Hang Lung Mathematics Awards
Vol. 7 (2016)

Hang Lung Mathematics Awards 2016

Silver Award

On the Iterated Circumcentres Conjecture and its Variants

Team members: Tsz Fung Yu, Tsz Chun Wong, Janice Ling
Teacher: Mr. Ho Fung Lee
School: Pui Ching Middle School

Hang Lung Mathematics Awards
Vol. 7 (2016), pp. 23–61

ON THE ITERATED CIRCUMCENTRES CONJECTURE AND ITS VARIANTS

TEAM MEMBERS
Tsz Fung Yu, Tsz Chun Wong, Janice Ling

TEACHER
Mr. Ho Fung Lee

SCHOOL
Pui Ching Middle School

Abstract. We study the *Iterated Circumcentres Conjecture* proposed by Goddyn in 2007: Let $P_1, P_2, P_3, \ldots$ be a sequence of points in $\mathbb{R}^d$ such that for every $i \geq d+2$ the points $P_{i-1}, P_{i-2}, \ldots, P_{i-d-1}$ are distinct, lie on a unique sphere, and further, P_i is the center of this sphere. If this sequence is periodic, then its period must be $2d+4$. We focus on cases of $d=2$ and $d=3$ and obtain partial results on the conjecture. We also study the sequence and prove its geometrical properties. Furthermore, we propose and look into several variants of the conjecture, namely the *Skipped Iterated Circumcentres Conjecture and the Spherical Iterated Circumcentres conjecture.*

1. Introduction

The problem we are going to investigate is based on a conjecture on iterated circumcentres from Open Problem Garden, proposed by Prof. Luis Goddyn.

Definition 1. *Let $P_1, P_2, P_3, \ldots$ be a sequence of points in $\mathbb{R}^d$. The sequence is called Iterated Circumcentres Sequence(ICS) if it fulfils the following: for every $i \geq d+2$, the points $P_{i-1}, P_{i-2}, \ldots, P_{i-d-1}$ lie on a unique sphere, and further, P_i is the center of this sphere.*

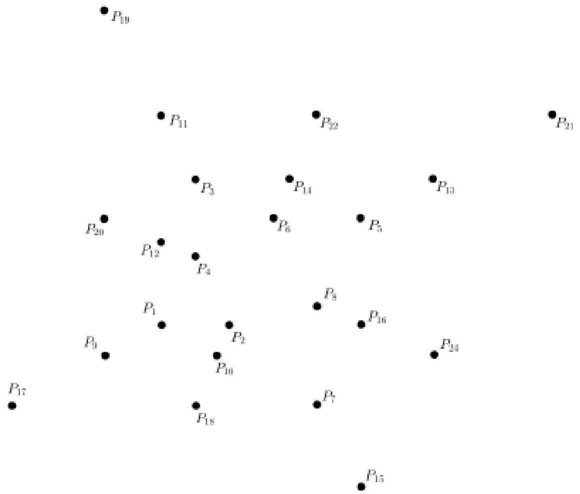


FIGURE 1. An example of ICS $\{P_i\}$ for $i = 1, 2, \ldots, 24$ and $d = 2$.

Conjecture 2. *If the ICS is periodic, its period must be* $2d + 4$.

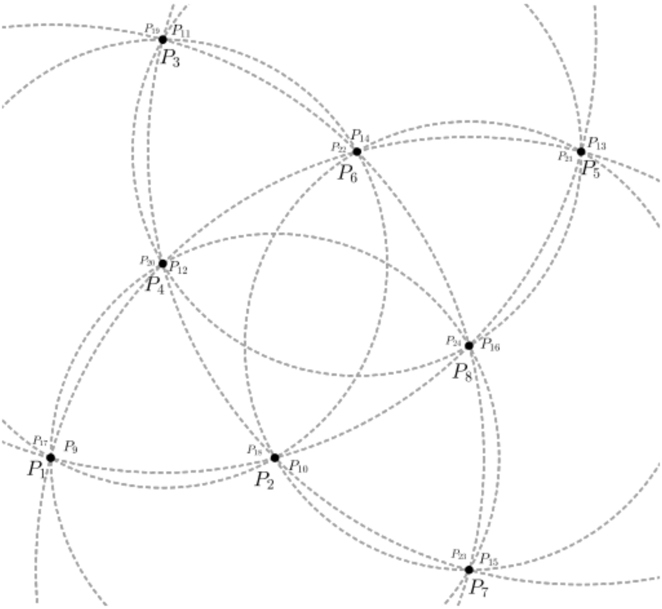


FIGURE 2. An example of a periodic ICS with period 8 and $d = 2$.

At the first glance, the conjecture seems to be quite simple. Yet, the problem has remained unsolved for years. Driven by curiosity, we started to investigate this problem. Other than periodicity, we discovered that when the sequence is not periodic, there are some very interesting phenomena, for example some points happen to be collinear, as shown in Figure 1. As we went deeper, we found that the sequence demonstrates some special geometrical patterns which remain unchanged even after we slightly alter the sequence. Finding this conjecture worth looking into, we decided to work on it.

In our study, we will focus on cases of $d = 2$ and $d = 3$. In Section 2, we focus on plane geometry. We study a Canadian Mathematical Olympiad (CMO) question set up by Prof. Goddyn, who proposed the conjecture, and further investigate the periodicity and geometrical properties of the sequence. We also propose and study some other variants, namely the 2D 1-skipped Iterated Circumcentres Sequence and 2D 2-skipped Iterated Circumcentres Sequence. As for Section 3 and Section 4, we focus on solid geometry and spherical geometry respectively, studying the original sequence and its variants in 3D. In this report, we mainly prove our lemmas and theorems by mathematical means, sometimes with the aid of Maple. [See reviewer's comment (2)]

2. Plane Geometry

2.1. 2D Iterated Circumcentres Sequence

We will first focus on the simplest case, which is $d = 2$. Here, we would like to find out when the sequence is periodic, and when it is periodic, whether its period is $2 \cdot 2 + 4 = 8$, as mentioned in the conjecture of Prof. Goddyn [2].

We define the 2D Iterated Circumcentres Sequence.

Definition 3. *Let $P_1, P_2, P_3, \ldots$ be a sequence of points in $\mathbb{R}^2$. The sequence is called the 2D Iterated Circumcentres Sequence(2D ICS) if P_i is the circumcentre of $\Delta P_{i-1}P_{i-2}P_{i-3}$ for every $i \geq 4$.*

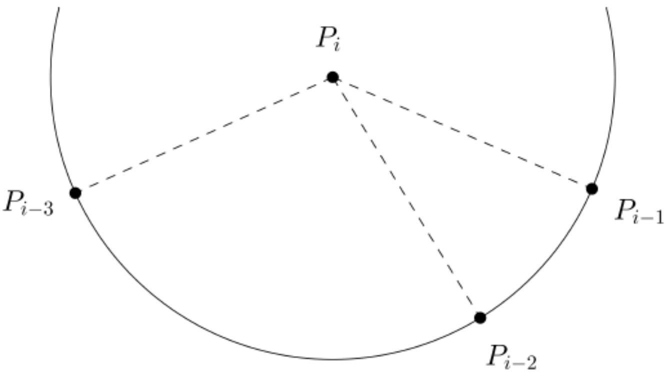


FIGURE 3. P_i is the circumcentre of $\Delta P_{i-1}P_{i-2}P_{i-3}$.

By the above definition of 2D ICS, we found that the sequence may not be well-defined, as shown in Example 4 and Example 5 below:

Example 4. *When P_1, P_2, P_3 are collinear, P_4 is not well-defined.*

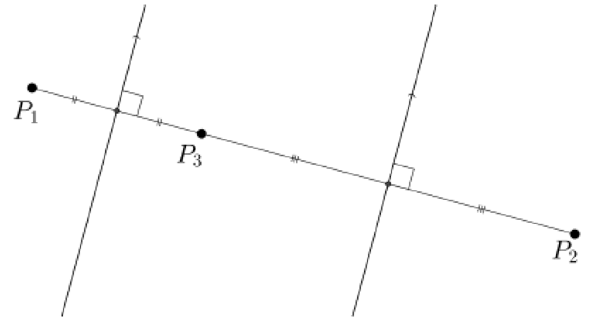


FIGURE 4. P_1, P_2, P_3 are collinear and the perpendicular bisectors of P_1P_3 and P_2P_3 are shown.

Referring to Figure 4, we can see that the point P_4 will not be well-defined. The circumcentre of a triangle is the point where the perpendicular bisectors of the triangle meet, but the perpendicular bisectors in this example are parallel. In other words, they do not intersect each other.

Therefore, when P_1, P_2, P_3 are collinear, P_4 is not well-defined.

Example 5. *When $\angle P_2P_1P_3 = 90°$, P_5 is not well-defined.*

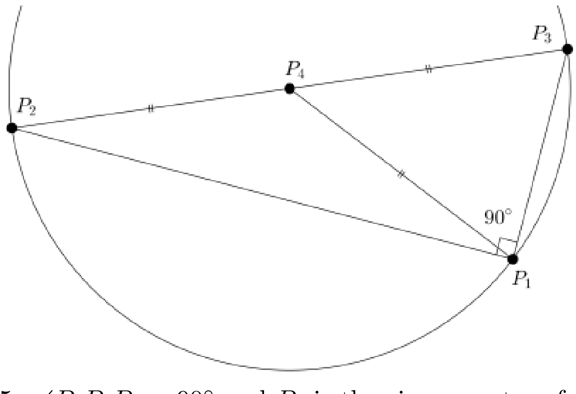


FIGURE 5. $\angle P_2P_1P_3 = 90°$ and P_4 is the circumcentre of $\Delta P_1P_2P_3$

As shown in Figure 5, the points P_2, P_3, P_4 are collinear, which leads to the perpendicular bisectors of these three points not intersecting each other.
The situation above is similar to that in Example 4, which means that when P_1, P_2, P_3 are collinear, P_5 is not well-defined.

As we found that in some cases the 2D ICS is not well-defined, before moving on to the studies of 2D ICS, we would like to first find out the condition such that the 2D ICS must be well-defined.

Theorem 6. *The 2D ICS $\{P_i\}$ is well-defined if P_1, P_2 and P_3 are not collinear and $\angle P_2P_1P_3 \neq 90°$.*

Proof. Let P_i, P_{i+1}, P_{i+2} be some points such that

(i) P_i, P_{i+1}, P_{i+2} are not collinear;
(ii) $\angle P_{i+1}P_iP_{i+2} \neq 90°$.

Suppose that $P_{i+1}, P_{i+2}, P_{i+3}$ are collinear.
As P_{i+3} is the perpendicular bisector of $P_{i+1}P_{i+2}$, the only possible position of P_{i+3} is the mid-point of $P_{i+1}P_{i+2}$. In other words, $P_{i+1}P_{i+3}P_{i+2}$ is a diameter of a circle. However, by $\angle$s is the same segment, $\angle P_{i+1}P_iP_{i+2} = 90°$, which causes contradiction.
$\therefore P_{i+1}, P_{i_2}, P_{i+3}$ are not collinear. [See reviewer's comment (3)]

Suppose that $\angle P_{i+2}P_{i+1}P_{i+3} = 90°$.
As $P_{i+3}P_{i+2}$ and $P_{i+3}P_{i+1}$ are radii of the same circle, $\Delta P_{i+1}P_{i+2}P_{i+3}$ is an isosceles triangle. Therefore,

$$\angle P_{i+1}P_{i+2}P_{i+3} = \angle P_{i+2}P_{i+1}P_{i+3} = 90° \text{ (base } \angle\text{s, isos. } \Delta)$$
$$\angle P_{i+1}P_{i+2}P_{i+3} + \angle P_{i+2}P_{i+1}P_{i+3} + \angle P_{i+1}P_{i+3}P_{i+2} = 180° \text{ } (\angle \text{ sum of } \Delta)$$

By solving the above equations, we have $\angle P_{i+1}P_{i+3}P_{i+2} = 0°$, which means that $P_{i+1}, P_{i+2}, P_{i+3}$ are collinear, thus causing contradiction.
$\therefore \angle P_{i+2}P_{i+1}P_{i+3} \neq 90°$. [See reviewer's comment (3)]

By the principle of mathematical induction, if P_1, P_2, P_3 are not collinear and $\angle P_3P_1P_2 \neq 90°$, for any positive integers i, P_i, P_{i+1}, P_{i+2} are not collinear.
As for any positive integers i, if P_i, P_{i+1}, P_{i+2} are not collinear, there exists an unique circumcentre, which is also the position of P_{i+3}.
As a result, if P_1, P_2, P_3 are not collinear and $\angle P_3P_1P_2 \neq 90°$, the 2D ICS $\{P_i\}$ is well-defined. □

In all the studies from now on, we assume that the 2D ICS is well-defined with the conditions stated in Theorem 6 satisfied.

