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MEAN SHADOW OF ROTATING OBJECTS

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Abstract. In this paper, we conduct an analysis of the problem concerning the mean shadow cast by rotating objects. The original problem was introduced by Cauchy in 1832. He proposed solutions for the 2-D and 3-D scenarios in 1842 and 1850 respectively. In the original problem, the shadow was formed by orthogonal projection. In 2022, the problem was revisited under the 3-D scenario of a light source with finite distance above the rotation center. Instead of 3-D scenario, we focus on the 2-D case and generalize the problem by placing the light source arbitrarily. We derive explicit formulae of the mean shadow. With these formulae, we provide a numerical method to compute the mean shadow, which surpasses the conventional simulation.

Contents

1. Introduction

1.1. **Motivation.** This investigation is motivated by [1], a YouTube video which proposes the following question:

Question 1.1. *Find the average area of a cube's shadow.*

The average area of the shadow is taken over all possible orientations of the cube rotating about its center. The shadow is formed by an infinite light source positioned above the cube.

In the video, two distinctive solutions are presented. One solution involves concrete computations while the other makes use of observation and symmetry. Finally, the result is generalized from cubes to all 3-D convex objects.

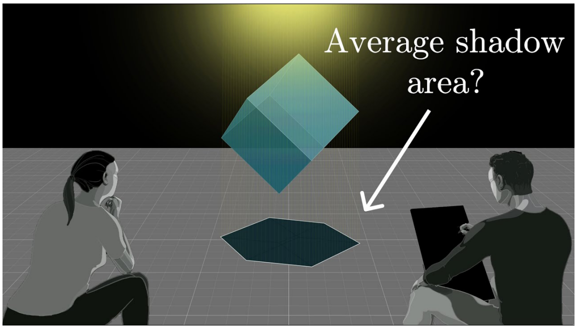

FIGURE 1. Thumbnail of the YouTube video from 3Blue1Brown ([1])

To gain a deeper understanding towards related problems, literature review is carried beforehand.

1.2. **Literature review.** Three results are found regarding the mean shadow problem. The first two concerns orthogonal projection (light source of infinite distance) while the third extends the realm to light source of finite distance. These results are listed in the following subsections.

1.2.1. *Infinite light source.* The classical problems concerning mean shadow in $\mathbb{R}^2$ and $\mathbb{R}^3$ were proposed by Cauchy in [2] and [3]. The following questions and theorems summerize Cauchy's findings.

Question 1.2.

- *In 2-D space, find the average length of the shadow of a rotating convex object, where the shadow is formed by an infinite light source positioned above the object.*
- *In 3-D space, find the average area of the shadow of a rotating convex object, where the shadow is formed by an infinite light source positioned above the object.*

Theorem 1.3 (Cauchy, [2] & [3])**.**

- *For 2-D space, the average length of the shadow of the convex object is $\frac{1}{\pi}$ times its perimeter.*
- *For 3-D space, the average area of the shadow of the convex object is $\frac{1}{4}$ times its surface area.*

Sketch of Proof.

- For 2-D space, the object is first decomposed into an infinite number of infinitesimal segments. Considering a single line segment of length ℓ rotating in space, we can readily compute its mean shadow:

$$\frac{1}{2\pi}\int_0^{2\pi} |\ell \cos\theta| \, \mathrm{d}\theta = \frac{2}{\pi} \cdot \ell$$

  Consequently, the mean shadow of the infinitesimal segments can be reassembled into the mean shadow of the object:

$$\text{Mean shadow} = \frac{1}{2} \cdot \left(\frac{2}{\pi} \cdot \text{Perimeter}\right) = \frac{1}{\pi} \cdot \text{Perimeter}$$

  Here, we have to divide by 2 because any straight line passing through a convex object must intersect the edge of the shape twice (except for tangent lines).

- For 3-D space, the object is decomposed into an infinite number of infinitesimal subrectangles. Considering a single subrectangle of area A rotating in space, we can readily compute its mean shadow:

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} |A \cos\theta| \cdot \frac{|2\pi \cos\theta|}{2\pi} \, \mathrm{d}\theta = \frac{1}{2} \cdot A$$

  Consequently, the mean shadow of the infinitesimal subrectangles can be reassembled into the mean shadow of the object:

$$\text{Mean shadow} = \frac{1}{2} \cdot \left(\frac{1}{2} \cdot \text{Surface area}\right) = \frac{1}{4} \cdot \text{Surface area}$$

□

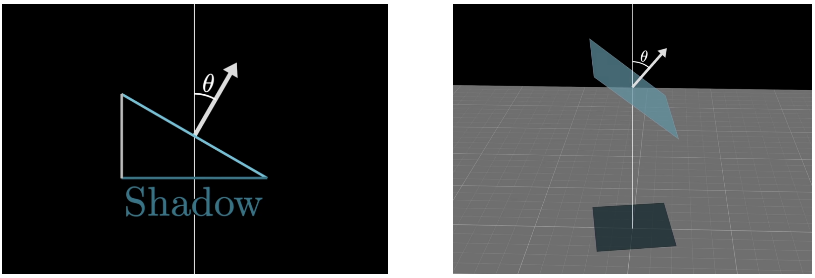

FIGURE 2. Shadow of line segment or subrectangle, captured from [1]

Note 1.4. *We can easily verify Theorem 1.3 through a circle in 2-D space and a sphere in 3-D space. Notice that their shadow remain the same under all possible orientations.*

- *For the circle with radius r in 2-D space,*

$$\frac{\textit{average length of shadow}}{\textit{perimeter}} = \frac{2r}{2\pi r} = \frac{1}{\pi}.$$

- *For the sphere with radius r in 3-D space,*

$$\frac{\textit{average area of shadow}}{\textit{surface area}} = \frac{\pi r^2}{4\pi r^2} = \frac{1}{4}.$$

Having succeeded in both $\mathbb{R}^2$ and $\mathbb{R}^3$, a general formula to $\mathbb{R}^n$ was proved by T. Bonnesen, H. Minkowski and T. Kubota in [4], [5] and [6] respectively. This theorem posits that the mean orthogonal projection of a convex object is equal to its surface area multiplied by a dimension-dependent constant.

Theorem 1.5 (Cauchy's Surface Area Formula,[7])**.** *Denote μ_{n-1} the $(n-1)$-dimensional Lebesgue measure on $\mathbb{R}^{n-1}$. Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, and $\mathbb{B}^{n-1}$ be the unit ball in $\mathbb{R}^{n-1}$.*

Given a convex body $K \subset \mathbb{R}^n$ and $u \in \mathbb{S}^{n-1}$, denote $K|u^\perp$ the orthogonal projection of K onto the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ perpendicular to u. Furthermore, denote $S(K)$ the volume of the surface of K. We have

$$S(K) = \frac{1}{\mu_{n-1}(\mathbb{B}^{n-1})} \int_{\mathbb{S}^{n-1}} \mu_{n-1}(K|u^\perp)\, \mathrm{d}u\,.$$

Note 1.6. *Theorem 1.5 tells us that the ratio between the mean shadow of K and the the volume of the surface of K can be found by*

$$\frac{\mu_{n-1}(\mathbb{B}^{n-1})}{\mu_n(\mathbb{S}^{n-1})}.$$

The detailed proof can be found in [7].

1.2.2. *Finite light source.* While the two results mentioned above only deals with orthogonal projection (light source of infinite distance), one may wonder what happens if the projection is non-orthogonal, i.e. the light source is of finite distance. This leads us to the following question:

Question 1.7. *In 3-D space, find the average area of the shadow of a cube rotating about its center, where the shadow is formed by a finite light source positioned above its center.*

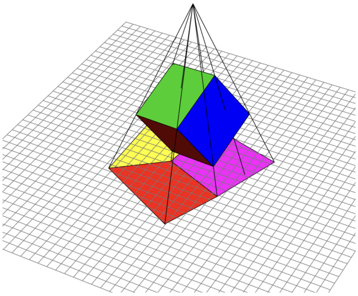

FIGURE 3. An illustration of the question, captured from [8]

A solution to Question 1.7 is provided in [8], a personal website of G. Egan. He derives an expression of the average shadow cast by a unit cube. Taking one step further, he extends this result to all regular polyhedra.

Theorem 1.8 ([8]). *In 3-D space, the average area of the shadow casted by a regular m-sided polyhedron rotating around the center with circumradius r, center height h, face at distance p from center and light source at height δ right above center is given by*

$$(h+\delta)^2\left[\frac{m}{2}\operatorname{arctan2}\left(y=2\sin\frac{2\pi}{m}\sqrt{(p^2-\delta^2)(p^2+r^2-\delta^2)}, x=-r^2-(2p^2+r^2-2\delta^2)\cos\frac{2\pi}{m}\right)-\pi\right].$$

With Theorem 1.8, Egan plots the following graph for various regular polyhedra and sphere with $r = h = 1$:

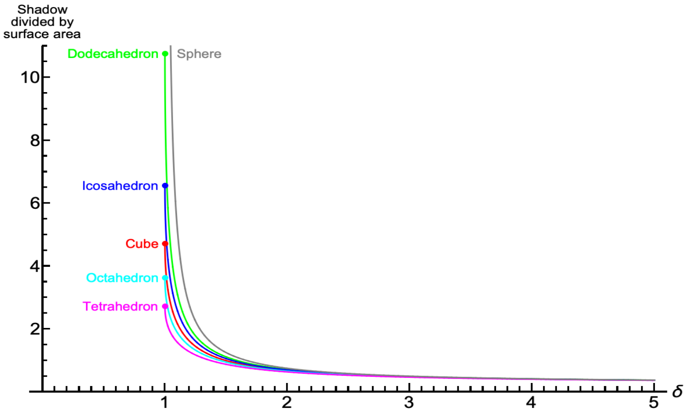

FIGURE 4. Variation of mean shadow to surface area w.r.t δ, captured from [8]

Note 1.9. *From the graph, we can see that as $\delta \to \infty$, the ratio of mean shadow to the surface area of all objects appears to converge to $\frac{1}{4}$, which matches the result of Theorem 1.3.*

The key ideas that Egan introduces are to consider a single face of the polyhedron and cut it into rectangles. In this paper, we recognize the importance of Egan's findings and adopt his approach.

1.3. **Setting Up the Main Question.** Our study is based on the 2-D analogue of Egan's work. However, we aim to construct a system that can accommodate more irregular shapes and arbitrary rotation centers. We also permit the free movement of the light source. We begin with assumptions and definitions before stating the main question of this paper:

Assumption 1.10.

- *Any part of the object is above the ground.*
- *Any part of the object is under the light source.*

These assumptions ensure the shadow is cast properly. Next, we introduce the following notations for the position of the light source and the rotation center.

Definition 1.11 ($\mathfrak{R}$ and h)**.** *Denote $\mathfrak{R}$ as the* ***rotation center*** *of the object. Let h be the* ***height*** *of $\mathfrak{R}$ above the ground.*

Definition 1.12 ($\mathfrak{S}$ and δ)**.** *Denote $\mathfrak{S}$ as* ***the position of the light source****. Let δ be the* ***vertical distance*** *between $\mathfrak{R}$ and $\mathfrak{S}$.*

Definition 1.13 (Mean shadow)**.** *The average length of the shadow of an object is called the mean shadow.*

We now propose the main question of this paper.

Question 1.14. *In 2-D space, find the mean shadow of a convex object rotating about $\mathfrak{R}$, where the shadow is formed by a light source located at $\mathfrak{S}$.*

In section 2, we consider the conventional case of the light source being directly above the center of the regular polygon rotating about it's center.

In section 3.1, we permit the light source to deviate from the position above the center of the polygon. In section 3.2, we let the regular polygon to no longer rotate about it's center but an arbitrary rotation point. In section 3.3, we extend to any convex shape. By systematically lifting the constraints imposed on the system, we have managed to provide solutions expressed in explicit formulas for each stage. In the end, we discuss practical applications of our findings and point out potential directions for future research, suggesting feasible explorations that could further build upon the foundation laid by our thesis.

