Hang Lung Mathematics Awards
Vol. 6 (2014)

Hang Lung Mathematics Awards 2014

Honorable Mention

A General Formula to Check the Divisibility by All Odd Divisors and its Extensions

Team Members: Chun Kit Du, Tung Him Lam, Hok Leung Chan, Chi Ming Ng, Kai Yin Ng
Teacher: Mr. Kwok Tai Wong
School: S.K.H. Lam Woo Memorial Secondary School

Hang Lung Mathematics Awards
Vol. 6 (2014), pp. 185–225

A GENERAL FORMULA TO CHECK THE DIVISIBILITY BY ALL ODD DIVISORS AND ITS EXTENSIONS

TEAM MEMBERS
Chun Kit Du, Tung Him Lam, Hok Leung Chan,
Chi Ming Ng, Kai Yin Ng

TEACHER
Mr. Kwok Tai Wong

SCHOOL
S.K.H. Lam Woo Memorial Secondary School

Abstract. The paper places much emphasis on the method of, without using division, checking the divisibility of an integer by an odd divisor. In part A, it mainly focuses on getting the general way to perform the divisibility test by an algorithm using the unit digit and the rest of truncated digits of the dividend. Parts B and C are extensions of part A. In part B, it attaches the importance on using the last two or more digits of the dividend and so the divisibility test is not just restricted to the ones digit. While parts A and B direct at the method of verifying the divisibility of a number, part C mainly concentrates on finding out the quotient without performing division algorithm. This unique method of division is discovered in the process of investigation in part A.

INTRODUCTION

In primary school education, students are taught the divisibility tests of three kinds of divisors:

1. Small primes (e.g. 2, 3, 5 and 11).
2. Small integral powers of 2 (e.g. 4 and 8).
3. Small composites using the concept of relative prime (e.g. 6 and 12).

As students with interests and abilities in mathematics, we (the students in this project) are naturally interested to know, is there a way to check the divisibility by 7? 13? Is it possible to generalize a way that can check any positive number? This is the motivation of this project.

Afterwards, we learned more about basic number theory, we know that:

4 To check the divisibility by 2^n or 5^n (where n is a positive integer), we only need to check the last n digits of the dividend. And

5 For 7 as the divisor, the difference between 2 times the unit digit and the remaining leading truncated number is also divisible by 7. If the product of 4 and the last digit is added to the rest of a number, the number is the multiple of 13 when the sum is divisible by 13.

The last result interested us. Is it always possible to do the divisibility check like 7 and 13? We have the following postulate: A large dividend can be separated into the unit digit and the truncated digits. The unit digit is multiplied by a '*code*', which depends on the divisor, then is added to truncated digits. If this smaller number is divisible by the divisor, the original larger dividend is also divisible by the divisor. Is this algorithm applicable to all divisors? And is it possible to generalize a formula to find the '*code*' of all divisors? Sources on internet either provide methods for some specific divisors (such as 19), or guess a general pattern (for 23, 33, ...) without a proof. It drives us to investigate a general formula to check the divisibility for all divisors.

For even divisors, we can write the divisor as $2^r \cdot s$, where r is a positive integer and s is an odd number. As 2^r and s are relatively prime, and we have already known the method of checking the divisibility by 2^r, it is sufficient to check all divisors if we can find the method of checking all odd divisors.

Furthermore, for odd divisors s, if the unit digit of the divisor is 5, we can find write $s = 5^t \cdot u$ where t is a positive integer and u is an odd number which is not divisible by 5 (i.e. the unit digit is not 5). As we have already known the method of checking the divisibility by 5^t, it is sufficient to check all odd divisors (and hence all divisors) if we can find the method of checking divisors with 1, 3, 7 or 9 as the unit digit. Of course, finding a method of all odd prime divisors is sufficient to check all divisors, but the method introduced in this paper is applicable to all odd divisors with 1, 3, 7 or 9 as the unit digit.

In the part A of this paper, we develop a general formula which can find the 'code' of all divisors ending with 1, 3, 7 or 9.

After the discovery presented in part A, we further explore the topic. We found that not only can the method be applied to the ones digit, but it can also be applied to two or more last digits being cut, and the code is unexpectedly simple - it equals to the original code (for cutting single digit) to the power of number of digits being cut. The results and proofs will be discussed in part B. [See reviewer's comment (2)]

In the process of checking divisibility in part A, we discovered that the quotient can be obtained in the process of truncated the number, and hence we can find the quotient without actually performing the division. This could be regarded as the quick division of a given number. The methodology will be discussed in part C.

A. A GENERAL FORMULA TO CHECK DIVISIBILITY BY ALL ODD DIVISORS

For a denary positive integer, N, it can always be expressed as

$$N = 10T + U$$

where T and $U \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Thus, U is the unit digit while T represents the remaining truncated number (definitions of U and T).

Often is it known that divisors can be both positive and negative. However, they are restricted to positive ones in the paper as the negative divisors are just the negation of the positive divisors. [See reviewer's comment (3)] Similarly, negative dividends are not discussed here. With the defined T and U, it is now going to study the divisibility tests for odd divisors, excluding multiples of 5, case by case.

1. ON DIVISOR WITH 1 AS THE UNIT DIGIT

In accordance with the expression of a decimal positive integer, a divisor with 1 in the ones place, π_1, can be expressed as

$$\pi_1 = 10\tau + 1$$

for some integer τ_1. Were τ_1 zero, π_1 would be equal to 1, which is trivial. Hence, τ_1 can only be positive.

Now, for a positive integer N, it is written as

$$N = 10T + U.$$

The number N is then rewritten as follows:

$$\begin{aligned} N &= 10T + U \\ &= 10\tau T - 10\tau T + T + 9T + U \\ &= (10\tau + 1)T - (10\tau - 9)T + U \\ &= (10\tau + 1)T - (10\tau - 9)T + (10\tau^2 - 9\tau)U - (10\tau^2 - 9\tau)U + U \\ &= (10\tau + 1)T - (10\tau - 9)(T - \tau U) - (10\tau^2 - 9\tau - 1)U \\ &= (10\tau + 1)T - (10\tau - 9)(T - \tau U) + (10\tau + 1)(1 - \tau)U \\ &= (10\tau + 1)(T + (1 - \tau)U) - (10\tau - 9)(T - \tau U). \end{aligned}$$

Since $(10\tau + 1)(T + (1 - \tau)U)$ where T, τ and U are integers, is a multiple of $(10\tau + 1)$, i.e. π_1, the number, N, is divisible by π_1 if and only if $(10\tau - 9)(T - \tau U)$ is a multiple of π_1. Note that

$$\pi_1 = 10\tau + 1 = (10\tau - 9) + 2 \times 5.$$

As $(10\tau - 9)$ has 1 as the unit digit, 2 or 5 cannot be a factor of $(10\tau - 9)$. Hence, $(10\tau + 1)$ and $(10\tau - 9)$ are coprime.

To verify whether the number, N, is divisible by π_1, it can be achieved by examining if π_1 is a factor of $(T - \tau U)$, which is the difference between τ_1 times the unit digit and the rest number.

Let $N_1 = (T - \tau U)$, which is also an integer[1], can be expressed similarly as

$$N_1 = 10T_1 + U_1.$$

The expression could then be expanded as before. What we would get is below:

$$N_1 = (10\tau + 1)(T_1 + (1 - \tau)U_1) - (10\tau - 9)(T_1 - \tau U_1).$$

Again, as $(10\tau + 1)(T_1 + (1 - \tau)U_1)$ is divisible by π_1, π_1 is a factor of N_1 if and only if $(10\tau - 9)(T_1 - \tau U_1)$ is a multiple of π_1. As what have been observed here, the process of subtracting the product of τ and the ones digit from the remaining truncated number could be continued. This would keep on reducing the number of digit of a figure and so lead to an easier way to check the divisibility of the figure by the given divisor.

For instance, to determine whether 208537 is divisible by 31, which is an odd number (in this case, $\tau = 3$, $T = 20853$ and $U = 7$):

1. $20853 - 7 \times 3 = 20832.$
2. $2083 - 2 \times 3 = 2077.$
3. $207 - 7 \times 3 = 186.$
4. $18 - 6 \times 3 = 0.$

Since 0 is a multiple of 31, 208537 is divisible by 31. In fact,

$$208537 = 7 \times 31^3.$$

Briefly concluded, to check the divisibility of an odd divisor with 1 in the unit place, we could multiply the unit digit of the number by the divisor with the ones digit dropped and subtract it from the remaining truncated number.

2. ON DIVISORS WITH 3 AS THE UNIT DIGIT

As what have been done in part A§1, an odd number with 3 in the unit place would be written as:

$$\pi_3 = 10\tau + 3$$

for some non-negative integer τ.

Then, a number N is expressed as

$$N = 10T + U$$

[1]If $(T - \tau U)$ is negative, we may consider $N_1 = \tau U - T$ without affecting the whole discussion. Moreover, if $T - \tau U$ becomes negative, the value of $\tau U - T$ should be small enough to perform direct checking of the divisibility. If $T - \tau U = 0$, as 0 is divisible by π_1, N is also divisible by π_1.

and expanded as below

$$\begin{aligned}
N &= 10T + U \\
&= 10\tau T - 10\tau T + 3T + 7T + U \\
&= (10\tau + 3)T - (10\tau - 7)T + U \\
&= (10\tau + 3)T - (10\tau - 7)T + (30\tau^2 - 11\tau)U - (30\tau^2 - 11\tau)U + U \\
&= (10\tau + 3)T - (10\tau - 7)T + (30\tau^2 - 11\tau - 6)U - (30\tau^2 - 11\tau - 7)U \\
&= (10\tau + 3)T - (10\tau - 7)T + (10\tau + 3)(3\tau - 2)U - (10\tau - 7)(3\tau + 1)U \\
&= (10\tau + 3)(T + (3\tau - 2)U) - (10\tau - 7)(T + (3\tau + 1)U).
\end{aligned}$$

Since 3 is the ones digit of $(10\tau - 7)$, where τ is an integer, 2 or 5 cannot be the factor of $(10\tau - 7)$. With the same argument presenting in part A§1, to determine if a number is a multiple of an integer with 3 in the ones place, the number with the ones digit truncated is first multiplied by 3 and then is added to one. Then, if the product between the sum and the ones digit of the number is divisible by the divisor, the number is a multiple of it. The process, as proved in part A§1, can be continued till the value of the product is small enough to perform direct checking.

Here is a quick example, to check whether 146969 is divisible by 53, observe that $\tau = 5$ and $(3\tau + 1) = 16$.

1. $14696 + 9 \times 16 = 14840.$
2. $1484 + 0 \times 16 = 1484.$
3. $148 + 4 \times 16 = 212.$
4. $21 + 2 \times 16 = 53.$

Therefore 146969 is divisible by 53. In this example we can see that ending 0 can be ‘crossed out’ in the algorithm.

3. ON DIVISORS WITH 7 AS THE UNIT DIGIT

Similar to the above 2 parts, A§1 and A§2, a divisor with 7 as the ones digit could be shown as:

$$\pi_7 = 10\tau + 7$$

for some non-negative integer τ.