2.1.1. Proof by Prof. Luis Goddyn

We found a similar problem in Canadian Mathematical Olympiad(CMO) 2001 [1], which is actually also set up by Prof. Goddyn. The question and the solution below has been converted into one-based for readers' convenience.

Question 7 (2001Q5a)**.** *Prove that the points $P_2, P_6, P_{10}, P_{14}, \ldots$ are collinear.*

Solution 8.

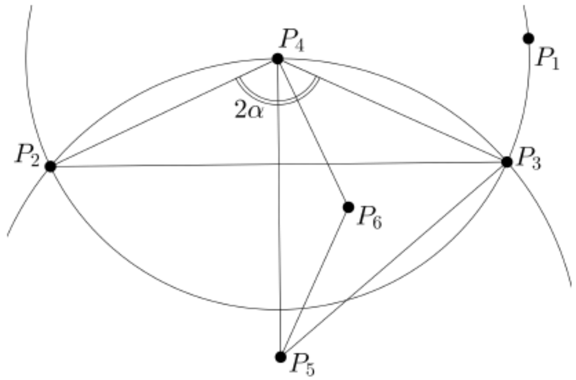


FIGURE 6. An example of 2D ICS $\{P_i\}$ for $i = 1, 2, \ldots, 6$ with $\angle P_2P_4P_3 = 2\alpha$

Let $\angle P_2P_4P_3 = 2\alpha$. As $\Delta P_2P_3P_4$ is isosceles, we have that

$$P_2P_3 = 2\sin\alpha.$$

The line P_4P_5 is the perpendicular bisector of P_2P_3. Since $\Delta P_3P_4P_5$ is isosceles, we can calculate its length,

$$P_4P_5 = \frac{P_3P_4/2}{\cos\alpha} = \frac{1}{2\cos\alpha}.$$

As P_6 is the circumcentre of $\Delta P_3P_4P_5$, we have $\angle P_4P_6P_5 = 2\angle P_4P_3P_5 = 2\angle P_3P_4P_5 = 2\alpha$. The isosceles triangle $\Delta P_4P_5P_6$ is therefore similar to $\Delta P_2P_3P_4$. As $P_4P_5 \perp P_2P_3$, we have $\angle P_2P_4P_6 = 90°$. Furthermore, the ratio $P_4P_6 \colon P_2P_4$ equals r where

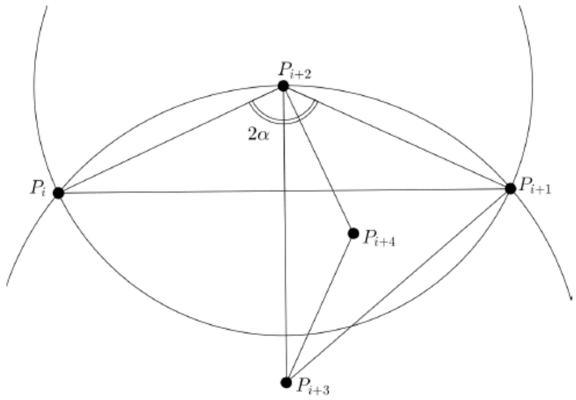
$$r = \frac{P_4P_5}{P_2P_3} = \frac{1}{(2\sin\alpha)(2\cos\alpha)} = \frac{1}{2\sin(2\alpha)}$$

By the same argument, we see that each $\angle P_iP_{i+2}P_{i+4}$ is a right angle with $P_{i+2}P_{i+4} \colon P_iP_{i+2} = r$. Thus the points $P_2, P_4, P_6, \ldots$ lie on a logarithmatic spiral of ratio r and period 4. It follows that $P_2, P_6, P_{10}, \ldots$ are collinear. [See reviewer' comment (4)]

In the following studies of 2D ICS, $\angle P_2P_4P_3 = 2\alpha$ and $r = \frac{1}{2\sin 2\alpha}$.
Also, by $\angle$ at *centre* twice $\angle$ at $\odot^{\text{ce}}$, $\angle P_2P_1P_3 = \alpha$.
Based on the above solution, we proposed the following lemmas:

Lemma 9. *$\angle P_iP_{i+2}P_{i+1} = \angle P_{i+2}P_{i+4}P_{i+3}$ for $i \geq 2$ and* $\angle P_iP_{i+2}P_{i+1} = \begin{cases} 2\alpha, & \text{if } i = 2, 4, 6\ldots \\ 180° - 2\alpha, & \text{if } i = 3, 5, 7\ldots \end{cases}$.

Proof.



Let $\angle P_iP_{i+2}P_{i+1} = 2\theta$, where $i \geq 2$.
As $P_{i+2}P_{i+3}$ is the perpendicular bisector of P_iP_{i+1}, we have that

$$\angle P_{i+1}P_{i+2}P_{i+3} = \theta$$
$$\because P_iP_{i+2} = P_{i+2}P_{i+1}$$
$$\therefore \angle P_{i+2}P_{i+1}P_{i+3} = \theta \text{ and } \angle P_{i+1}P_{i+3}P_{i+2} = 180^\circ - 2\theta$$

As $P_{i+3}P_{i+4}$ is the perpendicular bisector of $P_{i+1}P_{i+2}$, we have that

$$\angle P_{i+2}P_{i+3}P_{i+4} = 90^\circ - \theta$$
$$\because P_{i+1}P_{i+3} = P_{i+3}P_{i+2}$$
$$\therefore \angle P_{i+3}P_{i+2}P_{i+4} = 90^\circ - \theta \text{ and } \angle P_{i+2}P_{i+4}P_{i+3} = 2\theta$$

As a result, $\angle P_iP_{i+2}P_{i+1} = \angle P_{i+2}P_{i+4}P_{i+3} = 2\theta$ for integers $i \geq 2$.
By the principle of mathematical induction and substituting $2\theta = \angle P_2P_4P_3 = 2\alpha$, we have that $P_iP_{i+2}P_i = 2\alpha$ for even integers $i \geq 2$.

Similarly, by the principle of mathematical induction and substituting $2\theta = \angle P_3P_5P_4 = 180^\circ - 2\alpha$, we have that $P_iP_{i+2}P_{i+1} = 180^\circ - 2\alpha$ for odd integers $i \geq 3$.

$$\therefore \angle P_iP_{i+2}P_{i+1} = \begin{cases} 2\alpha, & \text{if } i = 2, 4, 6 \ldots \\ 180^\circ - 2\alpha, & \text{if } i = 3, 5, 7 \ldots \end{cases}.$$

□

Lemma 10. $\Delta P_iP_{i+1}P_{i+2} \sim \Delta P_{i+2}P_{i+3}P_{i+4}$ *for* $i \geq 2$.

Proof. For $i \geq 2$, by Lemma 9, we have that $\angle P_iP_{i+2}P_{i+1} = \angle P_{i+2}P_{i+4}P_{i+3}$

$$\because P_iP_{i+2} = P_{i+2}P_{i+1} \text{ and } P_{i+2}P_{i+4} = P_{i+4}P_{i+3}$$
$$\therefore \Delta P_iP_{i+1}P_{i+2} \sim \Delta P_{i+2}P_{i+3}P_{i+4}$$

□

Lemma 11. *$\{P_{2k}\}$ and $\{P_{2k+1}\}$ lie on two spirals, where $\angle P_iP_{i+2}P_{i+4} = 90°$.*

Remark 12. *Note that P_1 may not lie on any of the two spirals.*

Lemma 13. *$P_i, P_{i+4}.P_{i+8}, \ldots$ are collinear for $i \geq 2$.*

2.1.2. Periodicity of the 2D ICS

In Lemma 11, we have found out that $P_i, P_{i+2}, P_{i+4}, \ldots$ lie on a logarithmatic spiral. By studying the properties of this spiral, we want to find out when the sequence will be periodic, and what the period of the sequence is.

Lemma 14. *P_iP_{i+4} is perpendicular to $P_{i+2}P_{i+6}$. [See reviewer's comment (5)]*

Proof.

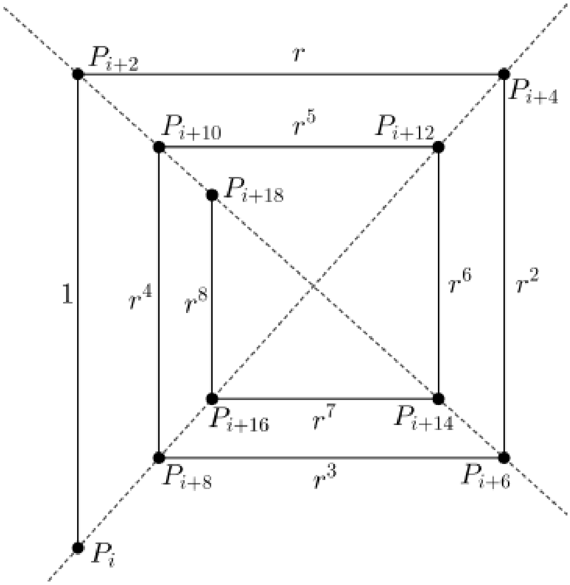


FIGURE 7. $P_i, P_{i+2}, P_{i+4}, \ldots, P_{i+18}$ form a square logarithmatic spiral.

$$\text{slope of } P_2P_6 = \frac{1}{r}, \text{ slope of } P_2P_{10} = \frac{1-r^2}{r-r^3} = \frac{1}{r}, \ldots$$
$$\text{slope of } P_4P_8 = \text{ slope of } P_4P_{12} = \ldots = -r$$

Therefore, the following is true for all positive even numbers i:

$$\text{slope of } P_iP_{i+4} \times \text{ slope of } P_{i+2}P_{i+6} = -1$$

Similarly, we have

$$\text{slope of } P_3P_7 = \text{ slope of } P_3P_{11} = \dots$$
$$\text{slope of } P_5P_9 = \text{ slope of } P_5P_{13} = \dots$$

Therefore, the following is true for any positive integers $i \geq 2$:

$$\text{slope of } P_iP_{i+4} \times \text{ slope of } P_{i+2}P_{i+6} = -1$$

□

Since the two lines are perpendicular to each other, the spiral will become a square if and only if the ratio r equals to 1 or -1. When the spiral becomes a square, the sequence will be periodic with period 8.

Theorem 15. *$P_1, P_2, P_3, \dots$ is periodic if and only if $\angle P_2P_1P_3 = 15°$ or $75°$ or $105°$ or $165°$*

Proof. The sequence is periodic if and only if the spiral becomes a square, and the spiral becomes a square if and only if $r = 1$ or -1.

Recalling that $\angle P_2P_1P_3 = \alpha$ and $r = \dfrac{1}{2\sin 2\alpha}$,

$$r = \frac{1}{2\sin 2\alpha} = 1 \text{ or } -1$$
$$\therefore \angle P_2P_1P_3 = \alpha = 15° \text{ or } 75° \text{ or } 105° \text{ or } 165°$$

□

2.1.3. Point of Convergence

In the previous section, we have studied the cases of $r = 1$ or -1. Here, we focus only on cases of $r < 1$, where the two spirals are converging. [See reviewer's comment (6)]

We want to prove what Prof. Goddyn has omitted, but we found not that trivial - that the two spirals have the same point of convergence, which is the point where the four diagonal lines are concurrent.

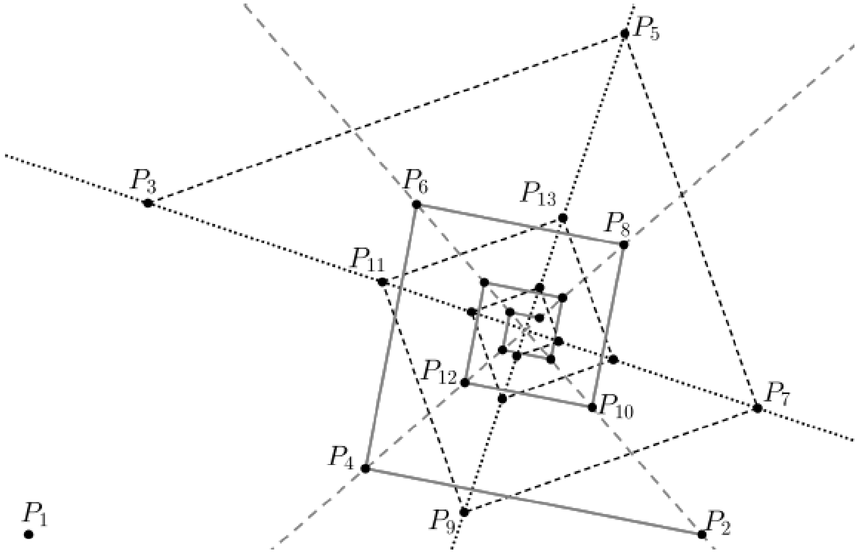


FIGURE 8. The two spirals $P_2, P_4, P_6, \ldots$ and $P_3, P_5, P_7, \ldots$ with first six points each and the four lines: $P_2P_6P_{10}\ldots$, $P_3P_7P_{11}\ldots$, $P_4P_8P_{12}\ldots$, $P_5P_9P_{13}\ldots$.

