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ON THE NUMBER OF MULTIPLICATIVE TYPE $\mathbf{E}_n$ FRIEZES

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Abstract. As a complementary development of the nature of frieze structures, we partially solve the Fontaine-Plamondon conjecture on the enumeration of multiplicative $\mathbf{E}_n$ friezes through an elementary combinatorial approach. Incorporating with the concept of quiddity sequences, we explicitly show the number of distinct $\mathbf{E}_6$ friezes are 70, with substantial evidence to support the number of $\mathbf{E}_7$ and $\mathbf{E}_8$ friezes to be 440 and 1694, respectively.

Contents

1. Introduction

Coxeter frieze patterns, introduced by H.S.M. Coxeter in 1971, are infinite arrays of numbers satisfying multiplicative recurrence relations, which were discovered to be constructed from convex polygon triangulations. These patterns show the profound connections between geometrical realization, combinatorics, and representation theory. Recently, their incorporation into cluster algebras [FZ01] with interpretations concerning Grassmannians $Gr(k,n)$ and moduli spaces significantly elevated their prominence in algebraic geometry and mathematic physics. For interested readers, please refer to [CC73][Mor15].

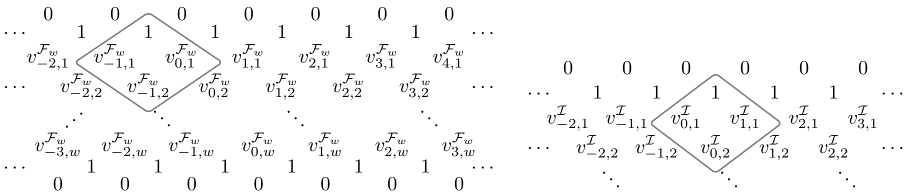
In this paper, we employ three classes of frieze patterns, namely closed friezes [Cox71], open friezes [BPT15] and multiplicative $\mathbf{E}_n$ friezes.

Definition 1.1 (Closed and open friezes)**.** *A closed frieze $\mathcal{F}_w$ of positive integers with a finite width w (number of non-trivial rows) defines an array $(v_{i,j}^{\mathcal{F}_w})_{i,j\in\mathbb{Z},-1\leq j\leq w+2}$ of shifted infinite rows such that $v_{i,-1}^{\mathcal{F}_w} = v_{i,w+2}^{\mathcal{F}_w} = 0$, $v_{i,0}^{\mathcal{F}_w} = v_{i,w+1}^{\mathcal{F}_w} = 1$ for all i, satisfying the diamond rule for all combinations of entries:*

$$v_{i,j}^{\mathcal{F}_w} v_{i+1,j}^{\mathcal{F}_w} - v_{i+1,j-1}^{\mathcal{F}_w} v_{i,j+1}^{\mathcal{F}_w} = 1$$

Similarly, an open frieze $\mathcal{I}$ of positive integers defines an array $\left(v_{i,j}^{\mathcal{I}}\right)_{i,j\in\mathbb{Z},j\geq-1}$ of shifted infinite rows, preserving the diamond rule of $\mathcal{F}_w$ with an infinite amount of rows, in absence of the rows of 1s and 0s bounded from below.

Definition 1.2 (Multiplicative $\mathbf{E}_n$ friezes)**.** *A multiplicative type $\mathbf{E}_n$ frieze of positive integers (denoted as $\mathcal{M}$) defines an alternative array $\left(v_{i,j}^{\mathcal{M}}\right)_{i,j\in\mathbb{Z},1\leq j\leq n}$ of shifted*



Closed frieze $\mathcal{F}_w$ with width w Open frieze $\mathcal{I}$

FIGURE 1. Visualization of closed and open friezes respectively, with a diamond marked respectively

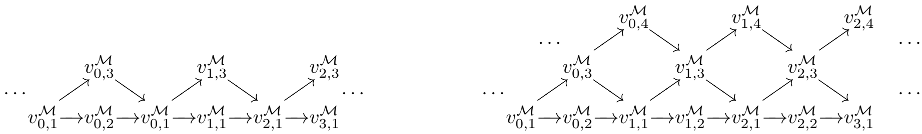
infinite rows, with $3 \leq n \leq 8$*, satisfying the frieze rule relation of:*

$$v_{i,j}^{\mathcal{M}} v_{i+1,j}^{\mathcal{M}} = \prod_{v \in V} v + 1$$

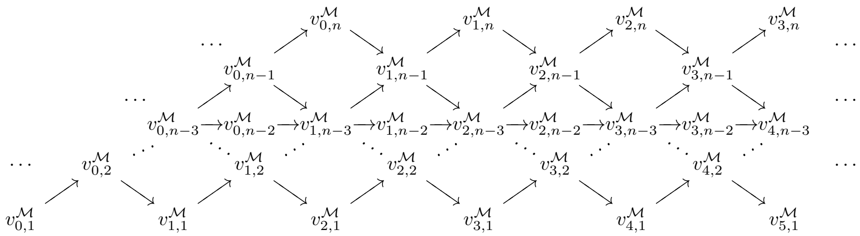
where V *is a set of entries of* $\mathcal{M}$ *satisfying*

$$V = \left\{ v_{h,k}^{\mathcal{M}} \,\middle|\, v_{h,k}^{\mathcal{M}} \to v_{i+1,j}^{\mathcal{M}}, h \in \mathbb{Z}, k \in \{1, 2, \dots, n\} \right\}$$

The following presents the linkage defining V *for multiplicative* $\mathbf{E}_n$ *friezes:*



Type $\mathbf{E}_3$ frieze Type $\mathbf{E}_4$ frieze



Type $\mathbf{E}_n, n \in \{5, 6, 7, 8\}$ frieze

FIGURE 2. Visualization of the combination of arrows for multiplicative $\mathbf{E}_n$ friezes

Remark. *For* $n = 5$*, the visualizations of* $v_{i,2}^{\mathcal{M}}$ *and* $v_{i,n-3}^{\mathcal{M}}$ *are redundant and are integrated for unity.*

Generally speaking, an *infinite* quiddity sequence of a frieze $\mathcal{N}$ is denoted by $a_i = v_{i,1}^{\mathcal{N}}$, where $i \in \mathbb{Z}$.

Proposition 1.3 (Periodicity of closed and multiplicative $\mathbf{E}_n$ friezes)**.** *A closed or multiplicative type* $\mathbf{E}_n$ *frieze, where* $3 \leq n \leq 8$*, is periodic.*

Proof. [Cox71] and [Mor15] proved the periodicity of closed friezes and multiplicative $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ friezes respectively. The remaining cases follow the isomorphic property stated by [Bae98], where $\mathbf{E}_3 \cong \mathbf{A}_1 \times \mathbf{A}_2$, $\mathbf{E}_4 \cong \mathbf{A}_4$ and $\mathbf{E}_5 \cong \mathbf{D}_5$, where periods of type $\mathbf{A}_n$ and $\mathbf{D}_n$ multiplicative friezes are proven in [Mor15]. ■

Definition 1.4 (Quiddity sequence). *Considering periodicity, the quiddity sequence $\mathcal{Q}_{\mathcal{T}}$ of a closed or multiplicative type $\mathbf{E}_n$ frieze $\mathcal{T}$, with maximal period of the type k, is defined as: $(q_i)_{i=\{1,\ldots,k\}} = (v_{i,1}^{\mathcal{T}})_{i=\{1,\ldots,k\}}$.*

We further denote a quiver of a multiplicative type $\mathbf{E}_n$ frieze $\mathcal{M}$ as $\Gamma_i(\mathcal{M}) := \left(v_{i,1}^{\mathcal{M}}, v_{i,2}^{\mathcal{M}}, v_{i,3}^{\mathcal{M}}, \ldots, v_{i,n}^{\mathcal{M}}\right) \in \mathbb{Z}^n$. Then, considering the period k of $\mathcal{M}$, we could say $\Gamma_i(\mathcal{M}) = \Gamma_{i+k}(\mathcal{M})$.

Conjecture. *(Interpretation of Fontaine-Plamondon conjecture, [FP16])*

(a) The number of friezes for multiplicative type $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ are 70, 440 and 1694 respectively.
(b) The number of quivers for multiplicative type $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ are 868, 4400 and 26952 respectively.

Remark. *The paper adopts an unified framework of quivers and friezes for enumeration, where the commonly referenced terminology of **friezes** is represented as **quivers** in our paper.*

Throughout this paper, we establish correspondences between the uniqueness of quivers, quiddity sequences, and friezes of multiplicative type $\mathbf{E}_n$, and their closed or open frieze counterparts. Section 2 introduces general properties of all three types of friezes, followed by the enumeration of type $\mathbf{E}_{3,4}$ friezes in Section 3. Section 4 to 5 detail enumeration methodologies to enumerate *unit-present* and *unit-free* quiddity sequences for types $\mathbf{E}_{5,6,7,8}$ in Section 4 and 5.

Definition 1.5 (Unit-present and unit-free quiddity sequences). *For a quiddity sequence $\mathcal{Q}_{\mathcal{M}}$ with period k of a multiplicative type $\mathbf{E}_n$ frieze $\mathcal{M}$, it is said to be unit-present if $\exists i \in \{1, \ldots, k\}$ s.t. $q_i = 1$. Else, it is said to be unit-free if $\forall i \in \{1, \ldots, k\}$, $q_i \geq 2$.*

We then combine all established theories to complete the full enumeration of type $\mathbf{E}_{5,6}$ friezes, and a partial enumeration for type $\mathbf{E}_{7,8}$.

2. General frieze properties

In this section, we introduce properties exhibited by closed, open and multiplicative type $\mathbf{E}_n$ friezes. We first prove an unified diagonal rule, before narrowing the focus to closed friezes, discussing their periodicity, glide-reflection symmetry and correspondence to polygonal triangulations with respect to their quiddity sequences under the cut-and-glue operation. Lastly, we characterize the periodicity, global succession and unit entry substitution properties of multiplicative type $\mathbf{E}_n$ friezes.

2.1. **Closed and open friezes.** A key feature of closed and open friezes is that the diagonals satisfy linear recurrence relations with entries given by the quiddity sequence.

Proposition 2.1 (Diagonal rule)**.** *Let $\mathcal{G}$ be a closed or open frieze, with entries $v^{\mathcal{G}}_{i,j}$ where $i \in \mathbb{Z}$, $j \in \mathbb{N}$. Then,*

$$v^{\mathcal{G}}_{i,j} = v^{\mathcal{G}}_{i,j-1} v^{\mathcal{G}}_{i+j-1,1} - v^{\mathcal{G}}_{i,j-2} \tag{2.1}$$

$$v^{\mathcal{G}}_{i,j} = v^{\mathcal{G}}_{i+1,j-1} v^{\mathcal{G}}_{i,1} - v^{\mathcal{G}}_{i+2,j-2} \tag{2.2}$$

Proof. We prove (2.1) by induction. When $j = 1$:

$$\begin{aligned} \text{R.H.S.} &= v^{\mathcal{G}}_{i,0} v^{\mathcal{G}}_{i,1} - v^{\mathcal{G}}_{i,-1} \\ &= v^{\mathcal{G}}_{i,1} \qquad (\text{notice that } v^{\mathcal{G}}_{i,0} = 1 \text{ and } v^{\mathcal{G}}_{i,-1} = 0) \\ &= \text{L.H.S.} \end{aligned}$$

Assume (2.1) holds for some positive integer k. For all $i \in \mathbb{Z}$:

$$v^{\mathcal{G}}_{i+1,k} = v^{\mathcal{G}}_{i+1,k-1} v^{\mathcal{G}}_{i+k,1} - v^{\mathcal{G}}_{i+1,k-2}$$

Notice that $v^{\mathcal{G}}_{i,k} = v^{\mathcal{G}}_{i,k-1} v^{\mathcal{G}}_{i+k-1,1} - v^{\mathcal{G}}_{i,k-2}$ also holds, we can hence yield:

$$\begin{aligned} \text{L.H.S.} &= v^{\mathcal{G}}_{i,k+1} \\ &= \frac{v^{\mathcal{G}}_{i,k} v^{\mathcal{G}}_{i+1,k} - 1}{v^{\mathcal{G}}_{i+1,k-1}} \qquad (\text{by diamond rule}) \\ &= v^{\mathcal{G}}_{i+k-1,1} v^{\mathcal{G}}_{i,k-1} v^{\mathcal{G}}_{i+k,1} - v^{\mathcal{G}}_{i,k-2} v^{\mathcal{G}}_{i+k,1} + \frac{v^{\mathcal{G}}_{i,k-2} v^{\mathcal{G}}_{i+1,k-2} - v^{\mathcal{G}}_{i+k-1,1} v^{\mathcal{G}}_{i,k-1} v^{\mathcal{G}}_{i+1,k-1} - 1}{v^{\mathcal{G}}_{i+1,k-1}} \\ &= v^{\mathcal{G}}_{i,k} v^{\mathcal{G}}_{i+k,1} - v^{\mathcal{G}}_{i,k-1} \\ &= \text{R.H.S.} \end{aligned}$$

Hence, (2.1) is proved. The argument of (2.2) follows an analogous construction, mutatis mutandis. ∎

Example 2.2 (Diagonal rule application)**.** *Consider the following closed frieze of width 4:*

$$\begin{array}{ccccccccccccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & 1 & & 4 & & 1 & & \color{red}{\mathbf{2}} & & 3 & & 1 & & \color{blue}{\mathbf{3}} & & 1 & & 4 & & 1 & & 2 & & 3 & & 1 & & 3 & \\
\cdots & & 3 & & 3 & & 1 & & \color{blue}{\mathbf{5}} & & 2 & & \color{red}{\mathbf{2}} & & 2 & & 3 & & 3 & & 1 & & 5 & & 2 & & 2 & & \cdots \\
 & 5 & & 2 & & 2 & & 2 & & \color{blue}{\mathbf{3}} & & \color{red}{\mathbf{3}} & & 1 & & 5 & & 2 & & 2 & & 2 & & 3 & & 3 & & 1 & \\
\cdots & & 3 & & 1 & & 3 & & 1 & & \mathbf{t} & & 1 & & 2 & & 3 & & 1 & & 3 & & 1 & & 4 & & 1 & & \cdots \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

From Proposition 2.1, we can yield $\mathbf{t} = 2(3) - 2 = 3(3) - 5 = 4$.

The closed frieze in Example 2.2 reveals two additional important properties, namely *periodicity* and *invariance under glide reflection*.

Theorem 2.3 (Periodicity of closed friezes, [Cox71])**.** *Rows in a closed frieze of width w are periodic with period dividing $w + 3$.*

The period w is called the *order* of the closed frieze. It is implied by a stronger symmetry, namely the glide reflection that involves reflection about a line and translation along it.

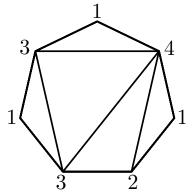
Theorem 2.4 (Glide symmetry, [Cox71]). *For a closed frieze $\mathcal{F}_w$, and its entries $(v_{i,j}^{\mathcal{F}_w})_{i,j\in\mathbb{Z},-1\leq j\leq w+2}$, it is invariant under a glide reflection, in which:*

$$\forall j \in \{0, 1, \ldots, w+1\} \qquad v_{i,j}^{\mathcal{F}_w} = v_{i+j+1,w-j+1}^{\mathcal{F}_w}$$

In other words, closed friezes have a triangular fundamental domain with vertical reflection and horizontal translation, a property demonstrated in Example 2.2. Next, we characterize the *degree sequence* arising from polygonal triangulations, before corresponding it to the quiddity sequence of a closed frieze.

Definition 2.5 (Degree sequence). *For all vertices of a triangulated w-gon, the number of triangles incident to each vertex determines a w-periodic degree sequence $\mathcal{D}_w := (\nu_1, \nu_2, \nu_3, ..., \nu_w)$ ordered clockwise. A cyclic degree sequence is the equivalence class of such sequences under cyclic permutation.*

Theorem 2.6 (Triangulation-closed frieze correspondence, [Cox71]). *A degree sequence $\mathcal{D}_{w+3}$ arises from a triangulation of a $(w+3)$-gon if and only if it coincides with the quiddity sequence $\mathcal{Q}_{\mathcal{F}_w} = (v_{i,1}^{\mathcal{F}_w}, \ldots, v_{i,w+3}^{\mathcal{F}_w})$ of a closed integral frieze of width w.*



$$\begin{array}{ccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & 1 & & 4 & & 1 & & 2 & & 3 & & 1 & & 3 & & 1 & \\
\cdots & & 3 & & 3 & & 1 & & 5 & & 2 & & 2 & & 2 & & \cdots \\
 & 5 & & 2 & & 2 & & 2 & & 3 & & 3 & & 1 & & 5 & \\
\cdots & & 3 & & 1 & & 3 & & 1 & & 4 & & 1 & & 2 & & \cdots \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \cdots
\end{array}$$

FIGURE 3. Triangulated heptagon of degree sequence $(1, 4, 1, 2, 3, 1, 3)$ and its corresponding closed frieze

Lemma 2.7 (Cardinality of cyclic degree sequence). *The number of distinct cyclic sequences of a convex m-gon (denoted as T_m) is given by:*

$$T_m = \frac{1}{m} \sum_{d|m} \phi(d) t(d)$$

where

$$t(d) = \begin{cases} C_{m-2} & \text{if } d = 1, \\ \left(\dfrac{m}{d}\right) C_{\frac{m}{d}-1} & \text{if } d > 1 \text{ and } \dfrac{m}{d} \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

with C_n denoting the n-th Catalan number and ϕ is the Euler's totient function.

Proof. Let X be a set of all degree sequences arising from a convex m-gon. By [Cox71], we have $|X| = C_{m-2}$, where C_n is the n-th Catalan number. Consider a cyclic group $G \cong \mathbb{Z}/m\mathbb{Z}$ acting on X through rotational symmetries of the m-gon. By Burnside's Lemma, the number of distinct cyclic degree sequences T_m equals to the average number of fixed points in X under the *action* of G.

$$T_m = \frac{1}{|G|}\sum_{g\in G}|X^g| = \frac{1}{m}\sum_{g\in G}|X^g|$$

where $|X^g|$ denotes the total number of degree sequences fixed by the rotation (group element) g.

We can hence evaluate the sum by partitioning G by order d of its elements, thus analyzing the fixed point sets X^g. Note such sets are non-empty only if $d \mid m$, with rotational symmetries of order d for a m-gon requiring so. Here, we denote $t(d)$ to be the number of degree sequences fixed by any rotation of order d. For each $d \mid m$, we have the following three cases:

Case for $d = 1$: Such identity rotation fixes all triangulations, so $t(1) = C_{m-2}$.
Case for $d = m$: No fixed triangulation admits full rotational symmetry, hence $t(m) = 0$.
Case for $1 < d < m$: Such fixed triangulations (of a regular m-gon without losing generality) must have d-fold rotational symmetry, which can be realized with a central regular d-gon, yielding $\frac{m}{d}$ distinct choices. The m-gon is hence partitioned into d congruent (that of identical triangulation) $\left(\frac{m}{d}+1\right)$-gons, permitting a further $C_{\frac{m}{d}-1}$ choices.

Further note there are $\phi(d)$ elements of order d for each $d \mid m$. Multiplying this result would hence yield the cardinality of cyclic degree sequences. ■

Developing on triangulation correspondence, we establish the cut-and-glue operation by demonstrating a general inductive structure of closed friezes. We capture the equivalent algebraic form of the geometric operation as a foundation tool for our original approach.

Theorem 2.8 (Cut-and-glue)**.** *A quiddity sequence $\mathcal{Q}_{\mathcal{F}_{w-1}}$ induces an integral closed frieze $\mathcal{F}_{w-1}$ of width $w-1$ iff $\mathcal{Q}_{\mathcal{F}_w} = v_{1,1}^{\mathcal{F}_{w-1}}, v_{2,1}^{\mathcal{F}_{w-1}}, \ldots, v_{k-1,1}^{\mathcal{F}_{w-1}}+1, 1, v_{k,1}^{\mathcal{F}_{w-1}}+1, \ldots, v_{w+2,1}^{\mathcal{F}_{w-1}}$, by adding 1 to two consecutive entries cyclically, induces an integral closed frieze $\mathcal{F}_w$ of width w, where $k \in \{1, \ldots, w+2\}$.*

Remark. *For boundary cases of $k = 1$ and $k = w+2$, 1 would be added to $v_{1,1}^{\mathcal{F}_{w-1}}$ and $v_{w+2,1}^{\mathcal{F}_{w-1}}$ respectively.*

Proof. Consider the integral closed friezes $\mathcal{F}_{w-1}$ and $\mathcal{F}_w$, where $w > 1$: From Theorem 2.3, their quiddity sequences $\mathcal{Q}_{\mathcal{F}_{w-1}}$ and $\mathcal{Q}_{\mathcal{F}_w}$ are $w+2$ and $w+3$ periodic respectively.
From Figure 4, let $v_{i+1,w}^{\mathcal{F}_w} = 1$. By diamond rule, all entries $v_{r,s}^{\mathcal{F}_w}$ where $r \in \{i, \ldots, i+w\}$, $s \in \mathbb{Z}$ and $r+s \leq i+w+1$ align with the entries of $\mathcal{F}_{w-1}$ at their corresponding positions. Hence, this infers only 3 entries of $\mathcal{Q}_{\mathcal{F}_w}$ will change, that is $v_{i,1}^{\mathcal{F}_w}$, $v_{i+w+1,1}^{\mathcal{F}_w}$

$$
\begin{array}{ccccccccccccccccc}
 & 0 & & 0 & & 0 & & \cdots & & 0 & & 0 & & 0 & & 0 & \\
\cdots & & 1 & & 1 & & \cdots & & \cdots & & 1 & & 1 & & 1 & & \cdots \\
 & v^{\mathcal{F}_w}_{i,1} & & v^{\mathcal{F}_{w-1}}_{k,1} & & v^{\mathcal{F}_{w-1}}_{k+1,1} & & \cdots & & v^{\mathcal{F}_{w-1}}_{k+w-1,1} & & v^{\mathcal{F}_{w-1}}_{k+w,1} & & v^{\mathcal{F}_w}_{i+w+1,1} & & v^{\mathcal{F}_w}_{i+w+2,1} & \\
\cdots & & v^{\mathcal{F}_w}_{i,2} & & v^{\mathcal{F}_{w-1}}_{k,2} & & \cdots & & \cdots & & v^{\mathcal{F}_{w-1}}_{k+w-1,2} & & v^{\mathcal{F}_w}_{i+w,2} & & v^{\mathcal{F}_w}_{i+w+1,2} & & \cdots \\
 & & \iddots & & \ddots & & & \cdots & & \iddots & & & \ddots & & \ddots & & \\
 & v^{\mathcal{F}_w}_{i-1,w-2} & & v^{\mathcal{F}_w}_{i,w-2} & & v^{\mathcal{F}_{w-1}}_{k,w-2} & & v^{\mathcal{F}_{w-1}}_{k+1,w-2} & & v^{\mathcal{F}_{w-1}}_{k+2,w-2} & & \cdots & & v^{\mathcal{F}_w}_{i+w,w-2} & & v^{\mathcal{F}_w}_{i+w+1,w-2} & \\
\cdots & & v^{\mathcal{F}_w}_{i-1,w-1} & & v^{\mathcal{F}_w}_{i,w-1} & & v^{\mathcal{F}_{w-1}}_{k,w-1} & & v^{\mathcal{F}_{w-1}}_{k+1,w-1} & & \cdots & & \cdots & & v^{\mathcal{F}_w}_{i+w,w-1} & & \cdots \\
 & v^{\mathcal{F}_w}_{i-2,w} & & v^{\mathcal{F}_w}_{i-1,w} & & v^{\mathcal{F}_w}_{i,w} & & \mathbf{1} & & v^{\mathcal{F}_w}_{i+2,w} & & \cdots & & v^{\mathcal{F}_w}_{i+w-1,w} & & v^{\mathcal{F}_w}_{i+w,w} & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & \cdots & & \cdots & & 1 & & \cdots \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & \cdots & & 0 & & 0 &
\end{array}
$$

FIGURE 4. Frieze $\mathcal{F}_w$; correspondence of entries between $\mathcal{F}_w$ and $\mathcal{F}_{w-1}$

and $v^{\mathcal{F}_w}_{i+w+2,1}$, with the remaining w entries overlapping with $\mathcal{F}_{w-1}$. Now we recall Proposition 2.1, yielding:

$$
v^{\mathcal{F}_{w-1}}_{k-1,1} = \frac{v^{\mathcal{F}_{w-1}}_{k+1,w-2} + v^{\mathcal{F}_{w-1}}_{k-1,w}}{v^{\mathcal{F}_{w-1}}_{k,w-1}} \qquad v^{\mathcal{F}_{w-1}}_{k+w,1} = \frac{v^{\mathcal{F}_{w-1}}_{k+1,w-2} + v^{\mathcal{F}_{w-1}}_{k+1,w}}{v^{\mathcal{F}_{w-1}}_{k+1,w-1}}
$$

$$
v^{\mathcal{F}_w}_{i,1} = \frac{v^{\mathcal{F}_w}_{i+2,w-2} + v^{\mathcal{F}_w}_{i,w}}{v^{\mathcal{F}_w}_{i+1,w-1}} \qquad v^{\mathcal{F}_w}_{i+w+1,1} = \frac{v^{\mathcal{F}_w}_{i+2,w-2} + v^{\mathcal{F}_w}_{i+2,w}}{v^{\mathcal{F}_w}_{i+2,w-1}}
$$

Notice that $v^{\mathcal{F}_{w-1}}_{k+1,w-1} = v^{\mathcal{F}_w}_{i+2,w-1}$, $v^{\mathcal{F}_{w-1}}_{k+1,w-2} = v^{\mathcal{F}_w}_{i+2,w-2}$ and $v^{\mathcal{F}_{w-1}}_{k,w-1} = v^{\mathcal{F}_w}_{i+1,w-1}$ as illustrated in Figure 4. We can thus evaluate the differences of $v^{\mathcal{F}_{w-1}}_{k+w,1}$ and $v^{\mathcal{F}_w}_{i+w+1,1}$; $v^{\mathcal{F}_{w-1}}_{k-1,1}$ and $v^{\mathcal{F}_w}_{i,1}$, that is, $\dfrac{v^{\mathcal{F}_w}_{i,w} - 1}{v^{\mathcal{F}_w}_{i+1,w-1}}$ and $\dfrac{v^{\mathcal{F}_w}_{i+2,w} - 1}{v^{\mathcal{F}_w}_{i+2,w-1}}$. Both are equal to 1 by the diamond rule, concluding $v^{\mathcal{F}_w}_{i+w+1,1} = v^{\mathcal{F}_{w-1}}_{k+w,1} + 1$ and $v^{\mathcal{F}_w}_{i,1} = v^{\mathcal{F}_{w-1}}_{k-1,1} + 1$. By Theorem 2.4, we yield $v^{\mathcal{F}_w}_{i+w+2,1} = 1$, completing the proof of the glue operation. The proof of the cutting operation is of similar structure. ■

2.2. **Multiplicative type $\mathbf{E}_n$ friezes.** A crucial property for the enumeration of multiplicative type $\mathbf{E}_n$ friezes is their *periodicity*, explicit periods building on Proposition 1.3 are as follows:

Theorem 2.9 (Periodicity of multiplicative type $\mathbf{E}_n$ friezes, [Mor15])**.** *The periods of a multiplicative type $\mathbf{E}_n$ frieze, denoted $p_m(\mathbf{E}_n)$, where $3 \le n \le 8$, are:*

Type	$\mathbf{E}_3$	$\mathbf{E}_4$	$\mathbf{E}_5$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
$p_m(\mathbf{E}_n)$	*6*	*7*	*10*	*14*	*10*	*16*

Remark. *The proof of $p_m(\mathbf{E}_7) = 10$ and $p_m(\mathbf{E}_8) = 16$ is presented in the Appendix*

Lemma 2.10 (Propagation of integrality)**.** *Let $\mathcal{G}$ be a closed or open frieze, with $(v^{\mathcal{G}}_{i,j})_{i,j\in\mathbb{Z},j\ge -1}$ be its entries. Denote $D^{\mathcal{R}}_{i,j} := (v^{\mathcal{G}}_{i,k})_{k=1,2,\ldots,j}$ and $D^{\mathcal{L}}_{i,j} := (v^{\mathcal{G}}_{h,k})_{h+k=i+j,\,k=1,2,\ldots,j}$. If (a) $D^{\mathcal{R}}_{i,j} \in \mathbb{Z}^j$ and $D^{\mathcal{R}}_{i+1,j-1} \in \mathbb{Z}^{j-1}$ or (b) $D^{\mathcal{L}}_{i,j} \in \mathbb{Z}^j$ and $D^{\mathcal{L}}_{i,j-1} \in \mathbb{Z}^{j-1}$, then for all integers h, k where $h+k \le i+j$ and $i \le h \le i+j-1$, all entries $v^{\mathcal{G}}_{h,k}$ are integers.*

Proof. Leveraging Proposition 2.1, we isolate $v^{\mathcal{G}}_{i,j-2}$, $v^{\mathcal{G}}_{i+2,j-2}$ respectively.

$$v^{\mathcal{G}}_{i+2,j-2} = v^{\mathcal{G}}_{i+1,j-1}v^{\mathcal{G}}_{i,1} - v^{\mathcal{G}}_{i,j} \qquad v^{\mathcal{G}}_{i,j-2} = v^{\mathcal{G}}_{i,j-1}v^{\mathcal{G}}_{i+j-1,1} - v^{\mathcal{G}}_{i,j}$$

For (a): $v^{\mathcal{G}}_{i+2,j-2}$ is an entry of $D^{\mathcal{R}}_{i+2,j-2}$. Since $v^{\mathcal{G}}_{i+1,j-1}$, $v^{\mathcal{G}}_{i,1}$ and $v^{\mathcal{G}}_{i,j}$ are integers by assumption, so is $v^{\mathcal{G}}_{i+2,j-2}$. Iterating along $D^{\mathcal{R}}_{i+2,j-2}$ (and all $D^{\mathcal{R}}_{m,n}$ with $m+n = i+j$) propagates integrality of the array.

For (b): $v^{\mathcal{G}}_{i,j-2}$ is an entry of $D^{\mathcal{L}}_{i,j-2}$. $v^{\mathcal{G}}_{i,j-1}$, $v^{\mathcal{G}}_{i+j-1,1}$ and $v^{\mathcal{G}}_{i,j}$ are integers by assumption, so is $v^{\mathcal{G}}_{i,j-2}$. Iterating along $D^{\mathcal{L}}_{i,j-2}$ (and all $D^{\mathcal{L}}_{i,j-k}$ with $k = \{3, \ldots, j-1\}$) propagates integrality of the array. ■

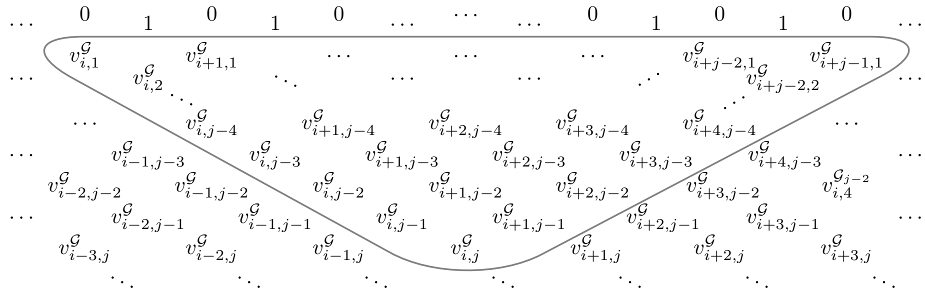


FIGURE 5. Frieze $\mathcal{G}$ with integer entries arising from Lemma 2.10 enclosed under the indicated region

Next, we establish the global successive property. Surprisingly, although it is likely known, we are unaware of any proofs through elementary means.

