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ON THE VARIATIONS OF THE BRACHISTOCHRONE CURVE

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Abstract. In this research report we will provide our own derivation of the classical Brachistochrone Curve. Afterwards, we will study some variations of the Brachistochrone Problem with two gravitational forces, friction and finally some concepts from non-Newtonian mechanics.

Keywords. Brachistochrone, calculus of variations, Euler-Lagrange equation, special relativity

Contents

1. Introduction

One of the very first problems in variational calculus is the Brachistochrone problem posed by Johann Bernoulli in 1696: Suppose there are two non-identical points, A and B, in a vertical plane with A at a height no shorter than B. Along which path does an object released from rest at A travel to B in the shortest possible time if the object is subject to gravitational and normal reaction forces only? It is now well known that the Brachistochrone Curve, the solution to the Brachistochrone problem, is (part of) a cycloid.

In this research report we will provide our own derivation of the classical Brachistochrone Curve by using an approach that has striking similarities with the Snell's Law in physics governing the refraction of light. This does not come as a surprise, however, since by Fermat's Principle Snell's Law in fact describes the path in which light travels fastest.

In the subsequent sections, we will find the Brachistochrone Curves under physical conditions different than that of the original Brachistochrone problem. In order to solve these variations, we will apply the Euler-Lagrange differential equation multiple times, a common technique used in variational calculus.

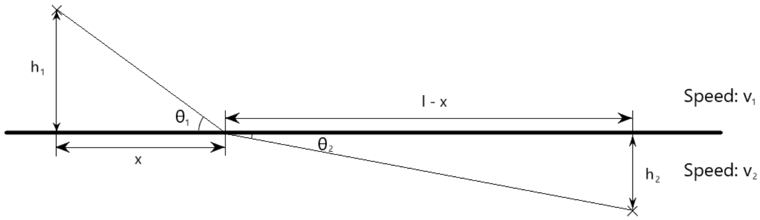
The first variation to be studied is the case where there are two constant gravitational forces of equal magnitude in mutually perpendicular directions. This slightly complicates the magnitude of velocity, giving a more complicated result. The approach is similar to most of the modern solutions to the original Brachistochrone problem – finding the magnitude of the velocity using the law of conservation of energy, and applying the Euler-Lagrange equation directly.

After that, we will study the Brachistochrone Curve under a single constant downward gravitational force with friction. We will first do a simpler approximation of the frictional force by ignoring the curvature, followed by a more complex computation of a more physically-accurate frictional force, obtaining two Brachistochrone curves (that look extremely similar). The first approach considers the free-body diagram of the object, and finds the net force on the object by the assumption that the net force points in the direction of motion. The second approach, having to consider the curvature, uses an approach with vectors similar to that of the derivation of the equations of circular motion in physics. In both approaches, the net force and hence the velocity is obtained, and subsequently the Euler-Lagrange equations are applied.

Afterwards, as there have been recent improvements to Newtonian mechanics in the past century, we attempt to model Brachistochrone curves in non-Newtonian situations, including using some concepts from special relativity and considering the case where an object free falls in hyperbolic motion.

2. Deriving the Brachistochrone Curve

We first show the an optimal path to travel between two points in two different zones separated by a horizontal boundary. Let the zones have height h_1 and h_2, with horizontal distance travelled l and speeds v_1 and v_2. Assume that the change of speed when crossing the boundary happens instantaneously and the paths are straight lines in their respective regions.



total travel time $t_{tt} = \dfrac{\sqrt{h_1^2 + x^2}}{v_1} + \dfrac{\sqrt{h_2^2 + (l-x)^2}}{v_2}$

$$\frac{dt_{tt}}{dx} = \frac{x}{v_1\sqrt{h_1^2 + x^2}} - \frac{l-x}{v_2\sqrt{h_2^2 + (l-x)^2}}$$

$$\frac{d^2t_{tt}}{dx^2} = \frac{\sqrt{h_1^2+x^2} - \frac{x^2}{\sqrt{h_1^2+x^2}}}{v_1(h_1^2+x^2)} + \frac{\sqrt{h_2^2+(l-x)^2} - \frac{(l-x)^2}{\sqrt{h_2^2+(l-x)^2}}}{v_2(h_2^2+(l-x)^2)}$$

$$= \frac{h_1^2}{v_1(h_1^2+x^2)^{\frac{3}{2}}} + \frac{h_2^2}{v_2(h_2^2+(l-x)^2)^{\frac{3}{2}}} > 0$$

Solving $\dfrac{dt_{tt}}{dx} = 0$, we get $\dfrac{\cos\theta_1}{v_1} = \dfrac{\cos\theta_2}{v_2}$, which is a condition to minimize the total time of travel. Note that this equation is extremely similar to Snell's Law in physics.

Hence, to minimize the total time of travel, $\dfrac{\cos\theta}{v}$,where $\tan\theta = \dfrac{dy}{dx}, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, should be constant at each point on the optimal path.

Suppose that A, the point at which the object is released from rest, has y-coordinate 0. By conservation of energy, the magnitude of the velocity of the object at (x, y) is given by $\sqrt{-2gy}$.

Alternatively, one can consider the component of gravitational force that contributes to motion, which is given by $g\sin\theta = g(\dfrac{dy}{ds})$. Then

$$\frac{d^2s}{dt^2} = g(\frac{dy}{ds})$$
$$\frac{ds}{dt}d(\frac{ds}{dt}) = (g)dy$$

Integrating, $\frac{v^2}{2} = gy + K$ for some constant K.
As $v|_{y=0} = 0, K = 0$; and as $y < 0$, speed of object is again given by $\sqrt{-2gy}$.