After plotting the points on a coordinate plane, we observed the patterns as shown in Figure 8. We therefore suspected the following:

(i) The four lines P_2P_6, P_3P_7, P_4P_8 and P_5P_9 are concurrent at a point.
(ii) The two spirals $P_2, P_4, P_6, \ldots$ and $P_3, P_5, P_7, \ldots$ are converging to the point where the four lines are concurrent.

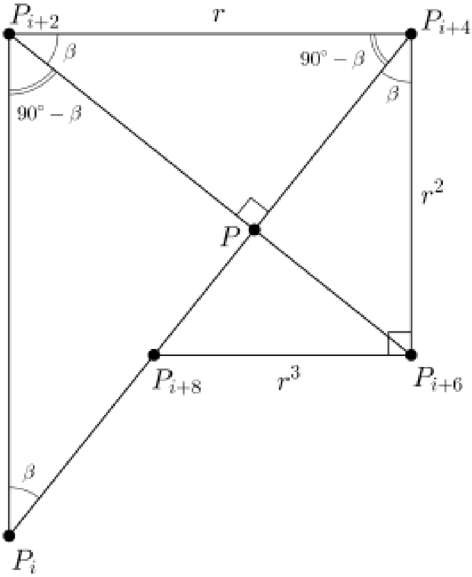
We want to prove that the two statements above are true. Before moving on to the proofs, we would like to first define the points P_{odd} and P_{even}:

Definition 16. *P_{even} is the point of intersection of lines $P_2P_6P_{10}\ldots$ and $P_4P_8P_{12}\ldots$.*

Definition 17. *P_{odd} is the point of intersection of lines $P_3P_7P_{11}\ldots$ and $P_5P_9P_{13}\ldots$.*

Lemma 18. *ΔPP_iP_{i+2} is similar to $\Delta PP_{i+2}P_{i+4}$, where $i \geq 2$ and* $P = \begin{cases} P_{even}, & \text{if } i = 2,4,6\ldots \\ P_{odd}, & \text{if } i = 3,5,7\ldots \end{cases}$.

Proof.



Let $\angle PP_iP_{i+2} = \beta$.
By Lemma 14, $P_iPP_{i+2} = \angle P_{i+2}PP_{i+4} = 90°$

$$\angle P_iP_{i+2}P = 90° - \beta$$

By Lemma 11, $\angle P_iP_{i+2}P_{i+4} = 90°$

$$\angle P_{i+4}P_{i+2}P = \beta$$
$$\angle P_{i+2}P_{i+4}P = 90° - \beta$$
$$\angle P_iP_{i+2}P = \angle P_{i+2}P_{i+4}P = 90° - \beta$$
$$\angle PP_iP_{i+2} = \angle PP_{i+2}P_{i+4} = \beta$$
$$\therefore \Delta PP_iP_{i+2} \sim \Delta PP_{i+2}P_{i+4} \text{ (AAA)}$$

□

Lemma 19. *If $P_i, P_{i+2}, P_{i+4}, \ldots$ lie on the vertices of a square logarithmic spiral, they will converge to P, where $i \geq 2$ and* $P = \begin{cases} P_{even}, & \text{if } i = 2,4,6\ldots \\ P_{odd}, & \text{if } i = 3,5,7\ldots \end{cases}$.

Proof. By Lemma 18, we have

$$\frac{P_{i+2}P}{P_iP} = \frac{P_{i+2}P_{i+4}}{P_iP_{i+2}} = r \text{ and } P_{i+2n}P = r^n \cdot P_iP$$
$$\because \lim_{n\to\infty} r^n = 0 \text{ and } r < 1$$
$$\therefore \lim_{n\to\infty} P_{i+2n}P = 0$$

□

Lemma 20. *$\Delta P_4P_5P_{odd}$ is similar to $\Delta P_8P_9P_{odd}$.*

Proof. By Lemma 10, $\Delta P_3P_4P_5$ is similar to $\Delta P_5P_6P_7$ and $\Delta P_7P_8P_9$.
Therefore, $\angle P_3P_5P_4 = \angle P_5P_7P_6 = \angle P_7P_9P_8$.
By Lemma 18, $\Delta P_{\text{odd}}P_3P_5$ is similar to $P_{\text{odd}}P_7P_9$.

$$\begin{gathered}
\angle P_3P_5P_{\text{odd}} = \angle P_7P_9P_{\text{odd}} \ \text{ (corr. } \angle\text{s, } \sim \Delta\text{s)} \\
\angle P_3P_5P_{\text{odd}} - \angle P_3P_5P_4 = \angle P_7P_9P_{\text{odd}} - \angle P_7P_9P_8 \\
\angle P_4P_5P_{\text{odd}} = \angle P_8P_9P_{\text{odd}} \\
P_3P_5 = P_4P_5 \ \text{ (radii)} \\
P_7P_9 = P_8P_9 \ \text{ (radii)} \\
\frac{PP_5}{PP_9} = \frac{P_3P_5}{P_7P_9} = \frac{P_4P_5}{P_8P_9} \ \text{ (corr. sides, } \sim \Delta\text{s)} \\
\Delta P_4P_5P_{\text{odd}} \sim \Delta P_8P_9P_{\text{odd}} \ \text{ (ratio of 2 sides, inc. } \angle) \\
\therefore \Delta P_4P_5P_{\text{odd}} \sim \Delta P_8P_9P_{\text{odd}}
\end{gathered}$$

□

Lemma 21. *P_4, P_{odd} and P_8 are collinear.*

Proof. By Lemma 20, $\Delta P_4P_5P_{\text{odd}}$ is similar to $\Delta P_8P_9P_{\text{odd}}$.

$$\angle P_4P_{\text{odd}}P_5 = \angle P_8P_{\text{odd}}P_9 \ \text{ (corr. } \angle\text{s, } \sim \Delta\text{s)}$$

Considering line $P_5P_{\text{odd}}P_9$,

$$\begin{gathered}
\angle P_4P_{\text{odd}}P_5 + \angle P_4P_{\text{odd}}P_9 = 180^\circ \\
\angle P_4P_{\text{odd}}P_9 + \angle P_4P_{\text{odd}}P_9 = 180^\circ \\
\therefore P_4P_{\text{odd}}P_8 \text{ is a straight line. (Converse of adj.}\angle\text{s on st.line)}
\end{gathered}$$

□

Theorem 22. *The two spirals $P_i, P_{i+2}, P_{i+4}, \ldots$ and $P_{i+1}, P_{i+3}, P_{i+5}, \ldots$ have the same point of convergence, which is the point where the four diagonal lines P_2P_6, P_3P_7, P_4P_8 and P_5P_9 are concurrent.*

Proof. By Lemma 21, P_4P_8 is concurrent with P_3P_7 and P_5P_9 at P_{odd}. Similarly, P_6P_{10} is also concurrent with P_3P_7 and P_5P_9 at P_{odd}.
By Lemma 13, P_i, P_{i+4}, P_{i+8} are collinear for $i \geq 2$.
Therefore, $P_iP_{i+4}, P_{i+1}P_{i+5}, P_{i+2}P_{i+6}$ and $P_{i+3}P_{i+7}$ are concurrent at P_{odd}.
By definition of P_{even} and P_{odd}, we have $P_{\text{even}} = P_{\text{odd}}$. From now on, we use P to denote the point of intersection of the four diagonal lines.
By Lemma 19, the two spirals both converge to P. □

2.1.4. Divergence

Finally, we will focus on cases of $r > 1$, where the two spirals are diverging. [See reviewer's comment (7)]

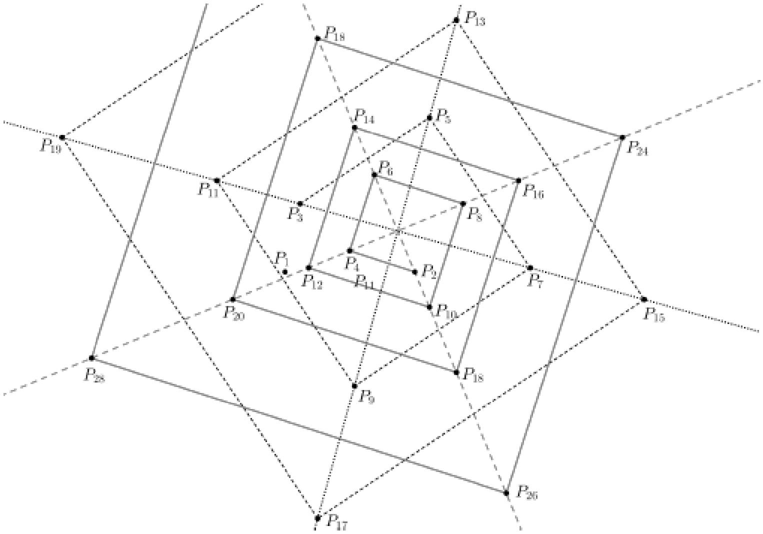


FIGURE 9. The two spirals $P_2, P_4, P_6, \ldots$ and $P_3, P_5, P_7, \ldots$ are diverging.

2.1.5. Illustrative Example

To better present what we have proved, we will give an illustrative example.
In the example, we have $P_1(-1, 0), P_1(1, 0)$ and $P_1(0.375, 0.5)$.
By definition of α. we have

$$\alpha = \angle P_2P_1P_3 \approx 19.98311^\circ$$

$$r = \frac{1}{2\sin 2\alpha} = \frac{1}{2\sin(2 \cdot 19.98311^\circ)} \approx 0.77840$$

As $r \approx 0.77840 < 1$, it is expected that the two spirals will converge to the point P, where P is the point of intersection of the four diagonal lines P_2P_6, P_3P_7, P_4P_8 and P_5P_9. The figure below also matches our expectation.

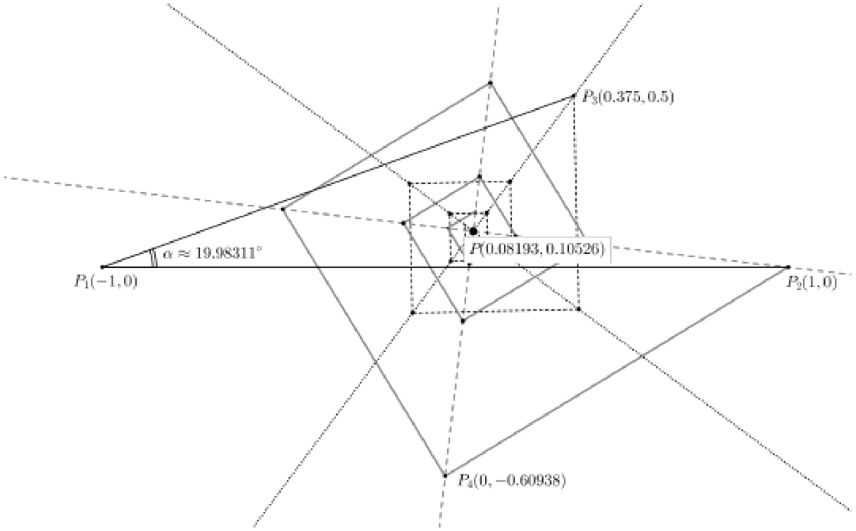


FIGURE 10. The example of the 2D ICS, with $P_1(-1,0), P_1(1,0)$ and $P_1(0.375, 0.5)$

2.2. 2D 1-skipped Iterated Circumcentres Sequence

After studying the 2D ICS, we try to make some changes to the original sequence while maintaining its interesting geometrical properties. We came up with the 2D 1-skipped Iterated Circumcentres Sequence.

Definition 23. *Let $P_1, P_2, P_3, \ldots$ be a sequence of points in $\mathbb{R}^2$. The sequence is called the 2D 1-skipped Iterated Circumcentres Sequence (2D 1-skipped ICS) if P_i is the circumcentre of $\Delta P_{i-2}P_{i-3}P_{i-4}$ for every $i \geq 5$.*

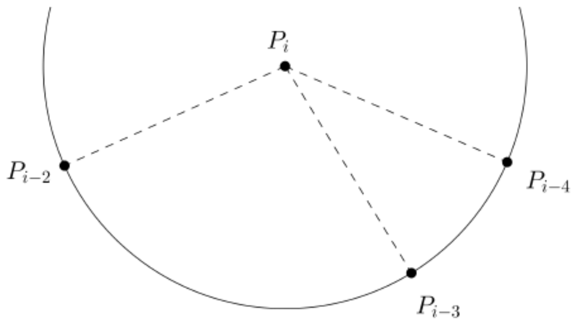


FIGURE 11. P_i is the circumcentre of $\Delta P_{i-2}P_{i-3}P_{i-4}$

We suspected that the 2D 1-skipped ICS may have similar properties with the 2D ICS which we have studied in the previous section. Therefore, we would like to start with investigating the periodicity of this sequence and find out whether they are really similar

Inspired by Prof. Goddyn [3], we found that it may be easier for us to study the properties of 2D 1-skipped ICS if we start with a special case.

