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Bronze Award

Orchard Visibility Problem

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ORCHARD VISIBILITY PROBLEM

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Abstract. In this paper, we discuss the generalization of the orchard visibility problem – from that of grid shapes to that of the shapes of the trees. We will even take a look at the problem of the visibility problem on a spherical surface and 3-D space.

1. Introduction

Our interest towards this problem originated from a book called Crimes and Mathdemeanors, which is a collection of stories about a young detective solving mysteries in mathematical ways. In one of the stories, a witness claimed that he saw the crime scene amongst a pile of columns. The young detective proved him lying using Minkowski's theorem. We shall then take a look at the problem itself.

Given a circular orchard with regularly spaced trees, located on a square lattice, the Orchard Visibility Problem is a problem concerning how thick the trees must grow in order to block the view from the centre completely.

This problem was originally posed by G. Polya in 1918. He and R. Honsberger soon reached the result that the minimum radius of the trees must be in a particular range. However, the definite value remained unsolved.

In 1986, however, T.T. Allen succeeded in obtaining a definite value for the minimum radius. Still, his proof was still not good enough for two reasons: First, his proof contained some minor flaws yet to be corrected; second, he still couldn't obtain the general solution to other variants, which we will present thoroughly in this paper.

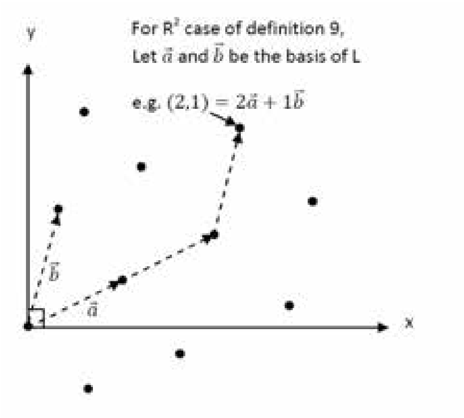
We will generalize in two directions, grid shape and tree shape. We will investigate the problem from square grid to parallelogram grid and hexagonal grid. The tree shape will be also generalized to all convex set, and alternative algorithms are also given for some special shapes. Also, we will investigate the problem where trees are planted on the vertices of a regular polyhedron, of which the light rays run along the circum sphere. Finally, we will look at the problem in 3-D space.

2. Definition

1. **Circular Trees**
   A circular tree with radius on point (x, y) is given by
   $$C(x,y,r) = \{(m,n) : \sqrt{(m-x)^2+(n-y)^2} \le r\}$$
   [See reviewer's comment (3)]
2. **Perpendicular distance between point A and line OB**
   $$d(A,OB) = \min\{AC : C \in OB\}$$
3. **Closed Set in $\mathbb{R}^n$**
   A set is closed in $\mathbb{R}^n$ if and only if it contains all of its limit points.
4. **Convex set in $\mathbb{R}^n$**
   If a set of points D is convex, $\forall x, y \in D, \{(1-t)x + ty : t \in [0,1]\} \subset D$
   [See reviewer's comment (4)]
   Note: All convex sets are assumed, in our paper, to be closed
5. **Breadth of a convex set in $\mathbb{R}^2$**
   $W(C,\theta) =$ distance between l_1 and l_2 where $l_1, l_2 \cap C \neq \emptyset$, C is only on one side of l_1 and l_2, slope $(l) = \cot\theta$.
6. **Diameter of convex set C**
   $$\text{Diameter }(C) = \max\{d(x,y) : x, y \in C\}$$
7. **Radius of convex set C**
   $$\frac{\text{Diameter }(C)}{2}$$
8. **Set C with central symmetry about X**
   $$X + A \in C \implies X - A \in C$$
9. **Lattice L in $\mathbb{R}^n$**
   $$L = \left\{ (x_1, x_2, \ldots, x_n) \;\middle|\; \begin{array}{l} x_1, x_2, \ldots, x_n \in \mathbb{Z}, \text{ where} \\ (x_1, x_2, \ldots, x_n) = \displaystyle\sum_{i=1}^{n} x_i(y_{i1}, y_{i2}, \ldots, y_{in}) \text{ and} \\ x_i \in \mathbb{Z}\ \forall i = 1, 2, \ldots, n \text{ and } (y_{i1}, y_{i2}, \ldots, y_{in}) \\ \text{are all linear independent and all} \\ (y_{i1}, y_{i2}, \ldots, y_{in}) \text{ are basis of } L \end{array} \right\}$$
   [See reviewer's comment (5)]
10. **Lattice point**
    Every element of lattice L

11. **Unit grid area of L in $\mathbb{R}^2$**
The area of the smallest parallelogram with vertices on L

12 **Primitive area of L in $\mathbb{R}^2$**
The area of the smallest triangle with vertices on L



13 **Coprime lattice point in $\mathbb{R}^n$**
A lattice point of L with coordinates $(a_1, a_2, a_3, ..., a_n)$ with

$$\gcd(a_1, a_2, a_3, ..., a_n) = 1$$

14 **Non-coprime lattice point in $\mathbb{R}^n$**
A non-origin lattice point which is not a coprime lattice point

15 **Packing density $\delta(\mathbf{C})$**
The maximum fraction of the volume filled by a given collection of C

16 **Packing density $\delta(\mathbf{C}, \boldsymbol{\Lambda})$**
The fraction of the volume filled by a given collection of C when C is arranged in Λ

17 **Visible area**
The total area of the region visible from the origin

18 **Orchard D in $\mathbb{R}^2$**
Convex set in $\mathbb{R}^2$ containing lattice points $(1,0), (0,1), (-1,0)$ and $(0,-1)$

19 **Orchard Visibility Problem for circular trees in circular orchard in $\mathbb{R}^2$**
For a set $\Lambda' = \{(x, y) : x, y \in \mathbb{Z}; (x, y) \neq O; \sqrt{x^2 + y^2} \leq R\}$, there is a circle $C(x, y, r)$ centered at each point in Λ'. [See reviewer's comment (6)] The problem is about finding the value of p so that $\forall r \geq p$, $\forall\theta$, $\exists(x, y) \in \Lambda'$, $\exists k \geq 0$, $(k\cos\theta, k\sin\theta) \in C(x, y, r)$ and $\forall r$ where $0 < r < p$, $\exists\theta$, $\forall(x, y) \in \Lambda'$, $\forall k \geq 0$, $(k\cos\theta, k\sin\theta) \notin C(x, y, r)$.

20 **General orchard problem in $\mathbb{R}^2$**
For a set $\Lambda' = \{(x, y) : x, y \in \mathbb{Z}; (x, y) \neq O; \sqrt{x^2 + y^2} \leq R\}$, there is a convex shape fC at every point in Λ' where C is a convex basic tree, where $f \in \mathbb{R}^+$. The problem will then become finding the value of p so that $\forall f \geq \frac{p}{\text{radius(C)}}$, $\forall\theta$, $\exists k \geq 0$, $(k\cos\theta, k\sin\theta) \in fC + \Lambda'$ and $\forall f$ where $0 < f < \frac{p}{\text{radius(C)}}$, $\exists\theta$, $\forall k \geq 0$, $(k\cos\theta, k\sin\theta) \notin fC + \Lambda'$. [See reviewer's comment (7)]

3. Circular trees

3.1. On a square grid

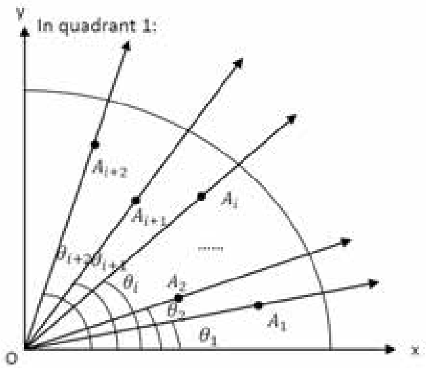
Observation:

Refer to definition 18.

For every orchard D in $\mathbb{R}^2$, all the coprime lattice points of D can be ordered in the following way:

$$A_1, A_2, A_3, \ldots$$

where $\theta_i > \theta_j$ when $i > j$ and θ_i is the anti-clockwise angle made from positive x-axis to A_i



Lemma 1. *Let $A(x_1, y_1)$ and $B(x_1, y_1)$ be two adjacent coprime lattice points in a convex domain D. Then $C(x_1 + x_2, y_1 + y_2)$ is outside D.*

Proof. Assume, on the contrary, that C belongs to D.
Case 1: C is a coprime lattice point.
This directly contradicts the premise that A and B lie on two consecutive rays.

$$\therefore \text{Case 1 does not stand}$$

Case 2: C is a non-coprime lattice point.

$$\Longrightarrow \text{ OC contains a coprime lattice point}$$

which again contradicts our argument that A and B lie on two consecutive rays.

$$\therefore \text{Case 2 does not stand}$$

Therefore, C is outside D, and the proof is complete. □

Lemma 2. *Let $A(x_1, y_1)$ and $B(x_1, y_1)$ be two adjacent coprime lattice points in a convex domain D. Then $C(x_1 + x_2, y_1 + y_2)$ is also a comprime lattice point.*

Proof. Assume, on the contrary, that C is not a coprime lattice point.

$$\Longrightarrow \exists M,\ M \text{ is a coprime lattice point on } OC$$

$$\Longrightarrow M = \left(\frac{x_1 + x_2}{\gcd(x_1 + x_2, y_1 + y_2)}, \frac{y_1 + y_2}{\gcd(x_1 + x_2, y_1 + y_2)} \right)$$

Now $\gcd(x_1+x_2, y_1+y_2) \in \mathbb{N}$ and $\gcd(x_1+x_2, y_1+y_2) \geq 2$ [See reviewer's comment (8)]

Case 1: $\gcd(x_1 + x_2, y_1 + y_2) = 2$
Then $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$, which is the mid-point of A and B. By the definition of a convex domain, M lies inside D, which contradicts our argument that A and B lie on two consecutive rays.

$$\therefore \text{Case 1 does not stand}$$

Case 2: $\gcd(x_1 + x_2, y_1 + y_2) \neq 2$
Then M lies nearer to the origin than the M in case 1. Again, by the definition of a convex domain, all the M's in this case lie in D, which contradicts our argument that A and B lie on two consecutive rays.

$$\therefore \text{Case 2 does not stand}$$

Therefore, C is a coprime lattice point. □

Theorem 3. *(Refer to definition 18.) For any orchard $D \subset \mathbb{R}^2$, the minimal radius p of each tree such that the visibility to the region outside D from the origin is completely blocked can be given by the formula $p = \dfrac{1}{d}$, where d is the distance from O to the closest coprime lattice point outside D.*

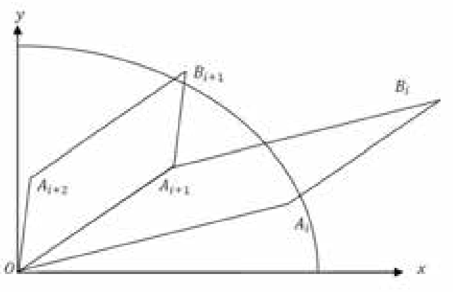
[See reviewer's comment (9)]

Proof.

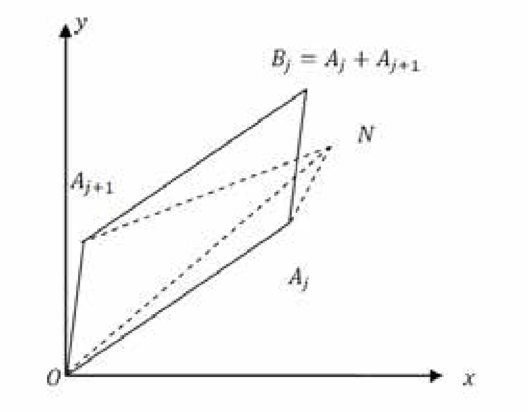
For every two adjacent coprime lattice points in D, namely A_i and A_{i+1}, construct a parallelogram $OA_iA_{i+1}B_i$, where $B_i = A_i + A_{i+1}$. By Pick's Theorem, $\text{Area}(OA_iA_{i+1}B_i) = 1$

$$\therefore \quad d(A_i, OB_i) = d(A_{i+1}, OB_i) = \frac{1}{OB_i}$$

$$\therefore \quad p = \max\left(\frac{1}{OB_i}\right) = \frac{1}{\min(OB_i)}$$



Let N be the closest coprime lattice point outside D then $\exists j$, ON is between OA_j and OA_{j+1}



In $\triangle OA_jN$, there are no other lattice points in the orchard since A_j and A_{j+1} are adjacent. And there are no other lattice points outside the orchard since N is already the closest. So by Pick's theorem, $(\triangle OA_jN) = \frac{1}{2}$

Assume $B_j \neq N$

By Pick's theorem,

$$\text{Area}(\triangle OA_jB_j) = \text{Area}(\triangle OA_jN) = \frac{1}{2}$$

$$\therefore OA_j // B_jN$$

Similarly,

$$\text{Area}(\triangle OA_{j+1}B_j) = \text{Area}(\triangle OA_{j+1}N) = \frac{1}{2}$$

$$\therefore OA_{j+1} // B_jN$$

$$\therefore OA_j // OA_{j+1}$$

So contradiction occurs.

$$\text{i.e. } B_j = N$$

Sub $\min(OB_i) = d$, [See reviewer's comment (10)]

$$p = \frac{1}{d}$$

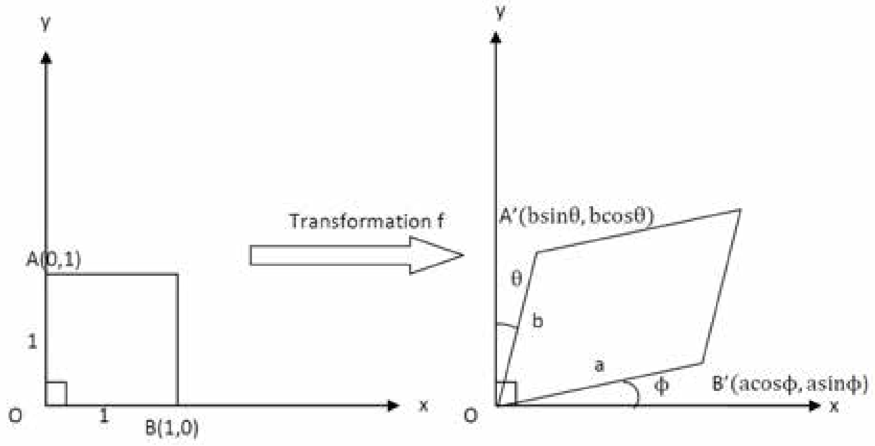
□

3.2. On a parallelogram grid

[See reviewer's comment (11)]

Lemma 4. *Any parallelogram lattice can be transformed from a square lattice using the transformation matrix:*

$$\begin{pmatrix} a\cos\phi & b\sin\theta \\ a\sin\phi & b\cos\theta \end{pmatrix}$$



Proof. Note that the two basic vectors that generate all lattice points in the square lattice are $(1,0)$ and $(0,1)$. [See reviewer's comment (12)] Then:

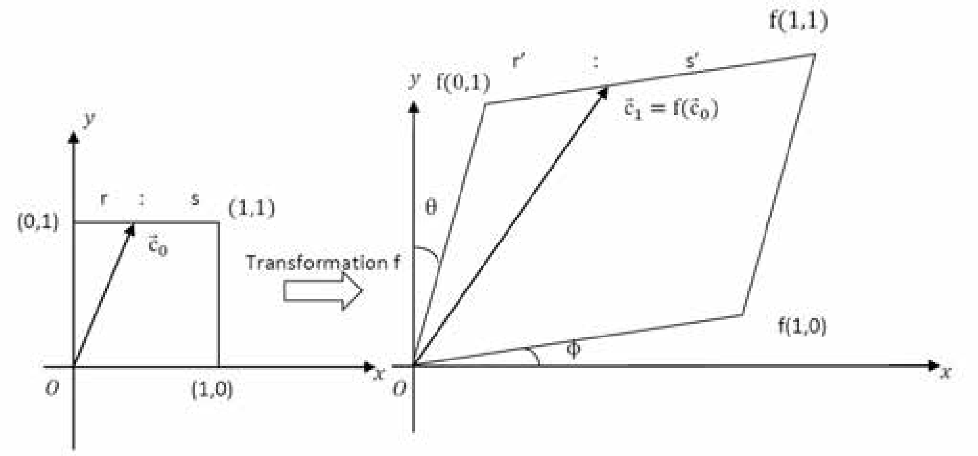
$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\cos\phi\\a\sin\phi\end{pmatrix} \text{ and } f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\sin\theta\\b\cos\theta\end{pmatrix}$$

$$\begin{aligned}
\because \quad & \text{f is a linear transformation.} \\
\therefore \quad f\begin{pmatrix}x\\y\end{pmatrix} &= f\left[x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right] \\
&= xf\begin{pmatrix}1\\0\end{pmatrix} + yf\begin{pmatrix}0\\1\end{pmatrix} \\
&= \begin{pmatrix}ax\cos\phi + by\sin\theta\\ax\sin\phi + by\cos\theta\end{pmatrix} \\
f\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}a\cos\phi + b\sin\theta\\a\sin\phi + b\cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}
\end{aligned}$$

$$\therefore \text{f can be considered as the matrix } \begin{pmatrix} a\cos\phi & b\sin\theta \\ a\sin\phi & b\cos\theta \end{pmatrix}$$

$\square$

Lemma 5. *Refer to the figure below.*



The transformation matrix

$$\begin{pmatrix} a\cos\phi & b\sin\theta \\ a\sin\phi & b\cos\theta \end{pmatrix}$$

ensures that $r : s = r' : s'$.

