclined from F_i and F_j* respectively, i.e.

$$\angle(A_X, F_i) = \angle(A_Y, F_j) \tag{1}$$

where A_X is directed towards V_j in $\angle(A_X, F_i)$ and A_Y is directed towards V_i in $\angle(A_Y, F_j)$. (see Figure 8)

Definition 9 is natural as it matches the meaning of the word "isogonal" (equal angles). However, it is rather clunky and hard to work with. The first step out of the box is the following that will eventually rephrase the meaning of equally inclined hyperplanes in terms of reflection in Lemma 13:

Definition 10. (Angle bisector of extended facets) The *angle bisector of F_i and F_j* $(i \neq j)$ is defined as the unique hyperplane $B_{i,j}$ which passes through $F_{i,j}$ and with $\mathbf{n}_i - \mathbf{n}_j$ as a normal vector. Also let $C_{i,j}$ denote the $(n-1)$-dimensional vector subspace parallel to $B_{i,j}$ (i.e. the orthogonal complement of $\mathbf{n}_i - \mathbf{n}_j$). These two notations will be adopted throughout.

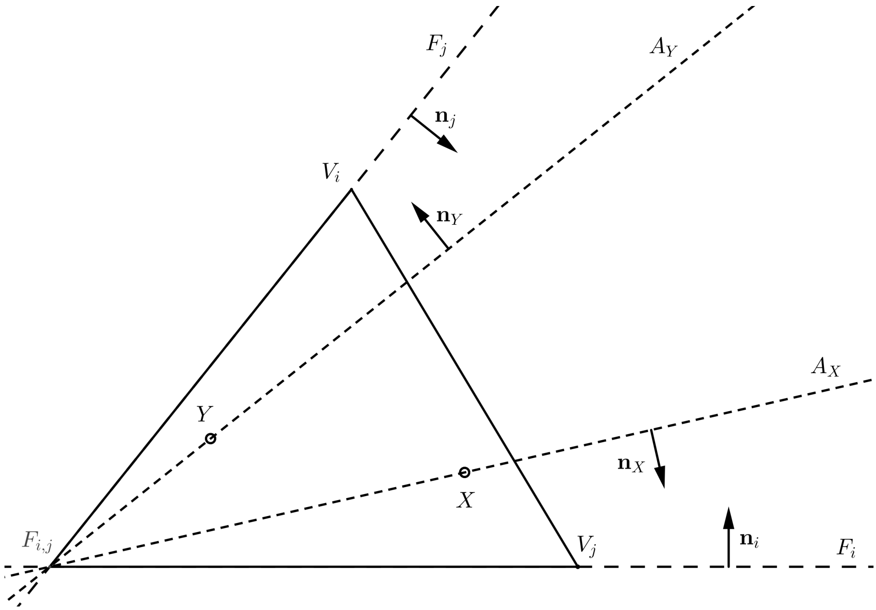

FIGURE 8. $\mathbf{n}_X$ and $\mathbf{n}_Y$ are the unit normal vectors of A_X and A_Y directed towards V_j and V_i respectively.

The word "angle bisector" is justified by that

$$\begin{aligned}\angle(B_{i,j}, F_i) &= \pi - \arccos\left\langle \frac{\mathbf{n}_j - \mathbf{n}_i}{\|\mathbf{n}_j - \mathbf{n}_i\|}, \mathbf{n}_i \right\rangle = \pi - \arccos \frac{\langle \mathbf{n}_i, \mathbf{n}_j\rangle - 1}{\|\mathbf{n}_j - \mathbf{n}_i\|} \\ &= \pi - \arccos\left\langle \frac{\mathbf{n}_i - \mathbf{n}_j}{\|\mathbf{n}_j - \mathbf{n}_i\|}, \mathbf{n}_j \right\rangle = \angle(B_{i,j}, F_j)\end{aligned}$$

where $B_{i,j}$ is directed towards V_j in $\angle(B_{i,j}, F_i)$ and is directed towards V_i in $\angle(B_{i,j}, F_j)$.

The following two lemmas will be useful in proving Lemma 13:

Lemma 11. (Reflectional symmetry due to angle bisector) $\mathbf{n}_i$ *and* $\mathbf{n}_j$ *are reflections of each other across* $C_{i,j}$.

Proof. Using the definition of reflection, we have

$$\begin{aligned}\mathrm{refl}_{C_{i,j}}(\mathbf{n}_i) &= \mathbf{n}_i - 2\left\langle \mathbf{n}_i, \frac{\mathbf{n}_i - \mathbf{n}_j}{\|\mathbf{n}_i - \mathbf{n}_j\|}\right\rangle \frac{\mathbf{n}_i - \mathbf{n}_j}{\|\mathbf{n}_i - \mathbf{n}_j\|} = \mathbf{n}_i - \frac{2\langle \mathbf{n}_i, \mathbf{n}_i - \mathbf{n}_j\rangle (\mathbf{n}_i - \mathbf{n}_j)}{\|\mathbf{n}_i - \mathbf{n}_j\|^2} \\ &= \mathbf{n}_i - \frac{2(1 - \langle \mathbf{n}_i, \mathbf{n}_j\rangle)(\mathbf{n}_i - \mathbf{n}_j)}{2 - 2\langle \mathbf{n}_i, \mathbf{n}_j\rangle} = \mathbf{n}_i - (\mathbf{n}_i - \mathbf{n}_j) = \mathbf{n}_j.\end{aligned}$$

□

Lemma 12. (Normal vectors of hyperplanes containing $F_{i,j}$) *Let H be a hyperplane containing $F_{i,j}$, and suppose $\mathbf{n}$ is a non-zero normal vector of H. Then, $\mathbf{n}$ has a unique representation in the form*

$$\mathbf{n} = \alpha \mathbf{n}_i + \beta \mathbf{n}_j \tag{2}$$

where $\alpha, \beta \in \mathbb{R}$ are not both zero. Furthermore, if H contains an interior point of Δ, then:

- *if $\mathbf{n}$ is the unit normal vector of H directed towards V_i, then $\alpha > 0 > \beta$.*
- *if $\mathbf{n}$ is the unit normal vector of H directed towards V_j, then $\alpha < 0 < \beta$.*

Proof. Take some arbitrary $S \in F_{i,j}$ and consider the vector subspaces $F_i - \mathbf{s}$, $F_j - \mathbf{s}$, $H - \mathbf{s}$ and $F_{i,j} - \mathbf{s}$. Let $W = (F_{i,j} - \mathbf{s})^\perp$. Since $F_{i,j} - \mathbf{s}$ is a subset of each of $F_i - \mathbf{s}$, $F_j - \mathbf{s}$, $H - \mathbf{s}$, by a well-known property of orthogonal complement, each of $(F_i - \mathbf{s})^\perp$, $(F_j - \mathbf{s})^\perp$, $(H - \mathbf{s})^\perp$ is a subset of W. In particular, $\mathbf{n}_i, \mathbf{n}_j, \mathbf{n} \in W$ (see Figure 9). Furthermore, as W is 2-dimensional and $\mathbf{n}_i, \mathbf{n}_j$ are non-parallel, $\{\mathbf{n}_i, \mathbf{n}_j\}$