2. Groundwork

To begin with, we focus on the most fundamental convex objects, regular polygons.

Question 2.1. *Find the mean shadow of the regular m-sided polygon with circumradius r and centered at $\mathfrak{R}$.*

An illustrative example is shown below:

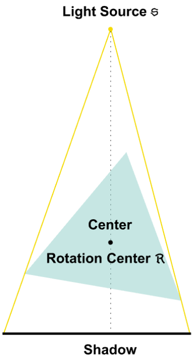

Figure 5. Illustrative example for Question 2.1

To solve the problem, we introduce the coordinate system. Without loss of generality, set $\mathfrak{R}$ as the origin. i.e., $\mathfrak{R}(0,0)$ and $\mathfrak{S}(0,\delta)$. Meanwhile, we also consider the projection of objects onto the x-axis instead of the ground (located on the line $y=-h$). The following definition arises:

Definition 2.2 (Projection and Shadow)**.** *In the coordinated system, we call the projection of an object from the light source to the x-axis a* ***projection*** *and call the projection of an object from the light source to the ground a* ***shadow****.*

Figure 6 illustrates the difference between a projection and a shadow. From the similar triangles formed, their length always differed by a ratio of $\frac{h+\delta}{\delta}$.

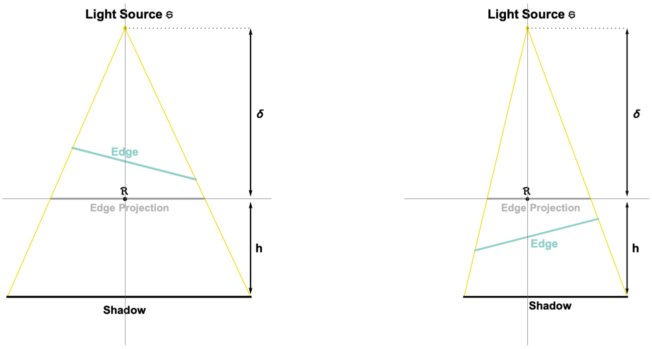

Figure 6. Illustrations of projection and shadow

A basic workflow is established to solve **Question 2.1**. It is also applied in further section to solve a more general problem. The basic workflow is demonstrated below:

(1) Consider only an edge of the object.

(2) Find an expression of the length of the projection of the edge at a certain angle of rotation.

(3) Compute the anti-derivative for the expression.

(4) Integrate the expression from 0 to 2π, and divide by 2π to obtain the mean projection over all possible orientations.

(5) Reassemble all edges to find the mean shadow of the complete object.

To start with, the following definition defines symbols related to the position of an edge.

Definition 2.3 (η and σ)**.** *Placing an edge horizontally directly above the origin,*

- *denote η as the distance between the midpoint of the edge and the origin.*
- *denote σ as half of the length of the edge.*

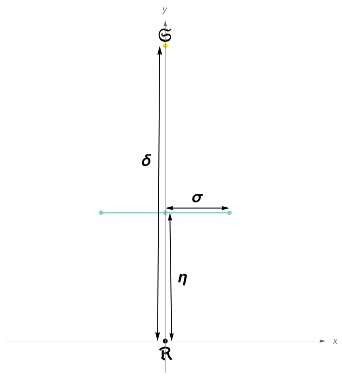

FIGURE 7. Illustration of η and σ

Definition 2.4 (θ). *Denote $\theta \in [0, 2\pi)$ as the angle of rotation of an edge in anti-clockwise direction.*

FIGURE 8. Illustration of the angle of rotation θ

With these symbols, we find an expression for the edge's projection at a certain angle of rotation.

Theorem 2.5 ($S_1(\theta)$). *The length of the edge projection at θ is $|S_1(\theta)|$, where* $S_1(\theta) = \dfrac{2\delta\sigma(\delta\cos\theta - \eta)}{(\delta - \eta\cos\theta)^2 - \sigma^2\sin^2\theta}$.

Proof. Notice that the coordinates of endpoints of the edge are

$$(\sigma\cos\theta - \eta\sin\theta, \sigma\sin\theta + \eta\cos\theta) \quad \text{and} \quad (-\sigma\cos\theta - \eta\sin\theta, -\sigma\sin\theta + \eta\cos\theta).$$

Their projections on the x-axis are then given respectively by

$$\left(\frac{\delta(\sigma\cos\theta - \eta\sin\theta))}{\delta - (\sigma\sin\theta + \eta\cos\theta)}, 0\right) \quad \text{and} \quad \left(\frac{\delta(-\sigma\cos\theta - \eta\sin\theta))}{\delta - (-\sigma\sin\theta + \eta\cos\theta))}, 0\right).$$

Hence, the edge projection at rotation angle θ can be found by

$$\left|\frac{\delta(\sigma\cos\theta - \eta\sin\theta))}{\delta - (\sigma\sin\theta + \eta\cos\theta)} - \frac{\delta(-\sigma\cos\theta - \eta\sin\theta))}{\delta - (-\sigma\sin\theta + \eta\cos\theta))}\right| = \left|\frac{2\delta\sigma(\delta\cos\theta - \eta)}{(\delta - \eta\cos\theta)^2 - \sigma^2\sin^2\theta}\right|.$$

□

Lemma 2.6 ($I_1(\theta)$). *An anti-derivative of $S_1(\theta)$ is given by*

$$I_1(\theta) = 2\delta \operatorname{artanh} \frac{\sigma\sin\theta}{\delta - \eta\cos\theta}.$$

Proof. Applying a tangent half-angle substitution $t = \tan\frac{\theta}{2}$,

$$\int S_1(\theta)\,\mathrm{d}\theta = 2\delta\sigma \int \frac{\delta \cdot \frac{1-t^2}{1+t^2} - \eta}{(\delta - \eta \cdot \frac{1-t^2}{1+t^2})^2 - \sigma^2 \cdot \frac{4t^2}{(1+t^2)^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= -4\delta\sigma \int \frac{(\delta+\eta)t^2 + (\delta-\eta)}{[(\delta+\eta)t^2 + 2\sigma t + (\delta-\eta)][(\delta+\eta)t^2 - 2\sigma t + (\delta-\eta)]}\,\mathrm{d}t\,.$$

Using the technique of partial fractions, we have

$$\begin{aligned}\int S_1(\theta)\,\mathrm{d}\theta &= 2\delta\int\frac{(\delta+\eta)t+\sigma}{(\delta+\eta)t^2+2\sigma t+(\delta-\eta)}\,\mathrm{d}t - 2\delta\int\frac{(\delta+\eta)t-\sigma}{(\delta+\eta)t^2-2\sigma t+(\delta-\eta)}\,\mathrm{d}t\\ &= \delta\ln\left|(\delta+\eta)t^2+2\sigma t+(\delta-\eta)\right| - \delta\ln\left|(\delta+\eta)t^2-2\sigma t+(\delta-\eta)\right| + C\\ &= \delta\ln\left|\frac{\delta-\eta\cos\theta+\sigma\sin\theta}{\delta-\eta\cos\theta-\sigma\sin\theta}\right| + C\\ &= 2\delta\,\mathrm{artanh}\,\frac{\sigma\sin\theta}{\delta-\eta\cos\theta} + C.\end{aligned}$$

In the last step, notice that the argument in the logarithm is always positive and we have used the following identity for inverse hyperbolic tangent:

$$2\,\mathrm{artanh}(x) = \ln\frac{1+x}{1-x}.$$

□

Having completed the calculation of the anti-derivative of $S_1(\theta)$, we determine the signs of $S_1(\theta)$.

Lemma 2.7. *Let*

$$\theta_1 = \arccos\frac{\eta}{\delta} \quad \textit{and} \quad \theta_2 = 2\pi - \arccos\frac{\eta}{\delta}.$$

- *If* $\theta \in [0,\theta_1)\cup(\theta_2,2\pi)$, *then* $S_1(\theta) > 0$.
- *If* $\theta \in (\theta_1,\theta_2)$, *then* $S_1(\theta) < 0$.

Proof. Notice that $S_1(\theta)$ is a continuous function.

We only need to consider the sign of $\delta\cos\theta-\eta$, as all other factor in $S_1(\theta)$ are always positive. Solving $\delta\cos\theta-\eta=0$ for $\theta\in[0,2\pi)$, we have $\theta=\theta_1$ or θ_2. Since $0<\theta_1<\pi<\theta_2<2\pi$, the result can then be verified by testing the signs of $\delta\cos\theta-\eta$ at $\theta=0,\pi$ and 2π. □

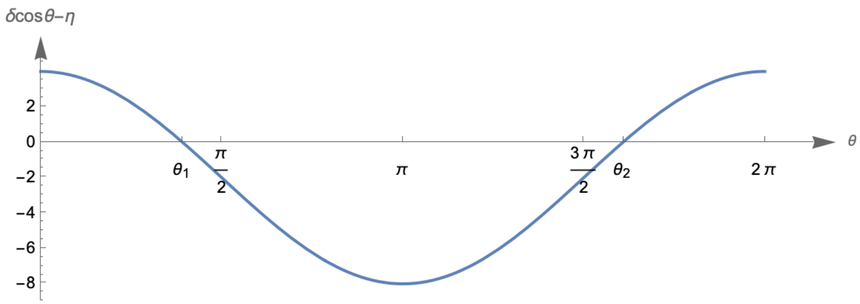

FIGURE 9. Plot of $\delta\cos\theta-\eta$ for $\theta\in[0,2\pi)$ for $\eta=2$, and $\delta=6$

Theorem 2.8. *The mean shadow of the edge is given by*

$$\frac{4(h+\delta)}{\pi}\operatorname{artanh}\frac{\sigma}{\sqrt{\delta^2-\eta^2}}.$$

Proof. We first compute the mean projection of the edge. By **Theorem 2.5** and **Lemma 2.7**,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|S_1(\theta)|\,\mathrm{d}\theta &= \frac{1}{2\pi}\int_0^{\theta_1}S_1(\theta)\,\mathrm{d}\theta - \frac{1}{2\pi}\int_{\theta_1}^{\theta_2}S_1(\theta)\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{\theta_2}^{2\pi}S_1(\theta)\,\mathrm{d}\theta\\ &= \frac{1}{2\pi}\left[(I_1(\theta_1)-I_1(0))+(I_1(\theta_1)-I_1(\theta_2))+(I_1(2\pi)-I_1(\theta_2))\right]\\ &= \frac{1}{\pi}(I_1(\theta_1)-I_1(\theta_2))\\ &= \frac{2\delta}{\pi}\left(\operatorname{artanh}\frac{\sigma\sin\theta_1}{\delta-\eta\cos\theta_1}-\operatorname{artanh}\frac{\sigma\sin\theta_2}{\delta-\eta\cos\theta_2}\right)\end{aligned}$$

Note that we have

$$\sin\theta_1 = -\sin\theta_2 = \sqrt{1-\frac{\eta^2}{\delta^2}} \quad\text{and}\quad \cos\theta_1 = \cos\theta_2 = \frac{\eta}{\delta}.$$

Together with $\operatorname{artanh}(-x) = -\operatorname{artanh}(x)$,

$$\frac{1}{2\pi}\int_0^{2\pi}|S_1(\theta)|\,\mathrm{d}\theta = \frac{4\delta}{\pi}\operatorname{artanh}\frac{\sigma\sqrt{1-\frac{\eta^2}{\delta^2}}}{\delta-\eta\cdot\frac{\eta}{\delta}} = \frac{4\delta}{\pi}\operatorname{artanh}\frac{\sigma}{\sqrt{\delta^2-\eta^2}}$$

The desired result can be obtained by multiplying the constant $\frac{h+\delta}{\delta}$. □

For the regular m-sided polygon as in **Question 2.1**, we have

$$\eta = r\cos\frac{\pi}{m} \quad\text{and}\quad \sigma = r\sin\frac{\pi}{m}.$$

Substituting the above relation in **Theorem 2.8** and multiplying the result by $\frac{m}{2}$, **Question 2.1** is answered by the following theorem:

Theorem 2.9. *In 2-D space, the mean shadow of the regular m-sided polygon with circumradius r and centered at $\mathfrak{R}$ is*

$$\frac{2m(h+\delta)}{\pi}\operatorname{artanh}\frac{r\sin\frac{\pi}{m}}{\sqrt{\delta^2-r^2\cos^2\frac{\pi}{m}}}.$$

3. The Generalized Problem

3.1. **Parameters Altering Edge Projection.** In this section, we continue to focus on an edge while introducing new degrees of freedom to the system, and talk about how the edge projection formula changes with new parameters added.