Proof. By the point of division formula,

$$\overrightarrow{c_0} = \frac{r}{r+s}(1,1) + \frac{s}{r+s}(0,1)$$

Now, after transformation, by Lemma 4, $\overrightarrow{c_0}$ becomes:

$$\begin{aligned} &\begin{pmatrix} a\cos\phi & b\sin\theta \\ a\sin\phi & b\cos\theta \end{pmatrix} \left[\begin{pmatrix} \frac{r}{r+s} \\ \frac{s}{r+s} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{s}{r+s} \end{pmatrix} \right] \\ &= \frac{r}{r+s} \begin{pmatrix} a\cos\phi + b\sin\theta \\ a\sin\phi + b\cos\theta \end{pmatrix} + \frac{s}{r+s} \begin{pmatrix} b\sin\theta \\ b\cos\theta \end{pmatrix} \\ &= \frac{r}{r+s} f(1,1) + \frac{s}{r+s} f(0,1) \end{aligned}$$

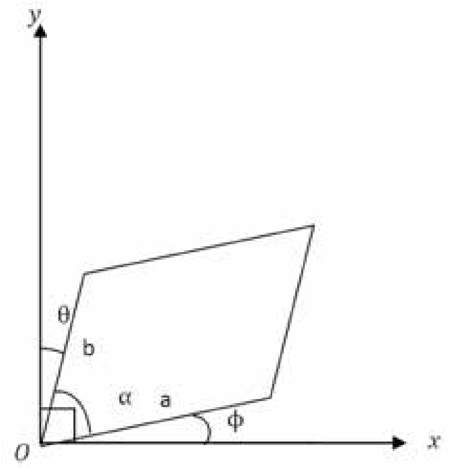
And we see that the ratio $r' : s'$ is :

$$\frac{r}{r+s} : \frac{s}{r+s} = r : s$$

$\square$

Lemma 6. *(Generalized Pick's theorem): For a figure F whose vertices are parallelogram lattice points, we have:* [See reviewer's comment (13)]

Let I, B and θ and ϕ be the interior lattice points and boundary lattice points of F, and the acute angles between two adjacent sides of a unit grid parallelogram respectively. [See reviewer's comment (14)] *Let a and b be adjacent sides of a parallelogram.*



The area of F is given by:

$$\left(I + \frac{B}{2} - 1\right) ab \sin\alpha$$

Proof.

Property 1:

No. of sides of F in one parallelogram $\leq$ no. of sides constituting F

Property 2:

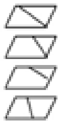
If there are n sides of F cutting a parallelogram, the parallelogram will be divided into $(n+1)$ regions.

By property 2, there may be more than one line cutting a parallelogram. However, the lines must not intersect internally, for that would contradict to our assumption that the figure is one whose vertices lie on lattice points. Therefore, adding one more line means dividing the figure into two, and adding n more lines means dividing the figure into $(n+1)$

Note:

If there is only one line cutting a parallelogram, then it is only one of four cases:

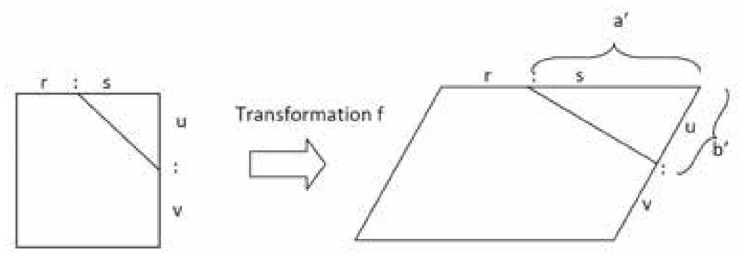
1. 2 triangles
2. 1 triangle and 1 quadrilateral
3. 1 triangle and 1 pentagon or
4. 2 quadrilaterals



Case 1, 2 and 3:

Observe that each triangle has two sides that are either the adjacent sides of the parallelogram or part of the adjacent sides of the parallelogram.

By lemma 5, we know that the ratio doesn't change no matter whether the grid is a square or a parallelogram, i.e.:



We can let the two sides of the triangle a' and b', where $0 < a' <\leq a$ and $0 < b' \leq b$.

$$\implies a' = \frac{s}{r+s}a, \quad b' = \frac{u}{u+v}b$$

$$\begin{aligned}
\text{Area of the triangle in square grid} &= \frac{1}{2}\left(\frac{s}{r+s}\right)\left(\frac{u}{u+v}\right) \\
&= \frac{1}{2}\left(\frac{us}{(r+s)(u+v)}\right) \\
\text{Area of the triangle in parallelogram grid} &= \frac{1}{2}a'b'\sin(90^\circ - \theta - \phi) \\
&= \frac{1}{2}\left(\frac{s}{r+s}\right)a\left(\frac{u}{u+v}\right)b\sin\alpha \\
&= \frac{1}{2}\left(\frac{s}{r+s}\right)\left(\frac{u}{u+v}\right)ab\sin\alpha
\end{aligned}$$

and the multiplying factor is $ab\sin\alpha$

$\therefore$ The area of the rest of the parallelogram must have a multiplying factor of $ab\sin\alpha$ as well since the area of the whole figure has a multiplying factor of $ab\sin\alpha$.

Case 4:

Again by Lemma 5,

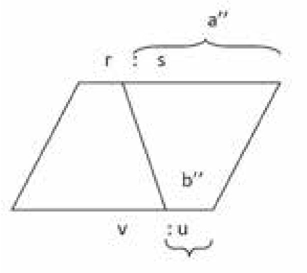
Observe that the quadrilaterals must be trapeziums.

We can let 2 opposing sides be a'' and b'', where $0 < a'' \leq a$ and $0 < b'' \leq a$.

$$\implies a'' = \frac{s}{r+s}a, \quad b' = \frac{u}{u+v}a$$

Area of trapezium in a square grid

$$\alpha = \frac{1}{2}(\frac{s}{r+s} + \frac{u}{u+v})$$



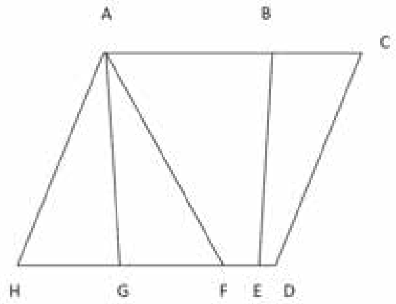
Area of trapezium in a parallelogram grid

$$= \frac{1}{2}(a'' + b'')b\sin(90^\circ - \theta - \phi)$$
$$= \frac{1}{2}(\frac{s}{r+s}a + \frac{u}{u+v}a)b\sin\alpha$$
$$= \frac{1}{2}(\frac{s}{r+s} + \frac{u}{u+v})ab\sin\alpha$$ and the multiplying factor is again $ab\sin\alpha$

Again, the area of the rest of the parallelogram must have a multiplying factor of $ab\sin\alpha$ as well since the area of the whole figure has a multiplying factor of $ab\sin\alpha$.

Now, from property 2, adding one more line means adding one more division into the figure [See reviewer's comment (15)], which means that there will be no complex figures will be formed in which no lines lie on the boundary of the parallelogram. Therefore, we can tackle cases in which there is more than one line cutting the parallelogram by considering there are separate independent lines cutting it. And we can obtain the same result in the theorem.

Demonstration:



Consider BE. By the above argument,

$$\text{the area of } AHEB = \text{original area of } AHEB \times ab\sin\alpha \tag{1}$$

Consider AG. By the above argument,

$$\text{the area of } AHG = \text{original area of } AHG \times ab\sin\alpha \tag{2}$$

By (1) and (2), the area of $ABEG$ = original area of $ABEG$ $\times ab\sin\alpha$

(Similarly for the remaining AFG and $ABEF$) Therefore, the lemma holds. □

Theorem 7. *Refer to definition 18. For any orchard $D \subset \mathbb{R}^2$, visibility to the region outside D from O is completely blocked by trees planted on a parallelogram lattice points in D with minimal radius $p = \dfrac{G}{d}$, where G is the unit grid area (by definition 11) and d is the distance from O to the closest coprime lattice point D. [See reviewer's comment (16)]*

Proof. [See reviewer's comment (17)]
For every two adjacent coprime lattice points in D, namely A_i and A_{i+1}, construct a parallelogram $OA_iA_{i+1}B_i$, where $B_i = A_i + A_{i+1}$. By Lemma 4,

$$\text{Area}(OA_iA_{i+1}B_i) = G$$

$$\therefore \qquad d(A_i, OB_i) = d(A_{i+1}, OB_i) = \frac{G}{OB_i}$$

$$\therefore \qquad p = \max\left(\frac{G}{OB_i}\right) = \frac{G}{\min(OB_i)}$$

Let N be the closest coprime lattice point outside D and ON is between OA_j and OA_{j+1} $\exists j$.

Assume $B_j \neq N$.

By Lemma 4,

$$\text{Area}(\triangle OA_jB_j) = \text{Area}(\triangle OA_jN) = \frac{G}{2}$$

$$\therefore OA_j // B_jN$$

Similarly,

$$\therefore OA_{j+1} // B_jN$$

$$\therefore OA_j // OA_{j+1}$$

So contradiction occurs.

$$\text{i.e. } B_j = N$$

Sub $\min(OB_i) = d$,

$$p = \frac{G}{d}$$

□

3.3. Removability of lattice points

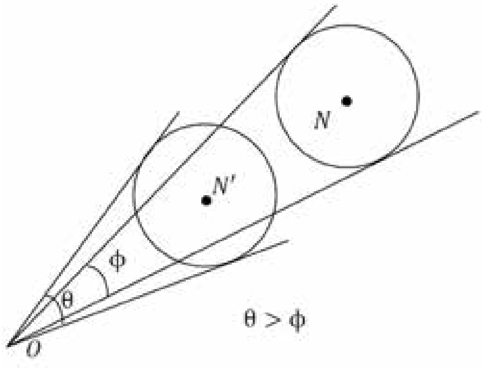
In this part, we are going to discuss the removability of lattice points, meaning that we are going to investigate what kind of lattice point can be removed without affecting the visibility when $r = p$.

Theorem 8. *In a circular orchard, a lattice point is removable if and only if it is a non-coprime lattice point when $r = p$.*

Proof.

'If' case:

The angle formed by two tangents from the origin tends to decrease with the increase of the distance of the circle centre and the origin. From the graph [See reviewer's comment (18)], we can see that θ is greater than ϕ.



A non-coprime lattice point N can be removed because there is a coprime lattice point N' lying on ON with a larger angle made by the tangents.

'Only if' case:

Since a lattice point is either a coprime one or a non-coprime one, except the origin, our approach to the proof can be narrowed into a problem of whether a coprime lattice point can be removed from the lattice.

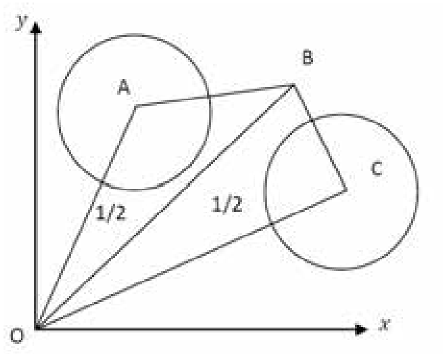
For any 3 adjacent coprime lattice point, namely A, B and C, by Pick's theorem, $\text{Area}(\triangle OAB) = \text{Area}(\triangle OBC) = \frac{1}{2}$. To make B removable, the circles with A and B as centres respectively must at least touch OB. In fact,

$$d(A, OB) = d(C, OB) = \frac{1}{OB}.$$

Assume $p \geq \frac{1}{OB}$, then $d \leq OB$ which is impossible for a circular orchard.

$$\therefore p < \frac{1}{OB}.$$

Therefore only non-coprime lattice points are removable.



□

3.4. Visible Area

[See reviewer's comment (19)]

Theorem 9. [See reviewer's comment (20)]

$$\textit{Visible Area} \leq \frac{n(P)}{2} - \frac{1}{2}(n(P)-2)\pi r^2$$

where $n(P)$ is the number of coprime lattice points in orchard D and $r = p = \frac{1}{d}$.

Proof. We have previously proved that adjacent coprime lattice points form a triangle with O with area=1/2. We have that number of these triangles $= n(P)$. And all points in these triangles (except the trees and possible shadow) are visible from the origin. [See reviewer's comment (21)]

Consider one of the triangles, namely OAB, where A and B are adjacent coprime lattice points.

Visible area between OA and OB from origin

$$= \frac{1}{2} - \text{tree areas inside } OAB - S_i + S_i' \quad \text{[See reviewer's comment (22)]}$$

where

$$\text{tree areas inside } OAB = \frac{\pi - \theta_i}{2\pi}\pi r^2 = \frac{1}{2}(\pi - \theta)r^2$$

Also $S = S'$ when the common tangent of the two circle passes through the origin. When the r increases, S increases and S' decreases.

$$\therefore S \geq S'$$

$\therefore$ Total Visible Area

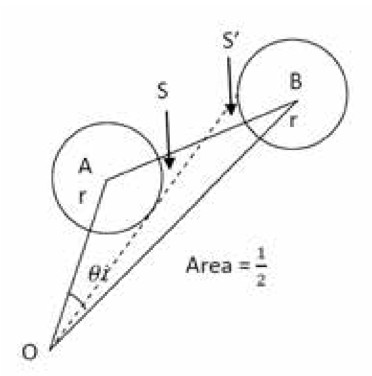
$$= \sum_{i=1}^{n(P)} \frac{1}{2} - \frac{1}{2}(\pi - \theta)r^2 - S_i + S_i'$$

$$\leq \frac{n(P)}{2} - \frac{1}{2}(n(P)\pi - \sum_{i=1}^{n(P)} \theta_i)r^2$$

$$\leq \frac{n(P)}{2} - \frac{1}{2}(n(P)\pi - 2\pi)r^2$$

$$\leq \frac{n(P)}{2} - \frac{1}{2}(n(P) - 2)\pi r^2$$

[See reviewer's comment (23)]



Note: We know that [8]

$$\lim_{R\to\infty} \frac{n(P)}{\pi R^2} = \frac{6}{\pi^2}$$

$\therefore$ If D is a circle with radius R,

$$\lim_{R\to\infty} \frac{\text{Total Visible Area}}{\text{Orchard Area}} \leq \lim_{R\to\infty} \frac{\frac{n(P)}{2} - \frac{1}{2}(n(P) - 2)\pi r^2}{\pi R^2}$$

$$= \lim_{R\to\infty} \frac{n(P)}{2\pi R^2} - \frac{(n(P) - 2)\pi}{2\pi R^2 d^2}$$

$$= \frac{3}{\pi^2} \quad \text{[See reviewer's comment (24)]}$$

$\square$

3.5. On a hexagon grid

Theorem 10. *The minimal radius of trees can be found by:*

$$p = \frac{\sqrt{2\sqrt{3}A}}{6},$$

where A is the area of a hexagon.