A number N is expanded:

$$\begin{aligned}
N &= 10T + U \\
&= 10\tau T - 10\tau T + 7T + 3T + U \\
&= (10\tau + 7)T - (10\tau - 3)T + U \\
&= (10\tau + 7)T - (10\tau - 3)T + (30\tau^2 + 11\tau)U - (30\tau^2 + 11\tau)U + U \\
&= (10\tau + 7)T - (10\tau - 3)T + (30\tau^2 + 11\tau - 6)U - (30\tau^2 + 11\tau - 7)U \\
&= (10\tau + 7)T - (10\tau - 3)T + (10\tau - 3)(3\tau + 2)U - (10\tau + 7)(3\tau - 1)U \\
&= (10\tau + 7)(T + (1 - 3\tau)U) - (10\tau - 3)(T - (3\tau + 2)U).
\end{aligned}$$

With the similar argument in Parts A§1 and A§2, a number is divisible by a divisor with 7 in the unit place if and only if $T - (3\tau + 2)U$ is divisible by that divisor. [See reviewer's comment (4)] In other words, the divisibility of the number can be determined by the product of the unit digit of the number, and the sum of 3 times the divisor with the ones digit truncated and 2. As what has been proved in part A§1, the process of the multiplication can be carried on till the number is small enough for direct checking.

Another quick example, to check whether 146969 is divisible by 47, noticed that $-(3\tau + 2) = -(3 \times 4 + 2) = -14$.

1. $14696 - 9 \times 14 = 14570$.
2. Ending 0 is 'crossed out'.
3. $145 - 7 \times 14 = 47$.

Therefore 146969 is divisible by 47.

4. ON DIVISORS WITH 9 AS THE UNIT DIGIT

Again, for a divisor with 9 as the unit digit, it is going to be expressed as:

$$\pi_9 = 10\tau + 9$$

for some non-negative integer τ.

For a number N,

$$N = 10T + U.$$

It is rewritten as:

$$\begin{aligned}
N &= 10T + U \\
&= 10\tau T - 10\tau T + 9T + T + U \\
&= (10\tau + 9)T - (10\tau - 1)T + U \\
&= (10\tau + 9)T - (10\tau - 1)T + (10\tau^2 + 9\tau)U - (10\tau^2 + 9\tau - 1)U \\
&= (10\tau + 9)(T + \tau U) - (10\tau - 1)T - (10\tau - 1)(\tau + 1)U \\
&= (10\tau + 9)(T + \tau U) - (10\tau - 1)(T + (\tau + 1)U).
\end{aligned}$$

Being akin to the proof in the above three parts, a number is divisible by a divisor with the unit digit 9 if and only if the product of the ones digit of the number and the sum of 1 and the divisor with the unit digit dropped is divisible by that divisor. Again, the process can continue for the product obtained. [See reviewer's comment (5)]

Using 146969 as example again, to check whether it is divisible by 59, observed that $+(\tau + 1) = +(5 + 1) = 6$.

1. $14696 + 9 \times 6 = 14750$.
2. $147 + 5 \times 6 = 177$. (0 is crossed out after step 1).
3. $17 + 7 \times 6 = 59$.

Therefore 146969 is divisible by 59. In fact, $146969 = 47 \times 53 \times 59$.

5. ON GENERAL CASE FOR DIVISORS WITH 1,3,7,9 AS THE UNIT DIGIT

As derived in the above parts, the divisibility tests are mainly the processes of verifying whether a number ending with 1, 3, 7 or 9 is a factor of a given number by examining the divisibility of a product of a constant and the unit digit of the number. To address conveniently, the constants mentioned before would be termed as the "*code*".

In the above parts, the *code* for the divisors with different numbers in the ones place varies. For simplification, a method to obtain the *code* will be introduced here. Before the derivation of the method, let us have a look on the expansion of a number, being shown in the above 4 parts so far:

$$\begin{aligned}
N &= (10\tau + 1)(T + (1 - \tau)U) - (10\tau - 9)(T - \tau U), \\
N &= (10\tau + 3)(T + (3\tau - 2)U) - (10\tau - 7)(T + (3\tau + 1)U), \\
N &= (10\tau + 7)(T + (1 - 3\tau)U) - (10\tau - 3)(T - (3\tau + 2)U), \\
N &= (10\tau + 9)(T + \tau U) - (10\tau - 1)(T + (\tau + 1)U).
\end{aligned}$$

It is observed that a number can be generally written as

$$N = (10\tau + i)(T + \tau U) - (10\tau + i - 10)(T + (A\tau + B)U)$$

where i is the unit digit of the odd divisor. The term $(A\tau + B)$ is actually the *code*.

When $i = 1, A = -1$ and $B = 0$.

When $i = 3, A = 3$ and $B = 1$.

When $i = 7, A = -3$ and $B = -2$.

When $i = 9, A = 1$ and $B = 1$.

To express A in terms of i, assume that A is a polynomial and let

$$A(i) = ai^3 + bi^2 + ci + d$$

It is known that

$$A(1) = -1, A(3) = 3, A(7) = -3, A(9) = 1. \qquad (*)$$

Thus, we have

$$\begin{cases}
A(1) = -1 = a + b + c + d, \\
A(3) = \quad 3 = 27a + 9b + 3c + d, \\
A(7) = -3 = 343a + 49b + 7c + d, \\
A(9) = \quad 1 = 729a + 81b + 9c + d.
\end{cases}$$

The system of equations is rewritten as follows:

$$\left(\begin{array}{cccc|c}
1 & 1 & 1 & 1 & -1 \\
27 & 9 & 3 & 1 & 3 \\
343 & 49 & 7 & 1 & -3 \\
729 & 81 & 9 & 1 & 1
\end{array}\right).$$

Solving the equations with Cramer's rule, the result is obtained as below (the operation is shown in APPENDIX I):

$$\begin{cases} a = \dfrac{7}{48}, \\[2ex] b = -\dfrac{35}{16}, \\[2ex] c = \dfrac{425}{48}, \\[2ex] d = -\dfrac{125}{16}. \end{cases}$$

For B, it is observed that

$$B = \frac{1 + iA}{10}.$$

Hence, to get the value of the code, $(A\tau + B)$, we can find A by

$$A(i) = \frac{7}{48}i^3 - \frac{35}{16}i^2 + \frac{425}{48}i - \frac{125}{16}$$

and then calculate the value of B.

Though this formula successfully merge the *code* of all odd divisors into one single formula, it is quicker and more convenient to use the result in (*) to perform quick divisibility check.

As a final numerical example of this part, we would like to check whether 24208893 (the phone number of school office) is divisible by 397 (the street number of school address). Now $A(7) = -3$ and $B(7) = -2$, so the *code* is $-3 \times 39 - 2 = -119$.

1. $2420889 - 3 \times 119 = 2420532$
2. $242053 - 2 \times 119 = 241815$
3. $24181 - 5 \times 119 = 23586$
4. $2358 - 6 \times 119 = 1644$
5. $164 - 4 \times 119 = -312$

Since 312 is not divisible by 397, so 24208893 is not divisible by 397.

We may compare our method to the ordinary division:

```
        60979
397 )24208893
     2382
      3888
      3573
       3159
       2779
        3803
        3573
         230
```

The following points can be noted:

1. The calculation using the *code* would be easier than the ordinary division. Both methods do a subtraction and a multiplication for 5 times[2] , but the ordinary division multiplies a digit by 397, while the code method multiplies a digit by the *code* of 397, which is 119. The magnitude of the *code* of divisor with unit digit 1 or 9 (which are $-\tau$ and $\tau+1$ respectively) would be significantly smaller than the original divisor ($10\tau+1$ or $10\tau+9$), while the magnitude of the *code* of divisor with unit digit 3 or 7 ($3\tau+1$ and $-3\tau-2$ respectively) are still smaller than the original divisor. Furthermore, for the divisors with unit digits 3 or 9, the *code* is positive, so additions (instead of subtractions) are performed, which is easier to calculate. Furthermore, in ordinary division sometimes we need to 'guess' the quotient by trial and error. For example, in the second last step when 3159 is divided by 397, students may not instantly know whether the quotient should be 7 or 8. However this problem will not be faced by the code method.

2. In the ordinary division we can find the quotient in the process. Actually the quotient can also be (easily) obtained by code method. The algorithm of finding the quotient by code method will be introduced in part C in this paper.

3. In the final step of code method, the number -312 is obtained, which is *not congruent* to the remainder (230) when the dividend (24208893) is divided by the divisor (397), i.e. the numbers obtained by code method are not congruent under modulo of the divisor. As the algorithm of checking divisibility is usually investigated under modulo calculation, this may be the reason why the code method introduced in this paper is not commonly discovered.

[2]In our example, the ordinary division encountered a zero in the quotient, therefore only four operations of multiplications and subtractions are shown.

B. CONCERNING THE LAST SEVERAL DIGITS AND REMAINING TRUNCATED NUMBER

Above is presented the divisibility test of a number, using the last digit of dividend. [See reviewer's comment (6)] However, the test is not restricted to the last digit. Not only can it be employed to the unit digit, but it can also be employed to last several digits of the dividend.

1. ON LAST TWO DIGITS

Before looking in the method using the last several digits of the dividend, the case of using the last two units will be considered. For a denary positive integer, N, it can be expressed as

$$N = 100T + U$$

where T and $U = \{0, 1, 2, ..., 98, 99\}$. It should be noticed that the T and U in the expression are not the same as in part A.

When the divisor is with 1 as the unit digit, that is $\pi_1 = 10\tau + 1$, the number, N, is expressed as

$$\begin{aligned}
N &= 100T + U \\
&= 100\tau^2 T - 100\tau^2 T + T + 99T + U \\
&= (-100\tau^2 + 1)T + (100\tau^2 + 99)T + U \\
&= (10\tau + 1)(1 - 10\tau)T + (100\tau^2 + 99)T + (100\tau^4 + 99\tau^2)U \\
&\quad - (100\tau^4 + 99\tau^2)U + U \\
&= (10\tau + 1)(1 - 10\tau)T + (100\tau^2 + 99)(T + \tau^2 U) - (100\tau^4 + 99\tau^2 - 1)U \\
&= (10\tau + 1)(1 - 10\tau)T + (100\tau^2 + 99)(T + \tau^2 U) \\
&\quad - (\tau^2 + 1)(10\tau + 1)(10\tau - 1)U \\
&= (10\tau + 1)(1 - 10\tau)(T + (\tau^2 + 1)U) + (100\tau^2 + 99)(T + \tau^2 U).
\end{aligned}$$

Now, it is going to prove that $(100\tau^2 + 99)$ and $(10\tau + 1)$ are coprime. Assume $(100\tau^2 + 99)$ and $(10\tau + 1)$ has a common factor f_c, which is greater than 1. Hence,

$$\begin{cases} 100\tau^2 + 99 = xf_c \\ 10\tau + 1 = yf_c \end{cases}$$

where x and y are positive integers. Thus,

$$\begin{aligned} 100\tau^2 + 10\tau + 1 + 99 &= xf_c + yf_c \\ 10\tau(10\tau + 1) + 100 &= xf_c + yf_c \\ 10\tau y f_c + 100 &= xf_c + yf_c \\ 10\tau y + \frac{100}{f_c} &= x + y. \end{aligned}$$

Since, $10\tau y$ and $(x + y)$ are integers, by closure of integer set, $100/f_c$ must be an integer, which implies that f_c is a factor of 100, that is 1, 2, 5, 10, 20, 50 or 100. [See reviewer's comment (7)] However, it is impossible that $100\tau^2 + 99$ or $10\tau + 1$ has the common factor, being an odd number not with 5 in the unit place. Hence, f_c could only be 1 and contradiction occurs. As a result, $100\tau^2 + 99$ and $10\tau + 1$ are coprime.