3.1.1. *Arbitrary Position of Edge.* In the previous section, the edge is limited to be aligned with the center of rotation $\mathfrak{R}$. This constraint is now relaxed:

Definition 3.1 (τ)**.** *Placing the edge horizontally, denote τ as the horizontal distance between the midpoint of the edge and the origin.*

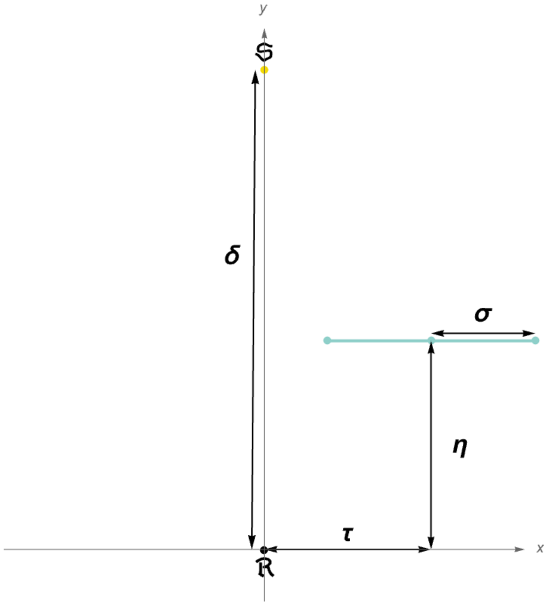

FIGURE 10. Illustration of τ

The following figures demonstrate the rotation of an edge:

(A) (B)

FIGURE 11. Illustration of the edge at different rotation angles

With these symbols, we then can compute the expression for the edge projection at a certain rotation degree θ.

Theorem 3.2 ($S_2(\theta)$)**.** *The length of the edge projection at θ is $|S_2(\theta)|$, where*

$$S_2(\theta) = \frac{2\delta\sigma(\delta\cos\theta - \eta)}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}.$$

Proof. Notice that the coordinates of endpoints of the edge are

$$(\tau\cos\theta + \sigma\cos\theta - \eta\sin\theta, \tau\sin\theta + \sigma\sin\theta + \eta\cos\theta)$$
$$\text{and}$$
$$(\tau\cos\theta - \sigma\cos\theta - \eta\sin\theta, \tau\sin\theta - \sigma\sin\theta + \eta\cos\theta).$$

Their projections on the x-axis are then given respectively by

$$\left(\frac{\delta(\tau\cos\theta + \sigma\cos\theta - \eta\sin\theta)}{\delta - (\tau\sin\theta + \sigma\sin\theta + \eta\cos\theta)}, 0\right) \quad \text{and} \quad \left(\frac{\delta(\tau\cos\theta - \sigma\cos\theta - \eta\sin\theta)}{\delta - (\tau\sin\theta - \sigma\sin\theta + \eta\cos\theta)}, 0\right).$$

Hence, the edge projection at rotation angle θ can be found by

$$\begin{aligned}&\left|\frac{\delta(\tau\cos\theta + \sigma\cos\theta - \eta\sin\theta)}{\delta - (\tau\sin\theta + \sigma\sin\theta + \eta\cos\theta)} - \frac{\delta(\tau\cos\theta - \sigma\cos\theta - \eta\sin\theta)}{\delta - (\tau\sin\theta - \sigma\sin\theta + \eta\cos\theta)}\right| \\ &= \left|\frac{2\delta\sigma(\delta\cos\theta - \eta)}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}\right|.\end{aligned}$$

□

Note 3.3. *Compare the formula for $S_1(\theta)$ and $S_2(\theta)$:*

$$S_1(\theta) = \frac{2\delta\sigma(\delta\cos\theta - \eta)}{(\delta - \eta\cos\theta)^2 - \sigma^2\sin^2\theta} \qquad S_2(\theta) = \frac{2\delta\sigma(\delta\cos\theta - \eta)}{(\delta - \eta\cos\theta \boldsymbol{- \tau\sin})^2 - \sigma^2\sin^2\theta}.$$

3.1.2. *Arbitrary Position of Light Source.* We also relax the constraint of the light source being right above the rotation center:

Definition 3.4 (λ)**.** *Let λ be the horizontal distance between $\mathfrak{R}$ and $\mathfrak{S}$.*

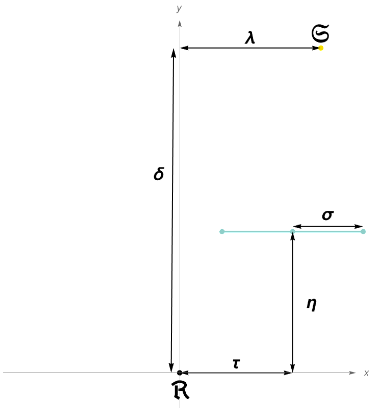

FIGURE 12. Illustration of λ

With these symbols, we then can compute the expression for the edge projection at a certain rotation degree θ.

Theorem 3.5 ($S_3(\theta)$). *The length of the edge projection at θ is $|S_3(\theta)|$, where*

$$S_3(\theta) = \frac{2\delta\sigma(\delta\cos\theta - \eta - \lambda\sin\theta)}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}.$$

Proof. Notice that the coordinates of endpoints of the edge are the same as those in the previous section

$$(\tau\cos\theta + \sigma\cos\theta - \eta\sin\theta, \tau\sin\theta + \sigma\sin\theta + \eta\cos\theta)$$

and

$$(\tau\cos\theta - \sigma\cos\theta - \eta\sin\theta, \tau\sin\theta - \sigma\sin\theta + \eta\cos\theta).$$

Their projections on the x-axis are then given respectively by

$$\left(\lambda - \delta\cdot\frac{\lambda - (\tau\cos\theta + \sigma\cos\theta - \eta\sin\theta)}{\delta - (\tau\sin\theta + \sigma\sin\theta + \eta\cos\theta)}, 0\right)$$

and

$$\left(\lambda - \delta\cdot\frac{\lambda - (\tau\cos\theta - \sigma\cos\theta - \eta\sin\theta)}{\delta - (\tau\sin\theta - \sigma\sin\theta + \eta\cos\theta)}, 0\right).$$

Hence, the edge projection at rotation angle θ can be found by

$$\begin{aligned}
&\left|\left(\lambda - \delta\cdot\frac{\lambda - (\tau\cos\theta + \sigma\cos\theta - \eta\sin\theta)}{\delta - (\tau\sin\theta + \sigma\sin\theta + \eta\cos\theta)}\right)\right. \\
&\qquad \left. - \left(\lambda - \delta\cdot\frac{\lambda - (\tau\cos\theta - \sigma\cos\theta - \eta\sin\theta)}{\delta - (\tau\sin\theta - \sigma\sin\theta + \eta\cos\theta)}\right)\right| \\
&= \left|\frac{2\delta\sigma(\delta\cos\theta - \eta - \lambda\sin\theta)}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}\right|.
\end{aligned}$$

□

Note 3.6. *Compare the formula of $S_2(\theta)$ and $S_3(\theta)$:*

$$S_2(\theta) = \frac{2\delta\sigma(\delta\cos\theta - \eta)}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}$$
$$S_3(\theta) = \frac{2\delta\sigma(\delta\cos\theta - \eta \boldsymbol{- \lambda} \mathbf{sin})}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}$$

3.2. **Mean Shadow Calculation and Assembling Full Shape.** In this section, we will be evaluating both the anti-derivative and the definite integrals corresponding to the edge projection formula calculated in the previous section and simplify the results. Upon doing so we can reassemble the edges into a complete shape.

3.2.1. *Anti-derivative.*

Lemma 3.7 ($(I_3(\theta))$)**.** *An anti-derivative of $S_3(\theta)$ is given by*

$$I_3(\theta) = 2\delta\Bigg(\operatorname{artanh}\frac{\sigma\sin\theta}{\delta - \eta\cos\theta - \tau\sin\theta}$$
$$+ \frac{\lambda}{\sqrt{\delta^2 - (\sigma-\tau)^2 - \eta^2}}\arctan\frac{(\delta+\eta)\tan\frac{\theta}{2} + \sigma - \tau}{\sqrt{\delta^2 - (\sigma-\tau)^2 - \eta^2}}$$
$$- \frac{\lambda}{\sqrt{\delta^2 - (\sigma+\tau)^2 - \eta^2}}\arctan\frac{(\delta+\eta)\tan\frac{\theta}{2} - \sigma - \tau}{\sqrt{\delta^2 - (\sigma+\tau)^2 - \eta^2}}\Bigg).$$

Proof. We write $S_3(\theta) = S_3'(\theta) + S_3''(\theta)$, where

$$S_3'(\theta) = \frac{2\delta\sigma(\delta\cos\theta - \eta)}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}$$

$$S_3''(\theta) = \frac{-2\delta\sigma\lambda\sin\theta}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}.$$

. . . .

To integrate $S_3'(\theta)$, apply tangent half-angle substitution $t = \tan\frac{\theta}{2}$:

$$\begin{aligned}
&\int S_3'(\theta)\,\mathrm{d}\theta \\
&= \int \frac{2\delta\sigma(\delta\cos\theta - \eta)}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}\,\mathrm{d}\theta \\
&= 2\delta\sigma \int \frac{\delta\cdot\frac{1-t^2}{1+t^2} - \eta}{(\delta - \eta\cdot\frac{1-t^2}{1+t^2} - \tau\cdot\frac{2t}{1+t^2})^2 - \sigma^2\cdot\frac{4t^2}{(1+t^2)^2}} \cdot \frac{2\,\mathrm{d}t}{1+t^2} \\
&= 4\delta\sigma \int \frac{\delta(1-t^2) - \eta(1+t^2)}{(\delta(1+t^2) - \eta(1-t^2) - 2\tau t)^2 - 4\sigma^2t^2}\,\mathrm{d}t \\
&= -4\delta\sigma \int \frac{(\delta+\eta)t^2 - \delta + \eta}{((\delta+\eta)t^2 + 2\sigma t - 2\tau t + \delta - \eta)\cdot((\delta+\eta)t^2 - 2\sigma t - 2\tau t + \delta - \eta)}\,\mathrm{d}t\,.
\end{aligned}$$

Using the technique of partial fraction, we have

$$\begin{aligned}
\int S_3'(\theta)\,\mathrm{d}\theta &= 2\delta \int \frac{(\delta+\eta)t + \sigma - \tau}{(\delta+\eta)t^2 + 2\sigma t - 2\tau t + \delta - \eta}\,\mathrm{d}t \\
&\quad - 2\delta \int \frac{(\delta+\eta)t - \sigma - \tau}{(\delta+\eta)t^2 - 2\sigma t - 2\tau t + \delta - \eta}\,\mathrm{d}t \\
&= \delta \ln \frac{(\delta+\eta)t^2 + 2\sigma t - 2\tau t + \delta - \eta}{(\delta+\eta)t^2 - 2\sigma t - 2\tau t + \delta - \eta} + C \\
&= \delta \ln \frac{\delta - \eta\cdot\frac{1-t^2}{1+t^2} + \sigma\cdot\frac{2t}{1+t^2} - \tau\cdot\frac{2t}{1+t^2}}{\delta - \eta\cdot\frac{1-t^2}{1+t^2} - \sigma\cdot\frac{2t}{1+t^2} - \tau\cdot\frac{2t}{1+t^2}} + C \\
&= 2\delta\,\operatorname{artanh} \frac{\sigma\sin\theta}{\delta - \eta\cos\theta - \tau\sin\theta} + C.
\end{aligned}$$

. . . .