Definition 24. *Let $P_1, P_2, P_3, \ldots$ be a sequence of points in $\mathbb{R}^2$. The sequence is called the Special Case of 2D 1-skipped ICS if P_4 lies on the perpendicular bisector of P_1P_2 and P_i is the circumcentre of $\Delta P_{i-2}P_{i-3}P_{i-4}$ for every $i \geq 5$.*

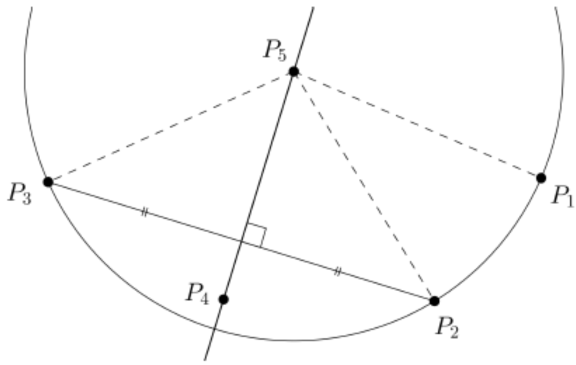


FIGURE 12. An example of the Special Case of 2D 1-skipped ICS.

[See reviewer's comment (8)]
Note that except at the very beginning of the sequence, the special case is actually the same as the general case.

Theorem 25. *The Special Case of 2D 1-skipped ICS $\{P_i\}$ is well-defined if the following conditions are satisfied:*

(a) P_1, P_2, P_3, P_4 are not concyclic;
(b) P_1, P_2, P_3 are not collinear;
(c) P_1, P_2, P_4 are not collinear;
(d) P_2, P_3, P_4 are not collinear;
(e) P_3 does not lie on the perpendicular bisector of P_1P_2;
(f) P_4 is not the circumcenter of $\Delta P_1, P_2, P_3$.

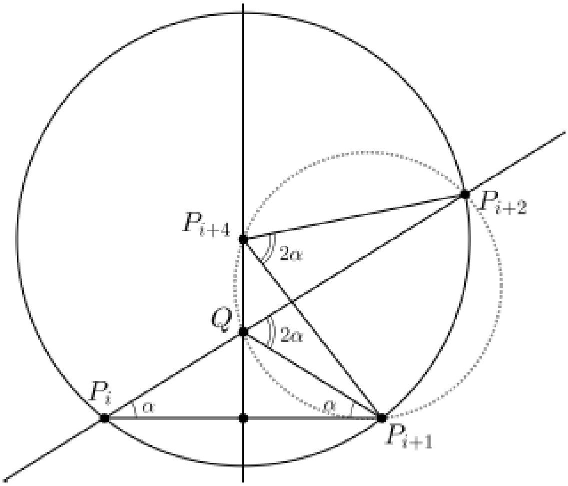
Proof. Let $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ be some points such that

(i) $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ are no concyclic;
(ii) P_i, P_{i+2}, P_{i+3} are not collinear;
(iii) P_i, P_{i+1}, P_{i+2} are not collinear;
(iv) $P_{i+1}, P_{i+2}, P_{i+3}$ are not collinear;

(v) P_{i+2} does not lie on the perpendicular bisector of P_iP_{i+1};
(vi) P_{i+3} is not the circumcenter of $\Delta P_i, P_{i+1}, P_{i+2}$.

Let Q be the point of intersection of P_iP_{i+2} and the perpendicular bisector of P_iP_{i+1}.
Let $\angle P_{i+2}P_{i+4}P_{i+1} = 2\alpha$.

$$\angle P_{i+2}P_iP_{i+1} = \alpha \ \ (\angle \text{ at } \textit{center} \text{ twice } \angle \text{ at } \odot^{\text{ce}})$$
$$\angle P_{i+3}P_iP_{i+1} = \angle P_{i+3}P_{i+1}P_i \ \ (\text{base } \angle\text{s, isos. } \Delta)$$
$$\angle P_{i+2}QP_{i+1} = 2\alpha \ \ (\text{ext. } \angle \text{ of } \Delta)$$
$$= \angle P_{i+2}P_{i+4}P_{i+1}$$
$$\therefore P_{i+1}, P_{i+2}, Q, P_{i+4} \text{ are concyclic. } (\angle\text{s is the same segment})$$



P_{i+3} and P_{i+4} lie on the perpendicular bisector of P_iP_{i+1}, and a line intersects a circle at at most 2 points. Therefore, $P_{i+1}, P_{i+2}, P_{i+3}, P_{i+4}$ are only concyclic if P_{i+3} is at the position of P_{i+4} or Q.

However, by (vi), P_{i+3} is not the circumcenter of $\Delta P_i, P_{i+1}, P_{i+2}$. In other words, it is not coincident with P_{i+4}. Also, by (iii), P_{i+3} does not lie on P_iP_{i+2}, which means it is not coincident with Q as well.
$\therefore P_{i+1}, P_{i+2}, P_{i+3}, P_{i+4}$ are not concyclic.
P_{i+3} and P_{i+4} lie on the perpendicular bisector of P_iP_{i+1}, therefore, $P_{i+1}, P_{i+3}, P_{i+4}$ are only collinear if P_{i+3} and P_{i+4} are coincident, which contradicts (vi).
$\therefore P_{i+1}, P_{i+3}, P_{i+4}$ are not collinear.
By (iv), we have that $P_{i+1}, P_{i+2}, P_{i+3}$ are not collinear.
$\therefore P_{i+1}, P_{i+2}, P_{i+3}$ are not collinear.
P_{i+3} and P_{i+4} lie on the perpendicular bisector of P_iP_{i+1}, and by (v), P_{i+2} does not lie on the perpendicular bisector of P_iP_{i+1}.
$\therefore P_{i+2}, P_{i+3}, P_{i+4}$ are not collinear.

P_{i+3} and P_{i+4} lie on the perpendicular bisector of P_iP_{i+1}, and P_{i+4} also lies on the perpendicular bisector of $P_{i+1}P_{i+2}$.
The perpendicular bisector of P_iP_{i+1} only intersects the perpendicular bisector of $P_{i+1}P_{i+2}$ at 1 point, and the point is P_{i+4}. In other words, P_{i+3} only lies on the perpendicular bisector of $P_{i+1}P_{i+2}$ if it is at the position of P_{i+4}.
However, by (vi), P_{i+3} is not coincident with P_{i+4}.
$\therefore P_{i+3}$ does not lie on the perpendicular bisector of $P_{i+1}P_{i+2}$.
P_{i+4} is the circumcenter of $\Delta P_iP_{i+1}P_{i+2}$, and by (i), $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ are not concyclic.
$\therefore P_{i+4}$ is not the circumcenter of $\Delta P_{i+1}, P_{i+2}, P_{i+3}$.

By the principle of mathematical induction, if initial conditions (a) to (f) are satisfied, then $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ are not concyclic, P_i, P_{i+1}, P_{i+2} are not collinear, P_i, P_{i+2}, P_{i+3} are not collinear, $P_{i+1}, P_{i+2}, P_{i+3}$ are not collinear, P_{i+2} does not lie on the perpendicular bisector of P_iP_{i+1} and P_{i+3} is not the circumcenter of $\Delta P_i, P_{i+1}, P_{i+2}$

As for any position integers i, if P_i, P_{i+1}, P_{i+2} are not collinear, there exists an unique circumcentre, which is also the position of P_{i+4}.

As a result, if the initial conditions of (a) to (f) are satisfied, the Special Case of 2D 1-skipped ICS $\{P_i\}$ is well-defined. $\square$

In all the studies from now on, we assume that the Special Case of 2D 1-skipped ICS is well-defined with the conditions stated in Theorem 25.

For general case of 2D 1-skipped ICS, the situation is similar, except that P_3, P_4, P_5 must not be collinear as well.

i.e. 2D 1-skipped ICS $\{P_i\}$ is well-defined if P_1, P_2, P_3, P_4 are not concyclic, P_1,P_2,P_3 are not collinear, P_1,P_2,P_4 are not collinear, P_2,P_3,P_4 are not collinear, P_3,P_4,P_5 are not collinear, P_3 does not lie on the perpendicular bisector of P_1P_2 and P_4 is not the circumcenter of $\Delta P_1, P_2, P_3$.

In all the studies from now on, we assume that the 2D 1-skipped ICS is well-defined with the conditions stated in Theorem 25 satisfied and P_3, P_4, P_5 are not collinear.

2.2.1. Periodicity of the 2D 1-skipped ICS

By rotation, translation and scaling, we let the initial conditions be

$$P_1(-1,0), P_2(1,0), P_3(a,b), P_4(0,c)$$

By Maple, we could find the coordinates of P_i in terms of a, b, c.[1]
As shown in Figure 13, we observe that $P_i, P_{i+6}, P_{i+12}, \dots$ are collinear. This

[1]The code of the program can be found in the appendix.

resembles the spirals we have come across in the 2D ICS. Therefore, we suspected that the 2D 1-skipped ICS is periodic with period 12.

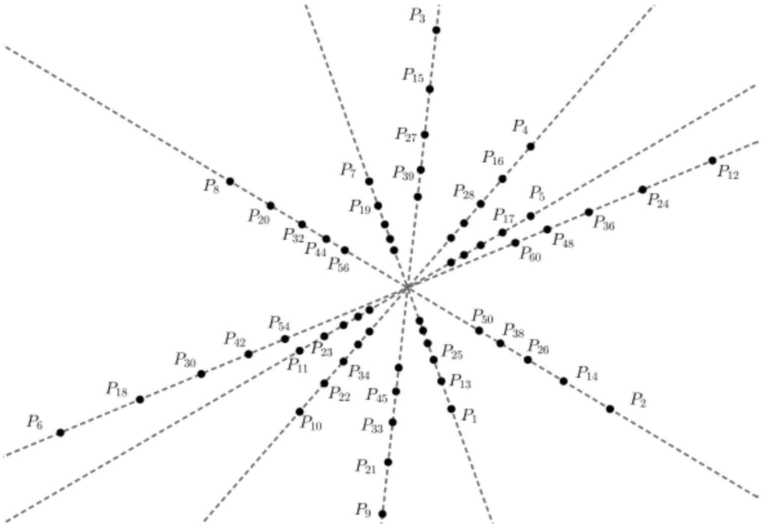


FIGURE 13. $P_i, P_{i+6}, P_{i+12}, \ldots$ are seemingly collinear.

We tried to fix the coordinates of P_1, P_2 and P_3, then translate P_4 along the perpendicular bisector of line P_1P_2. By setting the equation $P_1 = P_{13}$, we could find out the condition which the sequence has to satisfy such that it is periodic with period 12.
From the result of Maple, we obtained Theorem 26:

Theorem 26. *Given initial condition:* $P_1(-1,0), P_2(1,0), P_3(a,b), P_4(0,c)$*, the sequence is periodic if and only if the following equation is satisfied:*

$$c = \frac{b^3+5a^2b-6ab+b \pm \sqrt{7a^2b^4+22a^4b^2-44a^3b^2+22a^2b^2-a^6+4a^5-6a^4+4a^3-a^2}}{a^2+2a+1+b^2}$$

When we are studying 2D ICS, we found out that when P_1 and P_2 are fixed, there must exsist some P_3 such that the sequence is periodic. However, according to the equation above, we found that c may have no real roots. In other words, for some fixed positions of P_3, there may not exist any possible positions of P_4 such that the sequence is periodic with period 12.

Let $\Delta = 7a^2b^4 + 22a^4b^2 - 44a^3b^2 + 22a^2b^2 - a^6 + 4a^5 - 6a^4 + 4a^3 - a^2$

When $\Delta > 0$,
there are 2 possible values for c such that the sequence is periodic.
When $\Delta = 0$,
there is 1 possible value for c such that the sequence is periodic.

When $\Delta < 0$,
there are no possible values for c such that the sequence is periodic.
By completing the square and solving quadratic equation, we found that

$$\Delta > 0 \text{ when } -\sqrt{\frac{8\sqrt{2}-11}{7}} < \frac{b}{a-1} < \sqrt{\frac{8\sqrt{2}-11}{7}};$$

$$\Delta = 0 \text{ when } \frac{b}{a-1} = \pm\sqrt{\frac{8\sqrt{2}-11}{7}}.$$

Lemma 27. *If $\angle P_1P_2P_3 = \tan^{-1}(-\sqrt{\frac{8\sqrt{2}-11}{7}}) \approx 168°$ or $\tan^{-1}\sqrt{\frac{8\sqrt{2}-11}{7}} \approx 11.9°$, there is only 1 possible value for c such that the sequence is periodic.*

Proof. By rearranging $\frac{b}{a-1} = \pm\sqrt{\frac{8\sqrt{2}-11}{7}}$, we have

$$b = \pm\sqrt{\frac{8\sqrt{2}-11}{7}}(a+1) \mp 2\sqrt{\frac{8\sqrt{2}-11}{7}}$$

Since P_1P_2 is parallel to the x-axis, $\angle P_1P_2P_3 = \tan^{-1}(\pm\sqrt{\frac{8\sqrt{2}-11}{7}})$ □

As previously mentioned, the general case is actually as same as the special case except at the very beginning of the sequence.
i.e. P_5 lies on the perpendicular bisector of line P_2P_3.
In other words, all the lemmas and theorems obtained above can be applied to the general case by simply increasing the index i by 1.

