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Honorable Mention

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A SYNTHETIC APPROACH ON STUDYING THE MYSTERIOUS RIGHT KITE AND ITS APPLICATIONS ON CRYPTOGRAPHY IN RELATED TO POINCARÉ DISK MODEL IN THE VIEWS OF EUCLID GEOMETRY

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Abstract. In this study, we give a synthetic approach to the quadrilateral "Kite", or more specific to say, the "Right Kite". It is mainly based on the definitions, postulates (axioms), propositions (theorem and constructions) from *the Euclid's Elements*, which is known as one of the most successful and influential mathematical textbook attributed to the ancient Greek mathematician Euclid in Alexandria, Ptolemaic Egypt c, 300 BC.

Linked up with the definitions of a "Right Kite" and the lines which meet the boundary of a said circle orthogonally described in the Pointcaré Disk Model, we attempt to combine it in a mathematical task namely "Cryptography". The application of Poincaré Disk Model will be acted as a bridge to form a single key for encryption and decryption. Despite the single common trick we use, it leads to infinite possibilities by experiencing various and distinct mathematical skills in cryptography.

Last but not least, we would like to dedicate to the publication of Euclid's Elements and the discovery of Euclidean Geometry so that we can admire the beauty of Mathematics. Our ultimate goal is to lay the new insight into some of the most enjoyable and fascinating aspects of geometry regarding to the most unaware quadrilateral, Kite.

PART A

Traditional definitions of a Kite and Right (or Cyclic) Kite and its constructions

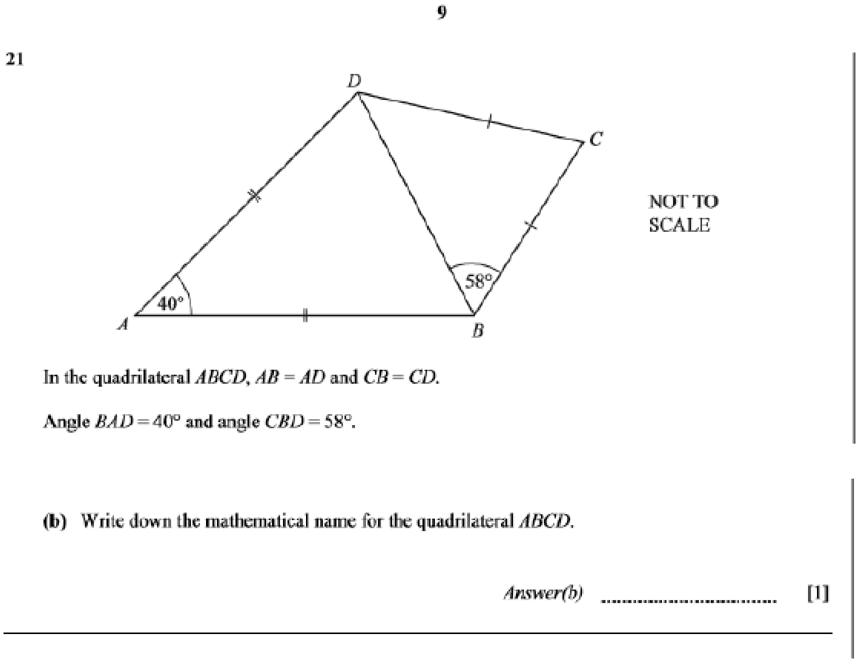
1. Introduction

1.1. Research motivation

Every children knows what a "kite" is, yet, a "kite" meant to them an flying object with a silk fabric sail, fine high-tensile silk flying line and resilient bamboo for a strong, lightweight framework.

Be that as it may, a different view occurs to those who have studied the Euclid's Geometry. "Kite" is a quadrilateral whose four sides can be grouped into two equal pair sides that are adjacent to each other, also its diagonals cross each other at right angles.

Sadly, "Cambridge International General Certificate of Secondary Education 0580 Mathematics November 2011 Principal Examiner Report for Teachers" revealed that the follow question about kite was less well answered. The result was about the same while the mentioned question appeared in an inter-school assessment in G.T. College (2014-15/Grade 9/Term 3). Many students failed to identify the figure below. Most students answered rhombus or scalene quadrilateral instead of kite.



In the quadrilateral *ABCD*, *AB* = *AD* and *CB* = *CD*.

Angle *BAD* = 40° and angle *CBD* = 58°.

(b) Write down the mathematical name for the quadrilateral *ABCD*.

Answer(b) .... [1]

FIGURE 1. IGCSE October/November 2011 Paper 1 (Core) 0580/13 Page 9 Q.21(b)

As the matter of fact, students are less familiar with the properties of kite and some students are even unaware that kite is a member of the quadrilateral family. After the mentioned incident we have encountered, we decided to conduct a synthetic approach in studying “kite” in a bid to let this beauty quadrilateral being brought to light.

Our constructions mainly use a Euclidean compass and straightedge and can be carried out by hand. Despite the use of non-Euclidean Poincaré Disk Model, all discussions in this study will only follow the views of Euclidean Geometry, using nothing more than the geometry concepts presented in New Senior Secondary geometry courses.

1.2. Different views on traditional kite

A traditional way to regard a kite as a quadrilateral with two distinct pairs of equal adjacent sides. Now, We are going to explore the multitude views of surprising relationships connected with kite and triangles in the followings. These views help us to construct certain kites later on.

<table>
<tr>
<td>View I:</td>
<td>
Kite is a formation of two distinct isosceles triangles with a common base, i.e. BD is the common base of the isosceles triangle ABD and CBD (see Figure 2).

Remarks: Obviously, a rhombus is formed if the two isosceles triangle are congruent with a common base BD overlapping each other.
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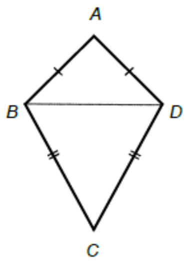

FIGURE 2
</td>
</tr>
<tr>
<td>View II:</td>
<td>
Kite is a formation of two congruent scalene triangles with one of the corresponding side overlapping each other. The overlapping side is also known as line of symmetry of the said kite.

AC is the common side of the congruent triangle pairs ΔABC & ΔADC (i.e. $\Delta ABC \equiv \Delta ADC$) and AC is the unique line of symmetry of kite $ABCD$, refer to Figure 3.

In this case, both adjacent angle pairs ($\angle BAC$ and $\angle DAC$, $\angle BCA$ and $\angle DCA$) along the line of symmetry should be acute angles, or otherwise,

(a) a dart is formed if one of the adjacent angles pair along the line of symmetry are the obtuse angles (i.e. $\angle BCA$ and $\angle DCA$ are obtuse angles), refer to Figure 4.
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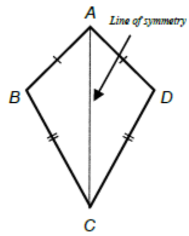


FIGURE 3

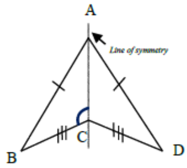
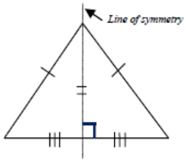
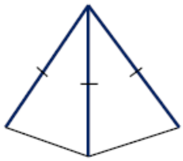
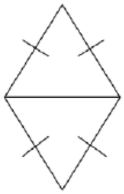
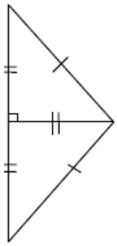
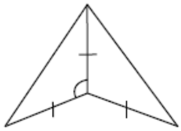


FIGURE 4
</td>
</tr>
</table>

	(b) a right triangle is formed if one of the adjacent angles pair along the line of symmetry is the right angles, refer to Figure 5 Since the angle sum of triangles should be less than two right angles (180°), the case of both adjacent angles equal to or greater than a right angle is negligible.	 FIGURE 5
View III:	A kite can be formed by combining two congruent isosceles acute triangles with the legs overlapped with each other, refer to Figure 6. Remarks: (i) If the overlapping side is the base of the isosceles triangles, a rhombus is obtained instead, refer to Figure 7. (ii) If we combine the legs of two congruent right isosceles triangles, an isosceles triangle is formes, refer to Figure 8. (iii) If we combine the legs of two congruent obtuse isosceles triangles, a dart is formed, refer to Figure 9.	 FIGURE 6  FIGURE 7  FIGURE 8  FIGURE 9

1.3. Constructions of Traditional kite

In secondary school, we have learnt the following construction skills which can be performed by straight-edge and Euclidean compass:

(i) Copying an angle;
(ii) Bisecting an angle;
(iii) Constructing a perpendicular bisector &
(iv) Constructing a perpendicular line passing through a point lying outside a line segment.

In general, we may regard the existence of a traditional kite equivalent to the successful of construction. Once you have constructed a quadrilateral fulfilled the definitions of kite, we can conclude that the kite exists accordingly.

Now, we will use the quadrilateral $ABCD$ as an example. Without loss of generosity, we may define the length of BC is greater than that of AB. In order to maintaining the corresponding adjacent sides are equal, we will set vertex A and C as the centre of circles with radii AB and CB respectively. In addition, the line segment AC is always perpendicular to the line segment BD.

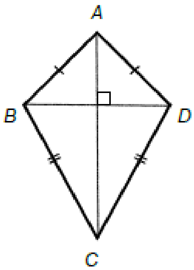


FIGURE 10

From View I, we may consider a kite is formed by two distinct isosceles triangles. We are able to construct a kite with certain adjacent sides.

Steps:

(i) Construct a line segment AC by a straightedge.
(ii) Draw an arc of radius AB from centre A.
(iii) Draw another arc of radius CB from centre C.

As centers A and C are arbitrary points on the line of symmetry AC, infinitely many kites of certain adjacent pairs, i.e. AB and CB can be obtained.

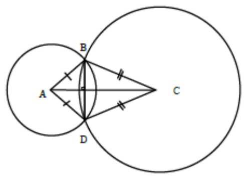


FIGURE 11

From View II, we may consider a kite is a formation of two congruent scalene triangles with one of the corresponding side overlapping each other. The overlapping side is also known as a line of symmetry.

