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A Geometric Approach to the Second Non-trivial Case of the Erdős-Szekeres Conjecture

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A GEOMETRIC APPROACH TO THE SECOND NON-TRIVIAL CASE OF THE ERDŐS-SZEKERES CONJECTURE

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Abstract. The Erdős-Szekeres conjecture, developed from the famous Happy-Ending Problem, hypothesizes on the number of points in general position needed on a plane to guarantee the existence of a convex n-gon. The research conducted aims to examine geometric characteristics of different constructions of points in general position, organized by number of points forming the convex hull of the set. This paper has explored the case of pentagons, reestablishing the previously proven result of the case using a geometrical approach in contrast to the combinatorial approaches generally adopted when exploring this problem. This paper also proves that the lower bound to the conjecture is not sharp under certain circumstances, an aspect never explored in the past. [See reviewer's comment (2)]

1. Introduction

1.1. Definition and Characteristics of Terms and Notations

This section will define terms and notations that are essential to the understanding of the paper, and note important characteristics of certain geometric structures.

Convex hull, the intersection of all convex sets containing a defined set of points on the Euclidean plane. In structures considered in this paper, it is a polygon with vertices at selected points in the defined set of points.

$x-y-z$ *configuration*, where x, y and z are integers, the convex hull organization of a defined set of points. For example, a *4-4-0 configuration* consists of eight points, 4 of which forming the convex hull of the set and the remaining 4 forming a convex quadrilateral; a *4-3-1 configuration* consists of 8 points, 4 of which forming the convex hull of the set, 3 of the remaining forming a triangle and the last positioned within the triangle. In structures considered in this paper, all variables except for the last have a value of at least 3.

$f(n)$, the number of points in general position on a plane required to guarantee the existence of a convex n-gon with vertices among the set of points.

1.2. Problem Background

The Erdős-Szekeres conjecture hypothesizes that $f(n) = 2^{n-2} + 1$. While the proposers have proven that $f(n) > 2^{n-2}$, the exact upper bound has not yet been proven for the general case, therefore unable to confirm the conjecture. Throughout the years, there have been efforts to establish an upper bound, which gradually lowered from the proposer's original proven bound of $f(n) \leq \binom{2n-4}{n=2} + 1$ to $f(n) \leq 2^{n+4n^{4/5}}$ by Andrew Suk in April, 2016[1]. [See reviewer's comment (3)] In addition to workings on the general case, there have also been efforts in proving the conjecture for small cases. The case of existence of a triangle is trivially solved. The cases of $f(4) = 5, f(5) = 9$ and $f(6) = 17$ are all mathematically proven[2,3], while the value of $f(n)$ for all $n \geq 7$ remains unknown.

1.3. Research aim and results

The majority of the above mentioned work is based on combinatorics and coordinate geometry, utilizing calculations of combination and slopes, as well as structures such as cups and caps. Yet, as this conjecture is categorized as a combinatorial geometry problem, it is reasonable to speculate whether a geometric approach would bring new insights to this problem. In particular, past approaches of more geometrically-inclined methods have led to the discovery of stronger upper bounds[4]. This research has therefore set to explore the second non-trivial cases with a geometric mindset, as the first non-trivial case is very well understood.

In addition to proving $f(5) = 9$ using a geometric approach, this paper will look at configurations with 8 points and check if they guarantee the existence of a convex pentagon, thus investigating on the question of whether the lower bound of the conjecture is always sharp. While the former presents a new approach to an established result, the latter provides insight on a new area of investigation regarding this conjecture - the configurations that lead to a sharp lower bound of the conjecture. [See reviewer's comment (2)]

[1]Suk, Andrew (2016),*On the Erdős-Szekeres convex polygon problem*, arXiv:1604.08657

[2]Kalbfleisch, J.D.; Kalbfleisch, J.G.; Stanton, R.G. (1970), "A combinatorial problem on convex regions", Proc. Louisiana Conf. Combinatorics, Graph Theory and Computing, Congressus Numerantium, 1, Baton Rouge, La.: Louisiana State Univ., pp. 180-188.