2.2.2. Transformation of the Points

After studying the periodicity of the sequence, we moved on to its geometrical properties. Note that we also started with the special case.

By plotting the sequence on a coordinate plane and translating the first four points, we obtained the graphs below.

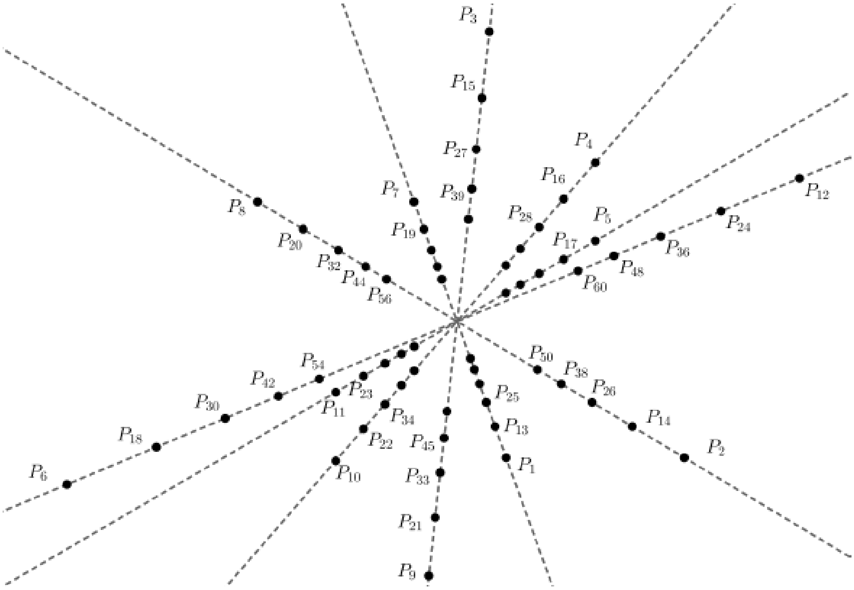


FIGURE 14. All the points seem to converge to a point.

We observed that the points are sometimes converging. In the previous section, the 2D ICS has a point of convergence, so we are curious about whether the points in the 2D 1-skipped ICS will also converge to a point.

Furthermore, we observed that the shapes formed by connecting some 6 points are similar, as indicated by the coloured lines in the following graph.

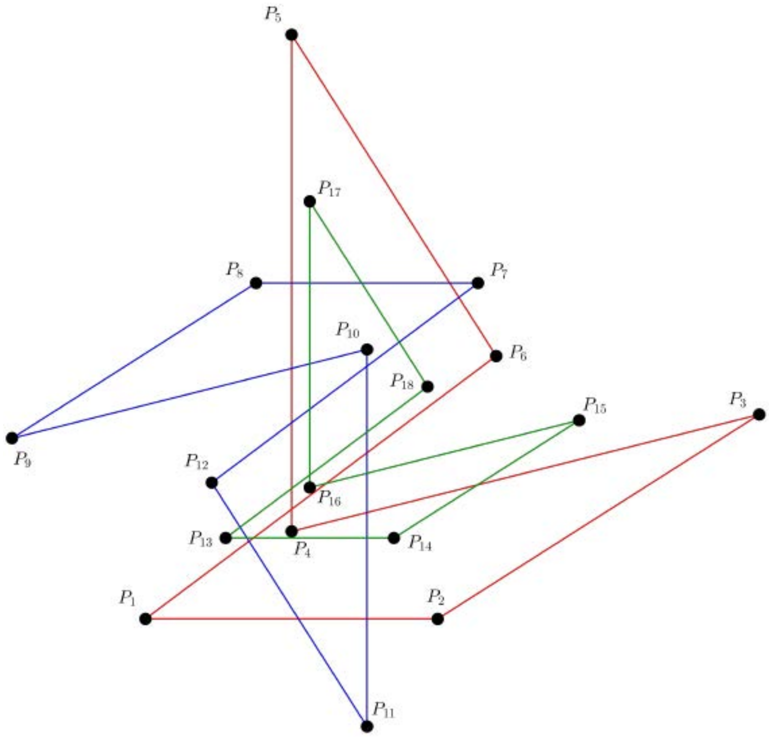


FIGURE 15. An illustration to our observation that the shapes formed by connecting some 6 points are similar.

Please note that the definition of P is different from the last subsection. Here, we define P as follow:

Definition 28. *P is the point of intersection of P_1P_7 and P_2P_8.*

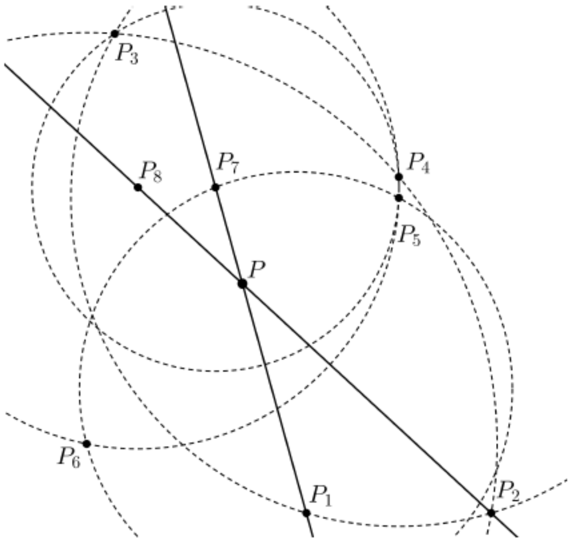


FIGURE 16. An illustration of P is the point of intersection of P_1P_7 and P_2P_8.

Lemma 29. *P_iP_{i+6} are concurrent at P for $1 \leq i \leq 6$.*

Proof. With the help of Maple, we have $PP_i + PP_{i+6} = P_iP_{i+6}$
In other words, P lies on segment P_iP_{i+6}.
Therefore, the six lines P_iP_{i+6} are concurrent for $1 \leq i \leq 6$. □

Lemma 30. *P_iP_{i+1} is perpendicular to $P_{i+3}P_{i+4}$.*

Proof. P_{i+3} is the circumcentre of P_{i-1}, P_i and P_{i+1}, therefore P_{i+3} lies on the perpendicular bisector of P_iP_{i+1}. Similarly, P_{i+4} is the circumcentre of P_i, P_{i+1} and P_{i+2}, therefore P_{i+4} also lies on the perpendicular bisector of P_iP_{i+1}.
$\therefore P_iP_{i+1} \perp P_{i+3}P_{i+4}$. □

Lemma 31. *P_iP_{i+1} is parallel to $P_{i+6}P_{i+7}$*

Proof. By Lemma 30, $P_iP_{i+1} \perp P_{i+3}P_{i+4}$, and $P_{i+3}P_{i+4} \perp P_{i+6}P_{i+7}$.
$\therefore P_iP_{i+1} // P_{i+6}P_{i+7}$ (int.$\angle$s supp.) □

Let r be $\dfrac{P_1P}{P_7P}$.

Lemma 32. *There is a constant r such that $\dfrac{P_iP}{P_{i+6}P} = r$ for $1 \leq i \leq 6$.*

Proof. By Lemma 31, $P_iP_{i+1}//P_{i+6}P_{i+7}$,

$$\Delta PP_iP_{i+1} \sim \Delta PP_{i+6}P_{i+7} \text{ (AAA)}$$
$$\Delta PP_1P_2 \sim \Delta PP_7P_8 \text{ (AAA)}$$
$$\frac{P_1P}{P_7P} = \frac{P_2P}{P_8P} = r$$
$$\Delta PP_2P_3 \sim \Delta PP_8P_9 \text{ (AAA)}$$
$$\frac{P_2P}{P_8P} = \frac{P_3P}{P_9P} = r$$
$$\text{Similarly, } \frac{P_1P}{P_7P} = \frac{P_2P}{P_8P} = \frac{P_3P}{P_9P} = \frac{P_4P}{P_{10}P} = \frac{P_5P}{P_{11}P} = \frac{P_6P}{P_{12}P} = r$$

□

To simplify the problem, we translated the graph to make P the origin.

Let $f(P_1, \ldots, P_6) = -r(P_1, \ldots, P_6)$.

Lemma 33. *$P_7, \ldots, P_{12}$ can be transformed from $P_1, \ldots, P_6$ by rotation by $180°$ and scaling by r.*

Proof.

$$f(P_1, \ldots, P_6) = -r(P_1, \ldots, P_6) = (-rP_1, \ldots, -rP_6)$$
$$\text{By Lemma 32, } (-rP_1, \ldots, -rP_6) = (P_7, \ldots, P_{12})$$
$$\text{Therefore, } f(P_1, \ldots, P_6) = (P_7, \ldots, P_{12})$$

□

Let $P_1', P_2', P_3', \ldots$ be a sequence of points. Define P_i' to be a point transformed from P_i by scaling by $\frac{1}{r}$ and rotation about P by $180°$.

Lemma 34. *P_i and P_{i+6}' are coincident.*

Proof.

$$\text{By definition, } (P_1', P_2', P_3', \ldots) = -r(P_1, P_2, P_3, \ldots)$$

By Lemma 33, $P_7, \ldots, P_{12}$ can be transformed from $P_1, \ldots, P_6$ by rotation about P by $180°$ and scaling by r.

$$\text{In other words, } (P_7, \ldots, P_{12}) = -r(P_1, \ldots, P_6)$$
$$\text{Therefore, } (P_1, \ldots, P_6) = (P_7', \ldots, P_{12}')$$

By definition, $P_1, P_2, P_3, \ldots$ is an 2D 1-skipped ICS. Since $P_1', P_2', P_3', \ldots$ are transformed from $P_1, P_2, P_3, \ldots$ by only scaling and rotation, $P_1', P_2', P_3', \ldots$ is also an 2D 1-skipped ICS.

By Theorem 25, 2D 1-skipped ICS is well-defined, therefore

$$(P_1, P_2, P_3, \ldots) = (P'_7, P'_8, P'_9, \ldots)$$

$\square$

Theorem 35. *$P_{6i+1}, \ldots, P_{6i+6}$ can be transformed from $P_1, \ldots, P_6$ by rotation about P by $(180° \times i)$ and scaling by r^i.*

Proof. Let $S(i)$: $(P_{6i+1}, \ldots, P_{6i+6}) = (-r)^i \cdot (P_1, \ldots, P_6)$.
By Lemma 33, $S(1)$ is true.
Assume $S(k)$ is true for some positive integer k.

i.e. $(P_{6k+1}, \ldots, P_{6k+6}) = (-r)^k \cdot (P_1, \ldots, P_6)$. (*)

$$\begin{aligned}\text{By definition, } (P_{6k+7}, \ldots, P_{6k+12}) &= -r(P'_{6k+7}, \ldots, P'_{6k+12}) \\ &= -r(P_{6k+1}, \ldots, P_{6k+6}) \text{ (By Lemma 34)} \\ &= (-r)^{k+1} \cdot (P_1, \ldots, P_6) \text{ (By (*))}\end{aligned}$$

$\therefore S(k+1)$ is also true.
$\therefore$ By the principle of mathematical induction, $S(i)$ is true for all positive integers i. $\square$

By Theorem 35, Lemma 32 is true for $6k+1 \leq i \leq 6k+6$, where k is any non-negative integers.
Furthermore, we obtain the following properties of 2D 1-skipped ICS:

Corollary 36. *$P_i, P_{i+6}, P_{i+12}, \ldots$ are collinear.*

Proof. By Theorem 35, during the transformation of P_i to P_{i+6}, P_{i+6} is rotated by 180°. Similarly, $P_{i+12}, P_{i+18}, \ldots$ are also rotated by 180°.
By converse of adj. $\angle$s on st. line, $P_i, P_{i+6}, P_{i+12}, \ldots$ are collinear. $\square$

Corollary 37. *$P_1, P_2, P_3, \ldots$ converge to P.*

Proof. By Theorem 35, $P_{6i+k} = P_k r^i$ for any positive integers k where $1 \leq k \leq 6$.