Theorem 2.11 (Global succession)**.** *For a multiplicative type $\mathbf{E}_n$ frieze $\mathcal{M}$ with its entries being $(v^{\mathcal{M}}_{i,j})_{i,j\in\mathbb{Z},j\in\{1,\ldots,n\}}$, if $\Gamma_i(\mathcal{M})$, $\Gamma_{i+1}(\mathcal{M}) \in \mathbb{N}^n$, then for all integers l, $\Gamma_{i+l}(\mathcal{M}) \in \mathbb{N}^n$.*

Proof. We omit the proof for $n \in \{3, 4\}$, as their enumeration follows directly from relations to closed friezes presented in Section 3. For $n \in \{5, 6, 7, 8\}$, in consideration of frieze rule, we construct three auxiliary open friezes $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ where

$$\mathcal{Q}_{\mathcal{I}_1} := (v^{\mathcal{M}}_{i,1})_{i=\{1,2,\ldots,p_m(\mathbf{E}_n)\}}$$
$$\mathcal{Q}_{\mathcal{I}_2} := (v^{\mathcal{M}}_{i,n-2})_{i=\{1,2,\ldots,p_m(\mathbf{E}_n)\}}$$
$$\mathcal{Q}_{\mathcal{I}_3} := (v^{\mathcal{M}}_{i,n})_{i=\{1,2,\ldots,p_m(\mathbf{E}_n)\}}$$

and for notational simplicity, we write $p_m(\mathbf{E}_n) = k$ in the following figure.
In $\mathcal{I}_1$, note $D^{\mathcal{R}}_{i,n-1} \in \mathbb{N}^{n-1}$ and $D^{\mathcal{R}}_{i+1,n-2} \in \mathbb{N}^{n-2}$. Lemma 2.10 implies $D^{\mathcal{R}}_{i+2,n-3} \in \mathbb{N}^{n-3}$, thus, $v^{\mathcal{M}}_{i+2,j} \in \mathbb{N}$ where $j = \{1, \ldots, n-3\}$. Similarly, in $\mathcal{I}_2$, we deduced $v^{\mathcal{M}}_{i+2,n-4}$ and $v^{\mathcal{M}}_{i+2,n-3}$ are positive integers, hence $D^{\mathcal{R}}_{i,3} \in \mathbb{N}^3$ and $D^{\mathcal{R}}_{i+1,2} \in \mathbb{N}^2$. We yield $v^{\mathcal{M}}_{i+2,n-2} \in \mathbb{N}$ via Lemma 2.10. Finally, in $\mathcal{I}_3$, $D^{\mathcal{L}}_{i-1,3} \in \mathbb{N}^3$ and $D^{\mathcal{L}}_{i-1,2} \in \mathbb{N}^2$ by assumption, thus $v^{\mathcal{M}}_{i-1,n}$ is a positive integer by Lemma 2.10. $D^{\mathcal{R}}_{i-1,4} \in \mathbb{N}^4$ and $D^{\mathcal{R}}_{i,3} \in \mathbb{N}^3$ $v^{\mathcal{M}}_{i+2,n-1}$, $v^{\mathcal{M}}_{i+2,n} \in \mathbb{N}$. Hence, $\Gamma_{i+2}(\mathcal{M}) \in \mathbb{N}^n$. Applying it for all integers i completes the proof. ■

$$
\begin{array}{c}
\cdots \quad 0 \quad 1 \quad \cdots \quad \cdots \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad \cdots \\
v^{\mathcal{M}}_{0,1} \quad v^{\mathcal{M}}_{1,1} \quad \cdots \quad v^{\mathcal{M}}_{k-1,1} \quad v^{\mathcal{M}}_{k,1} \quad v^{\mathcal{M}}_{k+1,1} \\
\cdots \quad v^{\mathcal{M}}_{0,2} \quad v^{\mathcal{M}}_{1,2} \quad \cdots \quad v^{\mathcal{M}}_{k-1,2} \quad v^{\mathcal{M}}_{k,2} \quad \cdots \\
\iddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots \\
v^{\mathcal{M}}_{-1,n-3} \quad v^{\mathcal{M}}_{0,n-3} \quad v^{\mathcal{M}}_{1,n-3} \quad \cdots \quad v^{\mathcal{M}}_{k-1,n-3} \quad v^{\mathcal{M}}_{k,n-3} \\
\cdots \quad v^{\mathcal{M}}_{-1,n-2}v^{\mathcal{M}}_{-1,n-1} \quad v^{\mathcal{M}}_{0,n-2}v^{\mathcal{M}}_{0,n-1} \quad v^{\mathcal{M}}_{1,n-2}v^{\mathcal{M}}_{1,n-1} \quad \cdots \quad v^{\mathcal{M}}_{k-1,n-2}v^{\mathcal{M}}_{k-1,n-1} \quad \cdots \\
\cdots \quad v^{\mathcal{M}}_{-1,n-1}v^{\mathcal{M}}_{0,n-1}+v^{\mathcal{M}}_{-1,n} \quad v^{\mathcal{M}}_{0,n-1}v^{\mathcal{M}}_{1,n-1}+v^{\mathcal{M}}_{0,n} \quad \cdots \quad \cdots \quad v^{\mathcal{M}}_{k-1,n-1}v^{\mathcal{M}}_{k,n-1}+v^{\mathcal{M}}_{k-1,n}
\end{array}
$$

Frieze $\mathcal{I}_1$

$$
\begin{array}{c}
\cdots \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad \cdots \\
v^{\mathcal{M}}_{0,n-2} \quad v^{\mathcal{M}}_{1,n-2} \quad v^{\mathcal{M}}_{2,n-2} \quad v^{\mathcal{M}}_{3,n-2} \\
\cdots \quad v^{\mathcal{M}}_{1,n-3} \quad v^{\mathcal{M}}_{2,n-3} \quad v^{\mathcal{M}}_{3,n-3} \quad \cdots \\
v^{\mathcal{M}}_{1,n-4}v^{\mathcal{M}}_{0,n-1} \quad v^{\mathcal{M}}_{2,n-4}v^{\mathcal{M}}_{1,n-1} \quad v^{\mathcal{M}}_{3,n-4}v^{\mathcal{M}}_{2,n-1} \quad v^{\mathcal{M}}_{4,n-4}v^{\mathcal{M}}_{3,n-1} \\
\iddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots
\end{array}
$$

Frieze $\mathcal{I}_2$

$$
\begin{array}{c}
\cdots \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad \cdots \\
v^{\mathcal{M}}_{-1,n} \quad v^{\mathcal{M}}_{0,n} \quad v^{\mathcal{M}}_{1,n} \quad v^{\mathcal{M}}_{2,n} \\
\cdots \quad v^{\mathcal{M}}_{0,n-1} \quad v^{\mathcal{M}}_{1,n-1} \quad v^{\mathcal{M}}_{2,n-1} \quad \cdots \\
v^{\mathcal{M}}_{0,n-3} \quad v^{\mathcal{M}}_{1,n-3} \quad v^{\mathcal{M}}_{2,n-3} \quad v^{\mathcal{M}}_{3,n-3} \\
\cdots \quad v^{\mathcal{M}}_{1,n-4}v^{\mathcal{M}}_{0,n-2} \quad v^{\mathcal{M}}_{2,n-4}v^{\mathcal{M}}_{1,n-2} \quad v^{\mathcal{M}}_{3,n-4}v^{\mathcal{M}}_{2,n-2} \\
\iddots \quad \ddots \quad \ddots \quad \ddots
\end{array}
$$

Frieze $\mathcal{I}_3$

FIGURE 6. Three auxiliary open friezes arising from $\mathcal{M}$

Remark. *It is sufficient to argue the positive integrality of such entries, since it is trivial that when* $\Gamma_i(\mathcal{M}), \Gamma_{i+1}(\mathcal{M}) \in \mathbb{R}_{>0}$, $\Gamma_{i+l}(\mathcal{M}) \in \mathbb{R}_{>0}$ *for all integers* l.

We now specialize to individual classes of $\mathbf{E}_n$-type friezes, analyzing their frieze rules and symmetries.

Proposition 2.12 (Properties of $\mathbf{E}_5$ friezes)**.** *For a multiplicative type* $\mathbf{E}_5$ *frieze* γ*, with its entries being* $\left(v^{\gamma}_{i,j}\right)_{i,j \in \mathbb{Z}, j=\{1,\ldots,5\}}$*, the following holds:*

(a) Periodicity (by Theorem 2.9): $\Gamma_i(\gamma) = \Gamma_{i+10}(\gamma)$ *for all integers* i*. (i.e. 10-periodic* $\mathcal{Q}_\gamma$*)*

(b) First-third row-ratio invariant: for all $m \in \mathbb{N}$ *and* $i \in \mathbb{Z}$*,*

$$
\begin{aligned}
v^{\gamma}_{i,1}v^{\gamma}_{i+2m-1,1} &= v^{\gamma}_{i-1,3}v^{\gamma}_{i+2m-2,3} \\
v^{\gamma}_{i,1}v^{\gamma}_{i+2m-1,3} &= v^{\gamma}_{i+2m,1}v^{\gamma}_{i-1,3}
\end{aligned}
\tag{2.3}
$$

(c) First-third row symmetry (by [Mor15]): for all $i \in \mathbb{Z}$*,*

$$
v^{\gamma}_{i,1} = v^{\gamma}_{i+4,3} \qquad\qquad v^{\gamma}_{i,3} = v^{\gamma}_{i+6,1} \tag{2.4}
$$

(d) Periodicity and expression on row $(v^{\gamma}_{i,5})_{i\in\mathbb{Z}}$: for all $i \in \mathbb{Z}$,

$$v^{\gamma}_{i,5} = v^{\gamma}_{i+5,5} = \frac{v^{\gamma}_{i,1} + v^{\gamma}_{i+8 \pmod{10},1}}{v^{\gamma}_{i+4 \pmod{10},1}} \tag{2.5}$$

For (2.3). We prove this by induction. When $m = 1$,

$$\text{L.H.S.} = v^{\gamma}_{i,1}v^{\gamma}_{i+1,1} = 1 + v^{\gamma}_{i,2}$$
$$\text{R.H.S.} = v^{\gamma}_{i-1,3}v^{\gamma}_{i,3} = 1 + v^{\gamma}_{i,2} = \text{L.H.S.}$$

Assume $v^{\gamma}_{i,1}v^{\gamma}_{i+2t-1,1} = v^{\gamma}_{i-1,3}v^{\gamma}_{i+2t-2,3}$ holds for some positive integer t. When $m = t + 1$,

$$\begin{aligned}\text{L.H.S.} &= v^{\gamma}_{i,1}v^{\gamma}_{i+2t+1,1}\\ &= v^{\gamma}_{i-1,3} \cdot \frac{1 + v^{\gamma}_{i+2t-1,2}}{v^{\gamma}_{i+2t-1,3}} \cdot \frac{1 + v^{\gamma}_{i+2t,2}}{v^{\gamma}_{i+2t-1,2}}\\ &= v^{\gamma}_{i-1,3}v^{\gamma}_{i+2t,3}\\ &= \text{R.H.S.}\end{aligned}$$

The second relation follows analogously from the first. ∎

For (2.5). By (2.4), $v^{\gamma}_{i,1} = v^{\gamma}_{i+4,3}$ and $v^{\gamma}_{i+1,1} = v^{\gamma}_{i+5,3}$. Consider the frieze rule of γ, we have:

$$v^{\gamma}_{i,1}v^{\gamma}_{i+1,1} = v^{\gamma}_{i+4,3}v^{\gamma}_{i+5,3} \qquad v^{\gamma}_{i,2} + 1 = v^{\gamma}_{i+5,2} + 1 \qquad v^{\gamma}_{i,2} = v^{\gamma}_{i+5,2}$$

Consider an open frieze $\mathcal{I}$ where $\mathcal{Q}_{\mathcal{I}} := (v^{\gamma}_{i,n-2})_{i=\{1,2,\ldots,10\}}$ (equivalent to $\mathcal{I}_3$ in Figure 6). Again, by (2.4), we can yield:

$$v^{\mathcal{I}}_{i-2,4} = v^{\gamma}_{i,1}v^{\gamma}_{i-1,3} = v^{\gamma}_{i+4,3}v^{\gamma}_{i+5,1} = v^{\mathcal{I}}_{i+3,4}$$

Thus, the row $(v^{\mathcal{I}}_{i,4})_{i\in\mathbb{Z}}$ has period 5, applying the diamond rule upward yields $v^{\gamma}_{i,5} = v^{\gamma}_{i+5,5}$. We then proceed to express $v^{\gamma}_{i,5}$ in terms of $\mathcal{Q}_{\gamma}$ By the frieze rule of γ and diamond rule:

$$v^{\mathcal{I}}_{i-3,5} = \frac{\left(v^{\gamma}_{i,1}v^{\gamma}_{i-1,3}\right)\left(v^{\gamma}_{i+1,1}v^{\gamma}_{i,3}\right) - 1}{v^{\gamma}_{i,2}} = v^{\gamma}_{i,2} + 2$$

Similarly, we can yield $v^{\mathcal{I}}_{i-2,5} = v^{\gamma}_{i+1,2} + 2$. Applying Proposition 2.1 on $v^{\mathcal{I}}_{i-3,6}$ gives:

$$v^{\gamma}_{i+2,5}\left(v^{\gamma}_{i,2} + 2\right) - v^{\gamma}_{i,1}v^{\gamma}_{i-1,3} = v^{\gamma}_{i-3,5}\left(v^{\gamma}_{i+1,2} + 2\right)v^{\gamma}_{i+2,1}v^{\gamma}_{i+1,3}$$

By the 5-periodic nature of the row $v^{\gamma}_{i,5}$ proved previously, we have the following:

$$v^{\gamma}_{i+2,5}\left(v^{\gamma}_{i,2} - v^{\gamma}_{i+1,2}\right) = v^{\gamma}_{i,1}v^{\gamma}_{i-1,3} - v^{\gamma}_{i+2,1}v^{\gamma}_{i+1,3}$$
$$v^{\gamma}_{i+2,5}\left(v^{\gamma}_{i,1}v^{\gamma}_{i+1,1} - v^{\gamma}_{i+1,1}v^{\gamma}_{i+2,1}\right) = v^{\gamma}_{i,1}v^{\gamma}_{i-1,3} - v^{\gamma}_{i+2,1}v^{\gamma}_{i+1,3}$$

Further note from (2.3), that $v^{\gamma}_{i,1}v^{\gamma}_{i+1,1} = v^{\gamma}_{i,3}v^{\gamma}_{i-1,3}$ and $v^{\gamma}_{i+1,1}v^{\gamma}_{i+2,1} = v^{\gamma}_{i,3}v^{\gamma}_{i+1,3}$. Substituting yields

$$v^{\gamma}_{i+2,5}v^{\gamma}_{i+1,1}\left(v^{\gamma}_{i,1} - v^{\gamma}_{i+2,1}\right) = \frac{v^{\gamma}_{i+1,1}\left[\left(v^{\gamma}_{i,1}\right)^2 - \left(v^{\gamma}_{i+2,1}\right)^2\right]}{v^{\gamma}_{i,3}}$$

$$v^{\gamma}_{i+2,5} = \frac{\left(v^{\gamma}_{i,1} + v^{\gamma}_{i+2,1}\right)\left(v^{\gamma}_{i,1} - v^{\gamma}_{i+2,1}\right)}{v^{\gamma}_{i,3}\left(v^{\gamma}_{i,1} - v^{\gamma}_{i+2,1}\right)} = \frac{v^{\gamma}_{i,1} + v^{\gamma}_{i+2,1}}{v^{\gamma}_{i+6,1}}$$

Shifting the index yields the expression of $v^{\gamma}_{i,5}$, completing the proof. ■

Remark. *Note considering the integrality of $v^{\gamma}_{i,5}$, we also yield the following for all integers i:*

$$v^{\gamma}_{i+4 \pmod{10},1} \,\Big|\, v^{\gamma}_{i,1} + v^{\gamma}_{i+8 \pmod{10},1} \tag{2.6}$$

Proposition 2.13 (Properties of $\mathbf{E}_6$ friezes)**.** *For a multiplicative type $\mathbf{E}_6$ frieze δ, with its entries being $\left(v^{\delta}_{i,j}\right)_{i,j\in\mathbb{Z},j=\{1,\ldots,6\}}$, the following holds:*

(a) Periodicity (by Theorem 2.9): $\Gamma_i(\delta) = \Gamma_{i+14}(\delta)$ for all integers i. (i.e. 14-periodic $\mathcal{Q}_\delta$)

(b) Row symmetry (by [Mor15]): for all $i \in \mathbb{Z}$,

$$\begin{aligned} v^{\delta}_{i,1} &= v^{\delta}_{i+5,6} & \qquad v^{\delta}_{i,6} &= v^{\delta}_{i+9,1} \\ v^{\delta}_{i,2} &= v^{\delta}_{i+6,5} & \qquad v^{\delta}_{i,5} &= v^{\delta}_{i+8,2} \end{aligned} \tag{2.7}$$

(c) Periodicity of row $(v^{\delta}_{i,3})_{i\in\mathbb{Z}}$ and $(v^{\delta}_{i,4})_{i\in\mathbb{Z}}$: for all $i \in \mathbb{Z}$,

$$v^{\delta}_{i,3} = v^{\delta}_{i+7,3} \qquad\qquad v^{\delta}_{i,4} = v^{\delta}_{i+7,4} \tag{2.8}$$

For (2.8). Consider an open frieze $\mathcal{I}$ with entries $(v^{\mathcal{I}}_{i,j})_{i,j\in\mathbb{Z},j\geq -1}$ where $\mathcal{Q}_{\mathcal{I}} := (v^{\delta}_{i,4})_{i=\{1,2,\ldots,14\}}$ (equivalent to $\mathcal{I}_2$ in Figure 6). By the frieze rule of δ, $v^{\mathcal{I}}_{i,4} = \dfrac{v^{\delta}_{i+2,2}v^{\delta}_{i+1,5}v^{\delta}_{i+3,2}v^{\delta}_{i+2,5} - 1}{v^{\delta}_{i+2,3}} = v^{\delta}_{i+3,1} + v^{\delta}_{i+1,6} + v^{\delta}_{i+3,1}v^{\delta}_{i+2,3}v^{\delta}_{i+1,6}$. Shifting yields $v^{\mathcal{I}}_{i+7,4} = v^{\delta}_{i+10,1} + v^{\delta}_{i+8,6} + v^{\delta}_{i+10,1}v^{\delta}_{i+9,3}v^{\delta}_{i+8,6}$. Applying diamond rule on $\mathcal{I}$ and (2.7) immediately gives:

$$\left(v^{\delta}_{i+2,2}v^{\delta}_{i+1,5}\right)\left(v^{\delta}_{i+3,2}v^{\delta}_{i+2,5}\right) - \left(v^{\delta}_{i+2,3}\right)\left(v^{\delta}_{i+3,1} + v^{\delta}_{i+1,6} + v^{\delta}_{i+3,1}v^{\delta}_{i+2,3}v^{\delta}_{i+1,6}\right) = 1$$

$$\left(v^{\delta}_{i+9,2}v^{\delta}_{i+8,5}\right)\left(v^{\delta}_{i+10,2}v^{\delta}_{i+9,5}\right) - \left(v^{\delta}_{i+9,3}\right)\left(v^{\delta}_{i+10,1} + v^{\delta}_{i+8,6} + v^{\delta}_{i+10,1}v^{\delta}_{i+9,3}v^{\delta}_{i+8,6}\right) = 1$$

$$v^{\delta}_{i+2,3} = \frac{\left(v^{\delta}_{i+2,2}v^{\delta}_{i+1,5}\right)\left(v^{\delta}_{i+3,2}v^{\delta}_{i+2,5}\right) - 1}{v^{\delta}_{i+3,1} + v^{\delta}_{i+1,6}\left(v^{\delta}_{i+2,2}v^{\delta}_{i+3,2} - 1\right)} = \frac{\left(v^{\delta}_{i+8,5}v^{\delta}_{i+9,2}\right)\left(v^{\delta}_{i+9,5}v^{\delta}_{i+10,2}\right) - 1}{v^{\delta}_{i+8,6} + v^{\delta}_{i+10,1}\left(v^{\delta}_{i+8,5}v^{\delta}_{i+9,5} - 1\right)} = v^{\delta}_{i+9,3}$$

This proves the 7-periodic property of $(v^{\delta}_{i,3})_{i\in\mathbb{Z}}$. We now consider the row $(v^{\mathcal{I}}_{i,3})_{i\in\mathbb{Z}}$ with (2.7):

$$v^{\mathcal{I}}_{i,3} = v^{\delta}_{i+2,2}v^{\delta}_{i+1,5} = v^{\delta}_{i+8,5}v^{\delta}_{i+9,2} = v^{\mathcal{I}}_{i+7,3}$$

Propagating diamond rule upward with 7-periodic $(v^{\mathcal{I}}_{i,3})_{i\in\mathbb{Z}}$ and $(v^{\mathcal{I}}_{i,2})_{i\in\mathbb{Z}}$ gives 7-periodic $(v^{\delta}_{i,4})_{i\in\mathbb{Z}}$. ■

3. Enumeration of $\mathbf{E}_{3,\ 4}$ friezes

In this section, we establish the correspondence between closed friezes and multiplicative type $\mathbf{E}_3$, $\mathbf{E}_4$ friezes, thus completing their enumerations. Using results from Section 2.1, we geometrically relate such friezes to polygonal triangulations. We also develop a number-theoretic framework that unveils an alternative enumeration approach for type $\mathbf{E}_3$ friezes. While computationally inefficient for friezes of higher $\mathbf{E}_n$ types, it remains valuable for developing unit-free quiddity sequences (refer to Section 5).

3.1. **Visualization to closed friezes and enumeration.** Consider the following frieze rule of multiplicative type $\mathbf{E}_3$ and $\mathbf{E}_4$ friezes α and β respectively.

$$\text{(3.1)} \quad \begin{cases} v^{\alpha}_{i,1}v^{\alpha}_{i+1,1} = 1 + v^{\alpha}_{i,2}v^{\alpha}_{i,3} \\ v^{\alpha}_{i,2}v^{\alpha}_{i+1,2} = 1 + v^{\alpha}_{i+1,1} \\ v^{\alpha}_{i,3}v^{\alpha}_{i+1,3} = 1 + v^{\alpha}_{i+1,1} \end{cases} \qquad \text{(3.2)} \quad \begin{cases} v^{\beta}_{i,1}v^{\beta}_{i+1,1} = 1 + v^{\beta}_{i,2}v^{\beta}_{i,3} \\ v^{\beta}_{i,2}v^{\beta}_{i+1,2} = 1 + v^{\beta}_{i+1,1} \\ v^{\beta}_{i,3}v^{\beta}_{i+1,3} = 1 + v^{\beta}_{i+1,1}v^{\beta}_{i,4} \\ v^{\beta}_{i,4}v^{\beta}_{i+1,4} = 1 + v^{\beta}_{i+1,3} \end{cases}$$

We can hence construct equivalent closed friezes $\mathcal{F}_3$ and $\mathcal{F}_4$ by the diamond rule shown in Section 1.

$$\begin{array}{ccccccccccccc}
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & v^{\alpha}_{i-1,2} & & v^{\alpha}_{i,2} & & v^{\alpha}_{i+1,2} & & v^{\alpha}_{i+2,2} & & v^{\alpha}_{i+3,2} & \\
\cdots & & v^{\alpha}_{i,1} & & v^{\alpha}_{i+1,1} & & v^{\alpha}_{i+2,1} & & v^{\alpha}_{i+3,1} & & \cdots \\
 & v^{\alpha}_{i-1,3} & & v^{\alpha}_{i,3} & & v^{\alpha}_{i+1,3} & & v^{\alpha}_{i+2,3} & & v^{\alpha}_{i+3,3} & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

Closed frieze $\mathcal{F}_3$ induced by α

$$\begin{array}{cccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & v^{\beta}_{i-1,4} & & v^{\beta}_{i,4} & & v^{\beta}_{i+1,4} & & v^{\beta}_{i+2,4} & & v^{\beta}_{i+3,4} & \\
\cdots & & v^{\beta}_{i,3} & & v^{\beta}_{i+1,3} & & v^{\beta}_{i+2,3} & & v^{\beta}_{i+3,3} & & \cdots \\
 & v^{\beta}_{i,1} & & v^{\beta}_{i+1,1} & & v^{\beta}_{i+2,1} & & v^{\beta}_{i+3,1} & & v^{\beta}_{i+4,1} & \\
\cdots & & v^{\beta}_{i,2} & & v^{\beta}_{i+1,2} & & v^{\beta}_{i+2,2} & & v^{\beta}_{i+3,2} & & \cdots \\
 & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & & 0 & & 0 & & 0 & & 0 & & \cdots
\end{array}$$

Closed frieze $\mathcal{F}_4$ induced by β

Figure 7. Realization of closed friezes from multiplicative type $\mathbf{E}_3$ and $\mathbf{E}_4$ friezes

By Theorem 2.6, $\mathcal{Q}_\alpha$ and $\mathcal{Q}_\beta$ coincides with the degree sequences $\mathcal{D}_6$ and $\mathcal{D}_7$ from hexagonal and heptagonal triangulations. Applying Lemma 2.7 for $m = 6$ and 7 completes the enumeration:

$$T_6 = \frac{1}{6}\sum_{d|6}\phi(d)t(d) = \frac{1}{6}(1\cdot 14 + 1\cdot 6 + 2\cdot 2 + 2\cdot 0) = 4$$

$$T_7 = \frac{1}{7}\sum_{d|7}\phi(d)t(d) = \frac{1}{7}(1\cdot 42 + 6\cdot 0) = 6$$

Remark. *Applying Lemma 2.4 on $\mathcal{F}_3$ and $\mathcal{F}_4$, we obtain the following:*

$$\text{(3.3)} \qquad v^{\alpha}_{i,2} = v^{\alpha}_{i+3,3} \qquad v^{\alpha}_{i,1} = v^{\alpha}_{i+3,1} \qquad v^{\beta}_{i,4} = v^{\beta}_{i+4,2} \qquad v^{\beta}_{i,3} = v^{\beta}_{i+4,1}$$

3.2. **Number theoretic approach on enumerating $\mathbf{E}_3$ friezes.** For a multiplicative type $\mathbf{E}_3$ frieze α, with entries $\left(v^{\alpha}_{i,j}\right)_{i,j\in\mathbb{Z},1\leq j\leq 3}$. From (3.1), we obtain:

$$v^{\alpha}_{1,2}v^{\alpha}_{1,3} = \frac{\left(v^{\alpha}_{1,1}+1\right)^2}{v^{\alpha}_{0,2}v^{\alpha}_{0,3}} \qquad v^{\alpha}_{0,1}v^{\alpha}_{1,1} = 1+\frac{\left(v^{\alpha}_{1,1}+1\right)^2}{v^{\alpha}_{1,2}v^{\alpha}_{1,3}} \qquad \left(v^{\alpha}_{1,1}\right)^2 - v^{\alpha}_{1,1}\left(v^{\alpha}_{1,2}v^{\alpha}_{1,3}v^{\alpha}_{0,1}-2\right) + v^{\alpha}_{1,2}v^{\alpha}_{1,3} + 1$$

By Vieta's formula, we can derive that the sum and product of roots are $v^{\alpha}_{1,2}v^{\alpha}_{1,3}v^{\alpha}_{0,1} - 2$ and $v^{\alpha}_{1,2}v^{\alpha}_{1,3}+1$ respectively. Taking (3.1) into account, we notice that the two roots are $v^{\alpha}_{1,1}$ and $v^{\alpha}_{2,1}$. This gives:

$$v^{\alpha}_{0,1}+v^{\alpha}_{1,1} = v^{\alpha}_{0,2}v^{\alpha}_{0,3}v^{\alpha}_{-1,1} - 2 \qquad v^{\alpha}_{0,1}v^{\alpha}_{1,1} = v^{\alpha}_{0,2}v^{\alpha}_{0,3}+1$$

Hence, shifting index would obtain

$$v^{\alpha}_{0,1}v^{\alpha}_{1,1}v^{\alpha}_{2,1} - v^{\alpha}_{0,1} - v^{\alpha}_{1,1} - v^{\alpha}_{2,1} - 2 = 0 \tag{3.4}$$

Without loss of generality, we assume $v^{\alpha}_{0,1} \leq v^{\alpha}_{1,1} \leq v^{\alpha}_{2,1}$, we then proceed to analyze the following cases:

Case 1: $v^{\alpha}_{0,1} \geq 3$**.** it follows that $v^{\alpha}_{0,1}v^{\alpha}_{1,1}v^{\alpha}_{2,1} \geq 9v^{\alpha}_{2,1}$, but $v^{\alpha}_{0,1}+v^{\alpha}_{1,1}+v^{\alpha}_{2,1}+2 \leq 3v^{\alpha}_{2,1}+2$, contradiction.
Case 2: $v^{\alpha}_{0,1} = 2$**.** this gives $v^{\alpha}_{0,1}v^{\alpha}_{1,1}v^{\alpha}_{2,1} = 4v^{\alpha}_{2,1}$, while $v^{\alpha}_{0,1}+v^{\alpha}_{1,1}+v^{\alpha}_{2,1}+2 \leq 3v^{\alpha}_{2,1}+2$, again a contradiction unless $v^{\alpha}_{0,1} = v^{\alpha}_{1,1} = v^{\alpha}_{2,1} = 2$.
Case 3: $v^{\alpha}_{0,1} = 1$**.** we have $\left(v^{\alpha}_{1,1}-1\right)\left(v^{\alpha}_{2,1}-1\right) = 4$. Solving yields $(v^{\alpha}_{1,1},\, v^{\alpha}_{2,1}) = (2,\, 5),\, (3,\, 3),\, (5,\, 2)$. Thus the only solutions (up to permutation) are $(2,\, 2,\, 2)$, $(1,\, 2,\, 5)$ and $(1,\, 3,\, 3)$.

Since $v^{\alpha}_{0,1}, v^{\alpha}_{1,1}, v^{\alpha}_{2,1}$ are consecutive entries and Theorem 2.3 gives $p_m(\mathbf{E}_3) = 6$, the permutative solutions yield six quiddity sequences by concatenation. Removing duplicates gives four distinct sequences: $(1,2,5,1,2,5)$, $(1,5,2,1,5,2)$, $(1,3,3,1,3,3)$, and $(2,2,2,2,2,2)$, inducing four distinct friezes.

4. Unit-present quiddity sequences: methodologies

This section develops a structural framework on the enumeration of *unit-present* quiddity sequences (where $\exists i \in \{1, 2 \ldots, p_m(\mathbf{E}_n)\}$ s.t. $q_i = 1$). In the same vein of Theorem 2.8, we conduct analysis under consecutive type $\mathbf{E}_n$ frieze structures. By combining Hamming weight analysis (base on the nature of entries), periodicity properties, and matrix multiplication, we demonstrate an efficient methodology of the counting procedure. We first define two notion related to unit entries, namely Hamming weight and canonical form of quiddity sequences, which will be used in subsequent theorems and definitions.