Therefore, to check the divisibility of a number by an odd divisor, it can be achieved by determining the divisibility of $(T + \tau^2 U)$. It is remarkable that the new *code* is actually the square of the *code* when unit digit is truncated.

For example, to check whether 208537 is divided by 31 by chopping last 2 digits. The code of chopping 1 digit is -3, so the code of chopping 2 digits is $(-3)^2 = 9$

1. $2085 + 37 \times 9 = 2418$,
2. $24 + 18 \times 9 = 186$.

By chopping two digits, it is possible to obtain a larger number in later steps[3], so it would be better to check whether the smaller number 186 is divisible by 31. Obviously $186 = 31 \times 6$, or we may use back the single-digit algorithm: $18 - 6 \times 3 = 0$ which also shows the divisibility. Therefore, we can choose to chop one or two last digits, which is particularly convenient if the tenth digit of the remaining number is 0.

When the unit digit of the divisor is 3, let the divisor be $\pi_3 = 10\tau + 3$, the number N is expanded as

---

[3]If we continue the 2-digit algorithm, $\begin{cases} 3. \quad 1 + 86 \times 9 = 775 \\ 4. \quad 7 + 75 \times 9 = 682 \\ 5. \quad 6 + 82 \times 9 = 744 \\ 6. \quad 7 + 44 \times 9 = 403 \\ 7. \quad 4 + 3 \times 9 = 31 \end{cases}$, we can achieve the same result, but obviously the calculation is redundant. Therefore, for a speedy division check, the number of digits being chopped should be selected wisely.

$$
\begin{aligned}
N &= 100T + U \\
&= 100\tau^2 T - 100\tau^2 T + 9T + 91T + U \\
&= (-100\tau^2 + 9)T + (100\tau^2 + 91)T + U \\
&= (10\tau + 3)(3 - 10\tau)T + (100\tau^2 + 91)T + (100\tau^2 + 91)(3\tau + 1)^2 U \\
&\quad - (100\tau^2 + 91)(3\tau + 1)^2 U + U \\
&= (10\tau + 3)(3 - 10\tau)T + (100\tau^2 + 91)(T + (3\tau + 1)^2 U) \\
&\quad - (10\tau + 3)(90\tau^3 + 33\tau^2 + 82\tau + 30)U \\
&= (10\tau + 3)((3 - 10\tau)T - (90\tau^3 + 33\tau^2 + 82\tau + 30)U) \\
&\quad + (100\tau^2 + 91)(T + (3\tau + 1)^2 U).
\end{aligned}
$$

Now, we try to prove $(100\tau^2 + 91)$ and $(10\tau + 3)$ are coprime. Assume $(100\tau^2 + 91)$ and $(10\tau + 3)$ has a common factor f_c, which is greater than 1. Hence,

$$10\tau + 3 \equiv 0 \pmod{f_c}$$

and so,

$$
\begin{aligned}
100\tau^2 + 91 &= 100\tau^2 + 60\tau + 9 - 60\tau + 82 \\
&= (10\tau + 3)^2 - 6(10\tau + 3) + 100 \\
&\equiv 100 \pmod{f_c}.
\end{aligned}
$$

Since $100 \equiv 0 \pmod{f_c}$ only when f_c is 1 or an even number or some integer with 5 in the unit place, while $(10\tau + 3)$ is an odd number with 3 as the unit digit, f_c could only be 1. Thus, there is a contradiction. Therefore, $100\tau^2 + 91$ and $10\tau + 3$ are coprime. As a result, to determine whether the odd divisor is the factor of a number, it can be achieved by checking the divisibility of $(T + (3\tau + 1)^2 U)$. It is noteworthy that the *code* is actually the square of the *code* when unit digit is truncated.

The result of the *code* is mathematically amazing, but it could be difficult to apply as a speedy divisibility check. For example, if we would like to chop two numbers to check whether a number is divisible by 53, the code would be

$$(3 \times 5 + 1)^2 = 16^2 = 256,$$

which is actually greater than the original divisor 53.

When the divisor is ending with 7, the number, N, is written as

$$\begin{aligned}
N &= 100T + U \\
&= 100\tau^2 T - 100\tau^2 T + 49T + 51T + U \\
&= (-100\tau^2 + 49)T + (100\tau^2 + 51)T + U \\
&= (10\tau + 7)(7 - 10\tau)T + (100\tau^2 + 51)T + (100\tau^2 + 51)(3\tau + 2)^2 U \\
&\quad - (100\tau^2 + 51)(3\tau + 2)^2 U + U \\
&= (10\tau + 7)(7 - 10\tau)T + (100\tau^2 + 51)(T + (3\tau + 2)^2 U) \\
&\quad - (10\tau + 7)(90\tau^3 + 57\tau^2 + 46\tau + 29)U \\
&= (10\tau + 7)((7 - 10\tau)T - (90\tau^3 + 57\tau^2 + 46\tau + 29)U) \\
&\quad + (100\tau^2 + 51)(T + (3\tau + 2)^2 U).
\end{aligned}$$

To prove that $(100\tau^2+51)$ and $(10\tau+7)$ are coprime, it is assumed that $(100\tau^2+51)$ and $(10\tau + 7)$ has a common factor f_c, which is greater than 1. Hence,

$$10\tau + 7 \equiv 0 \pmod{f_c}$$

and so,

$$\begin{aligned}
100\tau^2 + 51 &= 100\tau^2 + 140\tau + 49 - 140\tau + 2 \\
&= (10\tau + 7)^2 - 14(10\tau + 7) + 100 \\
&\equiv 100 \pmod{f_c}.
\end{aligned}$$

As $100 \equiv 0 \pmod{f_c}$ only when f_c is 1 or an even number or some integer with 5 in the unit place, while $(10\tau + 7)$ is an odd number with 7 in the unit place, f_c could only be 1. This is contradictory to the assumption. Thus, there is no common factor between $(10\tau + 7)$ and $(100\tau^2 + 51)$. Therefore, to verify whether a number is a multiple of the odd divisor, it can be achieved by examining the divisibility of $(T + (3\tau + 2)^2 U)$. It is also noticeable that the *code* is actually the square of the *code* when unit digit is truncated.

When the divisor is with 9 in the ones place, the number, N, can be shown as

$$\begin{aligned}
N &= 100T + U \\
&= 100\tau^2 T - 100\tau^2 T + 81T + 19T + U \\
&= (-100\tau^2 + 81)T + (100\tau^2 + 19)T + U \\
&= (10\tau + 9)(9 - 10\tau)T + (100\tau^2 + 19)T + (100\tau^2 + 19)(\tau + 1)^2 U \\
&\quad - (100\tau^2 + 19)(\tau + 1)^2 U + U \\
&= (10\tau + 9)(9 - 10\tau)T + (100\tau^2 + 19)(T + (\tau + 1)^2 U) \\
&\quad - (10\tau + 9)(10\tau^3 + 11\tau^2 + 2\tau + 2)U \\
&= (10\tau + 9)((9 - 10\tau)T - (10\tau^3 + 11\tau^2 + 2\tau + 2)U) \\
&\quad + (100\tau^2 + 19)(T + (\tau + 1)^2 U).
\end{aligned}$$

Now, assume $(100\tau^2 + 19)$ and $(10\tau + 9)$ has a common factor f_c, which is greater than 1. Hence,

$$10\tau + 9 \equiv 0 \pmod{f_c}$$

and so,

$$\begin{aligned}
100\tau^2 + 19 &= 100\tau^2 + 180\tau + 81 - 180\tau - 62 \\
&= (10\tau + 9)^2 - 18(10\tau + 9) + 100 \\
&\equiv 100 \pmod{f_c}.
\end{aligned}$$

As $100 \equiv 0 \pmod{f_c}$ only when f_c is 1 or an even number or some integer with 5 in the unit place, while $(10\tau + 9)$ is an odd number with 9 as the unit digit, f_c could only be 1. This contradicts with the assumption. Thus, $(10\tau + 9)$ and $(100\tau^2 + 19)$ are coprime. Therefore, to check whether a number is divisible by the odd divisor, we can determine the divisibility of $(T + (\tau + 1)^2 U)$. It should be conspicuous that the *code* is exactly the square of the *code* when unit digit is truncated.

As shown in the above 4 cases, when the last two digits of the dividend is truncated, the *code* for checking its divisibility would be the square of the *code* when only the unit digit is truncated. Hence, it is guessed that the power of the *code* is the same as the number of rightmost digits dropped. The conjecture will be proved in the following part.

2. ON LAST SEVERAL DIGITS

Considering the divisibility of a number by π_1 with last n digits truncated. [See reviewer's comment (8)]

Let n be a positive odd number.

$$\begin{aligned}
N &= 10^nT + U \\
&= 10^n\tau^nT - 10^n\tau^nT + T + (10^n - 1)T + U \\
&= (10^n\tau^n + 1)T + (-10^n\tau^n + 10^n - 1)T + U \\
&= ((10\tau)^n + 1)T + (-10^n\tau^n + 10^n - 1)T - \\
&\quad (-10^n\tau^{2n} + 10^n\tau^n - \tau^n)U + (-10^n\tau^{2n} + 10^n\tau^n - \tau^n)U + U \\
&= ((10\tau)^n + 1)T + (-10^n\tau^n + 10^n - 1)(T - \tau^nU) \\
&\quad + (-10^n\tau^{2n} + 10^n\tau^n - \tau^n + 1)U \\
&= ((10\tau)^n + 1)T + (-10^n\tau^n + 10^n - 1)(T - \tau^nU) \\
&\quad + ((10\tau)^n + 1)(1 - \tau^n)U \\
&= ((10\tau)^n + 1)(T + (1 - \tau^n)U) + (-(10\tau)^n + 10^n - 1)(T - \tau^nU).
\end{aligned}$$

Put $\tau = \dfrac{-1}{10}$ into $(10\tau)^n + 1$,

$$(10\tau)^n + 1 = (-1)^n + 1.$$

As n is an odd number, $(10\tau)^n + 1 = 0$ when $\tau = \dfrac{-1}{10}$. In other words, $(10\tau)^n + 1$ is divisible by $10\tau + 1$, that is π_1, by the remainder theorem. Now, assume $10\tau + 1$ and $(-(10\tau)^n + 10^n - 1)$ has a common factor f_c, which is greater than 1. Hence,

$$10\tau + 1 \equiv 0 \pmod{f_c}$$

and so,

$$-(10\tau)^n + 10^n - 1 \equiv -((10\tau)^n + 1) + 10^n \pmod{f_c}.$$

Since it is proved that $(10\tau)^n + 1$ is divisible by π_1, and we assumed that f_c is a factor of π_1, therefore $(10\tau)^n + 1$ is divisible by f_c. Hence

$$-(10\tau)^n + 10^n - 1 \equiv 10^n \pmod{f_c}.$$

As $10^n \equiv 0 \pmod{f_c}$ only when f_c is 1 or an even number or some integer with 5 in the unit place, while $(10\tau + 1)$ is an odd number with 1 as the unit digit, f_c could only be 1. This contradicts with the assumption. Thus, $(10\tau + 1)$ and $(-(10\tau)^n + 10^n - 1)$ are coprime. Therefore, to check whether a number is divisible by the odd divisor, we can determine the divisibility of $(T - \tau^nU)$. Hence, when last n number of the dividend is truncated, the *code* is exactly the n power of the *code* when unit digit is truncated.