Therefore, $\frac{\sqrt{-y}}{\cos\theta} = k_1$ for some real constant k_1 on the desired path.

$$\sqrt{-y} = k_1\sqrt{\frac{1}{1+(\frac{dy}{dx})^2}}$$
$$\frac{dy}{dx} = \sqrt{\frac{k_2+y}{-y}} \qquad \text{where } k_2 = k_1^2$$
$$\int\sqrt{\frac{-y}{k_2+y}}\,dy = \int dx$$

Substituting $y = \frac{k_2}{2}(\cos\phi - 1)$, $dy = \frac{-k_2}{2}\sin\phi d\phi$,

$$\int\frac{-k_2}{2}\sin\phi\sqrt{\frac{1-\cos\phi}{1+\cos\phi}}\,d\phi = x$$
$$-k_2\int\sin^2\frac{\phi}{2}\,d\phi = x$$
$$\frac{k_2}{2}(\sin\phi - \phi) = x + C_1$$

where C_1 is the constant of integration.

Parameterized,

$$\begin{cases} x = k(\sin\phi - \phi) + C \\ y = k(\cos\phi - 1) \end{cases}$$

where k and C are constants chosen to fit the starting and ending points.

3. Vertical and horizontal gravity

In the classic Brachistochrone problem it is assumed that the gravitational acceleration is constant with magnitude g in the negative y-direction. As an extension of the problem, what would be the path of minimal travel time if there were gravitational acceleration of magnitude g in both the positive x and negative y-direction?

Proceeding as before, the new velocity is now given by $\sqrt{2g(x-y)}$, and as we seek to optimise the total time t_t, we have:

$$\begin{aligned} t_t &= \int_A^B \frac{ds}{v} \\ &= \int_A^B \frac{\sqrt{dx^2+dy^2}}{\sqrt{2g(x-y)}} \\ &= \frac{1}{\sqrt{2g}} \int_{x_A}^{x_B} \frac{\sqrt{1+(y')^2}}{\sqrt{x-y}} dx \end{aligned}$$

Setting $F(x,y,y')$ as the integrand, where $t_t(y)$ is a functional in x, y, y', we apply the Euler-Lagrange equation:

$$\begin{aligned} \frac{\delta F}{\delta y} &= \frac{d}{dx}\left(\frac{\delta F}{\delta y'}\right) \\ \frac{\sqrt{1+y'^2}}{2(x-y)^{\frac{3}{2}}} &= \frac{2(x-y)y'' + y'(y'-1)((y')^2+1)}{2(x-y)^{\frac{3}{2}}((1+(y')^2)^{\frac{3}{2}}} \\ (1+y'^2)^2 &= 2(x-y)y'' + y'(y'-1)((y')^2+1) \\ \frac{(y'+1)(1+(y')^2)}{2y''} &= x-y \end{aligned}$$

Let $y' = \frac{dy}{dx} = -\tan\dfrac{\theta}{2}, x' = \dfrac{dx}{d\theta}, x'' = \dfrac{d^2x}{d\theta^2}, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2},$

$$\begin{aligned} \frac{(-\tan\frac{\theta}{2}+1)(1+\tan^2\frac{\theta}{2})}{2\frac{d}{dx}(-\tan\frac{\theta}{2})} &= x-y \\ \frac{(\tan\frac{\theta}{2}-1)(\sec^2\frac{\theta}{2})}{\sec^2\frac{\theta}{2}\frac{d\theta}{dx}} &= x-y \\ x'\left(\tan\frac{\theta}{2}-1\right) &= x-y \end{aligned}$$

Differentiating by θ,

$$\left(\frac{1}{2}\sec^2\frac{\theta}{2}\right)x' + \left(\tan\frac{\theta}{2} - 1\right)x'' = \left(\tan\frac{\theta}{2} + 1\right)x'$$
$$x'' + \left(\frac{\frac{1}{2}\sec^2\frac{\theta}{2} - \tan\frac{\theta}{2} - 1}{\tan\frac{\theta}{2} - 1}\right)x' = 0$$

Solving using integrating factor $I.F.$,

$$\begin{aligned}
\ln(I.F.) &= \int \frac{\frac{1}{2}\sec^2\frac{\theta}{2} - \tan\frac{\theta}{2} - 1}{\tan\frac{\theta}{2} - 1}\, d\theta \\
&= \int \frac{\frac{1}{2}\tan^2\frac{\theta}{2} - \tan\frac{\theta}{2} - \frac{1}{2}}{\tan\frac{\theta}{2} - 1}\, d\theta \\
&= \int \frac{tan\frac{\theta}{2}}{2}\, d\theta - \frac{1}{2}\int \frac{\tan\frac{\theta}{2} + 1}{\tan\frac{\theta}{2} - 1}\, d\theta \\
&= -\ln|\cos\frac{\theta}{2}| - \ln|\cos\frac{\theta}{2} - \sin\frac{\theta}{2}| + C \\
I.F. &= \frac{C_1}{(\cos\frac{\theta}{2})(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})} \\
\frac{dx}{d\theta} &= x' = -k\left(\cos\frac{\theta}{2}\right)\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) \\
\frac{dy}{d\theta} &= y'x' = k\left(\sin\frac{\theta}{2}\right)\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)
\end{aligned}$$

Let $\phi = 2\pi - \theta$,

$$\begin{aligned}
\frac{dx}{d\phi} &= -k\left(\cos\frac{\phi}{2}\right)\left(\sin\frac{\phi}{2} + \cos\frac{\phi}{2}\right) \\
\frac{dy}{d\phi} &= -k\left(\sin\frac{\phi}{2}\right)\left(\sin\frac{\phi}{2} + \cos\frac{\phi}{2}\right)
\end{aligned}$$

Integrating and parameterizing,

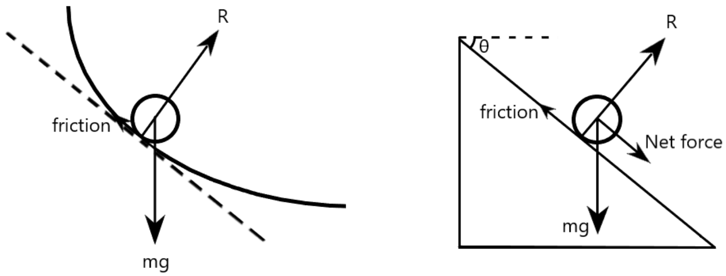
$$\begin{aligned}
x &= -\frac{k}{2}((\phi + \sin\phi) - (1 + \cos\phi)) + k_1 \\
y &= -\frac{k}{2}((\phi - \sin\phi) - (1 + \cos\phi)) + k_2
\end{aligned}$$

4. Deriving Brachistochrone with friction (approximation)

In this section we will consider first the friction due to normal reaction caused by the weight only. It is assumed that the object does not roll or experience rolling friction.