[3]Szekeres, G.; Peters, L. (2006), "Computer solution to the 17-point Erdős-Szekeres problem", *ANZIAM Journal*, **48** (02): 151-164, doi: 10.1017/S144618110000300X

[4]Suk, Andrew (2016), *On the Erdős-Szekeres convex polygon problem*, arXiv:1604.08657

1.4. Paper overview

The paper will first recall or prove several results that is essential in reaching conclusions in this paper. It will be followed by a purely geometric proof of the established $f(5) = 9$ result, in contrast to combinatorial approaches in the past. Lastly, there will be an analysis for configurations with 8 points to explore on the issue of the lower bound of the conjecture. [See reviewer's comment (2)]

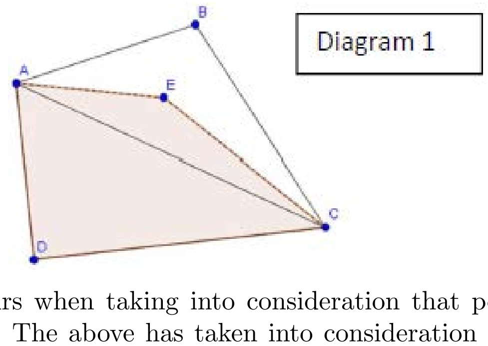
2. Mathematical proofs and analysis

2.1. Useful theorems and lemmas

Theorem 1 (the Happy Ending Problem)**.** *Any set of five points in the plane in general position has a subset of four points that form the vertices of a convex quadrilateral. In other words,* $f(4) = 5$.

Lemma 2. *For every 4-1 configuration, the point in the inner layer will form exactly 2 convex quadrilaterals with 3 of the 4 points forming the convex hull of the set of points. The 2-convex quadrilaterals formed shares exactly 1 edge, and that edge is one of the edges of the convex hull.*

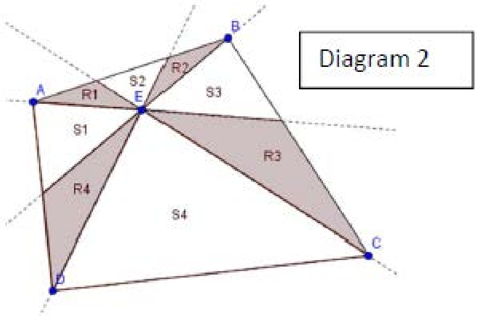
Proof. Suppose the *4-1* configuration consists of a quadrilateral $ABCD$ surrounding a single point E. Point E either lies in ΔABC or ΔACD. If point E lies in ΔABC, then as $\angle EAD < \angle BAD < 180°$, $\angle ECD < \angle BCD < 180°$, $\angle ADC < 180°$ (by definition of convex hull) and $\angle AEC < 180°$ (since point E lies within ΔABC), $AECD$ is a convex quadrilateral while $AECB$ is not (reflex$\angle AEC > 180°$), as demonstrated in Diagram 1. A similar case occurs if point E lies in ΔACD.



A similar case occurs when taking into consideration that point E either lies in ΔABD or ΔCBD. The above has taken into consideration all the 4 point combinations possible and confirms that exactly 2 forms a convex quadrilateral. As only one of the (A, B, C, E) or the (A, C, D, E) combination would form a convex quadrilateral, as with the (A, B, D, E) and (B, C, D, E) combination, the formed convex quadrilaterals must share an edge that forms the convex hull. This thereby completes the proof for Lemma 2. □

Lemma 3. *Among every 4-2 configuration in which a convex pentagon does not exist, the four points forming the convex hull of the set of points can be divided into two groups of two points in exactly one way, such that the two points in each group do not form the diagonal of the convex hull and that they form a convex quadrilateral with the remaining two points within the convex hull.*

Proof. Suppose the *4-2* configuration consists of a quadrilateral $ABCD$ surrounding two points E and F. Consider the position of point E within quadrilateral $ABCD$. By Lemma 2, it forms exactly 2 convex quadrilaterals with 3 of the 4 points A, B, C or D. Without loss of generality, we may assume that the two quadrilaterals are $AECD$ and $BCDE$, as demonstrated in Diagram 2.