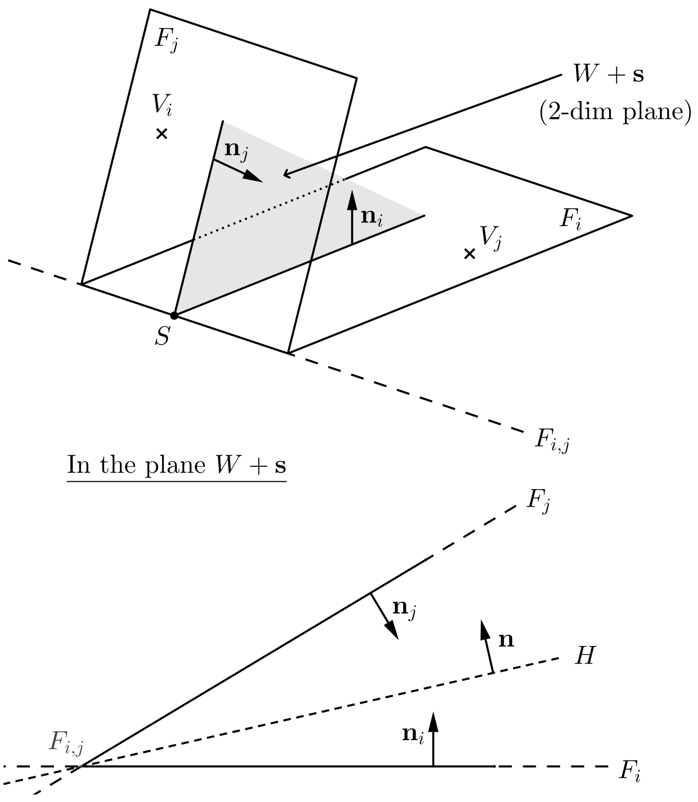

FIGURE 9.

actually forms a basis of W, implying the unique existence of real numbers α and

β, not both zero, satisfying (2). In the case where H contains an interior point of Δ, it follows that V_i and V_j lie on different sides of H; by taking the inner product with $\overrightarrow{SV_i}$, we obtain

$$\begin{aligned}\mathbf{n} &= \alpha\mathbf{n}_i + \beta\mathbf{n}_j \\ \left\langle \mathbf{n}, \overrightarrow{SV_i} \right\rangle &= \left\langle \alpha\mathbf{n}_i + \beta\mathbf{n}_j, \overrightarrow{SV_i} \right\rangle \\ \left\langle \mathbf{n}, \overrightarrow{SV_i} \right\rangle &= \alpha \left\langle \mathbf{n}_i, \overrightarrow{SV_i} \right\rangle + \beta \left\langle \mathbf{n}_j, \overrightarrow{SV_i} \right\rangle .\end{aligned}$$

Now $\left\langle \mathbf{n}_i, \overrightarrow{SV_i} \right\rangle = d_\pm(V_i, F_i) > 0$ and $\left\langle \mathbf{n}_j, \overrightarrow{SV_i} \right\rangle = d_\pm(V_i, F_j) = 0$, so α and $\left\langle \mathbf{n}, \overrightarrow{SV_i} \right\rangle$ have the same sign. So if $\mathbf{n}$ is towards V_i, then it follows that $\alpha > 0$ and if $\mathbf{n}$ is towards V_j, then it follows that $\alpha < 0$. Similarly, by taking the inner product with $\overrightarrow{SV_j}$, we obtain

$$\begin{aligned}\mathbf{n} &= \alpha\mathbf{n}_i + \beta\mathbf{n}_j \\ \left\langle \mathbf{n}, \overrightarrow{SV_j} \right\rangle &= \left\langle \alpha\mathbf{n}_i + \beta\mathbf{n}_j, \overrightarrow{SV_j} \right\rangle \\ \left\langle \mathbf{n}, \overrightarrow{SV_j} \right\rangle &= \alpha \left\langle \mathbf{n}_i, \overrightarrow{SV_j} \right\rangle + \beta \left\langle \mathbf{n}_j, \overrightarrow{SV_j} \right\rangle .\end{aligned}$$

Now $\left\langle \mathbf{n}_j, \overrightarrow{SV_j} \right\rangle = d_\pm(V_j, F_j) > 0$ and $\left\langle \mathbf{n}_i, \overrightarrow{SV_j} \right\rangle = d_\pm(V_j, F_i) = 0$, so β and $\left\langle \mathbf{n}, \overrightarrow{SV_j} \right\rangle$ have the same sign. So if $\mathbf{n}$ is towards V_j, then it follows that $\beta > 0$ and if $\mathbf{n}$ is towards V_i, then it follows that $\beta < 0$. □

Now we are ready to obtain new characterisations for a hyperplane being equally inclined from $F_{i,j}$:

Lemma 13. (New characterisations for being equally inclined from extended facets) *Let X and Y be points. Then, for any $i \neq j$, the following statements are equivalent:*

(a) *The reflection of X across $B_{i,j}$ lies on a hyperplane containing Y and $F_{i,j}$.* (see Figure 10)

(b) $d_\pm(X, F_i)d_\pm(Y, F_i) = d_\pm(X, F_j)d_\pm(Y, F_j)$. (see Figure 11)

(c) $\left\| \overrightarrow{YX'_i} \right\| = \left\| \overrightarrow{YX'_j} \right\|$. (see Figure 12)

(d) $x_i y_i d_\pm(V_i, F_i)^2 = x_j y_j d_\pm(V_j, F_j)^2$, *where $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ are the barycentric coordinates of X and Y with respect to Δ respectively.*

In particular, if X and Y are both interior points of Δ, then the above statements are also equivalent to the following:

(e) *A_X and A_Y are equally inclined from F_i and F_j respectively, i.e. X and Y satisfy (1) in Definition 9.*

Proof. For convenience, take some $S \in F_{i,j}$.

First we show that (a) and (b) are equivalent. Let X' be the reflection of X across $B_{i,j}$. Recall that by Lemma 12, a non-zero unit normal vector $\mathbf{n}$ of any hyperplane

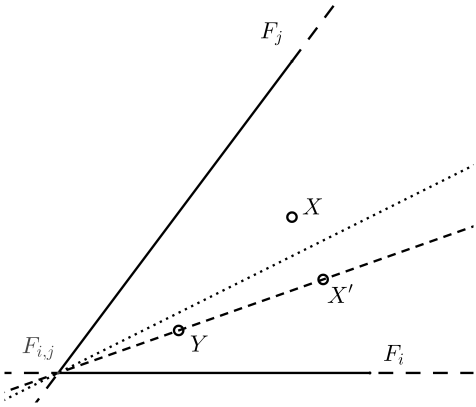

FIGURE 10.