Let n be a positive even number.

$$\begin{aligned}
N &= 10^nT + U \\
&= 10^n\tau^nT - 10^n\tau^nT + T + (10^n - 1)T + U \\
&= (-10^n\tau^n + 1)T + (10^n\tau^n + 10^n - 1)T + U \\
&= (-(10\tau)^n + 1)T + (10^n\tau^n + 10^n - 1)T \\
&\quad - (10^n\tau^{2n} + 10^nt^n - t^n)U - (10^n\tau^{2n} + 10^n\tau^n - \tau^n)U + U \\
&= ((-10\tau)^n + 1)T + (10^n\tau^n + 10^n - 1)(T - \tau^nU) \\
&\quad - (10^n\tau^{2n} + 10^n\tau^n - \tau^n - 1)U \\
&= ((-10\tau)^n + 1)T + (10^n\tau^n + 10^n - 1)(T - \tau^nU) \\
&\quad - ((10\tau)^n + 1)(1 + \tau^n)U \\
&= ((-10\tau)^n + 1)(T + (1 - \tau^n)U) + ((10\tau)^n + 10^n - 1)(T + \tau^nU).
\end{aligned}$$

Put $\tau = \dfrac{-1}{10}$ into $-(10\tau)^n + 1$,

$$-(10\tau)^n + 1 = -(-1)^n + 1.$$

As n is an even number, $-(10\tau)^n+1 = -1+1 = 0$ when $\tau = \dfrac{-1}{10}$. By the remainder theorem, $-(10\tau)^n + 1$ is divisible by $10\tau + 1$, that is π_1.

Now, assume $10\tau + 1$ and $((10\tau)^n + 10^n - 1)$ has a common factor f_c, which is greater than 1. Hence,

$$10\tau + 1 \equiv 0 \pmod{f_c}$$

and so,

$$\begin{aligned}
(10\tau)^n + 10^n - 1 &\equiv -(-(10\tau)^n + 1) + 10^n \pmod{f_c} \\
&\equiv 10^n \pmod{f_c}.
\end{aligned}$$

As $10^n \equiv 0 \pmod{f_c}$ only when f_c is 1 or an even number or some integer with 5 in the unit place, while $(10\tau + 1)$ is an odd number with 1 as the unit digit, f_c could only be 1. This contradicts with the assumption. Thus, $(10\tau + 1)$ and $((10\tau)^n+10^n-1)$ are coprime. Therefore, to check whether a number is divisible by the odd divisor, we can determine the divisibility of $(T - \tau^nU)$. Hence, combining the odd and even cases of n, when last n number of the dividend is truncated, the *code* is exactly the n power of the *code* when unit digit is truncated.

Considering the divisibility of a number by $\pi_3 = 10\tau+3$ with last n digits truncated. [See reviewer's comment (9)]

Let n be a positive odd number.

$$\begin{aligned}
N &= 10^nT + U \\
&= (10\tau)^nT - (10\tau)^nT + 3^nT + (10^n - 3^n)T + U \\
&= ((10\tau)^n + 3^n)T + (-(10\tau)^n + 10^n - 3^n)T + U \\
&= ((10\tau)^n + 3^n)T + (-(10\tau)^n + 10^n - 3^n)T \\
&\quad + (-(10\tau)^n + 10^n - 3^n)(3\tau + 1)^nU \\
&\quad - (-(10\tau)^n + 10^n - 3^n)(3\tau + 1)^nU + U \\
&= ((10\tau)^n + 3^n)T + (-(10\tau)^n + 10^n - 3^n)(T + (3\tau + 1)^nU) \\
&\quad - ((-(10\tau)^n + 10^n - 3^n)(3\tau + 1)^n - 1)U \\
&= ((10\tau)^n + 3^n)T + (-(10\tau)^n + 10^n - 3^n)(T + (3\tau + 1)^nU) \\
&\quad - ((-(10\tau)^n - 3^n)(3\tau + 1)^n + 10^n(3\tau + 1)^n - 1)U \\
&= ((10\tau)^n + 3^n)(T + (3\tau + 1)^n)U) - ((30\tau + 10)^n - 1)U \\
&\quad + (-(10\tau)^n + 10^n - 3^n)(T + (3\tau + 1)^nU).
\end{aligned}$$

Put $\tau = \dfrac{-3}{10}$ into $(10\tau)^n + 3^n$,

$$(10\tau)^n + 3^n = (-3)^n + 3^n.$$

As n is an odd number, $(10\tau)^n + 3^n = 0$ when $\tau = \dfrac{-3}{10}$. In other words, $(10\tau)^n + 3^n$ is divisible by $10\tau + 3$, that is π_3, by the remainder theorem.

Put $\tau = \dfrac{-3}{10}$ into $(30\tau + 10)^n - 1$, $(30\tau + 10)^n - 1 = (-9 + 10)^n - 1 = 0$. In other words, $(30\tau + 10)^n - 1$ is divisible by $10\tau + 3$, that is π_3, by the remainder theorem. Hence the first two terms in the expansion are both divisible by π_3.

Now, assume $10\tau + 3$ and $-(10\tau)^n + 10^n - 3^n$ has a common factor f_c, which is greater than 1. Hence,

$$10\tau + 3 \equiv 0 \pmod{f_c}$$

and so,

$$\begin{aligned}
-(10\tau)^n + 10^n - 3^n &\equiv -((10\tau)^n + 3^n) + 10^n \pmod{f_c} \\
&\equiv 10^n \pmod{f_c}.
\end{aligned}$$

As $10^n \equiv 0 \pmod{f_c}$ only when f_c is 1 or an even number or some integer with 5 in the unit place, while $(10\tau + 3)$ is an odd number with 3 as the unit digit, f_c could only be 1. Thus, $(10\tau + 3)$ and $(-(10\tau)^n + 10^n - 3^n)$ are coprime. Therefore, to check whether a number is divisible by the odd divisor, we can determine the divisibility of $(T + (3\tau + 1)^nU)$.

Let n be a positive even number.

$$\begin{aligned}N &= 10^nT + U\\
&= (10\tau)^nT - (10\tau)^nT + 3^nT + (10^n - 3^n)T + U\\
&= ((-10\tau)^n + 3^n)T + ((10\tau)^n + 10^n - 3^n)T + U\\
&= (-(10\tau)^n + 3^n)T + ((10\tau)^n + 10^n - 3^n)T + ((10\tau)^n + 10^n - 3^n)(3\tau + 1)^nU\\
&\quad - ((10\tau)^n + 10^n - 3^n)(3\tau + 1)^nU + U\\
&= (-(10\tau)^n + 3^n)T + ((10\tau)^n + 10^n - 3^n)(T + (3\tau + 1)^nU)\\
&\quad - (((10\tau)^n + 10^n - 3^n)(3\tau + 1)^n - 1)U\\
&= (-(10\tau)^n + 3^n)T + ((10\tau)^n + 10^n - 3^n)(T + (3\tau + 1)^nU)\\
&\quad - (((10\tau)^n - 3^n)(3\tau + 1)^n + 10^n(3\tau + 1)^n - 1)U\\
&= (-(10\tau)^n + 3^n)(T + (3\tau + 1)^n)U) - ((30\tau + 10)^n - 1)U\\
&\quad + (-(10\tau)^n + 10^n - 3^n)(T + (3\tau + 1)^nU).\end{aligned}$$

As just proven above, $-(10\tau)^n + 3^n$ and $(30\tau + 10)^n - 1$ are both divisible by π_3, and $(10\tau + 3)$ and $(-(10\tau)^n + 10^n - 3^n)$ are coprime. Therefore, to check whether a number is divisible by π_3, we can determine the divisibility of $(T - (3\tau + 1)^nU)$. Hence, combining the cases when n is odd and even, when last n number of the dividend is truncated, the *code* is exactly the n power of the *code* when unit digit is truncated. [See reviewer's comment (10)]

Considering the divisibility of a number by $\pi_7 = 10\tau+7$ with last n digits truncated. [See reviewer's comment (11)] Let n be a positive odd number

$$\begin{aligned}N &= 10^nT + U\\
&= (10\tau)^nT - (10\tau)^nT + 7^nT + (10^n - 7^n)T + U\\
&= ((10\tau)^n + 7^n)T + (-(10\tau)^n + 10^n - 7^n)T + U\\
&= ((10\tau)^n + 7^n)T + (-(10\tau)^n + 10^n - 7^n)T - (-(10\tau)^n + 10^n - 7^n)(3\tau + 2)^nU\\
&\quad + (-(10\tau)^n + 10^n - 7^n)(3\tau + 2)^nU + U\\
&= ((10\tau)^n + 7^n)T + (-(10\tau)^n + 10^n - 7^n)(T - (3\tau + 2)^nU)\\
&\quad + ((-(10\tau)^n + 10^n - 7^n)(3\tau + 2)^n + 1)U\\
&= ((10\tau)^n + 7^n)T + (-(10\tau)^n + 10^n - 7^n)(T - (3\tau + 2)^nU)\\
&\quad + ((-(10\tau)^n - 7^n)(3\tau + 2)^n + 10^n(3\tau + 2)^n + 1)U\\
&= ((10\tau)^n + 1)(T + (3\tau + 2)^n)U) - ((30\tau + 20)^n + 1)U\\
&\quad + (-(10\tau)^n + 10^n - 7^n)(T - (3\tau + 2)^nU).\end{aligned}$$

Put $\tau = \dfrac{-7}{10}$ into $(10\tau)^n + 7^n$,

$$(10\tau)^n + 7^n = (-7)^n + 7^n.$$

As n is an odd number, $(10\tau)^n+7^n=0$ when $\tau=\dfrac{-7}{10}$. In other words, $(10\tau)^n+7^n$ is divisible by $10\tau+7$, that is π_7, by the remainder theorem.

Put $\tau=\dfrac{-7}{10}$ into $(30\tau+20)^n+1$,

$$(30\tau+20)^n+1=(-21+20)^n+1=(-1)^n+1,$$

when n is odd, the expression equals to 0. Therefore, when n is odd, $(30\tau+20)^n+1$ is divisible by π_7, by the remainder theorem.