As the lemma only considers configurations where no convex pentagons exist, point F cannot rest in regions $R1, R2, R3$ or $R4$ (resting in $R1$ or $R3$ would result in A, C, D, E and F forming a convex polygon, and $R2$ or $R4$ would result in B, C, D, E and F forming a convex polygon), and can only rest in regions $S1, S2, S3$ or $S4$.

If F rests in $S1$, the division of points would be (A, B) and (C, D). Suppose BE meets AD at G. Since $\angle DFE < 180°$ and $\angle FEC < \angle AEC < 180°$, $FECD$ is a convex quadrilateral; since $\angle AFE < 180°$ and $\angle FEB < \angle BEG = 180°$, $AFEB$ is also a convex quadrilateral. But since reflex $\angle AFE > 180°$ and reflex$\angle BEF > 180°$, neither $AFED$ nor $BEFC$ are convex quadrilaterals, making (A, B) and (C, D) the only possible division. A similar situation occurs if F lies in $S3$. If F lies in $S2$ or $S4$, the division of points would be (A, D) and (B, C), in which the existence of the respective convex quadrilaterals can be proven with a method similar to the above.

Although the above proof only covers one of the possible positions of point E, it is done without loss of generality in accordance to lemma 2. Other possible positions of E can be proved accordingly. This thereby completes the proof of Lemma 3. □

Lemma 4. *Among any set of 5 points in a 3-2 configuration, there exists exactly 1 convex quadrilateral formed by the inner 2 points and 2 of the outer 3 points.*

Proof. By Theorem 1, there exists at least one convex quadrilateral among the said 5 points. It remains to prove that there exists exactly one convex quadrilateral. Suppose the said configuration is formed by a triangle ABC surrounding two points D and E. Points A, B and C cannot all be vertices of a convex quadrilateral, else it contradicts the assumption that the convex hull of the set is a triangle. Therefore, the convex quadrilateral consists of points D, E and two of A, B and C. Without loss of generality, we assume the convex quadrilateral is $ABDE$, as demonstrated in Diagram 3.

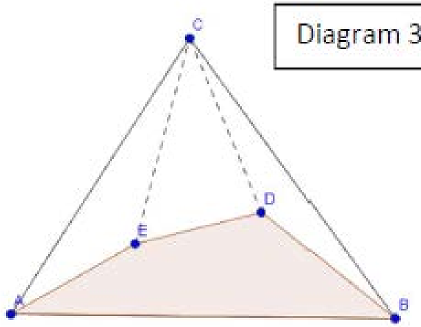


Diagram 3

Since $\angle AED < 180°$, reflex$\angle AED > 180°$, so $AEDC$ is not a convex quadrilateral. Similarly, reflex$\angle BDC > 180°$, so $BDEC$ is not a convex quadrilateral. Thus there can only be 1 convex quadrilateral among the 5 points, which is $AEDB$. This completes the proof of Lemma 4. □

2.2. The Geometric Proof of $f(5) = 9$

In this part, the four possible configurations of 9 points (*4-3-2, 4-4-1, 3-4-2, 3-3-3*) will be considered.

Lemma 5. *There must exist a convex pentagon whose vertices are among points in a 3-4-2 configuration.*

Proof. If a convex pentagon exists among the inward *4-2* configuration, the lemma is trivially proven. Suppose that is not the case. According to Lemma 3, there exists two different convex quadrilaterals within the inward *4-2* configuration, and they share the edge between the two innermost points. Suppose the four points in the second layer forms a quadrilateral $ABCD$, and the inner points are E and F, as shown in Diagram 4. Without loss of generality, we may assume the convex quadrilaterals formed are $ABFE$ and $EFDC$.