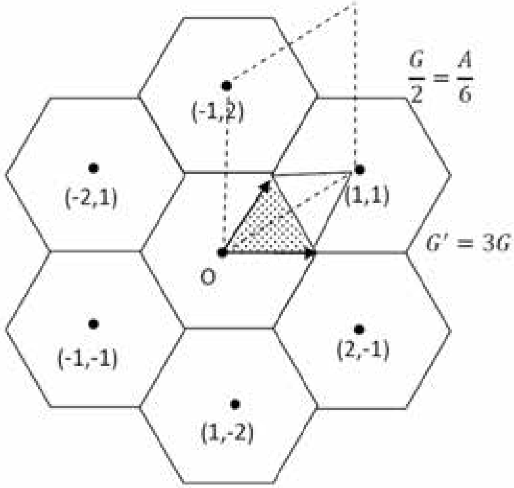
Proof.

A hexagonal grid is a rhombus lattice generated by $(1,0)$ and $(0,1)$ but all the central points are removed, which is another lattice generated by $(1,-2)$ and $(1,1)$.
[See reviewer's comment (25)]
Let G be the unit grid area of the rhombus lattice.

$$\because \frac{G}{2} = \frac{A}{6}$$

$$\therefore G = \frac{A}{3}$$

Actually the lattice of the central points has a unit grid area G' 3 times of the rhombus lattice. i.e. $G' = 3G$.



Assume there exist two adjacent central points with no non-central points in between. By Pick's Theorem, $G' = G$, which contradicts the previous result.

So we can deduce that two coprime centre points will at least be separated by one non-centre coprime lattice points in a circular orchard.

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three adjacent coprime lattice points in rhombus lattice where (x_2, y_2) is a central point. After (x_2, y_2) is removed, the minimum r required to block this area is $\frac{G}{\sqrt{x_2^2 + y_2^2}}$, which is obviously smaller than $\frac{G}{d}$ before removal.

The required $p = \max\left(\frac{G}{\sqrt{x_2^2 + y_2^2}}\right) = \frac{G}{\min(\sqrt{x_2^2 + y_2^2})}$ where (x_2, y_2) is a centre point. Obviously, there are 6 closest centre points, namely $(1,1)$, $(2,-1)$, $(1,-2)$, $(-1,-1)$, $(-2,1)$ and $(-1,2)$, which are the centres of the 6 adjacent hexagons. All of them have a distance of $\sqrt{3}l$ from the origin, where l is the length of a hexagon's side. Since $l = \sqrt{\frac{2A}{3\sqrt{3}}}, p = \frac{A}{3\sqrt{3}\sqrt{\frac{2A}{3\sqrt{3}}}} = \frac{\sqrt{2\sqrt{3}A}}{6}$, which is independent of R. □

4. Convex tree

Refer to definition 20 (General orchard problem):

For a set $\Lambda' = \{(x,y) : x,y \in \mathbb{Z}; (x,y) \neq O; \sqrt{x^2+y^2} \leq R\}$, there is a convex shape fC at every point in Λ' where C is a convex basic tree, where $f \in \mathbb{R}^+$. The problem will then become finding the value of p so that $\forall f \geq \frac{p}{\text{radius(C)}}$, $\forall\theta$, $\exists k \geq 0$, $(k\cos\theta, k\sin\theta) \in fC + \Lambda'$ and $\forall f$ where $0 < f < \frac{p}{\text{radius(C)}}$, $\exists\theta$, $\forall k \geq 0$, $(k\cos\theta, k\sin\theta) \notin fC + \Lambda'$.

4.1. General convex trees

Method I:

Only the following Lemma 11 is extracted from [1].

Lemma 11. *Let $C \subset \mathbb{R}^n$ be a convex set which is symmetrical about O, Λ be a lattice such that $\forall A_1, A_2 \in \Lambda, (C + A_1) \cap (C + A_2) = \emptyset$*

[See reviewer's comment (26)]

$$\forall v \in \mathbb{R}^n, \exists j \in \mathbb{Z}^+ \leq \frac{2^n\delta(C)}{\delta(C,\Lambda)}, \exists u \in \Lambda, jv \in int(C) + u$$

where $\delta(C)$ is the maximum packing density of C; $\delta(C,\Lambda)$ is the packing density of C under arrangement of Λ **(by definition 15,16)**.

Proof. Let $m = \frac{2^n\delta(C)}{\delta(C,\Lambda)}$ and $X = \cup_{i=0}^{m}(\Lambda + iv)$ [See reviewer's comment (27)].

If

$$\text{int}\left(\frac{1}{2}C + i_1v + u_1\right) \cap \text{int}\left(\frac{1}{2}C + i_2v + u_2\right) = \emptyset \quad \forall i_1, i_2 \in \mathbb{Z}^+ \cup 0$$

where

$$0 \leq i_1 < i_2 \leq m, u_1 \in \Lambda, u_2 \in \Lambda,$$

$$\begin{aligned}
\delta(\frac{1}{2}C, X) &= (m+1)\delta(\frac{1}{2}C, \Lambda) \\
&= \frac{(m+1)\delta(C,\Lambda)}{2^n} \\
&\text{Note: } \forall C \in \mathbb{R}^n, \text{ int(C) = C} \setminus \text{boundary(C)} \\
&> \frac{2^n\delta(C)}{\delta(C,\Lambda)} \cdot \frac{\delta(C,\Lambda)}{2^n} \\
&= \delta(C) \\
&= \delta(\frac{1}{2}C) \\
&\qquad \because \quad \delta(\frac{1}{2}C, X) \leq \delta(\frac{1}{2}C) \\
&\qquad \therefore \quad \text{Contradiction occurs.}
\end{aligned}$$

$$\therefore \text{int}\left(\frac{1}{2}C + i_1v + u_1\right) \cap \text{int}\left(\frac{1}{2}C + i_2v + u_2\right) \neq \emptyset \quad \exists 0 \leq i_1 < i_2 \leq m,$$
$$u_1 \in \Lambda, u_2 \in \Lambda$$

Assume $a \in \text{int}\left(\frac{1}{2}C + i_1v + u_1\right) \cap \text{int}\left(\frac{1}{2}C + i_2v + u_2\right)$,

$$a \in \text{int}\left(\frac{1}{2}C + i_1v + u_1\right) \text{ and } a \in \text{int}\left(\frac{1}{2}C + i_2v + u_2\right)$$

$$a = x_1 + i_1v + u_1 \text{ and } a = x_2 + i_2v + u_2, \text{ where } x_1, x_2 \in \frac{1}{2}C$$

$$\begin{aligned}
x_1 + i_1v + u_1 &= x_2 + i_2v + u_2, \\
(i_2 - i_1)v &= 2\frac{x_1 + (-x_2)}{2} + u_1 - u_2 \\
(i_2 - i_1)v &\in 2 \text{ int}(\frac{1}{2}C) + u_1 - u_2 \\
(i_2 - i_1)v &\in \text{ int}(C) + u_1 - u_2
\end{aligned}$$

Take $j = i_2 - i_1$, and $u = u_1 - u_2$, $j_v \in \text{int}(C) + u$. □

Theorem 12. *The range of the minimal radius of the convex symmetrical trees in orchard $D \subset \mathbb{R}^2$ is given by*

$$p \leq \frac{12}{R\max(\delta(C,\Lambda))}$$

where $R \geq 6$, $\max(\delta(C,\Lambda)) \geq \frac{1}{2}$ *and* $\max\{|A| : A \in C\} < 1$.

[See reviewer's comment (28)]

Proof. By Lemma 11,

$$\forall v \in \mathbb{R}^n, \exists j \in \mathbb{Z}^+ \leq \frac{2^n \delta(fC)}{\delta(fC,\Lambda)}, \exists u \in \Lambda, jv \in \operatorname{int}(fC) + u$$

$\because \delta(C) \leq 1$ and consider $n = 2$,

$$\forall v \in \mathbb{R}^2, \exists j \in \mathbb{Z}^+ \leq \frac{4}{\delta(fC,\Lambda)}, \exists u \in \Lambda, jv \in \operatorname{int}(fC) + u$$

To ensure $u \neq 0$, take $v \notin \operatorname{int}(fC)$. To ensure all the trees responsible for blockage lie inside the orchard, we need $|u| \leq R$. To ensure $|(\max(j)+1)v| \leq R$, take $\left|\left(\frac{4}{\delta(fC,\Lambda)} + 1\right) v\right| \leq R$. So $|v| \leq \frac{R}{\frac{4}{\delta(fC,\Lambda)} + 1}$. To ensure $\notin \operatorname{int}(fC)$, we can get $\max\{|A| : A \in fC\} < |v|$ [See reviewer's comment (29)]

$$\implies \max\{|A| : A \in fC\} \leq \frac{R}{\frac{4}{\delta(fC,\Lambda)} + 1},$$
$$f \max\{|A| : A \in C\} \leq \frac{R}{\frac{4}{f^2\delta(C,\Lambda)} + 1}$$

$$\delta(C,\Lambda) \max\{|A| : A \in C\} f^2 - R\delta(C,\Lambda) f + 4 \max\{|A| : A \in C\} \leq 0$$

$$\frac{R\delta(C,\Lambda) - \sqrt{R^2\delta^2(C,\Lambda) - 16\delta(C,\Lambda)\max^2\{|A| : A \in C\}}}{2\delta(C,\Lambda)\max\{|A| : A \in C\}}$$
$$\leq f$$
$$\leq \frac{R\delta(C,\Lambda) + \sqrt{R^2\delta^2(C,\Lambda) - 16\delta(C,\Lambda)\max^2\{|A| : A \in C\}}}{2\delta(C,\Lambda)\max\{|A| : A \in C\}}$$

Consider the left inequality,

$$\frac{R - R\sqrt{1 - \frac{16}{R^2\delta(C,\Lambda)}\max^2\{|A| : A \in C\}}}{2\max\{|A| : A \in C\}} \leq f$$

By Mean Value Theorem, when $f(x)$ is differentiable on $[0, x]$,

$$f(x) = f(0) + f'(a)x \quad \exists a \text{ where } 0 < a < x.$$

Using $f(x) = \sqrt{1-x}$ and $= \frac{16}{R^2\delta(C,\Lambda)}\max^2\{|A| : A \in C\}$, $\exists a$ where $0 < a < x$,

$$\frac{R - R\sqrt{1 - \frac{1}{2\sqrt{1-a}}\frac{16}{R^2\delta(C,\Lambda)}\max^2\{|A| : A \in C\}}}{2\max\{|A| : A \in C\}} \leq f,$$

$$\frac{4\max\{|A|:A\in C\}}{R\delta(C,\Lambda)\sqrt{1-a}}\leq f,$$
$$\min(f)\leq\frac{4\max\{|A|:A\in C\}}{R\delta(C,\Lambda)\sqrt{1-a}},$$
$$\min(f)\leq\frac{4\max\{|A|:A\in C\}}{R\delta(C,\Lambda)\sqrt{1-\frac{16}{R^2\delta(C,\Lambda)}\max^2\{|A|:A\in C\}}}$$

For $R\geq 6$, $\delta(C,\Lambda)\geq\frac{1}{2}$ and $\max\{|A|:A\in C\}<1$,

$$\min(f)\leq\frac{4\max\{|A|:A\in C\}}{R\delta(C,\Lambda)\sqrt{1-\frac{8}{9}}},$$
$$\min(f)\leq\frac{12\max\{|A|:A\in C\}}{R\delta(C,\Lambda)}$$

$\because$ Refer to the definition at the beginning of this chapter,

$$\min(f)=\frac{p}{\text{radius(C)}}\text{ and radius}(C)=\max\{|A|:A\in C\}$$

when C has central symmetry,

$$\therefore\quad\frac{p}{\max\{|A|:A\in C\}}\leq\frac{12\max\{|A|:A\in C\}}{R\delta(C,\Lambda)},$$
$$p\leq\frac{12}{R\delta(C,\Lambda)},$$
$$p\leq\frac{12}{R\max(\delta(C,\Lambda))},$$

which means that the initial tree, C, is enlarged to the greatest extent without overlapping each other with $\max\{|A|:A\in C\}<1$.

Similar method can be applied to circular trees directly. □

Theorem 13. *The range of the minimal radius of circular trees for complete blockage of visibility can be given by:*

$$p\leq\frac{1.65499}{R}$$

when $R\geq 3$.

Proof. By Lemma 11,

$$\forall v\in\mathbb{R}^n,\exists j\in\mathbb{Z}^+\leq\frac{2^n\delta(C)}{\delta(C,\Lambda)},\exists u\in\Lambda,jv\in\text{int}(C)+u$$

Sub $\delta(C,\Lambda) = \pi r^2$, $\delta(C) = \frac{\pi}{2\sqrt{3}}$, $n = 2$,

$$\forall v \in \mathbb{R}^2, \exists j \in \mathbb{Z}^+ \leq \frac{2}{\sqrt{3}r^2}, \exists u \in \Lambda, jv \in \operatorname{int}(C) + u$$

To ensure complete blockage, take

$$\left(\frac{2}{\sqrt{3}r^2} + 1\right) r \leq R$$

$$r^2 - rR + \frac{2}{\sqrt{3}} \leq 0$$

$$\frac{R - \sqrt{R^2 - \frac{8}{\sqrt{3}}}}{2} \leq r \leq \frac{R + \sqrt{R^2 - \frac{8}{\sqrt{3}}}}{2}$$

Consider $\dfrac{R - \sqrt{R^2 - \frac{8}{\sqrt{3}}}}{2} \leq r$, by Mean Value Theorem, when $f(x)$ is differentiable on $[0, x]$, $f(x) = f(0) + f'(a)x \ \exists\ a$ where $0 < a < x$.