By the construction skills (iv) mentioned above, we are able to construct a kite with certain diagonals perpendicular to each other, i.e. $AC \perp BD$. It follows that we are able to construct a kite with the given diagonals, refer to Figure 12. For the certain information of a triangle such as the side-angle-side of ΔABC is given, according to Book I, the proposition 4 from *the Euclid's Element*, a unique triangle can be constructed. We may then obtain a kite by drawing the mirror image (i.e. ΔADC). By extending the length of AC and BC of ΔABC to AC' and BC' respectively, we can obtain another distinct kites of side AB and $\angle ABC$ (i.e. kite $ABC'D'$ and kite $ABC''D''$), refer to Figure 13.

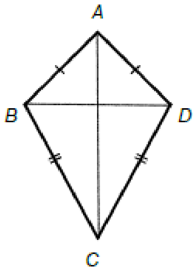


FIGURE 12

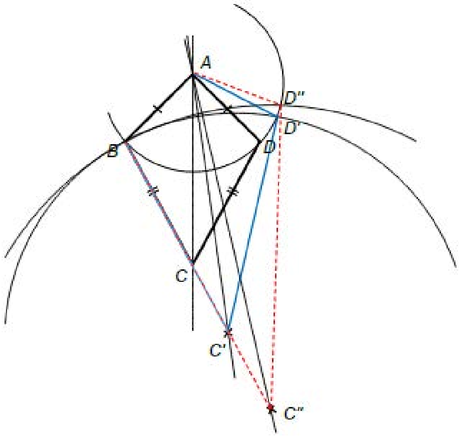
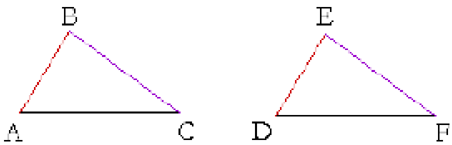


FIGURE 13

Euclid's Element BOOK I PROPOSITION 4
If two triangles have two sides equal to two sides respectively, and if the angles contained by those sides are also equal, then the remaining side will equal the remaining side, the triangles themselves will be equal areas, and the remaining angles will be equal, namely those that are opposite the equal sides.



Cited from http://www/themathpage.com/abooki/propI-4.htm

1.4. Area of a Traditional kite

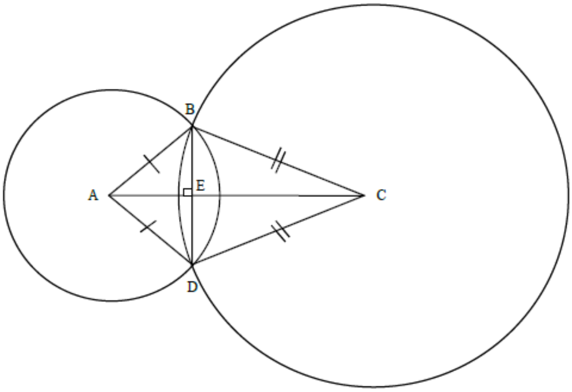


FIGURE 14

In view of the kite $ABCD$ is formed by the two congruent right triangles ABC and ADC with the line of symmetry AC. We can find the area of the kite as following:

$$\begin{aligned}&\textbf{Area of kite } \boldsymbol{ABCD}\\ =&\frac{1}{2}AC \times BE + \frac{1}{2}AC \times ED\\ =&\frac{1}{2}AC(BE + ED)\\ =&\frac{1}{2}(AC)(BD)\end{aligned}$$

Alternatively, if we regard the kite as the combining of two distinct isosceles triangles ABD and CBD, the area of the kite is as following:

$$\begin{aligned}&\textbf{Area of kite } \boldsymbol{ABCD}\\ =&\frac{1}{2}BD \times EC + \frac{1}{2}BD \times AE\\ =&\frac{1}{2}BD(AE + EC)\\ =&\frac{1}{2}(BD)(AC)\end{aligned}$$

No matter which approaches we consider, the area of a kite is found to be half of the product of both diagonals

1.5. Denoting a traditional kite

From Chapter 1.2 [View II], we define a kite as a formation of two congruent scalene triangles with one of the corresponding side overlapping each other. By the proposition 1.4 of *the Euclid's Elements, Book I*, two triangles are founded to be congruent when the two pair of sides and a pair of included angles of two triangles are correspondingly equal, i.e. S.A.S..

Now, we denote $\boldsymbol{a(\theta)b}$ to represent a traditional kite of sides $\boldsymbol{a}$ cm and $\boldsymbol{b}$ cm with an included angle $\boldsymbol{\theta}$, where $\boldsymbol{a(\theta)b}$ is known as congruent to $\boldsymbol{b(\theta)a}$**(i.e.** $\boldsymbol{a(\theta)b \equiv b(\theta)a}$**)** . It follows that we can construct a unique kite $\mathbf{3(75°)5}$ with sides $a = 3$ units, $b = 5$ units and an included angle $\boldsymbol{\theta} = 75°$ in the following with the aid of using a protractor:

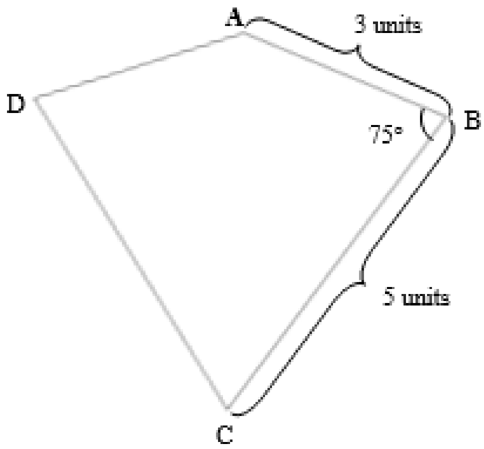


FIGURE 15. A traditional kite, i.e. $\mathbf{3(75°)5}$

Steps:

(1) Mark point A.
(2) Construct a 3 units long line segment by a ruler and mark point B at the end of line segment.
(3) Measure an angle of 75° by a protractor and produce a line segment to point C from point B.

Such that we have a scalene triangle ABC.

(4) With A as the centre and a radius of 3 units, draw an arc opposite to $\angle ABC$.
(5) With C as the centre and a radius of 5 units, draw an arc and mark point D at the intersection with the arc drawn in step (4).
(6) Joint the line segments from D to A as well as from D to C.

Now, a traditional kite $ABCD$ **3(75°)5 or 5(75°)3** is obtained.

2. Right Kite

2.1. The reason of choosing Right (or Cyclic) kite

Right kite is a specific type of kite, unlike the traditional one, one characteristic is added that a pair of opposite right angles facing to the line of symmetry of the kite is found.

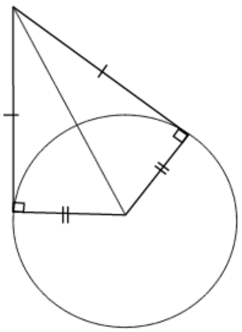


FIGURE 16

In the previous chapter, we have discussed the formation of kites. Kites can be divided into two congruent triangles by cutting along the line of symmetry [View II]. When it comes to the constructions of triangles, we can obtain those triangles by giving certain criteria including Side-Side-Side (S.S.S.), Side-Angle-Side (S.A.S), Angle-Side-Angle (A.S.A.), Angle-Angle-Side (A.A.S.), and Right-angled Hypotenuse-Side (R.H.S.).

However, in this study we have the limitations that we mainly focus on the constructions which can be carried out by hand with a Euclidean compass and straightedge, excluding the use of protractor. By the Construction skills (iv), we are able to construct a right angle. It follows that a right kite could be obtained either the adjacent sides of kites or both diagonals are given.

In addition, if we pay specific attention, right kite occurs commonly in the New Senior Secondary Mathematics curriculum, i.e. the application of tangent properties in Figure 17. We may see that the right kite $BOCT$ is found as OB and OC are equal to the radius of circle ABC. Radii OB and OC are perpendicular to tangents BT and CT respectively.

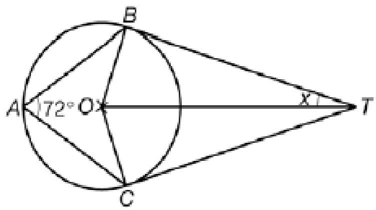


FIGURE 17

Another example that we have encountered in learning Mathematics in secondary school.

In the Figure 18, a circle is inscribed in a quadrilateral $ABCD$. P, Q, R and S are points of contact. Provided that $BC = DC$ and $\angle ACB = \angle ABD$, $ABCD$ is found to be a kite, a cyclic quadrilateral and a Right kite accordingly.

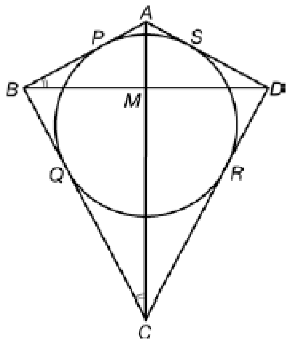


FIGURE 18

Proof.

$BC = CD$ (given)
$CQ = CR$ (tangent properties)
$\therefore CB - CQ = CD - CR$
$BQ = DR$
Since $BP = BQ$,
$DS = DR$ (tangent properties)
$\therefore BP = DS$
$AP = AS$ (tangent properties)
$\therefore AP + BP = AS + DS$
$AB = AD$
$\therefore ABCD$ is a kite

$\because AB = AD$
$\therefore \angle ABD = \angle ADB$
(base $\angle s$, isos. Δ)
$\angle ACB = \angle ABD$ (given)
$\therefore \angle ACB = \angle ADB$
$\therefore ABCD$ is a cyclic quadrilateral.
(converse of $\angle s$ in the same segment)

$AC = AC$ (common)
$AB = AD$ (proved)
$BC = DC$ (given)
$\therefore \Delta ABC \cong \Delta ADC$ (SSS)
$\therefore \angle ABC = \angle ADC$ (corr. $\angle s$, $\cong \Delta s$)
Since $\angle ABC + \angle ADC = 180^\circ$
(opp. $\angle s$, cyclic quad.)
$2\angle ADC = 180^\circ$
$\angle ADC = \underline{\underline{90^\circ}}$

$\because \Delta ABC \cong \Delta ADC$ (proved)
$\angle MAD = \angle MAB$ (corr. $\angle s, \cong \Delta s$)
$\angle ADB = \angle ABD$(base $\angle s$, isos. Δ)
$AD = AB$ (proved)
$\therefore \Delta ADM \cong \Delta ABM$ (ASA)
$\angle AMD = \angle AMB$ (corr. $\angle s, \cong \Delta s$)
$\because \angle AMD + \angle AMB = 180^\circ$ (adj. $\angle s$ on st. line)
$\therefore 2\angle AMD = 180^\circ$
$\angle AMD = 90^\circ$
$\therefore AC \perp BD$ and $ABCD$ is a Right kite.