Suppose A, the starting point, has coordinates $(0,0)$ and B, the end point, has coordinates (x_0, y_0), where $x_0 > 0, y_0 < 0$.

Since we consider the normal reaction caused only by the weight and ignore the normal reaction needed to take travel on a path with curvature, we can determine the forces acting on the object at each point by taking the net force to be in the tangential direction.



In other words, if the slope at any given point is $tan\theta$, where $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$ then the forces acting on the object would be equal to that if the object were sliding to the right on an inclined plane of slope $\tan\theta$.

$$tan\theta = \frac{dy}{dx}$$
$$ds = \sqrt{dx^2 + dy^2}$$

So, we have

$$\cos\theta = \frac{dx}{ds}$$
$$\sin\theta = \frac{dy}{ds}$$

Now, we consider the components of the forces parallel to the surface of the inclined plane, since we assumed that the net force would be parallel to the surface of the inclined plane

Taking the rightward direction as positive, it is easy to see that F_{weight}, the component of the weight parallel to the inclined surface is $mg\sin(-\theta)$. On the other hand, the component of the weight perpendicular to the inclined surface has magnitude $mg\cos\theta$, so the normal reaction will also have magnitude $mg\cos\theta$. Taking μ to be the coefficient of friction, $F_{friction}$ is equal to $-\mu mg\cos\theta$. The object is moving rightward so the velocity will always point leftward, hence the negative sign.

Let F_{tot} be the net force parallel to the surface of the inclined plane.

$$\begin{aligned}
F_{tot} &= F_{weight} + F_{friction} \\
m\frac{dv}{dt} &= -mg\frac{dy}{ds} - \mu mg\frac{dx}{ds} \\
\frac{dv}{dt} &= -g\left(\frac{dy}{ds} + \mu\frac{dx}{ds}\right) \\
\frac{dv}{dt} &= \frac{ds}{dt}\frac{dv}{ds} \\
&= v\frac{dv}{ds} \\
&= \frac{1}{2}\frac{d}{ds}(v^2) \\
\frac{d}{ds}\left(\frac{v^2}{2}\right) &= \frac{d}{ds}(-g(y+\mu x))
\end{aligned}$$

Since $v = 0$ at $(x, y) = (0, 0)$,

$$\begin{aligned}
v^2 &= -2g(y+\mu x) \\
v &= \sqrt{-2g(y+\mu x)}
\end{aligned}$$

The time taken is hence represented by the integral:

$$t = \int_0^{x_0} \sqrt{\frac{1+(y')^2}{-2g(y+\mu x)}}dx$$

We let $F(x, y, y') = \sqrt{\dfrac{1+(y')^2}{-2g(y+\mu x)}}$.

$$\frac{\partial F}{\partial y} = -\frac{1}{2(y+\mu x)}\sqrt{\frac{1+y'^2}{-2g(y+\mu x)}}$$
$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{-2g(y+\mu x)(1+y'^2)}}$$
$$\frac{d}{dx}(\frac{\partial F}{\partial y'}) = \frac{2y"(y+\mu x)(1+y'^2) - y'(y'+\mu)(1+y'^2) - 2y'^2y"(y+\mu x)}{2(y+\mu x)(1+y'^2)\sqrt{-2g(y+\mu x)(1+y'^2)}}$$

Applying the Euler-Lagrange equation:

$$\begin{aligned} &\frac{1}{2(y+\mu x)}\sqrt{\frac{1+y'^2}{-2g(y+\mu x)}} \\ &= \frac{2y"(y+\mu x)(1+y'^2) - y'(y'+\mu)(1+y'^2) - 2y'^2y"(y+\mu x)}{2(y+\mu x)(1+y'^2)\sqrt{-2g(y+\mu x)(1+y'^2)}} \\ \implies &-\sqrt{1+y'^2} \\ &= \frac{2y"(y+\mu x)(1+y'^2) - y'(y'+\mu)(1+y'^2) - 2y'^2y"(y+\mu x)}{(1+y'^2)\sqrt{(1+y'^2)}} \\ \implies -(1+y'^2)^2 &= 2y"(y+\mu x) - y'^2 - y'^4 - \mu y' - \mu y'^3 \\ \implies 0 &= 2y"(y+\mu x) + 1 + y'^2 - \mu y' - \mu y'^3 \\ \implies 0 &= 2y"(y+\mu x) + (1+y'^2)(1-\mu y') \end{aligned}$$

Rearranging the terms,

$$\begin{aligned} &(1+y'^2)(y'+\mu)(1-\mu y') = -2y"(y+\mu x)(y'+\mu) \\ \implies &(1+y'^2)(y'+\mu)(1-\mu y') = -2y"(y+\mu x)[\mu(1+y'^2) + y'(1-\mu y')] \\ \implies &2y'y"(y+\mu x)(1-\mu y)^{-2} + (1+y'^2)(y'+\mu)(1-\mu y')^{-2} \\ &= 2(1-\mu y')^{-3}(-\mu y")(1+y'^2)(y+\mu x) \\ \implies &\frac{d}{dx}\frac{(1+y'^2)(y+\mu x)}{(1-\mu y')^2} = 0 \end{aligned}$$

Let $\frac{(1+y'^2)(y+\mu x)}{(1-\mu y')^2} = -C$, where C is a real constant.