Using $f(x) = \sqrt{1 - x}$ and $x = \dfrac{8}{\sqrt{3}R^2}$, $\exists 0 < a < \dfrac{8}{\sqrt{3}R^2}$,

$$\frac{R - R\left(1 - \frac{1}{2\sqrt{1-a}} \frac{8}{\sqrt{3}R^2}\right)}{2} \leq r$$

$$\frac{2}{\sqrt{3}\sqrt{1-a}R} \leq r$$

Since $\dfrac{2}{\sqrt{3}\sqrt{1-a}R} \leq r$ can ensure complete blockage,

$$p \leq \frac{2}{\sqrt{3}\sqrt{1-a}R}$$

Since $R \geq 3$, we take

$$p \leq \frac{2}{\sqrt{3}\sqrt{1 - \frac{8}{\sqrt{3}3^2}}R}$$

$$p \leq \frac{1.65499}{R}$$

$\square$

Method II:

Theorem 14. *The range of the minimal radius p of the convex trees for complete blockage of visibility from the origin can be found by*

$$p \leq \frac{radius(C)}{R \min\{|A| : A \in boundary(C)\}}$$

Proof. Consider the inscribed circle of C with centre on the lattice point. When the visibility is completely blocked,

$$\text{radius}(fC) \geq p$$

When the inscribed circles are capable of complete blockage,

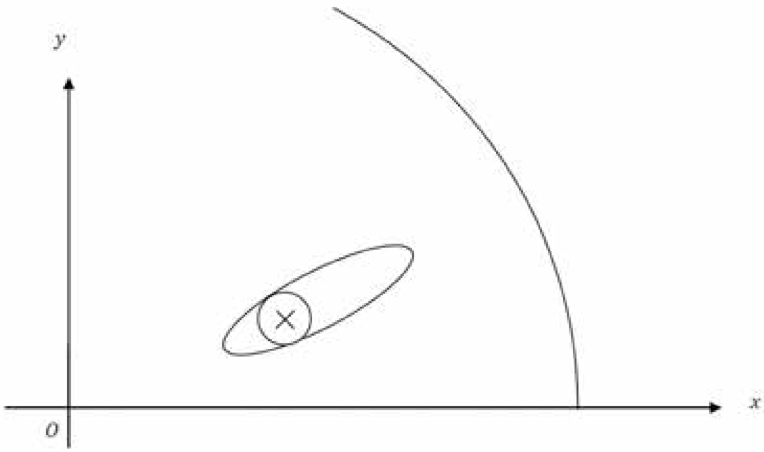
$$P_{\text{circular tree}} = \min\{|A| : A \in \text{boundary}(fC)\}$$

$$P_{\text{circular tree}} \frac{\text{radius}(fC)}{\min\{|A| : A \in \text{boundary}(fC)\}} \geq p$$

$$P_{\text{circular tree}} \frac{\text{radius}(C)}{\min\{|A| : A \in \text{boundary}(C)\}} \geq p$$

$$\because P_{\text{circular tree}} \leq \frac{1}{R}$$

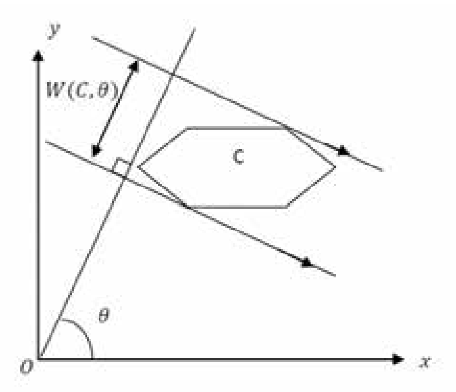
$$\therefore P_{\text{circular tree}} \geq \frac{\text{radius}(C)}{R \min\{|A| : A \in \text{boundary}(C)\}}$$



$\square$

Method III (Exact Value):

Definition 15. $W(C,\theta)$ *represents the breadth of a convex set C in angle θ.*



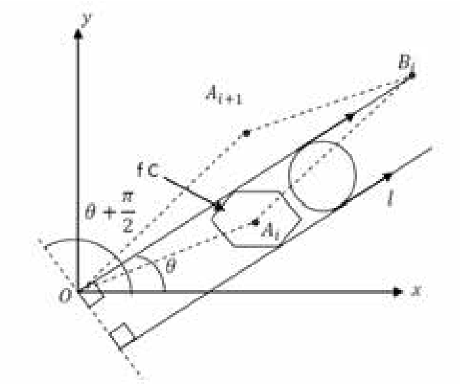
Theorem 16. *For all circular orchard D with radius R planted with convex trees C of central symmetry about the lattice point, the minimum value of radius($|fC|$) needed for complete blockage, namely p, can be calculated by*

$$p = \frac{2G radius(C)}{\min(OB_i \times W(C, \theta + \frac{\pi}{2}))}$$

where $B_i = A_i + A_{i+1}$ in which A_i and A_{i+1} are adjacent coprime lattice points of D, θ is the angle between OB_i and positive x-axis, and G is the unit grid area.

Proof.

Consider one of the parallelograms $OA_iA_{i+1}B_i$, construct line l so that $l//OB_i$ and l and OB_i are equidistant to A_i. For the tree C on A_i, when C is enlarged f times to touch OB_i, it also touch l since C is symmetrical. Since the distance between OB_i and l is $(fC, \theta + \frac{\pi}{2})$, fC is equivalent to a circular tree with $r = \dfrac{W(fC, \theta + \frac{\pi}{2})}{2}$ while determining the blockage of OB_i.
[See reviewer's comment (30)]



$$\frac{W(fC, \theta + \frac{\pi}{2})}{2} = \frac{G}{OB_i}$$

$$fW(fC, \theta + \frac{\pi}{2}) = \frac{2G}{OB_i}$$

$$f = \frac{2G}{OB_i \times W(C, \theta + \frac{\pi}{2})}$$

$$f\text{radius (C)} = \frac{2G \text{ radius (C)}}{OB_i \times W(C, \theta + \frac{\pi}{2})}$$

$$\max(f \text{ radius (C)}) = \max\left(\frac{2G \text{ radius (C)}}{OB_i \times W(fC, \theta + \frac{\pi}{2})}\right)$$

To ensure blockage of all B_i, take $p = \max(f \text{ radius (C)})$,

$$\therefore \quad p = \max\left(\frac{2G \text{ radius (C)}}{OB_i \times W(C, \theta + \frac{\pi}{2})}\right)$$

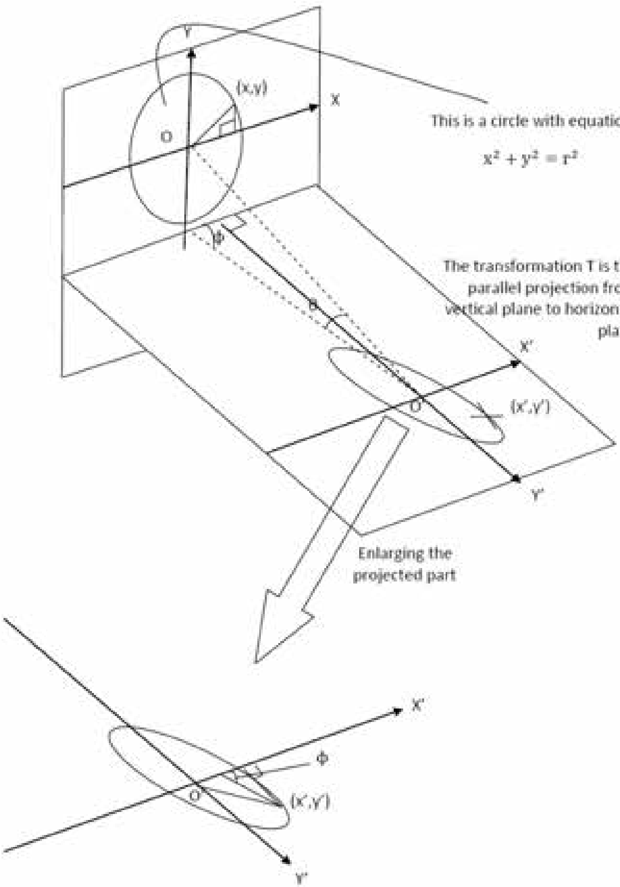
$$p = \frac{2G \text{ radius (C)}}{\min(OB_i \times W(C, \theta + \frac{\pi}{2}))}$$

□

4.2. Special cases

We will then try to obtain some simpler algorithms of exact p for some special cases.

Case I (Ellipses)



The above transformation T is a parallel projection from the vertical plane to the horizontal plane θ and ϕ are the angles specified in the figure.

Consider a circle C on the vertical plane with its centre on the origin with the x-axis parallel to the intersection between the vertical and the horizontal plane.

Assume that all the coordinates (x, y) and (x', y') are obtained from rectangular coordinate systems.

Let the equation of C be $x^2 + y^2 = r^2$.

Let $T(x, y) = (x', y')$

Then,

$$\begin{cases} x' = x + \dfrac{y}{\tan\theta}\cos\phi \\ \\ y' = \dfrac{y}{\tan\theta}\sin\phi \end{cases}$$

When $0^\circ < \phi < 90^\circ$, we have:

$$\begin{cases} x' = x - \dfrac{y'}{\tan\phi} \\ \\ y = \dfrac{\tan\theta}{\sin\phi}y' \end{cases}$$

Since $x^2 + y^2 = r^2$, we have

$$\left(x' - \frac{y'}{\tan\phi}\right)^2 + \left(\frac{\tan\theta}{\sin\phi}y'\right)^2 = r^2$$
$$x'^2 - \frac{2}{\tan\phi}x'y' + \left(\frac{1}{\tan^2\phi} + \frac{\tan^2\theta}{\sin^2\phi}\right)y'^2 = r^2$$

Let $a = 1, b = -\dfrac{2}{\tan\phi}, c = \dfrac{1}{\tan^2\phi} + \dfrac{\tan^2\theta}{\sin^2\phi}$ and $d = \sqrt{b^2 + (a-c)^2}$.

The above equation becomes:

$$ax^2 + bxy + cy^2 = r^2$$

$$\frac{a}{r^2}x^2 + \frac{b}{r^2}xy + \frac{c}{r^2}y^2 = 1 \qquad (\star)$$

Consider

$$\begin{aligned}\left(\frac{b}{r^2}\right)^2-4\left(\frac{a}{r^2}\right)\left(\frac{c}{r^2}\right)&=\frac{1}{r^2}\left(\frac{4}{\tan^2\phi}-4(1)\left(\frac{1}{\tan^2\phi}+\frac{\tan^2\theta}{\sin^2\phi}a\right)\right)\\&=-\frac{4\tan^2\theta}{r^2\sin^2\phi}\\&<0\ (\text{for }\theta\neq 0^\circ)\end{aligned}$$

By the theory of Linear Algebra for Quadratic Form, the graph of the transformed equation $(\star)$ is an ellipse. Let

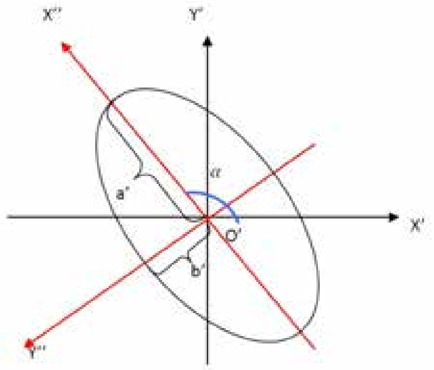
$$r^2\lambda_2=\frac{1}{2}(a+c+d)\text{ and }r^2\lambda_1=\frac{1}{2}(a+c-d)\tag{1}$$

We have $\lambda_2>\lambda_1$.

Also, by the theory of Linear Algebra for Quadratic Form, the equation of $(\star)$ can be transformed into

$$\lambda_1 x''^2+\lambda_2 y''^2=1\tag{$\star\star$}$$

Also, if we let α be the angle of the anti-clockwise rotation of the coordinate axes X' and Y' into the new coordinate system X' and Y' as the following figure: [See reviewer's comment (31)]



We have

$$\sin\alpha=\frac{b}{\sqrt{b^2+(a-c-d)^2}}\tag{2}$$

$$\text{and }\cos\alpha=\frac{a-c-d}{\sqrt{b^2+(a-c-d)^2}}\tag{3}$$

Now, we are going to show that if there is an ellipse on the horizontal $X'Y'$ plane with a' the longer semi-part, b' the shorter semi-part and α the angle between the longer semi-part and the X' axis in anti-clockwise direction, we can find a suitable parallel transformation that this ellipse is transformed into a circle. [See reviewer's comment (32)]

Firstly, by (1), we have

$$\lambda_2 + \lambda_1 = \frac{a}{r^2} + \frac{c}{r^2} \tag{4}$$

$$\lambda_2 - \lambda_1 = \frac{d}{r^2} \tag{5}$$

(4):

$$1 - c = 2 - r^2(\lambda_2 + \lambda_1)$$

(5):

$$r^2(\lambda_2 - \lambda_1) = \sqrt{b^2 + (1-c)^2} = \sqrt{b^2 + (2 - r^2(\lambda_2 + \lambda_1))^2}$$

$$r^4(\lambda_2 - \lambda_1)^2 - (2 - r^2(\lambda_2 + \lambda_1))^2 = b^2$$

$$4(1 - r^2\lambda_1)(r^2\lambda_2 - 1) = b^2$$

Also, by (5), we have

$$d = r^2(\lambda_2 - \lambda_1)$$

Then,

$$1 - c - d = 2 - r^2(\lambda_2 + \lambda_1) - r^2(\lambda_2 - \lambda_1)$$

$$1 - c - d = 2 - 2r^2\lambda_2 \tag{6}$$

By (2),

$$\sin^2\alpha = \frac{4(1 - r^2\lambda_1)(r^2\lambda_2 - 1)}{4(1 - r^2\lambda_1)(r^2\lambda_2 - 1) + 4(1 - r^2\lambda_2)^2}$$

$$\sin^2\alpha \times (r^2\lambda_2 - 1) \times [(1 - r^2\lambda_1) + (r^2\lambda_2 - 1)] = (1 - r^2\lambda_1)(r^2\lambda_2 - 1)$$

Then,

Case 1: $r^2\lambda_2 - 1 = 0$

By (6),

$$\begin{aligned} 1-c-d &= 0 \\ 1-c &= d \\ 1-c &= \sqrt{b^2+(1-c)^2} \\ b &= 0 \\ \phi &= 90^\circ \end{aligned}$$

And it contradicts to our assumption that $\phi < 90^\circ$.

Case 2: $r^2\lambda_2 - 1 \neq 0$

Then

$$\begin{aligned} \sin^2\alpha(r^2\lambda_2 - r^2\lambda_1) &= 1 - r^2\lambda_1 \\ r^2(\sin^2\alpha(\lambda_2-\lambda_1)+\lambda_2) &= 1 \\ r &= \frac{1}{\sqrt{\sin^2\alpha(\lambda_2-\lambda_1)+\lambda_1}} \end{aligned}$$

Since $\lambda_2 = \dfrac{1}{b'^2}, \lambda_1 = \dfrac{1}{a'^2}$

$$r = \frac{1}{\sqrt{\sin^2\alpha\left(\dfrac{1}{b'^2}-\dfrac{1}{a'^2}\right)+\dfrac{1}{a'^2}}}$$

Hence r can be obtained by a', b', α.

Also, by (4),

$$c = r^2\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right) - 1$$

$$\frac{1}{\tan^2\phi} + \frac{\tan^2\theta}{\sin^2\phi} = r^2\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right) - 1 \tag{7}$$

By (5),

$$d^2 = r^4(\frac{1}{b'^2}-\frac{1}{a'^2})^2$$

$$b^2 + \left(2 - r^2\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)^2\right)^2 = r^4\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)^2$$

$$b = -\sqrt{r^4\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)^2-\left(2-r^2\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)\right)^2}$$

$$\tan\phi = \frac{2}{\sqrt{r^4\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)^2-\left(2-r^2\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)\right)^2}}$$

Hence, ϕ can be obtained by a', b', α.

Also, by (7),

$$\tan^2\theta = \frac{r^2\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)-1-\frac{1}{\tan^2\phi}}{\frac{1}{\tan^2\phi}+1} \tag{8}$$

Hence, θ can be obtained by a', b', α. When $\phi = 90°$, $x = x'$ and $y = y'\tan\theta$.

$\therefore$ Clearly,

$$r = a' \tag{9}$$

$$\tan\theta = \frac{r}{b'} = \frac{a'}{b'} \tag{10}$$

And both r and θ can be found by a', b'.

If there are two ellipses E_1, E_2 on the $X'Y'$ plane with the same centre at the origin and they are similar, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$ for some k where a_1, a_2, b_1, b_2 are the semi-major axes and semi-minor axes of E_1, E_2 respectively.

By the inverse of the transformation T, let the circles C_1, C_2 be the images of the inverse of the transformation T from E_1, E_2 respectively and r_1, r_2 be the radii of C_1, C_2 respectively.

$\boxed{\text{When } 0° < \phi < 90°}$

By (4), (5), we have

$$r^2 = \frac{1}{2\lambda_2}(a+c+d)$$

Since, a, c, d are constants, then $r^2 = \frac{m}{\lambda_2}$ where m is a constant. We can have

$$r_1^2 = \frac{m}{\frac{1}{b_1^2}} = mb_1^2 \text{ and } r_2^2 = \frac{m}{\frac{1}{b_2^2}} = mb_2^2$$

$$\frac{r_1}{r_2} = \frac{b_1}{b_2} = k$$

Hence, when the ellipse is enlarged by a factor k, the corresponding circle formed by the inverse of the transformation T is also enlarged by the same factor k.