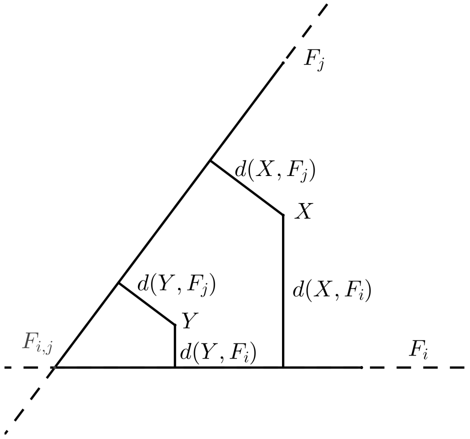

FIGURE 11.

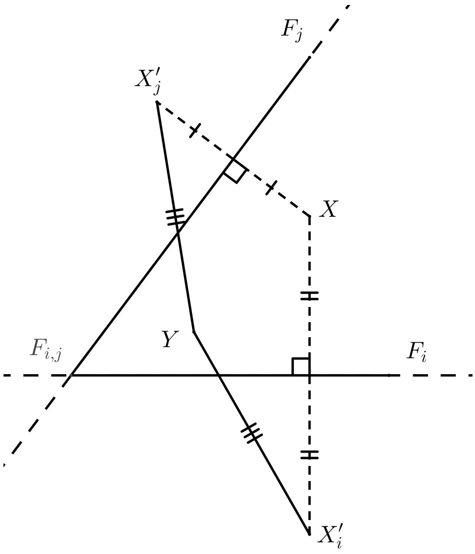

FIGURE 12.

H passing through $F_{i,j}$ can be represented in the form $\mathbf{n} = \alpha\mathbf{n}_i + \beta\mathbf{n}_j$, where α and β are not both zero. So actually (a) holds if and only if there exists some α and β not both zero such that $\left\langle \alpha\mathbf{n}_i + \beta\mathbf{n}_j, \overrightarrow{SX'} \right\rangle = 0$ and $\left\langle \alpha\mathbf{n}_i + \beta\mathbf{n}_j, \overrightarrow{SY} \right\rangle = 0$.

Now observe that

$$\begin{cases} \left\langle \alpha\mathbf{n}_i + \beta\mathbf{n}_j, \overrightarrow{SX'} \right\rangle = 0 \\ \left\langle \alpha\mathbf{n}_i + \beta\mathbf{n}_j, \overrightarrow{SY} \right\rangle = 0 \end{cases}$$

$$\begin{cases} \alpha \left\langle \mathbf{n}_i, \overrightarrow{SX'} \right\rangle + \beta \left\langle \mathbf{n}_j, \overrightarrow{SX'} \right\rangle = 0 \\ \alpha \left\langle \mathbf{n}_i, \overrightarrow{SY} \right\rangle + \beta \left\langle \mathbf{n}_j, \overrightarrow{SY} \right\rangle = 0 \end{cases}$$

$$\begin{cases} \alpha d_\pm(X', F_i) + \beta d_\pm(X', F_j) = 0 \\ \alpha d_\pm(Y, F_i) + \beta d_\pm(Y, F_j) = 0 \end{cases}$$

$$\begin{bmatrix} d_\pm(X', F_i) & d_\pm(X', F_j) \\ d_\pm(Y, F_i) & d_\pm(Y, F_j) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence the existence of α and β not both zero is equivalent to the above matrix having zero determinant; that is,

$$d_\pm(X', F_i)d_\pm(Y, F_j) = d_\pm(X', F_j)d_\pm(Y, F_i). \tag{3}$$

Moreover, by reflection symmetry one obtains $d_\pm(X', F_i) = \left\langle \mathbf{n}_i, \overrightarrow{SX'} \right\rangle = \left\langle \mathbf{n}_j, \overrightarrow{SX} \right\rangle = d_\pm(X, F_j)$ and $d_\pm(X', F_j) = \left\langle \mathbf{n}_j, \overrightarrow{SX'} \right\rangle = \left\langle \mathbf{n}_i, \overrightarrow{SX} \right\rangle = d_\pm(X, F_i)$, so (3) is equivalent to (b).

Then we show that (b) and (c) are equivalent. By using the fact that $\mathbf{x}'_i = \mathbf{x} - 2d_\pm(X, F_i)\mathbf{n}_i$, we perform the following manipulations:

$$\left\| \overrightarrow{YX_i} \right\|^2 = \left\| \overrightarrow{YX_j} \right\|^2$$

$$\left\| \mathbf{x}'_i - \mathbf{y} \right\|^2 = \left\| \mathbf{x}'_j - \mathbf{y} \right\|^2$$

$$\left\| \mathbf{x} - \mathbf{y} - 2d_\pm(X, F_i)\mathbf{n}_i \right\|^2 = \left\| \mathbf{x} - \mathbf{y} - 2d_\pm(X, F_j)\mathbf{n}_j \right\|^2$$

$$\begin{gathered} \left\| \mathbf{x} - \mathbf{y} \right\|^2 - 4d_\pm(X, F_i) \left\langle \mathbf{x} - \mathbf{y}, \mathbf{n}_i \right\rangle + 4d_\pm(X, F_i)^2 \\ = \left\| \mathbf{x} - \mathbf{y} \right\|^2 - 4d_\pm(X, F_j) \left\langle \mathbf{x} - \mathbf{y}, \mathbf{n}_j \right\rangle + 4d_\pm(X, F_j)^2 \end{gathered}$$

$$\begin{gathered} d_\pm(X, F_i)(\langle \mathbf{y} - \mathbf{x}, \mathbf{n}_i \rangle + d_\pm(X, F_i)) = d_\pm(X, F_j)(\langle \mathbf{y} - \mathbf{x}, \mathbf{n}_j \rangle + d_\pm(X, F_j)) \\ d_\pm(X, F_i)(\langle \mathbf{y} - \mathbf{x}, \mathbf{n}_i \rangle + \langle \mathbf{x} - \mathbf{s}, \mathbf{n}_i \rangle) = d_\pm(X, F_j)(\langle \mathbf{y} - \mathbf{x}, \mathbf{n}_j \rangle + \langle \mathbf{x} - \mathbf{s}, \mathbf{n}_j \rangle) \\ d_\pm(X, F_i)(\langle \mathbf{y} - \mathbf{s}, \mathbf{n}_i \rangle) = d_\pm(X, F_j)(\langle \mathbf{y} - \mathbf{s}, \mathbf{n}_j \rangle) \\ d_\pm(X, F_i)d_\pm(Y, F_i) = d_\pm(X, F_j)d_\pm(Y, F_j) \end{gathered}$$

hence (b) and (c) are equivalent.