$$\frac{1+y'^2}{(1-\mu y')^2} = -\frac{C}{y+\mu x}$$

Let $y' = \frac{dy}{dx} = -\cot\frac{\phi}{2}, -\frac{\pi}{2} < \phi < \frac{\pi}{2}$

$$\frac{1+\cot^2\frac{\phi}{2}}{(1+\mu\cot\frac{\phi}{2})^2} = -\frac{C}{y+\mu x}$$
$$\left(\frac{\csc\frac{\phi}{2}}{1+\mu\cot\frac{\phi}{2}}\right)^2 = -\frac{C}{y+\mu x}$$
$$y+\mu x = -C(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2})^2$$

Differentiating by ϕ,

$$-\cot\frac{\phi}{2}\frac{dx}{d\phi} - \mu\frac{dx}{d\phi} = -C(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2})(\cos\frac{\phi}{2}-\mu\sin\frac{\phi}{2})$$
$$dx = C(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2})\left(\frac{\cos\frac{\phi}{2}-\mu\sin\frac{\phi}{2}}{\cot\frac{\phi}{2}-\mu}\right)d\phi$$
$$dx = C\sin\frac{\phi}{2}(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2})d\phi$$

Integrating,

$$\int dx = \int C\sin\frac{\phi}{2}(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2})d\phi$$
$$x = \frac{C}{2}(\phi - sin\phi + \mu(k_x+\cos\phi))$$

Similarly,

$$\frac{dy}{d\phi} - \mu\tan\frac{\phi}{2}\frac{dy}{d\phi} = -C(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2})(\cos\frac{\phi}{2}-\mu\sin\frac{\phi}{2})$$
$$dy = -C(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2})\left(\frac{\cos\frac{\phi}{2}-\mu\sin\frac{\phi}{2}}{1-\mu\tan\frac{\phi}{2}}\right)d\phi$$
$$dy = -C(\cos\frac{\phi}{2})(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2})d\phi$$
$$\int dy = -\int C(\cos\frac{\phi}{2})(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2})d\phi$$
$$y = -\frac{C}{2}(k_y - \cos\phi + \mu(\phi+\sin\phi))$$

Substituting above results into $y+\mu x = -C\left(\sin\frac{\phi}{2}+\mu\cos\frac{\phi}{2}\right)^2$,

$$-\frac{C}{2}(k_y - \mu^2 k_x + (\mu^2 - 1)\cos\phi + 2\mu\sin\phi) = -C(\sin^2\frac{\phi}{2} + \mu\sin\phi + \mu^2\cos^2\frac{\phi}{2})$$

$$k_y - \mu^2 k_x + (\mu^2 - 1)\cos\phi = 2\sin^2\frac{\phi}{2} + 2\mu^2\cos^2\frac{\phi}{2}$$

Comparing coefficients,

$$\begin{cases} k_y - \cos\phi = 2\sin^2\dfrac{\phi}{2} \\ \mu^2(\cos\phi - k_x) = \mu^2\left(2\cos^2\dfrac{\phi}{2}\right) \end{cases}$$

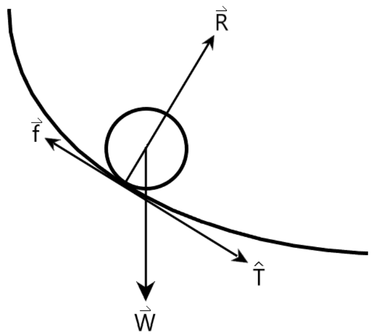
$$\therefore k_x = -1, k_y = 1$$

Therefore,

$$\begin{cases} x = \dfrac{C}{2}(\phi - \sin\phi - \mu(\cos\phi - 1)) \\ \\ y = \dfrac{C}{2}(\cos\phi - 1 - \mu(\phi + \sin\phi)) \end{cases}$$

5. Deriving the Brachistochrone Curve with friction (complete)

In the previous section, we have used an approximation of the frictional force by neglecting the effects of the curvature of the path. In this section, we shall take into consideration the curvature of the path and derive the Brachistochrone Curve with a more physically-accurate frictional force. It is assumed that the object does not roll or experience rolling friction.



We first consider the free body diagram of the moving object, which is to start from the origin and arrive at (x_0, y_0). Neglecting air resistance, the force diagram is as shown in the figure above, where R, f, w represent normal reaction due to the path, Coulomb friction with coefficient of friction μ and weight of the object respectively. Define $\hat{T}$ as the tangent vector, parameterized by $\hat{T} = (\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath}$, which is the unit vector in direction of the velocity $\hat{v}$, and $\hat{N}$ as the normal vector, parameterized by $\hat{N} = -(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath}$. Then, $\vec{R} = \left\|\vec{R}\right\| \hat{N}$, $\vec{f} = -\left\|\vec{f}\right\| \hat{T}$, $\vec{W} = -mg\hat{\jmath}$.

$$\begin{aligned}
\dot{\vec{v}} &= \frac{d}{dt}\vec{v} \\
&= \frac{d}{dt}\left(\|\vec{v}\|\,\hat{T}\right) \\
&= \frac{d\|\vec{v}\|}{dt}\hat{T} + \frac{d\hat{T}}{dt}\|\vec{v}\| \\
&= \frac{d\|\vec{v}\|}{dt}\hat{T} + \frac{d\hat{T}}{d\theta}\frac{d\theta}{dt}\|\vec{v}\| \\
&= \dot{v}\hat{T} + v\dot{\theta}\hat{N}
\end{aligned}$$

Resolving $\hat{\jmath}$ into tangential and normal directions,

$$\begin{aligned}
\hat{\jmath} &= (\hat{\jmath}\cdot\hat{T})\hat{T} + (\hat{\jmath}\cdot\hat{N})\hat{N} \\
&= \sin\theta\hat{T} + \cos\theta\hat{N}
\end{aligned}$$

By Newton's Second Law,

$$\begin{aligned}
m\dot{\vec{v}} &= \left\|\vec{R}\right\|\hat{N} - \left\|\vec{f}\right\|\hat{T} - mg\hat{\jmath} \\
m(\dot{v}\hat{T} + v\dot{\theta}\hat{N}) &= \left\|\vec{R}\right\|\hat{N} - \left\|\vec{R}\right\|\mu\hat{T} - mg\hat{\jmath} \\
m(\dot{v}\hat{T} + v\dot{\theta}\hat{N}) &= (-mg\cos\theta + \left\|\vec{R}\right\|)\hat{N} - (mg\sin\theta + \mu\left\|\vec{R}\right\|)\hat{T}
\end{aligned}$$