To integrate $S_3''(\theta)$, we also apply tangent half-angle substitution $t = \tan\frac{\theta}{2}$:

$$\begin{aligned}
&\int S_3''(\theta)\,\mathrm{d}\theta \\
&= -\int \frac{2\delta\sigma\lambda\sin\theta}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}\,\mathrm{d}\theta \\
&= -2\delta\sigma \int \frac{\lambda\cdot\frac{2t}{1+t^2}}{(\delta - \eta\cdot\frac{1-t^2}{1+t^2} - \tau\cdot\frac{2t}{1+t^2})^2 - \sigma^2(\frac{2t}{1+t^2})^2} \cdot \frac{2\,\mathrm{d}t}{1+t^2} \\
&= -8\delta\sigma\lambda \int \frac{t}{(\delta(1+t^2) - \eta(1-t^2) - 2\tau t)^2 - 4\sigma^2t^2}\,\mathrm{d}t \\
&= -8\delta\sigma\lambda \int \frac{t}{((\delta+\eta)t^2 + 2\sigma t - 2\tau t + \delta - \eta)\cdot((\delta+\eta)t^2 - 2\sigma t - 2\tau t + \delta - \eta)}\,\mathrm{d}t \\
&= 2\delta\lambda\left(\int \frac{\mathrm{d}t}{(\delta+\eta)t^2 + 2\sigma t - 2\tau t + \delta - \eta} - \int \frac{\mathrm{d}t}{(\delta+\eta)t^2 - 2\sigma t - 2\tau t + \delta - \eta}\right).
\end{aligned}$$

Using the technique of completing the square, we have

$$
\begin{aligned}
\int S_3''(\theta)\,\mathrm{d}\theta &= \frac{2\delta\lambda}{\delta+\eta}\left(\int \frac{\mathrm{d}t}{t^2+\frac{2(\sigma-\tau)}{\delta+\eta}t+\frac{\delta-\eta}{\delta+\eta}} - \int \frac{\mathrm{d}t}{t^2-\frac{2(\sigma+\tau)}{\delta+\eta}t+\frac{\delta-\eta}{\delta+\eta}}\right) \\
&= \frac{2\delta\lambda}{\delta+\eta}\left(\int \frac{\mathrm{d}(t+\frac{\sigma-\tau}{\delta+\eta})}{(t+\frac{\sigma-\tau}{\delta+\eta})^2+\frac{\delta^2-(\sigma-\tau)^2-\eta^2}{(\delta+\eta)^2}} - \int \frac{\mathrm{d}(t-\frac{\sigma+\tau}{\delta+\eta})}{(t-\frac{\sigma+\tau}{\delta+\eta})^2+\frac{\delta^2-(\sigma+\tau)^2-\eta^2}{(\delta+\eta)^2}}\right) \\
&= \frac{2\delta\lambda}{\delta+\eta}\left(\frac{1}{\sqrt{\frac{\delta^2-(\sigma-\tau)^2-\eta^2}{(\delta+\eta)^2}}}\arctan\frac{t+\frac{\sigma-\tau}{\delta+\eta}}{\sqrt{\frac{\delta^2-(\sigma-\tau)^2-\eta^2}{(\delta+\eta)^2}}}\right. \\
&\qquad \left. - \frac{1}{\sqrt{\frac{\delta^2-(\sigma+\tau)^2-\eta^2}{(\delta+\eta)^2}}}\arctan\frac{t-\frac{\sigma+\tau}{\delta+\eta}}{\sqrt{\frac{\delta^2-(\sigma+\tau)^2-\eta^2}{(\delta+\eta)^2}}}\right) + C \\
&= \frac{2\delta\lambda}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\arctan\frac{(\delta+\eta)t+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}} \\
&\qquad - \frac{2\delta\lambda}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\arctan\frac{(\delta+\eta)t-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}} + C \\
&= \frac{2\delta\lambda}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\arctan\frac{(\delta+\eta)\tan\frac{\theta}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}} \\
&\qquad - \frac{2\delta\lambda}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\arctan\frac{(\delta+\eta)\tan\frac{\theta}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}} + C.
\end{aligned}
$$

The result can be obtained by adding up the two indefinite integrals of $S_3'(\theta)$ and $S_3''(\theta)$. □

3.2.2. *Sign-changing Point.* Having completed the calculation of the anti-derivative within the absolute value, we need to deal with the absolute value. To begin with, we need to determine when $S_3(\theta)$ is positive.

Lemma 3.8. *Let*

$$
\begin{aligned}
\theta_1 &= \arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}} - \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}, \quad \textit{and} \\
\theta_2 &= 2\pi - \arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}} - \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}.
\end{aligned}
$$

- *If* $\theta \in [0,\theta_1)\cup(\theta_2,2\pi)$, *then* $S_3(\theta) > 0$.
- *If* $\theta \in (\theta_1,\theta_2)$, *then* $S_3(\theta) < 0$.

Proof. Notice that $S_3(\theta)$ is a continuous function.

We only need to consider $\delta\cos\theta - \eta - \lambda\sin\theta$ in

$$S_3(\theta) = \frac{2\delta\sigma(\delta\cos\theta - \eta - \lambda\sin\theta)}{(\delta - \eta\cos\theta - \tau\sin\theta)^2 - \sigma^2\sin^2\theta}$$

as all other factors are positive. We can determine when the sign of $S_3(\theta)$ may change by solving $\delta\cos\theta - \eta - \lambda\sin\theta = 0$ for $\theta \in [0, 2\pi)$:

$$\begin{aligned}
\delta\cos\theta - \eta - \lambda\sin\theta &= 0\\
\sqrt{\lambda^2+\delta^2}\left(\frac{\delta}{\sqrt{\lambda^2+\delta^2}}\cos\theta - \frac{\lambda}{\sqrt{\lambda^2+\delta^2}}\sin\theta\right) &= \eta\\
\cos\left(\arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}\right)\cos\theta - \sin\left(\arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}\right)\sin\theta &= \frac{\eta}{\sqrt{\lambda^2+\delta^2}}\\
\cos\left(\theta + \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}\right) &= \frac{\eta}{\sqrt{\lambda^2+\delta^2}}.
\end{aligned}$$

Notice that $\delta > \eta \geq 0$ by **Assumption 1.10**.

Therefore,

$$\begin{aligned}
0 &< \arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}} - \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}} < \pi < 2\pi,\\
0 &< \pi < 2\pi - \arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}} - \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}} < 2\pi.
\end{aligned}$$

As $y = \cos\left(\theta + \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}\right)$ has a period of 2π, the equation

$$\cos\left(\theta + \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}\right) = \frac{\eta}{\sqrt{\lambda^2+\delta^2}}$$

has at most two solutions for $\theta \in [0, 2\pi)$, which are

$$\begin{aligned}
\theta_1 &= \arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}} - \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}\\
\theta_2 &= 2\pi - \arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}} - \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}.
\end{aligned}$$

At last, the result can be verified by testing the signs of $\delta\cos\theta - \eta - \lambda\sin\theta$ at $\theta = 0, \pi$ and 2π.

When $\theta = 0$ and $\theta = 2\pi$, $\delta\cos\theta - \eta - \lambda\sin\theta = \delta - \eta > 0$. When $\theta = \pi$, $\delta\cos\theta - \eta - \lambda\sin\theta = -\delta - \eta < 0$. The result follows. □

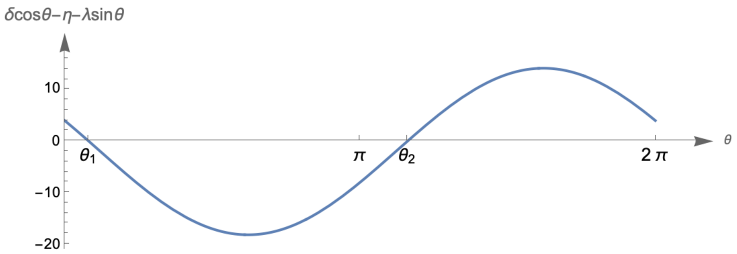

FIGURE 13. Plot of $\delta\cos\theta - \eta - \lambda\sin\theta$ for $\theta \in [0, 2\pi)$ for $\eta = 2$, $\delta = 6$, and $\lambda = 15$

3.2.3. *Definite Integral.*

Theorem 3.9. *The mean shadow of the edge is given by*

$$\frac{2(h+\delta)}{\pi}\Bigg(\operatorname{artanh}\frac{\sigma\sin\theta_1}{\delta-\eta\cos\theta_1-\tau\sin\theta_1}-\operatorname{artanh}\frac{\sigma\sin\theta_2}{\delta-\eta\cos\theta_2-\tau\sin\theta_2}$$
$$+\frac{\lambda}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\Big(-\frac{\pi}{2}+\arctan\frac{(\delta+\eta)\tan\frac{\theta_1}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}-\arctan\frac{(\delta+\eta)\tan\frac{\theta_2}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\Big)$$
$$-\frac{\lambda}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big(-\frac{\pi}{2}+\arctan\frac{(\delta+\eta)\tan\frac{\theta_1}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}-\arctan\frac{(\delta+\eta)\tan\frac{\theta_2}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big)\Bigg).$$

Proof. Mean edge projection is given by

$$\frac{1}{2\pi}\int_0^{2\pi}|S_3(\theta)|\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{\theta_1}S_3(\theta)\,\mathrm{d}\theta - \frac{1}{2\pi}\int_{\theta_1}^{\theta_2}S_3(\theta)\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{\theta_2}^{2\pi}S_3(\theta)\,\mathrm{d}\theta$$
$$= \frac{1}{2\pi}\Big(\int_0^{\theta_1}S_3(\theta)\,\mathrm{d}\theta + \int_{\theta_2}^{2\pi}S_3(\theta)\,\mathrm{d}\theta\Big) + \frac{1}{2\pi}\int_{\theta_2}^{\theta_1}S_3(\theta)\,\mathrm{d}\theta$$

....

The first part is

$$\frac{1}{2\pi}\Big(\int_0^{\theta_1}S_3(\theta)\,\mathrm{d}\theta + \int_{\theta_2}^{2\pi}S_3(\theta)\,\mathrm{d}\theta\Big)$$
$$= \frac{1}{2\pi}\left[(I_3(\theta_1)-I_3(0)) + (I_3(2\pi)-I_3(\theta_2))\right]$$
$$= \frac{1}{2\pi}\left[I_3(\theta_1)-I_3(\theta_2)\right]$$
$$= \frac{\delta}{\pi}\Bigg(\operatorname{artanh}\frac{\sigma\sin\theta_1}{\delta-\eta\cos\theta_1-\tau\sin\theta_1}-\operatorname{artanh}\frac{\sigma\sin\theta_2}{\delta-\eta\cos\theta_2-\tau\sin\theta_2}$$
$$+\frac{\lambda}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\Big(\arctan\frac{(\delta+\eta)\tan\frac{\theta_1}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}$$
$$-\arctan\frac{(\delta+\eta)\tan\frac{\theta_2}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\Big)$$

$$- \frac{\lambda}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big(\arctan \frac{(\delta+\eta)\tan\frac{\theta_1}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}} - \arctan \frac{(\delta+\eta)\tan\frac{\theta_2}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big)\Big).$$

. . . .