Definition 4.1 (Hamming weight). *For a sequence $S = \{s_1, s_2, \ldots, s_n\}$ (or its maximum non-repeating periodic part with period $p(S)$). The Hamming weight of S, denoted as $W_H(S)$, is defined as:*

$$W_H(S) = \sum_{i=1}^{n} v_i \qquad \text{where } v_i = \begin{cases} 1 & \text{if } s_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$W_H(S)$ is invariant under shifting for a periodic S. A quiddity sequence $\mathcal{Q}_{\mathcal{M}}$ arising from a frieze $\mathcal{M}$ is said to have r-th rank if $W_H(\mathcal{Q}_{\mathcal{M}}) = r$.

Definition 4.2 (Canonical form $\mathcal{C}(\mathcal{Q}_{\mathcal{M}})$). *For a multiplicative type $\mathbf{E}_n$ frieze $\mathcal{M}$ with quiddity sequence $\mathcal{Q}_{\mathcal{M}}$, a canonical form of $\mathcal{Q}_{\mathcal{M}}$, denoted $\mathcal{C}(\mathcal{Q}_{\mathcal{M}})$ is a cyclic shift:*

$$\mathcal{C}(\mathcal{Q}_{\mathcal{M}}) = (q_i)_{i=1}^{p_m(\mathbf{E}_n)} := \left(v_{j,1}^{\mathcal{M}}, v_{j+1,1}^{\mathcal{M}}, \ldots, v_{p_m(\mathbf{E}_n),1}^{\mathcal{M}}, v_{1,1}^{\mathcal{M}}, v_{2,1}^{\mathcal{M}}, \ldots, v_{j-1,1}^{\mathcal{M}}\right)$$

where $1 \leq j \leq p_m(\mathbf{E}_n)$ and $v_{j,1}^{\mathcal{M}} = 1$.

Next, we introduce the extended cut and glue operation, a key technique used to connect quiddity sequences of successive multiplicative type $\mathbf{E}_n$ friezes.

Theorem 4.3 (Extended cut and glue). *Let $\mathcal{Q}_{\mathcal{M}}$ be a quiddity sequence inducing a multiplicative type $\mathbf{E}_n$ frieze $\mathcal{M}$ ($4 \leq n \leq 7$), ordered as $(v_{j,1}^{\mathcal{M}}, v_{j+1,1}^{\mathcal{M}}, \ldots, v_{j+p_m(\mathbf{E}_n)-1,1}^{\mathcal{M}})$ for some $j \in \{1, 2, \ldots, p_m(\mathbf{E}_n)\}$. The gluing operation $\mathcal{Q}_{\mathcal{M}}^{j+}$, a bijective map $\mathcal{M} \to \mathcal{M}'$, constructs $\mathcal{Q}_{\mathcal{M}'}$ for a multiplicative type $\mathbf{E}_{n+1}$ frieze $\mathcal{M}'$ by prepending 1 to $\Gamma_j(\mathcal{M}) := (v_{j,1}^{\mathcal{M}}, \ldots, v_{j,n}^{\mathcal{M}})$, yielding $\Gamma_1(\mathcal{M}') := (1, v_{j,1}^{\mathcal{M}}, v_{j,2}^{\mathcal{M}}, \ldots, v_{j,n}^{\mathcal{M}})$. The cutting operation $\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})^{-}$, is the inverse map of glue, $\mathcal{M}' \to \mathcal{M}$.*

The explicit forms for $\mathcal{Q}^{j+}_{\mathcal{M}} = \mathcal{C}(\mathcal{Q}_{\mathcal{M}'})$ and $\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})^-$ are as follow:

$$\mathcal{Q}^{j+}_{\mathcal{M}} := \begin{cases} \big\{1, v^{\mathcal{M}}_{j,1}+1, v^{\mathcal{M}}_{j+1,1}, v^{\mathcal{M}}_{j+2,1}+v^{\mathcal{M}}_{j,4}, \\ \quad v^{\mathcal{M}}_{j+1,4}, v^{\mathcal{M}}_{j+2,4}, v^{\mathcal{M}}_{j+3,4}, \\ \quad v^{\mathcal{M}}_{j+4,1}+v^{\mathcal{M}}_{j+4,4}, v^{\mathcal{M}}_{j+6,1}, v^{\mathcal{M}}_{j+7,1}+1\big\} & \textit{if } n{=}4, \quad (4.1a) \\ \big\{1, v^{\mathcal{M}}_{j,1}+1, v^{\mathcal{M}}_{j+1,1}, v^{\mathcal{M}}_{j+2,1}, v^{\mathcal{M}}_{j,5}+v^{\mathcal{M}}_{j+3,1}, \\ \quad v^{\mathcal{M}}_{j+1,5}, v^{\mathcal{M}}_{j+2,5}, v^{\mathcal{M}}_{j+3,5}, v^{\mathcal{M}}_{j+4,5}, v^{\mathcal{M}}_{j+5,5}, \\ \quad v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+4,1}, v^{\mathcal{M}}_{j+7,1}, v^{\mathcal{M}}_{j+8,1}, v^{\mathcal{M}}_{j+9,1}+1\big\} & \textit{if } n{=}5, \quad (4.1b) \\ \big\{1, v^{\mathcal{M}}_{j,1}+1, v^{\mathcal{M}}_{j+1,1}, v^{\mathcal{M}}_{j+2,1}, v^{\mathcal{M}}_{j+3,1}, \\ \quad v^{\mathcal{M}}_{j+4,1}+v^{\mathcal{M}}_{j+9,1}, v^{\mathcal{M}}_{j+10,1}, v^{\mathcal{M}}_{j+11,1}, \\ \quad v^{\mathcal{M}}_{j+12,1}, v^{\mathcal{M}}_{j+13,1}+1\big\} & \textit{if } n{=}6, \quad (4.1c) \\ \big\{1, v^{\mathcal{M}}_{j,1}+1, v^{\mathcal{M}}_{j+1,1}, v^{\mathcal{M}}_{j+2,1}, v^{\mathcal{M}}_{j+3,1}, v^{\mathcal{M}}_{j+4,1}, \\ \quad v^{\mathcal{M}}_{j+5,1}+v^{\mathcal{M}}_{j,7}, v^{\mathcal{M}}_{j+1,7}, v^{\mathcal{M}}_{j+2,7}, v^{\mathcal{M}}_{j+3,7}, \\ \quad v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,1}, v^{\mathcal{M}}_{j+5,1}, v^{\mathcal{M}}_{j+6,1}, v^{\mathcal{M}}_{j+7,1}, \\ \quad v^{\mathcal{M}}_{j+8,1}, v^{\mathcal{M}}_{j+9,1}+1\big\} & \textit{if } n{=}7. \quad (4.1d) \end{cases}$$

Conversely, the explicit forms of $\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})^-$ are:

$$\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})^- := \begin{cases} \Big\{v^{\mathcal{M}'}_{1,1}-1, v^{\mathcal{M}'}_{2,1}, \frac{v^{\mathcal{M}'}_{8,1}(v^{\mathcal{M}'}_{9,1}-1)-1}{v^{\mathcal{M}'}_{4,1}}, \\ \frac{(v^{\mathcal{M}'}_{9,1}-1)(v^{\mathcal{M}'}_{1,1}-1)-1}{v^{\mathcal{M}'}_{5,1}}, \frac{v^{\mathcal{M}'}_{2,1}(v^{\mathcal{M}'}_{1,1}-1)-1}{v^{\mathcal{M}'}_{6,1}}, v^{\mathcal{M}'}_{8,1}, v^{\mathcal{M}'}_{9,1}-1\Big\} & n=4 \quad (4.2a) \\ \Big\{v^{\mathcal{M}'}_{1,1}-1, v^{\mathcal{M}'}_{2,1}, v^{\mathcal{M}'}_{3,1}, \\ \frac{v^{\mathcal{M}'}_{12,1}(v^{\mathcal{M}'}_{13,1}-1)}{v^{\mathcal{M}'}_{8,1}v^{\mathcal{M}'}_{12,1}-v^{\mathcal{M}'}_{3,1}}, \\ \frac{(v^{\mathcal{M}'}_{1,1}-1)(v^{\mathcal{M}'}_{8,1}v^{\mathcal{M}'}_{12,1}-v^{\mathcal{M}'}_{3,1})}{v^{\mathcal{M}'}_{9,1}(v^{\mathcal{M}'}_{8,1}v^{\mathcal{M}'}_{12,1}-v^{\mathcal{M}'}_{3,1})-v^{\mathcal{M}'}_{12,1}}, \\ \frac{(v^{\mathcal{M}'}_{13,1}-1)(v^{\mathcal{M}'}_{2,1}v^{\mathcal{M}'}_{6,1}-v^{\mathcal{M}'}_{11,1})}{v^{\mathcal{M}'}_{5,1}(v^{\mathcal{M}'}_{2,1}v^{\mathcal{M}'}_{6,1}-v^{\mathcal{M}'}_{11,1})-v^{\mathcal{M}'}_{2,1}}, \\ \frac{v^{\mathcal{M}'}_{2,1}(v^{\mathcal{M}'}_{1,1}-1)}{v^{\mathcal{M}'}_{2,1}v^{\mathcal{M}'}_{6,1}-v^{\mathcal{M}'}_{11,1}}, v^{\mathcal{M}'}_{11,1}, v^{\mathcal{M}'}_{12,1}, v^{\mathcal{M}'}_{13,1}-1\Big\} & n=5 \quad (4.2b) \\ \Big\{v^{\mathcal{M}'}_{1,1}-1, v^{\mathcal{M}'}_{2,1}, \dots, v^{\mathcal{M}'}_{10,7}, \\ v^{\mathcal{M}'}_{6,1}, v^{\mathcal{M}'}_{7,1}, v^{\mathcal{M}'}_{8,1}, v^{\mathcal{M}'}_{9,1}-1\Big\} & n=6 \quad (4.2c) \\ \Big\{v^{\mathcal{M}'}_{1,1}-1, v^{\mathcal{M}'}_{2,1}, v^{\mathcal{M}'}_{3,1}, v^{\mathcal{M}'}_{4,1}, v^{\mathcal{M}'}_{5,1}, \\ v^{\mathcal{M}'}_{11,1}, v^{\mathcal{M}'}_{12,1}, v^{\mathcal{M}'}_{13,1}, v^{\mathcal{M}'}_{14,1}, v^{\mathcal{M}'}_{15,1}-1\Big\} & n=7 \quad (4.2d) \end{cases}$$

To avoid redundant proofing, we identify a region of $\mathcal{M}'$ that retains entries from $\mathcal{M}$.

Lemma 4.4 (Region preservation)**.** *For a multiplicative type* $\mathbf{E}_n$ $(3 \leq n \leq 7)$ *frieze* $\mathcal{M}$*, with entries* $(v^{\mathcal{M}}_{i,j})_{i,j\in\mathbb{Z},j\in\{1,\dots,n\}}$*.* $\mathcal{Q}_{\mathcal{M}'} := \mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}})$*, inducing* $\mathcal{M}'$*, preserves entries of* $\mathcal{M}$ *in the specified region.*

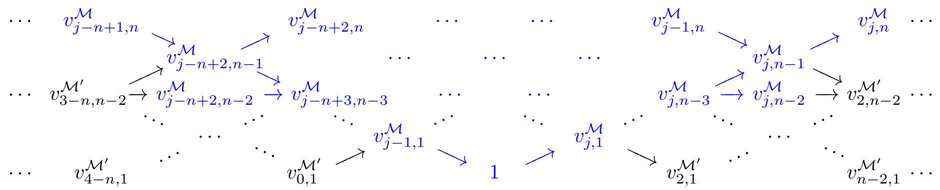


FIGURE 8. Entries of $\mathcal{M}'$ preserved from $\mathcal{M}$ (blue)

Proof. The proof is almost trivial by the frieze rule of $\mathcal{M}$ and $\mathcal{M}'$. ■

Proof of Theorem 4.3. The gluing operation $\mathcal{Q}^{i+}_{\mathcal{M}}$ is well-defined, uniquely induces $\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})$ of $\mathcal{M}'$ from $\mathcal{M}$. Surjectivity follows from $\Gamma_2(\mathcal{M}') \in \mathbb{N}^{n+1}$ and Theorem 2.11 by the frieze rule of $\mathcal{M}$. The proof proceeds by analyzing consecutive multiplicative type $\mathbf{E}_n$ friezes, by demonstrating the injectivity of the gluing operation. The cutting operation follows from the bijectivity nature established later. Here, we only prove (4.1a) and (4.2a), as remaining cases follow analogously (see Appendix for details).

For each part of the proof, we consider an open frieze $\mathcal{I}$ with entries $(v^{\mathcal{I}}_{i,j})_{i,j\in\mathbb{Z},j\geq -1}$, where $\mathcal{Q}_{\mathcal{I}} := (v^{\mathcal{M}'}_{k,1})_{k=\{j,\,j+1,\,\dots,\,j+p_m(\mathbf{E}_{n+1})-1\}}$, $v^{\mathcal{M}'}_{k,1} = v^{\mathcal{I}}_{k,1}$, with entries of $\Gamma_j(\mathcal{M})$ substituted and by Lemma 4.4:

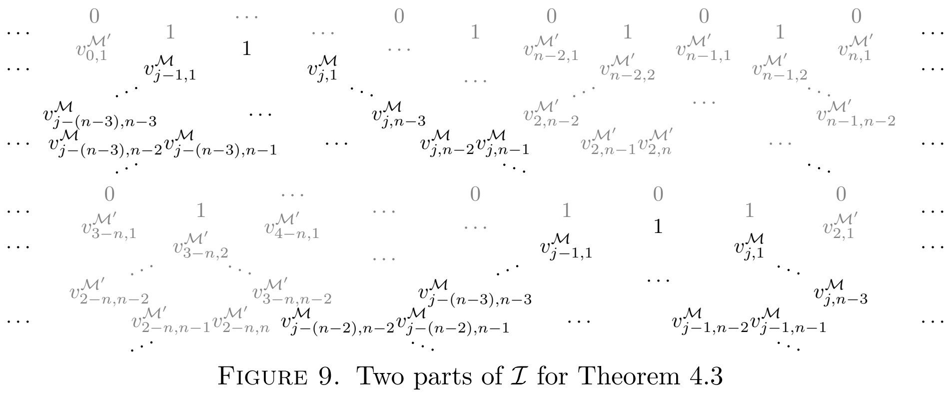


FIGURE 9. Two parts of $\mathcal{I}$ for Theorem 4.3

Generally, by the diamond rule on $\mathcal{I}$ under entries $(v^{\mathcal{I}}_{h,k})_{h+k\leq n-1,k\in\{1,\,\dots,\,n-2\}}$ and $(v^{\mathcal{I}}_{h,k})_{h+k\leq 2,k\in\{3-n,\,\dots,\,1\}}$ and the frieze rule of $\mathcal{M}$, through $p_m(\mathbf{E}_{n+1})$, we have the following:

$$(v^{\mathcal{M}'}_{1,1}, v^{\mathcal{M}'}_{2,1}, v^{\mathcal{M}'}_{3,1}, \dots, v^{\mathcal{M}'}_{n-1,1}) = (1, v^{\mathcal{M}}_{j,1}+1, v^{\mathcal{M}}_{j+1,1}, \dots, v^{\mathcal{M}}_{j+n-3,1})$$

$$\left(v^{\mathcal{M}'}_{p_m(\mathbf{E}_{n+1})+3-n,1}, \dots, v^{\mathcal{M}'}_{p_m(\mathbf{E}_{n+1}),1}\right) = (v^{\mathcal{M}}_{j-n+2,1}, \dots, v^{\mathcal{M}}_{j-1,1}+1)$$

For type $\mathbf{E}_4$ *to* $\mathbf{E}_5$ *(4.1a)(4.2a).* We have demonstrated the expression of $v^{\mathcal{M}'}_{2,1}$, $v^{\mathcal{M}'}_{3,1}$, $v^{\mathcal{M}'}_{9,1}$, $v^{\mathcal{M}'}_{10,1}$.

$$v^{\mathcal{M}'}_{2,1} = v^{\mathcal{M}}_{j,1} + 1 \qquad v^{\mathcal{M}'}_{3,1} = v^{\mathcal{M}}_{j+1,1} \qquad v^{\mathcal{M}'}_{9,1} = v^{\mathcal{M}}_{j+6,1} \qquad v^{\mathcal{M}'}_{10,1} = v^{\mathcal{M}}_{j+7,1} + 1$$

By (3.3) and Proposition 2.12 (2.4), we have $v^{\mathcal{M}'}_{7,1} = v^{\mathcal{M}'}_{1,3} = v^{\mathcal{M}}_{j,2} = v^{\mathcal{M}}_{j+3,4}$. Applying the relations iteratively gives $v^{\mathcal{M}'}_{5,1}$ and $v^{\mathcal{M}'}_{6,1}$:

$$v^{\mathcal{M}'}_{6,1} = v^{\mathcal{M}'}_{0,3} = \frac{1+v^{\mathcal{M}'}_{1,2}}{v^{\mathcal{M}'}_{1,3}} = \frac{1+v^{\mathcal{M}}_{j,1}}{v^{\mathcal{M}}_{j,2}} = v^{\mathcal{M}}_{j-1,2} = \underline{\underline{v^{\mathcal{M}}_{j+2,4}}} \qquad v^{\mathcal{M}'}_{0,5} = \frac{1+v^{\mathcal{M}'}_{1,4}}{v^{\mathcal{M}'}_{1,5}} = \frac{1+v^{\mathcal{M}}_{j,3}}{v^{\mathcal{M}}_{j,4}} = v^{\mathcal{M}}_{j-1,4}$$

$$v^{\mathcal{M}'}_{0,4} = \frac{1+v^{\mathcal{M}'}_{1,2}v^{\mathcal{M}'}_{0,5}}{v^{\mathcal{M}'}_{1,4}} = \frac{1+v^{\mathcal{M}}_{j,1}v^{\mathcal{M}}_{j-1,4}}{v^{\mathcal{M}}_{j,3}} = v^{\mathcal{M}}_{j-1,3} \qquad v^{\mathcal{M}'}_{0,2} = \frac{1+v^{\mathcal{M}'}_{1,1}v^{\mathcal{M}'}_{0,3}v^{\mathcal{M}'}_{0,4}}{v^{\mathcal{M}'}_{1,2}} = \frac{1+v^{\mathcal{M}}_{j-1,2}v^{\mathcal{M}}_{j-1,3}}{v^{\mathcal{M}}_{j,1}} = v^{\mathcal{M}}_{j-1,1}$$

$$v^{\mathcal{M}'}_{5,1} = v^{\mathcal{M}'}_{-1,3} = \frac{1+v^{\mathcal{M}'}_{0,2}}{v^{\mathcal{M}'}_{0,3}} = \frac{1+v^{\mathcal{M}}_{j-1,1}}{v^{\mathcal{M}}_{j-1,2}} = v^{\mathcal{M}}_{j-2,2} = \underline{\underline{v^{\mathcal{M}}_{j+1,4}}}$$

Applying diamond rule on $v^{\mathcal{I}}_{1,4}$, subsequently $v^{\mathcal{I}}_{2,3}$, $v^{\mathcal{I}}_{3,2}$ and $v^{\mathcal{I}}_{4,1}$, with (3.3) gives

$$v^{\mathcal{I}}_{1,4} = \frac{v^{\mathcal{M}'}_{1,3}v^{\mathcal{M}'}_{2,3}v^{\mathcal{M}'}_{1,4}v^{\mathcal{M}'}_{2,4}-1}{v^{\mathcal{M}'}_{2,2}} = 1 + \left(v^{\mathcal{M}}_{j+1,1} + v^{\mathcal{M}}_{j,2}v^{\mathcal{M}}_{j,3} + 1\right)v^{\mathcal{M}}_{j,4} = v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j+1,3} + v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j,1}v^{\mathcal{M}}_{j+1,1}$$

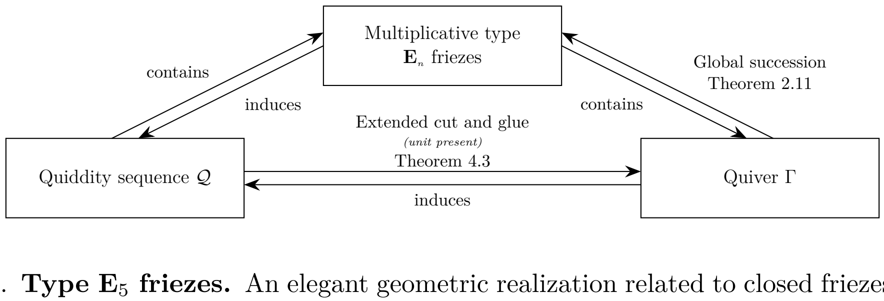
$$v^{\mathcal{I}}_{2,3} = \frac{1+\left(v^{\mathcal{M}}_{j+1,1}+v^{\mathcal{M}}_{j,2}v^{\mathcal{M}}_{j,3}\right)\left(v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j,1}v^{\mathcal{M}}_{j+1,1}\right)}{v^{\mathcal{M}}_{j,2}v^{\mathcal{M}}_{j,3}} = v^{\mathcal{M}}_{j+1,2}v^{\mathcal{M}}_{j+1,3} + v^{\mathcal{M}}_{j,2}v^{\mathcal{M}}_{j+1,2}v^{\mathcal{M}}_{j,4} + v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j+1,3} + v^{\mathcal{M}}_{j,2}v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,4}$$

$$v^{\mathcal{I}}_{3,2} = \frac{1+v^{\mathcal{M}}_{j+1,1}\left(v^{\mathcal{M}}_{j+1,2}v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j,2}v^{\mathcal{M}}_{j+1,2}v^{\mathcal{M}}_{j,4}+v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j,2}v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,4}\right)}{v^{\mathcal{M}}_{j+1,1}+v^{\mathcal{M}}_{j,2}v^{\mathcal{M}}_{j,3}} = v^{\mathcal{M}}_{j+1,2}v^{\mathcal{M}}_{j+1,3} + v^{\mathcal{M}}_{j+1,1}v^{\mathcal{M}}_{j,4}$$

$$v^{\mathcal{I}}_{4,1} = v^{\mathcal{M}'}_{4,1} = \frac{1+v^{\mathcal{M}}_{j+1,2}v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j+1,1}v^{\mathcal{M}}_{j,4}}{v^{\mathcal{M}}_{j+1,1}} = \underline{\underline{v^{\mathcal{M}}_{j+2,1} + v^{\mathcal{M}}_{j,4}}}$$

Expressing $v^{\gamma}_{8,1}$ follows analogously from $v^{\gamma}_{4,1}$, completing the expression of entries of $\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})$ by $\mathcal{M}$. ∎

Summarizing our developed techniques, the essence of the studies of type $\mathcal{E}_n$ friezes can be captured as the correspondences between type $\mathbf{E}_n$ friezes, quiddity sequences and quivers:



4.1. **Type $\mathbf{E}_5$ friezes.** An elegant geometric realization related to closed friezes is established to deduce the Hamming weight – a key enumeration tool – for quiddity sequences arising from multiplicative type $\mathbf{E}_5$ friezes, as follows:

Theorem 4.5 (Canonical $\mathbf{E}_5$ quiddity Hamming weight)**.** *For a multiplicative type* $\mathbf{E}_5$ *frieze* γ *with entries* $(v^{\gamma}_{i,j})_{i,j\in\mathbb{Z},1\leq j\leq 5}$. *Considering* $\mathcal{C}(\mathcal{Q}_{\gamma})$*:*

$$W_H(\mathcal{C}(\mathcal{Q}_\gamma)) = \begin{cases} 2 & \text{if } p(\mathcal{Q}_\gamma) = 5, \\ v^\gamma_{6,1} & \text{otherwise.} \end{cases}$$

Proof. From Theorem 4.3, we can rewrite the expression of $\mathcal{C}(\mathcal{Q}_\gamma)$ using the rows $(v^\beta_{i,3})_{i\in\mathbb{Z}}$ and $(v^\beta_{i,4})_{i\in\mathbb{Z}}$ from its corresponding type $\mathbf{E}_4$ frieze β with frieze rules, Proposition 2.1 and (3.3):
$\left\{1,\ v^\beta_{i+2,4}v^\beta_{i+3,4},\ v^\beta_{i+4,3},\ v^\beta_{i+2,4}v^\beta_{i+1,3},\ v^\beta_{i+1,4},\ v^\beta_{i+2,4},\ v^\beta_{i+3,4},\ v^\beta_{i+2,4}v^\beta_{i+4,3},\ v^\beta_{i+1,3},\ v^\beta_{i+1,4}v^\beta_{i+2,4}\right\}$
By the frieze rule of β, $v^\gamma_{2,1}$, $v^\gamma_{4,1}$, $v^\gamma_{8,1}$, $v^\gamma_{10,1} \neq 1$. We geometrically visualize the remaining entries of $\mathcal{Q}_\gamma$ into a closed frieze $\mathcal{F}_4$ induced by β. (equivalent to Figure 7) Following [Fel23], entries of $\mathcal{F}_w$ to diagonals of a $(w+3)$-gon, omitting boundary rows of 0s. Rows $(v^{\mathcal{F}_w}_{i,0})_{i\in\mathbb{Z}}$ and $(v^{\mathcal{F}_w}_{i,w+1})_{i\in\mathbb{Z}}$ represent boundary edges, where right-shifts induce clockwise rotation. Entries under a *diamond* satisfy the Ptolemy's identity with sides and diagonals of the corresponding triangulated $(w+3)$-gon being length 1.

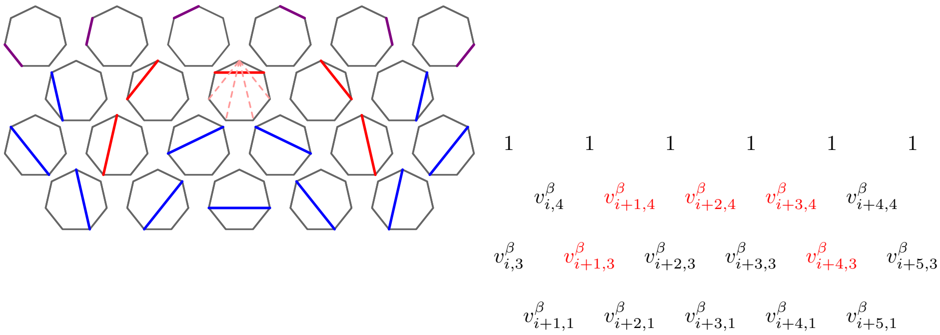


FIGURE 10. Visualization of entries of $\mathcal{C}(\mathcal{Q}_\gamma)$ (red) and corresponding entries of $\mathcal{F}_4$ under β

Considering the unit entry induced by Theorem 4.3, we analyze two cases. When $v^\beta_{i+2,4} = v^\gamma_{6,1} = 1$, $p(\mathcal{Q}_\gamma) = 5$ with no unit diagonals corresponding to entries of $\mathcal{Q}_\gamma$ in Figure 10 (i.e. no additional unit entries), yielding $W_H(\mathcal{C}(\mathcal{Q}_\gamma)) = 2$. When $v^\beta_{i+2,4} = v^\gamma_{6,1} \geq 2$, there are $(v^\beta_{i+2,4} - 1)$ diagonals incident to the corresponding vertex in $\mathcal{D}_7$ of $v^\beta_{i+2,4}$. Figure 10 show these diagonals correspond exactly to $v^\gamma_{3,4}$, $v^\gamma_{5,4}$, $v^\gamma_{7,4}$, $v^\gamma_{9,4}$, inferring $(v^\beta_{i+2,4} - 1)$ unit diagonals, thus $W_H(\mathcal{C}(\mathcal{Q}_\gamma)) = v^\beta_{i+2,4} = v^\gamma_{6,1}$. ∎

We further analyze the distribution of values of $v^\gamma_{6,1}$ for efficient enumeration, specifically that under polygonal triangulations, leveraging its equivalence to $v^\beta_{i+2,4}$ under $\mathcal{C}(\mathcal{Q}_\gamma)$.

Lemma 4.6 (Vertex incidence count)**.** *Let X denote the set of all degree sequences $\mathcal{D}_n := (\nu_1, \nu_2, \ldots \nu_n)$ arising from a convex n-gon. For any fixed positive integer $1 \leq k \leq n-2$, the number of vertices incident to exactly k triangles across all distinct triangulations of the n-gon is:*

$$|\{\mathcal{D}_n \in X \mid \exists i \in \mathbb{N} \ with \ \nu_i = k\}| = \frac{nk}{2n-k-4}\binom{2n-k-4}{n-k-2}$$

Proof. Fix a vertex v with exactly k incident triangles, implying exactly $(k-1)$ diagonals emanating from v. These dissect the n-gon into k *sub-polygons* with $s_1, s_2, \ldots, s_k \geq 2$ sides respectively, where $\sum_{i=1}^{n} s_i = n+k-2$. Note that the number of distinct cyclic degree sequences $\mathcal{D}_{s_i}$ arising from a sub-polygon of s_i sides is given by the Catalan number C_{s_i-2}. The total number of configurations (considering all ordered compositions $s_1, \ldots, s_k$ of $n+k-2$ and n choices of the initial vertex v) is

$$n \sum_{\substack{s_1+\cdots+s_k=n+k-2 \\ s_i \geq 2}} \prod_{j=1}^{k} C_{s_j-2} = n \sum_{\substack{t_1+\cdots+t_k=n-k-2 \\ t_i \geq 0}} \prod_{j=1}^{k} C_{t_j}$$

This gives the coefficient extraction under the generating function for generalized Catalan numbers C_n^k:

$$\begin{aligned} n[x^{n-k-2}]\left(\sum_{n=0}^{\infty} C_n x^n\right)^k &= \frac{nk}{2(n-k-2)+k}\binom{2(n-k-2)+k}{n-k-2} \\ &= \frac{nk}{2n-k-4}\binom{2n-k-4}{n-k-2} \end{aligned}$$

■

Remark. *Applications of Lemma 4.6 on the enumeration of type* $\mathbf{E}_5$ *friezes are discussed in Section 6.*

4.2. **Type $\mathbf{E}_{6,7,8}$ friezes.** To systematically store Hamming weight and periods across quiddity sequences, we introduce a matrix for each multiplicative $\mathbf{E}_n$ frieze type corresponding to unique canonical forms, building on Theorem 4.3.

Definition 4.7 (Unit and periodicity distribution matrices)**.** *Let* $Q_{\mathbf{E}_n} = \{\mathcal{Q}_{\mathcal{M}_1}, \mathcal{Q}_{\mathcal{M}_2}, \ldots, \mathcal{Q}_{\mathcal{M}_k}\}$ *be a list of all unique quiddity sequences arising from friezes* $\mathcal{M}_i$ *of multiplicative type* $\mathbf{E}_n$ *with a chosen order, where* $k = |Q_{\mathbf{E}_n}| + \left|Q^*_{\mathbf{E}_n}\right|$, $i \in \{1, 2, \ldots, k\}$.