$$\lim_{i \to \infty} r^i = 0$$

$\square$

Corollary 38. *$P_{6n+i}P_{6n+j}$ is parallel to $P_{6n+i+6}P_{6n+j+6}$ and $P_{6n+i+6}P_{6n+j}$ is parallel to $P_{6n+i+12}P_{6n+j+6}$, where $1 \leq i, j \leq 6$ and n is any non-negative integers.*

Proof. By Theorem 35, $P_{6+i}P = r \cdot P_iP$ and $P_{6+j}P = r \cdot P_jP$.

$$\angle P_iPP_j = \angle P_{6+i}PP_{6+j} \quad \text{(vert.opp. } \angle\text{s)}$$
$$\therefore \Delta PP_iP_j \sim \Delta PP_{6+i}P_{6+j} \quad \text{(ratio of 2 sides, inc } \angle\text{)}$$
$$\angle PP_iP_j = \angle PP_{6+i}P_{6+j}$$
$$\therefore P_iP_j // P_{6+i}P_{6+j}$$
$$\text{Similarly, } P_{6+i}P_j // P_{12+i}P_{6_j}$$

□

Corollary 39. *$P_i, P_j, P_{6+i}, P_{6+j}, P_{12+i}, P_{12+j}, \ldots$ form a parallelogram-like spiral with opposite side ratio r, where $1 \leq i, j \leq 6$.*

Proof. By Corollary 38, the opposite lines of the spiral are parallel. By Lemma 32, the points undergo scaling by r for each 6 points. □

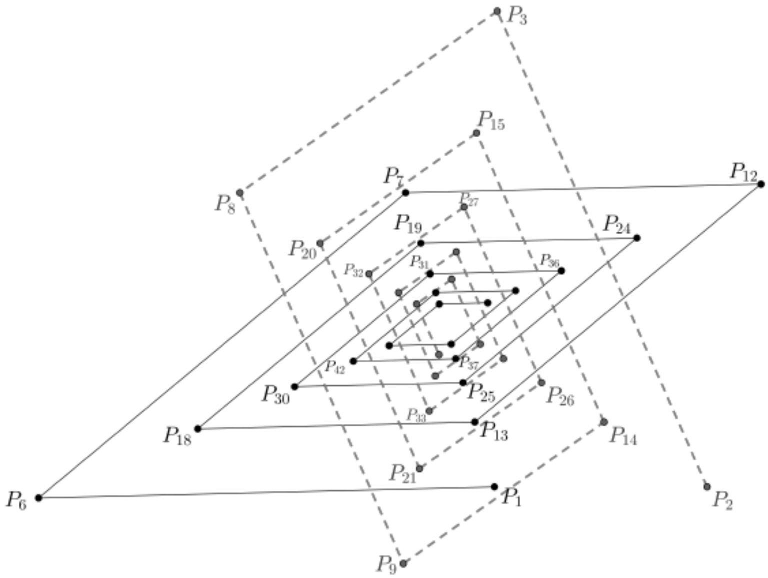


FIGURE 17. The two sprials formed by $P_1, P_6, P_7, P_{12}, \ldots$ and $P_2, P_3, P_8, P_9, \ldots$

Similar to the previous section, all the corollaries, lemmas and theorems above can be applied to the general case by increasing the indices by 1.

3. Solid Geometry

3.1. 3D Iterated Circumcentres Sequence

Definition 40. *Let $P_1, P_2, P_3, \ldots$ be a sequence of points in $\mathbb{R}^3$. The sequence is called the 3D Iterated Circumcentres Sequence(3D ICS) if P_i is the spherical circumcentre of $P_{i-1}, P_{i-2}, P_{i-3}, P_{i-4}$ for every $i \geq 5$.*

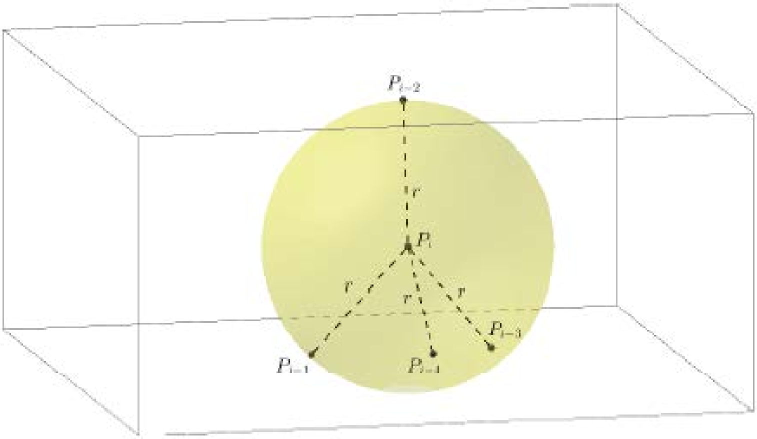


FIGURE 18. P_i is the circumcentre of $P_{i-1}, P_{i-2}, P_{i-3}, P_{i-4}$

Similar to 2D 1-skipped ICS, we will start with a special case.

Definition 41. *Let $P_1, P_2, P_3, \ldots$ be a sequence of points in $\mathbb{R}^3$. The sequence is called the Special Case of 3D ICS if P_3 lies on the plane perpendicularly bisecting P_1P_2, P_4 lies on the line equidistant to P_1, P_2, P_3, and P_i is the spherical circumcentre of $P_{i-1}, P_{i-2}, P_{i-3}, P_{i-4}$ for every $i \geq 5$.*

Theorem 42. *The 3D ICS $\{P_i\}$ is well-defined if the following conditions are satisfied:*

(a) P_1, P_2, P_3, P_4 are not coplanar;
(b) P_2, P_3, P_4, P_5 are not coplanar;
(c) Let C be the circumcenter of $\Delta P_2P_3P_4$ such that $P_1C \neq P_2C$;
(d) Let D be the circumcenter of $\Delta P_3P_4P_5$. $\angle P_3P_5P_4 \neq 90°$ if $P_2D = P_3D$.

Proof. Let $P_i, P_{i+1}, P_{i+2}, P_{i+3}, P_{i+4}$ be points such that

(i) $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ are not coplanar.
(ii) $P_{i+1}, P_{i+2}, P_{i+3}, P_{i+4}$ are not coplanar.
(iii) Let C be the circumcentre of $\Delta P_{i+1}P_{i+2}P_{i+3}$ such that $P_iC \neq P_{i+1}C$

(iv) Let D be the circumcentre of $\Delta P_{i+2}P_{i+3}P_{i+4}$. $\angle P_{i+2}P_{i+4}P_{i+3} \neq 90^\circ$ if $P_{i+1}D = P_{i+2}D$

By (ii), $P_{i+1}, P_{i+2}, P_{i+3}, P_{i+4}$ are not coplanar.

Suppose $P_{i+2}, P_{i+3}, P_{i+4}, P_{i+5}$ are coplanar.
Since $P_{i+3}, P_{i+4}, P_{i+5}$ lie on the plane perpendicular bisecting $P_{i+1}P_{i+2}$, we have $P_{i+5} = P_{i+3}$ or $P_{i+5} = P_{i+4}$, both of which cause contradiction.
$\therefore P_{i+2}, P_{i+3}, P_{i+4}, P_{i+5}$ are not coplanar.

Suppose that C is the circumcentre of $\Delta P_{i+2}P_{i+3}P_{i+4}$ such that $P_{i+1}C = P_{i+2}C$. Then, $C = P_{i+5}$, which makes $P_{i+2}, P_{i+3}, P_{i+4}, P_{i+5}$ coplanar, which causes contradiction.
$\therefore P_{i+1}C \neq P_{i+2}C$.

Suppose that D is the circumcentre of $\Delta P_{i+3}P_{i+4}P_{i+5}$ such that $\angle P_{i+3}P_{i+5}P_{i+4} = 90^\circ$ and $P_{i+2}D = P_{i+3}D$.
Since $P_{i+2}D = P_{i+3}D$, we have $D = P_{i+5}$. So, $P_{i+3}, P_{i+5}, P_{i+4}$ are collinear, which makes $P_{i+2}, P_{i+3}, P_{i+4}, P_{i+5}$ coplanar, which causes contradiction.
$\therefore \angle P_{i+3}P_{i+5}P_{i+4} \neq 90^\circ$ if $P_{i+2}D = P_{i+3}D$.
By the principle of mathematical induction, if initial conditions (a) to (d) are satisfied, $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ are not coplanar for all positive integers i.

For any positive integers i, if $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ are not coplanar, there exists a unique circumcentre which is the position of P_{i+4}

As a result, if the initial conditions (a) to (d) are satisfied, the 3D ICS $\{P_i\}$ is well-defined. □

In all the studies from now on, we assume that the 3D ICS and the Special Case of 3D ICS are well-defined with the conditions stated in Theorem 42 satisfied.

3.1.1. Transformation of the Points

We observe that in this case, P_i, P_{i+5} and P_{i+10} are collinear for $i \geq 1$. In order to prove our observation, we would like to find out some points or segments which can help us figure out the transformation of $P_i, P_{i+1}, P_{i+2}, P_{i+3}, P_{i+4}$ to $P_{i+5}, P_{i+6}, P_{i+7}, P_{i+8}, P_{i+9}$ and onwards. We also observe that the transformation involves rotation by 180° and scaling by r.

Let P be the intersection of P_1P_6 and P_2P_7, and let r be $\dfrac{P_1P}{P_6P}$.

Lemma 43. *P_iP_{i+5} are concurrent for $1 \leq i \leq 5$.*

Proof. With the help of Maple, we found that $PP_i + PP_{i+5} = P_iP_{i+5}$
In other words, P lies on segment P_iP_{i+5}.
Therefore, the five lines P_iP_{i+5} are concurrent for $1 \leq i \leq 5$. □

Lemma 44. *P_iP_{i+1} is parallel to $P_{i+5}P_{i+6}$.*

Proof. P_{i+2}, P_{i+3} and P_{i+4} lies on the perpendicular bisector of P_iP_{i+1}.
Also, P_iP_{i+1} and $P_{i+5}P_{i+6}$ are perpendicular to the plane $P_{i+2}P_{i+3}P_{i+4}$.
By int. $\angle$s supp., $P_iP_{i+1} // P_{i+5}P_{i+6}$. □

Lemma 45. *There is a constant r such that $\dfrac{P_iP}{P_{i+5}P} = r$ for $1 \leq i \leq 5$.*

Proof. By Lemma 44, $P_iP_{i+1} \; // \; P_{i+6}P_{i+7}$,
$\Delta PP_iP_{i+1} \sim \Delta PP_{i+5}P_{i+6}$ (AAA)
$\Delta PP_1P_2 \sim \Delta PP_6P_7$ with side ratio r,
Similar to Lemma 13, Lemma 45 is proved. □

To simplify the problem, we translated the graph to make P the origin.

Let $f(P_1, \ldots, P_5) = -r(P_1, \ldots, P_5)$.

Lemma 46. $f(P_1, \ldots, P_5) = (P_6, \ldots, P_{10})$

Proof.

$$f(P_1, \ldots, P_5) = -r(P_1, \ldots, P_5) = (-rP_1, \ldots, -rP_5)$$

$$\text{By Lemma 45, } (-rP_1, \ldots, -rP_5) = (P_6, \ldots, P_{10})$$

$$\text{Therefore,} f(P_1, \ldots, P_5) = (P_6, \ldots, P_{10})$$

□

Let $P'_1, P'_2, P'_3, \ldots$ be a sequence of points. Define P'_i to be a point transformed from P_i by scaling by $\frac{1}{r}$ and rotation about P by $180°$.

Lemma 47. $(P_1, P_2, P_3, \ldots) = (P'_6, P'_7, P'_8, \ldots)$

Proof.

$$\text{By definition, } (P'_1, P'_2, P'_3, \ldots) = -r(P_1, P_2, P_3, \ldots)$$

By Lemma 45 and Lemma 46, $P_6, \ldots, P_{10}$ can be transformed from $P_1, \ldots, P_5$ by rotation about P by $180°$ and scaling by r.

$$\text{In other word, } (P_6, \ldots, P_{10}) = -r(P_1, \ldots, P_5)$$

$$\text{Therefore, } (P_1, \ldots, P_5) = (P'_6, \ldots, P'_{10})$$

By definition, $P_1, P_2, P_3, \ldots$ is an 3D ICS. Since $P'_1, P'_2, P'_3, \ldots$ are transformed from $P_1, P_2, P_3, \ldots$ by only scaling and rotation, $P'_1, P'_2, P'_3, \ldots$ is also an 3D ICS.