For the second part,

$$\frac{1}{2\pi}\int_{\theta_2}^{\theta_1} S_3(\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{\theta 2}^{\theta_1} S_3'(\theta)\,\mathrm{d}\theta + \frac{1}{2\pi}\big(\int_{\pi}^{\theta_1} S_3''(\theta)\,\mathrm{d}\theta - \int_{\theta_2}^{\pi} S_3''(\theta)\,\mathrm{d}\theta\big).$$

The calculation of the first integral is already shown in the previous section, given by

$$\frac{\delta}{\pi}\Big(\operatorname{artanh}\frac{\sigma\sin\theta_1}{\delta-\eta\cos\theta_1-\tau\sin\theta_1} - \operatorname{artanh}\frac{\sigma\sin\theta_2}{\delta-\eta\cos\theta_2-\tau\sin\theta_2}\Big).$$

The rest of the expression involves an improper integral, which requires a special approach.

$$\begin{aligned}
&\frac{1}{2\pi}\big(\int_{\pi}^{\theta_1} S_3''(\theta)\,\mathrm{d}\theta - \int_{\theta_2}^{\pi} S_3''(\theta)\,\mathrm{d}\theta\big)\\
&= \frac{1}{2\pi}\Big(\big[\tfrac{2\delta\lambda}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\arctan\tfrac{(\delta+\eta)t+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\\
&\qquad - \frac{2\delta\lambda}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\arctan\frac{(\delta+\eta)t-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big]_{\infty}^{\tan\frac{\theta_1}{2}}\\
&\quad + \big[\tfrac{2\delta\lambda}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\arctan\tfrac{(\delta+\eta)t+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\\
&\qquad - \frac{2\delta\lambda}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\arctan\frac{(\delta+\eta)t-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big]_{\tan\frac{\theta_2}{2}}^{-\infty}\Big)\\
&= \frac{\delta}{\pi}\Big(\frac{\lambda}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\Big(-\pi + \arctan\frac{(\delta+\eta)\tan\frac{\theta_1}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\\
&\qquad - \arctan\frac{(\delta+\eta)\tan\frac{\theta_2}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\Big)\\
&\quad - \frac{\lambda}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big(-\pi + \arctan\frac{(\delta+\eta)\tan\frac{\theta_1}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\\
&\qquad - \arctan\frac{(\delta+\eta)\tan\frac{\theta_2}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big)\Big).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\theta_2}^{\theta_1} S_3(\theta)\,\mathrm{d}\theta\\
&=\frac{\delta}{\pi}\Big(\operatorname{artanh}\frac{\sigma\sin\theta_1}{\delta-\eta\cos\theta_1-\tau\sin\theta_1}-\operatorname{artanh}\frac{\sigma\sin\theta_2}{\delta-\eta\cos\theta_2-\tau\sin\theta_2}\\
&\quad+\frac{\lambda}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\Big(-\pi+\arctan\frac{(\delta+\eta)\tan\frac{\theta_1}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}-\arctan\frac{(\delta+\eta)\tan\frac{\theta_2}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\Big)\\
&\quad-\frac{\lambda}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big(-\pi+\arctan\frac{(\delta+\eta)\tan\frac{\theta_1}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}-\arctan\frac{(\delta+\eta)\tan\frac{\theta_2}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big)\Big).
\end{aligned}
$$

Finally, the result can be obtained by adding two results together, and multiplying by $\frac{h+\delta}{\delta}$. □

3.2.4. *Simplification.* In this section, we will simplify the mean shadow formula for an edge found in the previous section.

Note 3.10. *Note that $I_3(\theta)$ is given by*

$$
\begin{aligned}
&2\delta\Big(\operatorname{artanh}\frac{\sigma\sin\theta}{\delta-\eta\cos\theta-\tau\sin\theta}+\frac{\lambda}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\arctan\frac{(\delta+\eta)\tan\frac{\theta}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\\
&\quad-\frac{\lambda}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\arctan\frac{(\delta+\eta)\tan\frac{\theta}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}\Big).
\end{aligned}
$$

To further simplify the expression, we must find the cosine and sine values of θ_1 and θ_2,

$$
\begin{aligned}
\cos\theta_1 &= \cos(\arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}}-\arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}})\\
&=\cos(\arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}})\cos(\arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}})\\
&\quad+\sin(\arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}})\sin(\arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}})\\
&=\frac{\eta}{\sqrt{\lambda^2+\delta^2}}\cdot\frac{\delta}{\sqrt{\lambda^2+\delta^2}}+\sqrt{1-\frac{\eta^2}{\lambda^2+\delta^2}}\cdot\sqrt{1-\frac{\delta^2}{\lambda^2+\delta^2}}\\
&=\frac{\delta\eta+\lambda\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}.
\end{aligned}
$$

And similarly,

$$
\begin{aligned}
\sin\theta_1 &= \frac{-\lambda\eta+\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}\\
\cos\theta_2 &= \frac{\delta\eta-\lambda\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}\\
\sin\theta_2 &= \frac{-\lambda\eta-\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2},
\end{aligned}
$$

while the tangent value of $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$ is given by:

$$\begin{aligned}\tan\frac{\theta_1}{2} &= \tan\frac{\arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}} - \arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}}}{2} \\ &= \frac{\sin(\arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}}) - \sin(\arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}})}{\cos(\arccos\frac{\eta}{\sqrt{\lambda^2+\delta^2}}) + \cos(\arccos\frac{\delta}{\sqrt{\lambda^2+\delta^2}})} \\ &= \frac{\sqrt{1-\frac{\eta^2}{\lambda^2+\delta^2}} - \sqrt{1-\frac{\delta^2}{\lambda^2+\delta^2}}}{\frac{\eta}{\sqrt{\lambda^2+\delta^2}} + \frac{\delta}{\sqrt{\lambda^2+\delta^2}}} \\ &= \frac{\sqrt{\lambda^2+\delta^2-\eta^2}-\lambda}{\delta+\eta}.\end{aligned}$$

And similarly,

$$\tan\frac{\theta_2}{2} = \frac{-\sqrt{\lambda^2+\delta^2-\eta^2}-\lambda}{\delta+\eta}.$$

Now we can start to simplify the expression in **Theorem 3.9** one by one.

Lemma 3.11.

$$\begin{aligned}&\operatorname{artanh}\frac{\sigma\sin\theta_1}{\delta-\eta\cos\theta_1-\tau\sin\theta_1} - \operatorname{artanh}\frac{\sigma\sin\theta_2}{\delta-\eta\cos\theta_2-\tau\sin\theta_2} \\ &= \operatorname{artanh}\frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2}-\tau} + \operatorname{artanh}\frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2}+\tau}\end{aligned}$$

Proof. Substituting the previous trigonometric values in

$$\begin{aligned}&\operatorname{artanh}\frac{\sigma\sin\theta_1}{\delta-\eta\cos\theta_1-\tau\sin\theta_1} - \operatorname{artanh}\frac{\sigma\sin\theta_2}{\delta-\eta\cos\theta_2-\tau\sin\theta_2} \\ &= \operatorname{artanh}\frac{\sigma\cdot\frac{-\eta\lambda+\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}}{\delta-\eta\cdot\frac{\delta\eta+\lambda\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}-\tau\cdot\frac{-\eta\lambda+\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}} \\ &\quad - \operatorname{artanh}\frac{\sigma\cdot\frac{-\eta\lambda-\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}}{\delta-\eta\cdot\frac{\delta\eta-\lambda\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}-\tau\cdot\frac{-\eta\lambda-\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}}.\end{aligned}$$

As $0 < \theta_1 < \pi < \theta_2 < 2\pi$, $\sin\theta_1, \sin\theta_2 \neq 0$

$$\begin{aligned}
&\operatorname{artanh} \frac{\sigma \cdot \frac{-\eta\lambda+\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}}{\delta - \eta \cdot \frac{\delta\eta+\lambda\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2} - \tau \cdot \frac{-\eta\lambda+\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}} \\
&= \operatorname{artanh} \frac{\sigma}{(\delta - \eta \cdot \frac{\delta\eta+\lambda\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}) \cdot \frac{\lambda^2+\delta^2}{-\lambda\eta+\delta\sqrt{\lambda^2+\delta^2-\eta^2}} - \tau} \\
&= \operatorname{artanh} \frac{\sigma}{\frac{\delta(\lambda^2+\delta^2)-\eta(\delta\eta+\lambda\sqrt{\lambda^2+\delta^2-\eta^2})}{-\lambda\eta+\delta\sqrt{\lambda^2+\delta^2-\eta^2}} - \tau} \\
&= \operatorname{artanh} \frac{\sigma}{\frac{\delta(\lambda^2+\delta^2-\eta^2)-\lambda\eta\sqrt{\lambda^2+\delta^2-\eta^2})}{-\lambda\eta+\delta\sqrt{\lambda^2+\delta^2-\eta^2}} - \tau} \\
&= \operatorname{artanh} \frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2} \cdot \frac{-\lambda\eta+\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{-\lambda\eta+\delta\sqrt{\lambda^2+\delta^2-\eta^2}} - \tau} \\
&= \operatorname{artanh} \frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2} - \tau}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\operatorname{artanh} \frac{\sigma \cdot \frac{-\eta\lambda-\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}}{\delta - \eta \cdot \frac{\delta\eta-\lambda\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2} - \tau \cdot \frac{-\eta\lambda-\delta\sqrt{\lambda^2+\delta^2-\eta^2}}{\lambda^2+\delta^2}} \\
&= -\operatorname{artanh} \frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2} + \tau}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\operatorname{artanh} \frac{\sigma\sin\theta_1}{\delta - \eta\cos\theta_1 - \tau\sin\theta_1} - \operatorname{artanh} \frac{\sigma\sin\theta_2}{\delta - \eta\cos\theta_2 - \tau\sin\theta_2} \\
&= \operatorname{artanh} \frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2} - \tau} - \Big(-\operatorname{artanh} \frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2} + \tau}\Big) \\
&= \operatorname{artanh} \frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2} - \tau} + \operatorname{artanh} \frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2} + \tau}.
\end{aligned}$$

□

Lemma 3.12.

$$\begin{aligned}
&-\frac{\pi}{2} + \arctan \frac{(\delta+\eta)\tan\frac{\theta_1}{2} + \sigma - \tau}{\sqrt{\delta^2 - (\sigma-\tau)^2 - \eta^2}} - \arctan \frac{(\delta+\eta)\tan\frac{\theta_2}{2} + \sigma - \tau}{\sqrt{\delta^2 - (\sigma-\tau)^2 - \eta^2}} \\
&= \arctan \frac{\lambda(\sigma-\tau)}{\sqrt{(\lambda^2+\delta^2-\eta^2)(\delta^2-(\sigma-\tau)^2-\eta^2)}}.
\end{aligned}$$

Proof. By substituting values of $\tan\frac{\theta_1}{2}$ and $\tan\frac{\theta_2}{2}$,

$$\begin{aligned}
&\arctan\frac{(\delta+\eta)\tan\frac{\theta_1}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}-\arctan\frac{(\delta+\eta)\tan\frac{\theta_2}{2}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\\
&=\arctan\frac{(\delta+\eta)\cdot\frac{\sqrt{\lambda^2+\delta^2-\eta^2}-\lambda}{\delta+\eta}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}-\arctan\frac{(\delta+\eta)\cdot\frac{-\sqrt{\lambda^2+\delta^2-\eta^2}-\lambda}{\delta+\eta}+\sigma-\tau}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\\
&=\arctan\frac{\sqrt{\lambda^2+\delta^2-\eta^2}+(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}+\arctan\frac{\sqrt{\lambda^2+\delta^2-\eta^2}-(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}.
\end{aligned}$$

Simplifying this requires the arctan addition formula[9] denoted below:

$$\arctan x+\arctan y=\begin{cases}\arctan\frac{x+y}{1-xy} & \text{if } xy<1\\ \frac{\pi}{2} & \text{if } xy=1\text{, and } x>0 \text{ or } y>0\\ -\frac{\pi}{2} & \text{if } xy=1\text{, and } x<0 \text{ or } y<0\\ \pi+\arctan\frac{x+y}{1-xy} & \text{if } xy>1\text{, and } x>0 \text{ or } y>0\\ -\pi+\arctan\frac{x+y}{1-xy} & \text{if } xy>1\text{, and } x<0 \text{ or } y<0.\end{cases}$$

In this case,

$$\begin{aligned}
x&=\frac{\sqrt{\lambda^2+\delta^2-\eta^2}+(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}\\
y&=\frac{\sqrt{\lambda^2+\delta^2-\eta^2}-(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}.
\end{aligned}$$

First of all, in the case $xy\geq 1$, i.e. $x,y\neq 0$, as

$$\begin{gathered}
\sqrt{\lambda^2+\delta^2-\eta^2},\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}>0\\
\text{and}\quad(\sigma-\tau-\lambda)\text{ or }-(\sigma-\tau-\lambda)>0\\
x,y>0
\end{gathered}$$

Therefore, some cases can be eliminated and with $-\frac{\pi}{2}$ added, it becomes

$$-\frac{\pi}{2}+\arctan x+\arctan y=\begin{cases}-\frac{\pi}{2}+\arctan\frac{x+y}{1-xy} & \text{if } xy<1\\ 0 & \text{if } xy=1\\ \frac{\pi}{2}+\arctan\frac{x+y}{1-xy} & \text{if } xy>1\end{cases}$$

It is observed that a sign of symmetry is reached.

