Hang Lung Mathematics Awards
Vol. 6 (2014)

# Hang Lung Mathematics Awards 2014

## Honorable Mention

# On the Geometric Construction of Triangles and the Algebraic Interpretation of the Notion of Constructability

Team Members: Chi Cheuk Tsang, Ho Lung Tsui, Justin Chi Ho Tang, Ian Yu Young Kwan, Chun Hin Kwan
Teacher: Mr. Perrick King Bor Ching
School: St. Joseph's College

Hang Lung Mathematics Awards
Vol. 6 (2014), pp. 227–318


# ON THE GEOMETRIC CONSTRUCTION OF TRIAGNLES AND THE ALGEBRAIC INTERPRETATION OF THE NOTION OF CONSTRUCTABILITY

**TEAM MEMBERS**
Chi Cheuk Tsang, Ho Lung Tsui, Justin Chi Ho Tang,
Ian Yu Young Kwan, Chun Hin Kwan

**TEACHER**
MR. Perrick King Bor Ching

**SCHOOL**
ST. Joseph's College

Abstract. This study centres on the Euclidean construction of triangles from several given preconditions, and carries several major objectives surrounding this aim:

1. Devising a scheme to primarily distinguish cases in which Euclidean construction is impossible;
2. Seeking the simplest agenda in the construction of possible cases;
3. Giving a strict definition of Euclidean constructability;
4. Determining methods of rigorously proving inconstructability.

## INTRODUCTION

Geometric construction using Euclidean tools had its origins far beyond two millennia ago, and yet it remains one of the most interesting topics in both applied and theoretical mathematics. In particular, the construction of the triangle from only a few given conditions is one of the most common types of problems in this field. As lovers of the topic, we have long wondered whether a triangle can be constructed for any combination of these conditions, therefore we have done an investigation into under what conditions (and most importantly, the agenda through which) the triangle can be constructed, considering exactly three pieces of given information about its angles, side lengths, angle bisector length, median length, or altitude. In this study fifty-two sets of conditions are investigated.

The following notations facilitate our discussion:

*1. $A, B, C$ denote the lengths of the sides;*

2. *$a, b, c$ denote the sizes of the angles (opposite to sides $A, B, C$ respectively);*
3. *$IA, IB, IC$ denote the lengths of the angle bisectors (incident on sides $A, B, C$ respectively);*
4. *$MA, MB, MC$ denote the length of the medians (incident on sides $A, B, C$ respectively);*
5. *$HA, HB, HC$ denote the length of the altitudes (treating sides $A, B, C$ as the base respectively);*
6. *[x y z] denotes the set of given conditions $x, y$ and $z$.*

Then, for example, [A a IA] will denote the conditions of: given length of a side, size of the angle opposite this side, and length of angle bisector incident on this side. It is realised that such a specification of conditions is not unique, but by inspection the cases we have surveyed do not repeat.

For each case, we will either specify an agenda of construction (thus proving the set of conditions is 'constructible') or use algebra to show the quantity necessary for construction as impossible to be constructed using Euclidean tools (thus proving the set of conditions is 'inconstructible') An in-depth discussion of constructability may be found in the Appendix of this study.

It is assumed that the following basic construction techniques are known, and thus their methods of construction are omitted:

1. *Constructing a line segment equal to another given length;*
2. *Constructing a line segment of 'constructible' length as defined in the Appendix of this study;*
3. *Constructing an angle equal to another given angle;*
4. *Bisecting a given angle;*
5. *Construct the complement of a given angle;*
6. *Constructing a circle with given radius and centre;*
7. *Constructing the tangent to a given circle at a given point on its circumference;*
8. *Constructing a right-angled triangle with two sides / one side and the hypotenuse / one side and an acute angle / the hypotenuse and an acute angle given.*

**Case 1 [A B C]**

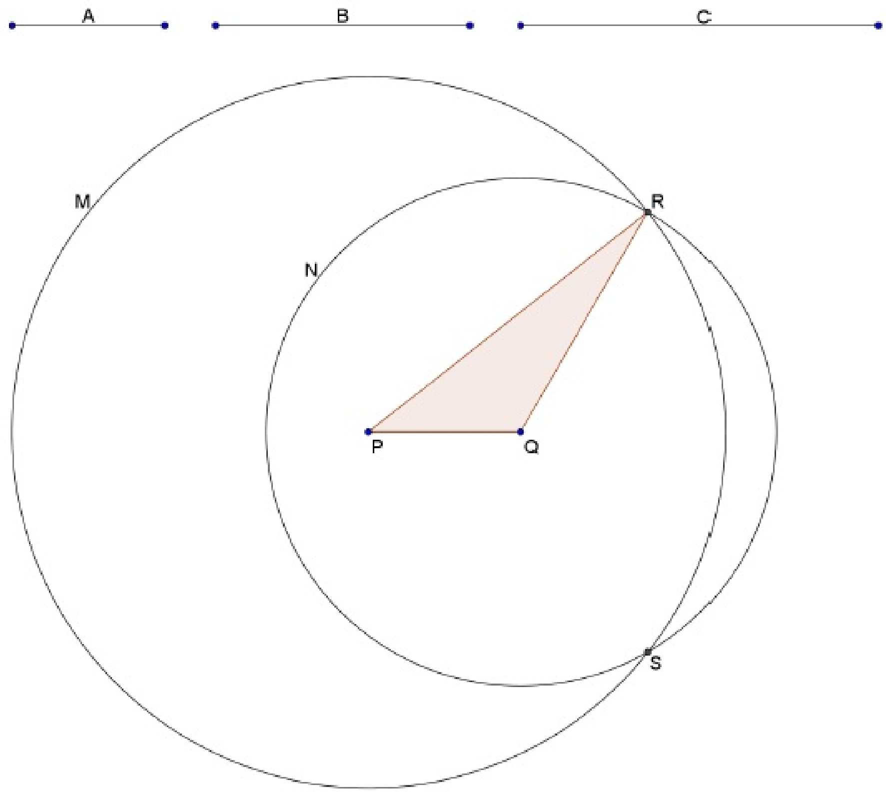
1. Construct $PQ = A$.

2. Construct circle $M$ with radius $C, P$ as centre. [See reviewer's comment (6)]

3. Construct circle $N$ with radius $B, Q$ as centre.

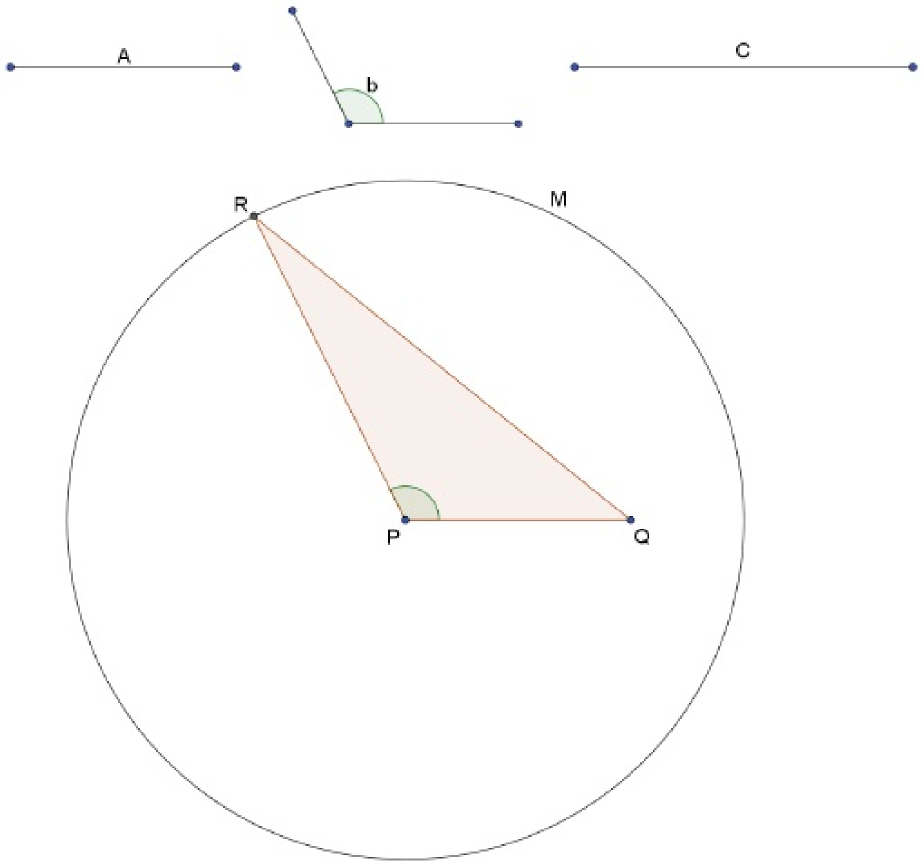
4. Denote intersections of $M$ and $N$ as $R$ and $S$.

5. Triangle $PQR$ is the required triangle.



**Case 2 [A b C]**

1. Construct $PQ = A$.
2. Construct circle $M$ with radius $C, P$ as centre.
3. Construct $R$ on $M$ such that angle $QPR =$ angle $b$
4. Triangle $PQR$ is the required triangle.



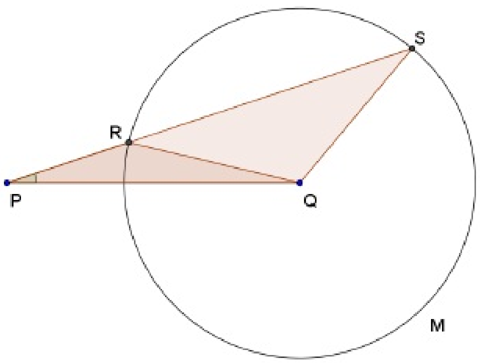
**Case 3 [A B a]**

1. Construct $PQ = B$.

2. Construct circle $M$ with radius $A, Q$ as centre.

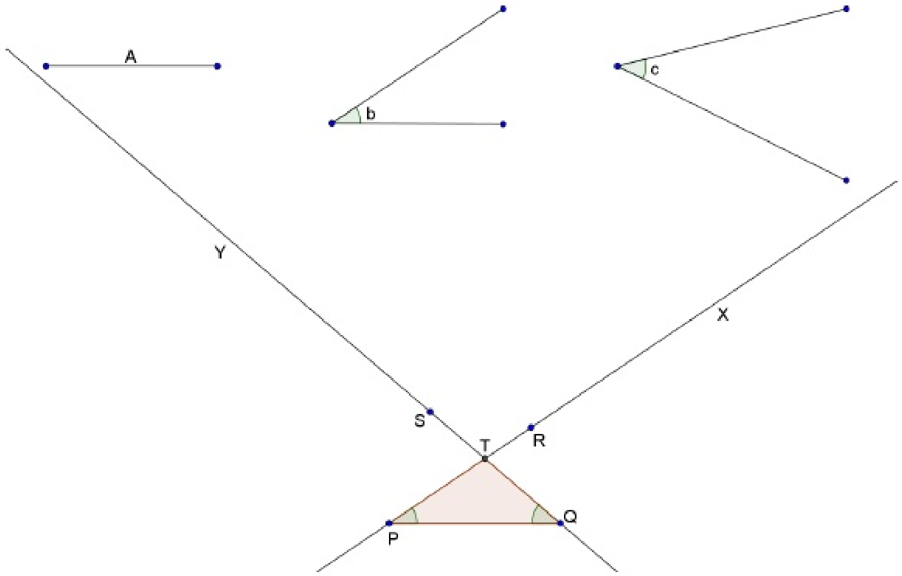
3. Construct $R$ and $S$ on $M$ such that angle $QPR$ = angle $QPS$ = angle $a$.

4. Triangles $PQR$ and $PQS$ are the required triangles.

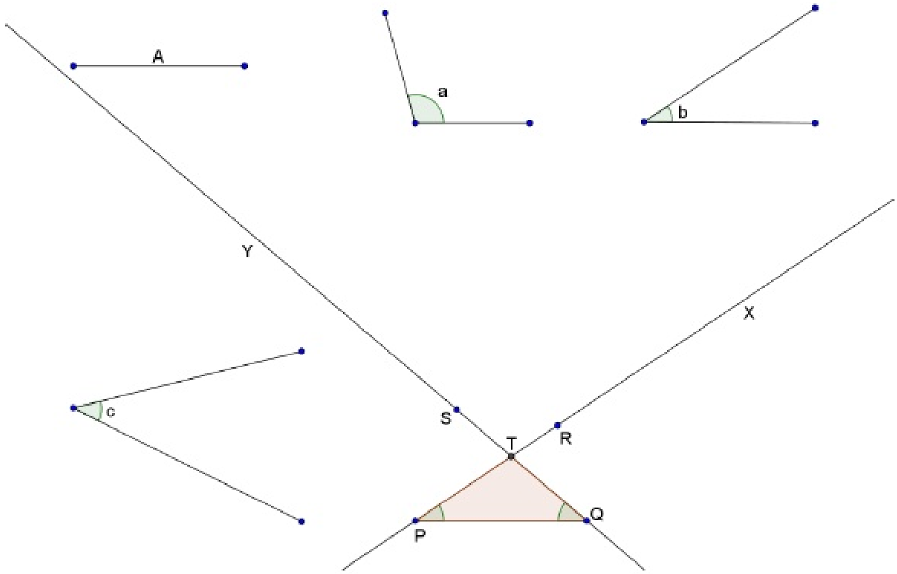
**Case 4 [A b c]**

1. Construct $PQ = A$.
2. Construct $R$ such that angle $QPR$ = angle $b$.
3. Construct $S$ such that angle $PQS$ = angle $c$ and is on the same side of $PQ$ as angle $QPR$.
4. Construct line $X$ through $PR$.
5. Construct line $Y$ through $QS$.
6. Denote intersection of $X$ and $Y$ as $T$.
7. Triangle $PQT$ is the required triangle.



**Case 5 [A a b]**

1. Construct complement angle $c$ of angle $(a+b)$.
2. Construct $PQ = A$.
3. Construct triangle $PQT$ using $PQ$, angle $b$ and angle $c$ as in case 4.
4. Triangle $PQT$ is the required triangle.



**Case 6 [a b c]**

There are an infinite number of similar triangles with angles $a, b$ and $c$.

The triangle is not fixed.

**Case 7 [A a I$A$]**

Consider triangle $PQR$ where $QR = A$, angle $QPR =$ angle $a$, angle bisector $PS = IA$.

Consider circumcircle $C$ of triangle $PQR$.

Since PS bisects angle $QPR$, $PS$ bisects minor arc $QR$.

Let this point of bisection be $T$.

Angle $RQT =$ angle $RPS =$ angle $QPS =$ angle $QRT$.

Hence triangle $RQT$ can be constructed.

By above, angle $SQT =$ angle $QPT$, angle $QTS =$ angle $PTQ$, hence triangle $SQT$ is similar to triangle $QPT$.

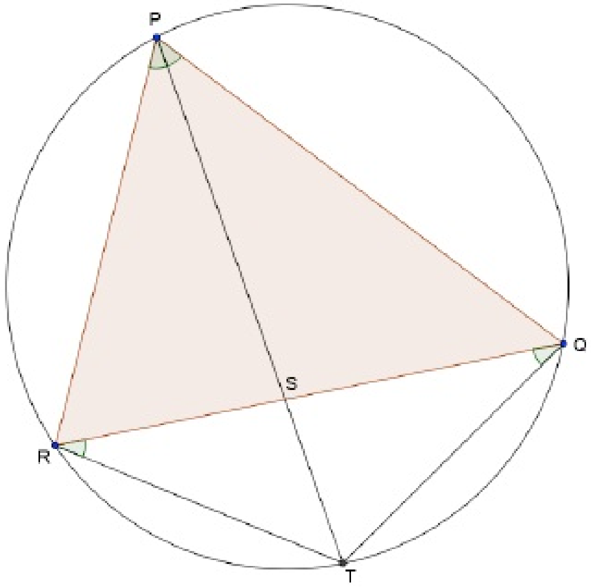
$$\frac{ST}{QT} = \frac{QT}{PT}$$

$$\frac{ST}{QT} = \frac{QT}{PS + ST}$$

$$ST^2 + PS \times ST - QT^2 = 0$$

$$ST = \sqrt{\left(\frac{PS}{2}\right)^2 + QT^2} - \frac{PS}{2} = \sqrt{\left(\frac{IA}{2}\right)^2 + QT^2} - \frac{IA}{2}.$$

Since triangle $RQT$ can be constructed, $QT$ is known, and $ST$ can also be constructed.




1. Construct triangle $Q'R'T'$ with $Q'R' = A$, angle $R'Q'T'$ = angle $Q'R'T'$ = angle $\frac{a}{2}$ as in case 4.

2. Construct right-angled triangle $DEF$ such that $DE = QT$, $EF = \frac{IA}{2}$.
[See reviewer's comment (7)]

3. Construct circle $M$ with radius $EF, F$ as centre.

4. Denote intersection of $M$ and $DF$ as $G$.

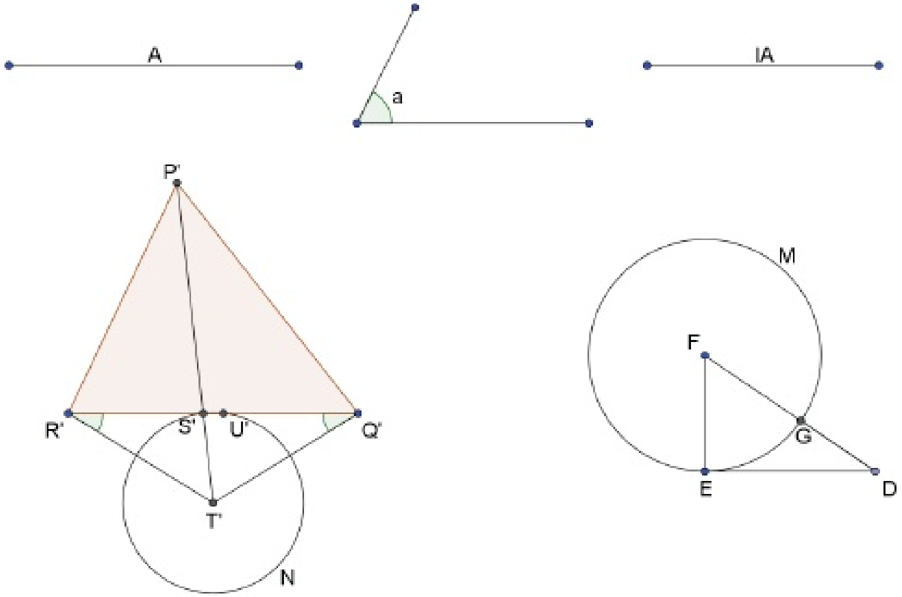
$$\therefore DG = \sqrt{\left(\frac{IA}{2}\right)^2 + Q'T'^2} - \frac{IA}{2}$$

5. Construct circle $N$ radius $DG, T'$ as centre.

6. Denote intersections of $N$ and $Q'R'$ as $S'$ and $U'$.

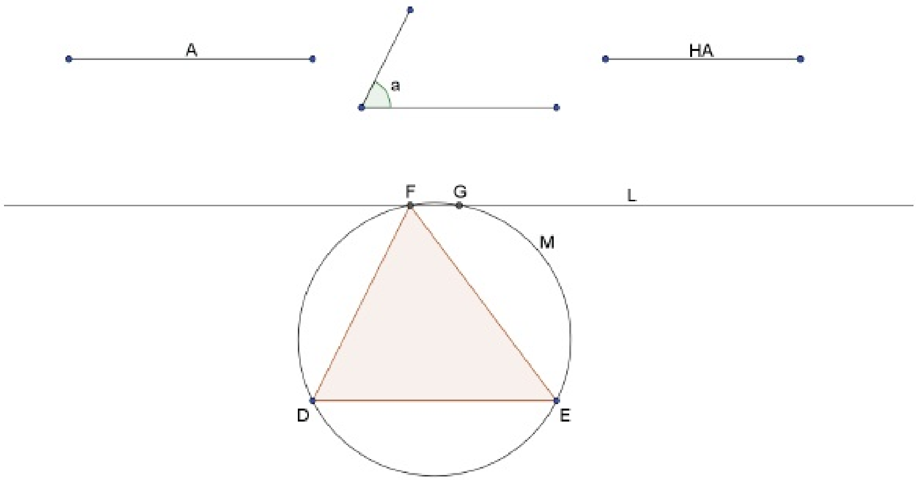
7. Extend $S'T'$ past $S'$ to $P'$ such that $P'S' = IA$.

8. Triangle $P'Q'R'$ is the required triangle.


A
a
IA
P'
R'
S'
U'
Q'
T'
N
M
F
G
E
D

**Case 8 [A a HA]**

1. Construct $DE = A$.
2. Construct line $L$ parallel to $DE$, separated by $HA$.
3. Construct circle $M$ passing through $D, E$ and including angle $a$ on circumference. [See reviewer's comment (8)]
4. Denote intersections of $M$ and $L$ as $F$ and $G$.
5. Triangle $DEF$ is the required triangle.