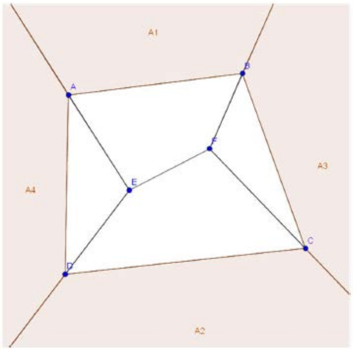


Diagram 4

If any of the three points in the convex hull of the set lie in $A1$ or $A2$, a convex pentagon would exist, proving the Lemma. Therefore, the only case left to consider is when all three points lie in either $A3$ or $A4$. That will result in at exactly 2 points lying in either $A3$ or $A4$, while the one remaining point lies in the other region, as three points lying in the same region would contradict the assumption that they form the convex hull. Without loss of generality, assume that 2 points G and H lie in $A4$, where G is nearer to A than H is, while a point I lies in $A3$. Since $\angle GAE, \angle HCE < 180°$, $\angle HGA < \angle HGI < 180°$ and $\angle GHC < \angle GHI < 180°$, $AGHCE$ forms a convex pentagon, completing the proof ot the said Lemma.

Although the case mentioned is only one of many possible cases, it is done without loss of generality in accordance with previous results or geometric symmetries, Therefore, the Lemma is true for all *3-4-2* configurations. □

[See reviewer's comment (4)]

Lemma 6. *There must exist a convex pentagon whose vertices are among a set of points in a 3-3-2 configuration.*

Proof. Suppose the inner *3-2* configuration consists of a triangle ABC surrounding points D and E. By Lemma 4, there exists a unique convex quadrilateral among these 5 points, and without loss of generality we assume it to be $ABDE$, as demonstrated in Diagram 5.

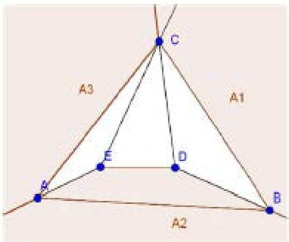


Diagram 5

Note that the area defined by rays DC and DB is considered $A1$ while that by DC and EA is area $A3$. Suppose the convex hull is a triangle FGH. If any of three points lie in $A2$, then a convex pentagon is formed by that point and A, E, D, B, proving the Lemma. The other case is when exactly 2 points lie in either $A1$ or $A3$ and 1 lie in the other; if 3 points lie in the same region, it would contradict the assumption that they form the convex hull of the set. If 2 points, say F and G (where F is nearer to C than G), lie in $A1$, then since $\angle GFC < \angle GFI < 180°$, $\angle FGB < \angle FGI < 180°$, F, G, B, D and C form a convex pentagon. Similarly, two points lie in $A3$, say F and G, then F, G, C, E and A would form a convex pentagon. This thereby completes the proof of Lemma 6. □

[See reviwer's comment (5)]

Lemma 7. *There must exist a convex pentagon whose vertices are among a set of points in a 4-3-1 configuration.*

Proof. Suppose that the inner *3-1* configuration consists of a triangle ABC surrounding a point D, as demonstrated in Diagram 6.

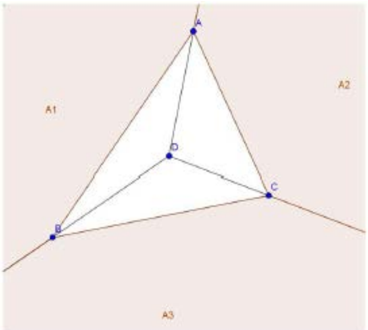


Diagram 6

Suppose the convex hull of the set is a quadrilateral $EFGH$, where each of these points lie in either $A1, A2$ or $A3$. Without loss of generality, we may assume that E lies in $A1$. If any of F, G or H lies in $A1$, say F, then since A and B are not in the convex hull of the set of points, the angles formed between E, F and A and E, F and B are smaller than 180 degrees, resulting in a convex pentagon formed by points A, E, F, B and D. The other case is where none of F, G, H lies in $A1$, so they either lie in $A2$ or $A3$.