□

Moreover, there is one more reason we choose to study a right kite. It is due to its simplicity. Requiring simple constructions so that we can discuss its existence and uniqueness.

After we equipped the skills of constructing a right angle, a right kite can be determined by the generation of the **similar** and **congruent right triangle** paris. Let's start with a right angle triangle, i.e. figure 19.

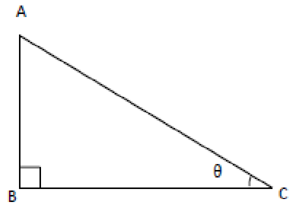


FIGURE 19

With the skills of Euclidean constructions, a right angle can be drawn by adding a line from point C to D where it meets the extending line segment AB at point D. Now we obtain a larger right triangle ACD with a right angle ACD.

Consider $\Delta ABC, \Delta CBD$ and ΔACD.
Obviously, $\angle ABC = \angle CBD = \angle ACD = 90°$.
As it is given that $\angle ACB = \theta$, we have
$\angle BCD = 90° - \theta$
$\angle BAC = 180° - 90° - \theta$ ($\angle$ sum of Δ)
$= 90° - \theta$
$\therefore \angle CAD = 90° - \theta$ (common angle)
Then, we have
$\angle CDB = 180° - 90° - \angle BCD$ ($\angle$ sum of Δ)
$= 90° - (90° - \theta)$
$= \theta$
$\angle ADC = \angle CDB = \theta$ (common angle)

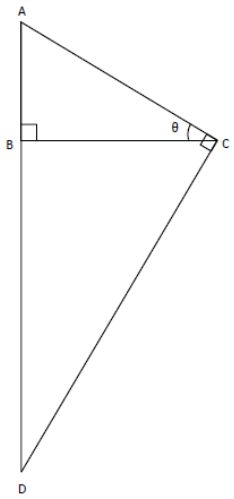


FIGURE 20

All in all, we have proved that
$\angle ABC \equiv \angle CBD \equiv \angle ACD$;
$\angle BAC \equiv \angle BCD \equiv \angle CAD$; &
$\angle ACB \equiv \angle CDB \equiv \angle ADC$,
which imply that $\Delta ABC \sim \Delta CBD \sim \Delta ACD$ (Equiangular).

Now, we attempt to copy the right triangle ACD along the common side AD so as to obtain a mirror image "right triangle AED" which is congruent to the right triangle ACD.

Steps:
(1) With A as the centre and of the radius AC, draw an arc opposite to $\angle ACB$.
(2) With D as the centre and of the radius DC, draw an arc and mark point E at the intersection with the arc drawn in step (1).
(3) Joint the line segments from E to A and from E to D.
A right kite $ACDE$ is obtained, see figure 21.

Consider ΔABC and ΔABE,
$AB = AB$ (common side)
$\angle ABC = \angle ABE = 90°$
$AC = AE$ (radii of common circle)
$\therefore \Delta ABC \equiv \Delta ABE$ (R.H.S.)

Similarly,
Consider ΔDBC and ΔDBE.
$DB = DB$ (common side)
$\angle DBC = \angle DBE = 90°$
$DC = DE$ (radii of common circle)
$\therefore \Delta DBC \equiv \Delta DBE$ (R.H.S.)

Since $AC = AE$;
$DC = DE$; &
$\angle ACD = \angle AED = 90°$;
therefore, the quadrilateral $ACDE$ is a right kite.

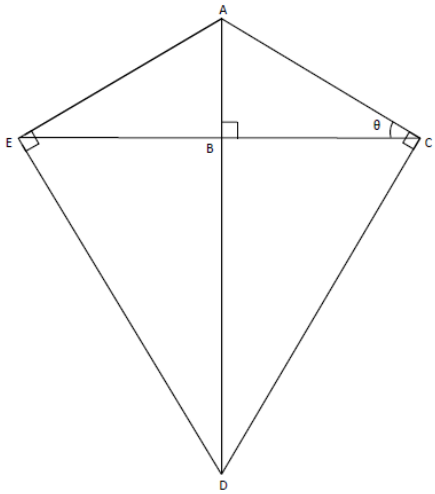


FIGURE 21

All in all, we can obtain a unique right kite $AEDC$ (see Figure 21) by giving the specific sides of AB and BC, where $AB \perp BC$ and $AB \neq BC$.

2.2. Area of a right (or cyclic) kite

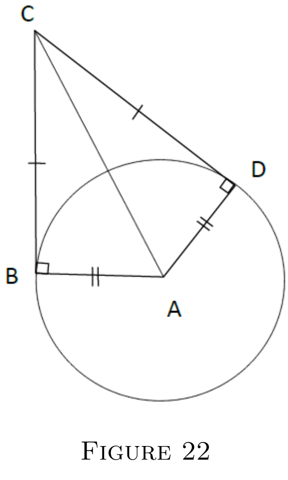


FIGURE 22

In view of the right kite $ABCD$, the area of the kite can be found more simplicity. Since the adjacent sides AB and BC (so as to AD and DC) met orthogonally, AB and AD are equal and BC and DC are equal which leads the formula of the area of the right kite $ABCD$ to be:

Area of kite $ABCD$

$$=\frac{1}{2}AB \times BC + \frac{1}{2}AD \times DC$$
$$=\frac{1}{2}AB \times BC + \frac{1}{2}AB \times BC$$
$$=AB \times BC$$

Simplicity always associates with the beauty of Mathematics. The formula of finding area of a right kite is more simple, comparing to that of a traditional kite.

2.3. Denoting a right kite and its constructions

From Chapter 1.5, we denote a traditional kite of sides a and b with an included angle $\boldsymbol{\theta}$ to be $\boldsymbol{a(\theta)b}$. For a right kite, the included angle $\boldsymbol{\theta}$ must be equal to $90°$. Therefore, we later on denote $\boldsymbol{a} \blacklozenge \boldsymbol{b}$ to represent a right kite of sides $\boldsymbol{a}$ and $\boldsymbol{b}$ with a right included angle, i.e. $\boldsymbol{a} \blacklozenge \boldsymbol{b}$ is equivalent to $\boldsymbol{a(90°)b}$. Therefore, we can construct a right kite with the given sides, i.e. $5 \blacklozenge 12$ as followings:

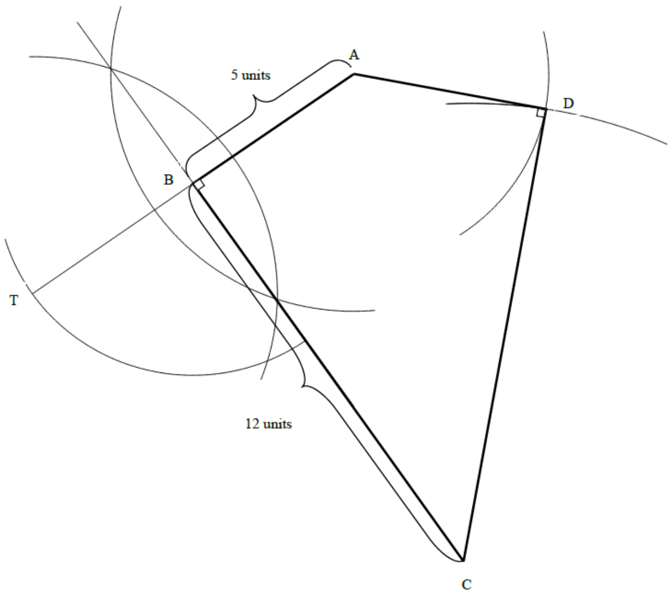


FIGURE 23. A right kite with adjacent sides 5 units and 12 units, **i.e.** $5 \blacklozenge 12$

Steps:

(1) Construct a 5 units line segment from point A to point B.
(2) With the centre B, draw an arc of radius 5 units which is opposite to point A.
(3) Extend the line segment AB and meet the arc at point T.
(4) Draw two arcs of sides greater than 5 units from centre T.
(5) Draw two arcs of the same radius of step (4) from centre A where it makes two intersection points with both pairs of arcs.
(6) Joint the intersection points of two pair of arcs and now a perpendicular bisector of AT is obtained.
(7) Produce the perpendicular bisector to form a 12 units long line segment and mark the end point as point C. Now, we have a right triangle ABC.
(8) With A as the centre and a radius of 5 units, draw an arc opposite to $\angle ABC$.
(9) With C as the centre and a radius of 12 units, draw an arc and mark point D at the intersection with the arc drawn in step (8).
(10) Joint the line segments from D to A and from D to C.

Construction of a right kite

In previous chapter, we have discussed how to construct a right kite by the knowledge of perpendicular bisector. Now, we move to construct a right kite with decided adjacent sides by using the properties of a circle, tangents and cyclic quadrilateral.