**Case 9 [A a MA]**

1. Construct $PQ = A$.

2. Construct complement angle $b$ of angle $a$.

3. Bisect angle $b$ to obtain angle $c$.

4. Construct triangle $PQT$ using $PQ$, angle $c$ and angle $c$ as in case 4.

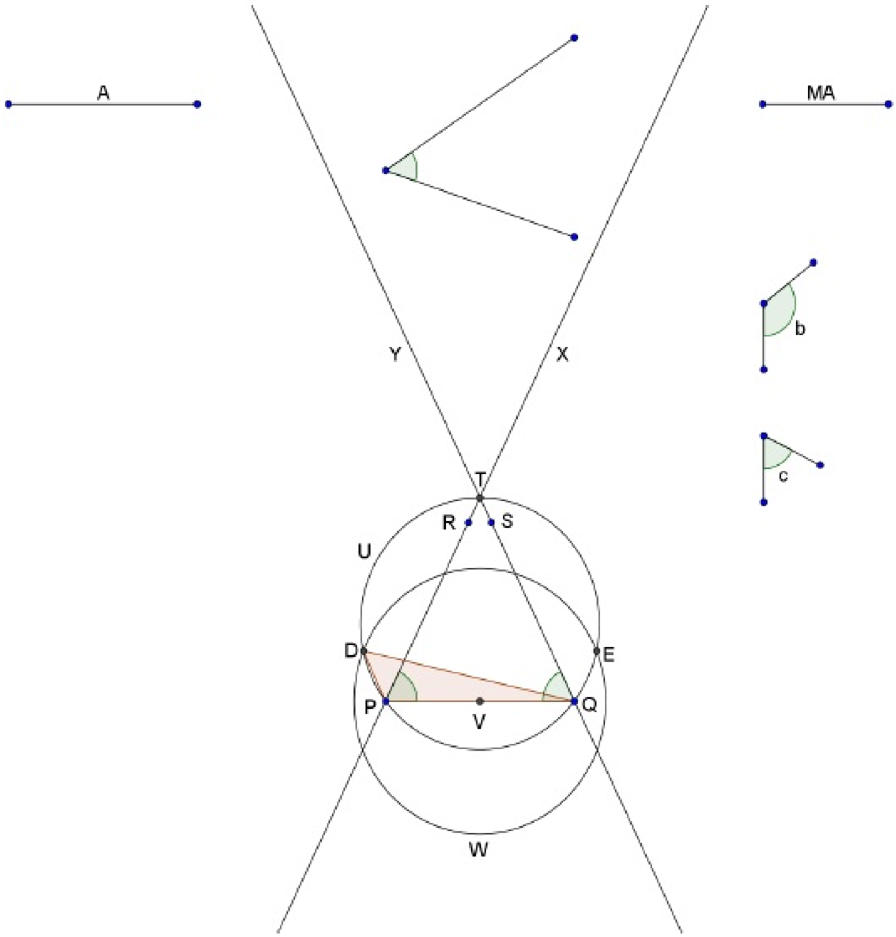
5. Construct circumcircle $U$ of triangle $PQT$.

6. Bisect $PQ$ at $V$.

7. Construct circle $W$ with radius $MA, V$ as centre.

8. Denote intersections of $U$ and $W$ as $D$ and $E$.

9. Triangle $DPQ$ is the required triangle.




[See reviewer's comment (9) and (10)]

### Case 10 [A b IA]

Consider triangle $PQR$, where $PQ = A, RS = IA$, angle $QPR =$ angle $b$.

Let $RS = x$, angle $PRQ = 2\theta$.

$$\therefore \begin{cases} PS = \dfrac{x \sin\theta}{\sin b}, \\ QS = \dfrac{x \sin\theta}{\sin(2\theta + b)}. \end{cases}$$

$$PQ = PS + QS = \frac{x \sin\theta}{\sin b} + \frac{x \sin\theta}{\sin(2\theta + b)},$$

$$\frac{PQ}{x} = \sin\theta \left( \frac{1}{\sin b} + \frac{1}{2\sin\theta\cos\theta\cos b + (1 - 2\sin^2\theta)\sin b} \right).$$

Let $y = \sin\theta$.

$$\frac{PQ}{x} = \sin\theta \left( \frac{1}{\sin b} + \frac{1}{2y\sqrt{1-y^2}\cos b + (1 - 2y^2)\sin b} \right).$$

Consider the case where $\begin{cases} PQ & = 2 \\ b & = \dfrac{\pi}{6} \\ x & = 1 \end{cases}.$

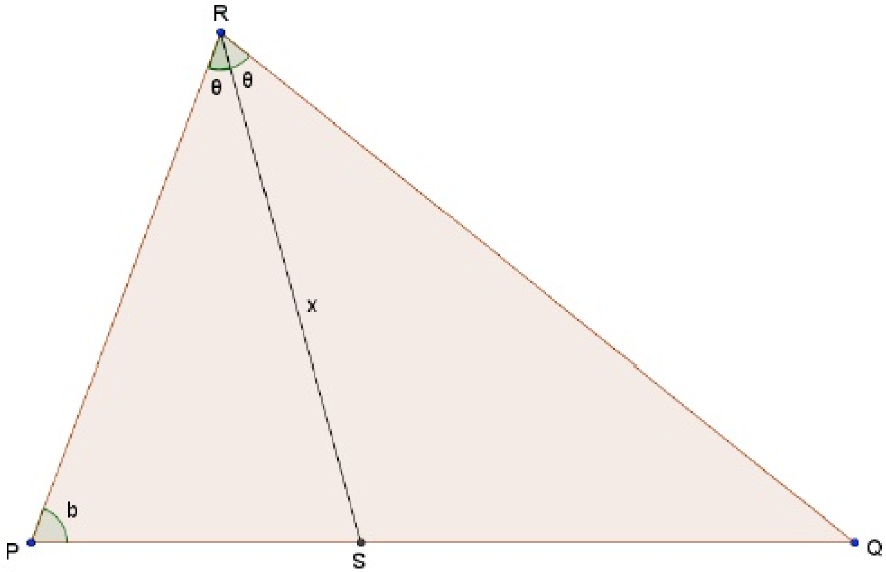
$$\therefore \quad 2 = y \left( 2 + \frac{1}{y\sqrt{3 - 3y^2} - y^2 - \dfrac{1}{2}} \right),$$

$$16y^6 - 32y^5 - 4y^4 + 36y^3 - 12y^2 - 4y + 1 = 0.$$

This polynomial has no constructible roots.

So $y$ cannot be constructed, and thus $\theta$ cannot be constructed.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]

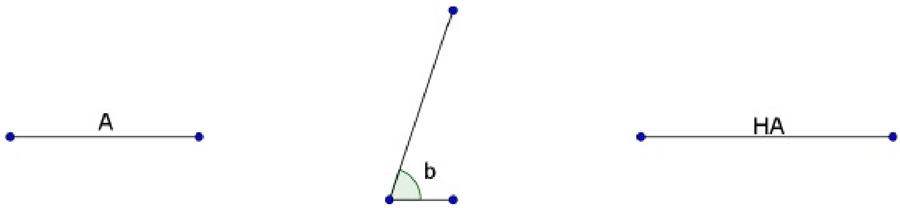

R
θ
θ
x
b
P
S
Q

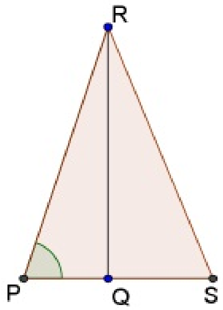
**Case 11 [A b HA]**

1. Construct right-angled triangle $PQR$ with angle $QPR$ = angle $b$ and $QR = HA$.

2. Extend $PQ$ past $Q$ to $S$ such that $PS = A$.

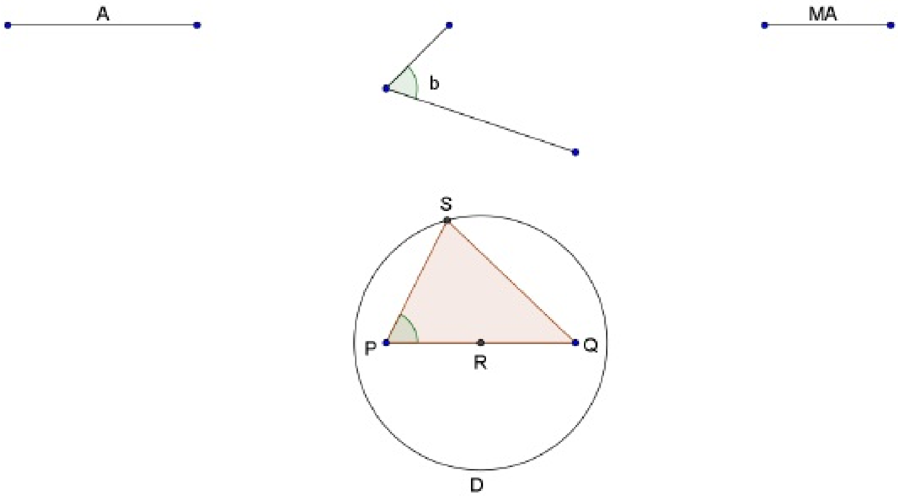
3. Triangle $PRS$ is the required triangle.



**Case 12 [A b MA]**

1. Construct $PQ = A$.
2. Bisect $PQ$ at $R$.
3. Construct circle $D$ with radius $MA, R$ as centre.
4. Construct $S$ on $D$ such that angle $PQS$ = angle $b$.
5. Triangle $PQS$ is the required triangle.



## Case 13 [A B IA]

Consider triangle $PQR$, where $PQ = A$, $RS = IA$, $PR = B$.

Let $RS = x$, angle $PRQ = 2\theta$.

$$\frac{PR}{PQ} = \frac{B}{A} = \frac{\sin b}{\sin 2\theta},$$

$$PS = \sqrt{PR^2 + x^2 - 2x\cos\theta \times PR} = \sqrt{A^2 + x^2 - 2Ax\cos\theta},$$

$$QS = x\frac{\sin\theta}{\sin b} = \frac{Ax}{2B\cos\theta},$$

$$PQ = PS + QS = \sqrt{PR^2 + x^2 - 2x\cos\theta \times PR} + \frac{Ax}{2B\cos\theta}.$$

Consider the case where $\begin{cases} A &= 2 \\ B &= 1\,. \\ x &= 1 \end{cases}$

$$\therefore \quad \sqrt{2 - 2\cos\theta} + \frac{1}{\cos\theta} = 2.$$

Let $y = \cos\theta$.

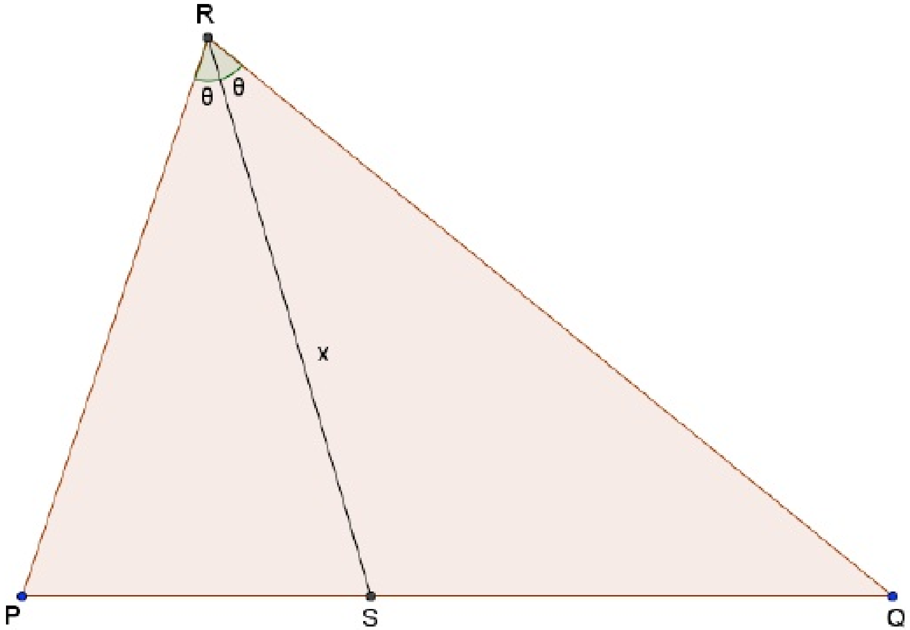
$$\therefore \quad \sqrt{2 - 2y} + \frac{1}{y} = 2,$$

$$2y^3 + 2y^2 - 4y + 1 = 0.$$

This polynomial has no constructible roots

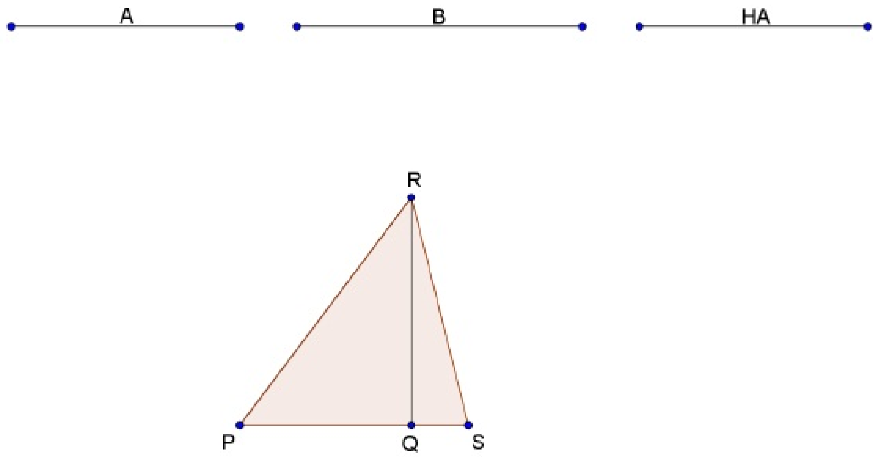
So $y$ cannot be constructed, and thus $\theta$ cannot be constructed.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]


R
θ
θ
x
P
S
Q

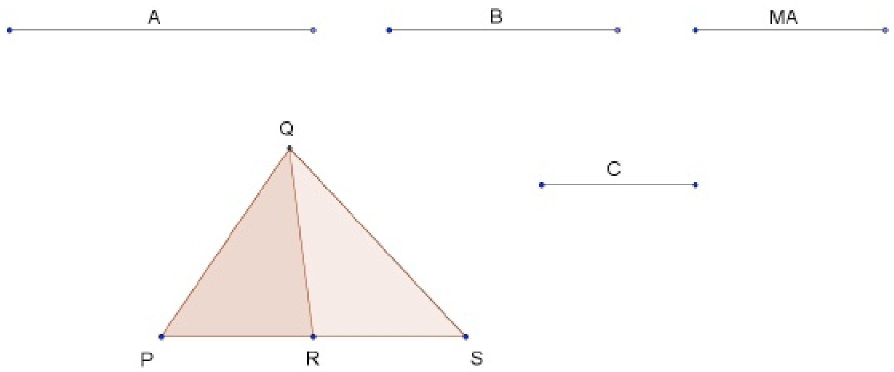
## Case 14 [A B HA]

1. Construct right-angled triangle $PQR$ with $QR = HA$ and $PR = B$.
2. Extend $PQ$ past $Q$ to $S$ such that $PS = A$.
3. Triangle $PRS$ is the required triangle.



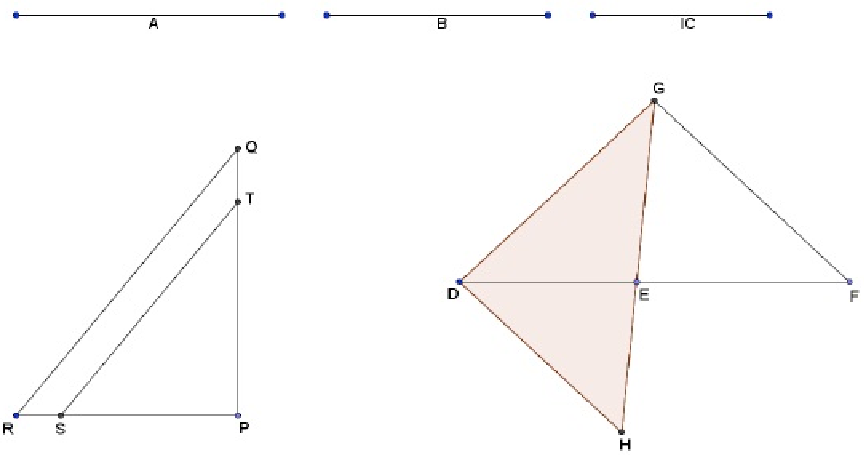
## Case 15 [A B MA]

1. Bisect $A$ to obtain $C$.
2. Construct triangle $PQR$ using $MA$, $B$ and $C$ as in case 1.
3. Extend $PR$ past $R$ to $S$ such that $PS = A$.
4. Triangle $PQS$ is the required triangle.



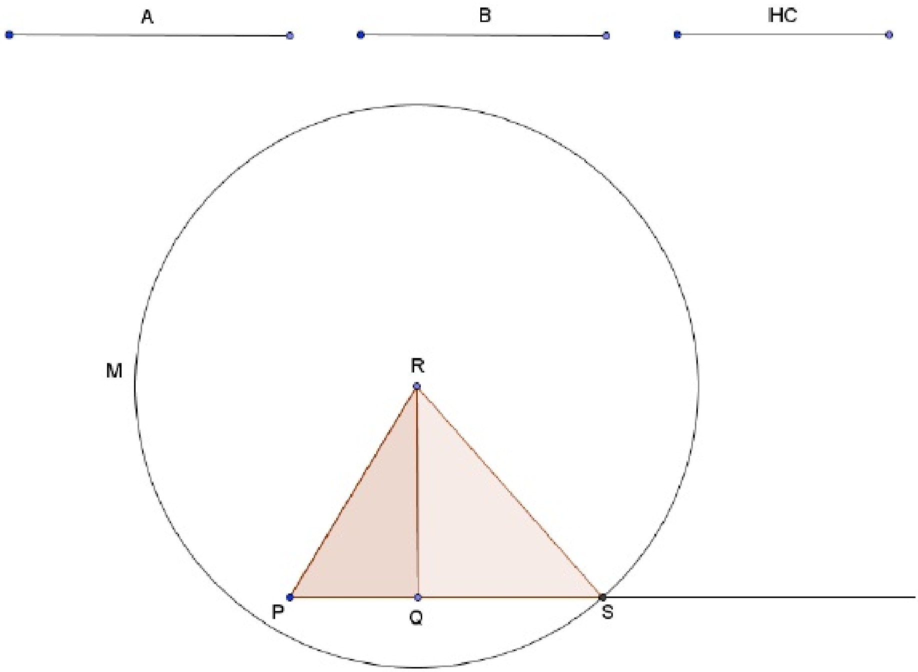
**Case 16 [A B IC]**

1. Construct right-angled triangle $QPR$ with $PQ = A$, $PR = B$.
2. Construct $S$ on $PR$ such that $PS = IC$.
3. Construct line $K$ through $S$ parallel to $QR$.
4. Denote intersection of $K$ and $PQ$ as $T$.
$$\therefore PT = \frac{A \times IC}{B}.$$
5. Construct $DE = IC$.
6. Extend $DE$ past $E$ to $F$ such that $EF = PT$.
7. Construct triangle $DFG$ using $DF, A$ and $A$ as in case 1.
8. Construct line $L$ through $D$ parallel to $GF$.
9. Extend $EG$ past $E$ to $H$ such that $H$ lies on $L$.
10. Triangle $GDH$ is the required triangle



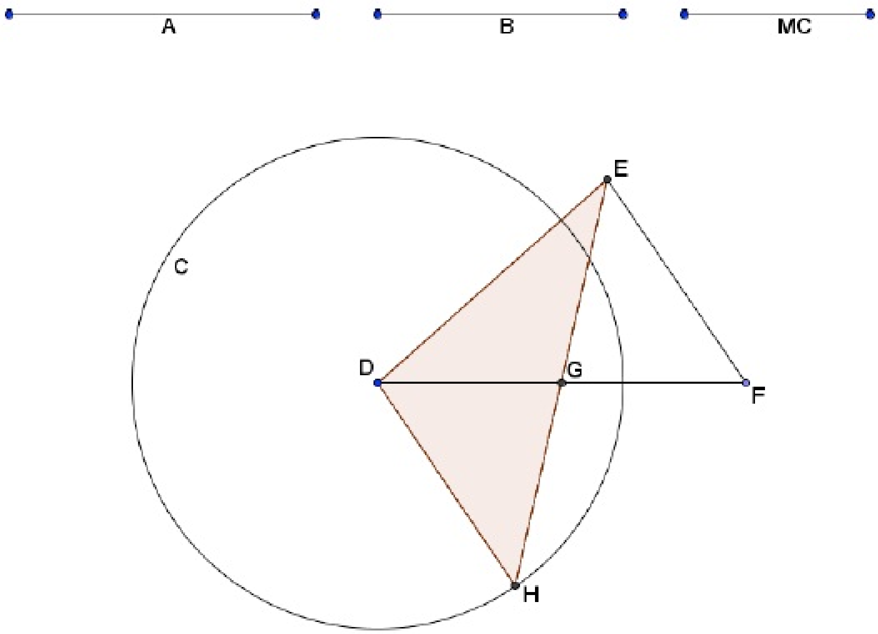
**Case 17 [A B HC]**

1. Construct right-angled triangle $PQR$ with $QR = HC$ and $PR = B$.
2. Construct circle $M$ with radius $A, R$ as centre.
3. Extend $PQ$ past $Q$ to $S$ such that $S$ lies on $M$.
4. Triangle $PRS$ is the required triangle.



## Case 18 [A B MC]

1. Construct triangle $DEF$ with $A, B$ and $2MC$ as in case 1.
2. Bisect $DF$ at $G$.
3. Construct circle $C$ with radius $B, D$ as centre.
4. Extend $EG$ past $G$ to $H$ such that $H$ lies on $C$.
5. Triangle $DEH$ is the required triangle.



## Case 19 [A HA IA]

1. Construct right-angled triangle $PQR$ with $PR = IA$ and $QR = HA$.

2. Extend $PQ$ past $Q$ to $S$ such that $RS$ is perpendicular to $PR$.

3. Let $PS = p$.

   Consider $T$ on $QS$ and extending $PQ$ past $P$ to $U$ such that triangle $RTU$ is the required triangle.