The further elimination of remaining cases can be achieved by a trick:

First of all, note that

$$\begin{aligned} x+y &= \frac{\sqrt{\lambda^2+\delta^2-\eta^2}+(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}} + \frac{\sqrt{\lambda^2+\delta^2-\eta^2}-(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}} \\ &= \frac{\sqrt{\lambda^2+\delta^2-\eta^2}}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}} > 0. \end{aligned}$$

Therefore,

$$\frac{x+y}{1-xy} > 0 \text{ if } xy < 1 \quad \text{and} \quad \frac{x+y}{1-xy} < 0 \text{ if } xy > 1.$$

Hence,

$$\begin{aligned} -\frac{\pi}{2} + \arctan x + \arctan y &= \begin{cases} -\frac{\pi}{2} + \frac{\pi}{2} - \arctan\frac{1-xy}{x+y} & \text{if } xy < 1 \\ 0 & \text{if } xy = 1 \\ \frac{\pi}{2} - \frac{\pi}{2} - \arctan\frac{1-xy}{x+y} & \text{if } xy > 1 \end{cases} \\ &= -\arctan\frac{1-xy}{x+y} = \arctan\frac{xy-1}{x+y}. \end{aligned}$$

At the end, the value of $\arctan\frac{xy-1}{x+y}$ can be found:

$$\begin{aligned} &\arctan\frac{xy-1}{x+y} \\ &= \arctan\frac{\frac{\sqrt{\lambda^2+\delta^2-\eta^2}+(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}} \cdot \frac{\sqrt{\lambda^2+\delta^2-\eta^2}-(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}} - 1}{\frac{\sqrt{\lambda^2+\delta^2-\eta^2}+(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}} + \frac{\sqrt{\lambda^2+\delta^2-\eta^2}-(\sigma-\tau-\lambda)}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}} \\ &= \arctan\frac{\frac{\lambda^2+\delta^2-\eta^2-(\sigma-\tau-\lambda)^2-(\delta^2-(\sigma-\tau)^2-\eta^2)}{\delta^2-(\sigma-\tau)^2-\eta^2}}{2\frac{\sqrt{\lambda^2+\delta^2-\eta^2}}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}} \\ &= \arctan\frac{\frac{\lambda^2-(\sigma-\tau-\lambda)^2+(\sigma-\tau)^2}{\delta^2-(\sigma-\tau)^2-\eta^2}}{2\frac{\sqrt{\lambda^2+\delta^2-\eta^2}}{\sqrt{\delta^2-(\sigma-\tau)^2-\eta^2}}} \\ &= \arctan\frac{\lambda(\sigma-\tau)}{\sqrt{(\lambda^2+\delta^2-\eta^2)(\delta^2-(\sigma-\tau)^2-\eta^2)}}. \end{aligned}$$

□

Applying similar methods as **Lemma 3.12**, the following part can also be simplified:

Lemma 3.13.

$$-\frac{\pi}{2}+\arctan\frac{(\delta+\eta)\tan\frac{\theta_1}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}-\arctan\frac{(\delta+\eta)\tan\frac{\theta_2}{2}-\sigma-\tau}{\sqrt{\delta^2-(\sigma+\tau)^2-\eta^2}}$$
$$=-\arctan\frac{\lambda(\sigma+\tau)}{\sqrt{(\lambda^2+\delta^2-\eta^2)(\delta^2-(\sigma+\tau)^2-\eta^2)}}$$

By adding up all the simplified terms, we deduce the simplified theorem of **Theorem 3.9**.

Theorem 3.14. *The mean shadow of the edge is given by*

$$=\frac{2(h+\delta)}{\pi}\Bigg(\operatorname{artanh}\frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2}-\tau}+\operatorname{artanh}\frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2}+\tau}$$
$$+\frac{\lambda}{\sqrt{\delta^2-\eta^2-(\sigma-\tau)^2}}\arctan\frac{\lambda(\sigma-\tau)}{\sqrt{(\delta^2-\eta^2-(\sigma-\tau)^2)(\lambda^2+\delta^2-\eta^2)}}$$
$$+\frac{\lambda}{\sqrt{\delta^2-\eta^2-(\sigma+\tau)^2}}\arctan\frac{\lambda(\sigma+\tau)}{\sqrt{(\delta^2-\eta^2-(\sigma+\tau)^2)(\lambda^2+\delta^2-\eta^2)}}\Bigg)$$

3.2.5. *Mean Shadow Theorem.* The previous results represented the mean shadow of line segments using 3 variables σ, η and τ only, which provided computational convenience, but it makes our formulae less straightforward. Therefore, in this section, we will use two relative position vectors A, B denoting the relative positions of the edge's two endpoints to $\mathfrak{R}$, and substitute σ, η and τ in terms of A, B, which points to our final mean shadow theorem.

And there is no reason to continue using λ, δ, therefore we will use $\mathfrak{R}_n$ and $\mathfrak{S}_n$ to denote the n-th coordinate of $\mathfrak{R}$ and $\mathfrak{S}$. Therefore, $\lambda = \mathfrak{S}_1$, $\delta = \mathfrak{S}_2$ and $h = \mathfrak{R}_2$.

Definition 3.15 ($\mathfrak{R}_n$, $\mathfrak{S}_n$)**.** $\mathfrak{R}_n$ *and* $\mathfrak{S}_n$ *denote the n-th coordinate of* $\mathfrak{R}$ *and* $\mathfrak{S}$*.*

Definition 3.16 (A, B)**.** $A(x_1, y_1)$*,* $B(x_2, y_2)$ *are relative position vectors denoting the relative positions of the edge's two endpoints to* $\mathfrak{R}$*.*

Definition 3.17. *For vectors* $A = (x_1, y_1)$*,* $B = (x_2, y_2)$*,*

$$\det(A,B)=\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}\quad \textit{and}\quad \langle A,B\rangle = x_1x_2+y_1y_2$$

Theorem 3.18 (Mean shadow theorem)**.**
Mean shadow of a line segment in $\mathbb{R}^2$

$$= \frac{2(\mathfrak{R}_2 + \mathfrak{S}_2)}{\pi}\Bigg(\operatorname{artanh}\frac{\sqrt{\|\mathfrak{S}\|^2\|A-B\|^2 - \det(A,B)^2}}{\|\mathfrak{S}\|^2 - \langle A,B\rangle}$$
$$+ \frac{\mathfrak{S}_1}{\sqrt{\mathfrak{S}_2^2 - \|A\|^2}}\arctan(\frac{\mathfrak{S}_1}{\sqrt{\mathfrak{S}_2^2 - \|A\|^2}} \cdot \frac{\|A\|^2 - \langle A,B\rangle}{\sqrt{\|\mathfrak{S}\|^2\|A-B\|^2 - \det(A,B)^2}})$$
$$+ \frac{\mathfrak{S}_1}{\sqrt{\mathfrak{S}_2^2 - \|B\|^2}}\arctan(\frac{\mathfrak{S}_1}{\sqrt{\mathfrak{S}_2^2 - \|B\|^2}} \cdot \frac{\|B\|^2 - \langle A,B\rangle}{\sqrt{\|\mathfrak{S}\|^2\|A-B\|^2 - \det(A,B)^2}})\Bigg)$$

Proof. First of all,

$$\sigma = \frac{\|A-B\|}{2}.$$

For η, the perpendicular distance from $\mathfrak{R}$ to the line segment, can be represented by

$$\frac{\det(A,B)}{\|A-B\|}.$$

We neglect the absolute value, as in **Theorem 3.14**, the formula only involves η^2.

By Pythagoras theorem,

$$\tau^2 = \left\|\frac{A+B}{2}\right\|^2 - \frac{\det(A,B)^2}{\|A-B\|^2}$$
$$= \frac{\|A+B\|^2\|A-B\|^2 - 4\det(A,B)^2}{4\|A-B\|^2}$$

Let ϕ be the angle between A and B, and using the fact that

$$\|A+B\|^2 = \|A\|^2 + \|B\|^2 - 2\|A\|\|B\|\cos\phi$$
$$\|A-B\|^2 = \|A\|^2 + \|B\|^2 + 2\|A\|\|B\|\cos\phi$$
$$\det(A,B)^2 = \|A\|^2\|B\|^2\sin^2\theta$$

Then the above expression can be simplified as:

$$\tau^2 = \frac{(\|A\|^2 + \|B\|^2)^2 - 4\|A\|^2\|B\|^2\cos^2\theta - 4\|A\|^2\|B\|^2\sin^2\theta}{4\|A-B\|^2}$$
$$= \frac{(\|A\|^2 - \|B\|^2)^2}{4\|A-B\|^2}.$$

As whether τ is positive or negative is not important due to the symmetry of the formula in **Theorem 3.14**, therefore,

$$\tau = \frac{\|A\|^2 - \|B\|^2}{2\|A-B\|}.$$

. . . .

Now, we can substitute these back into the original formula.

For the artanh part,

$$
\begin{aligned}
&\operatorname{artanh}\frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2}-\tau}+\operatorname{artanh}\frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2}+\tau}\\
&=\operatorname{artanh}\frac{\frac{\|A-B\|}{2}}{\sqrt{\lambda^2+\delta^2-\frac{\det(A,B)^2}{\|A-B\|^2}}-\frac{\|A\|^2-\|B\|^2}{2\|A-B\|}}\\
&\quad+\operatorname{artanh}\frac{\frac{\|A-B\|}{2}}{\sqrt{\lambda^2+\delta^2-\frac{\det(A,B)^2}{\|A-B\|^2}}+\frac{\|A\|^2-\|B\|^2}{2\|A-B\|}}\\
&=\operatorname{artanh}\frac{\|A-B\|^2}{2\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}-(\|A\|^2-\|B\|^2)}\\
&\quad+\operatorname{artanh}\frac{\|A-B\|^2}{2\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}+(\|A\|^2-\|B\|^2)}\\
&=\operatorname{artanh}\frac{4\|A-B\|^2\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}}{4(\lambda^2+\delta^2)\|A-B\|^2-4\det(A,B)^2-(\|A\|^2-\|B\|^2)^2+\|A-B\|^4}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&-4\det(A,B)^2-(\|A\|^2-\|B\|^2)^2+\|A-B\|^4\\
&=(\|A\|^2+\|B\|^2-2\|A\|\|B\|\cos\phi)^2-\|A\|^4-\|B\|^4\\
&\qquad+2\|A\|^2\|B\|^2-4\|A\|^2\|B\|^2\sin^2\phi\\
&=\|A\|^4+\|B\|^4+2\|A\|^2\|B\|^2\\
&\qquad-4\|A\|^3\|B\|\cos\phi-4\|A\|\|B\|^3\cos\phi+4\|A\|^2\|B\|^2\cos^2\phi\\
&\quad-\|A\|^4-\|B\|^4\\
&\qquad+2\|A\|^2\|B\|^2-4\|A\|^2\|B\|^2\sin^2\phi\\
&=8\|A\|^2\|B\|^2\cos^2\phi-4\|A\|^3\|B\|\cos\phi-4\|A\|\|B\|^3\cos\phi\\
&=4\langle A,B\rangle(2\|A\|\|B\|\cos\phi-\|A\|^2-\|B\|^2)\\
&=-4\|A-B\|^2\langle A,B\rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\operatorname{artanh}\frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2}-\tau}+\operatorname{artanh}\frac{\sigma}{\sqrt{\lambda^2+\delta^2-\eta^2}+\tau}\\
&=\operatorname{artanh}\frac{4\|A-B\|^2\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}}{4\|A-B\|^2(\lambda^2+\delta^2-\langle A,B\rangle)}\\
&=\operatorname{artanh}\frac{\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}}{\lambda^2+\delta^2-\langle A,B\rangle}.
\end{aligned}
$$