Without loss of generality, suppose F lies in $A2$, so by similar reasoning as above either a convex pentagon can be found in $A2$ among points A, C, D, F and one of G or H, or G and H both lies in $A3$. The latter case would result in a convex pentagon between D, B, C, G and H, thereby completing the proof of Lemma 7. □

[See reviewer's comment (6)]

Theorem 8. *Any set of nine points in the plane in general position has a subset of five points that form the vertices of a convex pentagon. In other words,* $f(5) = 9$.

Proof. As mentioned above, there are four possible configurations for the nine points in general position. The *4-3-2* and *4-4-1* configurations consist of an extra points on a *4-3-1* configuration, which is proven to contain a convex pentagon by Lemma 7; thus, both the above configurations must consist of a convex pentagon. The *3-4-2* configuration is proven by Lemma 5 to contain a convex pentagon. The *3-3-3* configuration consists of an extra point on a *3-3-2* configuration, which is proven to contain a convex pentagon by Lemma 6; thus, the *3-3-3* configuration must contain a convex pentagon. This exhausts all the cases for any set of nine points in general position, thereby proving Theorem 8. □

2.3. Analysis of 8-point configurations

In Section 2.2, we have proven that the *4-3-1* and *3-3-2* 8-point configurations can guarantee the existence of a convex pentagon. In this section, we will discuss the remaining two possible configurations.

Case 9. *The 3-4-1 configuration is not a sufficient condition for the existence of a convex pentagon.*

Proof. This is proven through an actual construction, as demonstrated in Diagram 7 and Table 1.

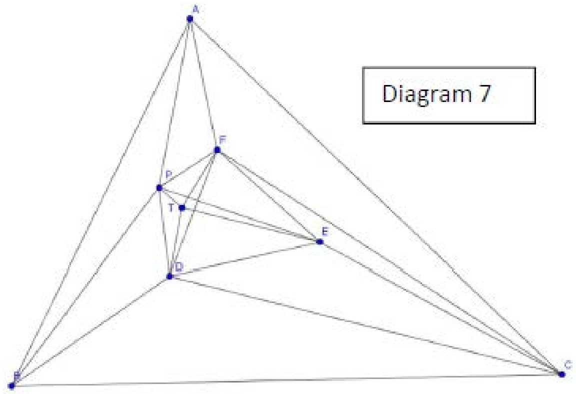


TABLE 1. Coordinates of points in Diagram 7*

Point	X-coordinate	Y-coordinate
A	7.5	4.7
B	1.34	-7.86
C	20.32	-7.46
D	6.8	-4.1
E	11.98	-2.96
F	8.44	0.2
P	6.44	-1.1
T	7.22	-1.79

A simple checking of all possible 5-point combinations yields the result that no convex pentagon exists. □

Case 10. *The 4-4 configuration is not a sufficient condition for the existence of a convex pentagon.*

Proof. This is proven through an actual construction, as demonstrated in Diagram 8 and Table 2.

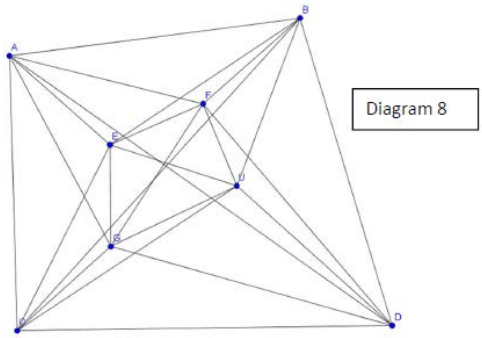


TABLE 2. Coordinates of points in Diagram 8*

Point	X-coordinate	Y-coordinate
A	2.82	2.58
B	13.92	4.02
C	3.12	-7.88
D	17.44	-7.68
E	6.66	-0.8
F	10.22	0.76
G	6.7	-4.68
U	11.5	-2.36

A simple checking of all possible 5-point combinations yields the result that no convex pentagon exists. □

[See reviewer's comment (7)]

3. Conclusion

In this paper, a purely geometric approach was used to investigate the second non-trival cases of the Erdős-Szekeres conjecture and reestablished the conclusion that any set of nine points in general position is sufficient to guarantee the existence of a convex pentagon with vertices among those points. The investigation also yielded the result that eight points are sufficient to guarantee the existence of a pentagon in two types of configurations, while it is not in two other types, which is surprising as the original proof Erdős and Szekeres provided on the lower bound of the conjecture only outlined one type of situation where the construction with one point fewer than the bound would fail[5]. As a whole, this paper approached a combinatorial geometry problem with a geometric method, in contrast to combinatorial means generally adopted in the past, and succeeded in proving the same results, providing insights to new possibilities in solving the conjecture or extensions of the conjecture.