   Let $PU = x$.

$$\therefore \begin{cases} PT = A - x \\ SU = x + p \\ ST = x + p - A \end{cases}.$$

   By considering the Sine Law,

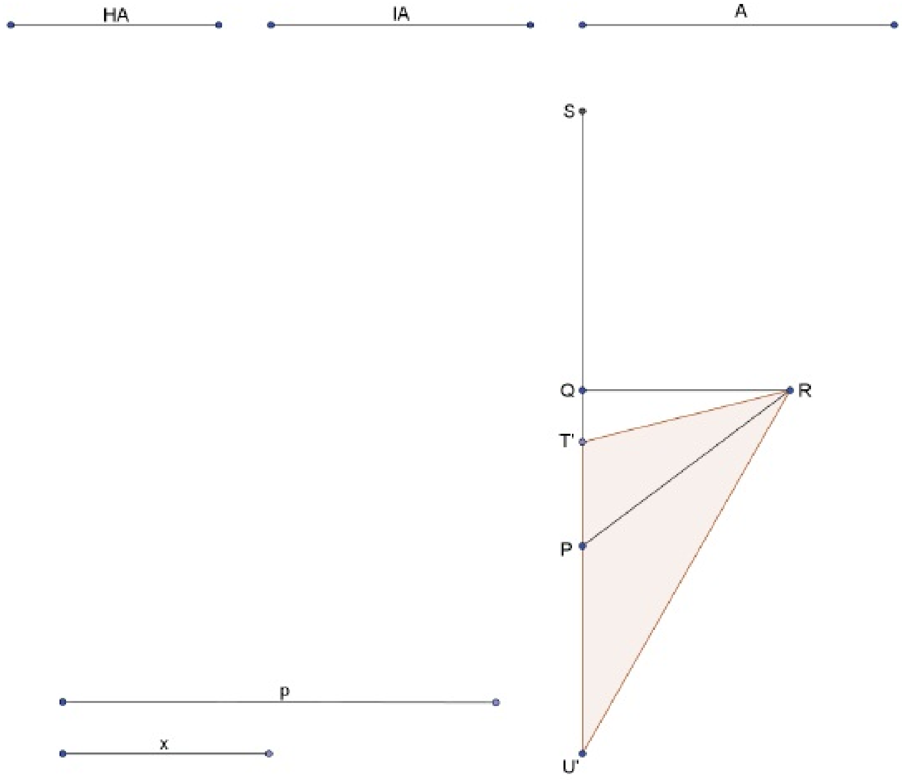
$$PU : PT = RU : RT = SU : ST$$
$$\therefore \frac{x}{A-x} = \frac{x+p}{x+p-A}$$
$$x = \frac{A - p + \sqrt{p^2 + A^2}}{2}.$$

   Thus $x$ is constructible from $A$ and $PS$.

4. Extend $PQ$ past $P$ to $U'$ such that $PU' = x$.

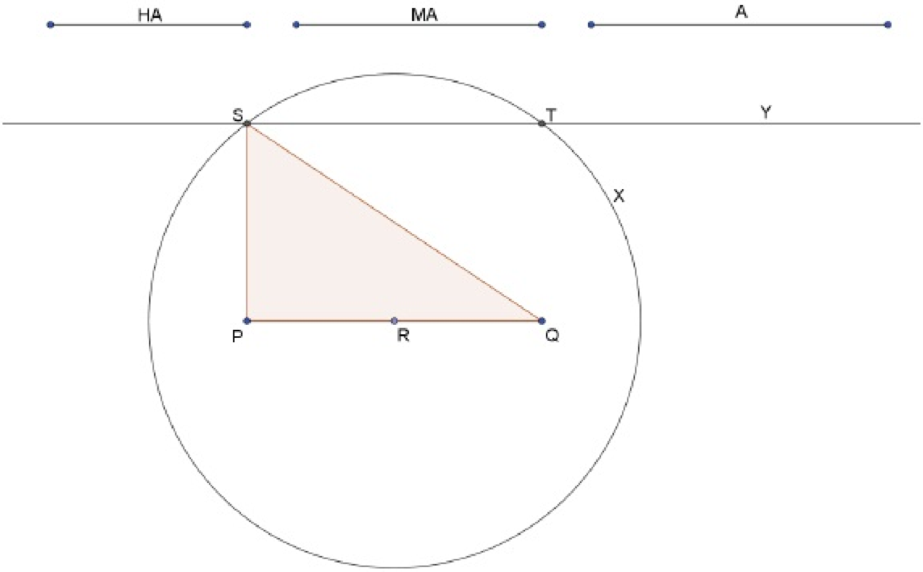
5. Construct $T'$ on PS such that $T'U' = A$.

6. Triangle $RT'U'$ is the required triangle.


HA
IA
A
S
Q
R
T'
P
p
x
U'

### Case 20 [A HA MA]

1. Construct $PQ = A$.
2. Bisect $PQ$ at $R$.
3. Construct circle $X$ with radius $MA, R$ as centre.
4. Construct line $Y$ parallel to $PQ$, separated by $HA$.
5. Denote intersections of $X$ and $Y$ as $S$ and $T$.
6. Triangle $PQS$ is the required triangle.



## Case 21 [A IA MA]

Consider the triangle $P'Q'R'$ with median $Q'S' = \dfrac{2MA}{A}$, angle bisector $Q'T' = \dfrac{2IA}{A}$ and $P'S' = 1$.

Let $S'T' = \dfrac{2z}{A}$ .

Let $U'$ be a point on $P'R'$ such that $Q'U' \perp P'R'$, and let $S'U' = x$, $Q'U' = y$.

$$\therefore \quad \begin{cases} x^2 + y^2 = \left(\dfrac{2MA}{A}\right)^2 \\ \dfrac{(1-x)^2 + y^2}{(x+1)^2 + y^2} = \left(\dfrac{1 - \dfrac{2z}{A}}{1 + \dfrac{2z}{A}}\right)^2 \\ \left(x - \dfrac{2z}{A}\right)^2 + y^2 = \left(\dfrac{2IA}{A}\right)^2 \end{cases}$$

$$\begin{cases} \dfrac{A^2 - 2A^2x + 4MA^2}{A^2 + 2A^2x + 4MA^2} = \left(\dfrac{A - 2z}{A + 2z}\right)^2 \\ z^2 + MA^2 - Axz = IA^2 \end{cases}$$

$$4z^4 - (A^2 + 4MA^2 + 4IA^2)z^2 + MA^2A^2 - IA^2A^2 = 0$$

$$z^2 = \frac{A^2 + 4MA^2 + 4IA^2 - \sqrt{(A^2 + 4MA^2 + 4IA^2)^2 - (4A\sqrt{MA^2 - IA^2})^2}}{8}$$

$$z = \frac{\sqrt{A^2 + 4MA^2 + 4IA^2 - \sqrt{(A^2 + 4MA^2 + 4IA^2)^2 - (4A\sqrt{MA^2 - IA^2})^2}}}{2\sqrt{2}}$$

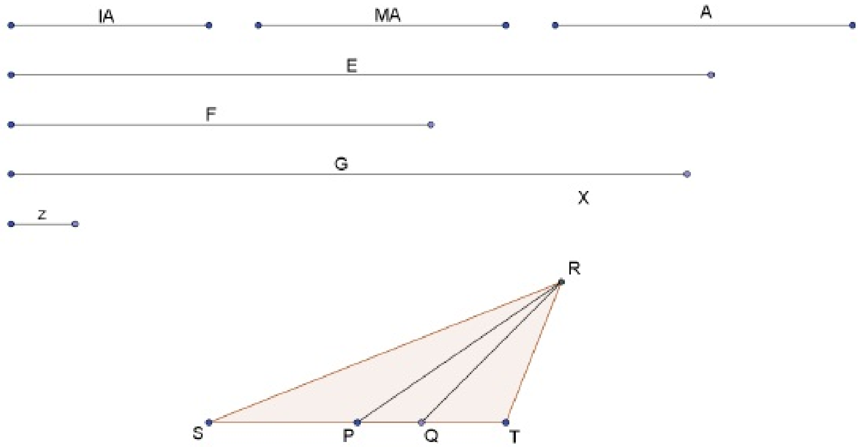
Let $\begin{cases} E = \sqrt{A^2 + 4MA^2 + 4IA^2} \\ F = 2\sqrt{A\sqrt{MA^2 - IA^2}} \end{cases}$ . Both of them are constructible.

$$\therefore z = \frac{\sqrt{E^2 - \sqrt{E^4 - F^4}}}{2\sqrt{2}} = \frac{\sqrt{E^2 - \sqrt{E^2 + F^2}\sqrt{E^2 - F^2}}}{2\sqrt{2}}.$$

Let $G = \sqrt{\sqrt{E^2 + F^2}\sqrt{E^2 - F^2}}$, which is constructible.

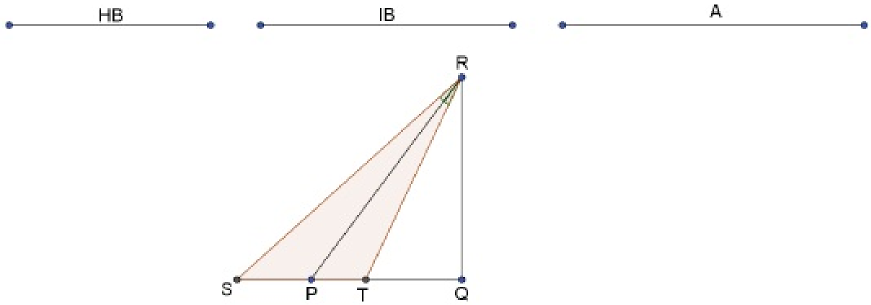
$\therefore z = \dfrac{\sqrt{E^2 - G^2}}{2\sqrt{2}}$, which is a constructible form.

1. Construct triangle $PQR$ using $z, IA$ and $MA$ as in case 1.
2. Extend $PQ$ past $P$ to $S$ such that $PS = \frac{1}{2}A$.
3. Extend $PQ$ past $Q$ to $T$ such that $PT = \frac{1}{2}A$.
4. Triangle $RST$ is the required triangle.



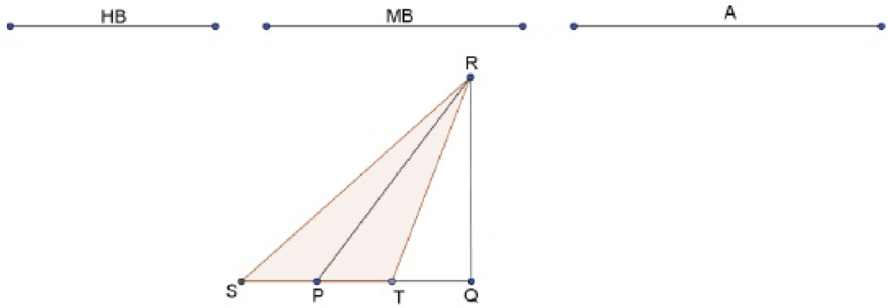
**Case 22 [A HB IB]**

1. Construct right-angled triangle $PQR$ with $PR = IB$ and $QR = HB$.
2. Extend $PQ$ past $P$ to $S$ such that $RS = A$.
3. Construct $T$ on $PQ$ such that angle $PRT$ = angle $PRS$.
4. Triangle $RST$ is the required triangle.



**Case 23 [A HB MB]**

1. Construct right-angled triangle $PQR$ with $PR = MB$ and $QR = HB$.
2. Extend $PQ$ past $P$ to $S$ such that $RS = A$.
3. Extend $PQ$ past $Q$ to $T$ such that $PT = PS$.
4. Triangle $RST$ is the required triangle.



## Case 24 [A IB MB]

Let $IB = x$, $MB = y$.

Let $\beta = \dfrac{B}{2}$.

**Case I: b < 90°.**

$$\because \begin{cases} \dfrac{b}{2} = \pi - c - \arcsin\left(\dfrac{\sin c}{x}A\right) \\ b = \arcsin\left(\dfrac{2\beta \sin c}{\sqrt{A^2 + 4\beta^2 - 4A\beta\cos c}}\right) \end{cases}$$

$$\therefore \arcsin\left(\frac{2\beta \sin c}{\sqrt{A^2 + 4\beta^2 - 4A\beta\cos c}}\right) = 2\pi - 2c - 2\arcsin\left(\frac{\sin c}{x}A\right).$$

**Case II: b ≥ 90°.**

$$\because \begin{cases} \dfrac{b}{2} = \pi - c - \arcsin\left(\dfrac{\sin c}{x}A\right) \\ b = \pi - \arcsin\left(\dfrac{2\beta \sin c}{\sqrt{A^2 + 4\beta^2 - 4A\beta\cos c}}\right) \end{cases}$$

$$\therefore \arcsin\left(\frac{2\beta \sin c}{\sqrt{A^2 + 4\beta^2 - 4A\beta\cos c}}\right) = 2c + 2\arcsin\left(\frac{\sin c}{x}A\right).$$

For both cases, taking sine of both sides and simplifying,

$$\begin{aligned} &\frac{\beta}{\sqrt{A^2 + 4\beta^2 - 4A\beta\cos c}} \\ &= -\frac{(2\cos^2 c - 1)A}{x}\sqrt{1 - \left(\frac{\sin c}{x}A\right)^2} - \left[1 - 2\left(\frac{\sin c}{x}A\right)^2\right]\cos c. \end{aligned}$$

Eliminating $c$,

$$\begin{aligned}
&\frac{\beta}{\sqrt{-A^2+2\beta^2+2y^2}} \\
&= -\frac{A}{x}\left[2\left(\frac{A^2+\beta^2-y^2}{2A\beta}\right)^2-1\right]\sqrt{1-\left(\frac{A}{x}\right)^2\left[1-\left(\frac{A^2+\beta^2-y^2}{2A\beta}\right)^2\right]} \\
&\quad -\frac{A^2+\beta^2-y^2}{2A\beta}\left\{1-2\left(\frac{A}{x}\right)^2\left[1-\left(\frac{A^2+\beta^2-y^2}{2A\beta}\right)\right]\right\}.
\end{aligned}$$

Consider the case in which $A=6, x=2, y=3$. Rearranging w.r.t. $\beta$,

$$\begin{aligned}
&\beta^6+(\sqrt{\beta^4-74\beta^2+729}-55)\beta^4-9(2\sqrt{\beta^4-74\beta^2+729}+165)\beta^2 \\
&+729(\sqrt{\beta^4-74\beta^2+729}+27)=-\frac{48\sqrt{2}\beta^4}{\sqrt{\beta^2-9}}
\end{aligned}$$

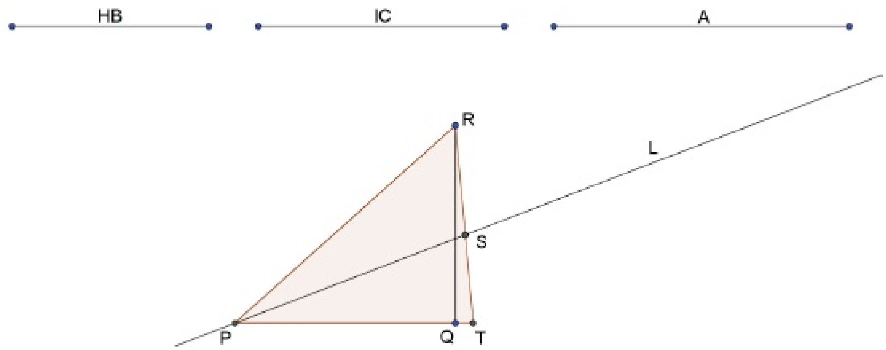
This equation has no constructible roots.

So $\beta$ cannot be constructed, and thus $B$ cannot be constructed.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]

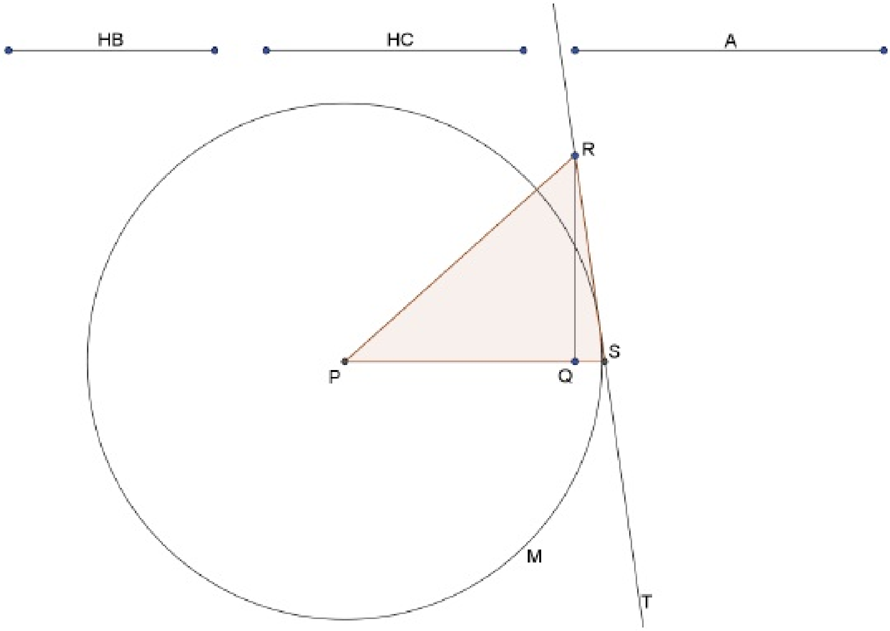
**Case 25 [A HB IC]**

1. Construct right-angled triangle $PQR$ with $PR = A$, $QR = HB$.
2. Bisect angle $QPR$ with line $L$.
3. Construct $S$ on $L$ such that $PS = IC$.
4. Extend $RS$ past $S$ to $T$ such that $P, Q$ and $T$ are collinear.
5. Triangle $PRT$ is the required triangle.



## Case 26 [A HB HC]

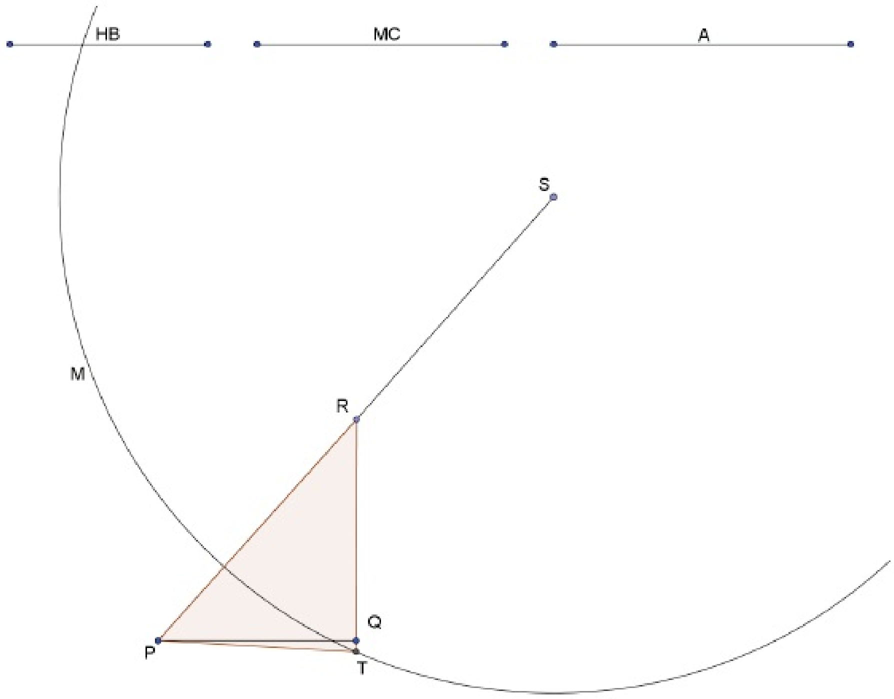
1. Construct right-angled triangle $PQR$ with $PR = A, QR = HB$.
2. Construct circle $M$ with radius $HC, P$ as centre.
3. Construct tangent $T$ to $M$ such that $T$ passes through $R$.
4. Extend $PQ$ past $Q$ to $S$ such that $S$ lies on $T$.
5. Triangle $PRS$ is the required triangle.




[See reviewer's comment (10)]

## Case 27 [A HB MC]

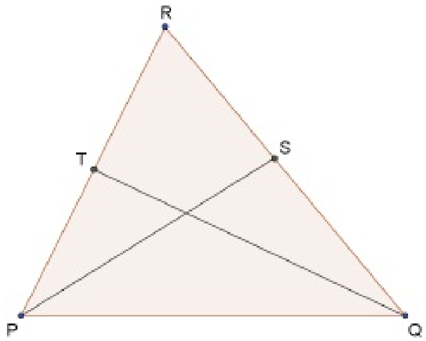
1. Construct right-angled triangle $PQR$ with $PR = A, PQ = HB$.
2. Extend $PR$ past $R$ to $S$ such that $RS = A$.
3. Construct circle $M$ with radius $2MC$, $S$ as centre.
4. Extend $QR$ past $Q$ to $T$ such that $T$ lies on $M$.
5. Triangle $PRT$ is the required triangle.




[See reviewer's comment (10)]

**Case 28 [A IB IC]**

Consider triangle $PQR$ with $PQ = A$, angle bisector $PS = IB$, angle bisector $QT = IC$.




Let $IB = x$, $IC = y$, $QS = m$, $PT = n$.

$$\because \begin{cases} \arccos \dfrac{m^2 + A^2 - x^2}{2Am} = 2 \arccos \dfrac{y^2 + A^2 - n^2}{2Ay} \\ \arccos \dfrac{n^2 + A^2 - y^2}{2An} = 2 \arccos \dfrac{y^2 + A^2 - m^2}{2Ax} \end{cases}$$

$$\therefore \begin{cases} \dfrac{m^2 + A^2 - x^2}{2Am} = 2 \left( \dfrac{y^2 + A^2 - n^2}{2Ay} \right)^2 - 1 \\ \dfrac{n^2 + A^2 - y^2}{2An} = 2 \left( \dfrac{y^2 + A^2 - m^2}{2Ax} \right)^2 - 1 \end{cases}$$

Consider the case in which $A = 10$, $x = 8$, $y = 7$.