Now, assume $10\tau+7$ and $(-(10\tau)^n+10^n-7^n)$ has a common factor f_c, which is greater than 1. Hence,

$$10\tau+7\equiv 0 \pmod{f_c}$$

and so,

$$\begin{aligned}-(10\tau)^n+10^n-7^n &\equiv -((10\tau)^n+7^n)+10^n \pmod{f_c}\\ &\equiv 10^n \pmod{f_c}.\end{aligned}$$

As $10^n\equiv 0 \pmod{f_c}$ only when f_c is 1 or an even number or some integer with 5 in the unit place, while $(10\tau+7)$ is an odd number with 7 as the unit digit, f_c could only be 1. Thus, $(10\tau+7)$ and $(-(10\tau)^n+10^n-7^n)$ are coprime. Therefore, to check whether a number is divisible by the odd divisor, we can determine the divisibility of $(T-(3\tau+2)^nU)$.

Consider when n is even.

$$\begin{aligned}N&=10^nT+U\\ &=(10\tau)^nT-(10\tau)^nT+7^nT+(10^n-7^n)T+U\\ &=(-(10\tau)^n+7^n)T+((10\tau)^n+10^n-7^n)T+U\\ &=(-(10\tau)^n+7^n)T+((10\tau)^n+10^n-7^n)T+((10\tau)^n+10^n-7^n)(-3\tau-2)^nU\\ &\quad-((10\tau)^n+10^n-7^n)(-3\tau-2)^nU+U\\ &=(-(10\tau)^n+7^n)T+((10\tau)^n+10^n-7^n)(T+(3\tau+2)^nU)\\ &\quad+((-(10\tau)^n+10^n-7^n)(3\tau+2)^n+1)U\\ &=(-(10\tau)^n+7^n)T+((10\tau)^n+10^n-7^n)(T+(3\tau+2)^nU)\\ &\quad+((-(10\tau)^n-7^n)(3\tau+2)^n+10^n(3\tau+2)^n+1)U\\ &=(-(10\tau)^n+7^n)(T+(3\tau+2)^n)U)-((30\tau+20)^n-1)U\\ &\quad+(-(10\tau)^n+10^n-7^n)(T+(3\tau+2)^nU).\end{aligned}$$

Put $\tau=\dfrac{-7}{10}$ into $-(10\tau)^n+7^n$,

$$-(10\tau)^n+7^n=-(-7)^n+7^n.$$

As n is an even number, $-(10\tau)^n+7^n=0$ when $\tau=\dfrac{-7}{10}$. In other words, $-(10\tau)^n+7^n$ is divisible by $10\tau+7$, that is π_7, by the remainder theorem.

Put $\tau = \dfrac{-7}{10}$ into $(30\tau+20)^n - 1$,

$$(30\tau+20)^n - 1 = (-21+20)^n - 1 = (-1)^n - 1,$$

when n is even, the expression equals to 0. Therefore, when n is even, $(30\tau+20)^n - 1$ is divisible by π_7, by the remainder theorem.

Now, assume $10\tau + 7$ and $(-(10\tau)^n + 10^n - 7^n)$ has a common factor f_c, which is greater than 1. Hence,

$$10\tau + 7 \equiv 0 \pmod{f_c}$$

and so,

$$\begin{aligned} -(10\tau)^n + 10^n - 7^n &\equiv -((10\tau)^n + 7^n) + 10^n \pmod{f_c} \\ &\equiv 10^n \pmod{f_c}. \end{aligned}$$

As proven above, $(10\tau + 7)$ and $(-(10\tau)^n + 10^n - 7^n)$ are coprime. Therefore, to check whether a number is divisible by the odd divisor, we can determine the divisibility of $(T + (3\tau + 2)^n U)$. Hence, combining the cases when n is odd and even, when last n number of the dividend is truncated, the *code* is exactly the n power of the *code* when unit digit is truncated. [See reviewer's comment (12)]

Considering the divisibility of a number by $\pi_9 = 10\tau+9$, with last n digits truncated.

Let n be a positive odd number

$$\begin{aligned} N &= 10^n T + U \\ &= (10\tau)^n T - (10\tau)^n T + 9^n T + (10^n - 9^n)T + U \\ &= ((10\tau)^n + 9^n)T + (-(10\tau)^n + 10^n - 9^n)T + U \\ &= ((10\tau)^n + 9^n)T + (-(10\tau)^n + 10^n - 9^n)T \\ &\quad + (-(10\tau)^n + 10^n - 9^n)(\tau+1)^n U - (-(10\tau)^n + 10^n - 9^n)(\tau+1)^n U + U \\ &= ((10\tau)^n + 9^n)T + (-(10\tau)^n + 10^n - 9^n)(T + (\tau+1)^n U) \\ &\quad - ((-(10\tau)^n + 10^n - 9^n)(\tau+1)^n - 1)U \\ &= ((10\tau)^n + 9^n)T + (-(10\tau)^n + 10^n - 9^n)(T + (\tau+1)^n U) \\ &\quad - ((-(10\tau)^n - 9^n)(\tau+1)^n + 10^n(\tau+1)^n - 1)U \\ &= ((10\tau)^n + 9^n)(T + (\tau+1)^n)U) - ((10\tau+10)^n - 1)U \\ &\quad + (-(10\tau)^n + 10^n - 9^n)(T + (\tau+1)^n U). \end{aligned}$$

Put $\tau = \dfrac{-9}{10}$ into $(10\tau)^n + 9^n$,

$$(10\tau)^n + 9^n = (-9)^n + 9^n.$$

As n is an odd number, $(10\tau)^n + 9^n = 0$ when $\tau = \dfrac{-9}{10}$. By the remainder theorem, $(10\tau)^n + 9^n$ is divisible by $10\tau + 9$, that is π_9.

Put $\tau = \dfrac{-9}{10}$ into $(10\tau + 10)^n - 1$,

$$(10\tau + 10)^n - 1 = (-9 + 10)^n - 1 = 1^n - 1 = 0.$$

In other words, $(10\tau + 10)^n - 1$ is divisible by π_9, by the remainder theorem.

Now, assume $10\tau + 9$ and $-(10\tau)^n + 10^n - 9^n$ has a common factor f_c, which is greater than 1. Hence,

$$10\tau + 9 \equiv 0 \pmod{f_c}$$

and so,

$$\begin{aligned} -(10\tau)^n + 10^n - 9^n &\equiv -((10\tau)^n + 9^n) + 10^n \pmod{f_c} \\ &\equiv 10^n \pmod{f_c}. \end{aligned}$$

As $10^n \equiv 0 \pmod{f_c}$ only when f_c is 1 or an even number or some integer with 5 in the unit place, while $(10\tau + 9)$ is an odd number with 9 as the unit digit, f_c could only be 1. Thus, $(10\tau+9)$ and $(-(10\tau)^n+10^n-9^n)$ are coprime. Therefore, to check whether a number is divisible by the odd divisor, we can determine the divisibility of $(T + (\tau + 1)^n U)$.

Let n be a positive even number

$$\begin{aligned} N &= 10^n T + U \\ &= (10\tau)^n T - (10\tau)^n T + 9^n T + (10^n - 9^n)T + U \\ &= (-(10\tau)^n + 9^n)T + ((10\tau)^n + 10^n - 9^n)T + U \\ &= (-(10\tau)^n + 9^n)T + ((10\tau)^n + 10^n - 9^n)T + ((10\tau)^n + 10^n - 9^n)(\tau + 1)^n U \\ &\quad - ((10\tau)^n + 10^n - 9^n)(\tau + 1)^n U + U \\ &= (-(10\tau)^n + 9^n)T + ((10\tau)^n + 10^n - 9^n)(T + (\tau + 1)^n U) \\ &\quad - (((10\tau)^n + 10^n - 9^n)(\tau + 1)^n - 1)U \\ &= (-(10\tau)^n + 9^n)T + ((10\tau)^n + 10^n - 9^n)(T + (\tau + 1)^n U) \\ &\quad - (((10\tau)^n - 9^n)(\tau + 1)^n + 10^n(\tau + 1)^n - 1)U \\ &= (-(10\tau)^n + 9^n)(T + (\tau + 1)^n)U) - ((10\tau + 10)^n - 1)U \\ &\quad + (-(10\tau)^n + 10^n - 9^n)(T + (\tau + 1)^n U). \end{aligned}$$

Put $\tau = \dfrac{-9}{10}$ into $-(10\tau)^n + 9^n$,

$$-(10\tau)^n + 9^n = -(-9)^n + 9^n.$$

As n is an even number, $-(10\tau)^n + 9^n = 0$ when $\tau = \dfrac{-9}{10}$. In other words, $-(10\tau)^n + 9^n$ is divisible by $10\tau + 9$, that is π_9, by the remainder theorem.

It is proven in the previous part that $(10\tau+10)^n - 1$ is divisible by π_9. Now, assume $10\tau + 9$ and $(-(10\tau)^n + 10^n - 9^n)$ has a common factor f_c, which is greater than 1.

Hence,

$$10\tau + 9 \equiv 0 \pmod{f_c}$$

and so,

$$\begin{aligned} -(10\tau)^n + 10^n - 9^n &\equiv -((10\tau)^n + 9^n) + 10^n \pmod{f_c} \\ &\equiv 10^n \pmod{f_c}. \end{aligned}$$

As $10^n \equiv 0 \pmod{f_c}$ only when f_c is 1 or an even number or some integer with 5 in the unit place, while $(10\tau + 9)$ is an odd number with 9 as the unit digit, f_c could only be 1. Thus, $(10\tau+9)$ and $(-(10\tau)^n + 10^n - 9^n)$ are coprime. Therefore, to check whether a number is divisible by the odd divisor, we can determine the divisibility of $(T + (\tau+1)^n U)$. Hence, combining the cases when n is odd and even, when last n number of the dividend is truncated, the *code* is exactly the n power of the *code* when unit digit is truncated. [See reviewer's comment (13)]

Here is an example of a quick checking process of divisibility by chopping several digits.

Question: Test whether 41479001 is divisible by 83.

Solution: The unit digit of the divisor is 3, therefore, from part A§2, the *code* is $+(3 \times 8 + 1) = +25$. The dividend to be tested ended with 001, therefore it could be quicker if we chop off last 3 digits. When 3 digits are chopped off, the *code* will be $(+25)^3 = +15625$. In most case, using such a large *code* may involve even more tedious calculation than simple division algorithm, but as the ending digits of the dividend to be tested is 001, it makes the calculation easier.

1. $41479 + 15625 \times 001 = 57104$.

So we now consider whether 57104 is divisible by 83. Now the tenth digit is 0, so it may be natural to consider chopping 2 digits, where the *code* will be $(+25)^2 = +625$.

2a. $571 + 625 \times 4 = 571 + 2500 = 3071$.

It is possible to continue the calculation using 3071. However, if only 1 digit to be chopped, the unit digit of the new number will be zero, so it could be faster:

2b. $5710 + 25 \times 4 = 5710 + 100 = 5810$.