Comparing coefficients of $\hat{T}$ and $\hat{N}$,

$$\begin{cases}
\dot{v} = -g\sin\theta - \mu r \\
v\dot{\theta} = -g\cos\theta - r \ \text{ where } \ r = \frac{\left\|\vec{R}\right\|}{m}
\end{cases}$$

Cancelling $|\vec{R}|$,

$$\dot{v} + \mu v\dot{\theta} + g(\sin\theta + \mu\cos\theta) = 0$$

And as $\dot{x} = v\cos\theta$

$$vv' + \mu v^2\theta' + g(\tan\theta + \mu) = 0 \text{ where } v' = \frac{dv}{dx} \text{ and } \theta' = \frac{d\theta}{dx}$$

Which is a constraint on an object rolling down a path.

Another constraint is:

$$\frac{dy}{dx} = \tan\theta$$

$$\int_0^{y_0} dy = \int_0^{x_0} \tan\theta \, dx$$

$$\int_0^{x_0} (\tan\theta)\, dx - y_0 = 0$$

$$\int_0^{x_0} \left(\tan\theta - \frac{y_0}{x_0}\right) dx = 0$$

The functional to be optimized is

$$T(v,\theta) = \int dt = \int \frac{\sec\theta}{v}\, dx$$

Using the method of Lagrange multiplers with the above two constraints,

we create a new functional which is to be optimized: $I(v,\theta) = \int_0^{x_0} F(v, v', \theta, \theta', x)\, dx$

$$= \int_0^{x_0} \frac{\sec\theta}{v} + \lambda\left(vv' + \mu v^2\theta' + g(\tan\theta + \mu)\right) + k\left(\tan\theta - \frac{y_0}{x_0}\right) dx$$

where λ and k are Lagrange multipliers.

As λ is a non-holonomic constraint while k is an isoperimetric constraint, λ is a function of x while k is a constant.

By a constrained and multivariable version of the Euler-Lagrange Equation on the dependent variables v, θ, λ:

$$\begin{cases} -\dfrac{\sec\theta}{v^2} + \lambda(v' + 2\mu v\theta') = (\lambda v)' \\ \dfrac{\sec\theta\tan\theta}{v^2} + (g\lambda + k)(\sec^2\theta) = (\mu\lambda v^2)' \\ vv' + \mu v^2\theta' + g(\tan\theta + \mu) = 0 \end{cases}$$

Note the repeated occurrence of $\tan\theta+\mu$, substitute $\tan\theta+\mu = h$ and $\dfrac{h'}{(h-\mu)^2+1} = \theta'$:

$$\begin{cases} -\dfrac{\sqrt{(h-\mu)^2+1}}{v^2} + \lambda(v' + 2\mu\dfrac{vh'}{(h-\mu)^2+1}) = (\lambda v)' \\ \dfrac{(h-\mu)\sqrt{(h-\mu)^2+1}}{v^2} + (g\lambda + k)((h-\mu)^2 - 1) = (\mu\lambda v^2)' \\ vv' + \mu\dfrac{v^2h'}{(h-\mu)^2+1} + gh = 0 \end{cases}$$

Writing in matrix form,

$$\begin{bmatrix} 0 & \frac{2\mu\lambda v}{(h-\mu)^2+1} & -v \\ \frac{2\mu\lambda v}{(h-\mu)^2+1} & 0 & \frac{\mu v^2}{(h-\mu)^2+1} \\ v & \frac{\mu v^2}{(h-\mu)^2+1} & 0 \end{bmatrix} \begin{bmatrix} v' \\ h' \\ \lambda' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{(h-\mu)^2+1}}{v^2} \\ \frac{h-\mu}{v\sqrt{(h-\mu)^2+1}} + g\lambda + k \\ -gh \end{bmatrix} \begin{matrix} ⓥ \\ ⓗ \\ (\lambda) \end{matrix}$$

$$-\mu v \text{ ⓥ } + 2\mu\lambda (\lambda) : 2\mu\lambda v v' + \mu v^2 \lambda' = -\frac{\mu}{v}(\sqrt{(h-\mu)^2+1}) + 2g\lambda h v$$

Compared with $((h-\mu)^2+1)$ⓗ and solving for v,

$$v = -\frac{h\sqrt{(h-\mu)^2+1}}{g\lambda(h^2+\mu^2+1) + k((h-\mu)^2+1)} \qquad ⊛$$

gv^2h ⓥ $+\sqrt{(h-\mu)^2+1}(\lambda)$
$= v'\sqrt{(h-\mu)^2+1} + \frac{\mu v}{(h-\mu)^2+1}(2g\lambda h v + \sqrt{(h-\mu)^2+1})h' - gv^2h\lambda' = 0 \qquad ⊕$

$$⊛' : v' = \frac{\begin{array}{c} hg\lambda'(h-\mu-1)(h-\mu+1)\left(h^2+\mu^2+1\right) \\ -h'\left(g\lambda\left(h\left(h\left(\mu h+\mu^2+3\right)-3\left(\mu^3+\mu\right)\right)+\mu^4-1\right)+k(-h+\mu-1)(-h+\mu+1)\left(-\mu h+\mu^2-1\right)\right) \end{array}}{\sqrt{-2\mu h+h^2+\mu^2-1}\left(g\lambda(h^2+\mu^2+1)+k(-2\mu h+h^2+\mu^2-1)\right)^2}$$

Substituting ⊛ and ⊛$'$ into ⊕,

$$\frac{\left(\mu^2+1\right)\left(-2\mu h^2+\mu^2+1\right)\left(gh\lambda' - h'(g\lambda+k)\right)}{\left(h^2(g\lambda+k) + (\mu^2+1)(g\lambda+k) - 2k\mu h\right)^2} = 0$$

Simplifying, $\dfrac{g\lambda'}{g\lambda+k} = \dfrac{h'}{h}$

Integrating, $\ln|g\lambda+k| = \ln|h| + C_1$ for some constant C, and

$$\lambda = C_1 h - \frac{k}{g}$$

for some constant C_1.