Now we show that (b) and (d) are equivalent. We make use of the fact that the barycentric coordinates of any point $X = (x_0, \ldots, x_n)$ with respect to Δ satisfies the following:

$$x_i = \frac{d_\pm(X, F_i)}{d_\pm(V_i, F_i)}. \tag{4}$$

This is because

$$\begin{aligned}\frac{d_\pm(X,F_i)}{d_\pm(V_i,F_i)} &= \frac{\langle \mathbf{x}-\mathbf{s},\mathbf{n}_i\rangle}{\langle \mathbf{v}_i-\mathbf{s},\mathbf{n}_i\rangle} = \frac{\left\langle \sum_{j=0}^n x_j(\mathbf{v}_j-\mathbf{s}),\mathbf{n}_i\right\rangle}{\langle \mathbf{v}_i-\mathbf{s},\mathbf{n}_i\rangle}\\ &= \frac{\sum_{j=0}^n x_j\langle \mathbf{v}_j-\mathbf{s},\mathbf{n}_i\rangle}{\langle \mathbf{v}_i-\mathbf{s},\mathbf{n}_i\rangle} = \frac{x_i\langle \mathbf{v}_i-\mathbf{s},\mathbf{n}_i\rangle}{\langle \mathbf{v}_i-\mathbf{s},\mathbf{n}_i\rangle} = x_i.\end{aligned}$$

Now using (4), we obtain

$$\begin{aligned}x_iy_id_\pm(V_i,F_i)^2 &= x_jy_jd_\pm(V_j,F_j)^2\\ \frac{d_\pm(X,F_i)}{d_\pm(V_i,F_i)}\frac{d_\pm(Y,F_i)}{d_\pm(V_i,F_i)}d_\pm(V_i,F_i)^2 &= \frac{d_\pm(X,F_j)}{d_\pm(V_i,F_j)}\frac{d_\pm(Y,F_i)}{d_\pm(V_i,F_j)}d_\pm(V_j,F_j)^2\\ d_\pm(X,F_i)d_\pm(Y,F_i) &= d_\pm(X,F_j)d_\pm(Y,F_j)\end{aligned}$$

as desired.

Finally we show that (a) and (e) are equivalent in the case where X and Y are interior points. Let $\mathbf{n}_X$ and $\mathbf{n}_Y$ be the unit normal vectors of A_X and A_Y such that A_X and A_Y are directed towards V_j and V_i respectively. By Lemma 12, we may write $\mathbf{n}_X = \alpha\mathbf{n}_i + \beta\mathbf{n}_j$ and $\mathbf{n}_Y = \gamma\mathbf{n}_i + \delta\mathbf{n}_j$, where $\alpha < 0 < \beta$ and $\gamma > 0 > \delta$. The main idea is to prove that both conditions are equivalent to $(\beta,\alpha) = (\gamma,\delta)$. Rewrite (e) as

$$\begin{aligned}\langle \mathbf{n}_X,\mathbf{n}_i\rangle &= \langle \mathbf{n}_Y,\mathbf{n}_j\rangle\\ \langle \alpha\mathbf{n}_i+\beta\mathbf{n}_j,\mathbf{n}_i\rangle &= \langle \gamma\mathbf{n}_i+\delta\mathbf{n}_j,\mathbf{n}_j\rangle\\ \alpha+\beta\langle \mathbf{n}_i,\mathbf{n}_j\rangle &= \gamma\langle \mathbf{n}_i,\mathbf{n}_j\rangle+\delta\end{aligned}$$

Also, note that $\|\mathbf{n}_X\| = \|\mathbf{n}_Y\| = 1$ implies $\alpha^2+\beta^2+2\alpha\beta\langle \mathbf{n}_i,\mathbf{n}_j\rangle = \gamma^2+\delta^2+2\gamma\delta\langle \mathbf{n}_i,\mathbf{n}_j\rangle = 1$. Hence $(p,q) = (\beta,\alpha)$ and $(p,q) = (\gamma,\delta)$ are both solutions to the following system:

$$\begin{cases} p\langle \mathbf{n}_i,\mathbf{n}_j\rangle + q = c, \text{ where } c = \langle \mathbf{n}_X,\mathbf{n}_i\rangle = \langle \mathbf{n}_Y,\mathbf{n}_j\rangle\\ 1 = p^2+q^2+2pq\langle \mathbf{n}_i,\mathbf{n}_j\rangle\\ p > 0 > q\end{cases}$$

However, combining the first two equations and rearranging gives $p^2 = \frac{1-c^2}{1-\langle \mathbf{n}_i,\mathbf{n}_j\rangle^2}$, so the unique positive value of p is $p = \sqrt{\frac{1-c^2}{1-\langle \mathbf{n}_i,\mathbf{n}_j\rangle^2}}$. It follows that the value of q is unique as well. Hence the first condition is equivalent to $(\beta,\alpha) = (\gamma,\delta)$. For (a), denote the $(n-1)$-dimensional vector subspace parallel to $B_{i,j}$ as $C_{i,j}$ and let $\mathbf{n}'_X$ be the reflection of $\mathbf{n}_X$ across $C_{i,j}$. Since $\mathbf{n}_i$ and $\mathbf{n}_j$ are reflections across $C_{i,j}$ (by Lemma 11), we have

$$\begin{aligned}\mathbf{n}'_X &= \mathrm{refl}_{C_{i,j}}(\mathbf{n}_X) = \mathrm{refl}_{C_{i,j}}(\alpha\mathbf{n}_i+\beta\mathbf{n}_j)\\ &= \alpha\mathrm{refl}_{C_{i,j}}(\mathbf{n}_i)+\beta\mathrm{refl}_{C_{i,j}}(\mathbf{n}_j) = \alpha\mathbf{n}_j+\beta\mathbf{n}_i = \beta\mathbf{n}_i+\alpha\mathbf{n}_j.\end{aligned}$$