By Theorem 42, 3D ICS is well-defined, so we have

$$(P_1, P_2, P_3, \ldots) = (P'_6, P'_7, P'_8, \ldots)$$

$\square$

Theorem 48. *$P_{5i+1}, \ldots, P_{5i+5}$ can be transformed from $P_1, \ldots, P_5$ by rotation about P by $(180^\circ \times i)$ and scaling by r^i.*

Proof. Let $S(i)$: $(P_{5i+1}, \ldots, P_{5i+5}) = (-r)^i \cdot (P_1, \ldots, P_5)$.
By Lemma 45 and Lemma 46, $S(1)$ is true.
Assume $S(k)$ is true for some positive integer k.
i.e. $(P_{5k+1}, \ldots, P_{5k+10}) = (-r)^k \cdot (P_1, \ldots, P_5)$. (*)

$$\begin{aligned}\text{By definition, } (P_{5k+6}, \ldots, P_{5k+10}) &= -r(P'_{5k+6}, \ldots, P'_{5k+10}) \\ &= -r(P_{5k+1}, \ldots, P_{5k+5}) \text{ (By Lemma 34)} \\ &= (-r)^{k+1} \cdot (P_1, \ldots, P_5) \text{ (By (*))}\end{aligned}$$

$\therefore S(k+1)$ is also true.
$\therefore$ By the principle of mathematical induction, $S(i)$ is true for all positive integers i. $\square$

By Theorem 48, we obtain the following properties of 3D ICS:

Corollary 49. *The points $P_i, P_{i+5}, P_{i+10}, \ldots$ are collinear.*

Proof. By Theorem 48, during the transformation of P_i to P_{i+5}, P_{i+5} is rotated by 180°. Similarly, $P_{i+10}, P_{i+15}, \ldots$ are also rotated by 180°.
By converse of adj. $\angle$s on st. line, $P_i, P_{i+5}, P_{i+10}, \ldots$ are collinear. $\square$

Corollary 50. *P is the point of convergence.*

Proof. By Theorem 48, $P_{5i+k} = P_k \cdot r^i$ for any positive integers k where $1 \leq k \leq 5$.

$$\lim_{i \to \infty} r^i = 0$$

$\square$

4. Further Experiments and Conjectures

In this section, we are going introduce some other variants of the ICS in which we failed to obtain significant result. We have done some numerical experiments so as to study their geometrical patterns.

4.1. 2D 2-skipped Iterated Circumcentres Sequence

4.1.1. Numerical Experiments

After studying the 2D 1-skipped ICS, we would continue with 2D 2-skipped Iterated Circumcentres Sequence.

Definition 51. *Let $P_1, P_2, P_3, \ldots$ be a sequence of points in $\mathbb{R}^2$. The sequence is called the 2D 2-skipped Iterated Circumcentres Sequence (2D 1-skipped ICS) if P_i is the circumcentre of $\Delta P_{i-3}P_{i-4}P_{i-5}$ for every $i \geq 6$.*

From the result we obtained from 2D ICS and 2D 1-skipped ICS, we thought that there will be points that are collinear in 2D 2-skipped ICS, and there will be a point of convergence, which is the intersection of the concurrent lines. However, after doing some numerical experiments, we found out our conjecture is not true. The points seem to be collinear for each 16 points, but by drawing straight lines, we found that the 16 points do not lie on the same straight line. In other words, they are not collinear.

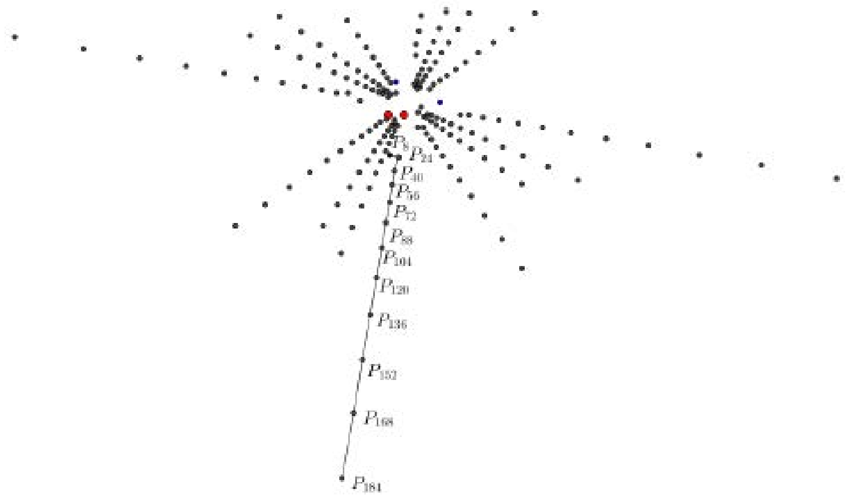


FIGURE 19. 2D 2-skipped ICS where points seem to be collinear.

By connecting each 16 points together in the figure, we see that points with larger indices are closer to being collinear, while points with smaller indices are more messy and do not seem collinear.

Other than points being seemingly collinear, we observed some other interesting patterns

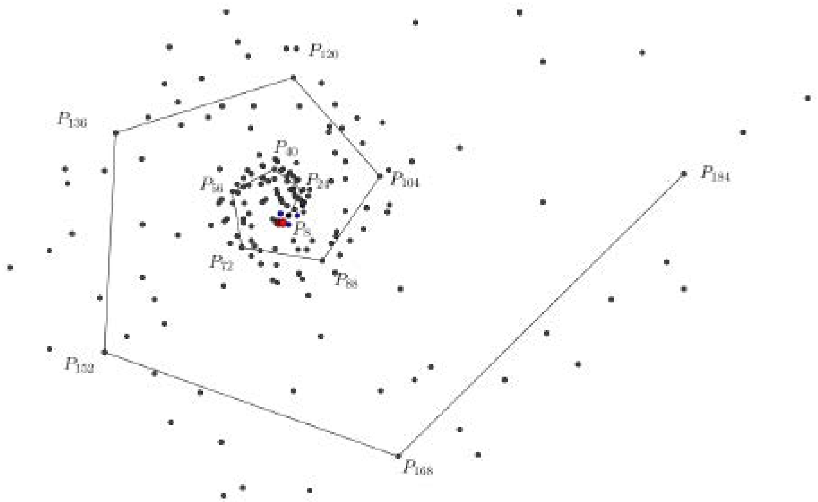


FIGURE 20. 2D 2-skipped ICS where points seems to be in a spiral.

We connected each 16 points together in the figure, and it seems that they are in a spiral. However, the angles of every consecutive 16 points are not the same.

There are also some figures which show a combination of the two properties stated above, but the figures turn out to be quite messy.

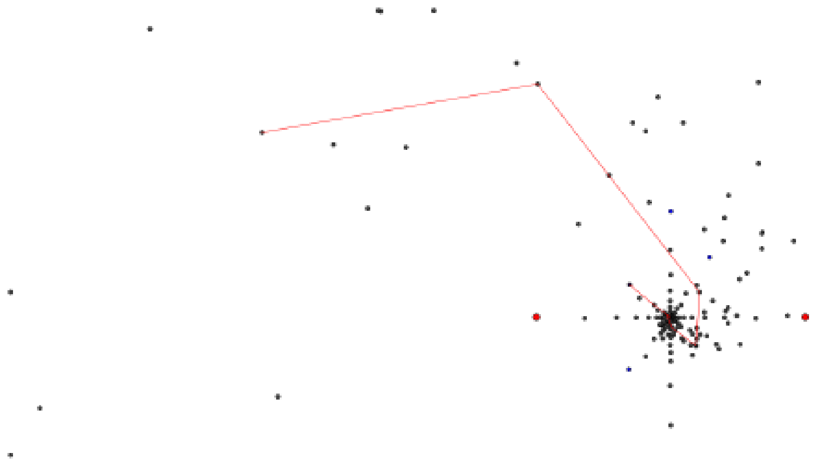


FIGURE 21. 2D 2-skipped ICS when points first seem to be collinear, then turned into a spiral.

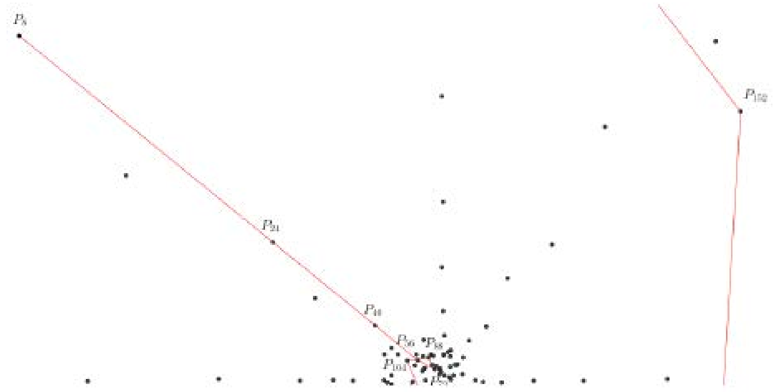


FIGURE 22. An enlarged version of Figure 21

The first figure is an enlarged figure of the second one. We observe that at first, the points seem to be collinear, and then they turned into a spiral.

Despite having discovered this interesting phenomenon, we are unable to prove why the points behave like that.

4.2. 3D 1-skipped Iterated Circumcentres Sequence

We also try to make some changes to the 3D ICS, and come up with the 3D 1-skipped Iterated Circumcentres Sequence.

Definition 52. *Let $P_1, P_2, P_3, \ldots$ be a sequence of points in $\mathbb{R}^3$. The sequence is called the 3D 1-skipped Iterated Circumcentres Sequence(3D 1-skipped ICS) if P_i is the spherical circumcentre of $P_{i-2}, P_{i-3}, P_{i-4}, P_{i-5}$ for every $i \geq 6$.*

4.2.1. Geometrical Patterns

By numerical experiment, we found that 3D 1-skipped ICS has lost the collinear property. We also observe a strange phenomenon - the points are behaving differently with different periods.

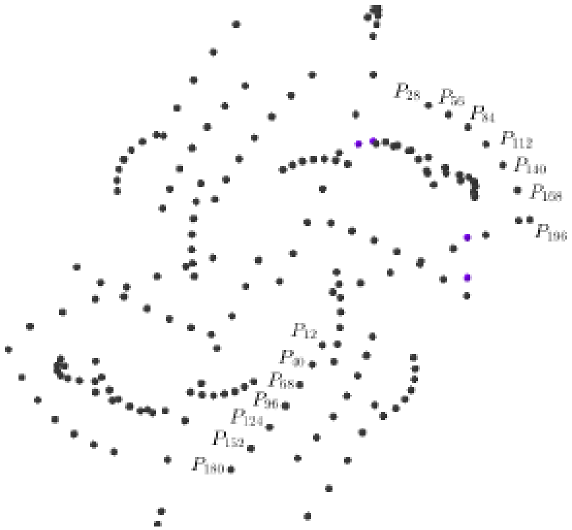


FIGURE 23. 3D 1-skipped ICS with period 28.

Above is an example that each 28 points seem to form a curve.

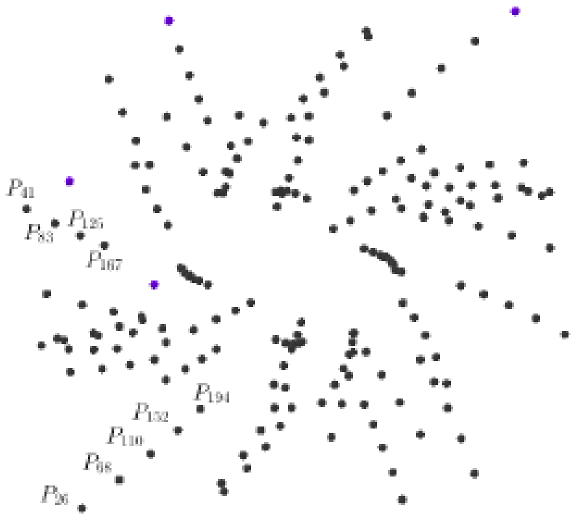


FIGURE 24. 3D 1-skipped ICS with period 42.

Above is an example that each 42 points seem to form a curve.

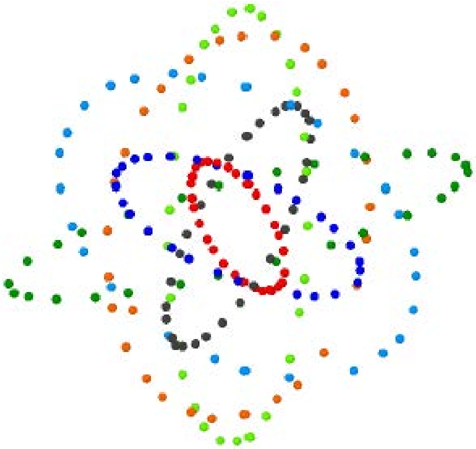


FIGURE 25. 3D 1-skipped ICS with unknown period.

Above is an example that points in the same sequence seem to have formed 7 different eclipses. By painting the points with different colours, We found out that the indices of the points with the same colour always differ by $7n$, where n is any positive integers. e.g. $P_1, P_8, P_{15}, \ldots$ forms the red eclipse, $P_2, P_9, P_{16}, \ldots$ forms the blue eclipse, and onwards.