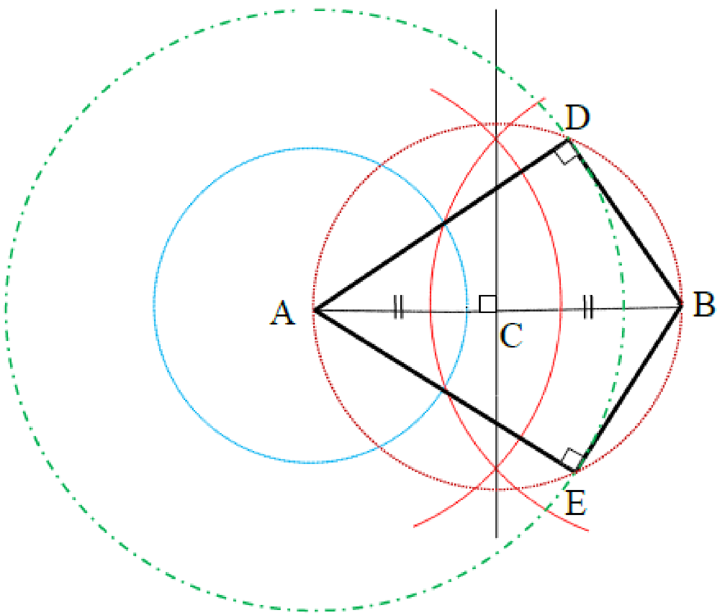


FIGURE 24

Steps:

(1) We construct a circle of centre A.
(2) We pick a point B outside the circle. (We are able to adjust the side of AB)
(3) Bisect the line segment AB and mark point C as the mid-point of AB.
(4) Construct a circle of centre C with radius CA or CB.
(5) Pick a point on the circle obtained in step (4) and mark it as point D.
(6) Construct an arc from centre A with radius AD, which is also intersecting the circle of step (4) at point E.
(7) Connect with line segment AD, DB, BE and EA.

Now, a right kite $ADBE(BD \blacklozenge DA)$ is constructed.

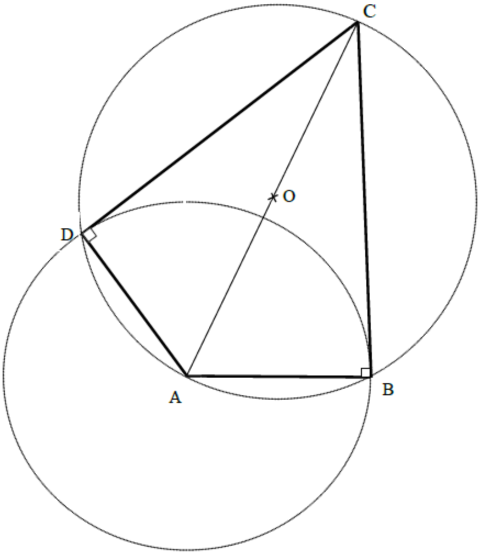


FIGURE 25

Steps:

(1) We construct a circle of centre O.
(2) We pick a line segment AB with point A and B on circle drawn in step (1). (We are able to adjust the side of AB)
(3) Construct a perpendicular line segment BC where C is a point on the same circle.
(4) Draw an arc of centre A with radius AB and intersect the circle at point D.
(5) Joint the line segment from point B to D and point A to D.

Now, a right kite $ADBE(AB \blacklozenge BC)$ is constructed.

2.4. Uniqueness of a Right kite

By using $a \blacklozenge b$ to denote a right kite, it consists of the properties of the proposition mentioning congruent triangle pairs S.A.S. in the *Euclid's Element*. The angle is equal to a right angle.

When it comes to the congruent or uniqueness of a quadrilateral, it should fulfilled the following criteria. The said quadrilaterals
(a) are of the same length of sides and the same perimeter.
(b) have the same corresponding angles.
(c) should completely cover each other when they are overlapping.
Then, It can also deduce that
$a \blacklozenge b \equiv c \blacklozenge d$ if and only if $[(a = c) \cap (b = d)] \cup [(a = d) \cap (b = c)]$.

*Thinking point:

(i) Could we construct a right kite with the given diagonals?
For example: A right kite with two diagonals of sides $2x$ cm and $2y$ cm

(ii) Is it a unique right kite?

To tackle question (i), we can use the proposition III.31 of the *Euclid's Elements* which state that in a circle the angle in the semicirlce is right.

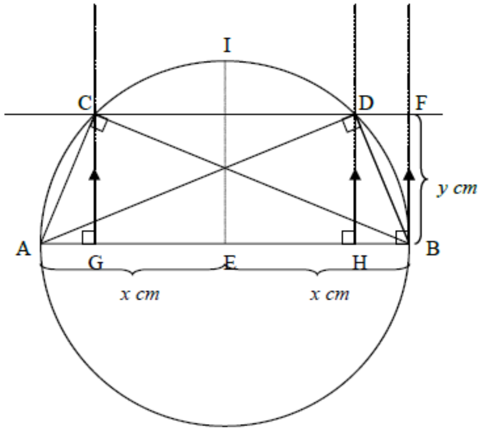


FIGURE 26

In figure 26, we have a circle of radius $x\ cm$ with diameter $AB = 2x\ cm$. Without loss of generosity, we may define that the value of x is greater than that of y. Then, we can construct the tangent BF where BF is perpendicular to AB. By constructing the parallel lines which intersect with the circle AIB at point C & D such that $GC//HD//BF$, we will obtain both right angles $\angle ACB$ and $\angle ACB$.

Then, we can construct an arc at centre A of the radius AC and intersect the circle AIB at J. After linking up with the line segments AJ and JB, a quadrilateral $ACBJ$ is constructed, details please refer to Figure 27. Obviously, $\Delta ACB \equiv \Delta AJB$(R.H.S.), the quadrilateral $ACBJ$ is a right kite. We can denote the right kite $ACBJ$ by $AC \blacklozenge CB$ or $AJ \blacklozenge JB$, where $AC = AJ$ and $CB = JB$.

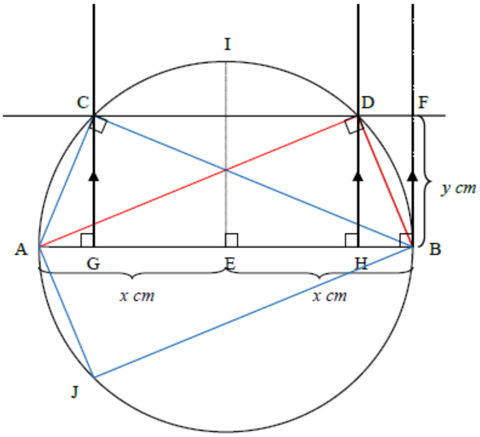


FIGURE 27

Similarly, we may construct an arc at centre B of the radius BD and intersect the same circle at point K. A right kite $BDAK$ is obtained, refer to Figure 28.

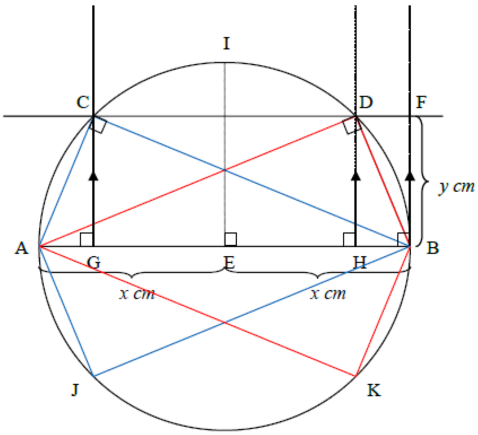


FIGURE 28

Now, we are going to discuss the uniqueness of the right kite. We will derive the solutions into two parts.

First, we want to prove the two right kites constructed above are congruent.

Adding the line segments CE and DE as shown in the Figure 29 below:

Consider ΔGCE and ΔHDE,
$CE = DE$ (radii of circle AIE)
$GC = HD = y\ cm$ (by construction)
$\angle CGE = \angle DHE = 90°$
(by construction)
$\therefore \Delta GCE \equiv \Delta HDE$ (R.H.S.)
Therefore, $GE = HE$ (corr. sides, $\cong \Delta s$)

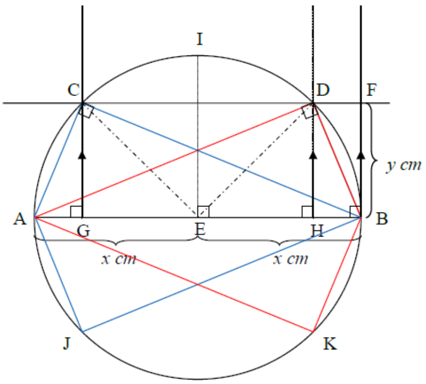


FIGURE 29

Consider ΔAGC and ΔBHD,
$\because AE = BE$ (radii of circle AIE)
$\&GE = HE$ (proved)
$\therefore AG = AE - GE = BE - HE = BH$
$GC = HD = y\ cm$ (by construction)
$\angle AGC = \angle BHD = 90°$ (by construction)

$\therefore \Delta AGC \equiv \Delta BHD$ (S.A.S.)
Therefore, $AC = BD$ (corr. sides, $\cong \Delta s$)

Consider ΔACB and ΔBDA,
$AC = BD$ (proved)
$AB = BA =$ diameter of circle AIE (common side)
$\angle ACB = \angle BDA = 90°$ (Angle in a semi-circle)
$\therefore \Delta ACB \equiv \Delta BDA$ (R.H.S.)
Therefore, $CB = DA$.

Since we have $AC = BD$ and $CB = DA$ and $\Delta ACB \equiv \Delta BDA$, we can deduce that $AC = BD = AJ = BK$ and $\Delta ACB \equiv \Delta AJB \equiv \Delta BDA \equiv \Delta BKA$.

It implies that $(AC \blacklozenge CB) \equiv (AJ \blacklozenge JB) \equiv (BD \blacklozenge DA) \equiv (BK \blacklozenge KA)$, or we may say $(CB \blacklozenge AC) \equiv (JB \blacklozenge AJ) \equiv (DA \blacklozenge BD) \equiv (KA \blacklozenge BK)$. Hence, the right kite $ACBJ$ is congruent to the right kite $BDAK$.

In addition, we found the line IE is the line of symmetry of the figure.

Second, we are interested in studying whether only the above two right kites ($ACBJ$ & $BDAK$) do exist.