Clearly, when there is a lattice L' on horizontal $X'Y'$ plane with equal elliptical tree centred at each lattice point where a', b', α are the semi-major axis, semi-minor axis and the angle between the semi-major axis and X' axis in anti-clockwise direction, and D' is the orchard, there is also a lattice L on vertical XY plane with equal circular tree centred at each lattice point where r is the radius of the circular tree and the corresponding orchard D. These lattice L, radius of each tree, and orchard D can be obtained by the inverse of the transformation T where ϕ, θ, r can be obtained by (8), (9) and (10).

Since we know that for orchard D with equal circular tree centred at all lattice points $\in D$, when $r = \dfrac{G}{f}$ where G is the area of primitive parallelogram of lattice L and f is the distance between the closet coprime lattice point outside D and the origin of XY plane, the visibility of the region outside D is completely blocked from the origin. [See reviewer's comment (33)] Since straight line on XY plane is transformed to straight line again on $X'Y'$ plane by the transformation T, then in the corresponding orchard D' with equal elliptical tree centred at all lattice points $\in D'$, the visibility of the region outside D' is also completely blocked from the origin of $X'Y'$ plane.

By (4), (5), when

$$\text{the semi-minor axis of these elliptical tree} = \sqrt{\frac{2}{a+c+d}}\frac{G}{f},$$
$$\text{the semi-major axis of these elliptical tree} = \sqrt{\frac{2}{a+c-d}}\frac{G}{f},$$

the visibility of the region outside D' from the origin in the $X'Y'$ plane is completely blocked by these elliptical trees.

$$\therefore p = \sqrt{\frac{2}{a+c-d}}\frac{G}{f}$$

$\boxed{\text{When } \phi = 90^\circ}$

By (9),

$$\frac{r_1}{r_2} = \frac{a_1}{a_2} = k$$

Hence, when the ellipse is enlarged by a factor k, the corresponding circle formed by the inverse of the transformation T is also enlarged by the same factor k.

Since we know that for orchard D with equal circular tree centred at all lattice points $\in D$, when $r = \frac{G}{f}$ where G is the area of primitive parallelogram of lattice L and f is the distance between the closet coprime lattice point outside D and the origin of XY plane, the visibility of the region outside D is completely blocked from the origin. Since straight line on XY plane is transformed to straight line again on plane $X'Y'$ by the transformation T, then in the corresponding orchard D' with equal elliptical tree centred at all lattice points $\in D'$, the visibility of the region outside D' is also completely blocked from the origin of $X'Y'$ plane.

$$a' = \frac{G}{f}, \qquad b' = \frac{G}{f\tan\theta}$$

$$\therefore p = \max\{a', b'\}$$

$$p = \max\{\frac{G}{f}, \frac{G}{f\tan\theta}\}$$

Case II (Squares):

Theorem 17. *The minimal radius of a square tree with sides parallel to the axes in an orchard $D \subset \mathbb{R}^2$ is given by*

$$p = \frac{\sqrt{2}}{t'}$$

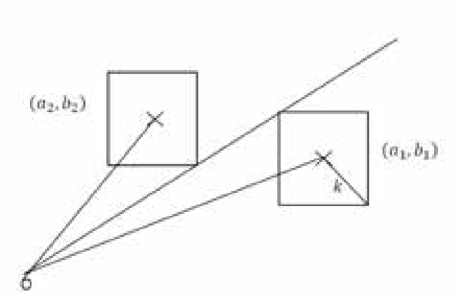
where $t' = \min\{|a_1|+|b_1|+|a_2|+|b_2|\}$ which is an integer. [See reviewer's comment (34)] *In particular, for circular orchard D with radius R,*

$$p = \frac{\sqrt{2}}{\lfloor\sqrt{R^2-1}\rfloor + 2}$$

Proof. Consider two adjacent coprime lattice points, namely (a_1, b_1) and (a_2, b_2), of which the former makes a smaller anti-clockwise angle with the positive x-axis.

Consider both points lying on the first quadrant.

Refer to the diagram.



Let k be the largest distance of the tree boundary from the centre.

The touch points are $\left(a_1 - \frac{k}{\sqrt{2}}, b_1 + \frac{k}{\sqrt{2}}\right)$ and $\left(a_2 + \frac{k}{\sqrt{2}}, b_2 - \frac{k}{\sqrt{2}}\right)$

Since these two points are collinear with the origin,

$$\frac{b_1 + \frac{k}{\sqrt{2}}}{a_1 - \frac{k}{\sqrt{2}}} = \frac{b_2 - \frac{k}{\sqrt{2}}}{a_2 + \frac{k}{\sqrt{2}}}$$

$$\therefore k = \frac{\sqrt{2}}{s}$$

where $s = a_1 + b_1 + a_2 + b_2$ and $a_1 b_2 - a_2 b_1 = 1$ since (a_1, b_1) and (a_2, b_2) are adjacent coprime lattice points. Then,

$$p_1 = \max(k) = \frac{\sqrt{2}}{t}$$

where $t = \min(a_1 + b_1 + a_2 + b_2)$.

Similarly, we get the following results:

1^{st} quadrant	$p_1 = \frac{\sqrt{2}}{t}$	$t = \min(a_1 + b_1 + a_2 + b_2)$
2^{nd} quadrant	$p_2 = \frac{\sqrt{2}}{t}$	$t = \min(-a_1 + b_1 - a_2 + b_2)$
3^{rd} quadrant	$p_3 = \frac{\sqrt{2}}{t}$	$t = \min(-a_1 - b_1 - a_2 - b_2)$
4^{th} quadrant	$p_4 = \frac{\sqrt{2}}{t}$	$t = \min(a_1 - b_1 + a_2 - b_2)$

Then, $p = \max(p_1, p_2, p_3, p_4)$.

By observation,

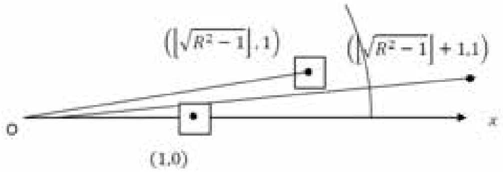
$$p = \frac{\sqrt{2}}{t'}$$

where $t' = \min\{|a_1| + |b_1| + |a_2| + |b_2|\}$ which is an integer.

Now we consider a circular orchard.

By Lemma 1 and Lemma 2, the vector sum is coprime and lies outside the orchard. And there are no coprime lattice points in quadrant 1 satisfying:

$$\begin{cases} x + y \leq \sqrt{R^2 - 1} + 1 \\ x^2 + y^2 > R^2 \end{cases}$$



$$\therefore t' > \sqrt{R^2 - 1} + 1$$

Consider the points and $(\lfloor\sqrt{R^2 - 1}\rfloor + 1, 1)$ and $(1, \lfloor\sqrt{R^2 - 1}\rfloor + 1)$.

They both have $|a_1|+|b_1|+|a_2|+|b_2| = \lfloor\sqrt{R^2-1}\rfloor+2$, which is the smallest integer larger than $\lfloor\sqrt{R^2-1}\rfloor+1$.

Hence, $t' = \lfloor\sqrt{R^2-1}\rfloor+2$. So we get

$$p = \frac{\sqrt{2}}{\lfloor\sqrt{R^2-1}\rfloor+2}$$

□

Case III (Equilateral triangles):

Theorem 18. *The minimal radius of an equilateral triangular tree is given by*

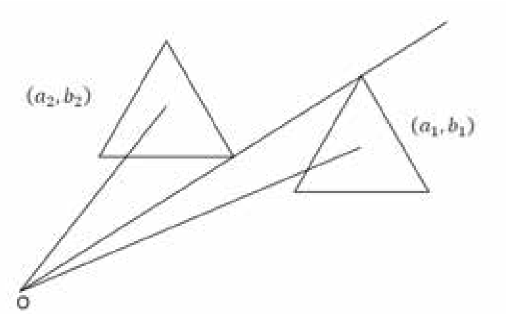
$$p = \max(p_1, p_2, p_3, p_4, p_5, p_6)$$

[See reviewer's comment (35)]

Proof. Consider two adjacent coprime lattice points, namely $M(a_1, b_1)$ and $N(a_2, b_2)$, of which the former makes a smaller anti-clockwise angle with the positive x-axis.

Let θ_1 and θ_2 be the anti-clockwise angle made from positive axis to M and N respectively.

Consider both angles lying between 0 and $\frac{\pi}{2}$



Refer to the diag.

Let k be the largest distance of the tree boundary from the centre.

The touch points are (a_1, b_1+k) and $(a_2+\frac{\sqrt{3}k}{2}, b_2-\frac{k}{2})$

Since the two points are collinear with the origin,

$$\frac{b_1+k}{a_1} = \frac{b_2-\frac{k}{2}}{a_2+\frac{\sqrt{3}k}{2}}$$

$$\therefore k = \frac{-s+\sqrt{8\sqrt{3}+s^2}}{2\sqrt{3}}$$

where $s = a_1+\sqrt{3}b_1+2a_2$.

Since k decreases with the increase of s, we get

$$p_1 = \frac{\sqrt{3}}{2}\max(k) = \frac{-t+\sqrt{8\sqrt{3}+t^2}}{4}$$

where $m = \min(a_1 + \sqrt{3}b_1 + 2a_2)$.

Similarly, we get the following results, calculation process is listed in the appendix:

$0 < \theta_1, \theta_2 < \frac{\pi}{3}$	$p_1 = \frac{-t+\sqrt{8\sqrt{3}+t^2}}{4}$	$t = \min(a_1 + \sqrt{3}b_1 + 2a_2)$
$\frac{\pi}{3} < \theta_1, \theta_2 < \frac{2\pi}{3}$	$p_2 = \frac{-t-\sqrt{-8\sqrt{3}+t^2}}{4}$	$t = \min(-a_1 - \sqrt{3}b_1 + a_2 - \sqrt{3}b_2)$
$\frac{2\pi}{3} < \theta_1, \theta_2 < \pi$	$p_3 = \frac{-t+\sqrt{8\sqrt{3}+t^2}}{4}$	$t = \min(-2a_1 - a_2 + \sqrt{3}b_2)$
$\pi < \theta_1, \theta_2 < \frac{4\pi}{3}$	$p_4 = \frac{-t-\sqrt{-8\sqrt{3}+t^2}}{4}$	$t = \min(2a_1 + a_2 + \sqrt{3}b_2)$
$\frac{4\pi}{3} < \theta_1, \theta_2 < \frac{5\pi}{3}$	$p_4 = \frac{-t+\sqrt{8\sqrt{3}+t^2}}{4}$	$t = \min(a_1 - \sqrt{3}b_1 - a_2 - \sqrt{3}b_2)$
$\frac{5\pi}{3} < \theta_1, \theta_2 < 2\pi$	$p_4 = \frac{-t-\sqrt{-8\sqrt{3}+t^2}}{4}$	$t = \min(-a_1 + \sqrt{3}b_1 - 2a_2)$

Then $p = \max(p_1, p_2, p_3, p_4, p_5, p_6)$.

Notes: If the two θ's lie on different regions, calculate both p and choose the smaller k.

$\square$

Case IV (Irregular triangles): [See reviewer's comment (36)]

Theorem 19. *The minimal radius of an irregular triangular tree is given by*

$$p = \frac{1}{2}\max\left\{\begin{matrix}\min\{f(x,y,i,j) : x, y \in \{v_1, v_2, v_3\}\} : \\ i, j \in \textit{adjacent comprime lattice points}\end{matrix}\right\}$$
$$\times \max\{|v_1 - v_2|, |v_2 - v_3|, |v_1 - v_3|\}$$

where v_1, v_2, and v_3 are the three vertices of the reference tree C with respect to the lattice point and $f(x,y,i,j)$ is the positive value of f so that

$$i + fx = k(j + fy)$$

[See reviewer's comment (37)]

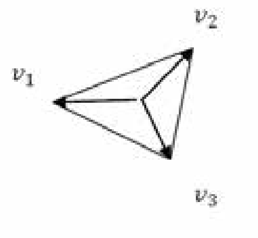
Proof.

For any two adjacent coprime lattice points, namely i and j, the enlargement factor for blockage between i and j is

$$m = \min\{f(x,y,i,j) : x, y \in \{v_1, v_2, v_3\}\}$$

where $f(x,y,i,j)$ is the positive value of f so that

$$i + fx = k(j + fy)$$

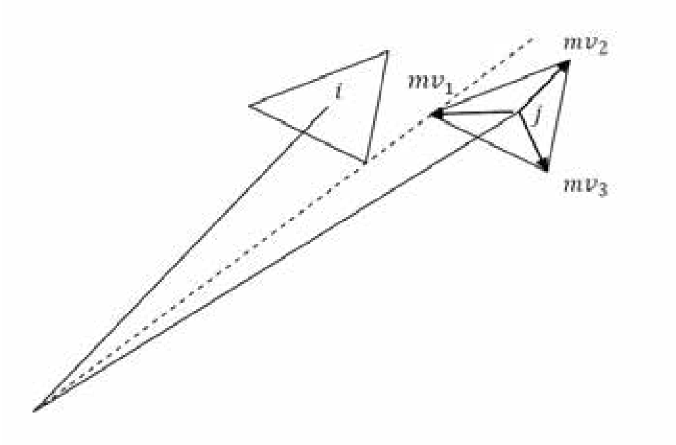


So for a more general case, the minimum enlargement factor is given by

$$f = \max\left\{\begin{matrix}\min\{f(x,y,i,j) : x,y \in \{v_1,v_2,v_3\}\} : \\ i,j \in \text{adjacent comprime lattice points}\end{matrix}\right\}$$

$$\because f = \frac{p}{\text{radius}(C)} \text{ and radius}(C) = \frac{1}{2}\max\{|v_1 - v_2|, |v_2 - v_3|, |v_3 - v_1|\}$$

$$\therefore p = \frac{1}{2}\max\left\{\begin{matrix}\min\{f(x,y,i,j) : x,y \in \{v_1,v_2,v_3\}\} : \\ i,j \in \text{adjacent comprime lattice points}\end{matrix}\right\} \times \max\{|v_1 - v_2|, |v_2 - v_3|, |v_1 - v_3|\}$$



□

Case V (Convex polygon):

Theorem 20. *The minimal radius of a convex polygon tree is given by*

$$p = \frac{1}{2}\max\left\{\begin{matrix}\min\{f(x,y,i,j) : x,y \in \{v_1,v_2,\ldots,v_n\}\} : \\ i,j \in \textit{adjacent comprime lattice points}\end{matrix}\right\} \times \max\{|v_a - v_b| : a,b \in \{1,2,\ldots,n\}\}$$

where $v_1, v_2, \ldots, v_n$ are the three vertices of the reference tree C with respect to the lattice point and $f(x,y,i,j)$ is the positive value of f so that

$$i + fx = k(j + fy)$$

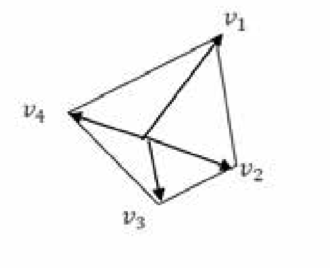
Proof.