(a) Denote $\mathcal{Q}^{j+}_{\mathcal{M}}$ *be a quiddity sequence of a multiplicative type* $\mathbf{E}_{n+1}$ *frieze by gluing* $\mathcal{Q}_{\mathcal{M}}$ *at the* j*-th entry, with* $j \in \{1, 2, \ldots, p_m(\mathbf{E}_n)\}$*, and* $\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}})$ *be its canonical form. The unit and periodicity distribution matrix,* $\mathfrak{D}(Q_{\mathbf{E}_{n+1}})$*, is defined as:*

$$\mathfrak{D}(Q_{\mathbf{E}_{n+1}}) := (d_{i,j})_{1 \leq i \leq k,\, 1 \leq j \leq p_m(\mathbf{E}_n)}$$

where

$$d_{i,j} = \left(W_H(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})),\, p(\mathcal{Q}_{\mathcal{M}_i}),\, p(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i}))\right)$$

(b) The unique canonical form distribution matrix, $\mathfrak{U}(Q_{\mathbf{E}_{n+1}})$*, is defined as:*

$$\mathfrak{U}(Q_{\mathbf{E}_{n+1}}) := (u_{i,j})_{1 \leq i \leq k,\, 1 \leq j \leq p_m(\mathbf{E}_n)}$$

where

$$u_{i,j} := \begin{cases} d_{i,j} & \text{if } j \leq p(\mathcal{Q}^{j+}_{\mathcal{M}_i}), \\ (0,0,0) & \text{otherwise.} \end{cases}$$

Proposition 4.8. *There exists an one-to-one correspondence between each non-zero entry in $\mathfrak{U}(Q_{\mathbf{E}_n})$ and an unique canonical form $\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}})$, where $\mathcal{M}$ is a multiplicative type $\mathbf{E}_n$ frieze, $j \in \{1, 2, \ldots, p_m(\mathbf{E}_n)\}$.*

Proof. By Definition 4.7 and Theorem 4.3, each $\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}})$ can be glued from an unique ordered $\mathcal{Q} \in Q_{\mathbf{E}_n}$ with $\mathfrak{D}(Q_{\mathbf{E}_{n+1}})$ enumerating all distinct gluing operations. Hence it suffices to show no entries in $\mathfrak{U}(Q_{\mathbf{E}_n})$ counts the Hamming weight and periodicity for the same canonical form. Also, by Theorem 4.3, a canonical form from multiplicative type $\mathbf{E}_{n+1}$ cuts to an *unique* $\mathcal{Q} \in Q_{\mathbf{E}_n}$ up to shifting. Hence, each $\mathcal{Q}$ induces $p(\mathcal{Q})$ unique canonical forms of multiplicative type $\mathbf{E}_{n+1}$. Definition 4.7(b) eliminates overcounting from the $\dfrac{p_m(\mathbf{E}_n)}{p(\mathcal{Q})}$ replicates, giving the bijection. ■

After ensuring the validity of the formulation of $\mathfrak{U}(Q_{\mathbf{E}_{n+1}})$, we then demonstrate the explicit enumeration methodology of unit-present quiddity sequences.

Theorem 4.9 (Unique quiddity formula)**.** *Let $\mathfrak{U}(Q_{\mathbf{E}_{n+1}}) = (u_{i,j})_{1 \le i \le k,\, 1 \le j \le p_m(\mathbf{E}_{n+1})}$ be an unique canonical form distribution matrix by ranking and arranging each $\mathcal{Q}_{\mathcal{M}} \in Q^{all}_{\mathbf{E}_n}$ in a fixed order, where $n \in \{4, 5, 6, 7\}$. Denote $Q^{*}_{\mathbf{E}_{n+1}}$ be the set of all unit-free quiddity sequences of type $\mathbf{E}_{n+1}$. Base on preceding knowledge on $Q^{all}_{\mathbf{E}_n}$, the total number of unique quiddity sequences, $\left|Q^{all}_{\mathbf{E}_{n+1}}\right|$, is given by:*

$$\left|Q^{all}_{\mathbf{E}_{n+1}}\right| = \left|Q^{*}_{\mathbf{E}_{n+1}}\right| + \sum_{r=1}^{p_m(\mathbf{E}_{n+1})} \sum_{d \mid p_m(\mathbf{E}_{n+1})} \frac{p_m(\mathbf{E}_{n+1})}{rd} \left|\{u_{i,j} \mid W_H(\mathcal{Q}^{j+}_{\mathcal{M}_i}) = r,\, p(\mathcal{Q}^{j+}_{\mathcal{M}_i}) = d\}\right|$$

The conjecture follows directly from Theorem 4.9, provided that all components are determined from preceding friezes and the analysis of unit-free quiddity sequences via. specialized methodologies.

Proof. It is obvious that the second, complex part of the formula enumerates for $|Q_{\mathbf{E}_{n+1}}|$. We dissect the proof of counting by ranks (i.e. distinct Hamming weights), that is:

(4.3)

$$\left|\{\mathcal{Q} \in Q_{\mathbf{E}_{n+1}} \mid W_H(\mathcal{Q}) = r\}\right| = \sum_{d \mid p_m(\mathbf{E}_{n+1})} \frac{p_m(\mathbf{E}_{n+1})}{rd} \left|\{u_{i,j} \mid W_H(\mathcal{Q}^{j+}_{\mathcal{M}_i}) = r,\, p(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})) = d\}\right|$$

Further dissecting the summation, we will first show:

$$\begin{aligned} &\left|\{\mathcal{Q} \in Q_{\mathbf{E}_{n+1}} \mid W_H(\mathcal{Q}) = r \text{ and } p(\mathcal{Q}) = d, \text{ where } d \mid p_m(\mathbf{E}_{n+1})\}\right| \\ &= \frac{p_m(\mathbf{E}_{n+1})}{rd} \left|\{u_{i,j} \mid W_H(\mathcal{Q}^{j+}_{\mathcal{M}_i}) = r,\, p(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})) = d\}\right| \end{aligned} \tag{4.4}$$

By Proposition 4.8, denote $U_r := \{u_{i,j} \mid W_H(\mathcal{Q}^{j+}_{\mathcal{M}_i} = r\}$. Hence, $|U_r|$ enumerates for the total number of *unique* canonical form in multiplicative type $\mathbf{E}_{n+1}$, each having Hamming weight r.

Notice that the total number of unique quiddity sequences of multiplicative type $\mathbf{E}_{n+1}$ with Hamming weight r should be larger than or equal to U_r
(i.e. $\left|\{\mathcal{Q} \in Q_{\mathbf{E}_{n+1}} \mid W_H(\mathcal{Q}) = r\}\right| \ge |U_r|$), since an unique unit-present quiddity sequence naturally carries multiple canonical form(s) through shifting itself.

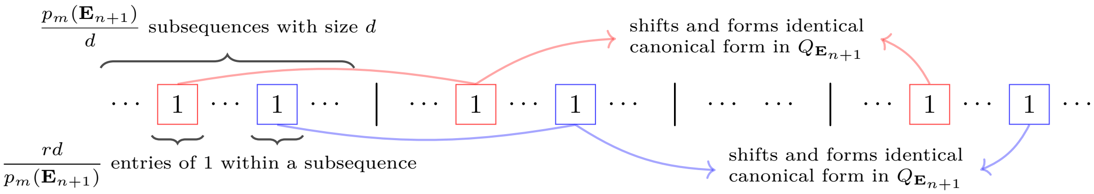


FIGURE 11. Visualization of a quiddity sequence in $Q_{\mathbf{E}_{n+1}}$ under maximum period $p_m(\mathbf{E}_{n+1})$

Consider a quiddity sequence $\mathcal{Q} \in Q_{\mathbf{E}_{n+1}}$, where $W_H(\mathcal{Q}) = r$ and $p(\mathcal{Q}) = d$. With the aid of Figure 11, $\mathcal{Q}$ induces $\dfrac{rd}{p_m(\mathbf{E}_{n+1})}$ distinct canonical forms in $Q_{\mathbf{E}_{n+1}}$. Enumerating each $\mathcal{Q} \in Q_{\mathbf{E}_{n+1}}$ once results in $|C|$, the unique count of canonical forms, inferring (4.4) holds.

Then, we proceed to sum all possible periods of friezes of multiplicative type $\mathbf{E}_{n+1}$, i.e. factors of $p_m(\mathbf{E}_{n+1})$, (quiddity sequences of different periods must be distinct), (4.3) hence follows naturally. Finally, summing up (4.3) based on the distribution of the Hamming weight r (quiddity sequences of different Hamming weights must also be distinct) completes the enumeration of $|Q_{\mathbf{E}_{n+1}}|$. ■

4.2.1. *$W_H(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}})$ by nature of entries.* With the increasing complexity of the canonical forms upon the gluing operation, we categorize their entries into two classes below to compute their respective Hamming weights. We also define a function that quantifies the presence of unit entries, to facilitate the classification of quiddity sequences.

Definition 4.10 (Inherited and emergent entries)**.** *For a quiddity sequence $\mathcal{Q}_{\mathcal{M}} := (q_1, \ldots, q_{p_m(\mathbf{E}_n)}) \in Q_{\mathbf{E}_n}$ arising from $\mathcal{M}$ gluing to $\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}})$ at the j-th entry, where $1 \le j \le p_m(\mathbf{E}_n)$ and $k = 4, 5, 6, 7$. We define the inherited entries $I(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}}))$ as the preserved entries in $\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}})$ after gluing from $\mathcal{Q}_{\mathcal{M}}$. The remaining entries in $\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}})$ are defined as emergent entries $E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}}))$. From Theorem 4.3:*

$$I(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}})) := \begin{cases} \{q_{\psi(m)}\}_{m\in\{1,2,7,8\}} & \text{if } k = 5, \\ \{q_{\psi(m)}\}_{m\in\{1,2,3,10,11,12\}} & \text{if } k = 6, \\ \{q_{\psi(m)}\}_{m\in\{1,\ldots,8\}} & \text{if } k = 7. \end{cases}$$

where $\psi(m) := (j + m) \bmod p_m(\mathbf{E}_n)$.

Proposition 4.11 (Hamming weight of $\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})$)**.** *For a quiddity sequence $\mathcal{Q}_{\mathcal{M}_i} \in Q_{\mathbf{E}_n}$ where $n \in \{4, 5, 6, 7\}$. $\forall i \in \{1, 2, \ldots, |Q^{all}_{\mathbf{E}_n}|\}$, $j \in \{1, 2, \ldots, p_m(\mathbf{E}_n)\}$:*

$$W_H(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})) = W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i}))) + W_H(I(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$$

Proof. By Definition 4.10, $E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i}))$ and $I(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i}))$ compliment each other in $\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})$. ■

Definition 4.12 (Kronecker delta)**.** *Define $\Delta_{i,j}$ as a function on i, j where*

$$\Delta_{i,j} = \begin{cases} 1 & \textit{if } i = j, \\ 0 & \textit{otherwise.} \end{cases}$$

Considering $W_H(I(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$, we can observe the entries enumerating the unit entries in $\mathcal{Q}_{\mathcal{M}_i} \in Q_{\mathbf{E}_n}$ followed by a j-unit shifting of binary indicator vector-checking based on n. Hence, the calculation of such Hamming weight under a fixed n can be formulated as a simple matrix multiplication.

Definition 4.13 (Circulant matrix)**.** *Let $\mathbf{m}$ be a row vector with length k acting as a filter. Then the circulant matrix $A_{\mathbf{m}}$ is defined as:*

$$(A_{\mathbf{m}})_{i,j} = \mathbf{m}_{(j-i) \bmod k}$$

For $n \in \{5, 6, 7\}$, define $A^n_{\mathbf{m}}$ with the following filters:

n	f
5	(0, 1, 1, 0, 0, 0, 1, 1, 0)
6	(0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0)
7	(0, 1, 1, 1, 1, 1, 1, 1, 1, 0)

Theorem 4.14 ($W_H(I(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$ criterion)**.** *Let $W^n = (w_{i,j})$ defined as $w_{i,j} = W_H(I(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$ and*

$$Q^n_{unit} = (q_{i,j}) = \begin{pmatrix} \Delta_{v^{\mathcal{M}_1}_{1,1},1} & \cdots & \cdots & \Delta_{v^{\mathcal{M}_t}_{1,1},1} \\ \Delta_{v^{\mathcal{M}_1}_{2,1},1} & \ddots & & \Delta_{v^{\mathcal{M}_t}_{2,1},1} \\ \vdots & & \ddots & \vdots \\ \Delta_{v^{\mathcal{M}_1}_{p_m(\mathbf{E}_n),1},1} & \cdots & \cdots & \Delta_{v^{\mathcal{M}_t}_{p_m(\mathbf{E}_n),1},1} \end{pmatrix} \quad \textit{where } t = \left|Q^{all}_{\mathbf{E}_n}\right| \textit{ and } n \in \{5, 6, 7\},$$

Then, $W^n = (A^n_{\mathbf{m}} \cdot Q^n_{unit})^{\mathsf{T}}$.

Proof. The proof follows from the formulation computing $W_H(I(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$ in each entry of W^n. ∎

We then proceed to establish conditions on $W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$ and its preceding frieze $\mathcal{M}$.

Theorem 4.15 ($W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$ criterion)**.** *For $\mathcal{Q}_{\mathcal{M}_i} := (q_1, \ldots, q_{p_m(\mathbf{E}_n)}) \in Q_{\mathbf{E}_n}$ gluing to $\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i}) \in Q_{\mathbf{E}_{n+1}}$ of a multiplicative type $\mathbf{E}_{n+1}$ frieze $\mathcal{M}'$, where $1 \le j \le p_m(\mathbf{E}_n)$ and $n \in \{5, 6, 7\}$. Then:*

$$W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i}))) = \begin{cases} W_H\left(\{v^{\mathcal{M}_i}_{j+1,5}, \ldots, v^{\mathcal{M}_i}_{j+5,5}\}\right) + 1 & \textit{if } n = 5, & (4.5a) \\ 1 & \textit{if } n = 6, & (4.5b) \\ W_H\left(\{v^{\mathcal{M}_i}_{j+1,7}, v^{\mathcal{M}_i}_{j+2,7}, v^{\mathcal{M}_i}_{j+3,7}\}\right) + 1 & \textit{if } n = 7. & (4.5c) \end{cases}$$

Proof. The proof is almost trivial for all cases by Theorem 4.3, following from $v^{\mathcal{M}'}_{1,1} = 1$ with some entries under $\mathcal{C}(\mathcal{Q}_{\mathcal{M}_i})$ being the sum of two entries in $\mathcal{M}$. ∎

Though not directly applicable, it is noteworthy that the Hamming weight of emergent entries of type $\mathbf{E}_6$ relates to *closed* friezes, as follows.

Proposition 4.16 (Type $\mathbf{E}_6$ $W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$ by predecessor types)**.** *Retaining the notation of Theorem 4.15 with $n = 5$, for the indicator*

$$\mathbf{1}_{c_k} = \begin{cases} 1 & \textit{if } v^{\mathcal{M}_i}_{k,1} + v^{\mathcal{M}_i}_{k+8 \pmod{10},1} = v^{\mathcal{M}_i}_{k+4 \pmod{10},1}, \\ 0 & \textit{otherwise.} \end{cases}$$

$$W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i}))) = \sum_{k=1}^{5} \mathbf{1}_{c_k} + 1$$

Furthermore, for $\mathcal{C}(\mathcal{Q}_{\mathcal{M}_i})^-$ ($\mathcal{Q}_{\mathcal{M}_i}$ is unit-present) inducing a multiplicative type $\mathbf{E}_4$ frieze $\mathcal{N}$ with $\mathcal{Q}_{\mathcal{N}} = (v^{\mathcal{N}}_{m,1},\, v^{\mathcal{N}}_{m+1,1},\, \dots\, ,\, v^{\mathcal{N}}_{m+6,1})$:

$$W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i}))) = W_H\left(\left\{v^{\mathcal{N}}_{m+3,4},\, v^{\mathcal{N}}_{m+4,4},\, v^{\mathcal{N}}_{m+5,4},\, v^{\mathcal{N}}_{m+6,4}\right\}\right)$$

Proof. By Theorem 4.15, $W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$ depends on $(v^{\mathcal{M}}_{j+k,5})_{k=1,\,2,\,3,\,4,\,5}$ and the glued unit entry. Recall Proposition 2.12 (2.5), $W_H((v^{\mathcal{M}}_{j+k,5})_{k=1,\,2,\,3,\,4,\,5})$ is invariant under cyclic shifting due to its 5-periodic nature. Without loss of generality, consider $W_H((v^{\mathcal{M}}_{k,5})_{k=1,\,2,\,3,\,4,\,5})$, which by Proposition 2.12 (2.5) satisfies:

$$v^{\mathcal{M}_i}_{k,1} + v^{\mathcal{M}_i}_{k+8 \pmod{10},1} = v^{\mathcal{M}_i}_{k+4 \pmod{10},1} \qquad \text{for } k = 1,\, 2,\, \dots\, ,\, 5$$

This completes the proof for $W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i})))$ by type $\mathbf{E}_5$ $\mathcal{Q}(\mathcal{M}_i)$. For $\mathcal{C}(\mathcal{Q}_{\mathcal{M}_i})^-$. Considering $\mathcal{C}(\mathcal{Q}^{m+}_{\mathcal{N}})$, substituting values of $\mathcal{M}_i$ into the equivalent closed frieze induced by $\mathcal{N}$:

$$\begin{array}{ccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & v^{\mathcal{N}}_{m-1,4} & & v^{\mathcal{N}}_{m,4} & & v^{\mathcal{M}_i}_{5,1} & & v^{\mathcal{M}_i}_{6,1} & & v^{\mathcal{M}_i}_{7,1} & & v^{\mathcal{N}}_{m+4,4} & & v^{\mathcal{N}}_{m+5,4} & \\
\cdots & & v^{\mathcal{N}}_{m,3} & & v^{\mathcal{M}_i}_{9,1} & \xleftrightarrow{\text{prod.}} & v^{\mathcal{M}_i}_{8,1} & & v^{\mathcal{M}_i}_{4,1} & \xleftrightarrow{\text{prod.}} & v^{\mathcal{M}_i}_{3,1} & & v^{\mathcal{N}}_{m+5,3} & & \cdots \\
 & v^{\mathcal{N}}_{m,1} & & v^{\mathcal{N}}_{m+1,1} & & v^{\mathcal{N}}_{m+2,1} & & v^{\mathcal{N}}_{m+3,1} & & v^{\mathcal{N}}_{m+4,1} & & v^{\mathcal{N}}_{m+5,1} & & v^{\mathcal{N}}_{m+6,1} & \\
\cdots & & v^{\mathcal{N}}_{m,2} & & v^{\mathcal{N}}_{m+1,2} & & v^{\mathcal{N}}_{m+2,2} & & v^{\mathcal{N}}_{m+3,2} & & v^{\mathcal{N}}_{m+4,2} & & v^{\mathcal{N}}_{m+5,2} & & \cdots \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \cdots
\end{array}$$

For each $k = 1,\, \dots\, ,\, 5$, $k = 1$ gives $v^{\mathcal{N}}_{m+2,4}v^{\mathcal{N}}_{m+3,4} + v^{\mathcal{N}}_{m+1,4}v^{\mathcal{N}}_{m+2,4} = v^{\mathcal{N}}_{m+2,4}$, implying $v^{\mathcal{N}}_{m+3,4} + v^{\mathcal{N}}_{m+1,4} = 1$, contradicting integrality. $k = 2$ gives $v^{\mathcal{M}_i}_{7,1} - v^{\mathcal{M}_i}_{3,1} = 1$, implying $v^{\mathcal{N}}_{m+4,4} = 1$ by diamond rule. $k = 3$ yields $v^{\mathcal{N}}_{m,4}v^{\mathcal{N}}_{m+1,4}v^{\mathcal{N}}_{m+2,4} - v^{\mathcal{N}}_{m+2,4} + v^{\mathcal{N}}_{m+2,4}v^{\mathcal{N}}_{m+3,4} = v^{\mathcal{N}}_{m+2,4}v^{\mathcal{N}}_{m+3,4}v^{\mathcal{N}}_{m+4,4} - v^{\mathcal{N}}_{m+2,4}$, simplifying gives $v^{\mathcal{M}_i}_{9,1} + v^{\mathcal{M}_i}_{7,1} = v^{\mathcal{M}_i}_{3,1}$. By Theorem 2.4, $v^{\mathcal{M}_i}_{7,1} + v^{\mathcal{N}}_{m+5,1} = v^{\mathcal{M}_i}_{3,1}$, so $v^{\mathcal{N}}_{m+5,4} = 1$ by Proposition 2.1. $k = 4$ gives $v^{\mathcal{M}_i}_{5,1} + v^{\mathcal{M}_i}_{3,1} = v^{\mathcal{M}_i}_{9,1}$. By Theorem 2.4, $v^{\mathcal{N}}_{m+1,1} = v^{\mathcal{M}_i}_{3,1}$. Proposition 2.1 gives $v^{\mathcal{M}_i}_{3,1} = v^{\mathcal{N}}_{m-1,4}v^{\mathcal{M}_i}_{9,1} - v^{\mathcal{M}_i}_{5,1}$. Theorem 2.9 implies $v^{\mathcal{N}}_{m-1,4} = v^{\mathcal{N}}_{m+6,4} = 1$. $k = 5$ gives $v^{\mathcal{N}}_{m,4}v^{\mathcal{N}}_{m+1,4}v^{\mathcal{N}}_{m+2,4} - v^{\mathcal{N}}_{m+2,4} + v^{\mathcal{N}}_{m+2,4} = v^{\mathcal{N}}_{m+1,4}v^{\mathcal{N}}_{m+2,4}$, reducing to $v^{\mathcal{N}}_{m,4} = v^{\mathcal{N}}_{m+7,4} = 1$ completes the proof. ■

4.2.2. *Periodicity classification.* Upon formulating the framework to compute $W_H(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_i}))$, we shift our focus to $p(\mathcal{Q}_{\mathcal{M}'})$, to prevent overcounting when $p(\mathcal{Q}_{\mathcal{M}'}) < p_m(\mathbf{E}_n)$.

Lemma 4.17 (Period criterion of $\mathbf{E}_n$ friezes)**.** *For a multiplicative type $\mathbf{E}_n$ frieze $\mathcal{M}$ ($n \in \{5,\, 6,\, 7\}$), when $\mathcal{Q}^{j+}_{\mathcal{M}}$ ($1 \leq j \leq p_m(\mathbf{E}_n)$) induces $\mathcal{Q}_{\mathcal{M}'}$ of a multiplicative type $\mathbf{E}_{n+1}$ frieze $\mathcal{M}'$, then:*

$$p(\mathcal{Q}_{\mathcal{M}'}) = \begin{cases} 7 \textit{ or } 14 & \textit{if } n{=}5, \quad (4.6a)\\ 10 & \textit{if } n{=}6, \quad (4.6b)\\ 4,\ 8 \textit{ or } 16 & \textit{if } n{=}7. \quad (4.6c)\end{cases}$$

Proof. Without loss of generality, we consider $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'}))$ in the remainder of the proof. If $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) = 1$, $v^{\mathcal{M}'}_{2,1} = v^{\mathcal{M}}_{j,1} + 1$, but $v^{\mathcal{M}}_{j,1} > 0$ implies $v^{\mathcal{M}'}_{1,1} \neq v^{\mathcal{M}'}_{2,1}$, contradicting $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) = 1$. Thus $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) \neq 1$.

For (a). By Theorem 2.9, $p_m(\mathbf{E}_6) = 14$, $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) \in \{2,\ 7,\ 14\}$. If $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) = 2$, $v^{\mathcal{M}'}_{2k+1,1} = 1$ for all $k \in \mathbb{Z}$, implying $v^{\mathcal{M}}_{i,5} + v^{\mathcal{M}}_{i+3,1} = 1$ by Theorem 4.3, contradicting positivity, thus $p(\mathcal{Q}_{\mathcal{M}'}) \in \{7,\ 14\}$.

For (b). By Theorem 4.3, $p_m(\mathbf{E}_7) = 10$, $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) \in \{2,\ 5,\ 10\}$. If $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) = 2$, $v^{\mathcal{M}'}_{2k+1,1} = 1$ and $v^{\mathcal{M}'}_{2k,1} = r$ for all $k \in \mathbb{Z}$, where $r \neq 2, 3$. We can thus consider an open frieze $\mathcal{I}$ with entries $(v^{\mathcal{I}}_{i,j})_{i,j \in \mathbb{Z}, j \geq -1}$, where $\mathcal{Q}_{\mathcal{I}} := (v^{\mathcal{M}'}_{k,1})_{k \in \{1,\,2,\,\ldots,\,10\}}$. By diamond rule, we obtain the following for all integers i:

$$v^{\mathcal{M}'}_{1,2} = \tfrac{1(r)-1}{1} = r-1 = v^{\mathcal{M}'}_{i,2} \qquad v^{\mathcal{M}'}_{1,3} = \tfrac{v^{\mathcal{M}'}_{1,2} v^{\mathcal{M}'}_{2,2} - 1}{r} = \tfrac{(r-1)^2-1}{r} = r-2 = v^{\mathcal{M}'}_{3,3}$$

$$v^{\mathcal{M}'}_{2,3} = \tfrac{v^{\mathcal{M}'}_{2,2} v^{\mathcal{M}'}_{3,2} - 1}{1} = r^2 - 2r \qquad v^{\mathcal{M}'}_{1,4} = \tfrac{v^{\mathcal{M}'}_{1,3} v^{\mathcal{M}'}_{2,3} - 1}{v^{\mathcal{M}'}_{2,2}} = \tfrac{(r-2)(r^2-2r)-1}{r-1} = r^2 - 3r + 1 = v^{\mathcal{M}'}_{i,4}$$

Applying frieze rule of $\mathcal{M}'$ on $v^{\mathcal{M}'}_{1,5} v^{\mathcal{M}'}_{1,6} v^{\mathcal{M}'}_{2,5} v^{\mathcal{M}'}_{2,6}$ gives:

$$\begin{aligned} v^{\mathcal{M}'}_{1,7} &= \tfrac{(v^{\mathcal{M}'}_{1,4} v^{\mathcal{M}'}_{2,4} - 1)(v^{\mathcal{M}'}_{2,4} v^{\mathcal{M}'}_{3,4} - 1) - (1 + v^{\mathcal{M}'}_{3,2} v^{\mathcal{M}'}_{2,4})(1 + v^{\mathcal{M}'}_{2,4})}{v^{\mathcal{M}'}_{2,4}(1 + v^{\mathcal{M}'}_{2,4})(1 + v^{\mathcal{M}'}_{3,2} v^{\mathcal{M}'}_{2,4})} \\ &= \tfrac{[(r^2-3r+1)^2-1]^2 - [(1+(r^2-3r+1)(r-1)][1+(r^2-3r+1)]}{(r^2-3r+1)[1+(r^2-3r+1)][1+(r-1)(r^2-3r+1)]} \\ &= r - \tfrac{2}{r-2} - 2 \end{aligned}$$

Notice $r \neq 2, 3$, with no integral solutions for $r < 2$ and $r \geq 5$. Hence, we let $r = 4$ and $v^{\mathcal{M}'}_{1,7} = 1$. Since the rows $v^{\mathcal{M}'}_{i,2}$ and $v^{\mathcal{M}'}_{i,4}$ are constant, $v^{\mathcal{M}'}_{i,7} = 1$ for all integers i. Consider the frieze rule of $\mathcal{M}'$, $v^{\mathcal{M}'}_{i,7} v^{\mathcal{M}'}_{i+1,7} = 1 + v^{\mathcal{M}'}_{i+1,6}$ implies $v^{\mathcal{M}'}_{i+1,6} = 0$, contradicting positivity. If $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) = 5$, then $v^{\mathcal{M}'}_{5k+1,1} = 1$ for all $k \in \mathbb{Z}$, implying $v^{\mathcal{M}}_{i+4,1} + v^{\mathcal{M}}_{i+9,1} = 1$, contradicting positive integrality, thus $p(\mathcal{Q}_{\mathcal{M}'}) = 10$.

For (c). By Theorem 4.3, $p_m(\mathbf{E}_8) = 16$, $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) \in \{2,\ 4,\ 8,\ 16\}$. If $p(\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})) = 2$, then $v^{\mathcal{M}'}_{2k+1,1} = 1$ for all $k \in \mathbb{Z}$, implying $v^{\mathcal{M}}_{i,7} + v^{\mathcal{M}}_{i+5,1} = 1$, contradicting positive integrality. Thus, $p(\mathcal{Q}_{\mathcal{M}'}) \in \{4,\ 8,\ 16\}$. ∎

Although computing the Hamming weight does not substantially increase the efficiency of acquiring necessary information for $\mathfrak{U}(\mathbf{E}_{n+1})$ because periods of quiddity sequences of type $\mathbf{E}_{n+1}$ usually reveals only when the gluing operation is performed. Yet, the elegance of this approach weight nevertheless reveals the beauty and connections between successive type $\mathbf{E}_n$ friezes.

5. Existence of unit-free quiddity

In this section, we characterize the unit-free quiddity sequences of multiplicative type $\mathbf{E}_n$ ($n \in \{5,6,7,8\}$) friezes, in which $\forall i \in \{1, \dots, p_m(\mathbf{E}_n)\}\, q_i \geq 2$.

We first establish some preliminaries. By the frieze rule of γ, we have the following:

$$v^\gamma_{i,2} = v^\gamma_{i,1}v^\gamma_{i+1,1} - 1 \qquad v^\gamma_{i+1,2} = v^\gamma_{i+1,1}v^\gamma_{i+2,1} - 1 \qquad v^\gamma_{i,2}v^\gamma_{i+1,2} = 1 + v^\gamma_{i+1,1}v^\gamma_{i,3}v^\gamma_{i,4}$$

$$(v^\gamma_{i,1}v^\gamma_{i+1,1} - 1)(v^\gamma_{i+1,1}v^\gamma_{i+2,1} - 1) = 1 + v^\gamma_{i+1,1}v^\gamma_{i,3}v^\gamma_{i,4}$$
$$v^\gamma_{i,1}v^\gamma_{i+1,1}v^\gamma_{i+2,1} - v^\gamma_{i,1} - v^\gamma_{i+2,1} = v^\gamma_{i,3}v^\gamma_{i,4}$$

Notice by (2.3), $v^\gamma_{i,1}v^\gamma_{i+1,1} = v^\gamma_{i,3}v^\gamma_{i-1,3}$. This gives

$$v^\gamma_{i,4} = v^\gamma_{i+2,1}v^\gamma_{i-1,3} - \frac{v^\gamma_{i,1}+v^\gamma_{i+2,1}}{v^\gamma_{i,3}} = v^\gamma_{i+2,1}v^\gamma_{i+5,1} - \frac{v^\gamma_{i,1}+v^\gamma_{i+2,1}}{v^\gamma_{i+6,1}}$$

Further let $v^\gamma_{i+5,1} = kv^\gamma_{i,1}$ for some positive constant k. By (2.3) and (2.4), we yield the following:

$$v^\gamma_{i+2,1}v^\gamma_{i-1,3} = v^\gamma_{i,1}v^\gamma_{i+1,3} \qquad v^\gamma_{i+2,1}v^\gamma_{i+5,1} = v^\gamma_{i,1}v^\gamma_{i+7,1} \qquad v^\gamma_{i+7,1} = kv^\gamma_{i+2,1}$$

Iterating the procedure yields the following:

$$v^\gamma_{i+2t,1} = kv^\gamma_{i+2t+5\,(\mathrm{mod}\ 10),1} \quad \text{for } t \in \{0,\ 1,\ 2,\ 3,\ 4\}$$

Furthermore, by the frieze rule of γ, we have the following:

$$v^\gamma_{i,4}v^\gamma_{i+1,4} = 1 + v^\gamma_{i+1,2}v^\gamma_{i,5} \qquad v^\gamma_{i,5}v^\gamma_{i+1,5} = 1 + v^\gamma_{i+1,4} \qquad v^\gamma_{i-1,5}v^\gamma_{i,5} = 1 + v^\gamma_{i,4}$$

Thus, by Proposition 2.12 (2.5), we can yield the following. For simplicity, indices are taken modulo 10:

$$v^\gamma_{i,4}v^\gamma_{i+1,4}v^\gamma_{i-1,5} = v^\gamma_{i+1,2}(v^\gamma_{i,4} + 1) + v^\gamma_{i-1,5}$$
$$v^\gamma_{i,1}v^\gamma_{i+1,1} - 1 + \frac{v^\gamma_{i+9,1}+v^\gamma_{i+7,1}}{v^\gamma_{i+3,1}} \equiv 0 \quad \left(\mathrm{mod}\ v^\gamma_{i+2,1}v^\gamma_{i+5,1} - \frac{v^\gamma_{i,1}+v^\gamma_{i+2,1}}{v^\gamma_{i+6,1}}\right)$$

(5.1)

$$ka_{2t+2}a_{2t+6} - 1 + \frac{a_{2t+2}+a_{2t+4}}{a_{2t+8}} \equiv 0 \quad \left(\mathrm{mod}\ ka_{2t}a_{2t+2} - \frac{a_{2t}+a_{2t+2}}{a_{2t+6}}\right) \quad \text{for } t \in \{0,\ 1,\ 2,\ 3,\ 4\}$$

where we denote $a_t := v^\gamma_{i+t,1}$.