Substituting into the first equation and rearranging w.r.t. $m$,

$$490m^2 - (n^4 - 298n^2 + 12401)m + 17640 = 0.$$

$$\therefore m = \frac{n^4 - 298n^2 + 12401 \pm \sqrt{n^8 - 596n^6 + 113606n^4 - 7390996n^2 + 119210401}}{980}.$$

Substituting into the second equation and rearranging w.r.t. $n$,

$$640(51+20n+n^2) = n\left[164-\left(\frac{n^4-298n^2+12401\pm\sqrt{n^8-596n^6+113606n^4-7390996n^2+119210401}}{980}\right)^2\right]^2$$

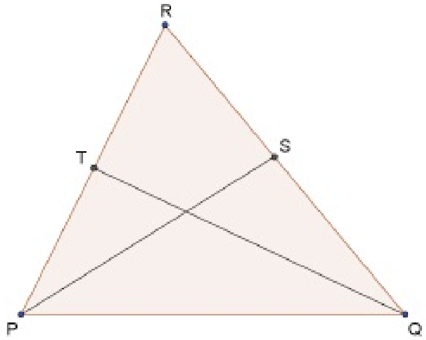
Except for $n = 3$ and $n = 17$, the other roots of $n$ are inconstructible.

However, these solutions of $n$ violate the triangle inequalities, so no suitable root of $n$ is constructible.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]

**Case 29 [A IB MC]**

Consider triangle $PQR$ with $PQ = A$, angle bisector $PS = IB$, median $QT = MC$.




Let $IB = x$, $MC = y$, $\theta =$ angle $\dfrac{d}{2}$, $z = \dfrac{C}{2}$.

$$\because \begin{cases} \sqrt{x^2+4z^2-4xz\cos\theta}+\sqrt{A^2+x^2-2Ax\cos\theta}=\sqrt{A^2+4z^2-4Az\cos 2\theta} \\ \cos 2\theta = \dfrac{A^2+z^2-y^2}{2Az} \\ \cos\theta = \sqrt{\dfrac{1}{2}\left(1+\dfrac{A^2+z^2-y^2}{2Az}\right)} \end{cases}$$

$$\therefore \quad \sqrt{x^2+4z^2-4xz\sqrt{\frac{1}{2}\left(1+\frac{A^2+z^2-y^2}{2Az}\right)}} + \sqrt{A^2+x^2-2Ax\sqrt{\frac{1}{2}\left(1+\frac{A^2+z^2-y^2}{2Az}\right)}} = \sqrt{-A^2+2z^2+2y^2}$$

Consider the case in which $A = 6, x = 5, y = 4$.

$$\sqrt{25+4z^2-20z\sqrt{\frac{z^2+12z+20}{24z}}}+\sqrt{61-60\sqrt{\frac{z^2+12z+20}{24z}}}=\sqrt{2z^2-4}$$

Except for $z = 2$, the other root of $z$ is inconstructible.

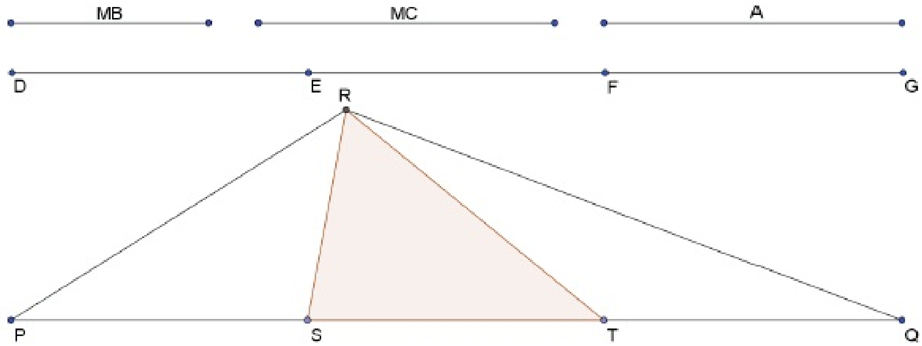
However, this solution of $z$ violates the triangle inequalities.

So no suitable root of $z$ is constructible.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]

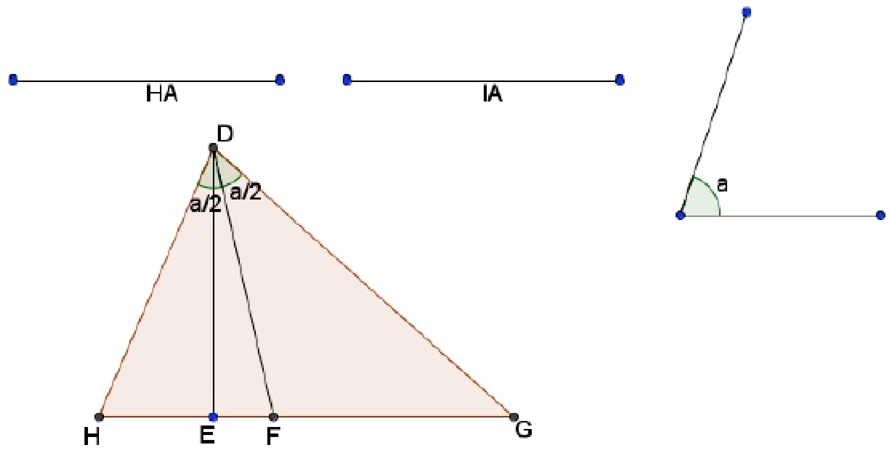
## Case 30 [A MB MC]

1. Construct $EF = A$.
2. Extend $EF$ past $F$ to $G$ such that $FG = A$.
3. Extend $EF$ past $E$ to $D$ such that $DE = A$.
4. Construct triangle $PQR$ using $DG$, $2MB$ and $2MC$ as in case 1.
5. Trisect $PQ$ at $S$ and $T$.
6. Triangle $RST$ is the required triangle.



## Case 31 [a HA IA]

1. Construct right-angled triangle $DEF$ with $DE = HA, DF = IA$.
2. Extend $EF$ past $F$ to $G$ such that angle $FDG$ = angle $\frac{a}{2}$.
3. Extend $EF$ past $E$ to $H$ such that angle $FDH$ = angle $\frac{a}{2}$.
4. Triangle $DHG$ is the required triangle.



## Case 32 [a HA MA]

Consider triangle $PQR$ with angle $PQR$ = angle $a$, height $HP = HA$, median $MP = MA$.

Consider circumcircle $C_1$ and circumcentre $O$ of triangle $PQR$.

Denote midpoint of minor arc $QR$ as $T$, midpoint of $QR$ as $M$.

$$\because \begin{cases} \angle QOR = 2\angle QPR \\ \angle QOR = 2\angle QOT \end{cases}$$

$$\therefore \angle QOT = \angle QPR = a$$

$$\because \begin{cases} \angle MRT = \dfrac{a}{2} \\ \angle MQT = \dfrac{a}{2} \end{cases}$$

We need to locate $Q, R$ to construct triangle $PQR$.

For this purpose, let $Q', T', R'$ be arbitrary points on $MQ$, $MR$, $MT$ respectively, such that angle $MR'T'$ = angle $MQ'T'$ = angle $\dfrac{a}{2}$.

$$\therefore \frac{MQ'}{MQ} = \frac{MR'}{MR} = \frac{MT'}{MT}$$

Let $P'$ on $MP$ satisfy $\dfrac{MP'}{MP} = \dfrac{MQ'}{MQ}$.

Consider circumcircle $C_2$ of triangle $Q'R'T'$.

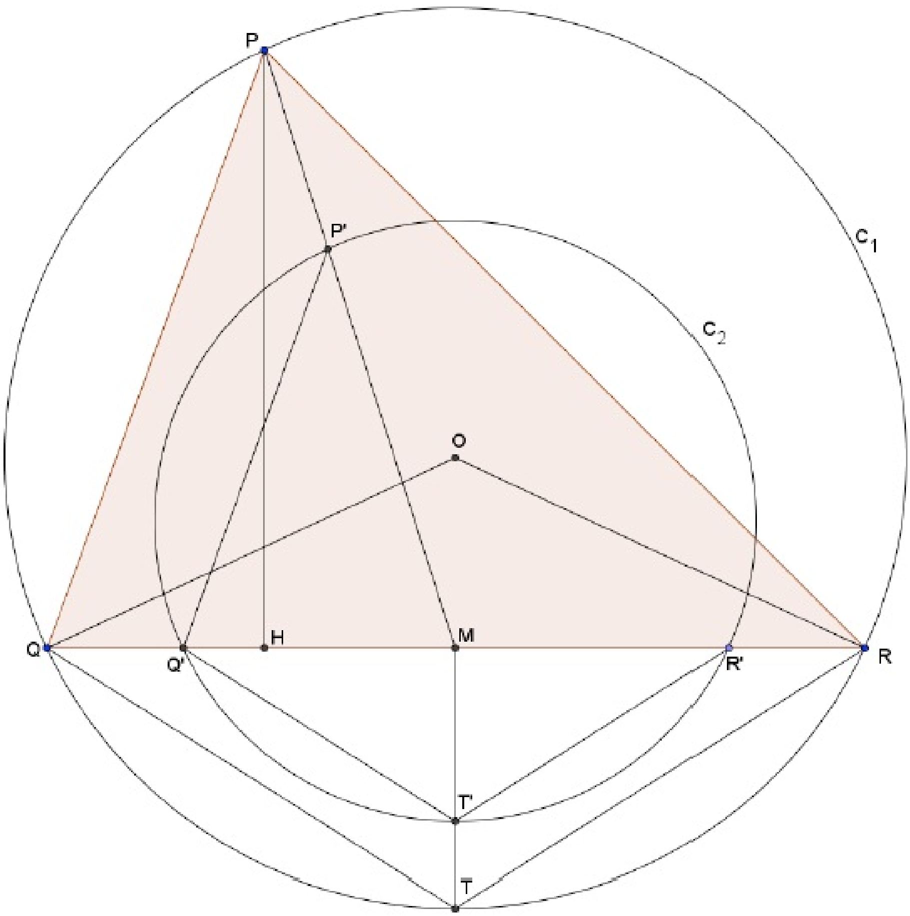
It can be shown that $P'$ also lies on $C_2$.

Triangle $MPQ$ is similar to triangle $MP'Q'$.

$PQ$ is parallel to $P'Q'$.

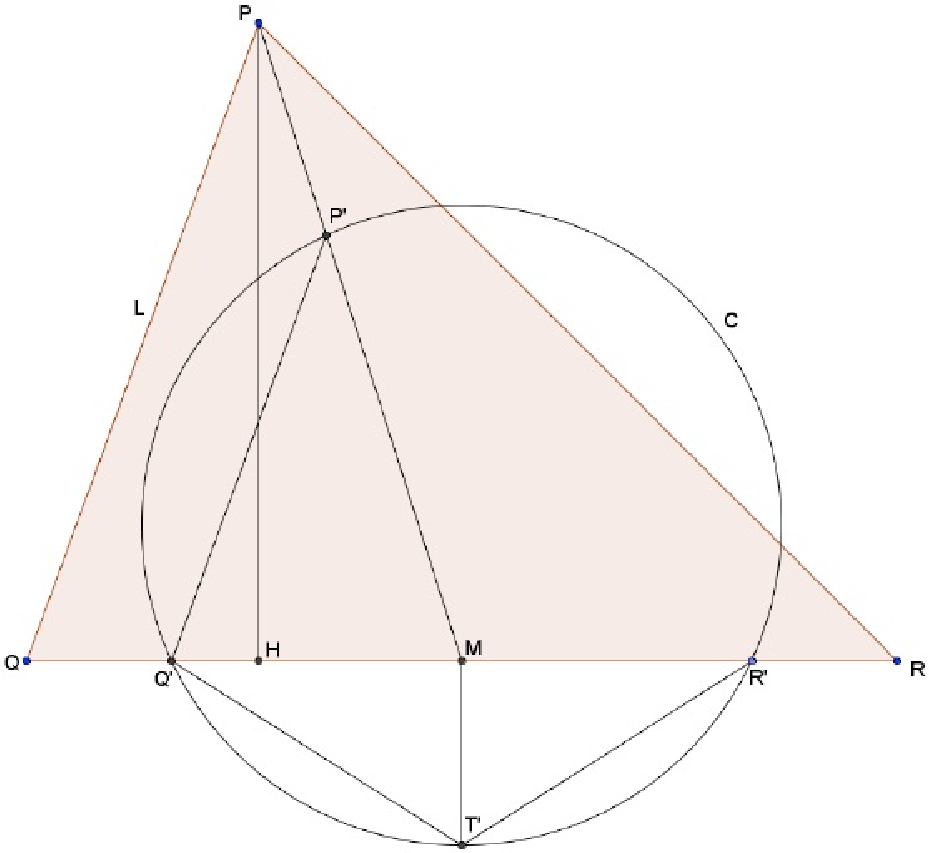
Hence by constructing line $L$ through $P$ parallel to $P'Q'$, the point of intersection of $L$ and $QR$ is $Q$.

Once $Q$ is located, $R$ is located.




1. Construct right-angled triangle $MHP$ with $HP = HA, MP = MA$.
2. Construct perpendicular $L_1$ to $HM$ at $M$.
3. Construct arbitrary $T'$ on $L_1$.
4. Extend $HM$ past $H$ to $Q'$ such that angle $MT'Q' = 90° - \dfrac{a}{2}$.
5. Extend $HM$ past $M$ to $R'$ such that angle $MT'R' = 90° - \dfrac{a}{2}$.
6. Construct circumcircle $C$ of triangle $Q'R'T'$.
7. Denote intersection of $C$ and $PM$ as $P'$.

8. Construct line $L$ through $P$ parallel to $P'Q'$.
9. Extend $MQ'$ past $Q'$ to $Q$ such that $Q$ lies on $L$.
10. Extend $MR'$ past $R'$ to $R$ such that $MR = MQ$.
11. Triangle $PQR$ is the required triangle.



## Case 33 [a IA MA]

Consider triangle $PQR$ with $PQ = A$, angle bisector $RU = IA = t$, median $RT = MA = m$ in a coordinate system where $R$ is the origin and $RU$ is along the positive x-axis.

Hence $U(t, 0)$.

The equation of $PR$ and $QR$ is $y^2 = \tan^2 \frac{a}{2} x^2$.

Let the slope of $PQ$ be $s$. Since $U$ lies on $PQ$, the equation of $PQ$ is $y = sx - st$ . Eliminating $y$ and rearranging w.r.t. $x$,

$$\left(s^2 - \tan^2 \frac{a}{2}\right) x^2 - 2s^2tx + s^2t^2 = 0.$$

The roots of this equation are the $x$-coordinates of $P$ and $Q$, so the midpoint of $PQ$ has $x$-coordinate equal to half the sum of roots, and substituting this into $y = sx - st$ yields the $y$-coordinate.

Hence the midpoint of $PQ$ is given by $(u, v) = \left( \dfrac{s^2t}{s^2 - \tan^2 \frac{a}{2}}, \dfrac{st \tan^2 \frac{a}{2}}{s^2 - \tan^2 \frac{a}{2}} \right)$.

Thus $v^2 = u^2 \tan^2 \frac{a}{2} - ut \tan^2 \frac{a}{2}$.

The midpoint of $PQ$ is simply $T$, so the distance of the midpoint of $PQ$ from $R$ must be $MA$.

Thus $u^2 + v^2 = m^2$.

Eliminating $v$ and rearranging w.r.t. $u$, $u^2 - ut \sin^2 \frac{a}{2} - m^2 \cos^2 \frac{a}{2} = 0$.

Sum of roots $= t \sin^2 \frac{a}{2} > 0$.

Product of roots $= -m^2 \cos^2 \frac{a}{2} < 0$.

From the product of roots, one is positive while the other is negative. From the sum of roots, the positive one has a greater magnitude that the negative one.

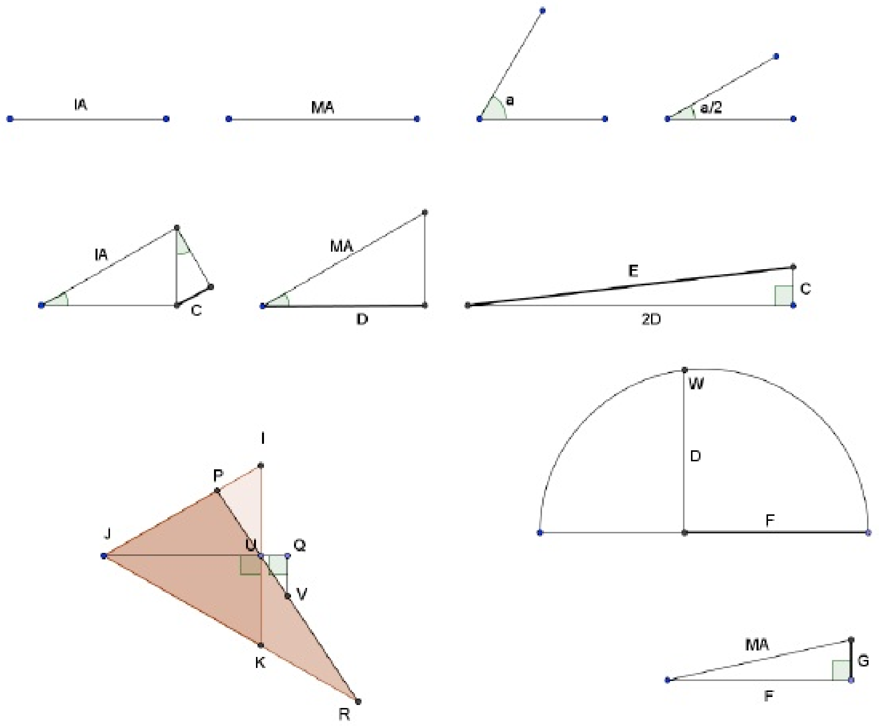
The positive root is taken for the value of $u$.

In order to facilitate the determination of the two roots, the sum and product of the absolute values of the roots are considered.

Sum of absolute roots $= \sqrt{\left(t \sin^2 \frac{a}{2}\right)^2 + 4\left(m \cos \frac{a}{2}\right)^2}$.

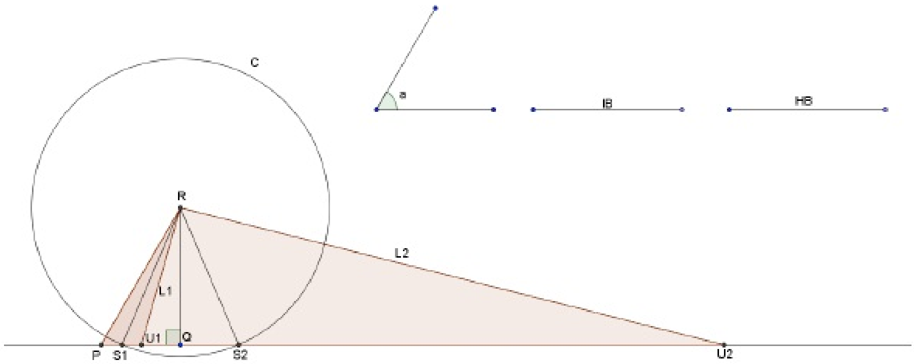
Product of absolute roots $= m^2 \cos^2 \dfrac{a}{2}$.

1. For convenience, let $IA = t, MA = m$.
2. Construct $C = t \sin^2 \dfrac{a}{2}$, $D = m \cos \dfrac{a}{2}$, $E = \sqrt{C^2 + 4D^2}$.
3. Construct semicircle of diameter $E$, find $W$ on the semicircle such that it is at a height $D$ to the diameter. The perpendicular of $W$ on the diameter divides the latter into two unequal portions.
4. Denote the longer portion as $F$.
5. Construct $G = \sqrt{m^2 - F^2}$.
6. Construct triangle $IJK$ with angle $IJK =$ angle $a$, height $JU = t$.
7. Extend $JU$ past $U$ to $Q$ such that $JQ = F$.
8. Construct $V$ such that VQ is perpendicular to JQ and VQ = G.
9. Extend $UV$ past $U$ to $P$ such that $P$ lies on $IJ$.
10. Extend $UV$ past $V$ to $R$ such that $J, K$ and $R$ are collinear.
11. Triangle $JPR$ is the required triangle.


IA
MA
a
a/2
IA
C
MA
D
E
C
2D
W
D
F
I
P
J
U
Q
V
K
R
MA
G
F

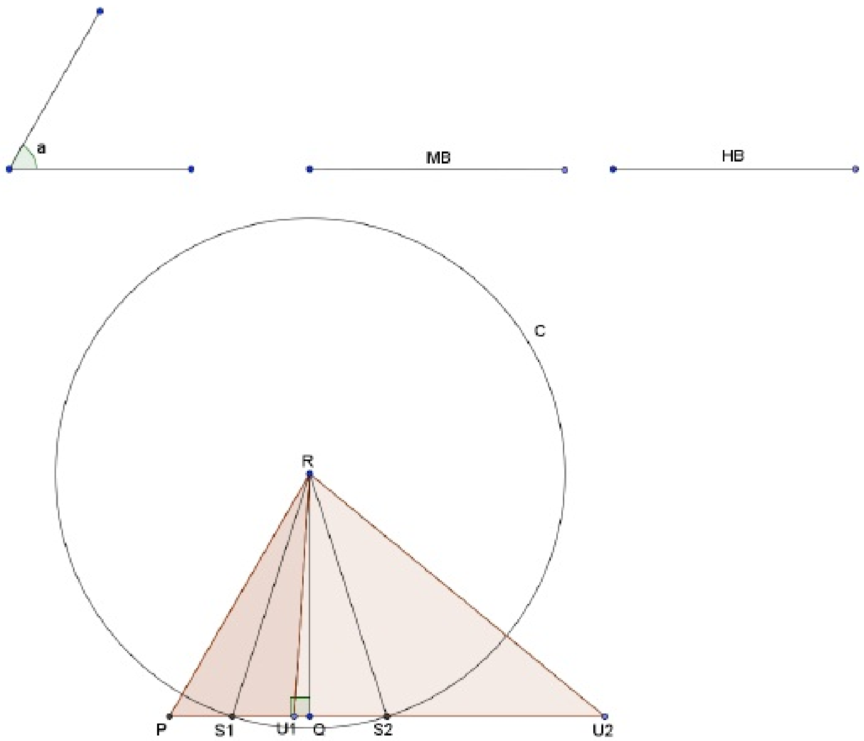
## Case 34 [a IB HB]

1. Construct right-angled triangle $PQR$ with angle $QPR =$ angle $a, QR = HB$.
2. Construct circle $C$ with radius $IB, R$ as centre.
3. Denote intersection of $C$ and $PQ$ as $S_1$.
4. Extend $PQ$ past $Q$ to $S_2$ such that $S_2$ lies on $C$.
5. Construct $U_1$ such that angle $S_1RU_1 =$ angle $PRS_1$.
6. Construct $U_2$ such that angle $S_2RU_2 =$ angle $PRS_2$.
7. Triangle $PRU_1$ and $PRU_2$ are the required triangles.