For the arctan part,

$$
\begin{aligned}
&\frac{\lambda}{\sqrt{\delta^2-\eta^2-(\sigma-\tau)^2}}\arctan\frac{\lambda(\sigma-\tau)}{\sqrt{(\delta^2-\eta^2-(\sigma-\tau)^2)(\lambda^2+\delta^2-\eta^2)}}\\
&\quad+\frac{\lambda}{\sqrt{\delta^2-\eta^2-(\sigma+\tau)^2}}\arctan\frac{\lambda(\sigma+\tau)}{\sqrt{(\delta^2-\eta^2-(\sigma+\tau)^2)(\lambda^2+\delta^2-\eta^2)}}\\
&=\frac{\lambda}{\sqrt{\delta^2-\|B\|^2}}\arctan\frac{\lambda(\frac{\|A-B\|}{2}-\frac{\|A\|^2-\|B\|^2}{2\|A-B\|})}{\sqrt{(\delta^2-\|B\|^2)(\lambda^2+\delta^2-\frac{\det(A,B)^2}{\|A-B\|^2})}}\\
&\quad+\frac{\lambda}{\sqrt{\delta^2-\|A\|^2}}\arctan\frac{\lambda(\frac{\|A-B\|}{2}+\frac{\|A\|^2-\|B\|^2}{2\|A-B\|})}{\sqrt{(\delta^2-\|A\|^2)(\lambda^2+\delta^2-\frac{\det(A,B)^2}{\|A-B\|^2})}}\\
&=\frac{\lambda}{\sqrt{\delta^2-\|A\|^2}}\arctan(\frac{\lambda}{\sqrt{\delta^2-\|A\|^2}}\cdot\frac{\frac{\|A-B\|^2}{2}+\frac{\|A\|^2-\|B\|^2}{2}}{\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}})\\
&\quad+\frac{\lambda}{\sqrt{\delta^2-\|B\|^2}}\arctan(\frac{\lambda}{\sqrt{\delta^2-\|B\|^2}}\cdot\frac{\frac{\|A-B\|^2}{2}-\frac{\|A\|^2-\|B\|^2}{2}}{\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}})\\
&=\frac{\lambda}{\sqrt{\delta^2-\|A\|^2}}\arctan(\frac{\lambda}{\sqrt{\delta^2-\|A\|^2}}\cdot\frac{\|A\|^2-\langle A,B\rangle}{\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}})\\
&\quad+\frac{\lambda}{\sqrt{\delta^2-\|B\|^2}}\arctan(\frac{\lambda}{\sqrt{\delta^2-\|B\|^2}}\cdot\frac{\|B\|^2-\langle A,B\rangle}{\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}}).
\end{aligned}
$$

....

Adding two results up, we can obtain the formula:

$$
\begin{aligned}
&\frac{2(h+\delta)}{\pi}\Bigg(\operatorname{artanh}\frac{\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}}{\lambda^2+\delta^2-\langle A,B\rangle}\\
&\quad+\frac{\lambda}{\sqrt{\delta^2-\|A\|^2}}\arctan(\frac{\lambda}{\sqrt{\delta^2-\|A\|^2}}\cdot\frac{\|A\|^2-\langle A,B\rangle}{\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}})\\
&\quad+\frac{\lambda}{\sqrt{\delta^2-\|B\|^2}}\arctan(\frac{\lambda}{\sqrt{\delta^2-\|B\|^2}}\cdot\frac{\|B\|^2-\langle A,B\rangle}{\sqrt{(\lambda^2+\delta^2)\|A-B\|^2-\det(A,B)^2}})\Bigg)
\end{aligned}
$$

Lastly, the final formula can be obtained by replacing h as $\mathfrak{R}_2$, λ as $\mathfrak{S}_1$ and δ as $\mathfrak{S}_2$. □

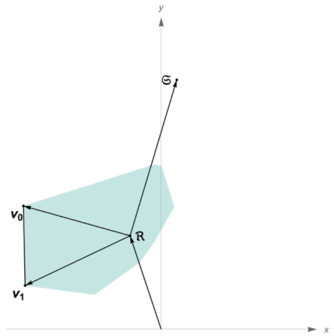

FIGURE 14. Iteration of all edges of a convex polygon

3.2.6. *Reassembling Complete Shape.*

Theorem 3.19. *For a convex polygon whose m vertices (represented in clockwise or anti-clockwise direction) have relative position vectors v_k for $k = 0, \ldots, m-1$ from $\mathfrak{R}$,*

Mean shadow

$$= \frac{\mathfrak{R}_2 + \mathfrak{S}_2}{\pi} \sum_{k=1}^{m} \Bigg(\operatorname{artanh} \frac{\sqrt{\|\mathfrak{S}\|^2 \|v_{k-1} - v_k\|^2 - \det(v_{k-1}, v_k)^2}}{\|\mathfrak{S}\|^2 - \langle v_{k-1}, v_k \rangle}$$
$$+ \frac{\mathfrak{S}_1}{\sqrt{\mathfrak{S}_2^2 - \|v_{k-1}\|^2}} \arctan\left(\frac{\mathfrak{S}_1}{\sqrt{\mathfrak{S}_2^2 - \|v_{k-1}\|^2}} \cdot \frac{\|v_{k-1}\|^2 - \langle v_{k-1}, v_k \rangle}{\sqrt{\|\mathfrak{S}\|^2 \|v_{k-1} - v_k\|^2 - \det(v_{k-1}, v_k)^2}} \right)$$
$$+ \frac{\mathfrak{S}_1}{\sqrt{\mathfrak{S}_2^2 - \|v_k\|^2}} \arctan\left(\frac{\mathfrak{S}_1}{\sqrt{\mathfrak{S}_2^2 - \|v_k\|^2}} \cdot \frac{\|v_k\|^2 - \langle v_{k-1}, v_k \rangle}{\sqrt{\|\mathfrak{S}\|^2 \|v_{k-1} - v_k\|^2 - \det(v_{k-1}, v_k)^2}} \right) \Bigg)$$

Since "a polygon inscribed in a convex curve must be convex" [10], by approximating a convex closed curve with a closed polygonal chain on the curve, we can deduce a formula for the mean shadow of a convex closed curve.

3.3. **Numerical Verification. Theorem 3.19** can be verified through simulation in *Mathematica*. We generated a set of 500 random scenarios containing a randomly placed light source, rotation center, and a random convex polygon.

(A) (B) (C)

Figure 15. Examples of randomly generated polygons

Utilizing *Mathematica*'s numerical integration to estimate the mean shadow, we compare estimation results with our results which is illustrated in the following plot. The full code is provided in the appendix (**Code 1**).

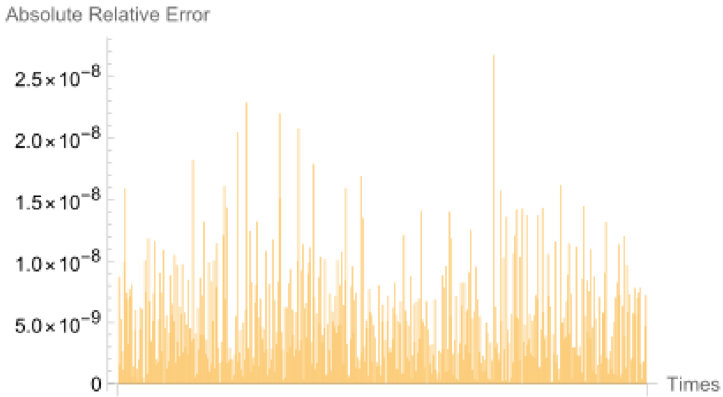

Figure 16. Absolute Relative error of *Mathematica* simulation from our proposed formula

From the graph, we can see that in all 500 randomly generated scenarios, the absolute relative errors are below 3.0×10^{-8}, which assures that our result is correct.

4. Analysis

4.1. **Symmetry in Our Formula.** In **Theorem 3.18**, we can see that the sign of $\mathfrak{S}_1$ does not matter as long as the absolute value is equal. Therefore we propose the following conjecture:

Conjecture 4.1. *In $\mathbb{R}^2$, with the same horizontal distance from the rotation center, whether the light source is on the left side or on the right side does not effect the mean shadow of an convex object.*

For a scenario like this:

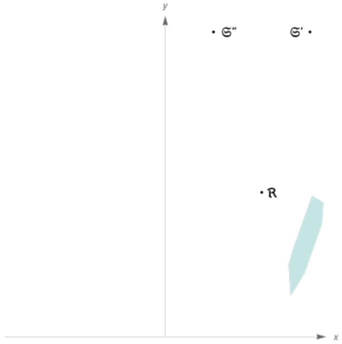

FIGURE 17. Illustration two light sources of same horizontal distance from rotation center

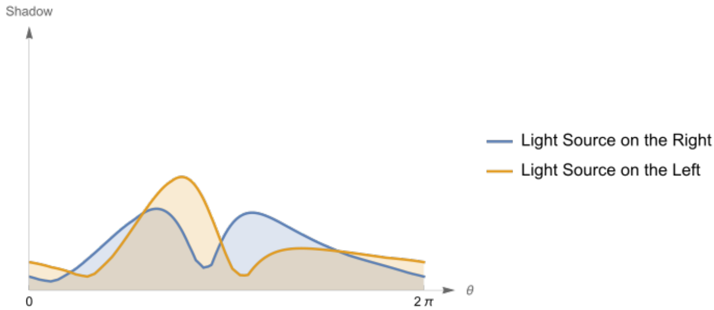

FIGURE 18. Graph of the length of the shadow against angle of rotation

We simulated two cases for the light source being on opposite sides of the rotation center with same horizontal distance. It can be seen that while the length of the shadow projected at a specific orientation varies with the two light source, the area under both curves are actually identical.

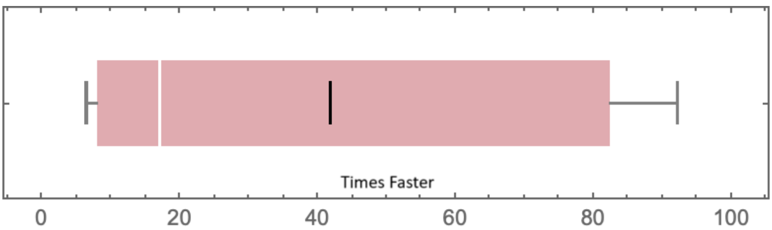

FIGURE 19. Our Method's Performance Compared With Method 1

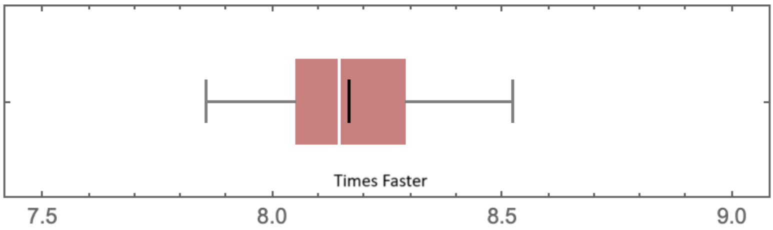

FIGURE 20. Our Method's Performance Compared With Method 2

4.2. **Performance Comparison.** We generated 50 different scenes to compare 3 methods about their speed to compute the mean shadow. The result is stated below.