With the line of symmetry IE, ΔAIE is congruent to ΔBIE. As we defined that $x > y > 0$, angle AIB must be an obtuse angle, refer to Figure 30. It follows that $90° < \angle AIB < 180°$. To study the angles ($\angle ANB, \angle ACB, \angle ALB, \angle AIB, \angle ADB, \angle AFB$, etc) formed with the fixed point AB to a variation point along NF where NF is always y cm equidistant to AB, we set Δz as the variation point moving from A to B and the corresponding angles formed be $\Delta\theta_1 + \Delta\theta_2$, i.e. when Δz is equal to 0 cm, ΔANB is formed and we have $\angle ANZ = \Delta\theta_1 = 0°$ and $\angle ZNB = \Delta\theta_2$ which is obviously an acute angle. As point z moving from A to B, we can predict the angle $\Delta\theta_1 + \Delta\theta_2$ is changing from acute angle to obtuse angle and end at the point F where $\angle AFB$ or we may call $\angle AFZ$ is an acute angle.

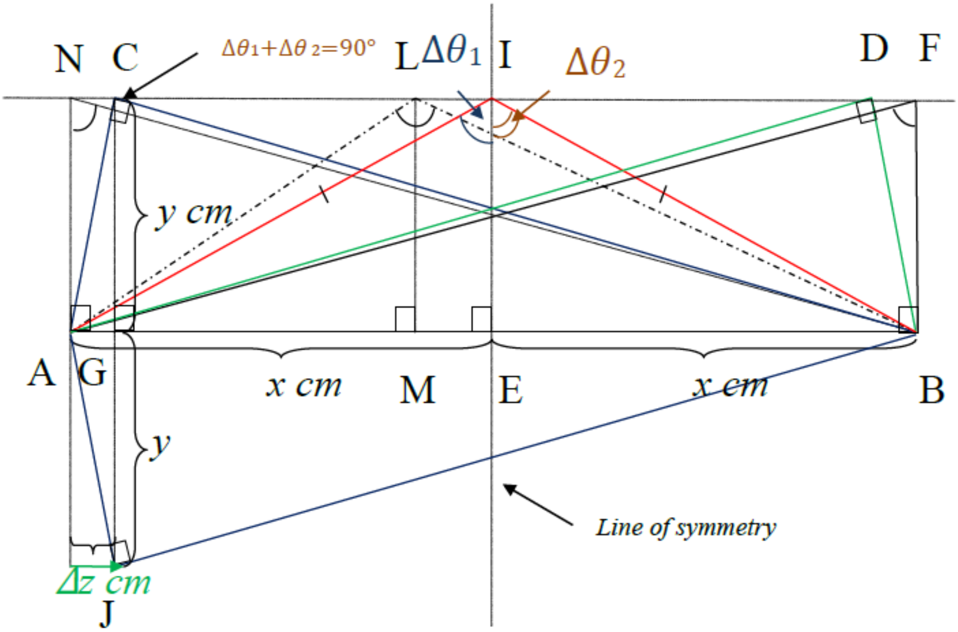


FIGURE 30

For x, y are constants and $x > y > 0$,

$$\tan\theta_1 = \frac{\Delta z}{y}$$
$$\tan\theta_2 = \frac{2x - \Delta z}{y}$$
$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$
$$\tan(\theta_1 + \theta_2) = \frac{\frac{\Delta z}{y} + \frac{2x - \Delta z}{y}}{1 - (\frac{\Delta z}{y} \times \frac{2x - \Delta z}{y})}$$
$$\tan(\theta_1 + \theta_2) = \frac{2xy}{y^2 - \Delta z(2x - \Delta z)}$$
$$\tan(\theta_1 + \theta_2) = \frac{2xy}{y^2 - 2x\Delta z + (\Delta z)^2}$$

Consider $\theta_1 + \theta_2 = 90°$, we have
$(\Delta z)^2 - 2x(\Delta z) + y^2 = 0$———(*)
For variable Δz,

$$\begin{aligned} \text{discriminant } \Delta \text{ of } (*) &= (-2x)^2 - 4y^2 \\ &= 4x^2 - 4y^2 \\ &= 4(x-y)(x+y) \end{aligned}$$

we have $(x - y > 0)$ and $(x + y > 0)$ $\because x > y > 0$
Therefore , discriminant Δ of (*) is greater than 0, which means Δz has two distinct real roots and Δz would occur at point G and H according to the previous finding.

For solving Δz,

$$(\Delta z)^2 - 2x(\Delta z) + y^2 = 0$$
$$\Delta z = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(y^2)}}{2(1)}$$
$$\Delta z = \frac{2x \pm \sqrt{4(x-y)(x+y)}}{2}$$
$$\Delta z = \underline{\underline{x \pm \sqrt{(x-y)(x+y)}}}$$

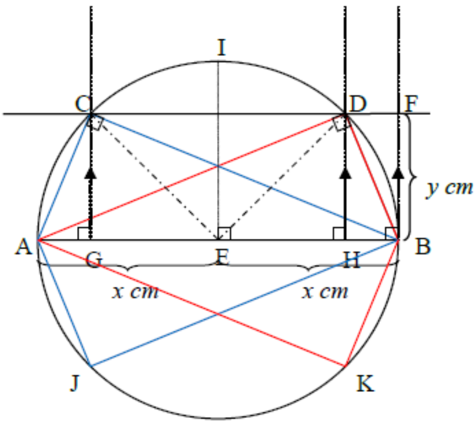


FIGURE 31. cited from page 18

All in all, we could conclude that a unique right kite is obtained for given both of the diagonals. We later on use $\boldsymbol{a} \boxtimes \boldsymbol{b}$ to denote the right kite with diagonals a and b.

3. Admiring the Beauty of Right Kite

3.1. Constructions of Spiral by Right kites with sides in Fibonacci sequence

i.e. Constructions of $1 \blacklozenge 1 \blacklozenge 2 \blacklozenge 3 \blacklozenge 5$

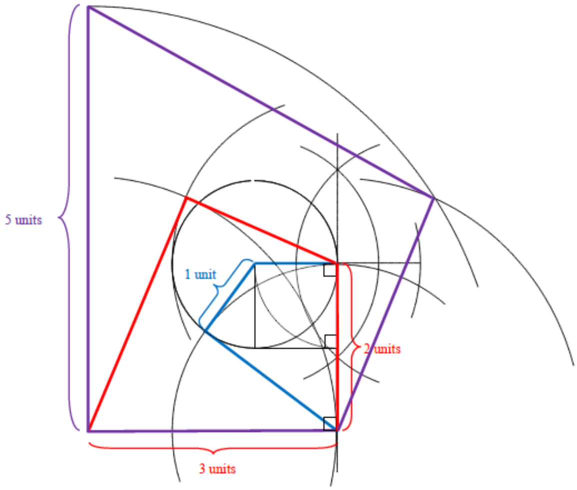


FIGURE 32

3.2. Constructions of Spiral by Right kites i.e. $1 \blacklozenge 1 \blacklozenge 2 \blacklozenge 3 \blacklozenge 5 \blacklozenge 8 \blacklozenge 13$

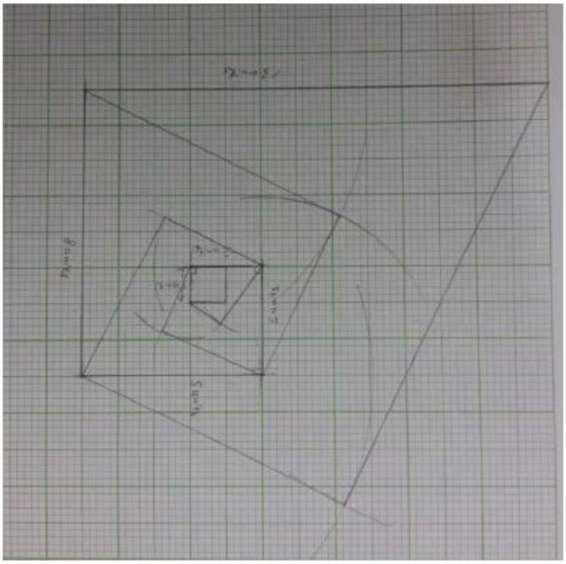


PHOTO 33

3.3. Constructions of kites with diagonals 2 and irrational numbers

By constructing the irrational numbers as shown in Figure 34, we are able to modify the figure and construct the kites with diagonals of sides 2 units and irrational number patterns in Figure 35.

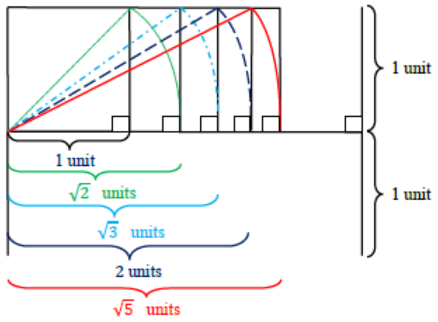


FIGURE 34

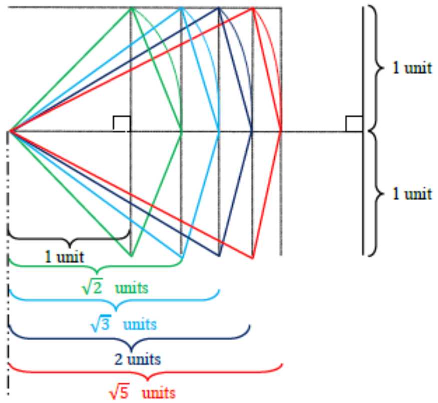


FIGURE 35

From Figure 35, we are able to construct the right kites with the diagonals of sides 2 units and $2\sqrt{x}$ units, $\forall x \in$ **Natural numbers** : $\{1, 2, 3, \ldots\}$, i.e. $\mathbf{2} \boxtimes \mathbf{2\sqrt{2}}$.