## Case 35 [a HB MB]

1. Construct right-angled triangle $PQR$ with angle $QPR$ = angle $a$, $QR = HB$.
2. Construct circle $C$ with radius $MB, R$ as centre.
3. Denote intersection of $C$ and $PQ$ as $S_1$.
4. Extend $PQ$ past $Q$ to $S_2$ such that $S_2$ lies on $C$.
5. Construct $U_1$ on $PQ$ such that $PS_1 = S_1U_1$.
6. Construct $U_2$ on $PQ$ such that $PS_2 = S_2U_2$.
7. Triangle $PRU_1$ and $PRU_2$ are the required triangles.



## Case 36 [a IB MB]

Consider triangle $PQR$ where angle $QPR =$ angle $a$, angle bisector $QS = IB$.

For convenience, let $IB = t$, $MB = m$, angle $PQS =$ angle $RQS = \dfrac{b}{2}$.

Set up a coordinate system with $Q$ as the origin and $QS$ lying on the positive x axis. Thus $S(t, 0)$.

The equation of $PR$ is $y = \tan\left(a + \dfrac{b}{2}\right)(x - t)$.

The equation of $QP$ and $QR$ is $y^2 = \tan^2 \dfrac{b}{2} x^2$.

The coordinates of $P$ and $R$ can be found by solving these two equations.

Eliminating $y$ and rearranging w.r.t. $x$,

$$\left[\tan^2\left(a+\frac{b}{2}\right) - \tan^2\frac{b}{2}\right]x^2 - 2t\tan^2\left(a+\frac{b}{2}\right)x + t^2\tan^2\left(a+\frac{b}{2}\right) = 0.$$

Let the midpoint of $PR$ be $U$.

The $x$-coordinate of $U$ is the half the sum of roots of above equation.

Let $U(u, v)$, then $(u, v) = \left(\dfrac{t\tan^2\left(a+\frac{b}{2}\right)}{\tan^2\left(a+\frac{b}{2}\right) - \tan^2\frac{b}{2}}, \dfrac{t\tan\left(a+\frac{b}{2}\right)\tan^2\frac{b}{2}}{\tan^2\left(a+\frac{b}{2}\right) - \tan^2\frac{b}{2}}\right)$.

Since $QU$ is the median on $PR$, $u^2 + v^2 = m^2$ .

$$\therefore \left[\frac{t\tan^2\left(a+\frac{b}{2}\right)}{\tan^2\left(a+\frac{b}{2}\right) - \tan^2\frac{b}{2}}\right]^2 + \left[\frac{t\tan\left(a+\frac{b}{2}\right)\tan^2\frac{b}{2}}{\tan^2\left(a+\frac{b}{2}\right) - \tan^2\frac{b}{2}}\right]^2 = m^2$$

Consider the case where $\begin{cases} t = 2 \\ a = \dfrac{\pi}{2} \\ m = 3 \end{cases}$.

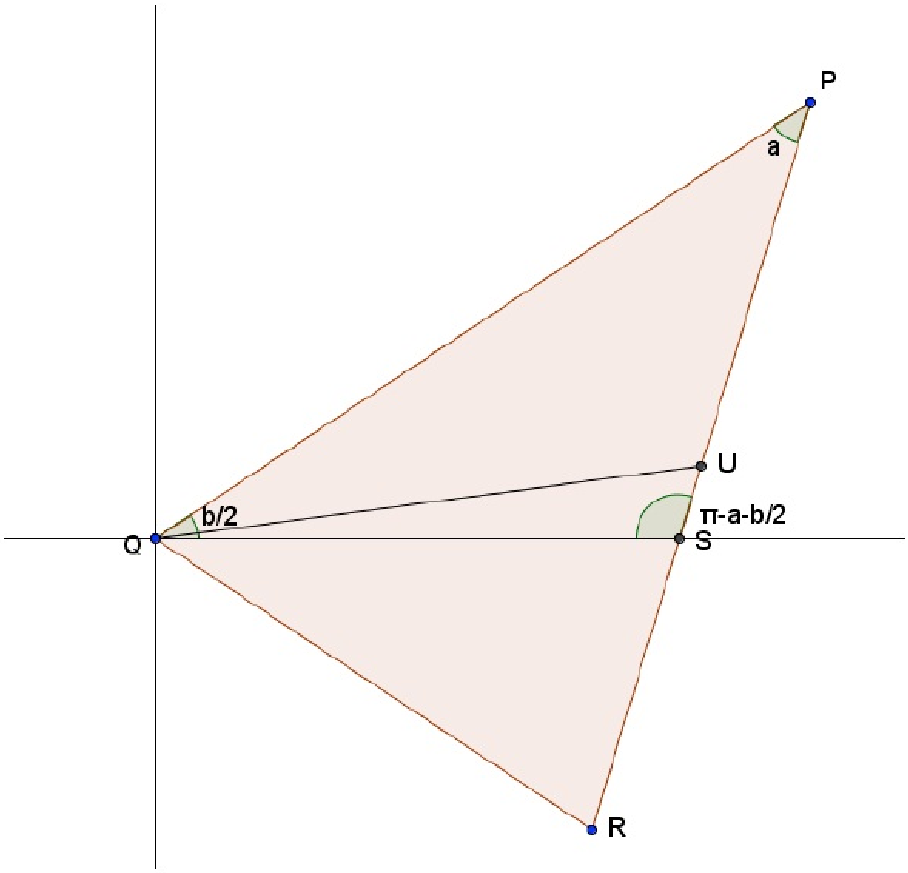
$$\therefore \left(\frac{2\cot^2\frac{b}{2}}{\cot^2\frac{b}{2} - \tan^2\frac{b}{2}}\right)^2 + \left(\frac{2\tan\frac{b}{2}}{\cot^2\frac{b}{2} - \tan^2\frac{b}{2}}\right)^2 = 9$$

Let $\beta = \tan\dfrac{b}{2}$. Substituting and rearranging w.r.t. $\beta$, $9\beta^8 - 4\beta^6 - 18\beta^4 + 5 = 0$.

This polynomial has no constructible roots.

So $\beta$ cannot be constructed, and thus $b$ cannot be constructed.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]



**Case 37 [a HB IC]**

Consider triangle $PQR$ with angle $QPR =$ angle $a$, height $QT = HB$, angle bisector $RS = IC$.

$$\begin{aligned} \because \quad & PQ = PS + QS \\ \therefore \quad & \frac{QT}{\sin a} = \frac{RS}{\sin a} \sin\frac{c}{2} + \frac{RS}{\sin(a+c)} \sin\frac{c}{2}. \end{aligned}$$

Consider the case where

$$\begin{cases} QT = 2 \\ a = \dfrac{\pi}{6} \\ RS = 3 \end{cases}.$$

$$\therefore 4 = 6\sin\frac{c}{2} + \frac{6}{\sqrt{\dfrac{1+\cos\frac{c}{2}}{2}} + \sqrt{3}\sqrt{\dfrac{1-\cos\frac{c}{2}}{2}}} \sin\frac{c}{2}$$

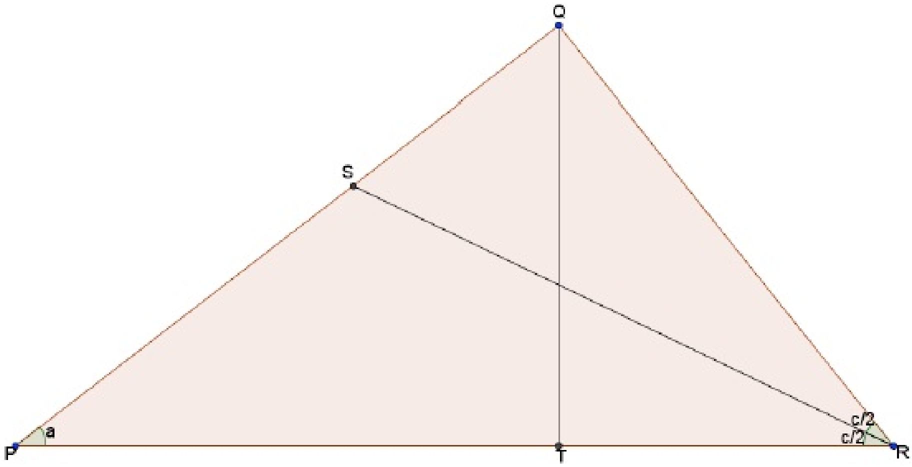
Let $u = \sin\dfrac{c}{2}$, substituting and rearranging w.r.t. $u$,

$$\begin{aligned} 26244u^{12} - 139968u^{11} + 306909u^{10} - 312984u^{9} + 30456u^{8} + 311904u^{7} \\ -391680u^{6} + 202848u^{5} + 1168u^{4} - 59328u^{3} + 31296u^{2} - 6192u + 576 = 0 \end{aligned}$$

This polynomial has no constructible roots.

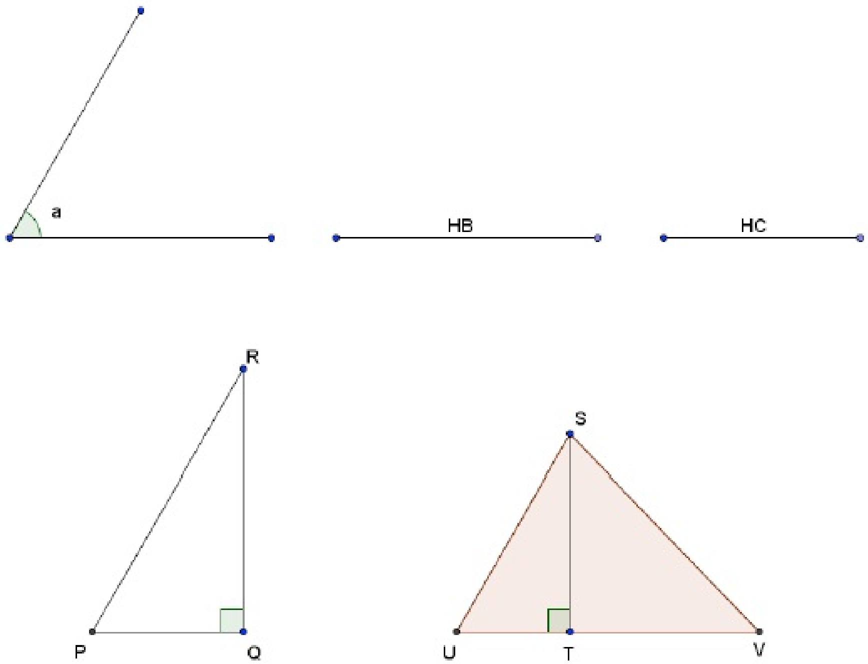
So $u$ cannot be constructed, and thus $c$ cannot be constructed.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]


Q
S
a
c/2
c/2
P
T
R

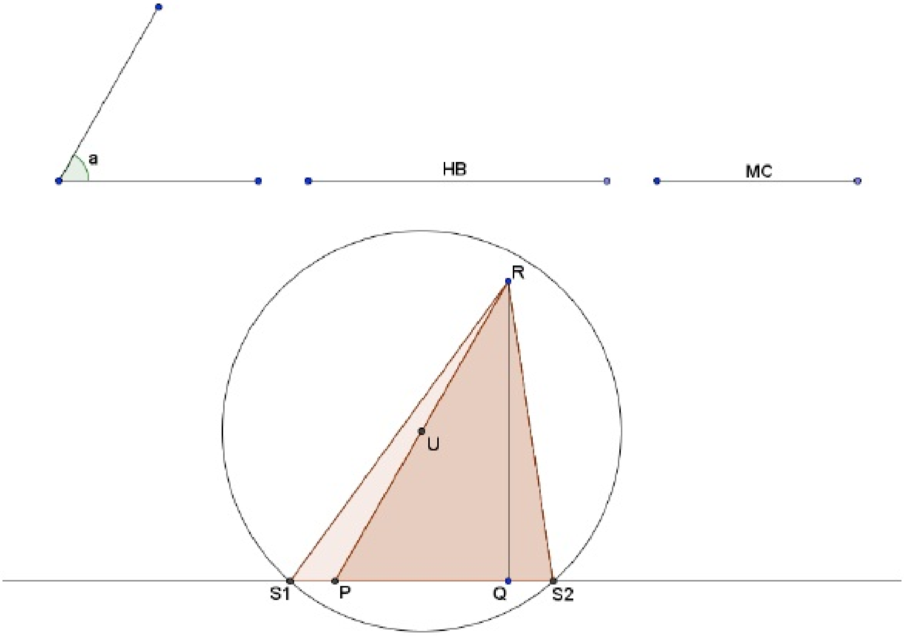
### Case 38 [a HB HC]

1. Construct right-angled triangle $PQR$ with angle $QPR$ = angle $a$, $QR = HB$.
2. Construct right-angled triangle $STU$ with angle $SUT$ = angle $a$, $ST = HC$.
3. Extend $UT$ past $T$ to $V$ such that $UV = PR$.
4. Triangle $USV$ is the required triangle.



**Case 39 [a HB MC]**

1. Construct right-angled triangle $PQR$ with angle $QPR =$ angle $a$, $QR = HB$.
2. Bisect $PR$ at $U$.
3. Construct circle $C$ with radius $MC, U$ at centre.
4. Extend $PQ$ past $P$ to $S_1$ such that $S_1$ lies on $C$.
5. Extend $PQ$ past $Q$ to $S_2$ such that $S_2$ lies on $C$.
6. Triangle $PRS_1$and $PRS_2$ are the required triangles.



**Case 40 [a IB IC]**

Consider triangle $PQR$ such that angle $QPR =$ angle $a$, angle bisector $RS = IB$, angle bisector $TQ = IC$.

Let angle $PQT =$ angle $RQT =$ angle $\theta$.

$$\therefore \frac{QT\sin(a+\theta)}{\sin(a+2\theta)} = \frac{RS\cos\left(\frac{a}{2}+\theta\right)}{\sin 2\theta}.$$

Consider the case where

$$\begin{cases} QT = 3 \\ a = \dfrac{\pi}{2} \\ SR = 2 \end{cases}.$$

$$\therefore 3\cos\theta\sin 2\theta = \sqrt{2}\cos 2\theta(\cos\theta - \sin\theta).$$

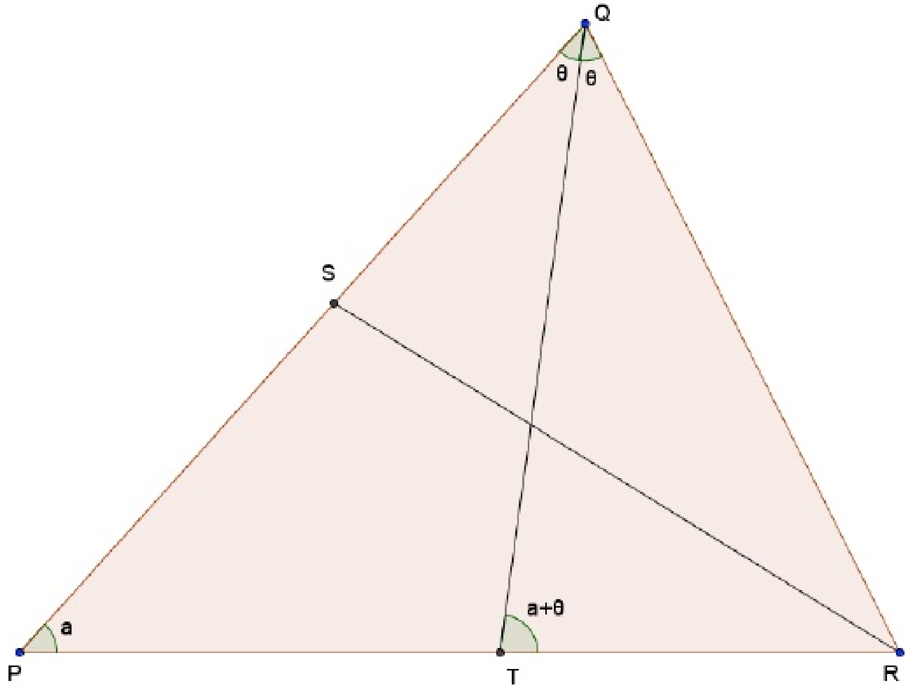
Let $u = \sin\theta$, substituting and rearranging w.r.t. $u$,

$$(13+6\sqrt{2})u^6 - (24+9\sqrt{2})u^4 + (12+3\sqrt{2})u^2 - \frac{1}{2} = 0.$$

This polynomial has no constructible roots.

So $u$ cannot be constructed, and thus $\theta$ cannot be constructed.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]


Q
θ
θ
S
a+θ
a
P
T
R

**Case 41 [a IB MC]**

Consider triangle $PQR$ such that angle $QPR =$ angle $a$, angle bisector $QT = IB$, median $RS = MC$.

Let angle $PQR =$ angle $b, PQ = C$.

$$\therefore \begin{cases} \dfrac{PQ}{\sin\left(a+\dfrac{b}{2}\right)} = \dfrac{QT}{\sin a} \\ \arcsin\left(\dfrac{PQ\sin a}{2RS}\right) + \arcsin\left(\dfrac{PQ\sin b}{2RS}\right) = \pi - a - b \end{cases}$$

$$\arcsin\left[\frac{QT\sin\left(a+\dfrac{b}{2}\right)}{2RS}\right] + \arcsin\left[\frac{QT\sin b\sin\left(a+\dfrac{b}{2}\right)}{2RS\sin a}\right] = \pi - a - b$$

Consider the case where

$$\begin{cases} QT = 2 \\ a = \dfrac{\pi}{2} \\ RS = 3 \end{cases}.$$

$$\therefore \arcsin\left(\frac{\cos\dfrac{b}{2}}{3}\right) + \arcsin\left(\frac{\sin b\cos\dfrac{b}{2}}{3}\right) = \frac{\pi}{2} - b$$

Taking sine of both sides,

$$\cos\frac{b}{2}\sqrt{9-\sin^2 b\cos^2\frac{b}{2}} + \sin b\cos\frac{b}{2}\sqrt{9-\cos^2\frac{b}{2}} = 9\cos b.$$

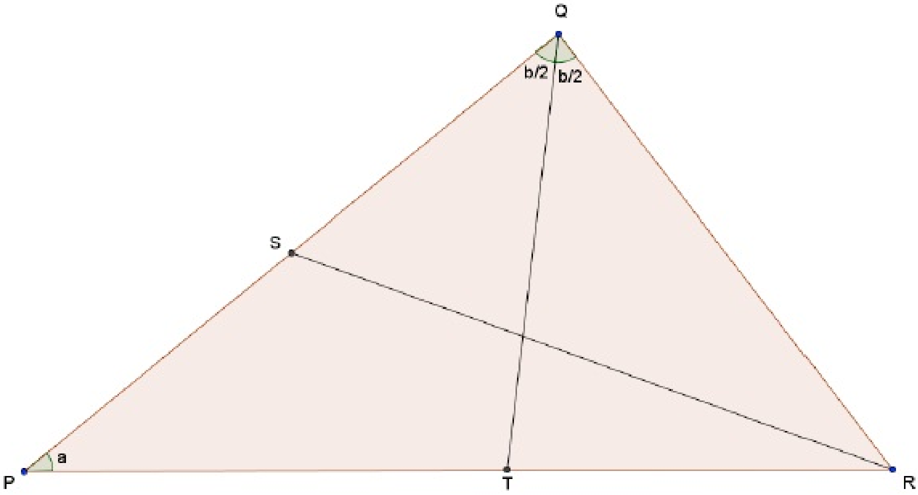
Let $u = \cos b$, substituting and rearranging w.r.t. $u$,

$$972u^6 + 9648u^5 - 115600u^4 - 20736u^3 + 22032u^2 = 0.$$

This polynomial has no constructible roots except for 0, which is absurd obviously.

So $u$ cannot be constructed, and thus $b$ cannot be constructed.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]


Q
b/2
b/2
S
a
P
T
R

### Case 42 [a MB MC]

Consider triangle $PQR$ with angle $QPR =$ angle $a$, median $RS = MB$, median $QT = MC$ (without loss of generality assume $MB < MC$).

Let $PQ = B$, $PR = C$.

$$\therefore \begin{cases} MC^2 = B^2 + \left(\dfrac{C}{2}\right)^2 - BC\cos a \\ MB^2 = C^2 + \left(\dfrac{B}{2}\right)^2 - BC\cos a \end{cases}$$

$$MB^2 - MC^2 = \frac{3}{4}C^2 - \frac{3}{4}B^2$$

$$B = \sqrt{C^2 + \frac{4}{3}(MC^2 - MB^2)}$$

Eliminating $B$ and rearranging w.r.t. $C$,

$$\cos a\sqrt{C^2 + \frac{4}{3}MC^2 - \frac{4}{3}MB^2}C - \frac{5}{4}C^2 = \frac{1}{3}MC^2 - \frac{4}{3}MB^2.$$

Let $E = \sqrt{\frac{4}{3}MB^2 - \frac{1}{3}MC^2}$, $F = \sqrt{\frac{4}{3}MC^2 - \frac{4}{3}MB^2}$, both of which are constructible.