We design an algorithm that for each edge of the polygon, uses our mean shadow theorem (**Theorem 3.18**) to compute the mean shadow of it, and then sums the results up.

Method 1 calculates the projection of all vertices of the polygon and takes the largest distance between the points of projection as the edge projection. The figure above shows that our method's performance can reach up to 90-fold of increase compared to this method.

Method 2 is an algorithm that for each edge of the polygon, use numerical integration to evaluate the mean shadow of it, and then sums up the results. The above figure shows that our method's performance can reach up to 8.5-fold of increase compared to this method.

The full code is provided in the appendix (**Code 2**).

5. Concave Shapes

We propose the following conjecture:

Conjecture 5.1. *For any 2-D concave polygon, its mean shadow of is equal to the mean shadow of its convex hull.*

(A) (B) (C)

FIGURE 21. Examples of randomly generated concave polygons and its convex hulls

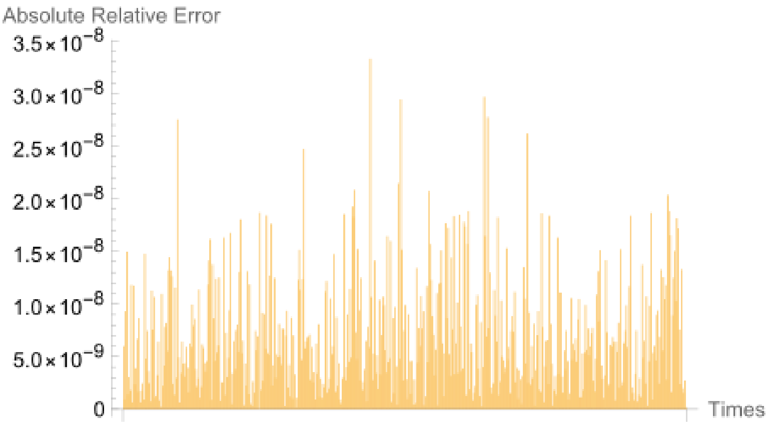

FIGURE 22. *Mathematica* simulation using our proposed formula on convex hull of concave polygons

By conducting numerical verification in *Mathematica*, we compared the values of the mean shadow of concave shapes with the mean shadow of their convex hull respectively. As shown in the figure above, the absolute relative error between the two values are all below 3.5×10^{-8}, so the conjecture is highly likely to be true. We leave the proof as an open problem.

APPENDIX

All computer codes below are written and running in *Mathematica* (Version 13.3.0.0):

Code 1:

```
ClearAll["Global`*"]
Off[NIntegrate::slwcon];
Off[NIntegrate::ncvb];

benchtimes = 500;
Error = {};
Special = {};
For[i = 1, i ≤ benchtimes, i++,
 R = {RandomReal[{-10, 10}], RandomReal[{20, 30}]};
 S = {RandomReal[{-10, 10}], RandomReal[{20, 30}]};
 m = RandomInteger[{3, 30}];
 poly = RandomPolygon[{"Convex", m}, DataRange → {RandomReal[{-10, 10}, 2], RandomReal[{-10, 10}, 2]}] // Normal;
 v = poly // First;
 poly = Translate[poly, R];

 result1 = NIntegrate[
   ((Max[#] - Min[#])/(2 π)) &[{(R + S)[[1]] - (R + S)[[2]] * ((R + S)[[1]] - #[[1]])/((R + S)[[2]] - #[[2]])} & /@ (Rotate[poly, θ, R] // CanonicalizePolygon // First)], {θ, 0, 2 π}, Method → {Automatic, "SymbolicProcessing" → 0},
   MinRecursion → 20, MaxRecursion → 20
  ];


 result2 = (R[[2]] + S[[2]])/π * Total[
    Function[{A, B},
       ArcTanh[Sqrt[Norm[S]^2 * Norm[A - B]^2 - Det[{A, B}]^2]/(Norm[S]^2 - Dot[A, B])]
        + S[[1]]/Sqrt[S[[2]]^2 - Norm[A]^2] ArcTan[S[[1]]/Sqrt[S[[2]]^2 - Norm[A]^2] * (Norm[A]^2 - Dot[A, B])/Sqrt[Norm[S]^2 * Norm[A - B]^2 - Det[{A, B}]^2]]
        + S[[1]]/Sqrt[S[[2]]^2 - Norm[B]^2] ArcTan[S[[1]]/Sqrt[S[[2]]^2 - Norm[B]^2] * (Norm[B]^2 - Dot[A, B])/Sqrt[Norm[S]^2 * Norm[A - B]^2 - Det[{A, B}]^2]]
      ][{#[[1]][[1]], #[[1]][[2]]}, {#[[2]][[1]], #[[2]][[2]]}] & /@ (Transpose@{v, RotateLeft@v})
   ];
 AppendTo[Error, Abs[(result1 - result2)/result1]];
]
BarChart[Error, AxesLabel → {"Times", "Absolute Relative Error"}]
```

Code 2:

```
In[ ]:= ClearAll["Global`*"]
Off[NIntegrate::slwcon];
Off[NIntegrate::ncvb];

R = {-2, 6};
S = {3, 10};

bench[] := (

  poly = RandomPolygon[{"Convex", 10}, DataRange → {RandomReal[{-5, 5}, 2], RandomReal[{-5, 5}, 2]}] // Normal;

  v = poly // First;
  poly = Translate[poly, R];

  {time1, result1} = RepeatedTiming[
    NIntegrate[
     ((Max[#] - Min[#])/(2 π)) &[{(R + S)[[1]] - (R + S)[[2]] * ((R + S)[[1]] - #[[1]])/((R + S)[[2]] - #[[2]])} & /@ (Rotate[poly, θ, R] // CanonicalizePolygon // First)],
     {θ, 0, 2 π}, Method → {Automatic, "SymbolicProcessing" → 0}
     ]
    ];
  {time2, result2} = RepeatedTiming[
    (R[[2]] + S[[2]])/(2 π) * Total[
      Function[{x1, y1, x2, y2},
         η = (x1 * y2 - x2 * y1)/Sqrt[(x1 - x2)^2 + (y1 - y2)^2];
         σ = Sqrt[(x1 - x2)^2 + (y1 - y2)^2]/2;
         τ = (x1^2 - x2^2 + y1^2 - y2^2)/(2 Sqrt[(x1 - x2)^2 + (y1 - y2)^2]);
         NIntegrate[(σ * Abs[S[[2]] Cos[θ] - η - S[[1]] Sin[θ]])/((S[[2]] - η Cos[θ] - τ Sin[θ])^2 - σ^2 Sin[θ]^2), {θ, 0, 2 π}, Method → {Automatic, "SymbolicProcessing" → 0}]
         ][#[[1]][[1]], #[[1]][[2]], #[[2]][[1]], #[[2]][[2]]] & /@ (Transpose@{v, RotateLeft@v})
      ]
    ];
  {time3, result3} = RepeatedTiming[
    (R[[2]] + S[[2]])/π * Total[
      Function[{A, B},
         ArcTanh[Sqrt[Norm[S]^2 * Norm[A - B]^2 - Det[{A, B}]^2]/(Norm[S]^2 - Dot[A, B])]
          + S[[1]]/Sqrt[S[[2]]^2 - Norm[A]^2] ArcTan[S[[1]]/Sqrt[S[[2]]^2 - Norm[A]^2] * (Norm[A]^2 - Dot[A, B])/Sqrt[Norm[S]^2 * Norm[A - B]^2 - Det[{A, B}]^2]]
          + S[[1]]/Sqrt[S[[2]]^2 - Norm[B]^2] ArcTan[S[[1]]/Sqrt[S[[2]]^2 - Norm[B]^2] * (Norm[B]^2 - Dot[A, B])/Sqrt[Norm[S]^2 * Norm[A - B]^2 - Det[{A, B}]^2]]
         ][{#[[1]][[1]], #[[1]][[2]]}, {#[[2]][[1]], #[[2]][[2]]}] & /@ (Transpose@{v, RotateLeft@v})
      ]
    ];
  {time1/time3, time2/time3}
  );

results = {{}, {}};
For[i = 1, i ≤ 50, i++, (
  result = bench[];
  AppendTo[results[[1]], result[[1]]];
  AppendTo[results[[2]], result[[2]]];
 )]
BoxWhiskerChart[results[[1]], "Mean", ChartStyle → {RGBColor[0.88, 0.67, 0.69]}]
BoxWhiskerChart[results[[2]], "Mean", ChartStyle → {RGBColor[0.79, 0.5, 0.5]}]
```

Code 3:

```
In[ ]:= ClearAll["Global`*"]
Error = {};
For[i = 1, i ≤ 500, i++,
 R = {RandomReal[{-10, 10}], RandomReal[{20, 30}]};
 S = {RandomReal[{-10, 10}], RandomReal[{20, 30}]};
 m = RandomInteger[{3, 30}];
 poly = RandomPolygon[m, DataRange → {RandomReal[{-10, 10}, 2], RandomReal[{-10, 10}, 2]}] // Normal;
 convexpoly = ConvexHullMesh[poly] // CanonicalizePolygon // Normal;
 poly = Translate[poly, R] // CanonicalizePolygon // Normal;

 result1 = NIntegrate[
   ((Max[#] - Min[#])/(2 π)) &[{(R + S)[[1]] - (R + S)[[2]] * ((R + S)[[1]] - #[[1]])/((R + S)[[2]] - #[[2]])} & /@ (Rotate[poly, θ, R] // CanonicalizePolygon // First)], {θ, 0, 2 π},
   Method → {Automatic, "SymbolicProcessing" → 0}, MinRecursion → 20, MaxRecursion → 20
  ];


 result2 = (R[[2]] + S[[2]])/π * Total[
    Function[{A, B},
       ArcTanh[Sqrt[Norm[S]^2 * Norm[A - B]^2 - Det[{A, B}]^2]/(Norm[S]^2 - Dot[A, B])]
       + S[[1]]/Sqrt[S[[2]]^2 - Norm[A]^2] ArcTan[S[[1]]/Sqrt[S[[2]]^2 - Norm[A]^2] * (Norm[A]^2 - Dot[A, B])/Sqrt[Norm[S]^2 * Norm[A - B]^2 - Det[{A, B}]^2]]
       + S[[1]]/Sqrt[S[[2]]^2 - Norm[B]^2] ArcTan[S[[1]]/Sqrt[S[[2]]^2 - Norm[B]^2] * (Norm[B]^2 - Dot[A, B])/Sqrt[Norm[S]^2 * Norm[A - B]^2 - Det[{A, B}]^2]]
      ][{#[[1]][[1]], #[[1]][[2]]}, {#[[2]][[1]], #[[2]][[2]]}] & /@ (Transpose@{convexpoly // First, RotateLeft@(convexpoly // First)})
   ];
 AppendTo[Error, Abs[(result2 - result1)/result1]];
]
BarChart[Error, AxesLabel → {"Times", "Absolute Relative Error"}]
```

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We would like to show our gratitude to Ms. Luo On Ki, Mr. Chan Tsz Hin, Mr. Yu Tsz Fung and Mr. Lee Ho Fung for providing useful comments. We also thank Mr. Lee Kwok Chu for teaching us integration.

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REVIEWERS' COMMENTS

The authors studied the two-dimensional mean shadow problem of a rotating convex polygon with a light source located at a finite distance. When the light source is from infinity, the problem was studied by Cauchy and a satisfactory formula was derived in any dimensions. The problem was recently studied (where the light has a special location), and the main contribution in this report is to derive a formula for the two-dimensional case.

Generally, reviewers regarded the problem was interesting, yet the authors may need to explain more about the geometric meaning of some integral formulae appeared in the article.