Substituting above expression into ⊛ and letting $A = gC_1$ and $B = A(\mu^2+1) - 2k\mu$, after some simplifications, we get:

$$v^2 = \frac{(h-\mu)^2+1}{(Ah^2+B)^2} \qquad ⑥$$

Differentiating by x,

$$vv' = -\frac{(h(Ah(h-3\mu)+2(\mu^2+1)A-B)+\mu B)}{(Ah^2+B)^3}h' \qquad ⑥'$$

Substituting ⑥ and ⑥$'$ into (λ),

$$g\,dx = \frac{Ah(h-4\mu)+2(\mu^2+1)A-B}{(Ah^2+B)^3}dh$$

and as $dy = \tan\theta dx$

$dy = (h-\mu)dx$

$$g\,dy = \frac{(h-\mu)(Ah(h-4\mu)+2(\mu^2+1)A-B)}{(Ah^2+B)^3}dh$$

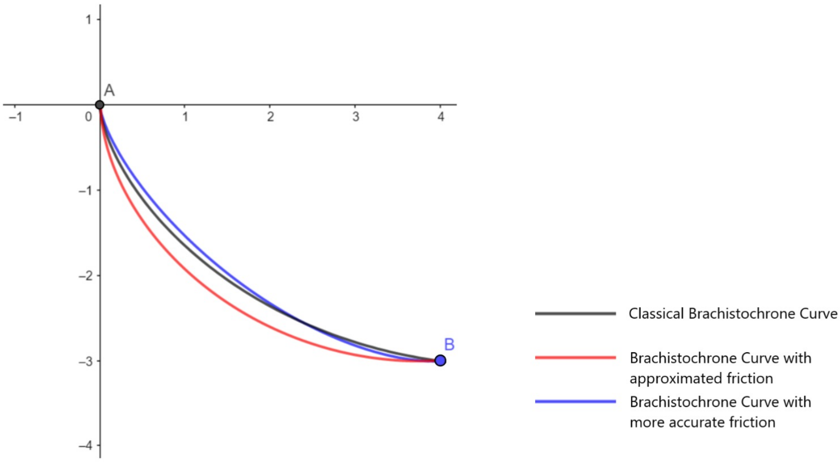
Integrating both sides, we have a parametrised solution:

$$gx = \frac{(3A(\mu^2+1)-B)\arctan(\sqrt{\frac{A}{B}}h)}{4\sqrt{A}B^{\frac{5}{2}}} + \frac{A(3A(\mu^2+1)-B)h^3+B(5A(\mu^2+1)-3B)h+4B^2\mu}{4B^2(Ah^2+B)^2} + C_2$$

$$gy = \frac{\mu(3A(\mu^2+1)+B)\arctan(\sqrt{\frac{A}{B}}h)}{4\sqrt{A}B^{\frac{5}{2}}} + \frac{A\mu(3A(\mu^2+1)+B)h^3+5B\mu(A(\mu^2+1)-B)h+2B^2(h^2+1)+6B^2\mu^2}{4B^2(Ah^2+B)^2} + C_3$$

where the constants C_2 and C_3 can fix the Brachistochrone curve onto the required starting point of the origin and ending point (x_0, y_0).

6. Comparison of Brachistochrone curves obtained



In this section we plot the curves obtained on the x-y coordinate plane, with $\mu = 0.34$. Observe that even when more and more concise calculations are made, the difference in the shape of the path is negligible, although the difficulty of solving for a solution increased greatly.

7. Approximation of the Brachistochrone curve under gravity with equations from special relativity

We have attempted to solve the Brachistochrone problem through concepts from Newtonian mechanics such as conservation of energy. However, it has been proven that Newtonian mechanics does not always hold true. In the following sections, we attempt to combine Newtonian mechanics and concepts from special relativity to find a more accurate solution to the Brachistochrone.

In special relativity, the relativistic momentum of an object is given by $p = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$, rather than $p = mv$ in classical Newtonian mechanics. In this section we shall take this equation and derive the Brachistochrone Curve for motion under this equation.

In this section we shall make several assumptions that are obscure in special relativity. First off, we will ignore the differences between "rest mass" and "relativistic mass", so that $mg = \frac{dp}{dt}$, where $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ and c is the speed of light in a vacuum. In addition, we shall assume that one's velocity does not dilate time. Again, we shall start from $A(0,0)$ and end at $B(x_0, y_0)$, where $y_0 < 0$.

To begin, consider an object under free fall constrained by the above equation,

$$\frac{d}{dt}\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = g$$

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = gt + C$$

Since the velocity is 0 at time $t = 0$,

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = gt$$

$$\frac{v^2}{1-\frac{v^2}{c^2}} = g^2t^2$$

$$v = \frac{cgt}{\sqrt{g^2t^2+c^2}}$$

$$-\frac{dy}{dt} = \frac{cgt}{\sqrt{g^2t^2+c^2}}$$

$$y = \frac{c^2}{g} - \frac{c}{g}\sqrt{g^2t^2+c^2}$$

$$t = \frac{\sqrt{g^2y^2-2c^2gy}}{cg}$$

$$v = \frac{c\sqrt{g^2y^2-2c^2gy}}{c^2-gy}$$

Since we are neglecting resistive forces like air resistance and friction, the normal force provided by the path does no work on the object. Hence, we can extend the above argument and therefore the velocity of the object is dependent on y and independent of x.