Also notice that $v^\gamma_{i-1,5} \,|\, 1 + v^\gamma_{i,4}$, in other words $\dfrac{v^\gamma_{i+2,1}+v^\gamma_{i+4,1}}{v^\gamma_{i+8,1}} \,\Big|\, 1 + kv^\gamma_{i+2,1}v^\gamma_{i,1} - \dfrac{v^\gamma_{i,1}+v^\gamma_{i+2,1}}{v^\gamma_{i+6,1}}$.

Next we are going to show for $\mathcal{Q}_\gamma$, if $\forall i \in \mathbb{Z}$, $q_i > 1$, then for any fixed $i \in \mathbb{Z}$, the 5-term subsequence $(v^\gamma_{i+2t,1})_{t=0,1,\dots,4}$ contains exactly one maximal element. Without loss of generality, $k \geq 1$. Assume for contradiction that there are $m > 1$ maximal elements, we proceed by case elimination:

For $m = 3, 4$ **or** 5**:** From (2.6), it immediately follows that all terms of the subsequence must be equal to some positive integer $n \geq 2$. Substituting into (5.1) for $t = 0$ gives:

$$kn^2 + 1 \equiv 0 \pmod{kn^2 - 2}$$

$$3 \equiv 0 \pmod{kn^2 - 2}$$

This requires $kn^2 = 1$ or 3, contradicting $n \geq 2$.

For $m = 2$**:** Without loss of generality, assume $v^\gamma_{i,1}$ is maximal. By (2.6), $v^\gamma_{i,1} \mid v^\gamma_{i+4,1} + v^\gamma_{i+6,1}$, if either $v^\gamma_{i+4,1}$ or $v^\gamma_{i+6,1}$ were maximal, all terms would be maximal, contradicting $m = 2$. Hence, we take the maximality of $v^\gamma_{i+2,1}$, by (2.6), $v^\gamma_{i+4,1} + v^\gamma_{i+6,1} = v^\gamma_{i+6,1} + v^\gamma_{i+8,1} = v^\gamma_{i,1} = v^\gamma_{i+2,1}$, forcing $v^\gamma_{i+4,1} = v^\gamma_{i+6,1} = v^\gamma_{i+8,1} = n$, $v^\gamma_{i,1} = v^\gamma_{i+2,1} = 2n$ for some positive integer $n \geq 2$. Substituting into (2.5) for $t = 0$ gives:

$$2n^2k - 1 + 3 \equiv 0 \pmod{4n^2k - 4}$$

$$2n^2k + 2 \geq 4n^2k - 4$$

$$n^2k \leq 3$$

This contradicts $n \geq 2$, the argument for $v^\gamma_{i+8,1}$ follows similarly by index shift of (5.1). Having established the necessary preliminaries, we now derive such unit-free quiddity sequences.

Theorem 5.1 ($\mathbf{E}_5$ unit-free quiddity existence)**.** *For a multiplicative type* $\mathbf{E}_5$ *frieze* γ*, with entries* $(v^\gamma_{i,j})_{i,j\in\mathbb{Z},j\in\{1,\ldots,5\}}$*. If* $\forall i \in \mathbb{Z}$*,* $q_i > 1$*, then* $\mathcal{Q}_\gamma$ *can only be permutations of* (2, 2, 2, 2, 4, 2, 2, 2, 2, 4).

Proof. Without loss of generality, assume $v^\gamma_{i,1}$ to be the *unique* maximal element of the subsequence $(v^\gamma_{i+2t,1})_{t=0}^4$. By (2.6), $v^\gamma_{i+4,1} + v^\gamma_{i+6,1} = v^\gamma_{i,1}$. Hence we substitute $t = 1$ for (5.1), this gives:

$$kv^\gamma_{i+4,1}v^\gamma_{i+8,1} - 1 + \tfrac{v^\gamma_{i+4,1}+v^\gamma_{i+6,1}}{v^\gamma_{i,1}} \equiv 0 \quad \left(\bmod\ kv^\gamma_{i+2,1}v^\gamma_{i+4,1} - \tfrac{v^\gamma_{i+2,1}+v^\gamma_{i+4,1}}{v^\gamma_{i+8,1}}\right)$$

$$kv^\gamma_{i+4,1}v^\gamma_{i+8,1} \equiv 0 \quad \left(\bmod\ kv^\gamma_{i+2,1}v^\gamma_{i+4,1} - \tfrac{v^\gamma_{i+2,1}+v^\gamma_{i+4,1}}{v^\gamma_{i+8,1}}\right)$$

We let the modulus to be M, it is obvious that:

$$M \mid kv^\gamma_{i+2,1}v^\gamma_{i+4,1}v^\gamma_{i+8,1}$$

Further note $kv^\gamma_{i+2,1}v^\gamma_{i+4,1}v^\gamma_{i+8,1} = Mv^\gamma_{i+8,1} + v^\gamma_{i+2,1} + v^\gamma_{i+4,1}$, substituting into above yields:

$$M \mid \left(v^\gamma_{i+2,1} + v^\gamma_{i+4,1}\right)$$

$$kv^\gamma_{i+2,1}v^\gamma_{i+4,1} \leq \left(v^\gamma_{i+2,1} + v^\gamma_{i+4,1}\right)\left(1 + \tfrac{1}{v^\gamma_{i+8,1}}\right)$$

$$k \leq \left(\tfrac{1}{v^\gamma_{i+2,1}} + \tfrac{1}{v^\gamma_{i+4,1}}\right)\left(1 + \tfrac{1}{v^\gamma_{i+8,1}}\right)$$

Notice $\dfrac{v^\gamma_{i+2,1}+v^\gamma_{i+4,1}}{v^\gamma_{i+8,1}} \mid 1+kv^\gamma_{i+2,1}v^\gamma_{i,1}-\dfrac{v^\gamma_{i,1}+v^\gamma_{i+2,1}}{v^\gamma_{i+6,1}}$ bounds a finite number of cases for $v^\gamma_{i+8,1}$ when $v^\gamma_{i+2,1}=v^\gamma_{i+4,1}$, similarly for $v^\gamma_{i+2,}>2$ or $v^\gamma_{i+4,1}>2$. A case-by-case analysis reveals the only solution is $v^\gamma_{i+2,1}=v^\gamma_{i+4,1}=v^\gamma_{i+6,1}=v^\gamma_{i+8,1}=2$, $v^\gamma_{i,1}=4$ and $k=1$. ■

Theorem 5.2 (**$\mathbf{E}_6$** unit-free quiddity non-existence)**.** *For a multiplicative type* $\mathbf{E}_6$ *frieze δ, with entries $(v^\delta_{i,j})_{i,j\in\mathbb{Z},j\in\{1,\dots,6\}}$. There exists no such quiddity sequences such that $\forall i\in\mathbb{Z}$, $q_i\geq 2$.*

Proof. Our proof reduces to studying 2-friezes. By [Bau+21] (Appendix B), every height-3 2-frieze contains a one unit entry. Moreover, [MOT11] (Section 3) gives a bijection with SL_3-friezes of width 3, implying their presence of unit entries. [Bau+21] (Corollary 4.16) proves a bijection between integral tame SL_3-friezes and mesh friezes in the Grassmannian cluster category $Gr(3,7)$ (type $\mathbf{E}_6$), showing every integral multiplicative type $\mathbf{E}_6$ frieze contains at least one unit entry.

We proceed to prove by contradiction assuming $q_i\geq 2$, first establishing lower bounds. The frieze rule of δ and Proposition 2.13 (2.7) gives $v^\delta_{i,2}$, $v^\delta_{i,5}\geq 3$ and $v^\delta_{i,6}\geq 2$. By assumption $v^\delta_{i+7,1}$, $v^\delta_{i+8,1}\geq 2$:

$$2v^\delta_{i,5}\geq 4v^\delta_{i,6}-2$$

$$v^\delta_{i,3}\geq 3v^\delta_{i,6}-2$$

Hence, $v^\delta_{i,3}\geq 4$. For $v^\delta_{i,4}$, using frieze rules of δ and Proposition 2.13 (2.7) yields:

$v^\delta_{i+1,1}=\frac{1+v^\delta_{i,2}}{v^\delta_{i,1}}$, $v^\delta_{i+2,1}=\frac{v^\delta_{i,1}+v^\delta_{i,3}}{v^\delta_{i,2}}$, $v^\delta_{i+7,1}=\frac{v^\delta_{i,3}+v^\delta_{i,6}}{v^\delta_{i,5}}$, $v^\delta_{i+8,1}=\frac{1+v^\delta_{i,5}}{v^\delta_{i,6}}$, $v^\delta_{i+9,1}=v^\delta_{i,6}$

Since $v^\delta_{i+1,1}$, $v^\delta_{i+2,1}$, $v^\delta_{i+7,1}$, $v^\delta_{i+8,1}$, $v^\delta_{i+9,1}\geq 2$, summing all induced inequalities gives $2v^\delta_{i,3}\geq v^\delta_{i,1}+v^\delta_{i,2}+v^\delta_{i,5}$. With $v^\delta_{i+3,1}\geq 2$, applying frieze rule of δ gives:

$$v^\delta_{i,2}+v^\delta_{i,4}v^\delta_{i,5}\geq 2v^\delta_{i,3}$$

$$v^\delta_{i,4}v^\delta_{i,5}\geq v^\delta_{i,1}+v^\delta_{i,5}$$

$$v^\delta_{i,5}(v^\delta_{i,4}-1)\geq v^\delta_{i,1}$$

Since $v^\delta_{i,1}\in\mathbb{N}$, hence $v^\delta_{i,4}\geq 2$ for all i. Ergo, $q_i\geq 2$ yields all entries in δ are greater than 1. This contradicts with the existence of unit entries, completing the proof. ■

Please refer to appendix for the explicit idea of proof of $\mathbf{E}_7,\mathbf{E}_8$ unit-free quiddity existence

6. Enumeration of type $\mathbf{E}_{5,6,7,8}$ friezes

In this section, we complete the enumeration of multiplicative type $\mathbf{E}_{5,6,7,8}$ friezes, by combining techniques demonstrated in Theorem 4.1 and the *unique quiddity formula* established in Theorem 4.9, with individual analysis of *unit-free* quiddity sequences demonstrated in Section 5.

6.1. **Type $\mathbf{E}_5$ friezes.** Each unit-present quiddity sequence gives unique shifted canonical form(s). The value of $v^{\gamma}_{6,1}$ corresponds to the number of triangles incident to a vertex in the geometric realization of Figure 10. Applying Lemma 4.6 ($n = 7$) enumerates all possible values of $v^{\gamma}_{6,1}$. For $v^{\gamma}_{6,1} = 1$ or 5, shifting gives identical 5-periodic and 2-periodic canonical forms. For $v^{\gamma}_{6,1} = 2, 3, 4$, we further divide by $v^{\gamma}_{6,1}$ for unique canonical forms per a single frieze. We divide the total by 7 to account for cyclic degree sequences, yielding

$$\frac{1}{7}\left(\frac{7(1)}{2(7)-1-4}\binom{10-1}{5-1} + \sum_{k=2}^{4}\left(\frac{7k}{k(10-k)}\binom{10-k}{5-k}\right) + \frac{7(5)}{2(7)-5-4}\binom{10-5}{5-5}\right) = 26$$

unique unit-present quiddity sequences for type $\mathbf{E}_5$ friezes. Accounting Theorem 5.2, it gives a 5-periodic frieze of permutation (2, 2, 2, 2, 4). This concludes the total number of unique $\mathbf{E}_5$ friezes are 27.

Furthermore, the number of unique quivers are:

$$10\sum_{k=2}^{4}\left(\frac{k}{k(10-k)}\binom{10-k}{5-k}\right) + 5\cdot\frac{1}{2(7)-1-4}\binom{10-1}{5-1} + 2\cdot\frac{5}{2(7)-5-4}\binom{10-5}{5-5} + 5 = 187$$

6.2. **Type $\mathbf{E}_{6,\,7,\,8}$ friezes.** We apply Theorem 4.9 on consecutive $\mathbf{E}_n$ types to enumerate unit-present quiddity sequences. We demonstrate an example of acquiring relevant information for the matrix $\mathfrak{U}(\mathbf{E}_7)$ base on a type $\mathbf{E}_6$ quiddity sequence. We consider $\mathcal{Q}_{\mathcal{M}_{51}} = (1, 3, 2, 6, 2, 4, 2, 1, 5, 1, 8, 3, 2, 3) \in Q_{\mathbf{E}_6}$. We showcase the derivation of the row $(d_{51,1}, d_{51,2}, \ldots, d_{51,14})$ in $\mathfrak{D}_{Q_{\mathbf{E}_7}}$ and $(u_{51,1}, u_{51,2}, \ldots, u_{51,14})$ in $\mathfrak{U}_{Q_{\mathbf{E}_7}}$.

We first consider $(d_{51,1}, d_{51,2}, \ldots, d_{51,14})$. For each entry, it is trivial that $p(\mathcal{Q}_{\mathcal{M}_{51}}) = 14$, with $p(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_{51}})) = 10$ for $1 \le j \le 14$ by Lemma 4.17, what is left to be computed is $W_H(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_{51}}))$. We separately compute the Hamming weight of *inherited* and *emergent* entries. By Theorem 4.15, $W_H(E(\mathcal{C}(\mathcal{Q}^{j+}_{\mathcal{M}_{51}}))) = 1$ for all j. We further consider the column of Q^6_{unit} under $\mathcal{M}_{51}$, and performing matrix multiplication to yield $\mathbf{w}_{51} = \big((W^6)_{51,1}, (W^6)_{51,2}, \ldots, (W^6)_{51,14}\big)$:

$$\mathbf{w}_{51} = \left(A^6_{\mathbf{m}} \cdot \begin{pmatrix} \Delta_{v^{\mathcal{M}_{51}}_{1,1},1} \\ \vdots \\ \Delta_{v^{\mathcal{M}_1}_{14,1},1} \end{pmatrix}\right)^{\intercal} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\end{pmatrix} \cdot (A^6_{\mathbf{m}})^{\intercal}$$
$$= \begin{pmatrix}0 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 3 & 2 & 2 & 0\end{pmatrix}$$

We then compute the row $(d_{51,1}, d_{51,2}, \ldots, d_{51,14})$ in $\mathfrak{D}(Q_{\mathbf{E}_7})$ by incrementing all entries in $\mathbf{w}_{51}$ by 1. Since $p(\mathcal{Q}^{j+}_{\mathcal{M}_{51}}) = 10$, $(u_{51,1}, u_{51,2}, \ldots, u_{51,14})$ in $\mathfrak{U}(Q_{\mathbf{E}_7})$ equals $(d_{51,1}, d_{51,2}, \ldots, d_{51,14})$. All rows in $\mathfrak{U}_{\mathbf{E}_{n+1}}$ from preceding friezes of $\mathbf{E}_n$ types follow analogously. Applying Theorem 4.9 (including *unit-free* quiddity sequences) completes the enumeration. Tables for the enumeration of $\mathbf{E}_{6,7,8}$ quiddity sequences (and quivers) with respective values of $\frac{p_m(\mathbf{E}_{n+1})}{rd}\left|u_{i,j}\,|\,W_H(\mathcal{Q}^{j+}_{\mathcal{M}_i})=r,\,p(\mathcal{Q}^{j+}_{\mathcal{M}_i})=d\right|$ are shown below:

$r \backslash d$	7	14
1	0	0
2	11	15
3	0	22
4	5	15
5	0	2

$\mathbf{E}_6$ friezes $= 70 + 0$
$= 70$

$r \backslash d$	7	14
1	0	0
2	77	210
3	0	308
4	5	210
5	0	28

$\mathbf{E}_6$ quivers $= 868 + 0$
$= 868$

$r \backslash d$	10
1	127
2	203
3	98
4	9
5	1

$\mathbf{E}_7$ friezes $\geq 438 + 2$
≥ 440

$r \backslash d$	10
1	1270
2	2030
3	980
4	90
5	10

$\mathbf{E}_7$ quivers $\geq 4380 + 20$
≥ 4400

$r \backslash d$	4	8	16
1	0	0	264
2	0	4	557
3	0	0	491
4	3	9	306
5	0	0	54
6	0	0	5
7	0	0	0
8	0	0	0

$\mathbf{E}_8$ friezes $\geq 1693 + 1$
≥ 1694

$r \backslash d$	4	8	16
1	0	0	4224
2	0	32	8912
3	0	0	7856
4	12	72	4896
5	0	0	864
6	0	0	80
7	0	0	0
8	0	0	0

$\mathbf{E}_8$ quivers $\geq 26948 + 4$
≥ 26952

Responding to the conjecture, we have explicitly demonstrated that the number of $\mathbf{E}_6$ friezes and quivers and 70 and 868 respectively, with strong evidence hinting for the conjectural count of $\mathbf{E}_7$ and $\mathbf{E}_8$ friezes and quivers to be 440 and 4400; 1694 and 26952 respectively. The counts of $|Q^*_{\mathbf{E}_{n+1}}|$ are obtained through experimentation.

Remark. *The paper adopts an unified framework of quivers and friezes for enumeration, where the commonly referenced terminology of **friezes** is represented as **quivers** in our paper.*

7. Post-research remarks

7.1. **Benchmarking against recent work.** In March 2025, [Zha25] presented a proof in arXiv for the enumeration of Dynkin type $\mathbf{E}_n$ friezes using diophantine geometry techniques , which coincidentally aligned with our research period. We summarize the methodologies both papers employed, their respective efficiency, philosophy and limitations:

	Approach of Zhang	Our approach
Core idea	*Transform* and solve: conversion to a more abstract field, diophantine geometry with pre-existing theorems	*Construct* and understand: hands-on tackling of the problem under the inductive framework of Theorem 4.3
Mathematical tools	Combination of cluster algebras and affine varieties, highly specialized knowledge	Accessible approach under combinatorial and geometric insights, Hamming weight and periodicity analysis
Enumerated aspect(s)	Total number of valid frieze patterns (in our terms, *quivers*)	Both configurations of friezes and quivers
Key contribution	Definitive evidence on enumeration using high-level theories	Structural blueprint, revealing the elegant combinatorial connection between successive type $\mathbf{E}_n$ friezes; $\mathbf{E}_5, \mathbf{E}_6$ complete enumeration without search
Computer reliance	Heavy reliance; brute force search by 64 GPUs and 1000 CPUs	Minimal reliance; analytical approach avoiding exhaustive searches
Limitations	Unable to unveil the elegance of unit-present quiddity sequences	Technical parts of *unit-free* quiddity sequences in higher types remain unsolved

7.2. **Further applications.** Nonetheless, our enumeration of multiplicative type $\mathbf{E}_n$ friezes through an elementary combinatorial analysis of quiddity sequences, hamming weights and periodicity, establishes deep, albeit indirect, connections to knot theory. Despite our approach being straightforward, the study of frieze structures is rooted in the field of *cluster algebras*, a structure that reveals a powerful unifying cross-field framework.

Recent research, such as the Oberwolfach report on "Cluster Algebras and Its Applications," [Bau+24] demonstrates a powerful connection between cluster algebras and knot theory. As an example, knot invariants like the Jones polynomial can be effectively studied and evaluated within the framework of cluster algebra. This link is crucial, as knot theory has found profound applications in the natural sciences. As highlighted in the work of De Witt Sumners [Sum95], knot theory is contributory in molecular biology for modeling the topology of DNA and enzyme action. Thus, our combinatorial results for the particular frieze patterns of type $\mathbf{E}_n$ enrich a mathematical theory that provides unexpected connections between seemingly distant fields.

8. Conclusion

Our paper's contribution is not in racing to the same answer as experts do, but in revealing the beautiful and intricate combinatorial structure that was hidden beneath the complex machinery of higher mathematics. We demonstrated that this conjecture can be almost completely solved by extracting insights from geometry and combinatorics. By distinguishing correspondence between quiddity sequences, friezes and quivers, we have added a new layer of understanding to the problem and showcased the power of creative, constructive mathematics.

9. Appendix

9.1. **Proof of Theorem 4.3 (cont).** We present the remaining proof for the extended cut and glue operation.

For type $\mathbf{E}_5$ *to* $\mathbf{E}_6$ *(4.1b)(4.2b).* Having expressed $v_{2,1}^{\mathcal{M}'}, \cdots, v_{4,1}^{\mathcal{M}'}, v_{12,1}^{\mathcal{M}'}, \cdots, v_{14,1}^{\mathcal{M}'}$. By Propositions 2.12 and 2.13 (2.5)(2.7), we immediately yield $v_{10,1}^{\mathcal{M}'} = v_{1,6}^{\mathcal{M}'} = v_{j,5}^{\mathcal{M}} = v_{j+5,5}^{\mathcal{M}}$. Further applying the relations recursively gives $v_{6,1}^{\mathcal{M}'}, v_{7,1}^{\mathcal{M}'}, v_{8,1}^{\mathcal{M}'}, v_{9,1}^{\mathcal{M}'}$:

$$v_{9,1}^{\mathcal{M}'} = v_{0,6}^{\mathcal{M}'} = \frac{1+v_{1,5}^{\mathcal{M}'}}{v_{1,6}^{\mathcal{M}'}} = \frac{1+v_{j,4}^{\mathcal{M}}}{v_{j,5}^{\mathcal{M}}} = v_{j-1,5}^{\mathcal{M}} = \underline{\underline{v_{j+4,5}^{\mathcal{M}}}} \qquad v_{0,5}^{\mathcal{M}'} = \frac{1+v_{0,6}^{\mathcal{M}'}v_{1,3}^{\mathcal{M}'}}{v_{1,5}^{\mathcal{M}'}} = \frac{1+v_{j-1,5}^{\mathcal{M}}v_{j,2}^{\mathcal{M}}}{v_{j,4}^{\mathcal{M}}} = v_{j-1,4}^{\mathcal{M}}$$

$$v_{8,1}^{\mathcal{M}'} = v_{-1,6}^{\mathcal{M}'} = \frac{1+v_{0,5}^{\mathcal{M}'}}{v_{0,6}^{\mathcal{M}'}} = \frac{1+v_{j-1,4}^{\mathcal{M}}}{v_{j-1,5}^{\mathcal{M}}} = v_{j-2,5}^{\mathcal{M}} = \underline{\underline{v_{j+3,5}^{\mathcal{M}}}} \qquad v_{0,4}^{\mathcal{M}'} = \frac{1+v_{1,3}^{\mathcal{M}'}}{v_{1,4}^{\mathcal{M}'}} = \frac{1+v_{j,2}^{\mathcal{M}}}{v_{j,3}^{\mathcal{M}}} = v_{j-1,3}^{\mathcal{M}}$$

$$v_{0,3}^{\mathcal{M}'} = \frac{1+v_{0,5}^{\mathcal{M}'}v_{0,4}^{\mathcal{M}'}v_{1,2}^{\mathcal{M}'}}{v_{1,3}^{\mathcal{M}'}} = \frac{1+v_{j-1,4}^{\mathcal{M}}v_{j-1,3}^{\mathcal{M}}v_{j,1}^{\mathcal{M}}}{v_{j,2}^{\mathcal{M}}} = v_{j-1,2}^{\mathcal{M}} \qquad v_{-1,5}^{\mathcal{M}'} = \frac{1+v_{-1,6}^{\mathcal{M}'}v_{0,3}^{\mathcal{M}'}}{v_{0,5}^{\mathcal{M}'}} = \frac{1+v_{j-2,5}^{\mathcal{M}}v_{j-1,2}^{\mathcal{M}}}{v_{j-1,4}^{\mathcal{M}}} = v_{j-2,4}^{\mathcal{M}}$$

$$v_{7,1}^{\mathcal{M}'} = v_{-2,6}^{\mathcal{M}'} = \frac{1+v_{-1,5}^{\mathcal{M}'}}{v_{-1,6}^{\mathcal{M}'}} = \frac{1+v_{j-2,4}^{\mathcal{M}}}{v_{j-2,5}^{\mathcal{M}}} = v_{j-3,5}^{\mathcal{M}} = \underline{\underline{v_{j+2,5}^{\mathcal{M}}}} \qquad v_{0,2}^{\mathcal{M}'} = \frac{1+v_{0,3}^{\mathcal{M}'}v_{1,1}^{\mathcal{M}'}}{v_{1,2}^{\mathcal{M}'}} = \frac{1+v_{j-1,2}^{\mathcal{M}}}{v_{j,1}^{\mathcal{M}}} = v_{j-1,1}^{\mathcal{M}}$$

$$v_{-1,4}^{\mathcal{M}'} = \frac{1+v_{0,3}^{\mathcal{M}'}}{v_{0,4}^{\mathcal{M}'}} = \frac{1+v_{j-1,2}^{\mathcal{M}}}{v_{j-1,3}^{\mathcal{M}}} = v_{j-2,3}^{\mathcal{M}} \qquad v_{-1,3}^{\mathcal{M}'} = \frac{1+v_{-1,5}^{\mathcal{M}'}v_{0,4}^{\mathcal{M}'}v_{0,2}^{\mathcal{M}'}}{v_{0,3}^{\mathcal{M}'}} = \frac{1+v_{j-2,4}^{\mathcal{M}}v_{j-2,3}^{\mathcal{M}}v_{j-1,1}^{\mathcal{M}}}{v_{j-1,2}^{\mathcal{M}}} = v_{j-2,2}^{\mathcal{M}}$$

$$v_{j-2,5}^{\mathcal{M}'} = \frac{1+v_{-2,6}^{\mathcal{M}'}v_{-1,3}^{\mathcal{M}'}}{v_{-1,5}^{\mathcal{M}'}} = \frac{1+v_{j-3,5}^{\mathcal{M}}v_{j-2,2}^{\mathcal{M}}}{v_{j-2,4}^{\mathcal{M}}} = v_{j-3,4}^{\mathcal{M}} \qquad v_{6,1}^{\mathcal{M}'} = v_{-3,6}^{\mathcal{M}'} = \frac{1+v_{-2,5}^{\mathcal{M}'}}{v_{-2,6}^{\mathcal{M}'}} = \frac{1+v_{j-3,4}^{\mathcal{M}}}{v_{j-3,5}^{\mathcal{M}}} = v_{j-4,5}^{\mathcal{M}} = \underline{\underline{v_{j+1,5}^{\mathcal{M}}}}$$

Then, we apply diamond rule on $v^{\mathcal{I}}_{1,5}$, $v^{\mathcal{I}}_{2,4}$, $v^{\mathcal{I}}_{3,3}$, $v^{\mathcal{I}}_{4,2}$ and $v^{\mathcal{I}}_{5,1}$, giving

$$\begin{aligned}
v^{\mathcal{I}}_{1,5} &= \frac{v^{\mathcal{M}'}_{1,4}v^{\mathcal{M}'}_{2,4}v^{\mathcal{M}'}_{1,5}v^{\mathcal{M}'}_{2,5}-1}{v^{\mathcal{M}'}_{2,3}} = \left(v^{\mathcal{M}}_{j+1,2}+v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,4}+1\right)v^{\mathcal{M}}_{j,5}+1 = v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,5}\left(v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,4}+1\right)\\
v^{\mathcal{I}}_{2,4} &= \frac{\left(v^{\mathcal{M}}_{j+1,2}+v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,4}\right)\left(v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j+1,4}+\left(v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,4}+1\right)v^{\mathcal{M}}_{j,5}\right)+1}{v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,3}} = \left(v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j,4}\right)\left(v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,5}\right)\\
v^{\mathcal{I}}_{3,3} &= \frac{v^{\mathcal{M}}_{j+1,2}\left(v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j,4}\right)\left(v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,5}\right)+1}{v^{\mathcal{M}}_{j+1,2}+v^{\mathcal{M}}_{j,3}v^{\mathcal{M}}_{j,4}} = v^{\mathcal{M}}_{j+1,2}v^{\mathcal{M}}_{j,5}+v^{\mathcal{M}}_{j+1,3}v^{\mathcal{M}}_{j+1,4}\\
v^{\mathcal{I}}_{4,2} &= \frac{1+v^{\mathcal{M}}_{j+2,1}\left(v^{\mathcal{M}}_{j+1,2}v^{\mathcal{M}}_{j,5}+v^{\mathcal{M}}_{j+1,3}v^{\mathcal{M}}_{j+1,4}\right)}{v^{\mathcal{M}}_{j+1,2}} = v^{\mathcal{M}}_{j+2,2}+v^{\mathcal{M}}_{j+2,1}v^{\mathcal{M}}_{j,5}\\
v^{\mathcal{I}}_{5,1} &= \frac{1+v^{\mathcal{M}}_{j+2,2}+v^{\mathcal{M}}_{j+2,1}v^{\mathcal{M}}_{j,5}}{v^{\mathcal{M}}_{j+2,1}} = \underline{\underline{v^{\mathcal{M}}_{j+3,1}+v^{\mathcal{M}}_{j,5}}}
\end{aligned}$$

Expressing $v^{\delta}_{11,1}$ is analogous to $v^{\delta}_{5,1}$, completing the expression of entries of $\mathcal{C}(\mathcal{Q}_{\mathcal{M}'})$ by $\mathcal{M}$. ∎

For type $\mathbf{E}_6$ *to* $\mathbf{E}_7$ *(4.1c)(4.2c).* We already show the expressions of $v^{\mathcal{M}'}_{17,1}, \cdots, v^{\mathcal{M}'}_{20,1}, v^{\mathcal{M}'}_{2,1}, \cdots, v$
Applying tedious frieze rules, we express $v^{\mathcal{M}'}_{6,1}$ by entries of $\mathcal{M}$ and demonstrate the periodicity of type $\mathbf{E}_7$.