On the other hand, (a) holds if and only if the reflection of $A_X-\mathbf{s}$ across $C_{i,j}$ is equal to $A_Y-\mathbf{s}$. Recall that reflection preserves inner products, so it also preserves orthogonality: hence (a) is actually equivalent to $(\mathrm{refl}_{C_{i,j}}(A_X-\mathbf{s}))^\perp = \mathrm{sp}(\{\mathbf{n}'_X\})$ and $(A_Y-\mathbf{s})^\perp = \mathrm{sp}(\{\mathbf{n}_Y\})$ being equal i.e. $\mathbf{n}'_X \parallel \mathbf{n}_Y$. Of course $\mathbf{n}'_X$ and $\mathbf{n}_Y$ are both unit vectors pointing towards V_i, so $\mathbf{n}'_X \parallel \mathbf{n}_Y$ if and only if $\mathbf{n}'_X = \mathbf{n}_Y$ i.e. $\beta\mathbf{n}_i + \alpha\mathbf{n}_j = \gamma\mathbf{n}_i + \delta\mathbf{n}_j$, which is equivalent to $(\beta, \alpha) = (\gamma, \delta)$. □

Letting Lemma 13 run through all $i \neq j$, we immediately obtain the first main theorem of this paper:

Theorem 14. (New characterisations for isogonal conjugate pairs) *Let X and Y be points. Then, the following statements are equivalent:*

(a) For all $i \neq j$, the reflection of X across $B_{i,j}$ lies on a hyperplane containing Y and $F_{i,j}$.
(b) $d_\pm(X, F_0)d_\pm(Y, F_0) = \cdots = d_\pm(X, F_n)d_\pm(Y, F_n)$.
(c) $\left\|\overrightarrow{YX'_0}\right\| = \cdots = \left\|\overrightarrow{YX'_n}\right\|$.
(d) $x_0y_0d_\pm(V_0, F_0)^2 = \cdots = x_ny_nd_\pm(V_n, F_n)^2$, where $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ are the barycentric coordinates of X and Y with respect to Δ respectively.

In particular, if X and Y are both interior points of Δ, then the above statements are also equivalent to the following:

(e) X and Y form an isogonal conjugate pair as in Definition 9.

Theorem 14 enables us to extend the scope of isogonal conjugation to non-interior points:

Definition 15. (Isogonal conjugate pair, isogonal conjugate of point) If two points X and Y satisfy any of the statements (a)—(d) in Theorem 14, we say that X and Y form an *isogonal conjugate pair*.

If Y is the unique point which forms an isogonal conjugate pair with X, and X is also the unique point which forms an isogonal conjugate pair with Y, then we say that X and Y are the *isogonal conjugate* of each other. In such case, we write $Y = X^*$.

The word "pair" is justified by that statements (b) and (d) in Theorem 14 are symmetric in X and Y. We have the following classification of points based on how they form isogonal conjugate pairs:

Theorem 16. (Number of points forming isogonal conjugate pairs) *Let X and Y be two points with barycentric coordinates $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ with respect to Δ respectively. Then:*

(a) If some of $x_0, \ldots, x_n$ are zero (i.e. X lies on some F_i), then X and Y form an isogonal conjugate pair if and only if for all $i = 0, \ldots, n$ at least one of x_i or y_i is zero. In such case, if X lies on exactly k of the extended facets, then Y lies on the remaining $(n + 1 - k)$ extended facets. The isogonal conjugate of X does not exist.

(b) If all of $x_0, \dots, x_n$ are non-zero and

$$\sum_{k=0}^{n} \frac{1}{d_\pm(V_k, F_k)^2 x_k} = 0, \tag{5}$$

then no points form an isogonal conjugate pair with X. The isogonal conjugate of X does not exist.

(c) Otherwise, the isogonal conjugate X^* of X exists, whose barycentric coordinates with respect to Δ are

$$\left(\frac{1}{d_\pm(V_0, F_0)^2 x_0} : \dots : \frac{1}{d_\pm(V_n, F_n)^2 x_n} \right).$$

This includes the case when X is an interior point of Δ (because the sum in the left-hand side of (5) is positive) (cf. [22]).

Proof. By Theorem 14, X and Y form an isogonal conjugate pair if and only if

$$x_0 y_0 d_\pm(V_0, F_0)^2 = \dots = x_n y_n d_\pm(V_n, F_n)^2. \tag{6}$$

Note that $d_\pm(V_0, F_0)^2, \dots, d_\pm(V_n, F_n)^2$ are all positive. Suppose Y forms an isogonal conjugate pair with X.

For case (a), the expressions in (6) must all be zero, hence

$$x_0 y_0 = \dots = x_n y_n = 0$$

and so for all $i = 0, \dots, n$, at least one of x_i or y_i is zero. (Note that for all i, all points on F_i form an isogonal conjugate pair with V_i.) Now suppose $X \in F_i$. If $Y \neq V_i$, then Y and V_i both form an isogonal conjugate pair with X. If $Y = V_i$, then for any point $Z \in F_i \setminus \{X\}$, we have that X and Z both form an isogonal conjugate pair with Y. In either case, X and Y are not the isogonal conjugate of each other as they are not the unique points which form an isogonal conjugate pair with each other.

For cases (b) and (c) we may rewrite (6) as

$$y_0 : \dots : y_n = \frac{1}{d_\pm(V_0, F_0)^2 x_0} : \dots : \frac{1}{d_\pm(V_n, F_n)^2 x_n}$$

so Y exists (and is unique) if and only if (5) does not hold, implying the desired claim. $\square$

2.2. $(2n+2)$**-point Hypersphere.** We now conclude this section with the following interesting result, which is the second main theorem of this paper:

Theorem 17. **(**$(2n+2)$**-point hypersphere)** *Suppose points X and Y form an isogonal conjugate pair. Then $X_0, \dots, X_n, Y_0, \dots, Y_n$ lie on a common hypersphere whose centre is the midpoint of X and Y (see Figure 13).*

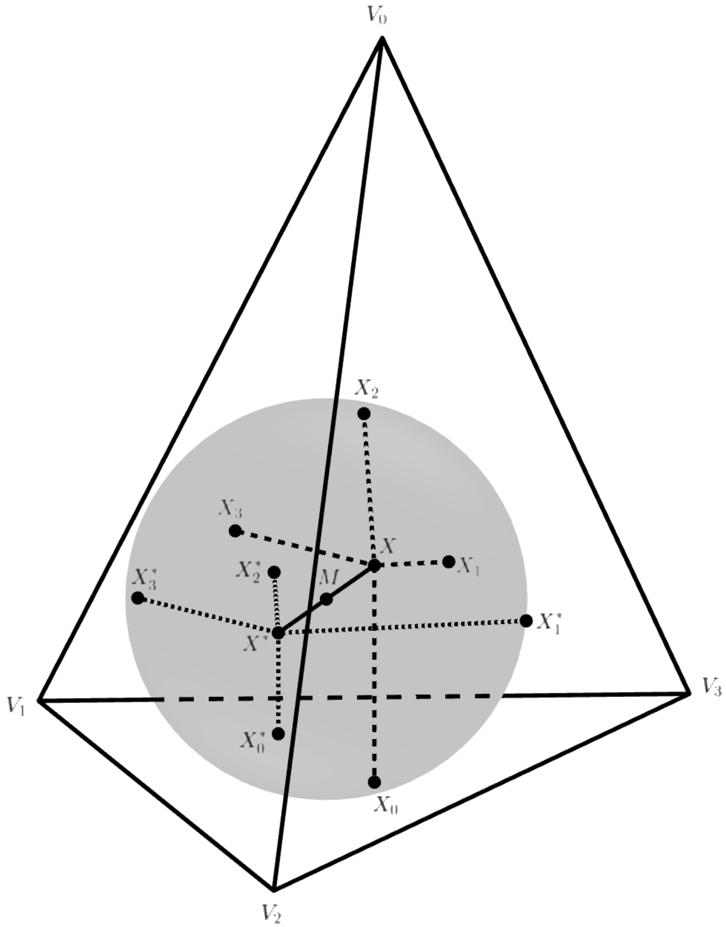

FIGURE 13.