Substituting and rearranging w.r.t. $C$,

$$\left(\frac{25}{16} - \cos^2 a\right)C^4 - \left(\frac{5}{2}E^2 + F^2\cos^2 a\right)C^2 + E^4 = 0.$$

Let $\alpha$ and $\beta$ be two distinct positive roots of $C$, with $\alpha > \beta$.

$$\therefore \begin{cases} \alpha^2 + \beta^2 = \dfrac{40E^2 + 16F^2\cos^2 a}{25 - 16\cos^2 a} \\ \alpha^2\beta^2 = \dfrac{16E^4}{25 - 16\cos^2 a} \end{cases}$$

Let $G = \sqrt{\alpha^2 + \beta^2}$, $H = \sqrt{\alpha\beta}$, both of which are constructible.

$$\therefore \begin{cases} 2\alpha = \sqrt{G^2 + 2H^2} + \sqrt{G^2 - 2H^2} \\ 2\beta = \sqrt{G^2 + 2H^2} - \sqrt{G^2 - 2H^2} \end{cases}$$

Select the suitable root for $C$, and the triangle can then be constructed.

1. Construct right-angled triangle $IJK$ with $IK = 2MB, IJ = MC$.

2. Extend $IJ$ past $J$ to $L$ on $IJ$ such that angle $JKL = 30°$. Thus $JL = E$.

3. Construct right-angled triangle $MNO$ with $MO = 2MC, MN = 2MB$.

4. Extend $MN$ past $N$ to $P$ such that angle $NOP = 30°$. Thus $NP = F$.

5. Construct right-angled triangle $QRS$ with $QR = 2E, RS = 6E$.

   Thus $QS = \sqrt{40}E$.

6. Construct right-angled triangle $SQT$ with $QT = 4F\cos a$.

   Thus $ST = \sqrt{40E^2 + 16F^2\cos^2 a}$.

7. Construct arbitrary $l$.

8. Construct semicircle $W_1$ with diameter $UV = 10l$.

9. Construct $X$ on $UV$ such that $UX = (5 - 4\cos a)l$.

10. Construct $Y$ on $W_1$ such that $XY$ is perpendicular to $UV$.

    Thus $XY = \sqrt{25 - 16\cos^2 a}l$.

11. Construct right-angled triangle $A_1B_1C_1$ with $A_1B_1 = l, B_1C_1 = XY$.

12. Extend $B_1C_1$ past $C_1$ to $D_1$ such that $B_1D_1 = ST$.

13. Extend $A_1B_1$ past $A_1$ to $E_1$ such that $A_1C_1$ is parallel to $D_1E_1$.

    Thus $B_1E_1 = G$.

14. Construct semicircle $W_2$ with diameter $I_1J_1$, in which $I_1K_1 = l, J_1K_1 = XY$.

15. Construct $K_1$ on $W_2$ such that $K_1L_1$ is perpendicular to $I_1J_1$.

    Thus $K_1L_1 = \sqrt[4]{25 - 16\cos^2 a}l$.

16. Construct right-angled triangle $M_1N_1O_1$ with $M_1N_1 = l$, $N_1O_1 = K_1L_1$.

17. Extend $N_1O_1$ past $O_1$ to $P_1$ such that $N_1P_1 = 2E$.

18. Extend $M_1N_1$ past $M_1$ to $Q_1$ such that $M_1O_1$ is parallel to $P_1Q_1$. Thus $N_1Q_1 = H$.

19. Construct $C' = \dfrac{\sqrt{G^2 + 2H^2} + \sqrt{G^2 - 2H^2}}{2}$.

20. Construct $C'' = \dfrac{\sqrt{G^2 + 2H^2} - \sqrt{G^2 - 2H^2}}{2}$.

21. Construct $C = C''$.

22. Construct $R_1S_1 = C$.

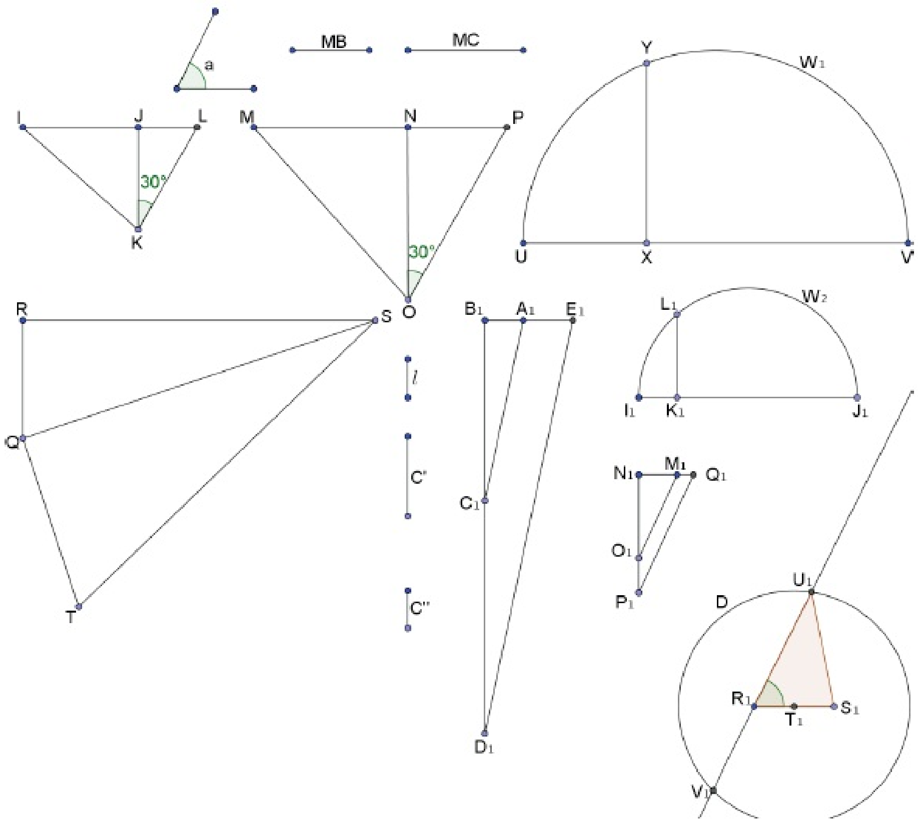
23. Construct line $L$ through $R_1$ such that it is inclined by angle a to $R_1S_1$.

24. Bisect $R_1S_1$ at $T_1$.

25. Construct circle $D$ with radius $MC, T_1$ as centre.

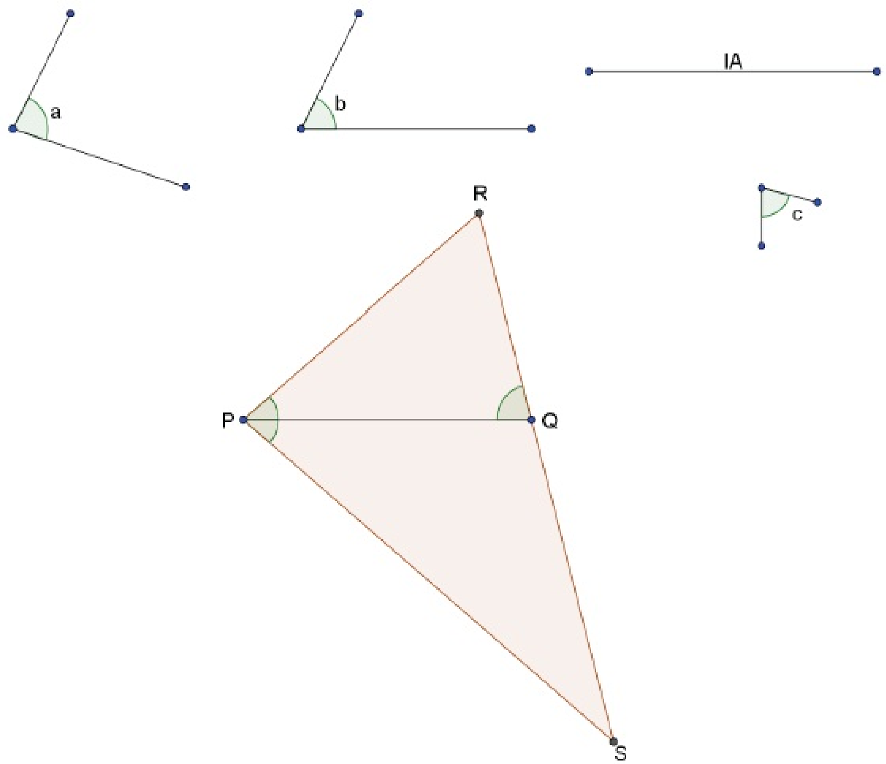
26. Denote intersections of $D$ and $L$ as $U_1$ and $V_1$.

27. Triangle $R_1S_1U_1$ is the required triangle.



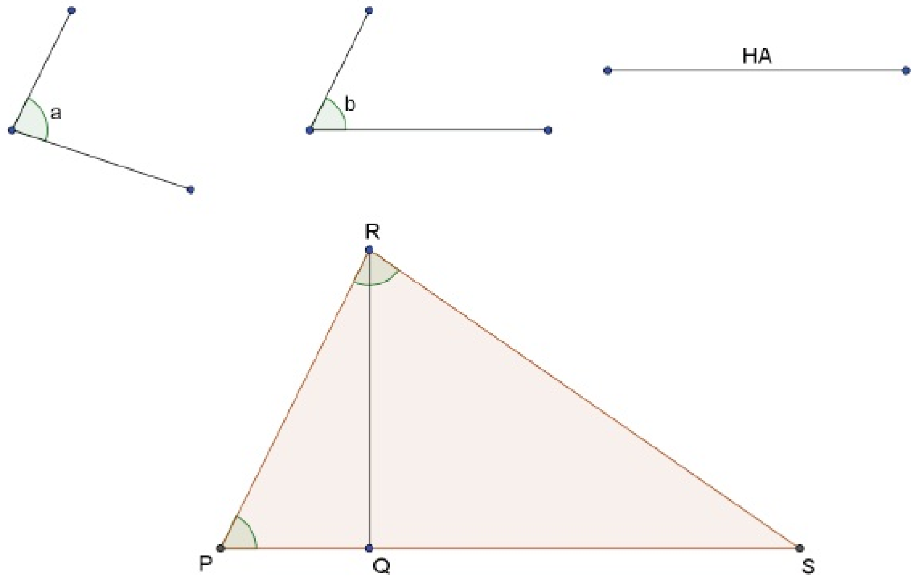
## Case 43 [a b IA]

1. Construct angle $c = \left(\frac{1}{2}a + b\right)$.
2. Construct triangle $PQR$ using IA, $\frac{1}{2}a$ and $c$ as in case 5.
3. Extend $QR$ past $Q$ to $S$ such that angle $RPS = a$.
4. Triangle $PRS$ is the required triangle.



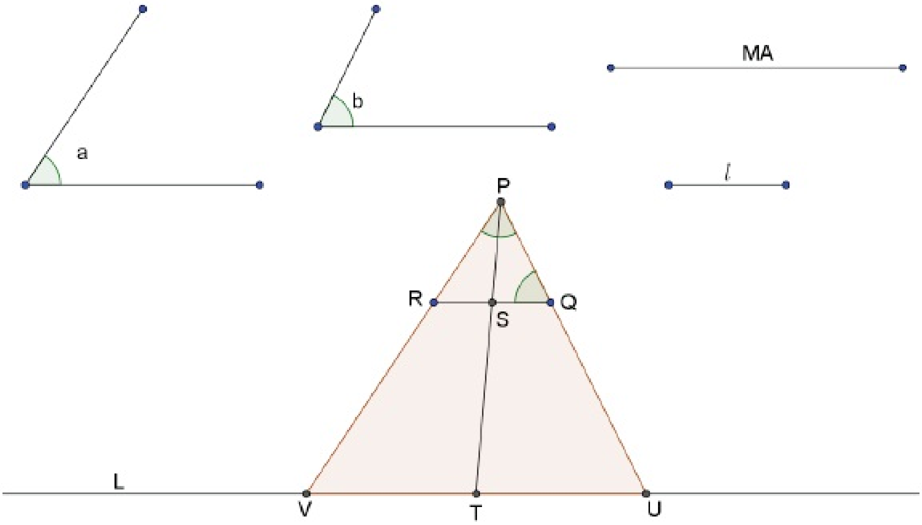
**Case 44 [a b HA]**

1. Construct right-angled triangle $PQR$ with $QR = HA$, angle $QPR = b$.
2. Extend $PQ$ past $Q$ to $S$ such that angle $PRS = a$.
3. Triangle $PRS$ is the required triangle.



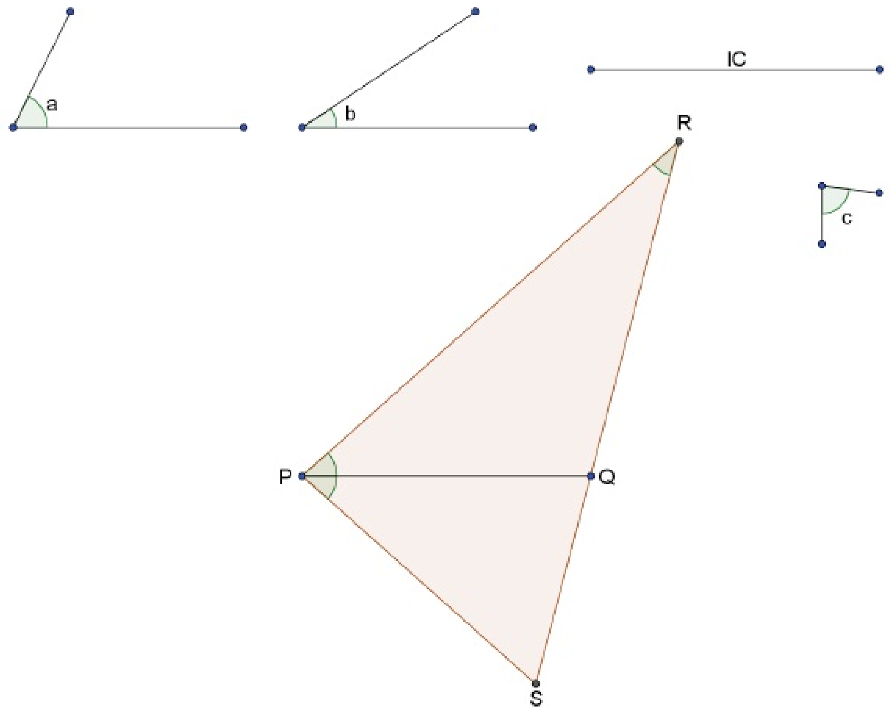
## Case 45 [a b MA]

1. Construct arbitrary l.
2. Construct triangle $PQR$ using $l, a$ and $b$ as in case 5.
3. Bisect $QR$ at $S$.
4. Extend $PS$ past $S$ to $T$ such that $PT = MA$.
5. Construct line $L$ through $T$ parallel to $QR$.
6. Extend $PQ$ past $Q$ to $U$ such that $U$ lies on $L$.
7. Extend $PR$ past $R$ to $V$ such that $V$ lies on $L$.
8. Triangle $PUV$ is the required triangle.



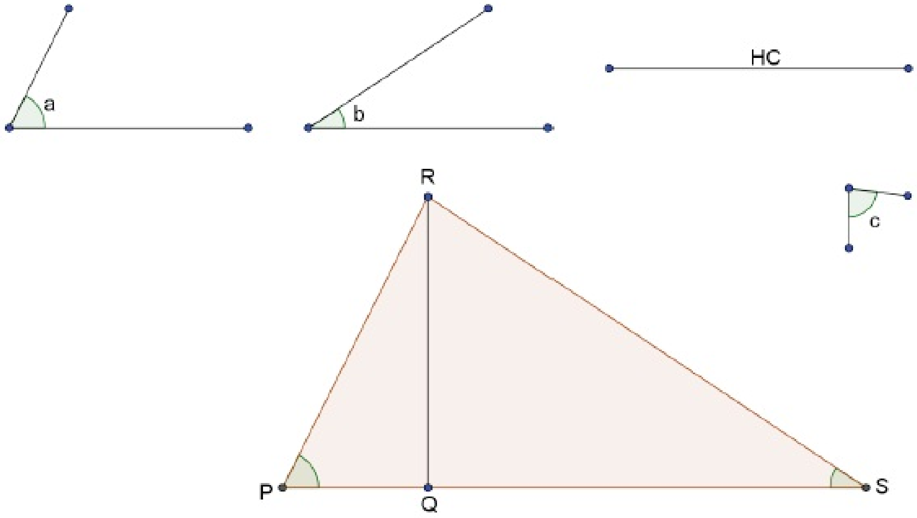
**Case 46 [a b IC]**

1. Construct complement angle $c$ of angle $(a+b)$.
2. Construct triangle $PRS$ using $c, b$ and $IC$ as in case 43.
3. Triangle $PRS$ is the required triangle.



## Case 47 [a b HC]

1. Construct complement angle $c$ of angle $(a+b)$.
2. Construct triangle $PRS$ using $c, b$ and $HC$ as in case 44.
3. Triangle $PRS$ is the required triangle.

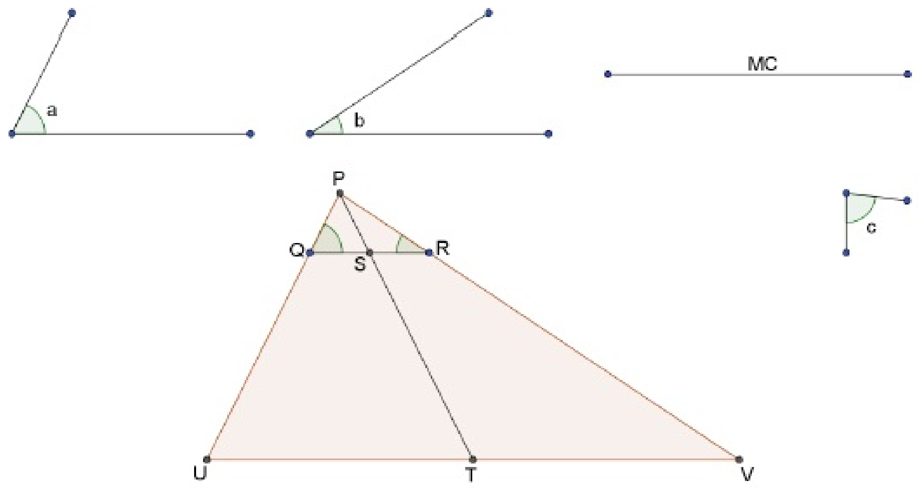


**Case 48 [a b MC]**

1. Construct complement angle $c$ of angle $(a + b)$.

2. Construct triangle $PUV$ using $c, b$ and $MC$ as in case 45.

3. Triangle $PUV$ is the required triangle.



## Case 49 [IA IB IC]

Richard Philip Baker, in his 1903 PhD dissertation entitled The Problem of the Angle-Bisectors, has demonstrated how the construction of a triangle from only its three angle bisectors given is impossible; however the whole problem is too complex to be shown here. It is now shown that construction is impossible for isosceles triangles, i.e. two distinct given bisectors.

Consider the constructed triangle $PQR$ with $PQ = QR$, $S, T, U$ lying on $PQ, QR, PR$ respectively such that $RS = PT = IA = IB$, $QU = IC$.

Let angle $QPR =$ angle $PRQ = \theta$.

Since $QU$ is also the perpendicular bisector of $PR$,

$$\begin{cases} \dfrac{PR}{2} = QR\cos\theta \\ \dfrac{PR}{QR} = 2\cos\theta \end{cases}.$$

Angle $QTP = \pi - \dfrac{3}{2}\theta$.

Considering triangle $PRT$, $\dfrac{PT}{\sin\theta} = \dfrac{PR}{\sin\left(\dfrac{3}{2}\theta\right)}$.

Also $\dfrac{QU}{QR} = \sin\theta$.

$$\therefore \left(\frac{QU}{PT}\right)^2 = \frac{\frac{1}{2} + \frac{3}{2}\cos\theta - 2\cos^2\theta}{4\cos^2\theta}$$

Let $x = \cos\theta, k = \dfrac{QU}{PT}$.

Substituting and rearranging w.r.t. $x$, $4x^3 + 8k^2x^2 - 3x - 1 = 0$.

Consider the case where

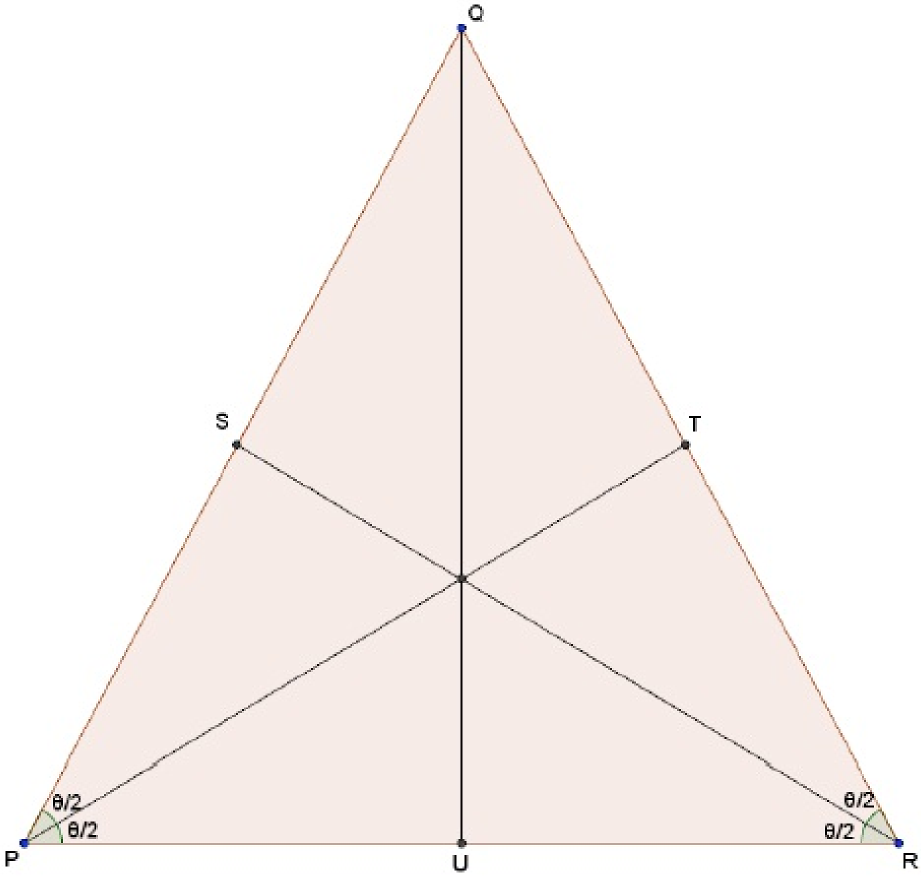
$$\begin{cases} QU = 3 \\ k = \dfrac{3}{2} \\ PT = 2 \end{cases}.$$

$$\therefore 4x^3 + 18x^2 - 3x - 1 = 0$$

This polynomial has no constructible roots.