For any two adjacent coprime lattice points, namely i and j, the enlargement factor for blockage between i and j is

$$m = \min\{f(x, y, i, j) : x, y \in \{v_1, v_2, \ldots, v_n\}\}$$

where $f(x, y, i, j)$ is the positive value of f so that

$$i + fx = k(j + fy)$$

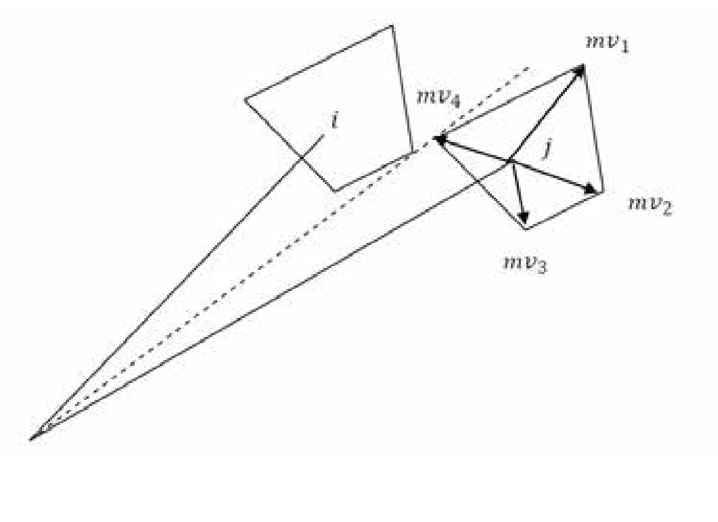


So for a more general case, the minimum enlargement factor is given by

$$f = \max\begin{Bmatrix} \min\{f(x, y, i, j) : x, y \in \{v_1, v_2, \ldots, v_n\}\} : \\ i, j \in \text{adjacent comprime lattice points} \end{Bmatrix}$$

$$\because f = \frac{p}{\text{radius}(C)} \text{ and } \text{radius}(C) = \frac{1}{2}\max\{|v_a - v_b| : a, b \in \{1, 2, \ldots, n\}\}$$

$$\therefore p = \frac{1}{2}\max\begin{Bmatrix} \min\{f(x, y, i, j) : x, y \in \{v_1, v_2, \ldots, v_n\}\} : \\ i, j \in \text{adjacent comprime lattice points} \end{Bmatrix}$$
$$\times \max\{|v_a - v_b| : a, b \in \{1, 2, \ldots, n\}\}$$



□

5. Sphere surface

We allocate the trees with the patterns of vertices of regular polyhedra.

There are 5 kinds of regular polyhedra:

1. Tetrahedron: 4 faces of equilateral triangle
2. Octahedron: 8 faces of equilateral triangle
3. Cube: 6 faces of square
4. Icosahedron: 20 faces of equilateral triangle
5. Dodecahedron: 12 faces of regular pentagon

For every regular polyhedron, there exists one circumsphere with radius R which can include all the vertices on its surface. Choose one of the vertices as the observer's position and plant trees (circles on the sphere surface) on all other vertices.

The problem concerns what is the minimum radius p (on the surface of the sphere) of trees needed to block the visibility of the observer from reaching the opposite point or the opposite tree if that is a vertex. We assume that light bends along the sphere's surface.

Due to its regular nature, choosing which vertex to be the observer's position will not affect the result.

Actually the problem becomes simpler if we group the vertices into rings: [See reviewer's comment (38)]

1. Tetrahedron: {1,3}
2. Octahedron: {1,4,1}
3. Cube: {1,3,3,1}
4. Icosahedron: {1,5,5,1}
5. Dodecahedron: {1,3,6,6,3,1}

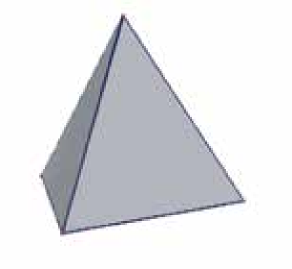
The numbers donate the number of vertices on the ring. The first "1" indicates the observation point. The target is the point opposite to the observation point on the sphere surface, which is the last "1" (if any). The blackened numbers are the rings responsible for blockage. Since spheres are symmetrical, a pair of blackened rings each on different hemisphere has the same radius but the vertices are placed alternatively. Let ϕ be the angle made by circumradii of 2 adjacent vertices of a ring.

Lemma 21. *The spherical distance d of two points P and Q on the surface of a sphere with center C and radius R can be given by $d = R\angle PCQ$.*

Proof. By the definition of radian. $\square$

Lemma 22. *Spherical law of cosines: $\cos c = \cos a \cos b + \sin a \sin b \cos C$ where a, b and c are 3 sides of any triangle on a unit sphere's surface and C is the opposite angle of side c.*

1. Tetrahedron $\{1, 3\}$



Theorem 23. *When trees are located at each vertex of a tetrahedron on a sphere surface except the observation point, we can get the following result:*

The minimal radius of the trees is given by

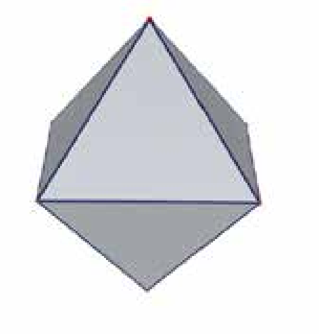
$$p = \frac{R\cot^{-1} 2\sqrt{2}}{2}$$

Proof. For tetrahedrons, the of the 2^{nd} ring is $\cot^{-1} 2\sqrt{2}$ [2]. By Lemma 21, the spherical distance of points on 2^{nd} ring is $R\cot^{-1} 2\sqrt{2}$. Since the lattice points form regular triangle and every light ray passing through the ring will pass through one of the spherical path, we only need to consider one of the spherical path. Moreover, the tangent of the mid-point of spherical distance is perpendicular to the light ray from the observer so p is half of the spherical distance between two consecutive vertices on the ring.

$$\therefore p = \frac{R\cot^{-1} 2\sqrt{2}}{2}$$

This is the minimal radius so that no light can reach the target point. □

2. Octahedron $\{1, 4, 1\}$



Theorem 24. *When trees are located at each vertex of an octahedron on a sphere surface except the observation point, we can get the following result:*

The minimal radius of the trees is given by

$$p = \frac{\pi R}{4}$$

Proof. For octahedron's 2^{nd} ring, $\phi = \frac{\pi}{2}$. By Theorem 1, the spherical distance of points on 2^{nd} ring is $\frac{\pi R}{2}$.

$$\therefore p = \frac{\pi R}{4}$$

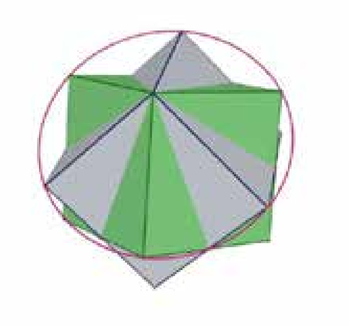
This is the minimal radius so that no light can reach the target point. □

3. Cube $\{1,3,3,1\}$

Theorem 25. *When trees are located at each vertex of a cube on a sphere surface except the observation point, we can get the following result:*

The minimal radius of the trees is given by

$$p \approx 0.49088R$$

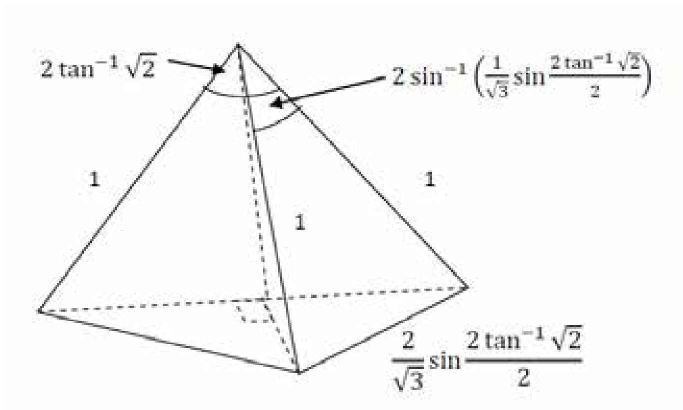


Proof. ϕ of cube's 2^{nd} and 3^{rd} ring $= 2\tan^{-1}\left(\frac{\sqrt{2}}{2} \div \frac{1}{2}\right) = 2\tan^{-1}\sqrt{2}$. Since the two rings are of same radius and their vertices are alternately located, we can reflect the 3^{rd} ring along the equator to form a combined ring with 6 points. Considering a tetrahedron, ϕ of the combined ring is

$$2\sin^{-1}\left(\frac{1}{\sqrt{3}}\sin\frac{2\tan^{-1}\sqrt{2}}{2}\right) \approx 0.98177$$

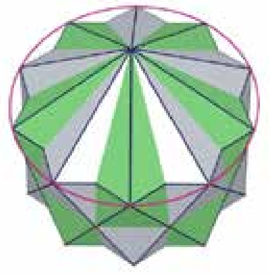
$$\therefore p \approx 0.49088R$$

This is the minimal radius so that no light can reach the target point. Note that it will be incorrect if we just divide the ϕ of an uncombined ring by 4.



$\square$

4. Icosahedron $\{1, 5, 5, 1\}$



Theorem 26. *When trees are located at each vertex of an icosahedron on a sphere surface except the observation point, we can get the following result: The minimal radius of the trees is given by*

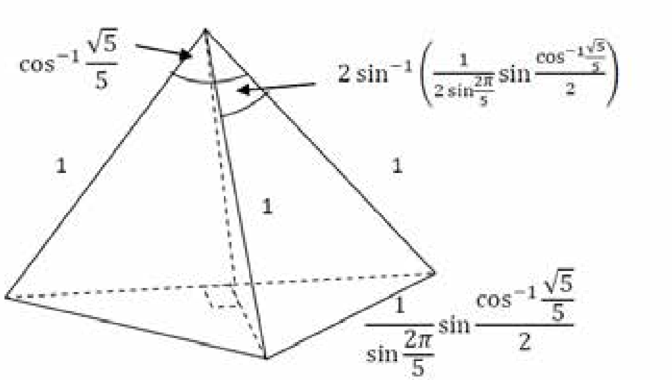
$$p \approx 0.28004R$$

Proof. Since the dihedral angle of dodecahedron $= \cos^{-1}\left(-\frac{\sqrt{5}}{5}\right)$ [3] and the dual polyhedron of a dodecahedron is an icosahedron [3], ϕ of icosahedron's 2^{nd} and 3^{rd} ring $= \pi - \cos^{-1}\left(-\frac{\sqrt{5}}{5}\right) = \cos^{-1}\left(\frac{\sqrt{5}}{5}\right)$. Since the two rings are of same radius and their vertices are alternately located, we can reflect the 3^{rd} ring along the equator to form a combined ring with 10 points. Considering a tetrahedron, ϕ of the combined ring is

$$2\sin^{-1}\left(\frac{1}{2\sin\frac{2\pi}{5}}\sin\frac{\cos^{-1}\frac{\sqrt{5}}{5}}{2}\right) \approx 0.56008$$

$$\therefore p \approx 0.28004R$$

This is the minimal radius so that no light can reach the target point.



□

5. Dodecahedron $\{1, 3, 6, 6, 3, 1\}$

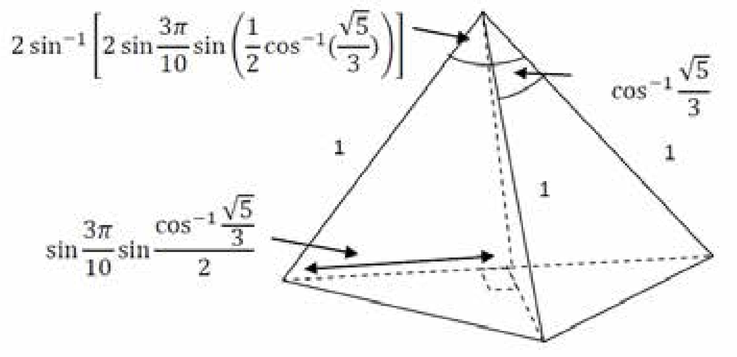
Theorem 27. *When trees are located at each vertex of a dodecahedron on a sphere surface except the observation point, we can get the following result:*

The minimal radius of the trees is given by

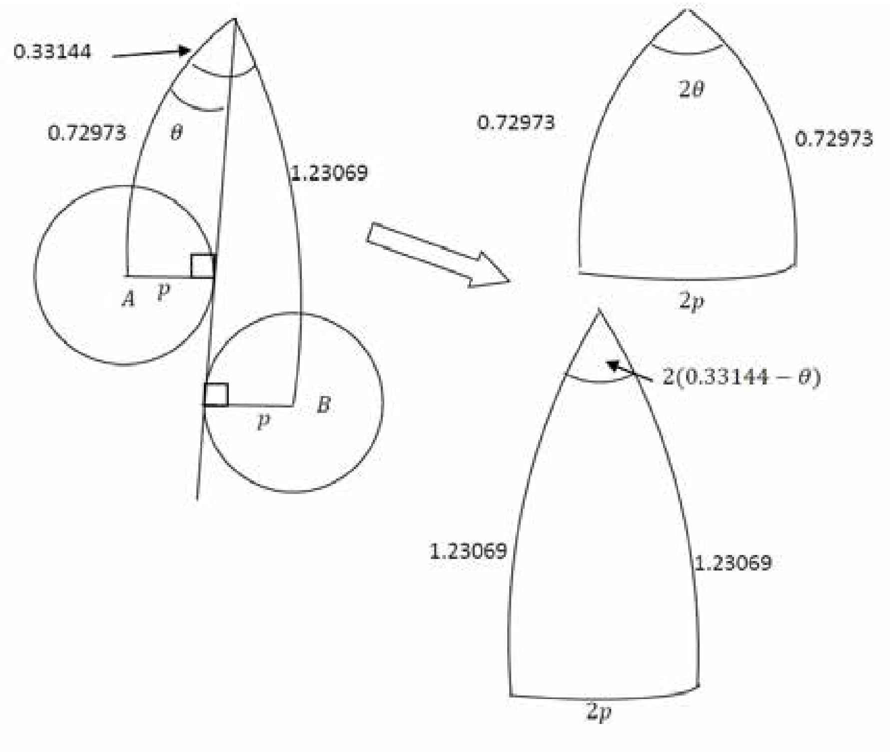
$$p \approx 0.12915R$$

Proof. Since the dihedral angle of icosahedrons $= \cos^{-1}\left(-\frac{\sqrt{5}}{3}\right)$ [4] and the dual polyhedron of an icosahedron is a dodecahedron [4], the angle of two nearest circumradii of a dodecahedron's vertices $= \pi - \cos^{-1}\left(-\frac{\sqrt{5}}{3}\right) = \cos^{-1}\left(\frac{\sqrt{5}}{3}\right) \approx 0.72973$. Considering a tetrahedron with its apex on the circumcentre,

$$\begin{aligned} &\text{the angle between two vertices on either side of a pentagon's diagonal} \\ = \ & 2\sin^{-1}\left[2\sin\frac{3\pi}{10}\sin\left(\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right)\right] \approx 1.23096. \end{aligned}$$



Reflect the 4^{th} and 5^{th} ring along the equator to combine with the 3^{rd} and 2^{nd} ring respectively. By observation, the largest gap is AB. We know that ≈ 0.72973, $OB \approx 1.23096$ and $\angle AOB$ on sphere ≈ 0.33144 through 3D model building on Cabri 3D.