We are astonished by this miraculous pattern. Unfortunately, we failed to account for the behaviour of the points, and was unable to investigate deeper.

4.3. 3D Spherical Iterated Circumcentres Sequence

Definition 53. *Let $P_1, P_2, P_3, \ldots,$ be a sequence of points in $\mathbb{R}^3$. The sequence is called the 3D Spherical Iterated Circumcentres Sequence (3D Spherical ICS) if $P_1, P_2, P_3, \ldots$ lie on the same sphere and P_i is the circumcentre of $P_{i-1}, P_{i-2}, P_{i-3}$ on the minor cap for every $i \geq 4$.*

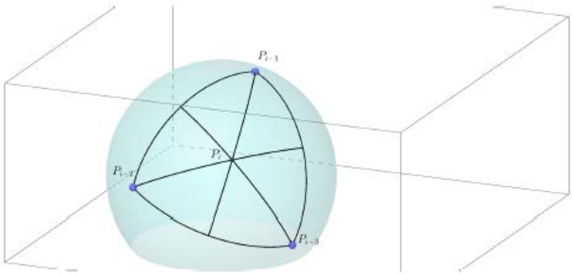


FIGURE 26. P_i is the circumcentre of $P_{i-1}, P_{i-2}, P_{i-3}$ on the minor cap.

4.3.1. Numerical Experiments

We observe that $P_3, P_4, P_7, P_8, P_{11}, P_{12}, \ldots$ are sometimes collinear, and when these points are collinear, the other points also happen to be collinear and are perpendicular to the line formed by linking $P_3, P_4, P_7, P_8, P_{11}, P_{12}, \ldots$

We try to use the way in 2D plane to solve this problem.

Lemma 54. *The theorem base $\angle$s, isos.Δ is applicable in spherical geometry while theorem $\angle$ at centre twice $\angle$ at $\odot^{\text{ce}}$ is not.*

Proof. Theorem $\angle$ sum of Δ is not applicable in spherical geometry. Therefore, theorem $\angle$ at *centre* twice $\angle$ at $\odot^{\text{ce}}$ is not applicable as well. $\square$

This lemma hints that the points may not be collinear.

We would like to investigate whether there are specific points that fulfill $\angle$ at *centre* twice $\angle$ at $\odot^{\text{ce}}$. However, we failed to find those points.

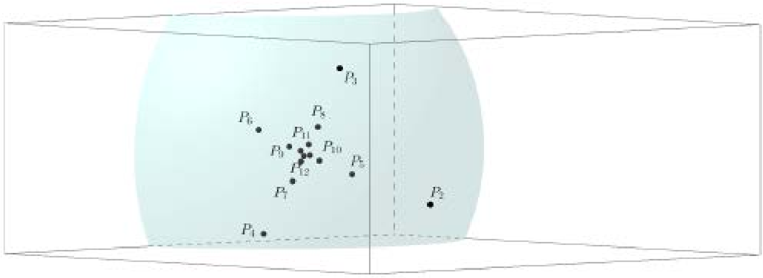


FIGURE 27. The points seems to be collinear.

5. Conclusion

We studied the Iterated Circumcentres Sequence in plane geometry and solid geometry. We also proposed a total of four variants, namely the 2D 1-skipped Iterated Circumcentres Sequence(2D 1-skipped ICS), 2D 2-skipped Iterated Circumcentres Sequence(2D 2-skipped ICS), 3D 1-skipped Iterated Circumcentres Sequence(3D 1-skipped ICS) and 3D Spherical Iterated Circumcentres Sequence(3D Spherical ICS).

For the 2D ICS,we have studied the its periodicity and proven Goddyn's conjecture. We also studied the properties of the sequence when it is converging or diverging. For the 2D 1-skipped ICS, we have studied its periodicity as well. Other than that, we have proven the relationship between each six points. For the 3D ICS, we have proven the relationship between each five points.

For the remained variants, we have done some numerical experiments and found out some interesting geometrical patterns. However, we are unable to account for the behaviours of the points.

Acknowledgement

We would like to show our gratitude to Prof. Luis Goddyn, who have proposed the conjecture on iterated circumcentres. His conjecture has inspired us a lot.
We also thank Mr. Yue Man Chung for assistance and comments which greatly improved our project.

Appendix A.

```
f := proc(n :: nonnegint)
option remember :
if n = 1 then
return -1;
elif n = 2 then
return 0;
elif n = 3 then
return 1;
elif n = 4 then
return 0;
elif n = 5 then
return a;
elif n = 6 then
return b;
elif n = 7 then
return 0;
elif n = 8 then
return c;
else
if n mod 2 = 1 then
A_x := f(n - 8); A_y := f(n - 7); B_x := f(n - 6); B_y := f(n - 5); C_x := f(n - 4); C_y := f(n - 3);
return ((A_x^2 + A_y^2)·(B_y - C_y) + (B_x^2 + B_y^2)·(C_y - A_y) + (C_x^2 + C_y^2)·(A_y - B_y)) / (2(A_x·(B_y - C_y) + B_x·(C_y - A_y) + C_x·(A_y - B_y)));
else
A_x := f(n - 9); A_y := f(n - 8); B_x := f(n - 7); B_y := f(n - 6); C_x := f(n - 5); C_y := f(n - 4);
return ((A_x^2 + A_y^2)·(C_x - B_x) + (B_x^2 + B_y^2)·(A_x - C_x) + (C_x^2 + C_y^2)·(B_x - A_x)) / (2(A_x·(B_y - C_y) + B_x·(C_y - A_y) + C_x·(A_y - B_y)));
end if;
end if;
end proc;
```

FIGURE 28. Program used to find P_i in terms of a, b, c.

$\frac{f(14)-f(2)}{f(13)-f(1)} \xrightarrow{\text{normalize}}$

$$\frac{(a^2+b^2-1+2cb)a}{a^2b+ca^2+4ba+b^3-c-cb^2+b} \quad (2)$$

$\frac{(f(26)-f(2))}{f(25)-f(1)} \xrightarrow{\text{normalize}}$

$$\frac{(a^2+b^2-1+2cb)a}{a^2b+ca^2+4ba+b^3-c-cb^2+b} \quad (3)$$

$f(25)=f(1) \xrightarrow{\text{normalize}}$

$$-\frac{1}{32}\frac{1}{b^2a^2(b-c+ac)}\left(14acb^2+2a^2cb^2+2c^2ab^2-10bc^2a+14bc^2a^2-18c^2ab^3+2bc^2a^2-4a^2b^2c+14b^4ac-9a^4cb^2-9a^4bc^2-15a^2b^4c+10a^2c^2b^2+3c^2b+2ba^2-6ab^2+ba^2-6ba^4-4a^2b^2-6b^4c-2c^2b^2+6c^2b^2-a^2c+2a^2c-3b^2c-2a^2c+2a^2b-4a^2b^2+a^4b+3a^2b^2+a^4c+3a^2b^2-6b^2a-3b^2c+3b^2c^2+b^2+2b^2+b^3-c^2b^4+2c^2a-2c^2a^2+c^2a^4-c^2\right)=-1 \quad (4)$$

$\xrightarrow{\text{isolate for } c}$

$$c=\frac{5ba^2+b^3-6ba+b+\sqrt{22b^2a^4+7a^2b^4-44b^2a^3+22a^2b^2-a^6-6a^4-a^2+4a^3+4a^5}}{1-2a+a^2+b^2} \quad (5)$$

$f(26)=f(2) \xrightarrow{\text{normalize}}$

$$-\frac{1}{32}\frac{1}{b^2a(b-c+ac)}\left(-4c^2b^2+22c^2ab^2-12acb+10a^2cb-4c^2ab-18a^2c^2b^2+2a^2c^2b-8a^2cb^2-8ba^4c-8acb^3+8ba^2c+a^6-a^2-b^2-c^2+2c^2a+2a^2-2a^5+2cb+10ab^2+2b^3c-2c^2a^2+2c^2b+a^4c^2-3a^4c^2+2b^2c^2+3b^2a^4+5a^2b^4-12b^2a^3-10ab^4+b^6\right)=0 \quad (6)$$

$\xrightarrow{\text{isolate for } c}$

$$c=\frac{5ba^2+b^3-6ba+b+\sqrt{22b^2a^4+7a^2b^4-44b^2a^3+22a^2b^2-a^6-6a^4-a^2+4a^3+4a^5}}{1-2a+a^2+b^2} \quad (7)$$

FIGURE 29. Result of the program.

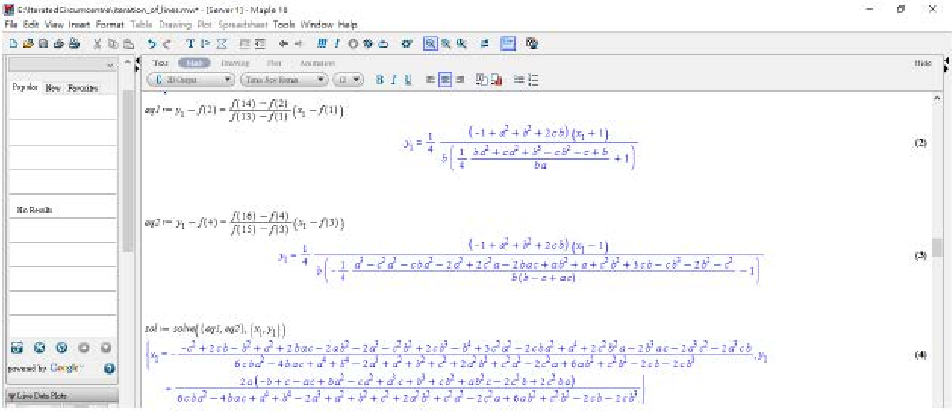
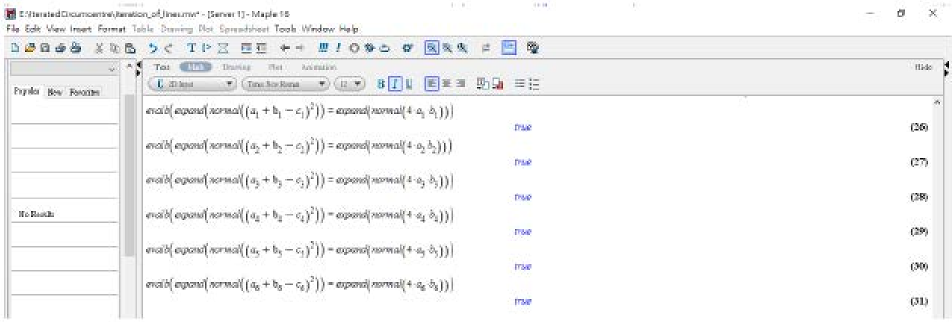


FIGURE 30. By solving equation of P_1P_7 and P_2P_8, we find the coordinates of P.



a_n, b_n, c_n are the square of the length of $P_nP, P_{n+6}P, P_nP_{n+6}$ respectively. We have to prove that $\sqrt{a_n} + \sqrt{b_n} = \sqrt{c_n}$. By rearranging terms, we have $(a_n + b_n - c_n)^2 = 4a_nb_n$. With the aid of Maple, we prove that the above equation is true for $1 \leq n \leq 6$.

For 3D cases, similar code is used.

REFERENCES

[1] Canadian Mathematical Society, *Problems and solution of Canadian Mathematical Olympiad 2001*, http://cms.math.ca/Competitions/CMO/solutions/sol_2001.pdf

[2] Goddyn, Luis A., *A conjecture on iterated circumcentres*, http://www.openproblemgarden.org/op/a_conjecture_on_iterated_circumcentres

[3] Goddyn, Luis A., *Circle Sequence: Making P(0) behave*, http://people.math.sfu.ca/~goddyn/Circles/whyRestriction.html

Reviewer's Comments

The reviewer has some comments about the presentation of this paper, as well as the notations and typos.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. Note that 2D 2-skipped Iterated Circumcentres Sequence is studied in Section 4, NOT in Section 2.
3. It is better to abandon the symbols $\because$ and $\therefore$ in the formal mathematical writing. Please replace them by the corresponding words.
4. It seems that "logarithmatic spiral" should be "logarithmic spiral". This also works in the remainder of the paper.
5. Lemma 14: Here i should be greater than or equals 2.
6. Here "$r < 1$" should be "$|r| < 1$" because α can greater than $90°$ in Theorem 15.
7. Here "$r > 1$" should be "$|r| > 1$".
8. Figure 12: Note that P_4 lies on the perpendicular bisector of P_1P_2 by the definition of Special Case of 2D 1-skipped ICS. Hence Figure 12 is wrong. Please modify it.
9. Since P_1 has more freedom in the definition of 2D 1-skipped ICS (a similar phenomenon is the Remark on Page 30), it seems that the main results involving P_1 in Section 2.2.2 just work for the Special Case of 2D 1-skipped ICS. The same logic also works for Section 3.1.1.