The time taken to travel from A to B is given by the integral:

$$t = \int_A^B \frac{ds}{v}$$
$$= \int_0^{x_0} \frac{c^2 - gy}{c} \sqrt{\frac{1 + y'^2}{g^2y^2 - 2c^2gy}} dx$$

Letting $F(x, y, y') = \frac{c^2 - gy}{c} \sqrt{\frac{1+y'^2}{g^2y^2 - 2c^2gy}}$, we apply once again the Euler-Lagrange equation. Since the expression does not contain x, i.e., $\frac{\partial F}{\partial x} = 0$, we can make use of the Beltrami Identity where for some constant R:

$$F - y' \frac{\partial F}{\partial y'} = \frac{R}{c}$$
$$\left(\frac{c^2 - gy}{c\sqrt{g^2y^2 - 2c^2gy}}\right)\left(\sqrt{1 + y'^2} - y' \frac{\partial}{\partial y'} \sqrt{1 + y'^2}\right) = \frac{R}{c}$$
$$\frac{1}{\sqrt{1 + y'^2}} = \frac{R\sqrt{g^2y^2 - 2c^2gy}}{c^2 - gy}$$
$$1 + y'^2 = \frac{(c^2 - gy)^2}{R^2(g^2y^2 - 2c^2gy)}$$

seperating variables and integrating,

$$x = \frac{\sqrt{\frac{c^4 + 2c^2g(R^2-1)y - g^2(R^2-1)y^2}{c^4R^2}} E\left(\sin^{-1}\left(\frac{\sqrt{\frac{g^2(R^2-1)}{c^4R^2}}(gy - c^2)}{g}\right) \middle| \left(\frac{R^2}{R^2-1}\right)\right)}{\sqrt{\frac{g^2(R^2-1)}{c^4R^2}} \sqrt{\frac{gy(2c^2-gy)}{c^4}} \sqrt{\frac{c^4 + 2c^2g(R^2-1)y - g^2(R^2-1)y^2}{gR^2y(gy-2c^2)}}} + C$$

where $E(f|m)$ is the elliptic integral of the second kind with parameter $m = k^2$ and C is the constant of integration. C and R can then be solved to fit the end points $(0, 0)$ and (x_0, y_0).

8. The Brachistochrone Curve under hyperbolic motion

In special relativity, an oft seen concept is that objects cannot move faster than light. However, in Newtonian mechanics, if one does not consider resistive forces, an object can accelerate indefinitely. Hence it is vital to have a kind of motion that models "free fall" better, which is the hyperbolic motion, with its proper acceleration given by α=γ^3a, where γ is the Lorentz factor. In this section, we attempt to model a Brachistochrone curve under the assumption that the object experiences proper normal acceleration rather than acceleration, and ”free fall” objects undergo hyperbolic motion.

Again we take the object to start from (0, 0) and ending at (x_0, y_0).

Our motivation is using conservation of energy in special relativity where there is no gravitational potential energy and we try to approximate this by hyperbolic motion and a variant of newtons second law to derive the velocity of the ball.

As all motion in this section is on a 2D-plane, the change in x^3, or the z-spacial coordinate, will always be 0, and will be omitted in this section.

We first consider a general relation between force and velocity in special relativity with some constraints.

Proposition. (A variant of Newton's second law) An object moving under a three-force in special relativity follows the relation given by $\vec{f_{net}} = m_0\gamma(\frac{v\gamma^2 a}{c^2}\vec{v}+\vec{a})$, where $\vec{f_{net}}$, $\vec{v}$ and $\vec{a}$ are the three-force, three-velocity and three-acceleration respectively, provided that the rest mass remains invariant.

Proof: In Newtonian mechanics, the force $\vec{F}$ is given by $\vec{F}_{net} = \dot{\vec{P}}$, where $\vec{P}$ is the momentum $m\vec{V}$, where $\vec{V}$ is the velocity. A corresponding three-momentum $\vec{p}$ in special relativity is given by $\vec{p} = \gamma m_0 \vec{v}$, where $\vec{v}$ is the relativistic three-velocity, m_0 is the rest mass and $\gamma = \gamma(v) = (1 - \frac{v^2}{c^2})^{-0.5}$ is the Lorentz factor. Then by a variant of Newton's Second Law, $\vec{f}_{net} = \dot{\vec{p}}$, where the dot represents a derivative to coordinate time and not proper time, we have:

$$\begin{aligned}\vec{f}_{net} &= \frac{d\vec{p}}{dt} \\ &= \frac{d}{dt}(\gamma m_0 \vec{v}) \\ &= m_0(\gamma\frac{d\vec{v}}{dt} + \frac{d\gamma}{dt}(\vec{v})) \\ &= m_0\gamma(\frac{v\gamma^2 a}{c^2}\vec{v} + \vec{a})\end{aligned}$$

and we are done.

As this section mainly concerns hyperbolic motion as aforementioned, and noting that when an object moves under vertical hyperbolic motion, the direction of $\vec{v}$ and $\vec{a}$ are the same, while the rest mass remains constant. So Proposition 8.1 can be simplified to

$$\vec{F}_{net} = m_0(\frac{v^2\gamma^2}{c^2} + 1)\gamma^2\vec{a}$$

In this case, we take the direction of acceleration to be $-\hat{j}$.

We consider the total energy of the object, which is the x^0 entry of the four-momentum vector. As we expect the object to fall downwards naturally, we assume that there should exist a downwards four-force which can provide potential energy for the object free falling.

Then the potential function $U(v)$ is given by:

$$\begin{aligned}\int f\,ds &= \int m_0 a(\frac{v^2\gamma^2}{c^2} + 1)\gamma\,ds \\ &= m_0 \int (\frac{v^2\gamma^2}{c^2} + 1)\gamma\,d\left(\frac{v^2}{2}\right) \\ &= -m_0c^2\sqrt{1 - \frac{v^2}{c^2}} + C = -\frac{m_0c^2}{\gamma} + C\end{aligned}$$

, where C is a constant. As the potential function must be used as a difference, C will always cancel out and we can take it as zero.

To find v in terms of y under hyperbolic motion, we use the equation of motion for hyperbolic motion which is given by

$(y + \frac{c^2}{\alpha})^2 - c^2t^2 = \frac{c^4}{\alpha^2}$, where y is the magnitude of displacement and t is coordinate time.