Future directions of research include looking at the third and fourth non-trivial cases (or ultimately, the general case) of this conjecture using a geometric method (as the third non-trivial case is currently proven with the aid of computer and the fourth remains unsolved), and looking into extensions of the conjecture, for example the types of configurations for which the lower bound of the conjecture is sharp. [See reviewer's comment (2)]

REFERENCES

[1] Erdős, P., Szekeres, G., *On some extremum problems in elementary geometry*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **3-4** (1961), 53-62.

[2] Kalbfleisch, J.D., Kalbfleisch, J.G. and Stanton, R.G., *A combinatorial problem on convex regions*, Proc. Louisiana Conf. Combinatorics, Graph Theory and Computing, 1970.

[3] Suk, Andrew, *On the Erdős-Szekeres convex polygon problem*, arXiv:1604.08657. 2016.

[4] Szekeres, G., Peters, L., *Computer solution to the 17-point Erdős-Szekeres problem*, ANZIAM Journal **48** (2006), 151-164.

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*As the figures were constructed in Geogebra, the coordinates are given in accordance with the values shown on Geogebra.

[5] Erdős, P.; Szekeres, G. (1961), *"On some extremum problems in elementary geometry",Ann. Univ. Sci. Budapest. Eötvös Sect. Math., 3-4: 53-62.* Reprinted in: *Erdős, P.*

Reviewer's Comments

This paper investigates the Erdős-Szekerer's conjecture, which gives a relationship between the number of points in a general-position point set and its largest convex polygon. More precisely, the conjecture states that the smallest number $f(n)$ of points for which any general position arrangement contains a convex subset of n points is $2^{n-2}+1$. The general case remains unproven. The first non-trivial case starts from $n = 4$, and it is now known that the conjecture is true up to $n = 6$.

This paper gives a geometric proof of the fact that $f(5) = 9$, i.e. the fact that a set of 9 points in general position guarantees the existence of a convex pentagon whose vertices are from this set. They make extensive use of the $x - y - z$ configuration to classify the distribution of points. E.g. a $4 - 3 - 1$ configuration consists of 8 points, in which 4 points form a convex quadratical, whose interior contains a triangle formed by 3 of the remaining points, and the remaining one point is contained in the interior of this triangle.

In the first two parts, they give the historical review and basic definitions. Part 2 is the main part of the paper. To study a 9-point-configuration, they first study the possible sub configurations ($4-2$, $4-1$ and $3-2$ configurations), they show in Lemmas 2, 3 and 4 that certain convex quadrilateral exists in these configurations. They then use these lemmas to prove the main result ($f(5) = 9$). The strategy is as follows. First of all, there are only four cases which need to be discussed: $3-4-2$, $3-3-3$, $4-3-2$ and $4-4-1$. They then show that the result is true in $3-4-2$, $3-3-2$ and $4-3-1$. Note that we have the following implication: $3-3-2 \Rightarrow 3-3-3$ and $4-3-1 \Rightarrow 4-3-2$ and $4-4-1$. So the proof is complete.