As the zero at the end can be chopped off, only consider 581 is sufficient, and

3. $58 + 25 \times 1 = 83$

which is obviously divisible by 83. Therefore 41479001 is divisible by 83. In fact, 41479001 is a *semiprime* which is a product of two primes 83 and 499747.

Briefly conclude, if we chop the rightmost n digits, the *code* will be the original *code* to be power n. Although the use of chopping more than 1 digit may not always make the checking process faster, the unexpectedly simple result does show the beauty of mathematics. [See reviewer's comment (14)]

C. GETTING ANOTHER FACTOR FROM A KNOWN FACTOR - A QUICKER WAY

The previous part has mainly emphasized on checking whether a number ending with 1, 3, 7 or 9 is a factor of a given number. Nevertheless, if the given number is a multiple of the given divisor, the quotient can be found easily from the divisibility test. The following pages will show how to obtain the quotient in the process of determining the divisibility.

1. ON DIVISOR WITH 1 AS THE UNIT DIGIT

Assume N_1 is a multiple of π_1, then

$$N_1 = 10T + U = (10\tau + 1)(10m_0 + r_0)$$

where m_0 and r_0 are non-negative integers and $r_0 < 10$.

$$\begin{aligned} N_1 &= 10T + U \\ &= (10\tau + 1)(10m_0 + r_0) \\ &= 10(10\tau m_0 + m_0 + r_0\tau) + r_0. \end{aligned}$$

Therefore, $(10\tau m_0 + m_0 + r_0\tau)$ represents the tenth digit while r_0 represent the unit digit. Hence,

$$\begin{aligned} T &= 10\tau m_0 + m_0 + r_0\tau, \\ U &= r_0. \end{aligned}$$

Next, in the process of checking the divisibility of N_1 by π_1, the ones digit, U, is multiplied by τ and then the product is subtracted from T. The difference is $N_{1,1}$, representing the integer obtained after the first time of the subtraction. Thus, we have

$$N_{1,1} = T + (-\tau)U$$

where $-\tau$ is the code. It is expressed as follows

$$\begin{aligned} N_{1,1} &= T + (-\tau)U \\ &= (10\tau m_0 + m_0 + r_0\tau) + (-\tau)r_0 \\ &= (10\tau + 1)m_0. \end{aligned}$$

The result is in fact coincides on the proof in part A§1, showing that $N_{1,1}$ is a multiple of π_1 if N_1 is a multiple of π_1. [See reviewer's comment (15)] But what is more remarkable is that m_0, which is the quotient, when N_1 is divided by π_1, with the ones digit dropped, is quotient when $N_{1,1}$ is divided by π_1.

Similarly, it is observable that

$$\begin{aligned} N_{1,1} &= (10\tau_1 + 1)m_0, \\ N_{1,2} &= (10\tau_1 + 1)m_1, \\ N_{1,3} &= (10\tau_1 + 1)m_2, \\ N_{1,4} &= (10\tau_1 + 1)m_3, \\ &\vdots \\ N_{1,n} &= (10\tau_1 + 1)m_{n-1}. \end{aligned}$$

Therefore, the quotient when N_1 is divided by π_1 could be found out from the process.

Review the checking process in part A§1

1. $20853 - 7 \times 3 = 20832$,
2. $2083 - 2 \times 3 = 2077$,
3. $207 - 7 \times 3 = 186$,
4. $18 - 6 \times 3 = 0$.

Now we can see that, when 208537 is divided by 31, the quotient is 6727.

2. ON DIVISOR WITH 3 AS THE UNIT DIGIT

Let N_3 be the multiple of π_3, then

$$N_3 = 10T + U = (10\tau + 3)(10m_0 + r_0)$$

where m_0 and r_0 are non-negative integers and $r_0 < 10$ and they are not necessarily same as part C§1.

$$\begin{aligned} N &= 10T + U \\ &= (10\tau + 3)(10m_0 + r_0) \\ &= 10(10\tau m_0 + 3m_0 + r_0\tau) + 3r_0. \end{aligned}$$

It is reminded that $3r_0$ is a single digit integer only when $r_0 \le 3$. For $r_0 = [4, 6]$, $3r_0$ is an integer between 10 and 19, while it is an integer between 20 and 29 for $r_0 = [7, 9]$. [See reviewer's comment (16)] Thus, we get

r_0	$0 \le r_0 \le 3$	$4 \le r_0 \le 6$	$7 \le r_0 \le 9$
T	$10\tau m_0 + 3m_0 + r_0\tau$	$10\tau m_0 + 3m_0 + r_0\tau + 1$	$10\tau m_0 + 3m_0 + r_0\tau + 2$
U	$3r_0$	$3r_0 - 10$	$3r_0 - 20$

For $r_0 =$	$N_{3,1} = T + (3\tau + 1)U$
$[0,3]$	$= (10\tau m_0 + 3m_0 + r_0\tau) + (3\tau + 1)(3r_0)$ $= 10\tau m_0 + 3m_0 + r_0\tau + 9r_0\tau + 3r_0$ $= (10\tau + 3)(m_0 + r_0)$
$[4,6]$	$= (10\tau m_0 + 3m_0 + r_0\tau + 1) + (3\tau + 1)(3r_0 - 10)$ $= 10\tau m_0 + 3m_0 + r_0\tau + 1 + 9r_0\tau + 3r_0 - 30\tau - 10$ $= 10\tau m_0 + 3m_0 + 10r_0\tau + 3r_0 - 30\tau - 9$ $= (10\tau + 3)(m_0 + r_0 + 3)$
$[7,9]$	$= (10\tau m_0 + 3m_0 + r_0\tau + 2) + (3\tau + 1)(3r_0 - 20)$ $= 10\tau m_0 + 3m_0 + r_0\tau + 2 + 9r_0\tau + 3r_0 - 60\tau - 20$ $= 10\tau m_0 + 3m_0 + 10r_0\tau + 3r_0 - 60\tau - 18$ $= (10\tau + 3)(m_0 + r_0 - 6)$

[See reviewer's comment (17)]

Now, for each possible value of U, we would like to find the value of r_0. However the calculations are rather repetitive and are presented in the APPENDIX II.

Therefore, grouping the results for each r_0 (see APPENDIX II), we have

U	r_0	m_0
0	0	+ 0
1	7	+ 1
2	4	+ 1
3	1	+ 1
4	8	+ 2
5	5	+ 2
6	2	+ 2
7	9	+ 3
8	6	+ 3
9	3	+ 3

As shown in the above table, U and r_0 correspond to each other in a one-to-one relationship, so r_0 could be found immediately when the unit digit of the number obtained in the process of checking one's divisibility is known. [See reviewer's comment (18)]

Take $13M_0 = 1111487$ with the units digit truncated as an example. In this case, $T = 111148, U = 7$ and $\tau = 1$. As proven in part A§2, the code is $+(3\tau + 1)$, which is $+(3\times 1)+1=+4$: [See reviewer's comment (19)]

1. $N_{5,1}=111148+4\times 7=111176$,
2. $N_{5,2}=11117+4\times 6=11141$,
3. $N_{5,3}=1114+4\times 1=1118$,
4. $N_{5,4}=111+4\times 8=143$,
5. $N_{5,5}=14+4\times 3=26$.

N_5 is eventually reduced to 26. To get the quotient, we have to get m and r from step 5 to step 1.

1. In step 5, as $26=13\times 2$, we get 2 as the single digit.
2. In step 4, we have 143 and so $U_4=3$. From the table, $r=1$ and m was added to 1. Hence, the quotient, whose unit digit dropped is the difference between the quotient obtained in last step and 1, is $(2-1)\times 10+1=11$.
3. In step 3, 1118 is obtained. As $U_3=8$, from the table, $r=6$ and m was added to 3. Thus, the quotient is $(11-3)\times 10+6=86$.
4. In step 2, 11141 is found. As $U_2=1$, looking at the table, $r=7$ and m was added to 1. Hence, the quotient is $(86-1)\times 10+7=857$.
5. In step 1, we have 111176. Since $U_1=6$, we get $r=2$ and m was added to 2. Thus, the quotient is $(857-2)\times 10+2=8552$.
6. Lastly, for 1111487, since $U_0=7$, so $r_0=9$ and 3 has to be subtracted from m. As a result, the quotient of 111148 when it is divided by 13 is $(8552-3)\times 10+9=85499$.

[See reviewer's comment (20)]

In the above algorithm, it is important that the quotient of a number could be calculated without really carrying out the division.

3. ON DIVISOR WITH 7 AS THE UNIT DIGIT

Assume N_7 is a multiple of π_7, then

$$N_7=10T+U=(10\tau+7)(10m_0+r_0)$$

where m_0 and r_0 are non-negative integers and $r_0<10$ and they are not necessarily same as part C§1 or part C§2.

$$\begin{aligned}N&=10T+U,\\&=(10\tau+7)(10m_0+r_0),\\&=10(10\tau m_0+7m_0+r_0\tau)+7r_0.\end{aligned}$$

Being akin to the proof in part C§2, the method of getting a quotient has to be considered case by case, from $r_0=0$ to $r_0=9$.

Grouping the results shown in APPENDIX III, we have

U	r_0	m_0
0	0	+0
1	3	+0
2	6	+0
3	9	+0
4	2	- 1
5	5	- 1
6	8	- 1
7	1	- 2
8	4	- 2
9	7	- 2

As shown in the above table, U and r_0 correspond to each other in a one-to-one relationship, so r_0 could be found immediately when the unit digit of the number obtained in the process of checking one's divisibility is known. [See reviewer's comment (18)]

Another quick example, when we check whether 146969 is divisible by 47 in part A§3, the steps are:

$$\begin{aligned}
&1. \quad 14696 - 9 \times 14 = 14570,\\
&2. \quad 1457 - 0 \times 14 = 1457,\\
&3. \quad 145 - 7 \times 14 = 47.
\end{aligned}$$

For step 3, the starting quotient is $47 \div 47 = 1$.

For step 2, $U_2 = 7$, from the table $r = 1$ and m was subtracted by 2, so the quotient is $(1 + 2) \times 10 + 1 = 31$.

For step 1, as step 2 is just 'crossing out zero', the quotient is obviously 310.

For the original 146969, $U = 9$, so $r = 7$ and m_0 was subtracted by 2, so the quotient of $146969 \div 47$ is $(310 \div 2) \times 10 + 7 = 3127$. [See reviewer's comment (21)]

4. ON DIVISOR WITH 9 AS THE UNIT DIGIT

Let π_9 be the factor of N_9, then

$$\begin{aligned}
N_9 &= 10T + U\\
&= (10\tau + 9)(10m_0 + r_0)\\
&= 10(10\tau m_0 + 9m_0 + r_0\tau) + 9r_0
\end{aligned}$$

where m_0 and r_0 are non-negative integers and $r_0 < 10$ and they are not necessarily same as part C§1 or part C§2. As derived in part A§4, the *code* would be $(\tau + 1)$.

For $r_0 = 0$,

$$\begin{aligned} U &= 0 \times 9 = 0, \\ N_{7,1} &= T + (\tau+1)U \\ &= (10\tau m_0 + 9m_0 + (0)\tau) + (\tau+1)(0) \\ &= (10\tau+9)m_0. \end{aligned}$$

Hence, the quotient of $N_{9,1}$, when it is divided by π_9, is m_0.