Proof. Let M be the midpoint of X and Y. Since X_i is the midpoint of X and $X_i{}'$,

$$\left\|\overrightarrow{MX_i}\right\| = \frac{1}{2}\left\|\overrightarrow{YX_i'}\right\|$$

the right-hand side is equal for all i by Theorem 14 (c). Furthermore,

$$\left\|\overrightarrow{MX_i}\right\| = \frac{1}{2}\left\|\overrightarrow{YX_i'}\right\| = \frac{1}{2}\left\|\overrightarrow{Y_i{}'X}\right\| = \left\|\overrightarrow{Y_iM}\right\|$$

where the second equality follows by reflectional symmetry. From this we conclude that $\left\|\overrightarrow{MX_0}\right\| = \cdots = \left\|\overrightarrow{MX_n}\right\| = \left\|\overrightarrow{MY_0}\right\| = \cdots = \left\|\overrightarrow{MY_n}\right\|$ and the result follows. $\square$

3. Isogonal Conjugate of Circumcentre

Definition 18. (Circumcentre) The circumcentre of Δ is the unique point (which will be denoted by O throughout) equidistant from the vertices of Δ, i.e. O satisfies

$$\left\|\overrightarrow{OV_0}\right\| = \cdots = \left\|\overrightarrow{OV_n}\right\|.$$

Note that O is not necessarily an interior point of Δ; it can lie outside Δ, and even lie on some of the extended facets F_i. Here we only discuss the case where O has an isogonal conjugate (i.e. O lies in case (c) of Theorem 16). The isogonal conjugate of O will be denoted by O^* throughout. First we will give yet another characterisation for isogonal conjugate pairs beside the ones given in Theorem 14, which will help us establish the main results related to O^*:

Lemma 19. (One more characterisation for isogonal conjugate pair) *Points X and Y form an isogonal conjugate pair if and only if*

$$\left\langle \overrightarrow{V_kY}, \overrightarrow{X_iX_j} \right\rangle = 0 \quad \textit{for all} \quad i, j \neq k. \tag{7}$$

Proof. Note that $\overrightarrow{X_i{}'X_j{}'} = 2\overrightarrow{X_iX_j}$ because X_i and X_j are the midpoints of $XX_i{}'$ and $XX_j{}'$ respectively. Furthermore, for any $i, j \neq k$, by reflectional symmetry we have that $\left\|\overrightarrow{V_kX_i{}'}\right\| = \left\|\overrightarrow{V_kX}\right\| = \left\|\overrightarrow{V_kX_j{}'}\right\|$. Thus

$$\begin{gathered}
\left\langle \overrightarrow{V_kY}, \overrightarrow{X_iX_j} \right\rangle = 0 \\
\left\langle \overrightarrow{V_kY}, \overrightarrow{X_i{}'X_j{}'} \right\rangle = 0 \\
\langle \mathbf{y} - \mathbf{v}_k, \mathbf{x}_i{}' \rangle = \langle \mathbf{y} - \mathbf{v}_k, \mathbf{x}_j{}' \rangle \\
\langle \mathbf{y} - \mathbf{v}_k, \mathbf{x}_i{}' \rangle - \frac{1}{2}\left\|\overrightarrow{V_kX_i{}'}\right\|^2 = \langle \mathbf{y} - \mathbf{v}_k, \mathbf{x}_j{}' \rangle - \frac{1}{2}\left\|\overrightarrow{V_kX_j{}'}\right\|^2 \\
\langle \mathbf{y}, \mathbf{x}_i{}' \rangle - \langle \mathbf{v}_k, \mathbf{x}_i{}' \rangle - \frac{1}{2}\left(\|\mathbf{v}_k\|^2 - 2\langle \mathbf{v}_k, \mathbf{x}_i{}' \rangle + \|\mathbf{x}_i{}'\|^2\right) \\
= \langle \mathbf{y}, \mathbf{x}_j{}' \rangle - \langle \mathbf{v}_k, \mathbf{x}_j{}' \rangle - \frac{1}{2}\left(\|\mathbf{v}_k\|^2 - 2\langle \mathbf{v}_k, \mathbf{x}_j{}' \rangle + \|\mathbf{x}_j{}'\|^2\right) \\
-\frac{1}{2}\|\mathbf{y}\|^2 + \langle \mathbf{y}, \mathbf{x}_i{}' \rangle - \frac{1}{2}\|\mathbf{x}_i{}'\|^2 = -\frac{1}{2}\|\mathbf{y}\|^2 + \langle \mathbf{y}, \mathbf{x}_j{}' \rangle - \frac{1}{2}\|\mathbf{x}_j{}'\|^2 \\
-\frac{1}{2}\|\mathbf{y} - \mathbf{x}_i{}'\|^2 = -\frac{1}{2}\|\mathbf{y} - \mathbf{x}_j{}'\|^2 \\
\left\|\overrightarrow{YX_i'}\right\| = \left\|\overrightarrow{YX_j'}\right\|
\end{gathered}$$

So (7) is actually equivalent to $\left\|\overrightarrow{YX_i'}\right\| = \left\|\overrightarrow{YX_j'}\right\|$ for all i, j, which is equivalent to X and Y forming an isogonal conjugate pair by Theorem 14 (c). $\square$

3.1. **Inexcentre of Pedal Simplex of O^*.**

Definition 20. (Pedal simplex) For any point X we define the pedal simplex (denoted by Δ_X throughout) of X to be the simplex whose vertices are the projections of X onto the extended facets of Δ, i.e.

$$\Delta_X = \operatorname{conv}(\{X_0, \dots, X_n\}).$$

The extended facet of Δ_X opposite to X_i will be denoted by $F_{X_i} = \operatorname{aff}(\{X_0, \dots, X_n\} \setminus \{X_i\})$, and its inward unit normal vector (directed towards X_i) will be denoted by $\mathbf{n}_{X_i}$. It will be the default unit normal vector whenever $d_\pm(P, F_{X_i})$ is considered.