So $x$ cannot be constructed, and thus $\theta$ cannot be constructed.

Since there is at least one case in which the triangle is not constructible yet it exists, the triangle cannot be constructed from these conditions alone in the general case. [See reviewer's comment (11)]



## Case 50 [HA HB HC]

**Lemma 1.**

$$\frac{1}{HA} : \frac{1}{HB} : \frac{1}{HC} = A : B : C.$$

*Proof.* Let the area of the required triangle be $S$.

$$\therefore 2S = A \times HA = B \times HB = C \times HC$$

$$\begin{cases} A : B = \dfrac{1}{HA} : \dfrac{1}{HB} \\ B : C = \dfrac{1}{HB} : \dfrac{1}{HC} \end{cases}$$

$$\therefore A : B : C = \frac{1}{HA} : \frac{1}{HB} : \frac{1}{HC}.$$

□

Consider the case in which $HA + HB > HC$.

Then triangle $PQR$ using $HA, HB$ and $HC$ can be constructed as in case 1.

Let the heights on $PQ, QR$ and $PR$ be $HA', HB'$ and $HC'$ respectively.

$$\because \begin{cases} A : B : C = \dfrac{1}{HA} : \dfrac{1}{HB} : \dfrac{1}{HC} \\ HA : HB : HC = \dfrac{1}{HA'} : \dfrac{1}{HB'} : \dfrac{1}{HC'} \end{cases}$$

$$\therefore A : B : C = HA' : HB' : HC'.$$

Triangle $P'Q'R'$ using $HA', HB'$ and $HC'$ can be constructed as in case 1.

Then the required triangle is similar to triangle $P'Q'R'$.

Consider the case in which $HA + HB < HC$.

Here triangle $PQR$ cannot be constructed as above due to violation of the triangle inequalities.

Instead we intend to construct triangle $PQR$ using $HA, HB$ and $\dfrac{HA \times HB}{HC}$ as in case 1.

We prove this is possible by verifying the triangle inequalities.

$$\frac{HA \times HB}{HC} < \frac{HA \times HB}{HA + HB} = \left(\frac{1}{HA} + \frac{1}{HB}\right)^{-1} < HA + HB$$

$$\therefore HA + HB > \frac{HA \times HB}{HC}.$$

$$\because \begin{cases} B + C > A \\ A : B : C = \dfrac{1}{HA} : \dfrac{1}{HB} : \dfrac{1}{HC} \\ A + C > B \end{cases}$$

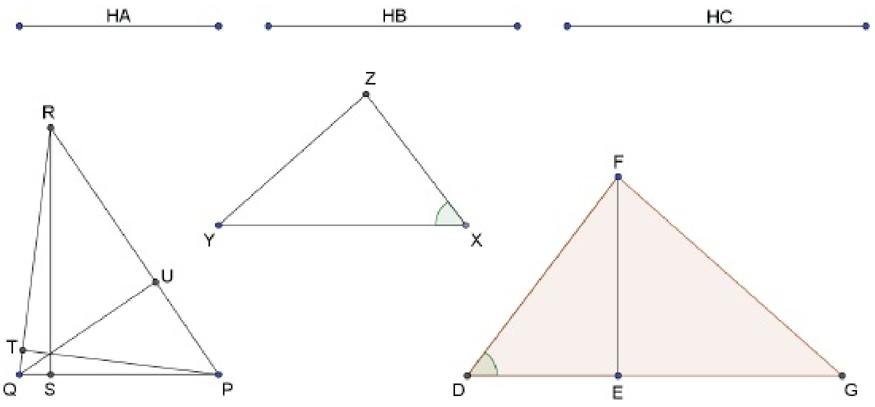
$$\therefore \begin{cases} \dfrac{1}{HB} + \dfrac{1}{HC} > \dfrac{1}{HA} \\ \\ \dfrac{1}{HA} + \dfrac{1}{HC} > \dfrac{1}{HB} \end{cases}$$

$$\begin{cases} HA + \dfrac{HA \times HB}{HC} > HB \\ \\ HB + \dfrac{HA \times HB}{HC} > HA \end{cases}$$

Therefore triangle $PQR$ satisfies the triangle inequalities and can be constructed.

**Case I: $HA + HB > HC$.**

1. Construct triangle $PQR$ using $HA, HB$ and $HC$ as in case 1.
2. Construct $S, T$ and $U$ on $PQ, QR$ and $PR$ respectively such that $RS$ is perpendicular to $PQ$, $PT$ is perpendicular to $QR$ and $QU$ is perpendicular to $PR$.
3. Construct triangle $XYZ$ using $RS, PT$ and $QU$ as in case 1.
4. Construct right-angled triangle $DEF$ with angle $EDF$ = angle $YXZ$ and $EF = HA$.
5. Extend $DE$ past $E$ to $G$ such that $G$ is at a perpendicular distance of $HC$ from $DF$.
6. Triangle $DFG$ is the required triangle.




**Case II:** $HA + HB < HC$**.**

1. Construct right-angled triangle $XYZ$ with $XY = HA, YZ = HC$.

2. Construct $V$ on $YZ$ such that $VY = HB$.

3. Construct $W$ on $XY$ such that $VW$ is parallel to $XZ$. Then $WY = \dfrac{HA \times HB}{HC}$.

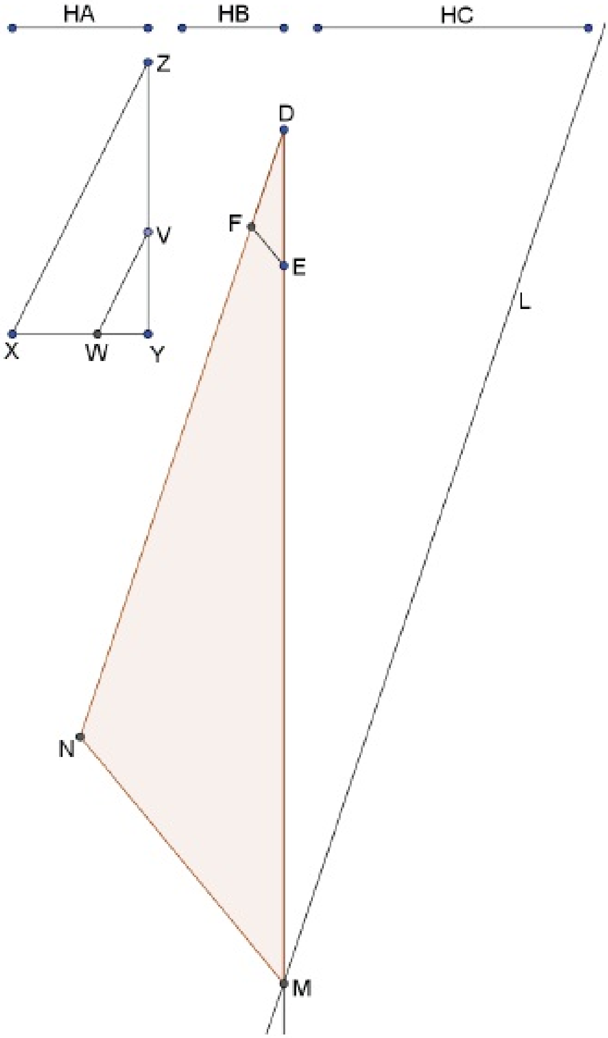
4. Construct triangle $DEF$ using $HA, HB$ and $\dfrac{HA \times HB}{HC}$ as in case 1, then the ratio between their lengths is $\dfrac{1}{HA} : \dfrac{1}{HB} : \dfrac{1}{HC} = A : B : C$.

5. Construct line $L$ parallel to and of distance $HC$ to $DF$.

6. Extend $DE$ beyond $E$ to $M$ such that $M$ lies on $L$.

7. Extend $DF$ beyond $F$ to $N$ such that $MN$ is parallel to $EF$.

8. Triangle $DMN$ is the required triangle.


HA
HB
HC
Z
D
V
F
E
L
X
W
Y
N
M

## Case 51 [MA MB MC]

Consider triangle $PQR$ and $S, T, U$ on sides $PQ, PR, QR$ respectively such that $QT = MA, RS = MB, PU = MC$.

Let the centroid of triangle $PQR$ be $M$.

Let line $L_1$ pass through $P$ such that it is parallel to $RS$.

Let line $L_2$ pass through $R$ such that it is parallel to $PU$.

Let $L_1$ and $L_2$ intersect at $V$.

$MPVR$ form a parallelogram, so $PV = MR = \frac{2}{3}MB$.

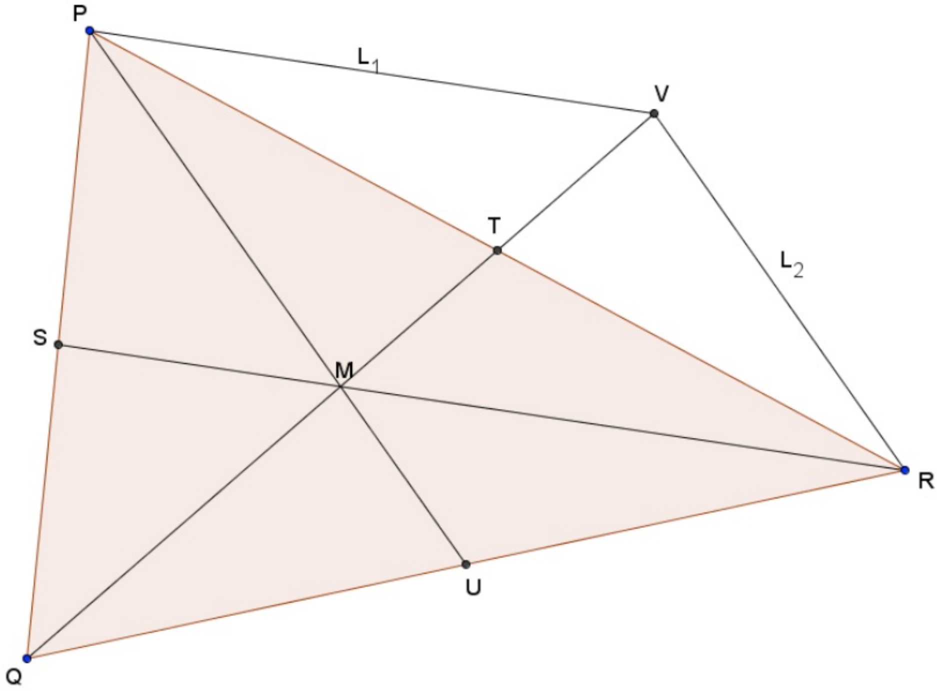
$$MP = \frac{2}{3}MC.$$

Triangle $PVT$ is congruent to triangle $RMT$, so $TV = MT = \frac{1}{3}QT$.

$$MV = \frac{2}{3}MA.$$

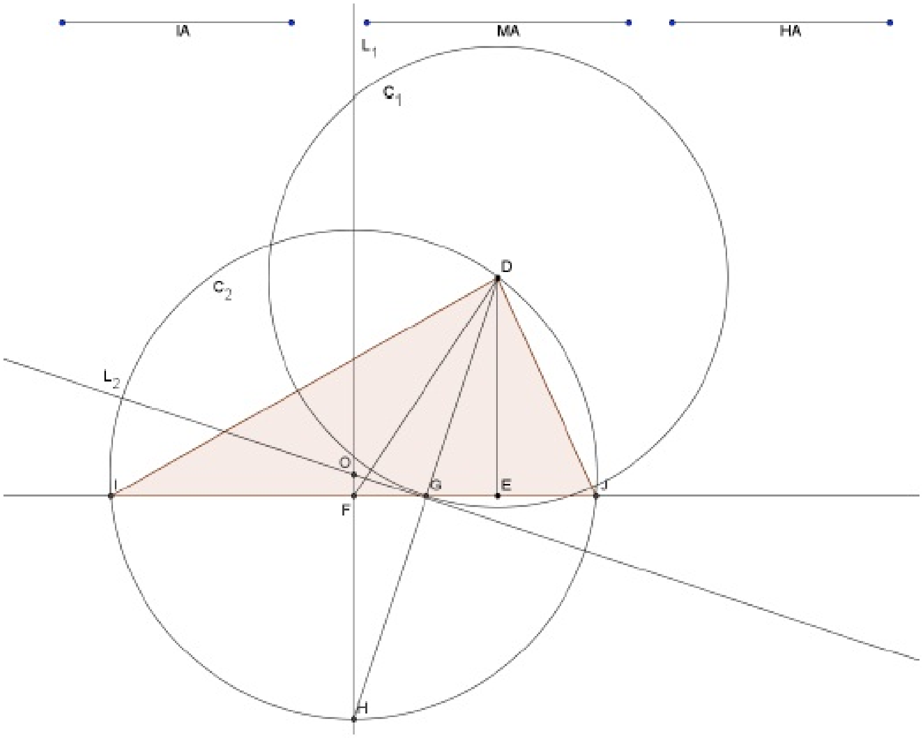
Hence triangle $PVM$ can be constructed from the given medians, and triangle $PQR$ can be constructed from triangle $PVM$.

1. Construct triangle $PMV$ using $\frac{2}{3}MA$, $\frac{2}{3}MB$ and $\frac{2}{3}MC$ as in case 1.
2. Bisect $MV$ at $T$.
3. Extend $PT$ past $T$ to $R$ such that $RT = PT$.
4. Extend $MP$ past $M$ to $U$ such that $PU = MA$.
5. Extend $RU$ past $U$ to $Q$ such that $QU = RU$.
6. Triangle $PQR$ is the required triangle.


P
L1
V
T
L2
S
M
R
U
Q

### Case 52 [IA MA HA]

1. Construct right-angled triangle $DEF$ with $DF = MA, DE = HA$.
2. Construct circle $C_1$ with radius $IA, D$ as centre.
3. Denote intersection of $C_1$ and $EF$ as $G$.
4. Construct line $L_1$ through $F$ perpendicular to $EF$.
5. Extend $DG$ past $G$ to $H$ such that $H$ lies on $L_1$.
6. Construct perpendicular bisector $L_2$ to $DH$.
7. Denote intersection of $L_1$ and $L_2$ be $O$.
8. Construct circle $C_2$ with radius $DO$, $O$ as centre.
9. Extend $EF$ past $E$ to $J$ such that $J$ lies on $C_2$.
10. Extend $EF$ past $F$ to $I$ such that $I$ lies on $C_2$.
11. Triangle $DIJ$ is the required triangle.


IA
MA
HA
L1
C1
C2
D
L2
O
I
G
E
J
F
H

## CONCLUSION

In this report, we have investigated into various possibilities in constructing triangles. We have come to the conclusion that despite tedious work that involves certain extent of computer usage, many triangles are actually proved to be constructible. We are glad that during the investigation, we have come across with various new techniques we have not been exposed to before and learnt a lot.

However, we also showed that some triangles are indeed not constructible and pointed out the relationship between geometric constructability and algebra. We understand the relation between these two seemingly distinct topics, and we wish to further investigate in this angle if possible in the future.

Nevertheless, we still have room for improvement and a lot to explore. We wish to find a simpler way to determine the constructability of triangles using less computer work, and we hope that in the future, we can learn more on the constructible numbers of straight edge and compasses.

All in all, we are contented with our report and have found this experience an exceptionally fruitful one.

## APPENDIX:

### Constructible Numbers and Related Polynomials

The following part of the study aims to use certain theorems and tools in Abstract Algebra to prove the inconstructibility of certain cases of triangles.

**Definition 2.** *A polynomial of a field $F$ is defined to be $\sum_{i=0}^{\infty} a_i x^i$, $a_i \in F$ and all but finite $a_i = 0$.*

**Definition 3.** *Let $R[x]$, associated with polynomial addition and multiplication, be the set of all polynomials with coefficient in a field $F$. $R[x]$ is a ring.*

[See reviewer's comment (12(c)(i))]

**Definition 4.** *$f(x) \in F[x]$ is irreducible iff $f(x) = g(x)h(x) \implies g(x)$ or $f(x)$ is a unit of $F[x]$.*

**Definition 5.** *An Ideal is an additive subgroup $N$ of a ring $R$ that satisfies the following property:*

$$aN \subseteq N,\ \ Nb \subseteq N,\ \ a, b \in R.$$

[See reviewer's comment (12(b)(i))]

**Definition 6.** *The ideal $\{ra \mid r \in R\}$ is the principal ideal generated by $a$, and is denoted onwards as $< a >$.*

**Definition 7.** *An ideal $M$ is a maximal ideal of a ring $R$ if*

$$M \subseteq N \subset R \implies M = N$$

[See reviewer's comment (12(b)(ii))]

**Definition 8.** *The unique monic polynomial $f(x) \in F[x]$ s.t. $f(\alpha) = 0$, $\alpha \in E$ and $p(\alpha) = 0 \implies f(x) \mid p(x)$. Then $f(\alpha)$ is called the irreducible polynomial and denoted as $irr(\alpha, F)$.*

[See reviewer's comment (12(b)(iii))]

**Definition 9.** *If $\alpha$ is algebraic over $F$, a simple extension field $F(\alpha)$ is the field $F[\alpha]$ / $irr(\alpha, F)$.*

[See reviewer's comment (12(b)(iv))]

**Theorem 10.** *(Evaluation Homomorphism for Field Theory) Let $F$ be a subfield of $E$, and let $\alpha \in E$, then the map $\phi_\alpha : F[x] \to E$ defined by*

$$\phi_\alpha(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n$$

*is a homomorphism.*

*Proof.* By definition of polynomial addition and multiplication, the proof is straight forward. □

**Theorem 11.** *If $F$ is a field, every ideal in $F[x]$ is principle.*

*Proof.* Let $N$ be an ideal in $F[x]$. Let $g(x)$ be the non-constant polynomial with smallest degree in $N$. Then by the division algorithm, any polynomial

$$f(x) = g(x)q(x) + r(x),$$

with the degree of $r(x)$ smaller than that of $g(x)$. Since $g(x)$ is already the smallest degree non-zero polynomial, degree of $r(x) = 0$, and thus $r(x) = 0$ and $f(x) = g(x)q(x)$, and therefore $N =< g(x) >$. □

**Theorem 12.** *An ideal $< p(x) >$ of $F[x]$ is maximal $\iff$ $p(x)$ is irreducible over $F$.*

*Proof.* $\Rightarrow$: Suppose $< p(x) >$ is maximal. Let $p(x) = f(x)g(x)$. Since $< p(x) >$ is a maximal ideal, it is a prime ideal. Therefore either $f(x)$ in $< p(x) >$ or $g(x)$ in $< p(x) >$. However since $p(x)$ is already the smallest degree non-zero polynomial for $< p(x) >$, either $f(x)$ or $g(x)$ has the same degree of $p(x)$. Therefore $p(x)$ is irreducible.

$\Leftarrow$: If $p(x)$ is irreducible, let $N$ be an ideal such that $< p(x) >\subseteq N \subseteq R$. Since $N$ is a principle ideal, $N =< g(x) >$. Then $p(x)$ is in $< g(x) >$, so $p(x) = g(x)f(x)$. Since $p(x)$ is irreducible, $g(x) = p(x)$ or $g(x)$ is unity. Then, $N =< p(x) >$ or $N = R$. Therefore $< p(x) >$ is maximal. □

**Theorem 13.** *(Kronecker's Theorem) Let $F$ be a field and $f(x)$ be a non-constant function in $F[x]$. Then there exists an extension field $E$ of $F$ such that there exists an element $\alpha \in E$ so that $f(\alpha) = 0$.*

*Proof.* Let $f(x)$ be factorized into irreducible polynomials. Say one of them is $p(x)$. Then $< p(x) >$ is a maximal ideal as proven above. Therefore we can consider the field $F[x]/ < p(x) >$.

Define mapping $\phi : F[x] \to F[x]/ < p(x) >$, $\phi(a) \to a+ < p(x) >$, $a \in F$, $\phi(x) \to x+ < p(x) >$,

$$\begin{aligned}&\phi(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)\\ =\ & a_0 + a_1(x+ < p(x) >) + a_2(x+ < p(x) >)^2 + \cdots + a_n(x+ < p(x) >)^n.\end{aligned}$$

We can see that there is an isomorphism $\phi(a) \to a+ < p(x) >$ between $F$ and a subfield of $F[x]/ < p(x) >$, meaning that the image of $F$ under $\phi$ is a subfield of $F[x]/ < p(x) >$. Therefore by Theorem 10, the mapping is a well-defined homomorphism. ($\alpha = x+ < p(x) >$ )

Now consider

$$\begin{aligned}\phi(p(x)) &= \phi(p_0 + p_1x + p_2x^2 + \cdots + p_nx^n)\\ &= p_0 + p_1(x+ < p(x) >) + p_2(x+ < p(x) >)^2 + \cdots + p_n(x+ < p(x) >)^n\\ &= p_0 + p_1x + p_2x^2 + \cdots + p_nx^n+ < p(x) >\\ &= p(x)+ < p(x) >\\ &=< p(x) >,\end{aligned}$$

which is the 0 of the field $F(x)/ < p(x) >$. Therefore, the theorem is proved. □

**Theorem 14.** *If $E$ is a simple extension field over $F$ and $\alpha$ is algebraic over $F$, and degree of irr$(\alpha, F) = n > 1$, then every element of $E$ can be expressed as $\beta = b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1}$.*

[See reviewer's comment (12(d))]

*Proof.* Since

$$\begin{aligned}& a_0 + a_1\alpha + a_2\alpha^2 + \cdots + \alpha^n = 0\\ \Longrightarrow\quad & \alpha^n = -(a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1}),\end{aligned}$$

therefore for any expressions with $\alpha^m$, $m > n$, it can be reduced to the form $b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1}$. □

The above theorem shows that $\{1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}\}$ forms a basis for $E$ over $F$. This allows us to come up to the following theorem, which is important to the inconstructibility of triangles.