For $1 \le r_0 \le 9$,

$$\begin{aligned} U &= 9r_0 - 10(r_0-1) = 10 - r_0, \\ N_{7,1} &= T + (\tau+1)U \\ &= (10\tau m_0 + 9m_0 + r_0\tau + r_0 - 1) + (\tau+1)(10-r_0) \\ &= 10\tau m_0 + 9m_0 + 10\tau + 9 \\ &= (10\tau+9)(m_0+1). \end{aligned}$$

Hence, the quotient of $N_{9,1}$, when it is divided by π_9, is (m_0+1).

Review the algorithm when we check whether 146969 is divisible by 59:

1. $14696 + 9 \times 6 = 14750$,
2. $1475 + 0 \times 6 = 1475$,
2. $147 + 5 \times 6 = 177$,
3. $17 + 7 \times 6 = 59$.

Now we can obtain the quotient:

1. In step 4, as $59 = 59 \times 1$, we get 1 as the single digit.
2. In step 3, we have 177 and $U_3 = 7$, so $r = 10 - 7 = 3$ and m was added to 1. Hence, the quotient, whose unit digit dropped is the difference between the quotient obtained in last step and 1, is $(1-1) \times 10 + 3 = 3$.
3. In step 2, 1475 is obtained. As $U_2 = 5$, so $r = 10 - 5 = 5$ and m was added to 1. Thus, the quotient is $(3-1) \times 10 + 5 = 25$.
4. In step 1 we have 14750, as $U_1 = 0$, $r = 0$ and m was unchanged. Hence the quotient is $25 \times 10 + 0 = 250$. (It is just 'crossing out zero').
5. Originally we have 146969, so $U = 9$ and $r = 10 - 9 = 1$, m was added to 1. Thus, the quotient is $(250-1) \times 10 + 1 = 2491$.

With some practice, the speed of calculating the quotient can be quite fast, especially when we compare it with the ordinary division by 59. [See reviewer's comment (22)]

5. CASE IN POINT

[See reviewer's comment (23)]

In general,

$$\begin{aligned} N_i &= \pi_i(10m_0 + r_0) = \pi_i M_0, \\ N_{i,1} &= \pi_i(10m_1 + r_1), \\ &\vdots \\ N_{i,n} &= \pi_i(10m_n + r_n), \end{aligned}$$

where M_0 is the quotient of N_i when it is divided by π_i. To know M_0, m_0, which will be shown in the operation of checking divisibility, and r_0, which could be observed from the unit digit, should be found. [See reviewer's comment (24)]

SUMMARY AND CONCLUSIONS

In this paper, we have found out the general method to verify the divisibility of a number by an odd divisor. It is mainly checking the divisibility of the sum or the difference of the product of the last several digits and a *code*. [See reviewer's comment (25)] The *code* for the unit digit is in the form of $(A\tau + B)$, where

$$A(i) = \frac{7}{48}i^3 - \frac{35}{16}i^2 + \frac{425}{48}i - \frac{125}{16} \text{ and } B = \frac{1 + iA}{10}$$

for the odd divisor with i in the unit place and $i \neq 5$. The *code* for last n digits truncated is actually the *code* for unit digit dropped to the power n. This is what have been proved in part B.

Not only have we sketched the general method to check the divisibility, but we have also discovered a way to get the quotient. Should we know that an odd number, without 5 as the unit digit, is a factor of the dividend, we will find out the quotient without really doing division in the process of checking divisibility. This has been shown in part C.

Instead of using modulo operation, the way we find the *code* is solely by rearranging and factorizing terms. By introducing this method, we hope that this approach might be inspiring for the research in prime numbers. [See reviewer's comment (26)] Nonetheless, we are happy to solve the problem which should have wondered a lot of primary school students, including us in the past. [See reviewer's comment (27)]

Last but not least, we sincerely thank our teacher, Mr. Wong, who has aided us a lot in the process, regardless of how hard at work is he. [See reviewer's comment (28)]

APPENDIX I

To solve

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -1 \\ 27 & 9 & 3 & 1 & 3 \\ 343 & 49 & 7 & 1 & -3 \\ 729 & 81 & 9 & 1 & 1 \end{array}\right)$$

$$a = \frac{\begin{vmatrix} -1 & 1 & 1 & 1 \\ 3 & 9 & 3 & 1 \\ -3 & 49 & 7 & 1 \\ 1 & 81 & 9 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ 343 & 49 & 7 & 1 \\ 729 & 81 & 9 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} -1 & 1 & 1 & 1 \\ 0 & 12 & 6 & 4 \\ 0 & 46 & 4 & -2 \\ 0 & 82 & 10 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -18 & -24 & -26 \\ 0 & -294 & -336 & -342 \\ 0 & -648 & -720 & -728 \end{vmatrix}} = \frac{\begin{vmatrix} -1 & 1 & 1 & 1 \\ 0 & 12 & 6 & 4 \\ 0 & 0 & -19 & -\frac{52}{3} \\ 0 & 0 & -31 & -\frac{76}{3} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -18 & -24 & -26 \\ 0 & 0 & 56 & \frac{248}{3} \\ 0 & 0 & 144 & 208 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -1 & 1 & 1 & 1 \\ 0 & 12 & 6 & 4 \\ 0 & 0 & -19 & -\frac{52}{3} \\ 0 & 0 & 0 & \frac{56}{19} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -18 & -24 & -26 \\ 0 & 0 & 56 & \frac{248}{3} \\ 0 & 0 & 0 & -\frac{32}{7} \end{vmatrix}}$$

$$= \frac{(-1) \times 12 \times (-19) \times \frac{56}{19}}{1 \times (-18) \times 56 \times \left(-\frac{32}{17}\right)}$$

$$= \frac{672}{4608}$$

$$= \frac{7}{48}.$$

$$
\begin{aligned}
b &= \frac{\begin{vmatrix} 1 & -1 & 1 & 1 \\ 27 & 3 & 3 & 1 \\ 343 & -3 & 7 & 1 \\ 729 & 1 & 9 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ 343 & 49 & 7 & 1 \\ 729 & 81 & 9 & 1 \end{vmatrix}} \\
&= \frac{\begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 30 & -24 & -26 \\ 0 & 340 & -336 & -342 \\ 0 & 730 & -720 & -728 \end{vmatrix}}{4608} \\
&= \frac{\begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 30 & -24 & -26 \\ 0 & 0 & -64 & -\frac{142}{3} \\ 0 & 0 & -136 & -\frac{286}{3} \end{vmatrix}}{4608} \\
&= \frac{\begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 30 & -24 & -26 \\ 0 & 0 & -64 & -\frac{142}{3} \\ 0 & 0 & 0 & \frac{21}{4} \end{vmatrix}}{4608} \\
&= \frac{1 \times 30 \times (-64) \times \frac{21}{4}}{4608} \\
&= \frac{-10080}{4608} \\
&= -\frac{35}{16}.
\end{aligned}
$$

$$
\begin{aligned}
c &= \frac{\begin{vmatrix} 1 & 1 & -1 & 1 \\ 27 & 3 & 3 & 1 \\ 343 & 49 & -3 & 1 \\ 729 & 81 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ 343 & 49 & 7 & 1 \\ 729 & 81 & 9 & 1 \end{vmatrix}} \\
&= \frac{\begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & -18 & 30 & -26 \\ 0 & -294 & 340 & -342 \\ 0 & -648 & 730 & -728 \end{vmatrix}}{4608} \\
&= \frac{\begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & -18 & 30 & -26 \\ 0 & 0 & -150 & \frac{248}{3} \\ 0 & 0 & -350 & 208 \end{vmatrix}}{4608} \\
&= \frac{\begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & -18 & 30 & -26 \\ 0 & 0 & -150 & \frac{248}{3} \\ 0 & 0 & 0 & \frac{136}{9} \end{vmatrix}}{4608} \\
&= \frac{1 \times (-18) \times (-150) \times \frac{136}{9}}{4608} \\
&= \frac{40800}{4608} \\
&= \frac{425}{16}.
\end{aligned}
$$

$$d = \frac{\begin{vmatrix} 1 & 1 & 1 & -1 \\ 27 & 9 & 3 & 3 \\ 343 & 49 & 7 & -3 \\ 729 & 81 & 9 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ 343 & 49 & 7 & 1 \\ 729 & 81 & 9 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & -18 & -24 & 30 \\ 0 & -294 & -336 & 340 \\ 0 & -648 & -720 & 730 \end{vmatrix}}{4608} = \frac{\begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & -18 & -24 & 30 \\ 0 & 0 & 56 & -150 \\ 0 & 0 & 144 & -350 \end{vmatrix}}{4608}$$

$$= \frac{\begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & -18 & -24 & 30 \\ 0 & 0 & 56 & -150 \\ 0 & 0 & 0 & \frac{250}{7} \end{vmatrix}}{4608}$$

$$= \frac{1 \times (-18) \times (56) \times \frac{250}{7}}{4608}$$

$$= \frac{-36000}{4608}$$

$$= -\frac{125}{16}.$$

APPENDIX II

Here is the detailed calculation of part C§2.

1. When $r_0 = 0$,

$$N_{3,1} = \pi_3(m_0 + 0),$$
$$U_1 = 0 \times 3 = 0.$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is m_0.

2. When $r_0 = 1$,

$$N_{3,1} = \pi_3(m_0 + 1),$$
$$U_1 = 1 \times 3 = 3.$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is $(m_0 + 1)$.

3. When $r_0 = 2$,

$$N_{3,1} = \pi_3(m_0 + 2),$$
$$U_1 = 2 \times 3 = 6.$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is $(m_0 + 2)$.

4. When $r_0 = 3$,

$$N_{3,1} = \pi_3(m_0 + 3),$$
$$U_1 = 3 \times 3 = 9.$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is $(m_0 + 3)$.

5. When $r_0 = 4$,

$$\begin{aligned} N_{3,1} &= \pi_3(m_0 + 4 - 3) \\ &= \pi_3(m_0 + 1), \\ U_1 &= 4 \times 3 - 10 = 2. \end{aligned}$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is $(m_0 + 1)$.

6. When $r_0 = 5$,

$$\begin{aligned} N_{3,1} &= \pi_3(m_0 + 5 - 3) \\ &= \pi_3(m_0 + 2), \\ U_1 &= 5 \times 3 - 10 = 5. \end{aligned}$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is $(m_0 + 2)$.

7. When $r_0 = 6$,

$$\begin{aligned} N_{3,1} &= \pi_3(m_0 + 6 - 3) \\ &= \pi_3(m_0 + 1), \\ U_1 &= 6 \times 3 - 10 = 8. \end{aligned}$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is $(m_0 + 3)$.

8. When $r_0 = 7$,

$$\begin{aligned} N_{3,1} &= \pi_3(m_0 + 7 - 6) \\ &= \pi_3(m_0 + 1), \\ U_1 &= 7 \times 3 - 20 = 1. \end{aligned}$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is $(m_0 + 1)$.