Lemma 21. (Existence of pedal simplex) *If X has an isogonal conjugate, then X has a pedal simplex, i.e. the points $X_0, \dots, X_n$ are affinely independent.*

Proof. Let X^* be the isogonal conjugate of X. Since X_i is the midpoint of XX_i', it is equivalent to show that $X_0', \dots, X_n'$ are affinely independent. Suppose they are not, i.e. they lie on a common hyperplane H. Then suppose there exists a point S equidistant to $X_0', \dots, X_n'$; then $S + \mathbf{n}$ is also equidistant to $X_0', \dots, X_n'$, where $\mathbf{n}$ is any normal vector of H. Thus $X_0', \dots, X_n'$ does not have a unique circumcentre. However, by Theorem 14, X^* is the unique point which is equidistant to $X_0', \dots, X_n'$, a contradiction. $\square$

Definition 22. (Inexcentre of pedal simplex) A point I is an inexcentre of Δ_X if its absolute distances to all the extended facets of Δ_X are equal, i.e.

$$d(I, F_{X_0}) = \cdots = d(I, F_{X_n}).$$

In particular, I is the incentre of Δ_X if

$$d_\pm(I, F_{X_0}) = \cdots = d_\pm(I, F_{X_n}).$$

After two lemmas, we can prove the third main theorem of the paper:

Lemma 23. *For any two interior points X and Y of Δ, $\left\langle \overrightarrow{V_iY}, \overrightarrow{X_jX_i} \right\rangle > 0$ for all $i \neq j$.*

Proof. Suppose the barycentric coordinates of Y with respect to Δ are $(y_0, \dots, y_n)$. Then

$$\begin{aligned}\left\langle \overrightarrow{V_iY}, \overrightarrow{X_jX_i} \right\rangle &= \left\langle \overrightarrow{V_iY}, \overrightarrow{X_jX} \right\rangle + \left\langle \overrightarrow{V_iY}, \overrightarrow{XX_i} \right\rangle \\ &= d_\pm(X, F_j) \left\langle \overrightarrow{V_iY}, \mathbf{n}_j \right\rangle + d_\pm(X, F_i) \left\langle \overrightarrow{V_iY}, -\mathbf{n}_i \right\rangle \\ &= d_\pm(X, F_j) d_\pm(Y, F_j) + d_\pm(X, F_i) \left\langle \mathbf{v}_i - \mathbf{y}, \mathbf{n}_i \right\rangle \\ &= d_\pm(X, F_j) d_\pm(Y, F_j) + d_\pm(X, F_i) \left\langle \left(\sum_{k=0}^{n} y_k \right) \mathbf{v}_i - \sum_{k=0}^{n} y_k \mathbf{v}_k, \mathbf{n}_i \right\rangle \\ &= d_\pm(X, F_j) d_\pm(Y, F_j) + d_\pm(X, F_i) \sum_{k=0}^{n} y_k \left\langle \mathbf{v}_i - \mathbf{v}_k, \mathbf{n}_i \right\rangle \\ &= d_\pm(X, F_j) d_\pm(Y, F_j) + d_\pm(X, F_i) \sum_{\substack{k=0 \\ k \neq i}}^{n} y_k d_\pm(V_i, F_i) \\ &> 0\end{aligned}$$

since all terms are positive. □

Lemma 24. *Let X be a point with isogonal conjugate X^*. Then for any i, $\overrightarrow{V_iX^*}$ is a scalar multiple of $\mathbf{n}_{X_i}$. In particular, if X is an interior point of Δ, then $\overrightarrow{V_iX^*}$ is a positive scalar multiple of $\mathbf{n}_{X_i}$.*

Proof. Recall by Lemma 19 that for any $j, k \neq i$ we have $\left\langle \overrightarrow{V_iX^*}, \overrightarrow{X_jX_k} \right\rangle = 0$. This implies that $\overrightarrow{V_iX^*}$ is actually orthogonal to the hyperplane containing $\{X_0, \dots, X_n\} \setminus \{X_i\}$, which is F_{X_i}. Hence it is parallel to the unit normal vector $\mathbf{n}_{X_i}$ of F_{X_i}, so it must be a scalar multiple of $\mathbf{n}_{X_i}$. If X is an interior point, by Lemma 23 we have $\left\langle \overrightarrow{V_iX^*}, \overrightarrow{X_jX_i} \right\rangle > 0$ for some $j \neq i$. However, $\left\langle \mathbf{n}_{X_i}, \overrightarrow{X_jX_i} \right\rangle > 0$ because $\mathbf{n}_{X_i}$ is directed towards X_i by definition. Hence $\overrightarrow{V_iX^*}$ and $\mathbf{n}_{X_i}$ must point in the same direction, so $\overrightarrow{V_iX^*}$ is a positive scalar multiple of $\mathbf{n}_{X_i}$. □

Theorem 25. (Isogonal conjugate of circumcentre as inexcentre of its pedal simplex) *The isogonal conjugate O^* of the circumcentre O, if it exists, is an inexcentre of its own pedal simplex. Furthermore, if X is an interior point of Δ, then it is the incentre of its own pedal simplex if and only if $X = O^*$.*

Proof. Note that $\overrightarrow{X_iX} = d_\pm(X, F_i)\mathbf{n}_i$ and $\overrightarrow{X_jX} = d_\pm(X, F_j)\mathbf{n}_j$. Also note that by Lemma 24, we can write $\overrightarrow{V_iX^*} = a_i \left\| \overrightarrow{V_iX^*} \right\| \mathbf{n}_{X_i}$ and $\overrightarrow{V_jX^*} = a_j \left\| \overrightarrow{V_jX^*} \right\| \mathbf{n}_{X_j}$ for some $a_i, a_j = \pm 1$ ($= 1$ when X^* is an interior point). Now $X^* = O$ if and only if