By spherical laws of cosines, solving

$$\cos^2 0.72973 + \sin^2 0.72973 \cos 2\theta = \cos^2 1.23069 + \sin^2 1.23096 \cos 2(0.33144 - \theta),$$

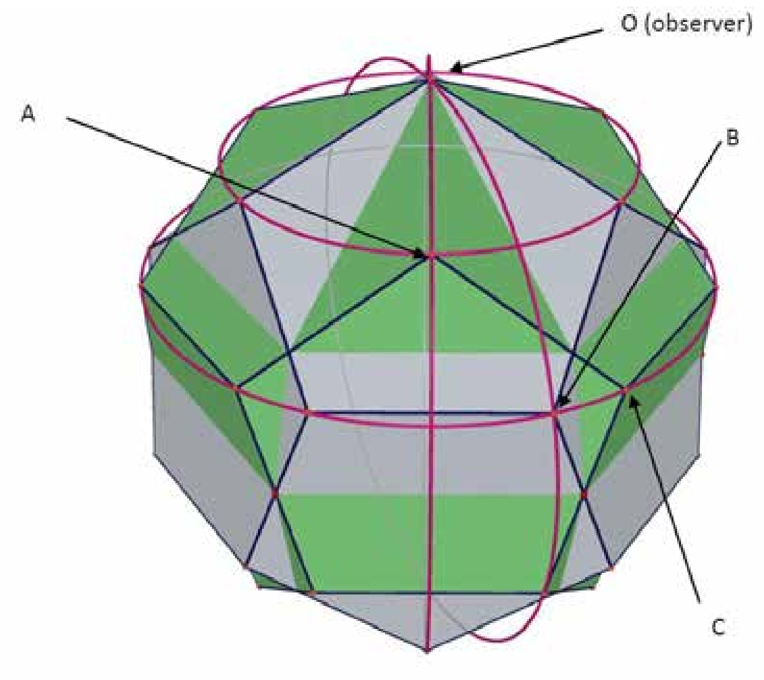
We get $\theta = 0.19441$.

So dividing $\angle AOB$ into two angles of 0.19441 and 0.13703 can achieve minimal radius.

$$p \approx \frac{R}{2} \cos^{-1}(\cos^2 0.72973 + \sin^2 0.72973 \cos 2 \times 0.19441)$$

$$\therefore p \approx 0.12915R$$

which is indeed larger than $0.12227R$, the p given by gap BC through Cabri 3D. So this is the minimal radius so that no light can reach the target point.


O (observer)
A
B
C

□

6. A glimpse of 3-D orchard problem

Problem:

Given a spherical orchard, what is the minimum radius of spherical trees p so that the visibility from the centre (the origin) of the sphere is blocked?

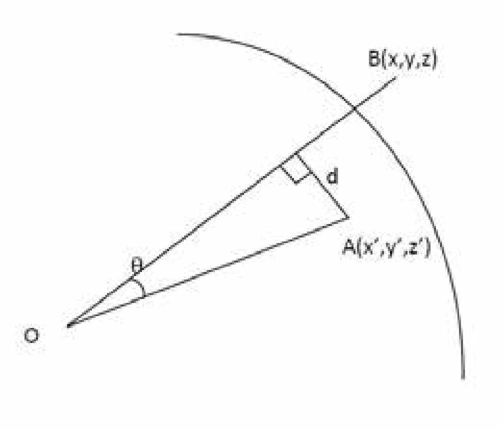
Note:

The lattice used here is obtained by the common Cartesian coordinate system.

Theorem 28. *Let (x, y, z) be the closest coprime lattice point beyond the spherical orchard.*

$$p > \frac{1}{\sqrt{x^2+y^2+z^2}}, \quad i.e. \quad p > \frac{1}{d}$$

Proof. Refer to the diagram below.



In general, for any point (x', y', z') which is not collinear with (x, y, z), its distance d from the line joining the centre and (x, y, z) can be given by:

$$d = |\overrightarrow{OA}| \sin\theta$$

$$d^2 = (x'^2 + y'^2 + z'^2)(1 - \cos^2\theta)$$

$$d^2 = (x'^2 + y'^2 + z'^2)\left[1 - \left[\frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}\right)^2\right]$$

$$d^2 = (x'^2 + y'^2 + z'^2)\left\{1 - \left[\frac{xx' + yy' + zz'}{(x'^2 + y'^2 + z'^2)(x^2 + y^2 + z^2)}\right]\right\}$$

$$d^2 = \frac{(x'^2 + y'^2 + z'^2)(x^2 + y^2 + z^2) - (xx' + yy' + zz')^2}{(x^2 + y^2 + z^2)}$$

$$d^2 = \frac{(z'y - y'z)^2 + (x'z - z'x)^2 + (y'x - x'y)^2}{x^2 + y^2 + z^2}$$

Since x, y, z, x', y' and z' are all integers,

$$d^2 \geq \frac{1}{x^2 + y^2 + z^2}$$

Now, we will show that the case where

$$d^2 = \frac{1}{x^2 + y^2 + z^2}$$

will lead to a contradiction and must be rejected.

For $d^2 = \dfrac{1}{x^2 + y^2 + z^2}$, either $z'y - y'z = 1$ or $x'z - z'x = 1$ or $y'x - x'y = 1$ while the other two are zero.

Without loss of generality, we let $z'y - y'z = 1$. Then $x'z - z'x = 0$ and $y'x - x'y = 0$.

$$\implies \frac{x}{x'} = \frac{z}{z'} \quad \text{and} \quad \frac{x}{x'} = \frac{y}{y'}$$

But that would mean that $\dfrac{y}{y'} = \dfrac{z}{z'}$.

$$\implies z'y - y'z = 0$$

which contradicts to our assumption that $z'y - y'z = 1$. Therefore,

$$d^2 > \frac{1}{x^2 + y^2 + z^2}$$

$$d > \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$p > \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

□

7. Conclusion

[See reviewer's comment (39)]

In this paper, we have developed several ways to approach the value of p and other properties in some of the diversified branches of the problem. We will present these results in a table:

Aspect	**Results**
Circular tree:	
Square grid	$p = \frac{1}{d}$
Parallelogram grid	$p = \frac{G}{d}$
Removability of trees	Removable iff non-coprime
Visible area	Visible Area $\leq \frac{n(P)}{2} - \frac{1}{2}(n(P) - 2)\pi r^2$
Hexagonal Grid	$p = \frac{\sqrt{2\sqrt{3}A}}{6}$
Convex trees:	
Method 1	$p \leq \frac{12}{R \max(\delta(C, \Lambda))}$
Method 2	$p \leq \frac{\text{radius}(C)}{R \min\{\lvert A\rvert : A \in \text{boundary } (C)\}}$
Method 3	$p = \frac{2G \text{ radius}(C)}{\min\{OB_i \times W(C, \theta + \frac{\pi}{2})\}}$
Ellipse trees	$\begin{cases} p = \sqrt{\frac{2}{a+c-d}}\frac{G}{f} & \text{when } \phi \neq \frac{\pi}{2} \\ p = \max\{\frac{G}{f}, \frac{G}{f\tan\theta}\} & \text{when } \phi = \frac{\pi}{2} \end{cases}$
Square trees	$p = \frac{\sqrt{2}}{\lfloor\sqrt{R^2 - 1}\rfloor + 2}$
Equilateral triangular trees	$p = \max\{p_1, p_2, p_3, p_4, p_5, p_6\}$
Irregular triangular trees	$p = \frac{1}{2}\max\{\min\{f(x, y, i, j) : x, y \in \{v_1, v_2, v_3\}\} :$ $i, j \in$ adjacent comprime lattice points$\}$ $\times \max\{\lvert v_1 - v_2\rvert, \lvert v_2 - v_3\rvert, \lvert v_1 - v_3\rvert\}$

Convex polygon trees	$p = \frac{1}{2}\max\{\min\{f(x,y,i,j) : x,y \in \{v_1,v_2,\dots,v_n\}\} :$ $i,j \in$ adjacent comprime lattice points$\}$ $\times \max\{\lvert v_a - v_b\rvert : a,b \in \{1,2,\dots,n\}\}$
Sphere surface:	
Tetrahedron lattice	$p = \dfrac{R\cot^{-1} 2\sqrt{2}}{2}$
Octahedron lattice	$p = \dfrac{\pi R}{4}$
Cube lattice	$p \approx 0.49088R$
Icosahedron lattice	$p \approx 0.28004R$
Dodecahedron lattice	$p \approx 0.12915R$
3-D orchard	$p > \dfrac{1}{d}$

Although we have investigated a few branches, there are still a lot of mysteries behind the problem to be unveiled. The following are some possible direction for future research:

1. Exact value for 3-D orchard
2. When trees are placed on irregular lattice
3. Light-reflecting trees (Pach's enchanted forest problem)

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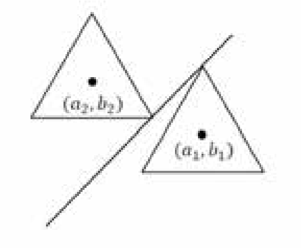
8. Appendix

Calculations of triangular p

1. $0 < \theta_1, \theta_2 < \dfrac{\pi}{3}$:



Touch points:

$(a_1, b_1 + k)$ and $\left(a_2 + \dfrac{\sqrt{3}}{2}k, b_2 - \dfrac{k}{2}\right)$

$$\frac{b_1 + k}{a_1} = \frac{b_2 - \dfrac{k}{2}}{a_2 + \dfrac{\sqrt{3}}{2}k}$$

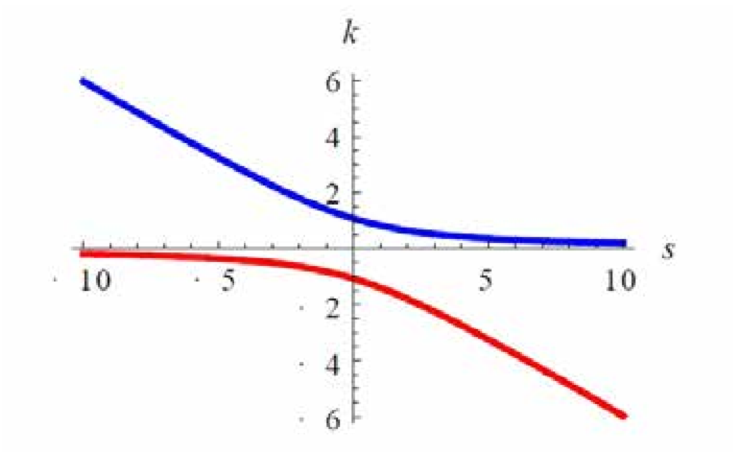
$$a_2 b_1 + \left(a_2 + \frac{\sqrt{3}}{2}b_1\right)k + \frac{\sqrt{3}}{2}k^2 = a_1 b_2 - \frac{a_1}{2}k$$

$$\frac{\sqrt{3}}{2}k^2 + \left(\frac{a_1}{2} + \frac{\sqrt{3}}{2}b_1 + a_2\right)k - 1 = 0$$

Let $a_1 + \sqrt{3}b_1 + 2a_2$ be s

$$\sqrt{3}k^2 + sk - 2 = 0$$

$$k = \frac{-s \pm \sqrt{s^2 + 8\sqrt{3}}}{2\sqrt{3}}$$



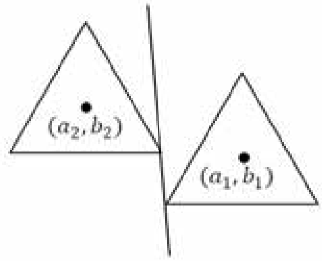
Refer to the diagram, since $k > 0$

$$k = \frac{-s + \sqrt{s^2 + 8\sqrt{3}}}{2\sqrt{3}}$$

$$p_1 = \frac{\sqrt{3}}{2}\max(k) = \frac{-t + \sqrt{t^2 + 8\sqrt{3}}}{4}$$

where $t = \min(a_1 + \sqrt{3}b_1 + 2a_2)$.

2. $\frac{\pi}{3} < \theta_1, \theta_2 < \frac{2\pi}{3}$:



Touch points:

$(a_1 - \frac{\sqrt{3}}{2}k, b_1 - \frac{k}{2})$ and $\left(a_2 + \frac{\sqrt{3}}{2}k, b_2 - \frac{k}{2}\right)$

$$\frac{b_1 - \frac{k}{2}}{a_1 - \frac{\sqrt{3}}{2}k} = \frac{b_2 - \frac{k}{2}}{a_2 + \frac{\sqrt{3}}{2}k}$$

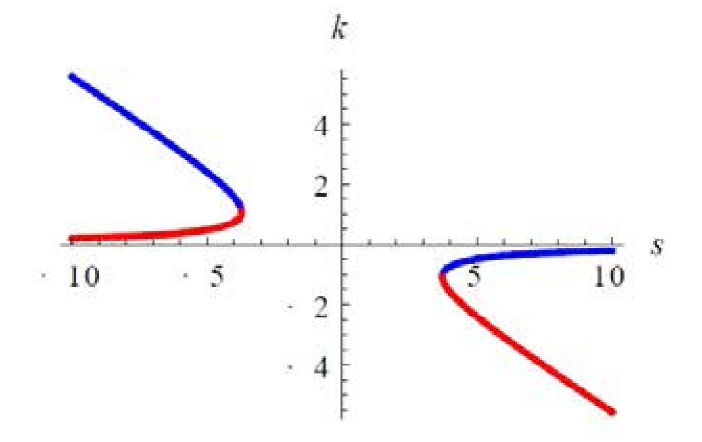
$$a_2b_1 - \left(\frac{1}{2}a_2 - \frac{\sqrt{3}}{2}b_1\right)k - \frac{\sqrt{3}}{4}k^2 = a_1b_2 + \left(-\frac{1}{2}a_1 - \frac{\sqrt{3}}{2}b_2\right)k + \frac{\sqrt{3}}{4}k^2$$

$$\frac{\sqrt{3}}{2}k^2 + \left(-\frac{1}{2}a_1 - \frac{\sqrt{3}}{2}b_1 + \frac{1}{2}a_2 - \frac{\sqrt{3}}{2}b_2\right)k + 1 = 0$$

Let $-a_1 - \sqrt{3}b_1 + a_2 - \sqrt{3}b_2$ be s

$$\sqrt{3}k^2 + sk + 2 = 0$$

$$k = \frac{-s \pm \sqrt{s^2 - 8\sqrt{3}}}{2\sqrt{3}}$$



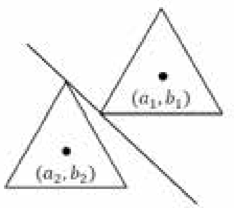
Refer to the diagram, since $k > 0$

$$k = \frac{-s - \sqrt{s^2 - 8\sqrt{3}}}{2\sqrt{3}}$$

$$p_2 = \frac{\sqrt{3}}{2}\max(k) = \frac{-t - \sqrt{t^2 - 8\sqrt{3}}}{4},$$

where $t = \max(-a_1 - \sqrt{3}b_1 + a_2 - \sqrt{3}b_2)$.

3. $\frac{2\pi}{3} < \theta_1, \theta_2 < \pi$:



Touch points:

$\left(a_1 - \frac{\sqrt{3}}{2}k, b_1 - \frac{k}{2}\right)$ and $(a_2, b_2 + k)$

$$\frac{b_1 - \frac{k}{2}}{a_1 - \frac{\sqrt{3}}{2}k} = \frac{b_2 + k}{a_2}$$

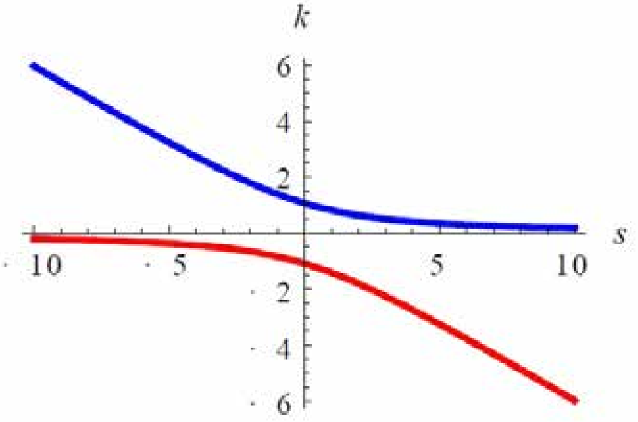
$$a_2 b_1 - \frac{a_2}{2}k = a_1 b_2 + \left(a_1 - \frac{\sqrt{3}}{2}b_2\right)k - \frac{\sqrt{3}}{2}k^2$$

$$\frac{\sqrt{3}}{2}k^2 + \left(-a_1 - \frac{a_2}{2} + \frac{\sqrt{3}}{2}b_2\right)k - 1 = 0$$

Let $-2a_1 - a_2 + \sqrt{3}b_2$ be s

$$\sqrt{3}k^2 + sk - 2 = 0$$

$$k = \frac{-s \pm \sqrt{s^2 + 8\sqrt{3}}}{2\sqrt{3}}$$



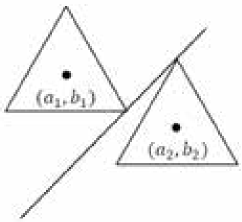
Refer to the diagram, since $k > 0$

$$k = \frac{-s + \sqrt{s^2 + 8\sqrt{3}}}{2\sqrt{3}}$$

$$p_3 = \frac{\sqrt{3}}{2} \max(k) = \frac{-t + \sqrt{t^2 + 8\sqrt{3}}}{4},$$

where $t = \min(-2a_1 - a_2 + \sqrt{3}b_2)$.