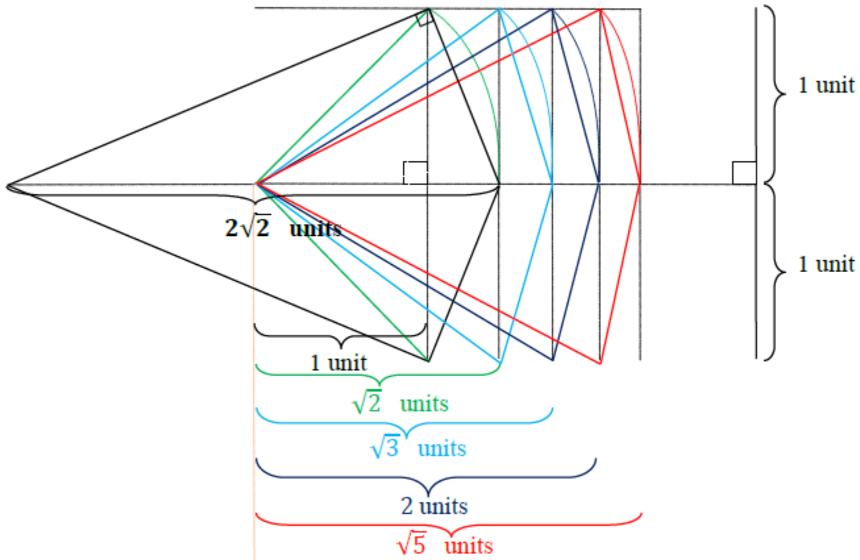


FIGURE 36

PART B

Applications on Right kite

4. Relationships of Poincaré Disk Model and Right kite

4.1. Introduction of Poincaré Disk Model

The Poincaré disk is a model for hyperbolic geometry in which a line is represented as an arc of a circle whose ends are perpendicular to the disk's boundary (and the diameter is also permitted). Two arcs which do not meet correspond to parallel rays, arcs which meet orthogonally correspond to perpendicular lines, and arcs which meet on the boundary are a pair of limit rays.

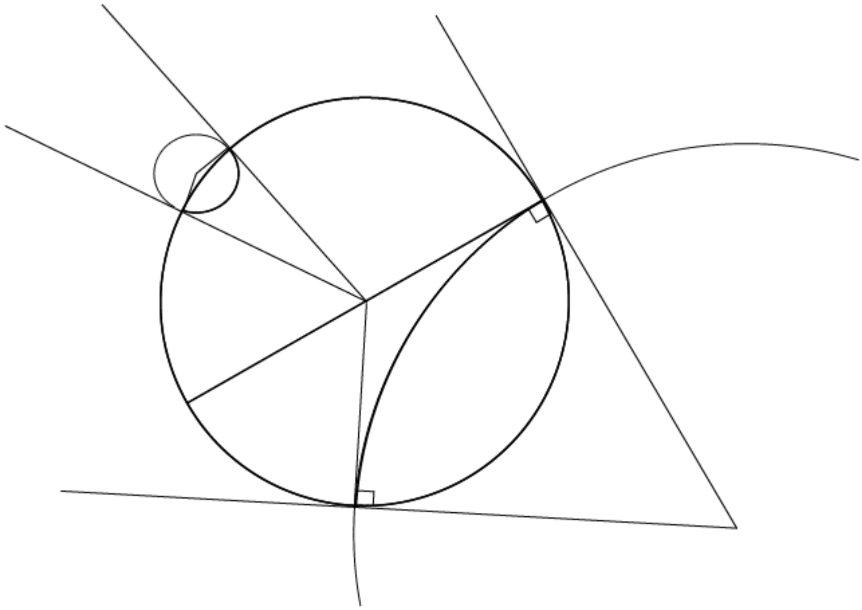


FIGURE 37

4.2. Relationships of a Poincaré Disk Model to the right kites

In secondary school, we have learnt the tangent properties where the tangents to a circle lead to a pair of right congruent triangles. For instance, the tangents of circle with centre A namely BC and DC formed two triangles ΔABC and ΔADC where $\Delta ABC \equiv \Delta ADC$, details refer to Figure 38. As BA and AD are the radii of circle with centre A, BA and AD are perpendicular to BC and DC respectively (tangent $\perp$ radius). As arc BD of circle with centre C is perpendicular to the boundary of the circle with centre A, arc BD is regard to the line in the Poincaré Disk. We also observe that a right kite $ABCD$ is formed combined with two congruent triangles ΔABC and ΔADC by the View II mentioned in Page 4 of this study. We conclude that every line "arc" of the Poincaré Disk will lead to a corresponding right kite, so as to the right kite $FGAH$ regarding to the line "arc" HG shown in the Figure 38 below.

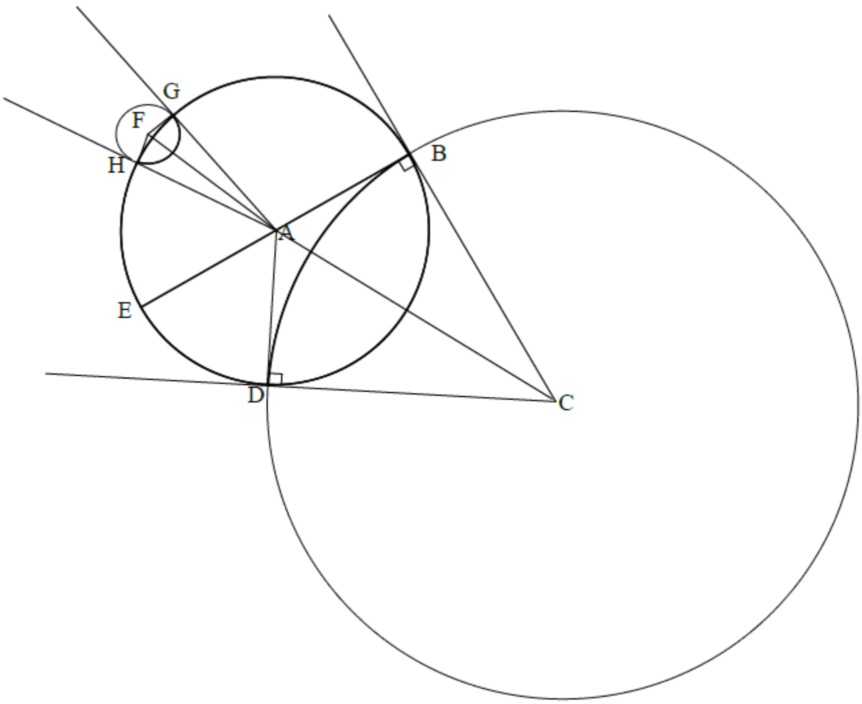


FIGURE 38

There is a limitation which the line of the Poincaré Disk formed by the diameter. Obviously, the two pair of tangents which perpendicular to the radii would be parallel to each other. The said tangents would not meet each other and therefore, neither congruent triangles nor right kite would be formed. It follows that the case does not link up with any right kite. Thus, we would neglect this exceptional situation.

Moreover, although the Poincaré Disk is a model for hyperbolic geometry, the application we use later on will only follow the plane geometry mentioned in the Euclid's Element.

5. Applications of Poincaré Disk Model in Cryptography

5.1. Introduction to Cryptography

Cryptography (or cryptology : the term mostly used in Mathematics) is the practice and study of techniques for secure communication in the presence of third parties. In the numerous fields of the information technology, cryptology is playing an important role. The types of cryptographic techniques are categorized based on the number of keys that are employed for encryption and decryption. Secret-key cryptography involves using the same key for encryption and decryption.

Now, we are going to introduce the use of Poincaré Disk Model in cryptography where the Poincaré Disk Model will act as a bridge (in form of the encoded data) and we will provide certain keys by means of focusing on the clockwise, anticlockwise, diagonals and adjacent sides of the right kites. The views are similar to the mathematical transformations.

5.2. Applications of Poincaré Disk Model in Cryptography

First, what do you think of the following figure? A circle with some arcs? Or a Poincaré Disk Model with few lines? [See reviewer's comment (3)]

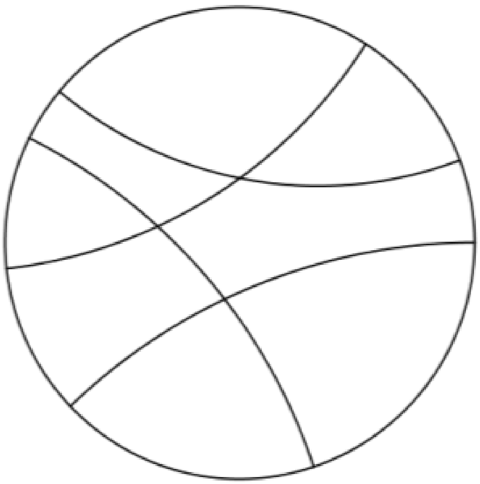


FIGURE 39

In this study, Figure 39 means a phrase or a secret message which only allow the person who knows the specific key to read it. It is an encoded data in the form of a Poincaré Disk Model. Before we start to decode this encryption, we need to have a simple table first, refer to Table 1.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

a	b	c	d	e	f	g	h	i	j	k	l	m
27	28	29	30	31	32	33	34	35	36	37	38	39
n	o	p	q	r	s	t	u	v	w	x	y	Z
40	41	42	43	44	45	46	47	48	49	50	51	52

space
0

TABLE 1

From Table 1, we simply define the capital letter "A" to "1" following the alphabetic order till "Z" to "26" and so as to the small letter "a" from "27" to "52" which is representing "z" respectively. The value "0" represents a space "␣".

At first, we start with a simple case where the message we passed has 8 digits. With the aid of Table 1, we can group all values into the alphabets and spaces.

For instance,
For any value X, we can simply compile the value X by "$X \pmod{53} \equiv 1 \mapsto A$". Therefore, "$A$" can be represented by any integers equal to the elements in the set $\{\ldots -105, -52, 1, 54, \ldots\}$. [See reviewer's comment (4)]

Now, we come to discuss the mechanism of decryption. Without loss of generosity, the Poincaré Disk Model's radius is of 27 units. Applying the coordinates geometry, we may set the disk with centre (0,0) in the coordinate plane. For the centre, we can find it out by constructing a right triangle with three vertices on the boundary of the Disk. The centre is the mid-point of the hypotenuse (bisection method). Then, we are able to construct the x-axis and y-axis where they are parallel to boundary of the rectangular A4 paper we used.
By constructing the right kite, we are able to obtain the four critical points A, B, C and D.