9. When $r_0 = 8$,

$$\begin{aligned} N_{3,1} &= \pi_3(m_0 + 8 - 6) \\ &= \pi_3(m_0 + 2), \\ U_1 &= 8 \times 3 - 20 = 4. \end{aligned}$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is $(m_0 + 2)$.

10. When $r_0 = 9$,

$$\begin{aligned} N_{3,1} &= \pi_3(m_0 + 9 - 6) \\ &= \pi_3(m_0 + 3), \\ U_1 &= 9 \times 3 - 20 = 7. \end{aligned}$$

Hence, the quotient of $N_{3,1}$, when it is divided by π_3, is $(m_0 + 3)$.

APPENDIX III

Here is the detailed calculation of part C§3. It is reminded that the *code* for an odd divisor, π_7, is $(-3\tau - 2)$.

1. When $r_0 = 0$,

$$\begin{aligned} U &= 0 \times 7 = 0, \\ N_{7,1} &= T + (-3\tau - 2)U \\ &= (10\tau m_0 + 7m_0 + (0)\tau) + (-3\tau - 2)(0) \\ &= 10\tau m_0 + 7m_0 \\ &= (10\tau + 7)m_0. \end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is m_0.

2. When $r_0 = 1$,

$$\begin{aligned} U &= 1 \times 7 = 7, \\ N_{7,1} &= T + (-3\tau - 2)U \\ &= (10\tau m_0 + 7m_0 + (1)\tau) + (-3\tau - 2)(7) \\ &= 10\tau m_0 + 7m_0 + \tau - 21\tau - 14 \\ &= 10\tau m_0 + 7m_0 - 20\tau - 14 \\ &= (10\tau + 7)(m_0 - 2). \end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is $(m_0 - 2)$.

3. When $r_0 = 2$,

$$\begin{aligned} U &= 2 \times 7 - 10 = 4, \\ N_{7,1} &= T + (-3\tau - 2)U \\ &= (10\tau m_0 + 7m_0 + (2)\tau + 1) + (-3\tau - 2)(4) \\ &= 10\tau m_0 + 7m_0 + 2\tau + 1 - 12\tau - 8 \\ &= 10\tau m_0 + 7m_0 - 10\tau - 7 \\ &= (10\tau + 7)(m_0 - 1). \end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is $(m_0 - 1)$.

4. When $r_0 = 3$,

$$\begin{aligned} U &= 3 \times 7 - 20 = 1, \\ N_{7,1} &= T + (-3\tau - 2)U \\ &= (10\tau m_0 + 7m_0 + (3)\tau + 2) + (-3\tau - 2)(1) \\ &= 10\tau m_0 + 7m_0 + 3\tau + 2 - 3\tau - 2 \\ &= 10\tau m_0 + 7m_0 \\ &= (10\tau + 7)m_0. \end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is m_0.

5. When $r_0 = 4$,

$$\begin{aligned} U &= 4 \times 7 - 20 = 8, \\ N_{7,1} &= T + (-3\tau - 2)U \\ &= (10\tau m_0 + 7m_0 + (4)\tau + 2) + (-3\tau - 2)(8) \\ &= 10\tau m_0 + 7m_0 + 4\tau + 2 - 24\tau - 16 \\ &= 10\tau m_0 + 7m_0 - 20\tau - 14 \\ &= (10\tau + 7)(m_0 - 2). \end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is $(m_0 - 2)$.

6. When $r_0 = 5$,

$$\begin{aligned} U &= 5 \times 7 - 30 = 5, \\ N_{7,1} &= T + (-3\tau - 2)U \\ &= (10\tau m_0 + 7m_0 + (5)\tau + 3) + (-3\tau - 2)(5) \\ &= 10\tau m_0 + 7m_0 + 5\tau + 3 - 15\tau - 10 \\ &= 10\tau m_0 + 7m_0 - 10\tau - 7 \\ &= (10\tau + 7)(m_0 - 1). \end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is $(m_0 - 1)$.

7. When $r_0 = 6$,

$$\begin{aligned} U &= 6 \times 7 - 30 = 2, \\ N_{7,1} &= T + (-3\tau - 2)U \\ &= (10\tau m_0 + 7m_0 + (6)\tau + 4) + (-3\tau - 2)(2) \\ &= 10\tau m_0 + 7m_0 + 6\tau + 4 - 6\tau - 4 \\ &= 10\tau m_0 + 7m_0 \\ &= (10\tau + 7)m_0. \end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is m_0.

8. When $r_0 = 7$,

$$\begin{aligned}
U &= 7 \times 7 - 40 = 9, \\
N_{7,1} &= T + (-3\tau - 2)U \\
&= (10\tau m_0 + 7m_0 + (7)\tau + 4) + (-3\tau - 2)(9) \\
&= 10\tau m_0 + 7m_0 + 7\tau + 4 - 27\tau - 18 \\
&= 10\tau m_0 + 7m_0 - 20\tau - 14 \\
&= (10\tau + 7)(m_0 - 2).
\end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is $(m_0 - 2)$.

9. When $r_0 = 8$,

$$\begin{aligned}
U &= 8 \times 7 - 50 = 6, \\
N_{7,1} &= T + (-3\tau - 2)U \\
&= (10\tau m_0 + 7m_0 + (8)\tau + 5) + (-3\tau - 2)(6) \\
&= 10\tau m_0 + 7m_0 + 8\tau + 5 - 18\tau - 12 \\
&= 10\tau m_0 + 7m_0 - 10\tau - 7 \\
&= (10\tau + 7)(m_0 - 1).
\end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is $(m_0 - 1)$.

10. When $r_0 = 9$,

$$\begin{aligned}
U &= 9 \times 7 - 60 = 3, \\
N_{7,1} &= T + (-3\tau - 2)U \\
&= (10\tau m_0 + 7m_0 + (9)\tau + 6) + (-3\tau - 2)(3) \\
&= 10\tau m_0 + 7m_0 + 9\tau + 6 - 9\tau - 6 \\
&= 10\tau m_0 + 7m_0 \\
&= (10\tau + 7)m_0.
\end{aligned}$$

Hence, the quotient of $N_{7,1}$, when it is divided by π_7, is m_0.

REFERENCES

[1] 陈文胜(2011), 《用割尾法判定百以内的质数整除》, http://blog.sina.com.cn/s/blog_4bdc6abb0100q4px.html .
[2] 徐剑(2003), 《整除判断专题学习网》, http://xujian.jdjy.cn/zhuti/zheng/index.htm .
[3] 汪昌隆, 《各種質數的倍判別法》, 各種質數的倍判別法(電子版), 育達學報, 22, 55 - 64 。
[4] P. M. Cohn, *Elements of Linear Algebra*, Chapman and Hall/CRC, 1994.
[5] A. Dunkel, *A Generalised Test for Divisibility*, The Mathematical Gazette **84** (2000) No. 499, pp. 79-81.

[6] M. Humphreys and N. Macharia, *Tests for Divisibility by 19. The Mathematical Gazette* **82** (1998) No. 495, pp. 475-477.
[7] A. Weil, *Basic Number Theory*, 3rd edition, Springer-Verlag, New York-Berlin, 1974.

Reviewer's Comments

General Comments

The flow of this report was well. The report first explained the motivation, and then stated the aim of each section clearly. The linkage between sections was also found. The organization of the report was good, but one needs some clarifications at certain parts.

The title of report is "A General Formula To Check Divisibility By All Odd Divisors And Its Extensions". What is that formula? The writing of abstract was not very well, since it was tedious. One may mention mainly two things: the divisibility check discussed in the report and the method of finding quotient. Those involving Parts A, B and C, in the original abstract, should be moved to the introduction.

The beginning was introduction. One suggests that the authors make a clearer picture about the organization of the report, and may add: *The report consists of three parts: Part A, Part B and Part C. In Part A, we develop.... Part B serves a generalization of part A. And part C discusses....* Another suggestion is that the authors should use simple English, and try to avoid using "As students with interests and abilities in mathematics" (second paragraph of the introduction section, line 1), "postulate" (fourth paragraph of the introduction section, line 2) and "methodology" (last paragraph of the introduction section, last line). Below are two rather unclear sentences in the introduction:
p.186, paragraph 3, lines 2 - 4: As 2^r... divisors.
p.186, paragraph 4, lines 3 - 6: As we... digit.

One may expect that the authors write down the definition, theorem and algorithm in a clear way. For example, on p.187, the definitions of U and T appeared in the first paragraph. These definitions were importance throughout the report, so the definitions should be stated in a clear manner. In this paper no main result (theorem/ algorithm) was written down explicitly. The readers may not be careful enough to notice the findings.

The whole report is difficult to follow. The reader may not recognize the aim of each section and sub-section.

The writing of proofs can be improved. A lot of isolated equations in the proofs without explaining the logical relationship between them were found. Don't abuse the footnote. On page 191 §5, the authors should conclude what they found before the first paragraph. One may write down four *codes* simply. On page 194, Point 1 carries too many points. On pages 197 to 207, 'Since $100 \equiv 0 \pmod{f_c}$... place,' this unclear phase appeared several times. In Part C, one suggests the authors recall the definitions of π_1, π_3....

It would be better if the report have a detailed normal citation system. Besides, naming for tables was missed.

For a report of such large number of pages, some typographical and minor mistakes are unavoidable. In the next section some spotted mistakes will be marked.

Mistakes

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. Change 'The results and proofs will be discussed in part B' to 'The aforementioned results and proofs will be presented in Part B'.
3. 'negation of the positive divisors' is used in a strange way.
4. Change 'that divisor' to 'π_7'.
5. 'Again, the ... obtained' is rather unclear.
6. What does 'above' refer?
7. 'closure of the integer set'– what does it mean?
8. 'Considering the ... truncated' is rather unclear.
9. 'Considering the ... truncated' is rather unclear.
10. Change 'odd and even' to 'odd or even'.
11. 'Considering the ... truncated' is rather unclear.
12. Change 'odd and even' to 'odd or even'.
13. Change 'odd and' to 'odd or'.
14. 'Although the ... mathematics' is meaningless.
15. 'The result is... part A §1' rather unclear.
16. Change '$[4, 6]$' and '$[7, 9]$ to '$4, 5, 6$' and '$7, 8, 9$' respectively.
17. The first table is not clear.
18. Change 'one-to-one relationship' to 'one-to-one correspondence'; 'so... is known' is rather unclear.
19. Change 'is $+(3 \times 1) + 4 = +4$' to 'is $+4(= +(3 \times 1) + 4)$'
20. In steps 2 & 4: 'the table' refer to which table?
21. Change 'is $(310 + 2) \times 10 + 7 = 3127$' to 'is $3127(= (310 + 2) \times 10 + 7)$'
22. 'With some... fast,' any scientific evidence?
23. The heading 'CASE IN POINT' is rather unclear.
24. 'To know... found' is rather unclear.
25. 'It is... code' is rather unclear.
26. Change 'By introducing... numbers' to 'We hope that the future direction on the research of prime numbers might be inspired by our method in this report'.
27. 'Nonetheless... past' should be included in the section of reflection.
28. 'Last... he' should be included in Acknowledgment.