**Theorem 15.** *If $E$ is a finite extension of field $F$, and $K$ a finite extension of field $E$, then $K$ is finite extension of $F$ and $[K : F] = [K : E][E : F]$. ($[M : N]$ is the dimension of $M$ over $N$ as a vector space)*

[See reviewer's comment (12(e))]

*Proof.* We would like to state here without rigorous proof that if the basis of $K$ over $E$ is $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ with $m$ elements and the basis of $E$ over $F$ is $\{1, \beta, \beta^2, \beta^3, \ldots, \beta^{n-1}\}$, then the basis of $K$ over $F$ is the basis with the $mn$ elements $\alpha_i\beta_j$. Then it is straightforward that $[K : F] = [K : E][E : F]$. □

**Theorem 16.** *If $\beta \in F(\alpha)$, then $deg(\beta, F) \mid deg(\alpha, F)$.*

[See reviewer's comment (12(f))]

*Proof.* $deg(\beta, F) = [F(\beta) : F]$, and $\deg(\alpha, F) = [F(\alpha) : F]$, and since

$$F \leq F(\alpha) \leq F(\beta),$$

from the above theorem, $deg(\beta, F) \mid deg(\alpha, F)$. □

We now proceed to proving the inconstructibility of certain cases of triangles.

First of all we need to prove the following things.

**Theorem 17.** *The set of all constructible numbers is a field under normal addition and multiplication.*

[See reviewer's comment (12(a))]

*Proof.* It has been shown in the earlier part of our studies that if $\alpha, \beta$ are constructible numbers, we can construct $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$, $\dfrac{\alpha}{\beta}$, thus the set of all constructible numbers is a field. □

To start with, we shall notice that $\mathbb{Q}$ is the fundamental field of constructible numbers. From $\mathbb{Q}$, we proceed to construct new extension fields to include other constructible numbers.

Now we should consider how new extension fields can be constructed from existing fields of constructible numbers through construction itself.

Notice via constructing, we are drawing a finite number of lines and circles and locating points of their intersection.

Hence it will be useful to consider points in a Cartesian plane, with coordinates in $\mathbb{Q}$, and expressing coordinates of points of intersection of lines and circles.

When drawing a line, we must be 'connecting' 2 points.

If the two points are $(a, b)$ and $(c, d)$ where $a, b, c, d \in \mathbb{Q}$, the line has equation

$$\frac{y-d}{x-c} = \frac{b-d}{a-c}$$
$$(b-d)x - (a-c)y = bc - ad,$$

where $b-d, a-c, bc-ad \in \mathbb{Q}$.

Hence the general equation for line is $Ax + By = C$.

When drawing a circle, we must be drawing it at one point as centre and the arc passing through another point.

If the centre is $(a, b)$ and the arc passes through $(c, d)$, where $a, b, c, d \in F$, the circle has equation

$$(x-a)^2 + (y-b)^2 = (c-a)^2 + (d-b)^2$$
$$x^2 - 2ax + y^2 - 2by + (2ac - c^2 + 2bd - d^2) = 0,$$

where $-2a, -2b, (2ac - c^2 + 2bd - d^2) \in \mathbb{Q}$.

Hence the general equation for the circle is $x^2 + Dx + y^2 + Ey + F = 0$

When finding intersection of 2 lines, we are solving

$$\begin{cases} A_1x + B_1y = C_1 \\ A_2x + B_2y = C_2 \end{cases}.$$

If the system has a solution it is

$$x = \frac{C_1B_2 - C_2B_1}{A_1B_2 - A_2B_1}, \quad y = \frac{A_1C_2 - A_2C_1}{A_1B_2 - A_2B_1},$$

where $\dfrac{C_1B_2 - C_2B_1}{A_1B_2 - A_2B_1}, \dfrac{A_1C_2 - A_2C_1}{A_1B_2 - A_2B_1} \in \mathbb{Q}$.

Whereas when finding intersection of line and circle, we are solving

$$\begin{cases} Ax + By = C \\ x^2 + Dx + y^2 + Ey + F = 0 \end{cases}.$$

$\left(\dfrac{C - By}{A}\right)^2 + D\left(\dfrac{C - By}{A}\right) + y^2 + Ey + F = 0$, which is a 2 degree polynomial (quadratic).

Whereas when finding intersection of 2 circles, we are solving

$$\begin{cases} x^2 + D_1x + y^2 + E_1y + F_1 = 0 \\ x^2 + D_2x + y^2 + E_2y + F_2 = 0 \end{cases}$$

$$(D_1 - D_2)x + (E_1 - E_2)y + (F_1 - F_2) = 0$$

$$\left(\frac{(F_2 - F_1) + (E_2 - E_1)y}{D_1 - D_2}\right)^2 + D_1\left(\frac{(F_2 - F_1) + (E_2 - E_1)y}{D_1 - D_2}\right) + y^2 + E_1y + F_1 = 0$$

Hence we are solving a quadratic in $y$, another 2 degree polynomial (quadratic).

Therefore, we can conclude that through intersection of two lines, a new field cannot be constructed. On the other hand, by intersecting a line and a circle or intersecting two circles, a new field $\S(\alpha_1)$ can be constructed, with $deg(\alpha, F) = 2$, (the extension field is constructed through a quadratic). [See reviewer's comment (12(c)(iii))]

Similarly, any other extension field of constructible numbers must be in the form $\S(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$, with $deg(\alpha, F) = 2$. Therefore we arrive to the final theorem.

**Theorem 18.** *If $\alpha$ is constructible, then $deg(\alpha, F) = 2^n$.*

[See reviewer's comment (12(c)(ii))]

*Proof.* $\alpha$ would be in some extension field $\S(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ with $deg(\alpha_i, F) = 2$. By Mathematical Induction,

$$\begin{aligned} & [\S(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) : \S] \\ = & [\S(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) : \S(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1})]\times \\ & [\S(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}) : \S(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-2})] \times \cdots \times [\S(\alpha_1) : \S] \\ = & 2^n. \end{aligned}$$

□

By the theorem above, we can determine the constructability of the triangles in different cases.

In case 10, $16y^6 - 32y^5 - 4y^4 + 36y^3 - 12y^2 - 4y + 1$ is an irreducible polynomial. Since the degree of the polynomial is 6, not a power of 2, the triangle is not constructible.

In case 13, $2y^3 + 2y^2 - 4y + 1$ is irreducible and of degree 3, and therefore it is inconstructible as well.

In case 24, the root of

$$\beta^6 + (\sqrt{\beta^4 - 74\beta^2 + 729} - 55)\beta^4 - 9(2\sqrt{\beta^4 - 74\beta^2 + 729} + 165)\beta^2 + 729(\sqrt{\beta^4 - 74\beta^2 + 729} + 27) = -\frac{48\sqrt{2}\beta^4}{\sqrt{\beta^2 - 9}}$$

is

$$\frac{1}{3}\sqrt{\frac{1}{3}\left(749 + \frac{21910 \times 5^{2/3}}{\sqrt[3]{\frac{1}{2}(5111419 + 94041i\sqrt{19})}} + 2^{2/3}\sqrt[3]{5(5111419 + 94041i\sqrt{19})}\right)},$$

which consists of a cube root, that is impossible to be constructible.

In case 28, the solutions of the equation

$$640(51 + 20n + n^2) = n\left[164 - \left(\frac{n^4 - 298n^2 + 12401 \pm \sqrt{n^8 - 596n^6 + 113606n^4 - 7390996n^2 + 119210401}}{980}\right)^2\right]^2$$

are the roots of the polynomials

$$5x^7 + 68x^6 - 1875x^5 - 23464x^4 + 194375x^3 + 2149668x^2 - 3874745x - 32000000.$$

Since the degree of the polynomial is 7, not a power of 2, n is inconstructible.

In case 29,

$$\sqrt{25 + 4z^2 - 20z\sqrt{\frac{z^2 + 12z + 20}{24z}}} + \sqrt{61 - 60\sqrt{\frac{z^2 + 12z + 20}{24z}}} = \sqrt{2z^2 - 4}$$

has only 2 as a solution, and therefore its inconstructible.

In case 36, the polynomial $9\beta^8 - 4\beta^6 - 18\beta^4 + 5$ can be factorized into

$$(\beta^2 + 1)(9\beta^6 - 13\beta^4 - 5\beta^2 + 5).$$

Since $i$ is clearly inconstructible, and degree of $(9\beta^6 - 13\beta^4 - 5\beta^2 + 5)$ is 6, not a degree of a power of 2, the triangle cannot be constructed as well.

In case 37, the polynomial

$$26244u^{12} - 139968u^{11} + 306909u^{10} - 312984u^9 + 30456u^8 + 311904u^7$$
$$-391680u^6 + 202848u^5 + 1168u^4 - 59328u^3 + 31296u^2 - 6192u + 576 = 0$$

is irreducible and of degree 12, and therefore $u$ is not constructible.

In case 40,

$$(13+6\sqrt{2})u^6-(24+9\sqrt{2})u^4+(12+3\sqrt{2})u^2-\frac{1}{2}=0.$$

is an irreducible polynomial of degree 6, and therefore $u$ is inconstructible.

In case 41,

$$972u^6+9648u^5-115600u^4-20736u^3+22032u^2=0,$$

it can be factorized into

$$u^2(972u^4+9648u^3-115600u^2-20736u+22032)=0.$$

The degree of the polynomial is 4. Now consider the basis formed by its roots. Let $\alpha$ be a root of the above equation. Then, $\S(\alpha)$ is a field and a vector space over $\S$ with basis $\{1,\alpha,\alpha^2,\alpha^3\}$.

[See reviewer's comment (12(c)(iv)]

If $\alpha$ is constructible, it must be in a field $\S(\beta_1,\beta_2)$, which is a vector space over $\S$ with basis $\{1,\beta_1,\beta_2,\beta_1\beta_2\}$. However, inspection of the roots show that it is impossible as the roots of the polynomials above involve cube roots, and therefore it is also inconstructible. [See reviewer's comment (12(h)]

In case 49, $4x^3+18x^2-3x-1$ is an irreducible polynomial of degree 3, and therefore it is inconstructible.

[See reviewer's comment (12(g))]

## REFERENCES

[1] Cut the knot, *Interactive Mathematics Miscellany and Puzzles*, http://www.cut-the-knot.org
[2] Yahoo! answers, *Constructing a triangle given length of 3 altitudes?*, https://answers.yahoo.com/question/index?qid=20110520005044AAQiQMO
[3] Wolfram Alpha, http://www.wolframalpha.com
[4] Fraleigh John B., *A First Course in Abstract Algebra*, $7^{th}$, Addison-Wesley, Boston, 2003
[5] M. S. Klamkin, Tom M. Apostol, Roger L. Creech, A. K. Austin, Zane C. Motteler, F. David Hammer, William F. Fox, Peter Orno, Robert Clark, John Ampe, Jerome C. Cherry, Howard Eves, Andreas N. Philippou, Philip Todd, Daniel A. Moran and A. Dickson Brackett, *Problems*, Mathematics Magazine **53**, No. 1, Jan., 1980, pp 49 - 54.

## Reviewer's Comments

This paper investigates the compass-and-straightedge construction of triangles with some given conditions. More precisely, given any three pieces of the following information:

1. Sides: $A$, $B$, $C$,
2. Angles at vertices $a$, $b$, and $c$, (opposite to sides $A$, $B$, $C$ respectively),
3. Angle bisectors of angles $a$, $b$, $c$, drawn to sides $A$, $B$, and $C$, respectively: $IA$, $IB$ , $IC$,
4. Medians to sides $A$, $B$, and $C$, respectively: $MA$, $MB$, $MC$,
5. Altitudes to sides $A$, $B$, and $C$, respectively: $HA$, $HB$, $HC$,

they try to prove or disprove the possibility of constructing such a triangle (and sometimes determine its uniqueness). For example, given two lengths and an angle, we can ask whether we can construct a triangle such that two sides equal the two given lengths and its included angle equals the given angle. Note that by a "given length", they mean that a line segment of that length is already drawn (similarly for the "given angle"). Say, using their definition, we can construct an equilateral triangle with side $\sqrt[3]{2}$, although it is known that $\sqrt[3]{2}$ is not a constructible number, i.e. cannot be constructed by using a straightedge and compass if we are given only the unit length.

Here are the reviewer's comments and suggestions about this paper.

1. The reviewer have comments on wordings, which have been amended in this paper.
2. The reviewer is not sure about the originality of this paper. It seems that the constructibility of triangles with these given conditions has already been completely determined, cf. the table on p. 241 of The Secrets of Triangles: A Mathematical Journey by Posamentier and Lehmann. In the same book the actual geometric constructions in some (but not all) cases are also given, but it does not offer proofs for the inconstructible cases.
3. It would be desirable to use the traditional convention that the capital letters $A$, $B$, $C$, etc, denote the angles (or vertices) and the lowercase letters $a$, $b$, $c$, etc denote the sides. The reviewer has never seen any textbook or paper which denotes an angle of a triangle by $a$ and its opposite side by $A$. As they also denote a point by a capital letter, it is quite confusing to use capital letters to also denote a length, whereas there is usually no confusion for using the same (capital) letter to denote both a vertex of a triangle and the angle at this vertex.
4. It would be much better to put a table listing the 52 cases, the possibility of construction, and the respective page number. The current form of the paper makes it very hard for the reader to locate any particular case. The table may look something like this:

| Case | Given data | Constuctible |
|---|---|---|
| 1 | [A B C] | Yes |
| 2 | [A b C] | Yes |
| ⋮ | ⋮ | ⋮ |

5. Considering all the possible triangle-construction possibilities modulo symmetry (e.g. [A B a] is essentially the same as [B C b]), there are actually 95 cases (cf. p. 240 of The Secrets of Triangles: A Mathematical Journey by Posamentier and Lehmann). As only 52 cases are investigated in this paper, the list is not complete. E.g. [MA IB IC], [MA IA IB], etc are missing.
6. Here and many other pages. It would better to change the phrase "with radius $C$, $P$ as centre" to "with radius $C$ and $P$ as centre". There are many other such instances by searching "as centre".
7. $DE = QT$ changed to $DE = Q'T'$.
8. The reviewer thinks the construction of such a circle needs more explanation. It can be constructed as follows. Construct an isosceles triangle with base $A$ and base angle $90^\circ - \dfrac{a}{2}$. The circumcircle of this triangle is then the desired circle.
9. The angle at the top of the page is not labelled. It should be labelled "$a$".
10. The figures just overlap with each other. The figure below should be lowered.
11. The sentence "Since there is at least one case in which the triangle is not constructible yet it exists, ..." is very clumsy. Perhaps it can be changed to: "We conclude that the triangle is not constructible in this case."
12. The appendix contains some basic knowledge in algebra which is mainly used to prove the impossibility to construct certain triangles with some given conditions. The main result is Theorem 18. However, there are a number of problems in this part which makes it not very satisfactory.
    (a) In the whole paper, the term "constructible numbers" is not defined, but in Theorem 17 they discuss their properties as if it has already been defined. This is not a minor issue because the term "constructible" can refer to either lengths (or points) or angles (although they are closely related).
    From the context, a "constructible number" means one which can be represented by a finite number of additions, subtractions, multiplications, divisions, and finite square root extractions of integers, and such numbers correspond to line segments which can be constructed using only straightedge and compass. They also do not give a proof or a reference for this non-trivial fact.
    (b) The definitions given in the appendix are not satisfactory. Let me give some examples.
        (i) It is not clear what the statement "$aN \subseteq N, Nb \subseteq N, \quad a, b \in R$" means. It should be changed to "$aN \subseteq N, Nb \subseteq N$ for all $a, b \in R$".
        (ii) The symbols $\subseteq$ and $\subset$ usually just mean "is a subset of" and do not indicate any algebraic structure (such as "is an ideal in"). The

definition can be changed for example to the following:
An ideal $M$ of a ring $R$ is a maximal ideal if for any ideal $N$ in $R$ containing $M$, we have either $N = M$ or $N = R$.

(iii) There is no indication what $E$ is. The whole sentence is ungrammatical. It can be changed for example to the following:
Let $E$ be a field extension of $F$ and $\alpha \in E$. Then a monic polynomial $f(x) \in F[x]$ is called the irreducible polynomial of $\alpha$ if $f(\alpha) = 0$ and $f(x)$ divides $p(x)$ whenever $p(\alpha) = 0$. Such a polynomial is necessarily unique and is denoted as $\mathrm{irr}(\alpha, F)$.

(iv) There is no indication where $\alpha$ comes from, and the term "algebraic over $F$" is not yet defined. The term $\mathrm{irr}(\alpha, F)$ should also be changed to $\langle \mathrm{irr}(\alpha, F) \rangle$. Assuming the term "algebraic over $F$" has already been defined, this can be rewritten for example as:
Let $E$ be a field extension of $F$ and $\alpha \in E$ is algebraic over $F$. Then we define the field $F(\alpha)$ to be the field $F[\alpha]/\langle \mathrm{irr}(\alpha, F) \rangle$. If the field extension $E$ satisfies $E = F(\alpha)$ for some $\alpha \in E$, then it is called a simple extension of $F$.

(c) There are many places where the symbols are not consistent or have not been defined. For example:
  (i) $R[x]$ should be $F[x]$.
  (ii) $\deg(\alpha, F)$ should be $\deg(\alpha, \mathbb{Q})$.
  (iii) It is indeed not very clear what they mean. E.g. what does it mean by "... through intersection of two lines, a new field cannot be constructed"? Same for the second setence. That being said, the reviewer thinks both § and $F$ are actually $\mathbb{Q}$.
  (iv) The reviewer thinks § should be $\mathbb{Q}$. But again the reviewer does not understand what this paragraph precisely means.

(d) Theorem 14 as stated is wrong, or at least no carefully stated. For example $E = \mathbb{Q}(2^{\frac{1}{4}})$ is a simple extension of $\mathbb{Q}$. If we take $\beta = 2^{\frac{1}{4}}$ and $\alpha = \beta^2 = \sqrt{2}$, then clearly $\alpha^2 - 2 = 0$ but $\beta \neq b_0 + b_1\sqrt{2}$. The correct statement should be:
Let $E = F(\alpha)$ be a simple extension field over $F$ and $\deg(\mathrm{irr}(\alpha, F)) = n$, then any element in $\beta$ in $E$ can be expressed as $\beta = b_0 + \cdots + b_{n-1}\alpha^{n-1}$.

(e) "$\alpha_i\beta_j$" should be changed to "$\alpha^i\beta^j$, $i \in \{0, \cdots, m-1\}$, $j \in \{0, \cdots, n-1\}$".

(f) The term $\deg(\alpha, F)$ has not been defined. It is defined as $\deg(\alpha, F) := \deg(\mathrm{irr}(\alpha, F))$. In the proof, $F \leq F(\alpha) \leq F(\beta)$ should be changed to $F \leq F(\beta) \leq F(\alpha)$. Indeed, they should also explain that the notation "$\leq$" means "is a subfield of".

(g) One of the weakest points in their argument is that often they claim that a certain polynomial is irreducible over $\mathbb{Q}$ and has degree which is not a power of 2, and then invoke Theorem 18 to conclude that their roots are not constructible numbers. In all cases, they do not offer a proof (e.g. by Eisenstein's criterion ) that the given polynomials are really irreducible over $\mathbb{Q}$. E.g. in Case 37, it is not clear to me why $26244u^{12} - 139968u^{11} + 306909u^{10} - 312984u^9 + 30456u^8 + 311904u^7 - 391680u^6 + 202848u^5 + 1168u^4 - 59328u^3 + 31296u^2 - 6192u + 576$ is irreducible over $\mathbb{Q}$. For the reviewer, it is already hard enough to factorize it into linear and quadratic factors (over $\mathbb{R}$), even by computers.

In a similar manner, in case 24, it is claimed that the roots of a certain algebraic equation (which can be made into a polynomial by rationlization) are inconstructible because "the root consists of an expression involving the cubic root". It seems that they find this expression by some software. This seems to be an invalid argument because there are multiple roots. Even if one of the roots is inconstructible, it does not mean all its roots are inconstructible.

(h) It is said that "inspection of the roots show that it is impossible as the roots of the polynomials above involve cube roots". The reviewer thinks they input the equation into a mathematical software to get this conclusion. The reviewer put it into Mathematica and did find an expression involving the cubic root. However, it also involves the imaginary number $i$. Numerical computation shows that the roots may be complex (which is not possible if the derivation is correct), but the imaginary part is extremely small, so this is perhaps due to roundoff or truncation error. However, this does raise a problem: since there must be a real root(s) to this equation, the answer the reviewer got from Mathematica cannot satisfactorily explain that the real roots are inconstructible. The reviewer thinks their explanation is not satisfactory.

13. The references part is poorly written. They just give four internet links and one standard undergraduate algebra textbook. One of the links is a mathematics puzzle page and one another is linked to the WolframAlpha site.

    First of all, they are not numbered and are not really cited anywhere in this paper. They do not even tell the reader how they are relevant. E.g. they don't give any hint how the link http://www.cut-the-knot.org is relavant. The authors are responsible to indicate how the references are relevant and the references should be removed if they are not cited.

Overall, the reviewer is confident that the geometric constructions they give are correct. On the other hand, the reviewer has doubts about the correctness of the proof on the inconstructibility of some of the triangles, as the algebraic argument seems to be at least incomplete. Nevertheless the reviewer thinks the conclusions on the constructibility in this paper should be correct, as can be checked against the table on p.241 of The Secrets of Triangles: A Mathematical Journey by Posamentier and Lehmann.