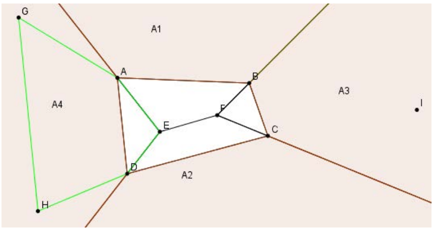
Here are my comments concerning the style and the mathematics in this paper.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. There are several occasions (p.63, p.64, p.72) where the author mentions the "lower bound" to the Erdős-Szekeres conjecture and whether this "lower bound" is "sharp" (p. 64, p. 72). I am a bit confused by these terms because the conjecture is that $f(n) = 2^{n-2} + 1$, which is not an inequality. Of course Erdős and Szekeres proved that $f(n) > 2^{n-2}$, which means 2^{n-2} is a lower bound for $f(n)$. So does the "lower bound to the conjecture" mean 2^{n-2}?

   I am not exactly sure what she means by the "sharpness" either, perhaps what she means is that there are configurations of 2^{n-2} points in which no convex n-gon is formed (already proved by Erdős-Szekeres, though the author seems to provide additional examples). Or does it just mean $f(n) = 2^{n-2} + 1$? And by "not sharp", perhaps she means there are some (perhaps non-trivial) configurations of 2^{n-2} or less points in which a convex n-gon is still guaranteed to exist, which is trivially true. In any case, I think the author should elaborate more, as these two terms appear in the introduction, abstract and conclusion (that sounds import) but never appear again in the main body.

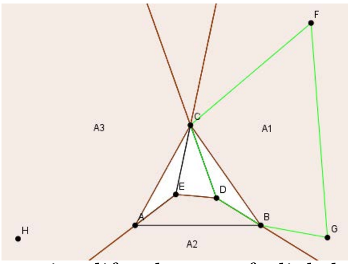
3. $\binom{2n-4}{n=2}$ should be $\binom{2n-4}{n-2}$.
4. Lemma 5.
   This lemma shows the result for the $3-4-2$ configuration. However, there is a typo which unfortunately appears repeatedly. In the second paragraph of the proof (below the diagram), all the "C" should be changed to "D" and so it is $AGHDE$, not $AGHCE$ that forms a convex pentagon. The proof seems correct to me, but I think the sentence "G is nearer to A than H is" is of no use here, and is possibly not true even if $AGHDE$ forms a convex pentagon, so this sentence should be deleted.

   Moreover, I suggest they should add the points G, H, I on the figure for easier understanding. For example, it may look something like this:



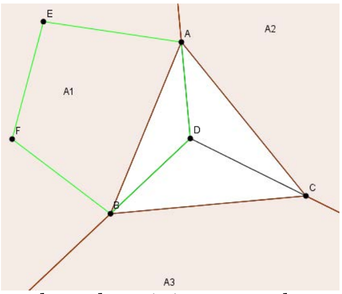
5. Lemma 6.
   My suggestion is similar to the previous one. Again there is a typo in the paragraph under the diagram: all the "I" should be changed to "H" (this time they state correctly that F, G, B, D, C forms a convex pentagon). I also don't see why they put the sentence "F is nearer to C than G", I think this is not necessary and is not true even if $FGBDC$ is a convex pentagon, so it should be deleted.

   Again I would suggest adding the points F, G, H in the diagram in this subcase, which may look something like this:



6. Lemma 7. First we can simplify the proof slightly by observing that by pigeonhole principle, two of the four points E, F, G, H must lie on the same A_i. So we can without loss of generality assume E and F lie on A_1, and the result is almost immediate in this case. Therefore the last paragraph "Without loss of generality, ..." can be deleted.

Again I think it is much better to add (at least some of) the points E, F, G, H in the diagram for better illustration. This may look something like this:



7. For Case 9 and 10, they show by giving actual examples that a $3-4-1$ configuration and a $4-4$ configuration cannot guarantee the existence of a convex pentagon. While their examples are quite plausible by mere inspection, I don't think it is really a "simple checking" as claimed. There are 56 such pentagons and should perhaps better checked by computer than by hand (it's easy for a human being to miss some pentagons and it's hard to keep track). If possible, I would suggest the author to provide a computer program and the output to show this claim.
8. I would suggest they put the cited references all in the last part and remove the footnote (but that's perhaps only a personal taste). Perhaps another way is to keep both.
9. Overall, I do not find many mistakes or typos in the paper.