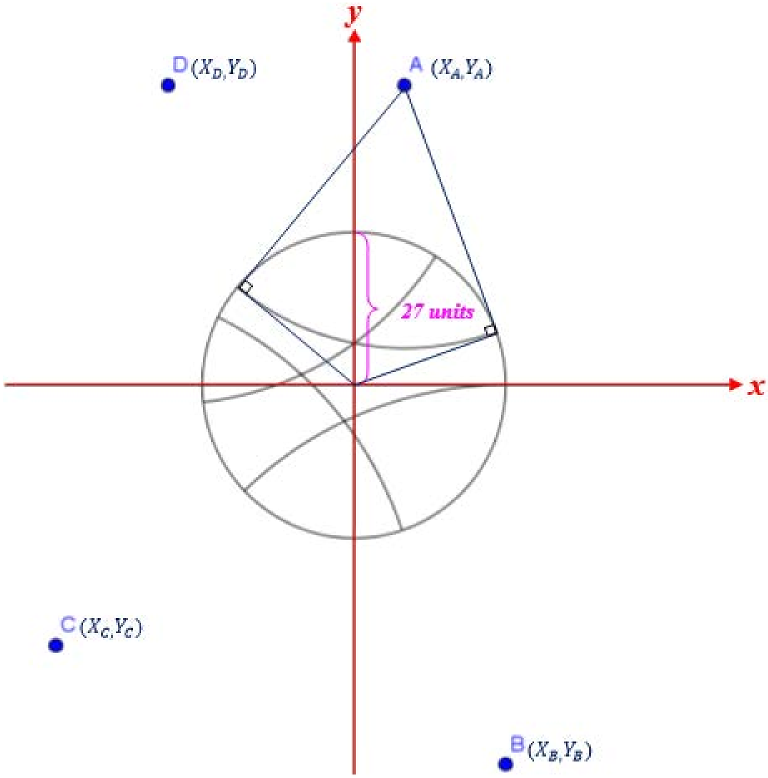


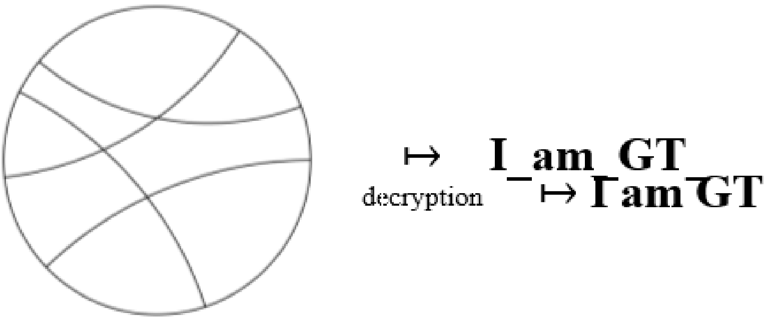
FIGURE 40

After that, we can study the corresponding coordinates of the said four points, i.e. $A(9, 53), B(27, -67), C(-53, -46)$ and $D(-33, 53)$. By dividing the coordinate plane into four quadrants, we refine the order from the four quadrants in the clockwise direction from A to D. Each point are now carrying two data and thus we can reform the phrase or sentence by decryption in the following order :$X_AY_AX_BY_BX_CY_CX_DY_D$.

Referring to Table 1, we now retrieve the $X_AY_AX_BY_BX_CY_CX_DY_D$ to be 9 53 27 -67 -53 -46 -33 53. By reducing or adding the multiple of 53 till we obtain the value of range from 0 to 52 matching with Table 1, we get $-67 + 2(53) = 39, -53 + 53 = 0, -46 + 53 = 7$, $-33 + 53 = 20$ and $53 - 53 = 0$. Finally, the value should be 9 53 27 39 0 7 20 0.

	X_A	Y_A	X_B	Y_B	X_C	Y_C	X_D	Y_D
Encrypted code	9	53	27	39	0	7	20	0
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
Wordings	I	_	a	m	_	G	T	_

The message "I am GT" finally obtained and it is equivalent to Figure 39.



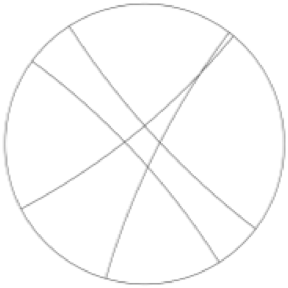
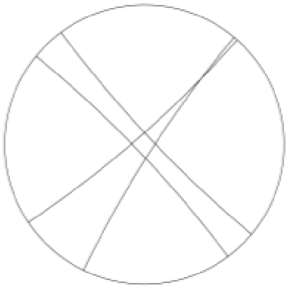
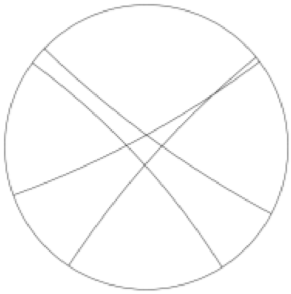
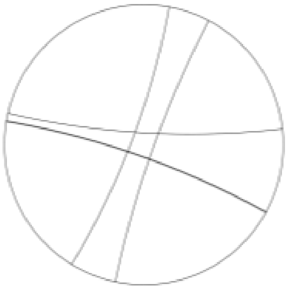
As there are infinitely many integers representing the alphabets, we can deduce infinitely many possible line patterns in Poincaré Disk to transform the same message "I am_GT_". Examples are shown in the following.

Now, it comes to the reverse process, encryption, seeing how we construct the disk by finding certain information. The range of the corresponding coordinates should fulfill the criteria listed below.

(i) The point must locate outside the Disk or there will be no possible line being constructed, i.e. $(|X_A|^2 + |Y_A|^2) > 27^2$;
(ii) Point A lines on quadrant A such that $X_A > 0$ and $Y_A > 0$;
(iii) Point B lines on quadrant B such that $X_B > 0$ and $Y_B < 0$;
(iv) Point C lines on quadrant C such that $X_C < 0$ and $Y_C < 0$;
(v) Point D lines on quadrant D such that $X_D < 0$ and $Y_D > 0$, where $A(X_A, Y_A)$, $B(X_B, Y_B), C(X_C, Y_C)\&D(X_D, Y_D)$ so that we can adjust the sequence according to the coordinates $X_AY_AX_BY_BX_CY_CX_DY_D$ in clockwise.

(mod 53)	$X_A > 0$	$Y_A > 0$	$X_B > 0$	$Y_B < 0$	$X_C < 0$	$Y_C < 0$	$X_D < 0$	$Y_D > 0$
Wordings	I	_	a	m	_	G	T	_
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
Encrypted code	9	53	27	39	0	7	20	0
Possible Choices	$9 + 53n$ $n > 0$	$53n^*$ $n > 0$	$27 + 53n$ $n \geq 0$	$39 - 53n$ $n \geq 2$	$-53n$ $n \geq 1$	$7 - 53n$ $n \geq 1$	$20 - 53n$ $n \geq 2$	$53n$ $n \geq 1$
Choices I	115	106	133	-67	-106	-99	-86	106
Choices II	168	159	186	-120	-159	-152	-139	159
Choices III	115	159	133	-67	-106	-99	-86	159
Choices IV	9	159	186	-67	-53	-152	-139	53

*Remarks: n can be equal to 0 if $(|X_A|^2 + |Y_A|^2) > 27^2$ referring to criteria (i).

$I\_am\_GT\_$	
Choices I (take $n = 2$)	
Choices II (take $n = 3$)	
Choices III	
Choices IV	

Apparently, it is unnecessary to pick the same value of "n" for completing the encryption. Such as Choices III and Choices IV, the coordinates are picked up by random. However, the value of n we choose is still worth to be studied. By observation, we may see that the points we choose is closer to the Poincaré Disk Model as long as the magnitude of n tends to 0.

In the previous case, the least value of n that fulfilled the criteria is 2. Therefore, we would like to name the Choice I with picking $n = 2$ as Rnak 1, so does Choice II with picking $n = 3$ as Rank 2.

When it comes to the application on cryptology, the level of standard in security can be upgraded if we restricted the communication coding by an increment in number of n. For example, person A are communicating with person B secretly, A starts to send the Poincaré Disk of Rank 1. Person B should reply the message by Poincaré Disk of Rank 2 and so on.

Later on, we would like to call the **G**raphical **T**ransformation method as **GT method** in short. Let's try one more and get the message from Figure 41 below.