$\left\|\overrightarrow{V_0X^*}\right\| = \cdots = \left\|\overrightarrow{V_nX^*}\right\|$, and for any $i \neq j$,

$$
\begin{aligned}
\left\|\overrightarrow{V_iX^*}\right\| &= \left\|\overrightarrow{V_jX^*}\right\| \\
\frac{d_\pm(X,F_j)d_\pm(X^*,F_j)}{\left\|\overrightarrow{V_iX^*}\right\|} &= \frac{d_\pm(X,F_i)d_\pm(X^*,F_i)}{\left\|\overrightarrow{V_jX^*}\right\|} \\
\frac{d_\pm(X,F_j)\left\langle\overrightarrow{V_iX^*},\mathbf{n}_j\right\rangle}{\left\|\overrightarrow{V_iX^*}\right\|} &= \frac{d_\pm(X,F_i)\left\langle\overrightarrow{V_jX^*},\mathbf{n}_i\right\rangle}{\left\|\overrightarrow{V_jX^*}\right\|} \\
\frac{\left\langle\overrightarrow{V_iX^*},d_\pm(X,F_j)\mathbf{n}_j\right\rangle}{\left\|\overrightarrow{V_iX^*}\right\|} &= \frac{\left\langle\overrightarrow{V_jX^*},d_\pm(X,F_i)\mathbf{n}_i\right\rangle}{\left\|\overrightarrow{V_jX^*}\right\|} \\
\frac{\left\langle a_i\left\|\overrightarrow{V_iX^*}\right\|\mathbf{n}_{X_i},\overrightarrow{X_jX}\right\rangle}{\left\|\overrightarrow{V_iX^*}\right\|} &= \frac{\left\langle a_j\left\|\overrightarrow{V_jX^*}\right\|\mathbf{n}_{X_j},\overrightarrow{X_iX}\right\rangle}{\left\|\overrightarrow{V_jX^*}\right\|} \\
a_i\left\langle\mathbf{n}_{X_i},\overrightarrow{X_jX}\right\rangle &= a_j\left\langle\mathbf{n}_{X_j},\overrightarrow{X_iX}\right\rangle \\
a_id_\pm(X,F_{X_i}) &= a_jd_\pm(X,F_{X_j}) \\
\implies d(X,F_{X_i}) &= d(X,F_{X_j})
\end{aligned}
$$

So if X and O are isogonal conjugates, then $\left\|\overrightarrow{V_0X^*}\right\| = \cdots = \left\|\overrightarrow{V_nX^*}\right\|$ implies $d(X,F_{X_0}) = \cdots = d(X,F_{X_n})$, so X is an inexcentre of its pedal simplex. If we restrict our view and consider interior points X only, then since $a_i = 1$ for all i we actually have $\left\|\overrightarrow{V_0X^*}\right\| = \cdots = \left\|\overrightarrow{V_nX^*}\right\| \iff d_\pm(X,F_{X_0}) = \cdots = d_\pm(X,F_{X_n})$, therefore X is the incentre of its own pedal simplex if and only if $X^* = O$ (i.e. $X = O^*$). $\square$

3.2. **Radical Centre of Facetal Circumhyperspheres.**

Definition 26. (Facetal circumhypersphere) The *facetal circumhypersphere* S_i of the facet opposite to V_i, i.e. $\mathrm{conv}(V \setminus \{V_i\})$, is the hypersphere centred at O_i with radius $R_i = \left\|\overrightarrow{O_iV_j}\right\|$ where $j \neq i$.

We conclude this section with the final main theorem of this paper:

Theorem 27. (Isogonal conjugate of circumentre as radical centre of facetal circumhyperspheres) *Let X be a point. Then,*

$$\mathrm{Pow}(X,S_0) = \cdots = \mathrm{Pow}(X,S_n) \tag{8}$$

if and only if $X = O^*$, where $\mathrm{Pow}(X,S_i) = \left\|\overrightarrow{O_iX}\right\|^2 - {R_i}^2$ is called the power of X with respect to S_i. In other words, O^* is the radical centre of $S_0,\ldots,S_n$.

Proof.

$$
\begin{aligned}
\operatorname{Pow}(X, S_i) &= \operatorname{Pow}(X, S_j) \\
\left\|\overrightarrow{O_iX}\right\|^2 - \left\|\overrightarrow{O_iV_k}\right\|^2 &= \left\|\overrightarrow{O_jX}\right\|^2 - \left\|\overrightarrow{O_jV_k}\right\|^2 \\
\left\langle \overrightarrow{O_iX} - \overrightarrow{O_iV_k}, \overrightarrow{O_iX} + \overrightarrow{O_iV_k} \right\rangle &= \left\langle \overrightarrow{O_jX} - \overrightarrow{O_jV_k}, \overrightarrow{O_jX} + \overrightarrow{O_jV_k} \right\rangle \\
\left\langle \overrightarrow{V_kX}, \overrightarrow{O_iX} + \overrightarrow{O_iV_k} \right\rangle &= \left\langle \overrightarrow{V_kX}, \overrightarrow{O_jX} + \overrightarrow{O_jV_k} \right\rangle \\
\left\langle \overrightarrow{V_kX}, \overrightarrow{O_iX} + \overrightarrow{O_iV_k} - \left(\overrightarrow{O_jX} + \overrightarrow{O_jV_k}\right) \right\rangle &= 0 \\
\left\langle \overrightarrow{V_kX}, 2\overrightarrow{O_iO_j} \right\rangle &= 0 \\
\left\langle \overrightarrow{V_kX}, \overrightarrow{O_iO_j} \right\rangle &= 0
\end{aligned}
$$

By considering all $i, j \neq k$, we find that (8) is actually equivalent to $\left\langle \overrightarrow{V_kX}, \overrightarrow{O_iO_j} \right\rangle = 0$ for all $i, j \neq k$. By Lemma 19, this is true if and only if $X = O^*$. $\square$

4. Closing Remarks

Besides having obtained the four main theorems in this paper, we also have some interesting observations during the course of research.

With some work, one can show that the circumhypersphere of Δ has the barycentric equation

$$\sum_{i<j} x_i x_j \left\|\overrightarrow{V_iV_j}\right\|^2 = 0.$$

It can be re-written into (5) if and only if $n = 2$, whose solution set is precisely the circumcircle of a triangle excluding the vertices. This explains certain degree of well-behavedness of O^* in two dimensions, as it is impossible for the circumcentre of a triangle to lie on the circumcircle. However, we still do not know whether O might lie in the solution set of (5) when $n > 2$ — if the answer is negative, then the assumption made in the first sentence of Theorem 25 could nearly be removed.

Furthermore, in order to study the solution set of the homogeneous equation (5) in $\mathbb{R}^{n+1}$, it can be seen as an object in the projective space $\mathbb{RP}^n$. Advanced knowledge and techniques from the latter may help us understand the solution set of (5).

Our research has been heavily dependent on the existence of an inner product. What if we consider the geometries, of simplex centres in particular, arising from only a norm? We learned that this field is called Minkowski geometry (see [14], [15], [17], [18] and [25]), which has great room for research.

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REVIEWERS' COMMENTS

This paper presents a discussion of generalizing the orthocentre of a triangle to the isogonal conjugate of the circumcentre of a simplex in high dimensions. The paper also covers discussion on two particular properties of the isogonal conjugate, e.g., whether the isogonal conjugate is the incentre or an excentre of its pedal simplex, and whether the isogonal conjugate is the radical centre of the facetal circumhyperspheres of the simplex.

The method used to derive the results was mainly vector and linear algebra, yet it requires some level of understanding of undergraduate level linear algebra, and that impresses some of the referees.