$$\begin{aligned}
v^{\mathcal{M}'}_{2,5} &= \frac{1+v^{\mathcal{M}'}_{2,4}}{v^{\mathcal{M}'}_{1,5}} = \frac{1+v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j,5}}{v^{\mathcal{M}}_{j,4}} = v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,5}\\
v^{\mathcal{M}'}_{2,6} &= \frac{1+v^{\mathcal{M}'}_{2,4}v^{\mathcal{M}'}_{1,7}}{v^{\mathcal{M}'}_{1,6}} = \frac{1+v^{\mathcal{M}}_{j,6}\left(v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j,5}\right)}{v^{\mathcal{M}}_{j,5}} = v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j,6}\\
v^{\mathcal{M}'}_{2,7} &= \frac{1+v^{\mathcal{M}'}_{2,7}}{v^{\mathcal{M}'}_{1,7}} = \frac{1+v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j,6}}{v^{\mathcal{M}}_{j,6}} = v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j,4}\\
v^{\mathcal{M}'}_{3,4} &= \frac{1+v^{\mathcal{M}'}_{3,3}v^{\mathcal{M}'}_{2,5}v^{\mathcal{M}'}_{2,6}}{v^{\mathcal{M}'}_{2,4}} = \frac{1+v^{\mathcal{M}}_{j+1,3}\left(v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,5}\right)\left(v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j,6}\right)}{v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j,5}} = v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+1,3}v^{\mathcal{M}}_{j,6}\\
v^{\mathcal{M}'}_{3,5} &= \frac{1+v^{\mathcal{M}'}_{3,4}}{v^{\mathcal{M}'}_{2,5}} = \frac{1+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+1,3}v^{\mathcal{M}}_{j,6}}{v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,5}} = \frac{v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j+1,5}}{v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,5}} = v^{\mathcal{M}}_{j+1,5}\\
v^{\mathcal{M}'}_{3,6} &= \frac{1+v^{\mathcal{M}'}_{3,4}v^{\mathcal{M}'}_{2,7}}{v^{\mathcal{M}'}_{2,6}} = \frac{1+\left(v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+1,3}v^{\mathcal{M}}_{j,6}\right)\left(v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j,4}\right)}{v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j,6}} = v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,6}\\
v^{\mathcal{M}'}_{3,7} &= \frac{1+v^{\mathcal{M}'}_{3,6}}{v^{\mathcal{M}'}_{2,7}} = \frac{1+v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,6}}{v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j,4}} = \frac{v^{\mathcal{M}}_{j,4}v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,6}}{v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j,4}} = v^{\mathcal{M}}_{j+1,4}\\
v^{\mathcal{M}'}_{4,3} &= \frac{1+v^{\mathcal{M}'}_{4,2}v^{\mathcal{M}'}_{3,4}}{v^{\mathcal{M}'}_{3,3}} = \frac{1+v^{\mathcal{M}}_{j+2,2}\left(v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+1,3}v^{\mathcal{M}}_{j,6}\right)}{v^{\mathcal{M}}_{j+1,3}} = v^{\mathcal{M}}_{j+2,3}+v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j,6}\\
v^{\mathcal{M}'}_{4,4} &= \frac{1+v^{\mathcal{M}'}_{4,3}v^{\mathcal{M}'}_{3,5}v^{\mathcal{M}'}_{3,6}}{v^{\mathcal{M}'}_{3,4}} = \frac{1+v^{\mathcal{M}}_{j+1,5}\left(v^{\mathcal{M}}_{j+2,3}+v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j,6}\right)\left(v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,6}\right)}{v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+1,3}v^{\mathcal{M}}_{j,6}} = v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}\\
v^{\mathcal{M}'}_{4,5} &= \frac{1+v^{\mathcal{M}'}_{4,4}}{v^{\mathcal{M}'}_{3,5}} = \frac{1+v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}}{v^{\mathcal{M}}_{j+1,5}} = \frac{v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j+1,5}}{v^{\mathcal{M}}_{j+1,5}} = v^{\mathcal{M}}_{j+2,2}+v^{\mathcal{M}}_{j+2,5}\\
v^{\mathcal{M}'}_{4,6} &= \frac{1+v^{\mathcal{M}'}_{4,4}v^{\mathcal{M}'}_{3,7}}{v^{\mathcal{M}'}_{3,6}} = \frac{1+v^{\mathcal{M}}_{j+1,4}\left(v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}\right)}{v^{\mathcal{M}}_{j+1,3}+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+1,6}} = v^{\mathcal{M}}_{j+2,3}\\
v^{\mathcal{M}'}_{4,7} &= \frac{1+v^{\mathcal{M}'}_{4,6}}{v^{\mathcal{M}'}_{3,7}} = \frac{1+v^{\mathcal{M}}_{j+2,3}}{v^{\mathcal{M}}_{j+1,4}} = v^{\mathcal{M}}_{j+2,4}\\
v^{\mathcal{M}'}_{5,2} &= \frac{1+v^{\mathcal{M}'}_{5,1}v^{\mathcal{M}'}_{4,3}}{v^{\mathcal{M}'}_{4,2}} = \frac{1+v^{\mathcal{M}}_{j+3,1}\left(v^{\mathcal{M}}_{j+2,3}+v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j,6}\right)}{v^{\mathcal{M}}_{j+2,2}} = v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j,6} = v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j,6}\\
v^{\mathcal{M}'}_{5,3} &= \frac{1+v^{\mathcal{M}'}_{5,2}v^{\mathcal{M}'}_{4,4}}{v^{\mathcal{M}'}_{4,3}} = \frac{1+\left(v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j,6}\right)\left(v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}\right)}{v^{\mathcal{M}}_{j+2,3}+v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j,6}} = v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+1,6}\\
v^{\mathcal{M}'}_{5,4} &= \frac{1+v^{\mathcal{M}'}_{5,3}v^{\mathcal{M}'}_{4,5}v^{\mathcal{M}'}_{4,6}}{v^{\mathcal{M}'}_{4,4}} = \frac{1+v^{\mathcal{M}}_{j+2,3}\left(v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+1,6}\right)}{v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}} = v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+2,3}
\end{aligned}$$

$$v^{\mathcal{M}'}_{5,5} = \frac{1+v^{\mathcal{M}'}_{5,4}}{v^{\mathcal{M}'}_{4,5}} = \frac{1+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+2,3}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,5}}{v^{\mathcal{M}}_{j+2,2}+v^{\mathcal{M}}_{j+2,5}} = \frac{v^{\mathcal{M}}_{j+2,2}v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+2,5}v^{\mathcal{M}}_{j+3,2}}{=} v^{\mathcal{M}}_{j+3,2}$$

$$v^{\mathcal{M}'}_{5,6} = \frac{1+v^{\mathcal{M}'}_{5,4}v^{\mathcal{M}'}_{4,7}}{v^{\mathcal{M}'}_{4,6}} = \frac{1+v^{\mathcal{M}}_{j+2,4}\left(v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+2,3}\right)}{v^{\mathcal{M}}_{j+2,3}} = v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+2,4}$$

$$v^{\mathcal{M}'}_{5,7} = \frac{1+v^{\mathcal{M}'}_{5,6}}{v^{\mathcal{M}'}_{4,7}} = \frac{1+v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+2,4}}{v^{\mathcal{M}}_{j+2,4}} = \frac{v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j+3,4}+v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j+3,1}}{=} v^{\mathcal{M}}_{j+3,4}+v^{\mathcal{M}}_{j+3,1}$$

$$v^{\mathcal{M}'}_{6,1} = \frac{1+v^{\mathcal{M}'}_{5,2}}{v^{\mathcal{M}'}_{5,1}} = \frac{1+v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j,6}}{v^{\mathcal{M}}_{j+3,1}} = \frac{v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+4,1}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j,6}}{v^{\mathcal{M}}_{j+3,1}} = v^{\mathcal{M}}_{j+4,1}+v^{\mathcal{M}}_{j,6} = \underline{\underline{v^{\mathcal{M}}_{j+4,1}+v^{\mathcal{M}}_{j+9,1}}} \quad \text{(by (2.7))}$$

$$v^{\mathcal{M}'}_{6,2} = \frac{1+v^{\mathcal{M}'}_{6,1}v^{\mathcal{M}'}_{5,3}}{v^{\mathcal{M}'}_{5,2}} = \frac{1+\left(v^{\mathcal{M}}_{j+4,1}+v^{\mathcal{M}}_{j,6}\right)\left(v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+1,6}\right)}{v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j,6}} = v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j+1,5}$$

$$v^{\mathcal{M}'}_{6,3} = \frac{1+v^{\mathcal{M}'}_{6,2}v^{\mathcal{M}'}_{5,4}}{v^{\mathcal{M}'}_{5,3}} = \frac{1+\left(v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j+1,5}\right)\left(v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+2,3}\right)}{v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+1,6}} = v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+2,3}$$

$$v^{\mathcal{M}'}_{6,4} = \frac{1+v^{\mathcal{M}'}_{6,3}v^{\mathcal{M}'}_{5,5}v^{\mathcal{M}'}_{5,6}}{v^{\mathcal{M}'}_{5,4}} = \frac{1+v^{\mathcal{M}}_{j+3,2}\left(v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+2,3}\right)\left(v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+2,4}\right)}{v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+2,3}} = v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4}$$

$$v^{\mathcal{M}'}_{6,5} = \frac{1+v^{\mathcal{M}'}_{6,4}}{v^{\mathcal{M}'}_{5,5}} = \frac{1+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4}}{v^{\mathcal{M}}_{j+3,2}} = \frac{v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+4,2}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4}}{v^{\mathcal{M}}_{j+3,2}} = v^{\mathcal{M}}_{j+4,2}+v^{\mathcal{M}}_{j+2,4}$$

$$v^{\mathcal{M}'}_{6,6} = \frac{1+v^{\mathcal{M}'}_{6,4}v^{\mathcal{M}'}_{5,7}}{v^{\mathcal{M}'}_{5,6}} = \frac{1+\left(v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4}\right)\left(v^{\mathcal{M}}_{j+3,4}+v^{\mathcal{M}}_{j+3,1}\right)}{v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+2,4}} = v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,4}+v^{\mathcal{M}}_{j+3,2}$$

$$v^{\mathcal{M}'}_{6,7} = \frac{1+v^{\mathcal{M}'}_{6,6}}{v^{\mathcal{M}'}_{5,7}} = \frac{1+v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,4}}{v^{\mathcal{M}}_{j+3,4}+v^{\mathcal{M}}_{j+3,1}} = \frac{v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+4,1}+v^{\mathcal{M}}_{j+3,4}+v^{\mathcal{M}}_{j+4,1}}{v^{\mathcal{M}}_{j+3,1}+v^{\mathcal{M}}_{j+3,4}} = v^{\mathcal{M}}_{j+4,1}$$

We can proceed to further express values of $\Gamma_{16}(\mathcal{M}')$ by entries of $\mathcal{M}$, using the frieze rules of $\mathcal{M}$:

$$v^{\mathcal{M}'}_{16,6} = \frac{1+v^{\mathcal{M}'}_{17,4}v^{\mathcal{M}'}_{16,7}}{v^{\mathcal{M}'}_{17,6}} = \frac{1+\left(v^{\mathcal{M}}_{j-3,4}+v^{\mathcal{M}}_{j-4,4}v^{\mathcal{M}}_{j-4,5}\right)v^{\mathcal{M}}_{j-5,6}}{v^{\mathcal{M}}_{j-4,5}} = \frac{v^{\mathcal{M}}_{j-4,5}v^{\mathcal{M}}_{j-5,5}+v^{\mathcal{M}}_{j-4,4}v^{\mathcal{M}}_{j-4,5}v^{\mathcal{M}}_{j-5,6}}{v^{\mathcal{M}}_{j-4,5}} = v^{\mathcal{M}}_{j-5,5}+v^{\mathcal{M}}_{j-4,4}v^{\mathcal{M}}_{j-5,6}$$

$$v^{\mathcal{M}'}_{16,5} = \frac{1+v^{\mathcal{M}'}_{17,4}}{v^{\mathcal{M}'}_{17,5}} = \frac{1+v^{\mathcal{M}}_{j-4,3}+v^{\mathcal{M}}_{j-4,4}v^{\mathcal{M}}_{j-4,5}}{v^{\mathcal{M}}_{j-4,4}} = \frac{v^{\mathcal{M}}_{j-4,4}v^{\mathcal{M}}_{j-5,4}+v^{\mathcal{M}}_{j-4,4}v^{\mathcal{M}}_{j-4,5}}{v^{\mathcal{M}}_{j-4,4}} = v^{\mathcal{M}}_{j-5,4}+v^{\mathcal{M}}_{j-4,5}$$

$$v^{\mathcal{M}'}_{16,4} = \frac{1+v^{\mathcal{M}'}_{17,3}v^{\mathcal{M}'}_{16,5}v^{\mathcal{M}'}_{16,6}}{v^{\mathcal{M}'}_{17,4}} = \frac{1+v^{\mathcal{M}}_{j-4,3}\left(v^{\mathcal{M}}_{j-5,4}+v^{\mathcal{M}}_{j-4,5}\right)\left(v^{\mathcal{M}}_{j-5,5}+v^{\mathcal{M}}_{j-4,4}v^{\mathcal{M}}_{j-5,6}\right)}{v^{\mathcal{M}}_{j-4,3}+v^{\mathcal{M}}_{j-4,4}v^{\mathcal{M}}_{j-4,5}} = v^{\mathcal{M}}_{j-5,4}v^{\mathcal{M}}_{j-5,5}+v^{\mathcal{M}}_{j-4,3}v^{\mathcal{M}}_{j-5,6}$$

$$v^{\mathcal{M}'}_{16,3} = \frac{1+v^{\mathcal{M}'}_{17,2}v^{\mathcal{M}'}_{16,4}}{v^{\mathcal{M}'}_{17,3}} = \frac{1+v^{\mathcal{M}}_{j-4,2}\left(v^{\mathcal{M}}_{j-5,4}v^{\mathcal{M}}_{j-5,5}+v^{\mathcal{M}}_{j-4,3}v^{\mathcal{M}}_{j-5,6}\right)}{v^{\mathcal{M}}_{j-4,3}} = \frac{v^{\mathcal{M}}_{j-4,3}v^{\mathcal{M}}_{j-5,3}+v^{\mathcal{M}}_{j-4,3}v^{\mathcal{M}}_{j-4,2}v^{\mathcal{M}}_{j-5,6}}{v^{\mathcal{M}}_{j-4,3}} = v^{\mathcal{M}}_{j-5,3}+v^{\mathcal{M}}_{j-4,2}v^{\mathcal{M}}_{j-5,6}$$

$$v^{\mathcal{M}'}_{16,2} = \frac{1+v^{\mathcal{M}'}_{17,1}v^{\mathcal{M}'}_{16,3}}{v^{\mathcal{M}'}_{17,2}} = \frac{1+v^{\mathcal{M}}_{j-4,1}\left(v^{\mathcal{M}}_{j-5,3}+v^{\mathcal{M}}_{j-4,2}v^{\mathcal{M}}_{j-5,6}\right)}{v^{\mathcal{M}}_{j-4,2}} = \frac{v^{\mathcal{M}}_{j-4,2}v^{\mathcal{M}}_{j-5,2}+v^{\mathcal{M}}_{j-4,2}v^{\mathcal{M}}_{j-4,1}v^{\mathcal{M}}_{j-5,6}}{v^{\mathcal{M}}_{j-4,2}} = v^{\mathcal{M}}_{j-5,2}+v^{\mathcal{M}}_{j-4,1}v^{\mathcal{M}}_{j-5,6}$$

$$v^{\mathcal{M}'}_{16,1} = \frac{1+v^{\mathcal{M}'}_{16,2}}{v^{\mathcal{M}'}_{17,1}} = \frac{1+v^{\mathcal{M}}_{j-5,2}+v^{\mathcal{M}}_{j-4,1}v^{\mathcal{M}}_{j-5,6}}{v^{\mathcal{M}}_{j-4,1}} = \frac{v^{\mathcal{M}}_{j-4,1}v^{\mathcal{M}}_{j-5,1}+v^{\mathcal{M}}_{j-4,1}v^{\mathcal{M}}_{j-5,6}}{v^{\mathcal{M}}_{j-4,1}} = v^{\mathcal{M}}_{j-5,1}+v^{\mathcal{M}}_{j-5,6}$$

By Proposition 2.13 (2.7)(2.8), and the 14-periodic nature of $\mathcal{M}$ by Theorem 2.9, we have

$$
\begin{aligned}
v^{\mathcal{M}'}_{16,1} &= v^{\mathcal{M}}_{j-5,1}+v^{\mathcal{M}}_{j-5,6} = v^{\mathcal{M}}_{j+9,1}+v^{\mathcal{M}}_{j+4,1} = v^{\mathcal{M}'}_{6,1} && \text{(by Theorem 2.9; (2.7)}\\
v^{\mathcal{M}'}_{16,2} &= v^{\mathcal{M}}_{j-5,2}+v^{\mathcal{M}}_{j-4,1}v^{\mathcal{M}}_{j-5,6} = v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+1,6} = v^{\mathcal{M}'}_{6,2} && \text{(by (2.7))}\\
v^{\mathcal{M}'}_{16,3} &= v^{\mathcal{M}}_{j-5,3}+v^{\mathcal{M}}_{j-4,2}v^{\mathcal{M}}_{j-5,6} = v^{\mathcal{M}}_{j+2,3}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+2,5} = v^{\mathcal{M}'}_{6,3} && \text{(by (2.7)(2.8))}\\
v^{\mathcal{M}'}_{16,4} &= v^{\mathcal{M}}_{j-5,4}v^{\mathcal{M}}_{j-5,5}+v^{\mathcal{M}}_{j-4,3}v^{\mathcal{M}}_{j-5,6} = v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3} = v^{\mathcal{M}'}_{6,4} && \text{(by (2.7)(2.8)}\\
v^{\mathcal{M}'}_{16,5} &= v^{\mathcal{M}}_{j-5,4}+v^{\mathcal{M}}_{j-4,5} = v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+4,2} = v^{\mathcal{M}'}_{6,5} && \text{(by (2.8))}\\
v^{\mathcal{M}'}_{16,6} &= v^{\mathcal{M}}_{j-5,5}+v^{\mathcal{M}}_{j-4,4}v^{\mathcal{M}}_{j-5,6} = v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,4} = v^{\mathcal{M}'}_{6,6} && \text{(by (2.7)(2.8))}\\
v^{\mathcal{M}'}_{16,7} &= v^{\mathcal{M}}_{j-5,6} = v^{\mathcal{M}}_{j+4,1} = v^{\mathcal{M}'}_{6,7} && \text{(by (2.7))}
\end{aligned}
$$

This gives $\Gamma_6(\mathcal{M}') = \Gamma_{16}(\mathcal{M}')$, ensuring the maximum period of *unit-present* quiddity sequence of multiplicative type $\mathbf{E}_7$ friezes is 10. ■

For type $\mathbf{E}_7$ *to* $\mathbf{E}_8$ *(4.1d)(4.2d).* Having expressed $v^{\mathcal{M}'}_{28,1}, \cdots, v^{\mathcal{M}'}_{32,1}, v^{\mathcal{M}'}_{2,1}, \cdots, v^{\mathcal{M}'}_{6,1}$, we now extend to $\Gamma_{11}(\mathcal{M}')$ by frieze rules, thus demonstrating the periodicity of type $\mathbf{E}_8$.

$$v^{\mathcal{M}'}_{2,6} = \frac{1+v^{\mathcal{M}'}_{2,5}}{v^{\mathcal{M}}_{1,6}} = \frac{1+v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j,6}}{v^{\mathcal{M}}_{j,5}} = \frac{v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j,6}}{v^{\mathcal{M}}_{j,5}} = v^{\mathcal{M}}_{j+1,5} + v^{\mathcal{M}}_{j,6}$$

$$v^{\mathcal{M}'}_{2,7} = \frac{1+v^{\mathcal{M}'}_{2,5}v^{\mathcal{M}'}_{1,8}}{v^{\mathcal{M}'}_{1,7}} = \frac{1+v^{\mathcal{M}}_{j,7}\left(v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j,6}\right)}{v^{\mathcal{M}}_{j,6}} = \frac{v^{\mathcal{M}}_{j,6}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j,6}v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j,7}}{v^{\mathcal{M}}_{j,6}} = v^{\mathcal{M}}_{j+1,6} + v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j,7}$$

$$v^{\mathcal{M}'}_{2,8} = \frac{1+v^{\mathcal{M}'}_{2,7}}{v^{\mathcal{M}'}_{1,8}} = \frac{1+v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j,7}}{v^{\mathcal{M}}_{j,7}} = \frac{v^{\mathcal{M}}_{j,7}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j,7}v^{\mathcal{M}}_{j,5}}{v^{\mathcal{M}}_{j,7}} = v^{\mathcal{M}}_{j+1,7} + v^{\mathcal{M}}_{j,5}$$

$$v^{\mathcal{M}'}_{3,5} = \frac{1+v^{\mathcal{M}'}_{3,4}v^{\mathcal{M}'}_{2,6}v^{\mathcal{M}'}_{2,7}}{v^{\mathcal{M}'}_{2,5}} = \frac{1+v^{\mathcal{M}}_{j+1,4}\left(v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j,6}\right)\left(v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j,7}\right)}{v^{\mathcal{M}}_{j+1,4}+v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j,6}} = v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+1,6} + v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j,7}$$

$$v^{\mathcal{M}'}_{3,6} = \frac{1+v^{\mathcal{M}'}_{3,5}}{v^{\mathcal{M}'}_{2,6}} = \frac{1+v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j,7}}{v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j,6}} = \frac{v^{\mathcal{M}}_{j+1,6}v^{\mathcal{M}}_{j,6}+v^{\mathcal{M}}_{j+1,6}v^{\mathcal{M}}_{j+1,5}}{v^{\mathcal{M}}_{j+1,5}+v^{\mathcal{M}}_{j,6}} = v^{\mathcal{M}}_{j+1,6}$$

$$v^{\mathcal{M}'}_{3,7} = \frac{1+v^{\mathcal{M}'}_{3,5}v^{\mathcal{M}'}_{2,8}}{v^{\mathcal{M}'}_{2,7}} = \frac{1+\left(v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j,6}\right)\left(v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j,5}\right)}{v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j,5}v^{\mathcal{M}}_{j,7}} = v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+1,7} + v^{\mathcal{M}}_{j+1,4}$$

$$v^{\mathcal{M}'}_{3,8} = \frac{1+v^{\mathcal{M}'}_{3,7}}{v^{\mathcal{M}'}_{2,8}} = \frac{1+v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j+1,4}}{v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j,5}} = \frac{v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j,5}+v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j,7}}{v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j,5}} = v^{\mathcal{M}}_{j+1,5}$$

$$v^{\mathcal{M}'}_{4,4} = \frac{1+v^{\mathcal{M}'}_{4,3}v^{\mathcal{M}'}_{3,5}}{v^{\mathcal{M}'}_{3,4}} = \frac{1+v^{\mathcal{M}}_{j+2,3}\left(v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j,7}\right)}{v^{\mathcal{M}}_{j+1,4}} = \frac{v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+1,4}v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j,7}}{v^{\mathcal{M}}_{j+1,4}} = v^{\mathcal{M}}_{j+2,4} + v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j,7}$$

$$v^{\mathcal{M}'}_{4,5} = \frac{1+v^{\mathcal{M}'}_{4,4}v^{\mathcal{M}'}_{3,6}v^{\mathcal{M}'}_{3,7}}{v^{\mathcal{M}'}_{3,5}} = \frac{1+v^{\mathcal{M}}_{j+1,6}\left(v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j,7}\right)\left(v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j+1,4}\right)}{v^{\mathcal{M}}_{j+2,5}v^{\mathcal{M}}_{j+2,6}+v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j+1,7}} = v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j+1,7} + v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}$$

$$v^{\mathcal{M}'}_{4,6} = \frac{1+v^{\mathcal{M}'}_{4,5}}{v^{\mathcal{M}'}_{3,6}} = \frac{1+v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j,7}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}}{v^{\mathcal{M}}_{j+1,6}} = \frac{v^{\mathcal{M}}_{j+1,6}v^{\mathcal{M}}_{j+2,6}+v^{\mathcal{M}}_{j+1,6}v^{\mathcal{M}}_{j+2,3}}{v^{\mathcal{M}}_{j+1,6}} = v^{\mathcal{M}}_{j+2,6} + v^{\mathcal{M}}_{j+2,3}$$

$$v^{\mathcal{M}'}_{4,7} = \frac{1+v^{\mathcal{M}'}_{4,5}v^{\mathcal{M}'}_{3,8}}{v^{\mathcal{M}'}_{3,7}} = \frac{1+v^{\mathcal{M}}_{j+1,5}\left(v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}\right)}{v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j+1,4}} = v^{\mathcal{M}}_{j+2,4}$$

$$v^{\mathcal{M}'}_{4,8} = \frac{1+v^{\mathcal{M}'}_{4,7}}{v^{\mathcal{M}'}_{3,8}} = \frac{1+v^{\mathcal{M}}_{j+2,4}}{v^{\mathcal{M}}_{j+1,5}} = \frac{v^{\mathcal{M}}_{j+1,5}v^{\mathcal{M}}_{j+2,5}}{v^{\mathcal{M}}_{j+1,5}} = v^{\mathcal{M}}_{j+2,5}$$

$$v^{\mathcal{M}'}_{5,3} = \frac{1+v^{\mathcal{M}'}_{5,2}v^{\mathcal{M}'}_{4,4}}{v^{\mathcal{M}'}_{4,3}} = \frac{1+v^{\mathcal{M}}_{j+3,2}\left(v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j,7}\right)}{v^{\mathcal{M}}_{j+2,3}} = \frac{v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,3}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j,7}}{v^{\mathcal{M}}_{j+2,3}} = v^{\mathcal{M}}_{j+3,3} + v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j,7}$$

$$v^{\mathcal{M}'}_{5,4} = \frac{1+v^{\mathcal{M}'}_{5,3}v^{\mathcal{M}'}_{4,5}}{v^{\mathcal{M}'}_{4,4}} = \frac{1+\left(v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j,7}\right)\left(v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}\right)}{v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j,7}} = v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+1,7} + v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+1,6}$$

$$v^{\mathcal{M}'}_{5,5} = \frac{1+v^{\mathcal{M}'}_{5,4}v^{\mathcal{M}'}_{4,6}v^{\mathcal{M}'}_{4,7}}{v^{\mathcal{M}'}_{4,5}} = \frac{1+v^{\mathcal{M}}_{j+2,4}\left(v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,7}\right)\left(v^{\mathcal{M}}_{j+2,6}+v^{\mathcal{M}}_{j+2,3}\right)}{v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+1,6}} = v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4} + v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,6}$$

$$v^{\mathcal{M}'}_{5,6} = \frac{1+v^{\mathcal{M}'}_{5,5}}{v^{\mathcal{M}'}_{4,5}} = \frac{1+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,6}}{v^{\mathcal{M}}_{j+2,6}+v^{\mathcal{M}}_{j+2,3}} = \frac{v^{\mathcal{M}}_{j+2,3}v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+2,6}v^{\mathcal{M}}_{j+3,3}}{v^{\mathcal{M}}_{j+2,6}+v^{\mathcal{M}}_{j+2,6}} = v^{\mathcal{M}}_{j+3,3}$$

$$v^{\mathcal{M}'}_{5,7} = \frac{1+v^{\mathcal{M}'}_{5,5}v^{\mathcal{M}'}_{4,8}}{v^{\mathcal{M}'}_{4,7}} = \frac{1+v^{\mathcal{M}}_{j+2,5}\left(v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,6}\right)}{v^{\mathcal{M}}_{j+2,4}} = \frac{v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+2,4}v^{\mathcal{M}}_{j+3,4}}{v^{\mathcal{M}}_{j+2,4}}$$

$$v^{\mathcal{M}'}_{5,8} = \frac{1+v^{\mathcal{M}'}_{5,7}}{v^{\mathcal{M}'}_{4,8}} = \frac{1+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+3,4}}{v^{\mathcal{M}}_{j+2,5}} = \frac{v^{\mathcal{M}}_{j+2,5}v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+2,5}v^{\mathcal{M}}_{j+3,5}}{v^{\mathcal{M}}_{j+2,5}} = v^{\mathcal{M}}_{j+3,2} + v^{\mathcal{M}}_{j+3,5}$$

$$v^{\mathcal{M}'}_{6,2} = \frac{1+v^{\mathcal{M}'}_{6,1}v^{\mathcal{M}'}_{5,3}}{v^{\mathcal{M}'}_{5,2}} = \frac{1+v^{\mathcal{M}}_{j+4,1}\left(v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j,7}\right)}{v^{\mathcal{M}}_{j+3,2}} = \frac{v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+4,2}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j,7}}{v^{\mathcal{M}}_{j+3,2}} = v^{\mathcal{M}}_{j+4,2} + v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j,7}$$

$$v^{\mathcal{M}'}_{6,3} = \frac{1+v^{\mathcal{M}'}_{6,2}v^{\mathcal{M}'}_{5,4}}{v^{\mathcal{M}'}_{5,3}} = \frac{1+\left(v^{\mathcal{M}}_{j+4,2}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j,7}\right)\left(v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+1,6}\right)}{v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j,7}} = v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+1,6} + v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+1,7}$$

$$v^{\mathcal{M}'}_{6,4} = \frac{1+v^{\mathcal{M}'}_{6,3}v^{\mathcal{M}'}_{5,5}}{v^{\mathcal{M}'}_{5,4}} = \frac{1+\left(v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+1,7}\right)\left(v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,6}\right)}{v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+1,6}} = v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+2,4} + v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+2,6}$$

$$v^{\mathcal{M}'}_{6,5} = \frac{1+v^{\mathcal{M}'}_{6,4}v^{\mathcal{M}'}_{5,6}v^{\mathcal{M}'}_{5,7}}{v^{\mathcal{M}'}_{5,5}} = \frac{1+v^{\mathcal{M}}_{j+3,3}\left(v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+2,6}\right)\left(v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+3,4}\right)}{v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,6}} = v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+3,4} + v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,5}$$

$$v^{\mathcal{M}'}_{6,6} = \frac{1+v^{\mathcal{M}'}_{6,5}}{v^{\mathcal{M}'}_{5,6}} = \frac{1+v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+3,4}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,5}}{v^{\mathcal{M}}_{j+3,3}} = \frac{v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+4,3}+v^{\mathcal{M}}_{j+3,3}v^{[M,j+4,1]}_{,}v^{\mathcal{M}}_{j+2,5}}{v^{\mathcal{M}}_{j+3,3}} = v^{\mathcal{M}}_{j+4,3} + v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+2,5}$$

$$v^{\mathcal{M}'}_{6,7} = \frac{1+v^{\mathcal{M}'}_{6,5}v^{\mathcal{M}'}_{5,8}}{v^{\mathcal{M}'}_{5,7}} = \frac{1+\left(v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+3,4}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,5}\right)\left(v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+3,5}\right)}{v^{\mathcal{M}}_{j+3,2}v^{\mathcal{M}}_{j+2,5}+v^{\mathcal{M}}_{j+3,4}} = v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3} + v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+3,5}$$

$$v^{\mathcal{M}'}_{6,8} = \frac{1+v^{\mathcal{M}'}_{6,7}}{v^{\mathcal{M}'}_{5,8}} = \frac{1+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3}+v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+3,5}}{v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+3,5}} = \frac{v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+3,5}}{v^{\mathcal{M}}_{j+3,2}+v^{\mathcal{M}}_{j+3,5}} = v^{\mathcal{M}}_{j+4,2}$$