Solving as $y < 0$,

$$t = \frac{\sqrt{-y(2c^2 - y\alpha)}}{c\sqrt{\alpha}}$$

Differentiating and simplifying, $v^2 = -\frac{c^2\alpha y(2c^2 - \alpha y)}{(\alpha y - c^2)^2}$ and $\gamma = 1 - \frac{\alpha y}{c^2}$

Therefore, substituting back, $U(v(y)) = U(y) = \frac{m_0c^4}{c^2 - \alpha y}$

Note that similar to gravitational potential energy, this form of potential energy is only dependent on the height of the object, and as height(y) decreases, $U(y)$ decreases.

Proposition. (Conservation of energy) In special relativity conservation of energy can be modelled, provided one considers the mass-energy equivalence.

By Proposition 7.2, considering the object at points $(0,0)$ and (x,y), the equation for conservation of energy is given by:

$$KE_{initial} + PE_{initial} = KE_{final} + PE_{final}$$

and as $v = 0$ at the origin, which is the starting point, it can be simplified to

$$U(y)|_{y=0} = \frac{1}{2}\gamma m_0 v^2 + U(y)$$

Solving for v^2,

$$v^2 = -\frac{2\alpha c^2 y}{(c^2 - \alpha y)^2}(\sqrt{(c^2 - \alpha y)^2 + \alpha^2 y^2} + \alpha y)$$

Then we optimise $\int d\tau = \int \frac{dt}{\gamma}$, which is the total proper time, and we rewrite it as

$I(y) = \int F(x, y, y')dx = \int \sqrt{\frac{1 + y'^2}{\frac{1}{v^2} - \frac{1}{c^2}}} dx$

The functional is to be optimised by the Euler-Lagrange equation. Since the functional to be integrated is independent of x, we apply the Beltrami identity and simplify to get:

$1 + y'^2 = k^2 q(y)$

where $q = q(y) = \frac{-y(\sqrt{(c^2 - \alpha y)^2 + \alpha^2 y^2} - \alpha y)}{(c^2 - \alpha y)^2}$, and we have $q > 0$ for all $y < 0$, and k is a constant.

Then we have a quadrature solution where

$$\int \frac{dy}{\sqrt{k^2 q - 1}} = x$$

$$\int \left(\frac{1}{k\sqrt{q}}\right)\left(1 - \frac{1}{k^2 q}\right)^{-0.5} = x$$

To approximate this integral, we expand the series around 0 up to the fifth term:

$$\frac{1}{k\sqrt{q}} = \frac{kc}{\sqrt{p}} + \frac{ka^2p^{3/2}}{4c^3} - \frac{ka^3p^{5/2}}{4c^5} + \frac{5ka^4p^{7/2}}{32c^7} + O(p^{9/2})$$

$$\left(1-\frac{1}{k^2q}\right)^{-0.5} = \frac{35c^8}{128k^8p^4} + \frac{5c^6}{16k^6p^3} + \frac{c^4\left(35a^2+24k^4\right)}{64k^8p^2} +$$

$$\frac{c^2\left(-35a^3+30a^2k^2+32k^6\right)}{64k^8p} + \left(\frac{3a^2\left(35a^2-20ak^2+16k^4\right)}{128k^8}+1\right) + O(p)$$

where $p=-y$.

Then multiplying the series together, we have $x = \int \frac{dy}{\sqrt{k^2q-1}}$

$$\approx -\frac{5c^9}{64k^8}p^{-\frac{7}{2}} - \frac{c^7}{8k^6}p^{-\frac{5}{2}} - \left(\frac{105a^2c^5}{256k^8}+\frac{c^5}{4k^4}\right)p^{-\frac{3}{2}} + \left(\frac{315a^3c^3}{256k^8}-\frac{35a^2c^3}{32k^6}-\frac{c^3}{k^2}\right)p^{-\frac{1}{2}}$$

$$+\sqrt{p}\left(\frac{4095a^4c}{2048k^8}-\frac{35a^3c}{32k^6}+\frac{15a^2c}{16k^4}+2c\right)$$

$$+p^{3/2}\left(-\frac{35a^5}{192ck^8}+\frac{85a^4}{768ck^6}-\frac{a^3}{16ck^4}+\frac{a^2}{12ck^2}\right)$$

$$+p^{5/2}\left(\frac{175a^6}{1024c^3k^8}-\frac{3a^5}{32c^3k^6}+\frac{39a^4}{640c^3k^4}-\frac{a^3}{20c^3k^2}+\frac{a^2}{10c^3}\right)$$

$$+p^{7/2}\left(-\frac{85a^7}{1024c^5k^8}+\frac{195a^6}{3584c^5k^6}-\frac{3a^5}{112c^5k^4}+\frac{5a^4}{224c^5k^2}-\frac{a^3}{14c^5}\right)$$

$$+p^{9/2}\left(\frac{175a^8}{6144c^7k^8}-\frac{25a^7}{1536c^7k^6}+\frac{5a^6}{384c^7k^4}+\frac{5a^4}{144c^7}\right) + C,$$

where C and k are constants to be chosen to match the endpoints required.

References

[1] Ashby, N.; Brittin, W. E.; Love, W. F.; and Wyss, W. *Brachistochrone with Coulomb Friction.* Amer. J. Phys. 43, 902-905, 1975.

[2] Jeffrey C. Hayen *Brachistochrone with Coulomb Friction* International Journal of Non-Linear Mechanics Vol. 40, Issue 8, 1057-1075, Octoer 2005

REVIEWERS' COMMENTS

Reviewers of this paper think that the choice of the topic is ambitious for a group of high school students. The problem about the Brachistochrone curve equation is classical. It is the particular curve joining two prescribed points in the plane minimizing the total travel time of a particle freely sliding due the influence of gravity. The authors derived the curve using the method of Calculus of Variations, and also investigated several more complicated variants which allow additional gravitational forces, relativistic forces, and friction forces. Reviewers also suggested some expository improvement of the paper, particularly on the use of numbering on equations and figures.