4. $\pi < \theta_1, \theta_2 < \dfrac{4\pi}{3}$:



Touch points:

$\left(a_1 + \dfrac{\sqrt{3}}{2}k, b_1 - \dfrac{k}{2}\right)$ and $(a_2, b_2 + k)$

$$\frac{b_1 - \dfrac{k}{2}}{a_1 + \dfrac{\sqrt{3}}{2}k} = \frac{b_2 + k}{a_2}$$

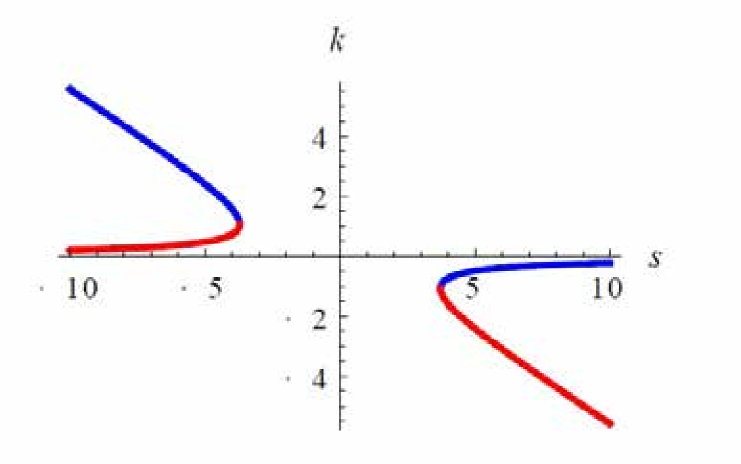
$$a_2 b_1 - \frac{1}{2}a_2 k = a_1 b_2 + \left(a_1 + \frac{\sqrt{3}}{2}b_2\right)k + \frac{\sqrt{3}}{2}k^2$$

$$\frac{\sqrt{3}}{2}k^2 + \left(a_1 + \frac{1}{2}a_2 + \frac{\sqrt{3}}{2}b_2\right)k + 1 = 0$$

Let $2a_1 + a_2 + \sqrt{3}b_2$ be s

$$\sqrt{3}k^2 + sk + 2 = 0$$

$$k = \frac{-s \pm \sqrt{s^2 - 8\sqrt{3}}}{2\sqrt{3}}$$



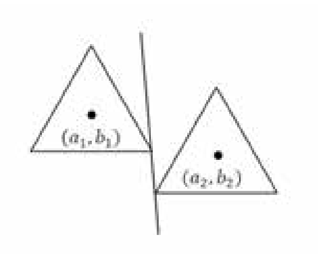
Refer to the diagram, since $k > 0$

$$k = \frac{-s - \sqrt{s^2 - 8\sqrt{3}}}{2\sqrt{3}}$$

$$p_4 = \frac{\sqrt{3}}{2}\max(k) = \frac{-t - \sqrt{t^2 - 8\sqrt{3}}}{4},$$

where $t = \max(2a_1 + a_2 + \sqrt{3}b_2)$.

5. $\dfrac{4\pi}{3} < \theta_1, \theta_2 < \dfrac{5\pi}{3}$:



Touch points:

$\left(a_1 + \dfrac{\sqrt{3}}{2}k, b_1 - \dfrac{k}{2}\right)$ and $\left(a_2 - \dfrac{\sqrt{3}}{2}k, b_2 - \dfrac{k}{2}\right)$

$$\frac{b_1 - \frac{k}{2}}{a_1 + \frac{\sqrt{3}}{2}k} = \frac{b_2 - \frac{k}{2}}{a_2 - \frac{\sqrt{3}}{2}k}$$

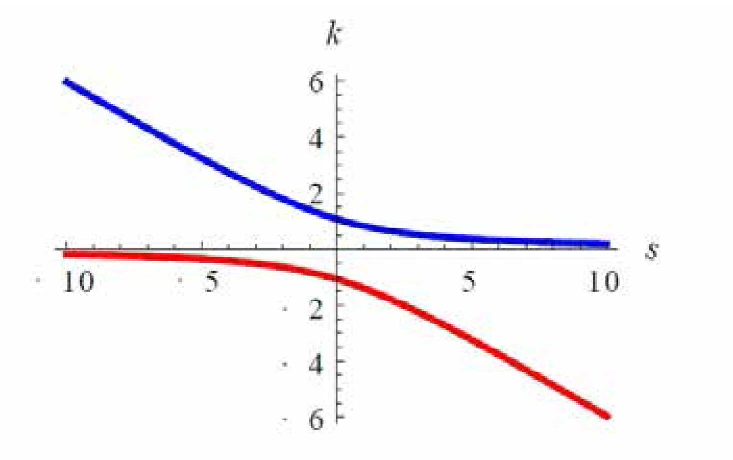
$$a_2 b_1 - \left(\frac{a_2}{2} + \frac{\sqrt{3}}{2}b_1\right)k + \frac{\sqrt{3}}{4}k^2 = a_1 b_2 - \left(\frac{a_1}{2} - \frac{\sqrt{3}}{2}b_2\right)k - \frac{\sqrt{3}}{4}k^2$$

$$\frac{\sqrt{3}}{2}k^2 + \left(\frac{a_1}{2} - \frac{\sqrt{3}}{2}b_1 - \frac{a_2}{2} - \frac{\sqrt{3}}{2}b_2\right)k - 1 = 0$$

Let $a_1 - \sqrt{3}b_1 - a_2 - \sqrt{3}b_2$ be s

$$\sqrt{3}k^2 + sk - 2 = 0$$

$$k = \frac{-s \pm \sqrt{s^2 + 8\sqrt{3}}}{2\sqrt{3}}$$



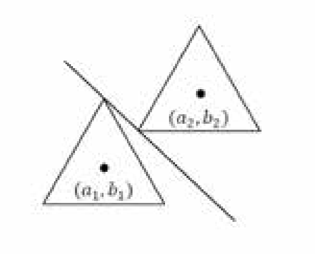
Refer to the diagram, since $k > 0$

$$k = \frac{-s + \sqrt{s^2 + 8\sqrt{3}}}{2\sqrt{3}}$$

$$p_5 = \frac{\sqrt{3}}{2}\max(k) = \frac{-t + \sqrt{t^2 + 8\sqrt{3}}}{4},$$

where $t = \min(a_1 - \sqrt{3}b_1 - a_2 - \sqrt{3}b_2)$.

5. $\dfrac{5\pi}{3} < \theta_1, \theta_2 < 2\pi$:



Touch points:

$(a_1, b_1 + k)$ and $\left(a_2 - \dfrac{\sqrt{3}}{2}k, b_2 - \dfrac{k}{2}\right)$

$$\frac{b_1 + k}{a_1} = \frac{b_2 - \dfrac{k}{2}}{a_2 - \dfrac{\sqrt{3}}{2}k}$$

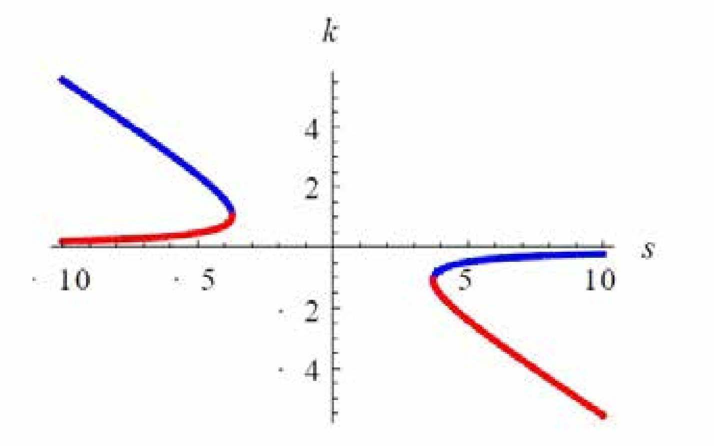
$$a_2b_1 + \left(a_2 - \frac{\sqrt{3}}{2}b_1\right)k - \frac{\sqrt{3}}{2}k^2 = a_1b_2 - \frac{a_1}{2}k$$

$$\frac{\sqrt{3}}{2}k^2 + \left(-\frac{1}{2}a_1 + \frac{\sqrt{3}}{2}b_1 - a_2\right)k + 1 = 0$$

Let $-a_1 + \sqrt{3}b_1 - 2a_2$ be s

$$\sqrt{3}k^2 + sk + 2 = 0$$

$$k = \frac{-s \pm \sqrt{s^2 - 8\sqrt{3}}}{2\sqrt{3}}$$



Refer to the diagram, since $k > 0$

$$k = \frac{-s - \sqrt{s^2 - 8\sqrt{3}}}{2\sqrt{3}}$$

$$p_6 = \frac{\sqrt{3}}{2}\max(k) = \frac{-t - \sqrt{t^2 - 8\sqrt{3}}}{4},$$

where $t = \max(-a_1 + \sqrt{3}b_1 - 2a_2)$.

Reviewer's Comments

Suppose that trees in circular shape of the same size are planted on lattice points of a square grid inside a circular orchard. The orchard visibility problem, which dates back to 1918 when G. Polya posed it and is the subject of the paper under review, asks for the minimum size of each tree required for the complete blockage of the view of the boundary of the orchard if one stands at the center of the orchard. In this paper, the authors discuss this problem and a number of its generalizations, and in each of them give (the range of) the minimum size of the trees required to ensure the complete blockage of the view of the boundary of the orchard. Their results are summarized in Section 7: Conclusion. The aspects of the original problem they generalize are

1. the arrangement of the trees: apart from the square grid case, they also consider parallelogram and hexagonal grids,
2. the shape of the trees: apart from circular trees, trees which are convex and in particular of simple geometric shapes (e.g. ellipses, squares and triangles) are considered,
3. the surface on which the orchard lies: in the original problem, the orchard is assumed to be planar. The authors consider the case where the orchard is a sphere and circular trees are planted on the vertices of a regular polytope the sphere circumscribes. Now the task becomes finding the minimum radius of the trees required to ensure that if one stands on one of the vertices, he/she cannot see the diametrically opposite point, assuming that light travels along the spherical surface along great circles.
4. the dimension of the orchard: they consider an orchard in the shape of a ball, which is three-dimensional, and trees also in the shape of a ball planted on lattice points of a cubical grid inside the orchard.

The authors are able to define their problems in precise mathematical language, order the topics they discuss in the paper with increasing generality, and conclude their paper with a table showcasing their results for the easy reference of the readers. However, the clarity of the exposition leaves much to be desired. Very often they start each section outright with a bunch of unmotivated lemmata/theorems without an introductory paragraph explaining what they will do in the section or an outline summarizing the strategy of their proofs. Connecting paragraphs between lemmata/theorems are virtually non-existent, and this often leaves me head-scratching as to what the authors want to do next. Moreover, many proofs appear to be skeletal: apart from a series of equations and formulae, no heuristic explanations in words or pictures are offered, making it hard for the readers to decipher what is going on (an example of this is the proof of Lemma 11). Occasionally, there are notation and terminology which are neither explained nor defined. In the following more specific problems on clarity, typos and grammar are listed.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. There are no suitable punctuations (e.g. colon or period) at the end of each line of equations and formulae.
3. Change '...radius r on point...' to '...'radius r centered at the point...'. Also, it is customary to use (x, y) to denote variable points rather than a specific point.
4. The sentence should read 'A set of points D is convex if...'.
5. The notation $(x_1, x_2, \cdots, x_n)$ is confusing. It usually means the Cartesian coordinates. However, it seems to the reviewer that the authors mean the coordinates are with respect to a general basis, not necessarily the standard orthonormal basis. The confusion actually persists later in sections dealing with both square grids and parallelogram grids.
6. 'For a set...' should read 'For the set...'. It is a good idea to illustrate Definition 19 (orchard visibility problem) with a picture.
7. The reviewer suggests that the authors explain why Definition 20 is the mathematical statement of the orchard visibility problem. They should at least point out that they are using the fact that light travels along straight lines, which is relevant to the mathematical statement.
8. Delete '$\gcd(x_1 + x_2, y_1 + y_2) \in \mathbb{N}$' as it is redundant (what follows is already sufficient).
9. The authors should state Pick's Theorem before stating Theorem 3 because it is used in the proof of Theorem 3.
10. 'Sub' should read 'Substitute'. Never use unexplained abbreviations.
11. The whole section can be greatly simplified. In particular, the proof of the generalized Pick's Theorem can be shortened significantly by noting that the linear transformation $T = \begin{pmatrix} a\cos\phi & b\sin\theta \\ a\sin\phi & b\cos\theta \end{pmatrix}$ leads to a change of area by the factor $\det T = ab\sin\alpha$.
12. '...two basic vectors...' should read '...two basis vector...'.
13. 'figure' should read 'polygon'. Pick's theorem gives the area of any polygon whose vertices are lattice points of the integer lattice.
14. Delete 'and θ and ϕ' because θ and ϕ does not appear in Pick's formula! They should instead define α.
15. Change 'figure' to 'parallelogram'.
16. '...coprime lattice point D' should read '...coprime lattice point outside D'.
17. The authors should point out that the proof is similar to that of Theorem 3
18. Change 'From the graph' to 'From the picture on the right'.
19. The authors should define visible area at the beginning of the section.
20. Add 'We have that' before the displayed formula. Never start a sentence with a mathematical expression!
21. Delete 'And'. Never start a sentence with 'and'.
22. The authors should define S_i and S'_i.
23. The last '$\leq$' should be changed to '$=$'.

24. The authors should explain why the term $\dfrac{(n(P)-2)\pi}{2\pi R^2 d^2}$ tends to 0 and what happens to d as R tends to infinity.
25. The notation $(1,0)$, $(0,1)$, $(-1,2)$...is confusing. The authors should point out that they are using a special basis for the hexagonal grid.
26. Add a full stop and 'Then' after $\varnothing$ to indicate that the statement that follows is the conclusion of the lemma.
27. It should be $m = \left\lfloor \dfrac{2^n \delta(C)}{\delta(C,\Lambda)} \right\rfloor$.
28. Theorem 12: With respect to which parameter is the maximum taken? The reviewer thinks it should be Λ. The authors should specify this.
29. It should be 'To ensure $v \notin$...'.
30. 'W' is missing. It should be in front of $(fC, \theta + \frac{\pi}{2})$. The authors should explain why 'fC is equivalent to a circular tree with...', which is not obvious to the reviewer.
31. Insert 'are transformed' between Y' and 'into'.
32. Instead of 'longer semi-part', 'shorter semi-part', it is customary to use semi-major axis and semi-minor axis.
33. '$\frac{G}{f}$' should be changed to '$\frac{G}{d}$'. It is confusing to use f, which here actually means the distance between the closest coprime lattice point outside the orchard and the origin, which is denoted by d in previous sections.
34. What are a_1, a_2, b_1, b_2? The authors should specify.
35. What are $p_1, \cdots, p_6$? The authors should define them.
36. It is more customary to say 'Scalene triangles'.
37. With respect to which parameter are maximum and minimum taken? What is a reference tree? The authors should specify.
38. What is the meaning of 'rings'?
39. For the convenience of the readers who only skim through the paper and read the main results directly, it is better to recall the definitions of various notations, or at least refer the readers to where they are defined.