$$v^{\mathcal{M}'}_{7,1} = \frac{1+v^{\mathcal{M}'}_{6,2}}{v^{\mathcal{M}'}_{6,1}} = \frac{1+v^{\mathcal{M}}_{j+4,2}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j,7}}{v^{\mathcal{M}}_{j+4,1}} = \frac{v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+5,1}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j,7}}{v^{\mathcal{M}}_{j+4,1}} = \underline{\underline{v^{\mathcal{M}}_{j+5,1} + v^{\mathcal{M}}_{j,7}}}$$

$$v^{\mathcal{M}'}_{7,2} = \frac{1+v^{\mathcal{M}'}_{7,1}v^{\mathcal{M}'}_{6,3}}{v^{\mathcal{M}'}_{6,2}} = \frac{1+\left(v^{\mathcal{M}}_{j+5,1}+v^{\mathcal{M}}_{j,7}\right)\left(v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+1,7}\right)}{v^{\mathcal{M}}_{j+4,2}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j,7}} = v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+1,7} + v^{\mathcal{M}}_{j+1,6}$$

$$v^{\mathcal{M}'}_{7,3} = \frac{1+v^{\mathcal{M}'}_{7,2}v^{\mathcal{M}'}_{6,4}}{v^{\mathcal{M}'}_{6,3}} = \frac{1+\left(v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+1,7}+v^{\mathcal{M}}_{j+1,6}\right)\left(v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+2,6}\right)}{v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+1,6}+v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+1,7}} = v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+2,6} + v^{\mathcal{M}}_{j+2,4}$$

$$v^{\mathcal{M}'}_{7,4} = \frac{1+v^{\mathcal{M}'}_{7,3}v^{\mathcal{M}'}_{6,5}}{v^{\mathcal{M}'}_{6,4}} = \frac{1+\left(v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+2,6}+v^{\mathcal{M}}_{j+2,4}\right)\left(v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+3,4}+v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,5}\right)}{v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+2,4}+v^{\mathcal{M}}_{j+4,2}v^{\mathcal{M}}_{j+2,6}} = v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+3,4} + v^{\mathcal{M}}_{j+3,3}v^{\mathcal{M}}_{j+2,5}$$

$$v_{7,5}^{\mathcal{M}'} = \frac{1+v_{7,4}^{\mathcal{M}'}v_{6,6}^{\mathcal{M}'}v_{6,7}^{\mathcal{M}'}}{v_{6,5}^{\mathcal{M}'}} = \frac{1+\left(v_{j+5,1}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+3,3}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}\right)\left(v_{j+4,3}^{\mathcal{M}}+v_{j+4,1}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}\right)\left(v_{j+4,1}^{\mathcal{M}}v_{j+3,3}^{\mathcal{M}}+v_{j+4,2}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}\right)}{v_{j+4,2}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+4,1}^{\mathcal{M}}v_{j+3,3}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}}$$
$$= v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}} + v_{j+4,1}^{\mathcal{M}}v_{j+5,1}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}} + v_{j+4,2}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}} + v_{j+4,1}^{\mathcal{M}}v_{j+3,3}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}$$

$$v_{7,6}^{\mathcal{M}'} = \frac{1+v_{7,5}^{\mathcal{M}'}}{v_{6,6}^{\mathcal{M}'}} = \frac{1+\left[\left(v_{j+4,3}^{\mathcal{M}}+v_{j+4,1}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}\right)\left(v_{j+3,3}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}\right)-1\right]}{v_{j+4,3}^{\mathcal{M}}+v_{j+4,1}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}} = v_{j+5,1}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}} + v_{j+3,3}^{\mathcal{M}}$$

$$v_{7,7}^{\mathcal{M}'} = \frac{1+v_{7,5}^{\mathcal{M}'}v_{6,8}^{\mathcal{M}'}}{v_{6,7}^{\mathcal{M}'}} = \frac{1+v_{j+4,2}^{\mathcal{M}}\left[\left(v_{j+4,3}^{\mathcal{M}}+v_{j+4,1}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}\right)\left(v_{j+3,3}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}\right)-1\right]}{v_{j+4,1}^{\mathcal{M}}v_{j+3,3}^{\mathcal{M}}+v_{j+4,2}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}} = v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}} + v_{j+4,2}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}$$

$$v_{7,8}^{\mathcal{M}'} = \frac{1+v_{7,7}^{\mathcal{M}'}}{v_{6,8}^{\mathcal{M}'}} = \frac{1+v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}+v_{j+4,2}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}}{v_{j+4,2}^{\mathcal{M}}} = \frac{v_{j+4,2}^{\mathcal{M}}v_{j+5,2}^{\mathcal{M}}+v_{j+4,2}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}}{v_{j+4,2}^{\mathcal{M}}} = v_{j+5,2}^{\mathcal{M}} + v_{j+2,5}^{\mathcal{M}}$$

$$v_{8,1}^{\mathcal{M}'} = \frac{1+v_{7,2}^{\mathcal{M}'}}{v_{7,1}^{\mathcal{M}'}} = \frac{1+v_{j+5,1}^{\mathcal{M}}v_{j+1,7}^{\mathcal{M}}+v_{j+1,6}^{\mathcal{M}}}{v_{j+5,1}^{\mathcal{M}}+v_{j,7}^{\mathcal{M}}} = \frac{v_{j+1,7}^{\mathcal{M}}v_{j+5,1}^{\mathcal{M}}+v_{j+1,7}^{\mathcal{M}}v_{j,7}^{\mathcal{M}}}{v_{j+5,1}^{\mathcal{M}}+v_{j,7}^{\mathcal{M}}} = \underline{\underline{v_{j+1,7}^{\mathcal{M}}}}$$

$$v_{8,2}^{\mathcal{M}'} = \frac{1+v_{8,1}^{\mathcal{M}'}v_{7,3}^{\mathcal{M}'}}{v_{7,2}^{\mathcal{M}'}} = \frac{1+v_{j+1,7}^{\mathcal{M}}\left(v_{j+5,1}^{\mathcal{M}}v_{j+2,6}^{\mathcal{M}}+v_{j+2,4}^{\mathcal{M}}\right)}{v_{j+5,1}^{\mathcal{M}}v_{j+1,7}^{\mathcal{M}}+v_{j+1,6}^{\mathcal{M}}} = \frac{v_{j+2,6}^{\mathcal{M}}v_{j+1,6}^{\mathcal{M}}+v_{j+2,6}^{\mathcal{M}}v_{j+5,1}^{\mathcal{M}}v_{j+1,7}^{\mathcal{M}}}{v_{j+1,6}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+1,7}^{\mathcal{M}}} = v_{j+2,6}^{\mathcal{M}}$$

$$v_{8,3}^{\mathcal{M}'} = \frac{1+v_{8,2}^{\mathcal{M}'}v_{7,4}^{\mathcal{M}'}}{v_{7,3}^{\mathcal{M}'}} = \frac{1+v_{j+2,6}^{\mathcal{M}}\left(v_{j+5,1}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+3,3}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}\right)}{v_{j+5,1}^{\mathcal{M}}v_{j+2,6}^{\mathcal{M}}+v_{j+2,4}^{\mathcal{M}}} = \frac{v_{j+3,4}^{\mathcal{M}}v_{j+2,4}^{\mathcal{M}}+v_{j+3,4}^{\mathcal{M}}v_{j+5,1}^{\mathcal{M}}v_{j+2,6}^{\mathcal{M}}}{v_{j+2,4}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+2,6}^{\mathcal{M}}} = v_{j+3,4}^{\mathcal{M}}$$

$$v_{8,4}^{\mathcal{M}'} = \frac{1+v_{8,3}^{\mathcal{M}'}v_{7,5}^{\mathcal{M}'}}{v_{7,4}^{\mathcal{M}'}} = \frac{1+v_{j+3,4}^{\mathcal{M}}\left[\left(v_{j+4,3}^{\mathcal{M}}+v_{j+4,1}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}\right)\left(v_{j+3,3}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}\right)-1\right]}{v_{j+5,1}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+3,3}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}} = v_{j+4,1}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}} + v_{j+3,5}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}$$

$$v_{8,5}^{\mathcal{M}'} = \frac{1+v_{8,4}^{\mathcal{M}'}v_{7,6}^{\mathcal{M}'}v_{7,7}^{\mathcal{M}'}}{v_{7,5}^{\mathcal{M}'}} = \frac{1+\left(v_{j+4,1}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+3,5}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}\right)\left(v_{j+5,1}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}+v_{j+3,3}^{\mathcal{M}}\right)\left(v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}+v_{j+4,2}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}\right)}{\left(v_{j+4,3}^{\mathcal{M}}+v_{j+4,1}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}\right)\left(v_{j+3,3}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}\right)-1}$$
$$= v_{j+4,2}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}} + v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}$$

$$v_{8,6}^{\mathcal{M}'} = \frac{1+v_{8,5}^{\mathcal{M}'}}{v_{7,6}^{\mathcal{M}'}} = \frac{1+v_{j+4,2}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}}{v_{j+5,1}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}+v_{j+3,3}^{\mathcal{M}}} = \frac{v_{j+4,3}^{\mathcal{M}}v_{j+3,3}^{\mathcal{M}}+v_{j+4,3}^{\mathcal{M}}v_{j+5,1}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}}{v_{j+5,1}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}+v_{j+3,3}^{\mathcal{M}}} = v_{j+4,3}^{\mathcal{M}}$$

$$v_{8,7}^{\mathcal{M}'} = \frac{1+v_{8,5}^{\mathcal{M}'}v_{7,8}^{\mathcal{M}'}}{v_{7,7}^{\mathcal{M}'}} = \frac{1+\left(v_{j+4,2}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}\right)\left(v_{j+5,2}^{\mathcal{M}}+v_{j+2,5}^{\mathcal{M}}\right)}{v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}+v_{j+4,2}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}} = v_{j+3,4}^{\mathcal{M}} + v_{j+3,5}^{\mathcal{M}}v_{j+5,2}^{\mathcal{M}}$$

$$v_{8,8}^{\mathcal{M}'} = \frac{1+v_{8,7}^{\mathcal{M}'}}{v_{7,8}^{\mathcal{M}'}} = \frac{1+v_{j+3,4}^{\mathcal{M}}+v_{j+3,5}^{\mathcal{M}}v_{j+5,2}^{\mathcal{M}}}{v_{j+5,2}^{\mathcal{M}}+v_{j+2,5}^{\mathcal{M}}} = \frac{v_{j+3,5}^{\mathcal{M}}v_{j+2,5}^{\mathcal{M}}+v_{j+3,5}^{\mathcal{M}}v_{j+5,2}^{\mathcal{M}}}{v^{\mathcal{M}}_{j+2,5+v_{j+5,2}^{\mathcal{M}}}} = v_{j+3,5}^{\mathcal{M}}$$

$$v_{9,1}^{\mathcal{M}'} = \frac{1+v_{8,2}^{\mathcal{M}'}}{v_{8,1}^{\mathcal{M}'}} = \frac{1+v_{j+2,6}^{\mathcal{M}}}{v_{j+1,7}^{\mathcal{M}}} = \frac{v_{j+2,7}^{\mathcal{M}}v_{j+1,7}^{\mathcal{M}}}{v_{j+1,7}^{\mathcal{M}}} = \underline{\underline{v_{j+2,7}^{\mathcal{M}}}}$$

$$v_{9,2}^{\mathcal{M}'} = \frac{1+v_{9,1}^{\mathcal{M}'}v_{8,3}^{\mathcal{M}'}}{v_{8,2}^{\mathcal{M}'}} = \frac{1+v_{j+2,7}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}}{v_{j+2,6}^{\mathcal{M}}} = \frac{v_{j+3,6}^{\mathcal{M}}v_{j+2,6}^{\mathcal{M}}}{v_{j+2,6}^{\mathcal{M}}} = v_{j+3,6}^{\mathcal{M}}$$

$$v_{9,3}^{\mathcal{M}'} = \frac{1+v_{9,2}^{\mathcal{M}'}v_{8,4}^{\mathcal{M}'}}{v_{8,3}^{\mathcal{M}'}} = \frac{1+v_{j+3,6}^{\mathcal{M}}\left(v_{j+4,1}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+3,5}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}\right)}{v_{j+3,4}^{\mathcal{M}}} = \frac{v_{j+3,4}^{\mathcal{M}}v_{j+4,4}^{\mathcal{M}}+v_{j+3,4}^{\mathcal{M}}v_{j+4,1}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}}{v_{j+3,4}^{\mathcal{M}}} = v_{j+4,1}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}} + v_{j+4,4}^{\mathcal{M}}$$

$$v_{9,4}^{\mathcal{M}'} = \frac{1+v_{9,3}^{\mathcal{M}'}v_{8,5}^{\mathcal{M}'}}{v_{8,4}^{\mathcal{M}'}} = \frac{1+\left(v_{j+4,1}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}+v_{j+4,4}^{\mathcal{M}}\right)\left(v_{j+4,2}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}\right)}{v_{j+4,1}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+3,5}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}} = v_{j+5,1}^{\mathcal{M}}v_{j+5,4}^{\mathcal{M}} + v_{j+4,2}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}$$

$$v_{9,5}^{\mathcal{M}'} = \frac{1+v_{9,4}^{\mathcal{M}'}v_{8,6}^{\mathcal{M}'}v_{8,7}^{\mathcal{M}'}}{v_{8,5}^{\mathcal{M}'}} = \frac{1+v_{j+4,3}^{\mathcal{M}}\left(v_{j+5,1}^{\mathcal{M}}v_{j+5,4}^{\mathcal{M}}+v_{j+4,2}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}\right)\left(v_{j+3,4}^{\mathcal{M}}+v_{j+3,5}^{\mathcal{M}}v_{j+5,2}^{\mathcal{M}}\right)}{v_{j+4,2}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+5,1}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}} = v_{j+5,2}^{\mathcal{M}}v_{j+4,4}^{\mathcal{M}} + v_{j+4,3}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}$$

$$v_{9,6}^{\mathcal{M}'} = \frac{1+v_{9,5}^{\mathcal{M}'}}{v_{8,6}^{\mathcal{M}'}} = \frac{1+v_{j+5,2}^{\mathcal{M}}v_{j+4,4}^{\mathcal{M}}+v_{j+4,3}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}}{v_{j+4,3}^{\mathcal{M}}} = \frac{v_{j+4,3}^{\mathcal{M}}v_{j+5,3}^{\mathcal{M}}+v_{j+4,3}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}}{v_{j+4,3}^{\mathcal{M}}} = v_{j+5,3}^{\mathcal{M}} + v_{j+3,6}^{\mathcal{M}}$$

$$v_{9,7}^{\mathcal{M}'} = \frac{1+v_{9,5}^{\mathcal{M}'}v_{8,8}^{\mathcal{M}'}}{v_{8,7}^{\mathcal{M}'}} = \frac{1+v_{j+3,8}^{\mathcal{M}}\left(v_{j+5,2}^{\mathcal{M}}v_{j+4,4}^{\mathcal{M}}+v_{j+4,3}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}\right)}{v_{j+3,4}^{\mathcal{M}}+v_{j+3,5}^{\mathcal{M}}v_{j+5,2}^{\mathcal{M}}} = \frac{v_{j+4,4}^{\mathcal{M}}v_{j+3,4}^{\mathcal{M}}+v_{j+4,4}^{\mathcal{M}}v_{j+5,2}^{\mathcal{M}}v_{j+3,5}^{\mathcal{M}}}{v_{j+3,4}^{\mathcal{M}}+v_{j+3,5}^{\mathcal{M}}v_{j+5,2}^{\mathcal{M}}} = v_{j+4,4}^{\mathcal{M}}$$

$$v_{9,8}^{\mathcal{M}'} = \frac{1+v_{9,7}^{\mathcal{M}'}}{v_{8,8}^{\mathcal{M}'}} = \frac{1+v_{j+4,4}^{\mathcal{M}}}{v_{j+3,5}^{\mathcal{M}}} = \frac{v_{j+3,5}^{\mathcal{M}}v_{j+4,5}^{\mathcal{M}}}{v_{j+3,5}^{\mathcal{M}}} = v_{j+4,5}^{\mathcal{M}}$$

$$v_{10,1}^{\mathcal{M}'} = \frac{1+v_{9,2}^{\mathcal{M}'}}{v_{9,1}^{\mathcal{M}'}} = \frac{1+v_{j+3,6}^{\mathcal{M}}}{v_{j+2,7}^{\mathcal{M}}} = \frac{v_{j+2,7}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}}{v_{j+2,7}^{\mathcal{M}}} = \underline{\underline{v_{j+3,7}^{\mathcal{M}}}}$$

$$v_{10,2}^{\mathcal{M}'} = \frac{1+v_{10,1}^{\mathcal{M}'}v_{9,3}^{\mathcal{M}'}}{v_{9,2}^{\mathcal{M}'}} = \frac{1+v_{j+3,7}^{\mathcal{M}}\left(v_{j+4,1}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}+v_{j+4,4}^{\mathcal{M}}\right)}{v_{j+3,6}^{\mathcal{M}}} = \frac{v_{j+3,6}^{\mathcal{M}}v_{j+4,1}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}+v_{j+3,6}^{\mathcal{M}}v_{j+4,6}^{\mathcal{M}}}{v_{j+3,6}^{\mathcal{M}}} = v_{j+4,1}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}} + v_{j+4,6}^{\mathcal{M}}$$

$$v_{10,3}^{\mathcal{M}'} = \frac{1+v_{10,2}^{\mathcal{M}'}v_{9,4}^{\mathcal{M}'}}{v_{9,3}^{\mathcal{M}'}} = \frac{1+\left(v_{j+4,1}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}+v_{j+4,6}^{\mathcal{M}}\right)\left(v_{j+5,1}^{\mathcal{M}}v_{j+5,4}^{\mathcal{M}}+v_{j+4,2}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}\right)}{v_{j+4,1}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}+v_{j+4,4}^{\mathcal{M}}}$$

$$v_{10,4}^{\mathcal{M}'} = \frac{1+v_{10,2}^{\mathcal{M}'}v_{9,5}^{\mathcal{M}'}}{v_{9,4}^{\mathcal{M}'}} = \frac{1+\left(v_{j+4,1}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}+v_{j+4,6}^{\mathcal{M}}\right)\left(v_{j+5,2}^{\mathcal{M}}v_{j+4,4}^{\mathcal{M}}+v_{j+4,3}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}\right)}{v_{j+5,1}^{\mathcal{M}}v_{j+5,4}^{\mathcal{M}}+v_{j+4,2}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}} = v_{j+3,7}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}} + v_{j+5,2}^{\mathcal{M}}v_{j+4,6}^{\mathcal{M}}$$

$$v_{10,5}^{\mathcal{M}'} = \frac{1+v_{10,4}^{\mathcal{M}'}v_{9,6}^{\mathcal{M}'}v_{9,7}^{\mathcal{M}'}}{v_{9,5}^{\mathcal{M}'}} = \frac{1+v_{j+4,4}^{\mathcal{M}}\left(v_{j+3,7}^{\mathcal{M}}v_{j+4,3}^{\mathcal{M}}+v_{j+5,2}^{\mathcal{M}}v_{j+4,6}^{\mathcal{M}}\right)\left(v_{j+5,3}^{\mathcal{M}}+v_{j+3,6}^{\mathcal{M}}\right)}{v_{j+5,2}^{\mathcal{M}}v_{j+4,4}^{\mathcal{M}}+v_{j+4,3}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}} = v_{j+5,3}^{\mathcal{M}}v_{j+4,6}^{\mathcal{M}} + v_{j+4,4}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}$$

$$v_{10,6}^{\mathcal{M}'} = \frac{1+v_{10,5}^{\mathcal{M}'}}{v_{9,6}^{\mathcal{M}'}} = \frac{1+v_{j+4,6}^{\mathcal{M}}v_{j+5,3}^{\mathcal{M}}+v_{j+4,4}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}}{v_{j+5,3}^{\mathcal{M}}+v_{j+3,6}^{\mathcal{M}}} = \frac{v_{j+4,6}^{\mathcal{M}}v_{j+5,3}^{\mathcal{M}}+v_{j+4,6}^{\mathcal{M}}v_{j+3,6}^{\mathcal{M}}}{v_{j+5,3}^{\mathcal{M}}+v_{j+3,6}^{\mathcal{M}}} = v_{j+4,6}^{\mathcal{M}}$$

$$v_{10,7}^{\mathcal{M}'} = \frac{1+v_{10,5}^{\mathcal{M}'}v_{9,8}^{\mathcal{M}'}}{v_{9,7}^{\mathcal{M}'}} = \frac{1+v_{j+4,5}^{\mathcal{M}}\left(v_{j+5,3}^{\mathcal{M}}v_{j+4,6}^{\mathcal{M}}+v_{j+4,4}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}\right)}{v_{j+4,4}^{\mathcal{M}}} = \frac{v_{j+4,4}^{\mathcal{M}}v_{j+5,4}^{\mathcal{M}}+v_{j+4,4}^{\mathcal{M}}v_{j+4,5}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}}{v_{j+4,4}^{\mathcal{M}}} = v_{j+5,4}^{\mathcal{M}} + v_{j+4,5}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}$$

$$v_{10,8}^{\mathcal{M}'} = \frac{1+v_{10,7}^{\mathcal{M}'}}{v_{9,8}^{\mathcal{M}'}} = \frac{1+v_{j+5,4}^{\mathcal{M}}+v_{j+4,5}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}}{v_{j+4,5}^{\mathcal{M}}} = \frac{v_{j+4,5}^{\mathcal{M}}v_{j+5,5}^{\mathcal{M}}+v_{j+4,5}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}}{v_{j+4,5}^{\mathcal{M}}} = v_{j+5,5}^{\mathcal{M}} + v_{j+3,7}^{\mathcal{M}}$$

$$v_{11,1}^{\mathcal{M}'} = \frac{1+v_{10,2}^{\mathcal{M}'}}{v_{10,1}^{\mathcal{M}'}} = \frac{1+v_{j+4,1}^{\mathcal{M}}v_{j+3,7}^{\mathcal{M}}+v_{j+4,6}^{\mathcal{M}}}{v_{j+3,7}^{\mathcal{M}}} = \frac{v_{j+3,7}^{\mathcal{M}}v_{j+4,7}^{\mathcal{M}}+v_{j+3,7}^{\mathcal{M}}v_{j+4,1}^{\mathcal{M}}}{v_{j+3,7}^{\mathcal{M}}} = \underline{\underline{v_{j+4,1}^{\mathcal{M}} + v_{j+4,7}^{\mathcal{M}}}}$$

$$
\begin{aligned}
v^{\mathcal{M}'}_{11,2} &= \frac{1+v^{\mathcal{M}'}_{11,1}v^{\mathcal{M}'}_{10,3}}{v^{\mathcal{M}'}_{10,2}} = \frac{1+\left(v^{\mathcal{M}}_{j+4,1}+v^{\mathcal{M}}_{j+4,7}\right)\left(v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+4,2}\right)}{v^{\mathcal{M}}_{j+4,1}v^{\mathcal{M}}_{j+3,7}+v^{\mathcal{M}}_{j+4,6}} = v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,2}\\
v^{\mathcal{M}'}_{11,3} &= \frac{1+v^{\mathcal{M}'}_{11,2}v^{\mathcal{M}'}_{10,4}}{v^{\mathcal{M}'}_{10,3}} = \frac{1+\left(v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,2}\right)\left(v^{\mathcal{M}}_{j+4,3}v^{\mathcal{M}}_{j+3,7}+v^{\mathcal{M}}_{j+5,2}v^{\mathcal{M}}_{j+4,7}\right)}{v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+4,6}+v^{\mathcal{M}}_{j+3,1}v^{\mathcal{M}}_{j+4,2}} = v^{\mathcal{M}}_{j+5,2}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,3}\\
v^{\mathcal{M}'}_{11,4} &= \frac{1+v^{\mathcal{M}'}_{11,2}v^{\mathcal{M}'}_{10,5}}{v^{\mathcal{M}'}_{10,4}} = \frac{1+\left(v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,2}\right)\left(v^{\mathcal{M}}_{j+5,3}v^{\mathcal{M}}_{j+4,6}+v^{\mathcal{M}}_{j+4,4}v^{\mathcal{M}}_{j+3,7}\right)}{v^{\mathcal{M}}_{j+4,3}v^{\mathcal{M}}_{j+3,7}+v^{\mathcal{M}}_{j+5,2}v^{\mathcal{M}}_{j+4,6}} = v^{\mathcal{M}}_{j+5,3}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,4}\\
v^{\mathcal{M}'}_{11,5} &= \frac{1+v^{\mathcal{M}'}_{11,4}v^{\mathcal{M}'}_{10,6}v^{\mathcal{M}'}_{10,7}}{v^{\mathcal{M}'}_{10,5}} = \frac{1+v^{\mathcal{M}}_{j+4,6}\left(v^{\mathcal{M}}_{j+5,3}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,4}\right)\left(v^{\mathcal{M}}_{j+5,4}+v^{\mathcal{M}}_{j+4,5}v^{\mathcal{M}}_{j+3,7}\right)}{v^{\mathcal{M}}_{j+5,3}v^{\mathcal{M}}_{j+4,6}+v^{\mathcal{M}}_{j+4,4}v^{\mathcal{M}}_{j+3,7}} = v^{\mathcal{M}}_{j+5,4}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,5}v^{\mathcal{M}}_{j+4,6}\\
v^{\mathcal{M}'}_{11,6} &= \frac{1+v^{\mathcal{M}'}_{11,5}}{v^{\mathcal{M}'}_{10,6}} = \frac{1+v^{\mathcal{M}}_{j+5,4}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,5}v^{\mathcal{M}}_{j+4,6}}{v^{\mathcal{M}}_{j+4,6}} = \frac{v^{\mathcal{M}}_{j+4,6}v^{\mathcal{M}}_{j+5,6}+v^{\mathcal{M}}_{j+4,6}v^{\mathcal{M}}_{j+4,5}}{v^{\mathcal{M}}_{j+4,6}} = v^{\mathcal{M}}_{j+4,5}+v^{\mathcal{M}}_{j+5,6}\\
v^{\mathcal{M}'}_{11,7} &= \frac{1+v^{\mathcal{M}'}_{11,5}v^{\mathcal{M}'}_{10,8}}{v^{\mathcal{M}'}_{10,7}} = \frac{1+\left(v^{\mathcal{M}}_{j+5,4}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,5}v^{\mathcal{M}}_{j+4,6}\right)\left(v^{\mathcal{M}}_{j+5,5}+v^{\mathcal{M}}_{j+3,7}\right)}{v^{\mathcal{M}}_{j+5,4}+v^{\mathcal{M}}_{j+4,5}v^{\mathcal{M}}_{j+3,7}} = v^{\mathcal{M}}_{j+4,6}+v^{\mathcal{M}}_{j+5,5}v^{\mathcal{M}}_{j+4,7}\\
v^{\mathcal{M}'}_{11,8} &= \frac{1+v^{\mathcal{M}'}_{11,7}}{v^{\mathcal{M}'}_{10,8}} = \frac{1+v^{\mathcal{M}}_{j+4,6}+v^{\mathcal{M}}_{j+5,5}v^{\mathcal{M}}_{j+4,7}}{v^{\mathcal{M}}_{j+5,5}+v^{\mathcal{M}}_{j+3,7}} = \frac{v^{\mathcal{M}}_{j+4,7}v^{\mathcal{M}}_{j+3,7}+v^{\mathcal{M}}_{j+4,7}v^{\mathcal{M}}_{j+5,5}}{v^{\mathcal{M}}_{j+3,7}+v^{\mathcal{M}}_{j+5,5}} = v^{\mathcal{M}}_{j+4,7}
\end{aligned}
$$

Thus, we express entries of $\Gamma_{27}(\mathcal{M}')$ by entries of $\mathcal{M}$, using $p(\mathcal{M})=10$ and frieze/-diamond rules:

$$
\begin{aligned}
v^{\mathcal{M}'}_{27,7} &= \frac{1+v^{\mathcal{M}'}_{28,5}v^{\mathcal{M}'}_{27,8}}{v^{\mathcal{M}'}_{28,7}} = \frac{1+v^{\mathcal{M}}_{j-6,7}\left(v^{\mathcal{M}}_{j-5,4}+v^{\mathcal{M}}_{j-5,5}v^{\mathcal{M}}_{j-5,6}\right)}{v^{\mathcal{M}}_{j-5,6}} = \frac{v^{\mathcal{M}}_{j-5,6}\left(v^{\mathcal{M}}_{j-6,6}+v^{\mathcal{M}}_{j-5,5}v^{\mathcal{M}}_{j-6,7}\right)}{v^{\mathcal{M}}_{j-5,6}} = v^{\mathcal{M}}_{j+4,6}+v^{\mathcal{M}}_{j+5,5}v^{\mathcal{M}}_{j+4,7} = v^{\mathcal{M}'}_{11,7}\\
v^{\mathcal{M}'}_{27,6} &= \frac{1+v^{\mathcal{M}'}_{28,5}}{v^{\mathcal{M}'}_{28,6}} = \frac{1+v^{\mathcal{M}}_{j-5,4}+v^{\mathcal{M}}_{j-5,5}v^{\mathcal{M}}_{j-5,6}}{v^{\mathcal{M}}_{j-5,5}} = v^{\mathcal{M}}_{j+4,5}+v^{\mathcal{M}}_{j+5,6} = v^{\mathcal{M}'}_{11,6}\\
v^{\mathcal{M}'}_{27,5} &= \frac{1+v^{\mathcal{M}'}_{28,4}v^{\mathcal{M}'}_{27,6}v^{\mathcal{M}'}_{27,7}}{v^{\mathcal{M}'}_{28,5}} = \frac{1+v^{\mathcal{M}}_{j-5,4}\left(v^{\mathcal{M}}_{j-6,5}+v^{\mathcal{M}}_{j-5,6}\right)\left(v^{\mathcal{M}}_{j-6,6}+v^{\mathcal{M}}_{j-5,5}v^{\mathcal{M}}_{j-6,7}\right)}{v^{\mathcal{M}}_{j-5,4}+v^{\mathcal{M}}_{j-5,5}v^{\mathcal{M}}_{j-5,6}} = v^{\mathcal{M}}_{j+5,4}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,5}v^{\mathcal{M}}_{j+4,6} = v^{\mathcal{M}'}_{11,5}\\
v^{\mathcal{M}'}_{27,4} &= \frac{1+v^{\mathcal{M}'}_{28,3}v^{\mathcal{M}'}_{27,5}}{v^{\mathcal{M}'}_{28,4}} = \frac{1+v^{\mathcal{M}}_{j-5,3}\left(v^{\mathcal{M}}_{j-5,4}v^{\mathcal{M}}_{j-6,7}+v^{\mathcal{M}}_{j-6,5}v^{\mathcal{M}}_{j-6,6}\right)}{v^{\mathcal{M}}_{j-5,4}} = \frac{v^{\mathcal{M}}_{j-5,4}\left(v^{\mathcal{M}}_{j-5,3}v^{\mathcal{M}}_{j-6,7}+v^{\mathcal{M}}_{j-6,4}\right)}{v^{\mathcal{M}}_{j-5,4}} = v^{\mathcal{M}}_{j+4,4}+v^{\mathcal{M}}_{j+5,3}v^{\mathcal{M}}_{j+4,7} = v^{\mathcal{M}'}_{11,4}\\
v^{\mathcal{M}'}_{27,3} &= \frac{1+v^{\mathcal{M}'}_{28,2}v^{\mathcal{M}'}_{27,4}}{v^{\mathcal{M}'}_{28,3}} = \frac{1+v^{\mathcal{M}}_{j-5,2}\left(v^{\mathcal{M}}_{j-6,4}+v^{\mathcal{M}}_{j-5,3}v^{\mathcal{M}}_{j-6,7}\right)}{v^{\mathcal{M}}_{j-5,3}} = \frac{v^{\mathcal{M}}_{j-5,3}\left(v^{\mathcal{M}}_{j-5,2}v^{\mathcal{M}}_{j-6,7}+v^{\mathcal{M}}_{j-6,3}\right)}{v^{\mathcal{M}}_{j-5,3}} = v^{\mathcal{M}}_{j+5,2}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,3} = v^{\mathcal{M}'}_{11,3}\\
v^{\mathcal{M}'}_{27,2} &= \frac{1+v^{\mathcal{M}'}_{28,1}v^{\mathcal{M}'}_{27,3}}{v^{\mathcal{M}'}_{28,2}} = \frac{1+v^{\mathcal{M}}_{j-5,1}\left(v^{\mathcal{M}}_{j-5,2}v^{\mathcal{M}}_{j-6,7}+v^{\mathcal{M}}_{j-6,3}\right)}{v^{\mathcal{M}}_{j-5,2}} = \frac{v^{\mathcal{M}}_{j-5,2}\left(v^{\mathcal{M}}_{j-5,1}v^{\mathcal{M}}_{j-6,7}+v^{\mathcal{M}}_{j-6,2}\right)}{v^{\mathcal{M}}_{j-5,2}} = v^{\mathcal{M}}_{j+5,1}v^{\mathcal{M}}_{j+4,7}+v^{\mathcal{M}}_{j+4,2} = v^{\mathcal{M}'}_{11,2}\\
v^{\mathcal{M}'}_{27,1} &= \frac{1+v^{\mathcal{M}'}_{27,2}}{v^{\mathcal{M}'}_{28,1}} = \frac{1+v^{\mathcal{M}}_{j-5,1}v^{\mathcal{M}}_{j-6,7}+v^{\mathcal{M}}_{j-6,2}}{v^{\mathcal{M}}_{j-5,1}} = \frac{v^{\mathcal{M}}_{j-5,1}\left(v^{\mathcal{M}}_{j-6,7}+v^{\mathcal{M}}_{j-6,1}\right)}{v^{\mathcal{M}}_{j-5,2}} = v^{\mathcal{M}}_{j+4,1}+v^{\mathcal{M}}_{j+4,7} = v^{\mathcal{M}'}_{11,1}
\end{aligned}
$$

Therefore, $\Gamma_{11}(\mathcal{M}') = \Gamma_{27}(\mathcal{M}')$, implying the maximum period of *unit-present* quiddity sequence of multiplicative type $\mathbf{E}_8$ friezes is 16. ■

9.2. **Type $\mathbf{E}_7, \mathbf{E}_8$ unit-free quiddity existence.** Without succeed on the proof of the existence of the unit-free quiddity sequence, we provide an argument potentially restricts the number of such friezes. By the frieze rules of ϵ, if any of $v^{\epsilon}_{i,2}, v^{\epsilon}_{i,3}, v^{\epsilon}_{i,4} = 1$, then q_i must contain a 1. Assuming $q_i \geq 2$, by consider an open frieze $\mathcal{I}$ with entries $(v^{\mathcal{I}}_{i,j})_{i,j\in\mathbb{Z},j\geq -1}$ gives lower bounds:

$$v^{\epsilon}_{i,2} \geq 3, \quad v^{\epsilon}_{i,3} \geq 4, \quad v^{\epsilon}_{i,4} \geq 5, \quad v^{\epsilon}_{i,5}v^{\epsilon}_{i,6} \geq 6, \quad v^{\epsilon}_{i,6}v^{\epsilon}_{i+1,6} + v^{\epsilon}_{i,7} \geq 7.$$

[Bau+21] suggests no mesh frieze for type $\mathbf{E}_7$ has all entries ≥ 2, implying no non-unitary multiplicative frieze. We thus bound the Hamming weight of $\{v^{\epsilon}_{i,7}, \ldots, v^{\epsilon}_{i+10,7}\}$ since other entries scale with q_k. We prove a key inequality when $q_k \geq 2$:

Proof. Let $S = \{q_k \in \mathbb{R}^6 : v^{\epsilon}_{i,1} \geq 2\}$ and define

$$v^{\epsilon}_{i,7}(q_k) = \frac{N(q_k)}{D(v^{\epsilon}_{i+1,1}, v^{\epsilon}_{i+2,1}, v^{\epsilon}_{i+3,1}, v^{\epsilon}_{i+4,1})},$$

where

$$N(q_k) = v^\epsilon_{i,1}v^\epsilon_{i+1,1}v^\epsilon_{i+2,1}v^\epsilon_{i+3,1}v^\epsilon_{i+4,1}v^\epsilon_{i+5,1} - v^\epsilon_{i,1}v^\epsilon_{i+1,1}v^\epsilon_{i+2,1}v^\epsilon_{i+3,1} - v^\epsilon_{i,1}v^\epsilon_{i+1,1}v^\epsilon_{i+2,1}v^\epsilon_{i+5,1} - v^\epsilon_{i,1}v^\epsilon_{i+1,1}v^\epsilon_{i+4,1}v^\epsilon_{i+5,1}$$
$$+ v^\epsilon_{i,1}v^\epsilon_{i+1,1} - v^\epsilon_{i,1}v^\epsilon_{i+3,1}v^\epsilon_{i+4,1}v^\epsilon_{i+5,1} + v^\epsilon_{i,1}v^\epsilon_{i+3,1} + v^\epsilon_{i,1}v^\epsilon_{i+5,1} - v^\epsilon_{i+2,1}v^\epsilon_{i+3,1}v^\epsilon_{i+4,1}v^\epsilon_{i+5,1} + v^\epsilon_{i+2,1}v^\epsilon_{i+3,1}+$$
$$v^\epsilon_{i+2,1}v^\epsilon_{i+5,1} + v^\epsilon_{i+4,1}v^\epsilon_{i+5,1} - 2,$$
$$D(q_k) = v^\epsilon_{i+1,1}v^\epsilon_{i+2,1}v^\epsilon_{i+3,1}v^\epsilon_{i+4,1} - v^\epsilon_{i+1,1}v^\epsilon_{i+2,1} - v^\epsilon_{i+1,1}v^\epsilon_{i+4,1} - v^\epsilon_{i+3,1}v^\epsilon_{i+4,1} + 2.$$

Then $v^\epsilon_{i,7}(P) = 1$ at $P = (2, 2, 2, 2, 2, 2)$. For $v^\epsilon_{i,1} \geq 2$,

$$\frac{\partial D}{\partial v^\epsilon_{i+1,1}} = v^\epsilon_{i+4,1}(v^\epsilon_{i+2,1}v^\epsilon_{i+3,1} - 1) - v^\epsilon_{i+2,1} \geq 4 > 0,$$

so $D \geq D(2, 2, 2, 2) = 6 > 0$. Thus, $v^\epsilon_{i,7} \geq 1 \iff F := N - D \geq 0$. We show $\partial_{v^\epsilon_{i,1}} F > 0$ on S for all i.
For $v^\epsilon_{i,1}$:

$$\partial_{v^\epsilon_{i,1}} F = v^\epsilon_{i+1,1}v^\epsilon_{i+2,1}v^\epsilon_{i+3,1}v^\epsilon_{i+4,1}v^\epsilon_{i+5,1} - v^\epsilon_{i+1,1}v^\epsilon_{i+2,1}v^\epsilon_{i+3,1} - v^\epsilon_{i+1,1}v^\epsilon_{i+2,1}v^\epsilon_{i+5,1} - v^\epsilon_{i+1,1}v^\epsilon_{i+4,1}v^\epsilon_{i+5,1} + v^\epsilon_{i+1,1}$$
$$- v^\epsilon_{i+3,1}v^\epsilon_{i+4,1}v^\epsilon_{i+5,1} + v^\epsilon_{i+3,1} + v^\epsilon_{i+5,1}$$

This is minimized at $(2, \dots, 2)$, yielding $\min \partial_{v^\epsilon_{i,1}} F = 6 > 0$.
For $v^\epsilon_{i+1,1}$:

$$\partial_{v^\epsilon_{i+1,1}} F = v^\epsilon_{i,1}v^\epsilon_{i+2,1}v^\epsilon_{i+3,1}v^\epsilon_{i+4,1}v^\epsilon_{i+5,1} - v^\epsilon_{i,1}v^\epsilon_{i+2,1}v^\epsilon_{i+3,1} - v^\epsilon_{i,1}v^\epsilon_{i+2,1}v^\epsilon_{i+5,1} - v^\epsilon_{i,1}v^\epsilon_{i+4,1}v^\epsilon_{i+5,1} + v^\epsilon_{i,1}$$
$$- v^\epsilon_{i+2,1}v^\epsilon_{i+3,1}v^\epsilon_{i+4,1} - v^\epsilon_{i+2,1} - v^\epsilon_{i+4,1}$$

At $(2, \dots, 2)$, $\min \partial_{v^\epsilon_{i+1,1}} F = 6 > 0$. By symmetry, $\min \partial_{v^\epsilon_{i,1}} F \geq 6 > 0$ for $i = 2, 3, 4, 5$.
So F is strictly increasing on S, with minimum $F(P) = 0$. Hence $v^\epsilon_{i,7}(a) \geq 1$, with equality only at P. ■

Next we should proceed on the bound on $\max(W_H(\{v^\epsilon_{i,7}, \dots, v^\epsilon_{i+10,7}\}))$ for $q_k \geq 2$ However, we conduct trials without succeed due to its high complexity.
Again, we provide a detailed argument establishing the existence of a unit-free quiddity sequence for the multiplicative type $\mathbf{E}_8$ and we denote it by ζ. By the propagation rules of the frieze ζ, if any of the entries $v^\zeta_{i,2}, v^\zeta_{i,3}, v^\zeta_{i,4}, v^\zeta_{i,5}$ equals 1, then the corresponding quiddity term q_i must contain a 1. Proceeding under the assumption that $q_i \geq 2$ for all i, the inequalities derived from the open frieze condition yield the following lower bounds:

$$v^\zeta_{i,2} \geq 3, \quad v^\zeta_{i,3} \geq 4, \quad v^\zeta_{i,4} \geq 5, \quad v^\epsilon_{i,5} \geq 6, \quad v^\zeta_{i,6}v^\zeta_{i,7} \geq 7, \quad v^\zeta_{i,7}v^\zeta_{i+1,7} + v^\zeta_{i,8} \geq 8.$$

Furthermore, it is established in [Bau+22] that the periodicity of a frieze arising from a unit-free quiddity sequence for $\mathbf{E}_8$ cannot be 2. Consequently, we posit that any solution must manifest with a periodicity greater than 2, and we investigate the possibility of a 4-periodic unit-free quiddity sequence. Should such a sequence exist, the search can be constrained by the following divisibility condition, which enhances computational efficiency:

$$v^{\zeta}_{i+1,2} | v^{\zeta}_{i,4} - v^{\zeta}_{i+1,2}$$

This condition is motivated by the result in [Bau+22], which demonstrates a constant difference between rows p and $p-2$ in an open frieze induced by a p-periodic unit-free quiddity sequence.

9.3. **Perquisite and derived data.** In preparation to apply Theorem 4.9 and to showcase its result, we present the set of data used for calculations. The set of *all* quiddity sequences and quivers can be found in *here*, the collection of $\mathfrak{D}(Q_{\mathbf{E}_{n+1}})$ and $\mathfrak{U}(Q_{\mathbf{E}_{n+1}})$ for $n = 5,\ 6,\ 7$ can be found in *here*.

9.3.1. *Quiddity sequences set* $Q_{\mathbf{E}_n}$.
A. $Q_{\mathbf{E}_5} = \{\mathcal{Q}_{\mathcal{M}_1},\ \mathcal{Q}_{\mathcal{M}_2},\ \ldots,,\ \mathcal{Q}_{\mathcal{M}_{27}}\}$

$\mathcal{Q}_{\mathcal{M}_1} = (1,2,2,6,4,2,1,4,3,8)$	$\mathcal{Q}_{\mathcal{M}_2} = (1,2,2,10,3,2,1,4,5,6)$	$\mathcal{Q}_{\mathcal{M}_3} = (1,2,3,4,3,2,1,6,2,6)$
$\mathcal{Q}_{\mathcal{M}_4} = (1,2,3,10,2,2,1,6,5,4)$	$\mathcal{Q}_{\mathcal{M}_5} = (1,2,4,6,2,2,1,8,3,4)$	$\mathcal{Q}_{\mathcal{M}_6} = (1,2,3,4,5,1,2,3,4,5)$
$\mathcal{Q}_{\mathcal{M}_7} = (1,2,3,7,4,1,2,3,7,4)$	$\mathcal{Q}_{\mathcal{M}_8} = (1,2,5,3,4,1,2,5,3,4)$	$\mathcal{Q}_{\mathcal{M}_9} = (1,2,5,8,3,1,2,5,8,3)$
$\mathcal{Q}_{\mathcal{M}_{10}} = (1,2,7,5,3,1,2,7,5,3)$	$\mathcal{Q}_{\mathcal{M}_{11}} = (1,3,1,6,3,3,1,3,2,9)$	$\mathcal{Q}_{\mathcal{M}_{12}} = (1,3,1,9,2,3,1,3,3,6)$
$\mathcal{Q}_{\mathcal{M}_{13}} = (1,3,2,3,2,3,1,6,1,6)$	$\mathcal{Q}_{\mathcal{M}_{14}} = (1,3,2,3,4,1,3,2,3,4)$	$\mathcal{Q}_{\mathcal{M}_{15}} = (1,3,2,5,3,1,3,2,5,3)$
$\mathcal{Q}_{\mathcal{M}_{16}} = (1,3,5,2,3,1,3,5,2,3)$	$\mathcal{Q}_{\mathcal{M}_{17}} = (1,3,5,7,2,1,3,5,7,2)$	$\mathcal{Q}_{\mathcal{M}_{18}} = (1,3,8,5,2,1,3,8,5,2)$
$\mathcal{Q}_{\mathcal{M}_{19}} = (1,4,1,4,2,4,1,4,1,8)$	$\mathcal{Q}_{\mathcal{M}_{20}} = (1,4,1,4,3,2,2,2,2,6)$	$\mathcal{Q}_{\mathcal{M}_{21}} = (1,4,1,6,2,2,2,2,3,4)$
$\mathcal{Q}_{\mathcal{M}_{22}} = (1,4,3,2,3,1,4,3,2,3)$	$\mathcal{Q}_{\mathcal{M}_{23}} = (1,4,3,5,2,1,4,3,5,2)$	$\mathcal{Q}_{\mathcal{M}_{24}} = (1,4,7,3,2,1,4,7,3,2)$
$\mathcal{Q}_{\mathcal{M}_{25}} = (1,5,1,5,1,5,1,5,1,5)$	$\mathcal{Q}_{\mathcal{M}_{26}} = (1,5,4,3,2,1,5,4,3,2)$	$\mathcal{Q}_{\mathcal{M}_{27}} = (2,2,2,2,4,2,2,2,2,4)$

B. $Q_{\mathbf{E}_6} = \{\mathcal{Q}_{\mathcal{M}_1}, \mathcal{Q}_{\mathcal{M}_2}, \ldots, \mathcal{Q}_{\mathcal{M}_{70}}\}$

$\mathcal{Q}_{\mathcal{M}_{1}}=(1,2,2,2,7,5,3,2,2,1,6,4,3,9)$
$\mathcal{Q}_{\mathcal{M}_{2}}=(1,2,2,2,12,4,3,3,1,2,5,4,5,7)$
$\mathcal{Q}_{\mathcal{M}_{3}}=(1,2,2,3,5,4,4,1,3,1,5,6,2,7)$
$\mathcal{Q}_{\mathcal{M}_{4}}=(1,2,2,3,13,3,4,2,1,3,4,6,5,5)$
$\mathcal{Q}_{\mathcal{M}_{5}}=(1,2,2,4,8,3,5,1,2,2,4,8,3,5)$
$\mathcal{Q}_{\mathcal{M}_{6}}=(1,2,2,3,5,7,2,2,2,1,9,3,4,6)$
$\mathcal{Q}_{\mathcal{M}_{7}}=(1,2,2,3,9,6,2,3,1,2,8,3,7,5)$
$\mathcal{Q}_{\mathcal{M}_{8}}=(1,2,2,5,4,6,3,1,3,1,8,5,3,5)$
$\mathcal{Q}_{\mathcal{M}_{9}}=(1,2,2,5,11,5,3,2,1,3,7,5,8,4)$
$\mathcal{Q}_{\mathcal{M}_{10}}=(1,2,2,7,7,5,4,1,2,2,7,7,5,4)$
$\mathcal{Q}_{\mathcal{M}_{11}}=(1,2,3,1,7,4,2,3,2,1,5,3,2,10)$
$\mathcal{Q}_{\mathcal{M}_{12}}=(1,2,3,1,11,3,2,4,1,2,4,3,3,7)$
$\mathcal{Q}_{\mathcal{M}_{13}}=(1,2,3,2,4,3,3,1,4,1,4,6,1,7)$
$\mathcal{Q}_{\mathcal{M}_{14}}=(1,2,3,2,13,2,3,2,1,4,3,6,3,4)$
$\mathcal{Q}_{\mathcal{M}_{15}}=(1,2,3,3,9,2,4,1,2,3,3,9,2,4)$
$\mathcal{Q}_{\mathcal{M}_{16}}=(1,2,3,2,4,7,1,3,2,1,10,2,3,5)$
$\mathcal{Q}_{\mathcal{M}_{17}}=(1,2,3,2,7,6,1,4,1,2,9,2,5,4)$
$\mathcal{Q}_{\mathcal{M}_{18}}=(1,2,3,5,3,6,2,1,4,1,9,5,2,4)$
$\mathcal{Q}_{\mathcal{M}_{19}}=(1,2,3,5,11,5,2,2,1,4,8,5,7,3)$
$\mathcal{Q}_{\mathcal{M}_{20}}=(1,2,3,8,8,5,3,1,2,3,8,8,5,3)$
$\mathcal{Q}_{\mathcal{M}_{21}}=(1,2,4,1,5,3,2,2,3,1,4,4,1,9)$
$\mathcal{Q}_{\mathcal{M}_{22}}=(1,2,4,1,11,2,2,3,1,3,3,4,2,5)$
$\mathcal{Q}_{\mathcal{M}_{23}}=(1,2,4,2,6,2,3,1,3,2,3,8,1,5)$
$\mathcal{Q}_{\mathcal{M}_{24}}=(1,2,4,1,5,5,1,4,2,1,7,2,2,7)$
$\mathcal{Q}_{\mathcal{M}_{25}}=(1,2,4,1,8,4,1,5,1,2,6,2,3,5)$
$\mathcal{Q}_{\mathcal{M}_{26}}=(1,2,4,3,3,4,2,1,5,1,6,6,1,5)$
$\mathcal{Q}_{\mathcal{M}_{27}}=(1,2,4,3,13,3,2,2,1,5,5,6,4,3)$
$\mathcal{Q}_{\mathcal{M}_{28}}=(1,2,4,5,10,3,3,1,2,4,5,10,3,3)$
$\mathcal{Q}_{\mathcal{M}_{29}}=(1,2,4,3,8,6,1,3,1,3,10,3,5,3)$
$\mathcal{Q}_{\mathcal{M}_{30}}=(1,2,4,7,5,6,2,1,3,2,10,7,3,3)$
$\mathcal{Q}_{\mathcal{M}_{31}}=(1,2,5,1,7,2,2,2,2,2,3,5,1,6)$
$\mathcal{Q}_{\mathcal{M}_{32}}=(1,2,6,1,4,4,1,3,3,1,6,3,1,7)$
$\mathcal{Q}_{\mathcal{M}_{33}}=(1,2,6,1,9,3,1,4,1,3,5,3,2,4)$
$\mathcal{Q}_{\mathcal{M}_{34}}=(1,2,6,3,5,3,2,1,4,2,5,9,1,4)$
$\mathcal{Q}_{\mathcal{M}_{35}}=(1,2,6,2,10,4,1,3,1,4,7,4,3,3)$
$\mathcal{Q}_{\mathcal{M}_{36}}=(1,2,6,5,7,4,2,1,3,3,7,10,2,3)$
$\mathcal{Q}_{\mathcal{M}_{37}}=(1,2,8,1,6,3,1,3,2,2,5,4,1,5)$
$\mathcal{Q}_{\mathcal{M}_{38}}=(1,3,1,3,6,4,5,2,2,1,5,3,5,8)$
$\mathcal{Q}_{\mathcal{M}_{39}}=(1,3,1,4,4,3,6,1,3,1,4,4,3,6)$
$\mathcal{Q}_{\mathcal{M}_{40}}=(1,3,1,4,10,2,6,2,1,3,3,4,7,4)$
$\mathcal{Q}_{\mathcal{M}_{41}}=(1,3,1,4,4,5,3,2,2,1,7,2,6,5)$
$\mathcal{Q}_{\mathcal{M}_{42}}=(1,3,1,6,3,4,4,1,3,1,6,3,4,4)$
$\mathcal{Q}_{\mathcal{M}_{43}}=(1,3,1,6,8,3,4,2,1,3,5,3,10,3)$
$\mathcal{Q}_{\mathcal{M}_{44}}=(1,3,2,1,7,3,3,4,2,1,4,2,3,11)$
$\mathcal{Q}_{\mathcal{M}_{45}}=(1,3,2,2,3,2,4,1,5,1,3,4,1,7)$
$\mathcal{Q}_{\mathcal{M}_{46}}=(1,3,2,2,11,1,4,2,1,5,2,4,3,3)$
$\mathcal{Q}_{\mathcal{M}_{47}}=(1,3,2,2,3,3,2,2,3,1,5,2,2,5)$
$\mathcal{Q}_{\mathcal{M}_{48}}=(1,3,2,2,7,2,2,3,1,3,4,2,4,3)$
$\mathcal{Q}_{\mathcal{M}_{49}}=(1,3,2,4,4,2,3,1,3,2,4,4,2,3)$
$\mathcal{Q}_{\mathcal{M}_{50}}=(1,3,2,2,3,5,1,4,2,1,9,1,4,4)$
$\mathcal{Q}_{\mathcal{M}_{51}}=(1,3,2,6,2,4,2,1,5,1,8,3,2,3)$
$\mathcal{Q}_{\mathcal{M}_{52}}=(1,3,2,6,9,3,2,2,1,5,7,3,8,2)$
$\mathcal{Q}_{\mathcal{M}_{53}}=(1,3,3,1,4,2,3,2,4,1,3,3,1,10)$
$\mathcal{Q}_{\mathcal{M}_{54}}=(1,3,3,2,6,1,4,1,3,3,2,6,1,4)$
$\mathcal{Q}_{\mathcal{M}_{55}}=(1,3,3,1,4,4,1,6,2,1,7,1,3,6)$
$\mathcal{Q}_{\mathcal{M}_{56}}=(1,3,3,4,2,3,2,1,7,1,6,4,1,4)$
$\mathcal{Q}_{\mathcal{M}_{57}}=(1,3,3,4,12,2,2,2,1,7,5,4,5,2)$
$\mathcal{Q}_{\mathcal{M}_{58}}=(1,3,3,10,5,4,2,1,3,3,10,5,4,2)$
$\mathcal{Q}_{\mathcal{M}_{59}}=(1,3,5,8,8,3,2,1,3,5,8,8,3,2)$
$\mathcal{Q}_{\mathcal{M}_{60}}=(1,3,6,1,8,2,1,5,1,4,5,2,2,3)$
$\mathcal{Q}_{\mathcal{M}_{61}}=(1,3,9,1,6,2,1,4,2,3,5,3,1,4)$
$\mathcal{Q}_{\mathcal{M}_{62}}=(1,4,1,6,2,3,3,1,4,1,6,2,3,3)$
$\mathcal{Q}_{\mathcal{M}_{63}}=(1,4,1,6,7,2,3,2,1,4,5,2,9,2)$
$\mathcal{Q}_{\mathcal{M}_{64}}=(1,4,2,2,4,1,5,1,4,2,2,4,1,5)$
$\mathcal{Q}_{\mathcal{M}_{65}}=(1,4,2,1,5,3,2,6,2,1,5,1,4,8)$
$\mathcal{Q}_{\mathcal{M}_{66}}=(1,4,2,3,2,2,3,1,7,1,4,3,1,5)$
$\mathcal{Q}_{\mathcal{M}_{67}}=(1,4,2,9,3,3,2,1,4,2,9,3,3,2)$
$\mathcal{Q}_{\mathcal{M}_{68}}=(1,4,5,7,7,2,2,1,4,5,7,7,2,2)$
$\mathcal{Q}_{\mathcal{M}_{69}}=(1,5,2,1,6,1,5,3,2,2,2,2,2,7)$
$\mathcal{Q}_{\mathcal{M}_{70}}=(1,5,3,8,4,2,2,1,5,3,8,4,2,2)$

9.3.2. *Matrices* $\mathfrak{D}(Q_{\mathbf{E}_{n+1}})$ *and* $\mathfrak{U}(Q_{\mathbf{E}_{n+1}})$.

A. $\mathfrak{D}(Q_{\mathbf{E}_6}) = (d_{i,j})_{1\leq i\leq 27,\, 1\leq j\leq 10}$

$$\begin{pmatrix}
(2,10,14) & (2,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (2,10,14) & (2,10,7) & (4,10,14) & (4,10,14) \\
(2,10,14) & (2,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (2,10,7) & (2,10,14) & (4,10,14) & (4,10,14) \\
(3,10,14) & (3,10,14) & (4,10,7) & (4,10,14) & (4,10,14) & (4,10,7) & (3,10,14) & (3,10,14) & (5,10,14) & (5,10,14) \\
(2,10,14) & (2,10,7) & (3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (2,10,14) & (2,10,14) & (4,10,14) & (4,10,14) \\
(2,10,7) & (2,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (2,10,14) & (2,10,14) & (4,10,14) & (4,10,14) \\
(2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) \\
(2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) \\
(3,5,14) & (3,5,14) & (4,5,7) & (5,5,14) & (4,5,14) & (3,5,14) & (3,5,14) & (4,5,7) & (5,5,14) & (4,5,14) \\
(2,5,14) & (2,5,7) & (3,5,14) & (4,5,14) & (3,5,14) & (2,5,14) & (2,5,7) & (3,5,14) & (4,5,14) & (3,5,14) \\
(2,5,7) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (2,5,7) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) \\
(3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) & (2,10,14) & (2,10,7) & (4,10,14) & (4,10,14) \\
(3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) & (2,10,7) & (2,10,14) & (4,10,12) & (4,10,14) \\
(4,10,14) & (4,10,14) & (4,10,7) & (4,10,14) & (4,10,14) & (4,10,7) & (4,10,14) & (4,10,14) & (5,10,14) & (5,10,14) \\
(3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,14) & (3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,14) \\
(3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,7) & (3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,7) \\
(3,5,14) & (3,5,14) & (4,5,7) & (5,5,14) & (4,5,14) & (3,5,14) & (3,5,14) & (4,5,7) & (5,5,14) & (4,5,14) \\
(2,5,14) & (2,5,7) & (3,5,14) & (4,5,14) & (3,5,14) & (2,5,14) & (2,5,7) & (3,5,14) & (4,5,14) & (3,5,14) \\
(2,5,7) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (2,5,7) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) \\
(4,10,14) & (4,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) \\
(4,10,14) & (4,10,14) & (4,10,14) & (4,10,14) & (4,10,7) & (4,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) \\
(4,10,14) & (4,10,14) & (4,10,14) & (4,10,14) & (4,10,14) & (4,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,7) \\
(3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,14) & (3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,14) \\
(3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,7) & (3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,7) \\
(2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) \\
(3,2,14) & (3,2,14) & (3,2,14) & (3,2,14) & (3,2,14) & (3,2,14) & (3,2,14) & (3,2,14) & (3,2,14) & (3,2,14) \\
(2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) \\
(2,5,14) & (2,5,14) & (2,5,7) & (2,5,14) & (2,5,14) & (2,5,14) & (2,5,14) & (2,5,7) & (2,5,14) & (2,5,14)
\end{pmatrix}$$

B. $\mathfrak{U}(Q_{\mathbf{E}_6}) = (u_{i,j})_{1\leq i\leq 27,\, 1\leq j\leq 10}$

$$\begin{pmatrix}
(2,10,14) & (2,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (2,10,14) & (2,10,7) & (4,10,14) & (4,10,14) \\
(2,10,14) & (2,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (2,10,7) & (2,10,14) & (4,10,14) & (4,10,14) \\
(3,10,14) & (3,10,14) & (4,10,7) & (4,10,14) & (4,10,14) & (4,10,7) & (3,10,14) & (3,10,14) & (5,10,14) & (5,10,14) \\
(2,10,14) & (2,10,7) & (3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (2,10,14) & (2,10,14) & (4,10,14) & (4,10,14) \\
(2,10,7) & (2,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (2,10,14) & (2,10,14) & (4,10,14) & (4,10,14) \\
(2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(3,5,14) & (3,5,14) & (4,5,7) & (5,5,14) & (4,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(2,5,14) & (2,5,7) & (3,5,14) & (4,5,14) & (3,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(2,5,7) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) & (2,10,14) & (2,10,7) & (4,10,14) & (4,10,14) \\
(3,10,14) & (3,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) & (2,10,7) & (2,10,14) & (4,10,14) & (4,10,14) \\
(4,10,14) & (4,10,14) & (4,10,7) & (4,10,14) & (4,10,14) & (4,10,7) & (4,10,14) & (4,10,14) & (5,10,14) & (5,10,14) \\
(3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,7) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(3,5,14) & (3,5,14) & (4,5,7) & (5,5,14) & (4,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(2,5,14) & (2,5,7) & (3,5,14) & (4,5,14) & (3,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(2,5,7) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(4,10,14) & (4,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) \\
(4,10,14) & (4,10,14) & (4,10,14) & (4,10,14) & (4,10,7) & (4,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,14) \\
(4,10,14) & (4,10,14) & (4,10,14) & (4,10,14) & (4,10,14) & (4,10,14) & (3,10,14) & (3,10,14) & (4,10,14) & (4,10,7) \\
(3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(3,5,14) & (3,5,14) & (4,5,14) & (5,5,14) & (4,5,7) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(3,2,14) & (3,2,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(2,5,14) & (2,5,14) & (3,5,14) & (4,5,14) & (3,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \\
(2,5,14) & (2,5,14) & (2,5,7) & (2,5,14) & (2,5,14) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0) & (0,0,0)
\end{pmatrix}$$

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EDITOR'S COMMENTS

This paper works on the Fontaine–Plamondon conjecture, which concerns counting multiplicative E_n friezes, using an elementary combinatorial approach based on quiddity sequences. The team confirms that there are exactly 70 distinct E_6 friezes, and gives strong evidence that the counts for E_7 and E_8 are 440 and 1694. Finding an elementary, hands-on route to confirm a count in this area of combinatorics—which sits close to cluster algebras—is a meaningful achievement, and the split into unit-present and unit-free quiddity sequences is a clean organizing idea.

The work connects to some quite advanced modern mathematics—Coxeter friezes, cluster algebras, and Grassmannians—and reduces it to a manageable enumeration with careful casework, periodicity arguments, and reductions such as $E_3 \cong A_1 \times A_2$. The team also compares their results against recent work, which shows good awareness of the wider field. Overall, the team has made interesting progress on a topic that is unusually advanced for secondary school students.