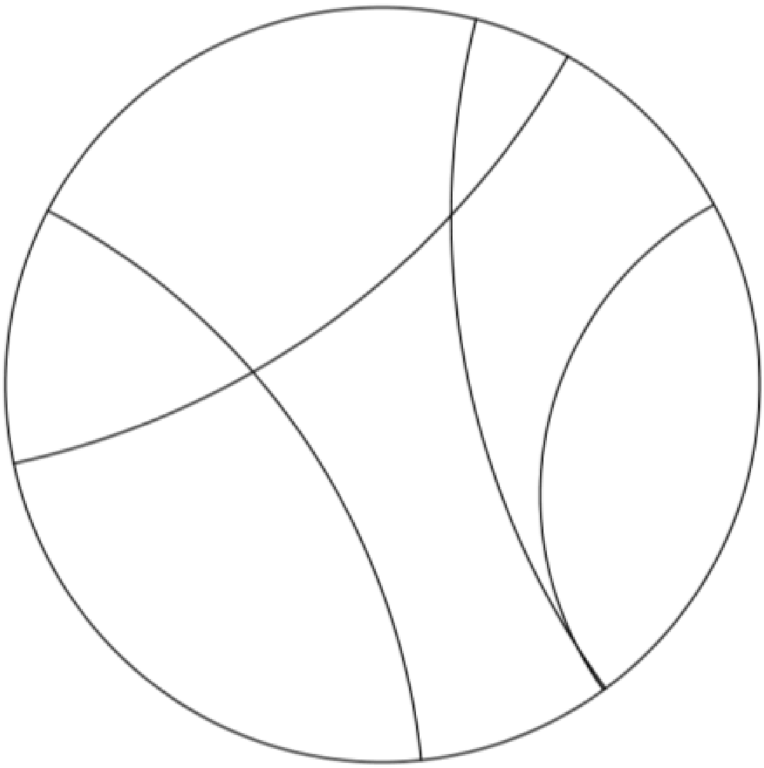
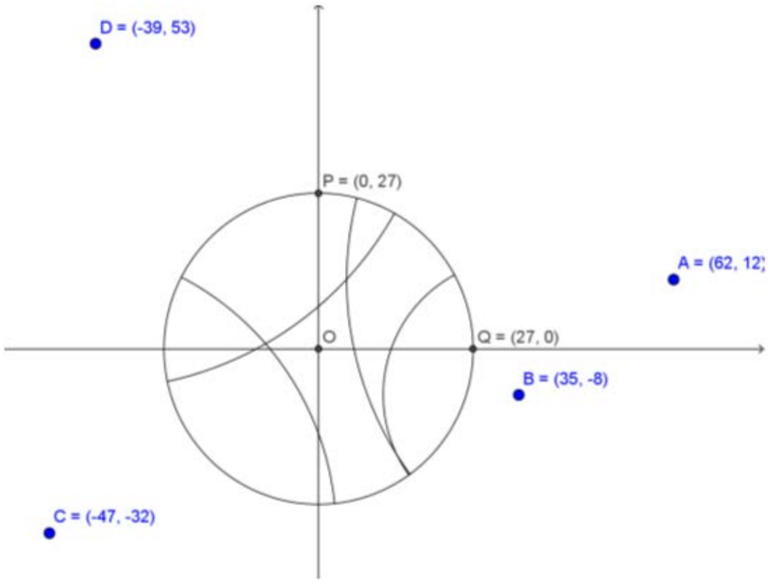


FIGURE 41

After finding the corresponding centre of the arcs, we have:

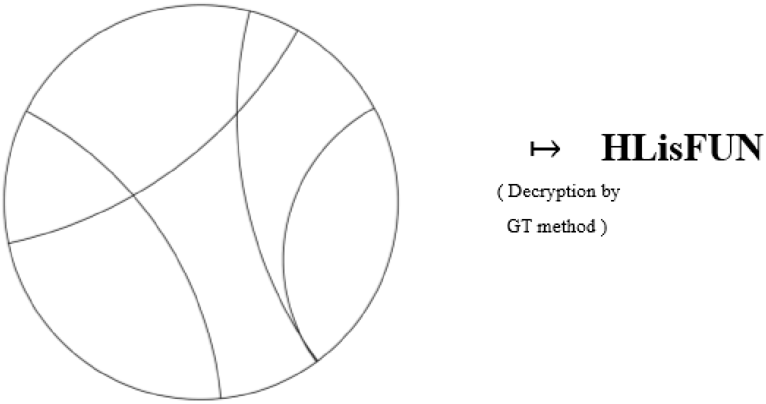


From Point A to D, the coordinates are $(62, 12), (35, -8), (-47, -32), (-39, 53)$ respectively. In order to reading the data from Table 1 effectively, we can rearrange the value to the range from 0 to 52.
Therefore, we have

$$X_A : 62 \bmod 53 \equiv 9;$$
$$Y_B : -8 \bmod 53 \equiv 45;$$
$$X_C : -47 \bmod 53 \equiv 6;$$
$$Y_C : -32 \bmod 53 \equiv 21;$$
$$X_D : -39 \bmod 53 \equiv 14;$$
$$Y_D : 53 \bmod 53 \equiv 0.$$

(mod 53)	X_A	Y_A	X_B	Y_B	X_C	Y_C	X_D	Y_D
Encrypted code	9	12	35	45	6	21	14	0
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
Wordings	H	L	i	s	F	U	N	_

Finally, we retrieve the message "HLisFUN_".



Although we can still read the meaning from the message distinguish by the capital letters and the small letters, we attempt to further study and overcome the limitation bounded by only 8 digits spacing.

The reason it caused the 8 digits is owning to the four regions we divided as shown Figure 42. [See reviewer's comment (5)]

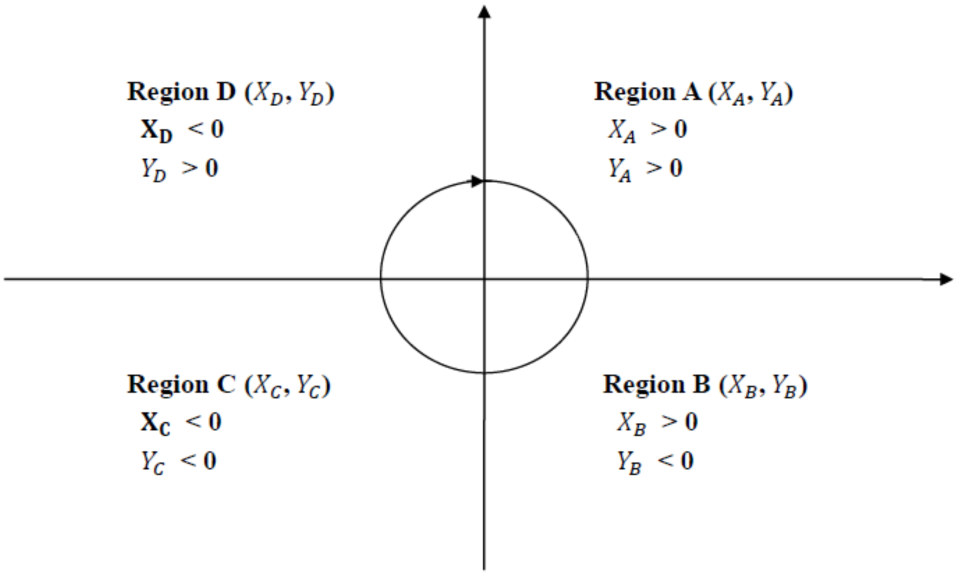


Figure 42

In a bid to enlarge the digit spacing, we now introduce a 16-digits Graphical Transformation method. [See reviewer's comment (6)] The trick is dividing the coordinates plane into 8 regions by adding two lines of symmetry, $x = y$ and $x = -y$.

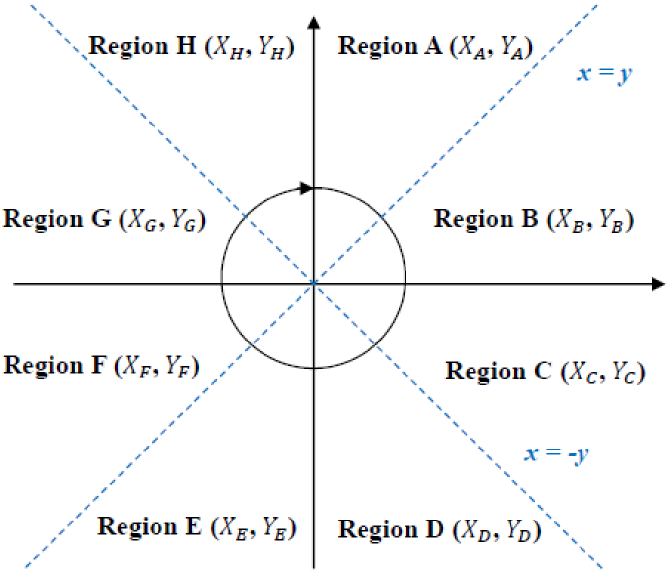


FIGURE 43

After that, we have to pick up the values we needed carefully and every data should fulfill the conditions I & II listed below and in the following table. The conditions are mainly following the linear programming.

Conditions I: $(|X_n|^2 + |Y_n|^2) > 27^2$, where $n = A, B, C, D, E \ldots, H$ as the points should outside the Disk.

Regions (coordinates)	**Conditions II**				
Region A(X_A, Y_A)	$X_A > 0$	and	$Y_A > 0$	and	$X_A < Y_A$
Region B(X_B, Y_B)	$X_B > 0$	and	$Y_B > 0$	and	$X_B > Y_B$
Region C(X_C, Y_C)	$X_C > 0$	and	$Y_C < 0$	and	$X_C > -Y_C$
Region D(X_D, Y_D)	$X_D > 0$	and	$Y_D < 0$	and	$X_D < -Y_D$
Region E(X_E, Y_E)	$X_E < 0$	and	$Y_E < 0$	and	$X_E > Y_E$
Region F(X_F, Y_F)	$X_F < 0$	and	$Y_F < 0$	and	$X_F < Y_F$
Region G(X_G, Y_G)	$X_G < 0$	and	$Y_G > 0$	and	$X_G < -Y_G$
Region H(X_H, Y_H)	$X_H < 0$	and	$Y_H > 0$	and	$X_H > -Y_H$

	X_A	Y_A	X_B	Y_B	X_C	Y_C	X_D	Y_D	X_E
Words	H	a	n	g	_	L	u	n	g
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
Table code	8	27	40	33	0	12	47	40	33
mod 53	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
Encrypted code	8	27	40	33	53	-41	47	-66	-20

	Y_E	X_F	Y_F	X_G	Y_G	X_H	Y_H
Words	_	i	s	_	f	u	n
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
Table code	0	35	45	0	32	47	40
mod 53	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
Encrypted code	-53	-71	-8	-53	32	-6	40

Now, we can construct a Poincaré Disk Model of radius 27 units with the lines (arcs) where the corresponding centre are $A(8,27)$, $B(40,33)$, $C(53,-41)$, $D(47,-66)$, $E(-20,-53)$, $F(-71,-8)$, $G(-53,32)$ & $H(-6,40)$. Afterwards, we will see that the corresponding points are clearly distributed in the 8 regions so that we can allow the receptive party to decode the Figure in clockwise starting from point A. For details please refer to Figure 44.

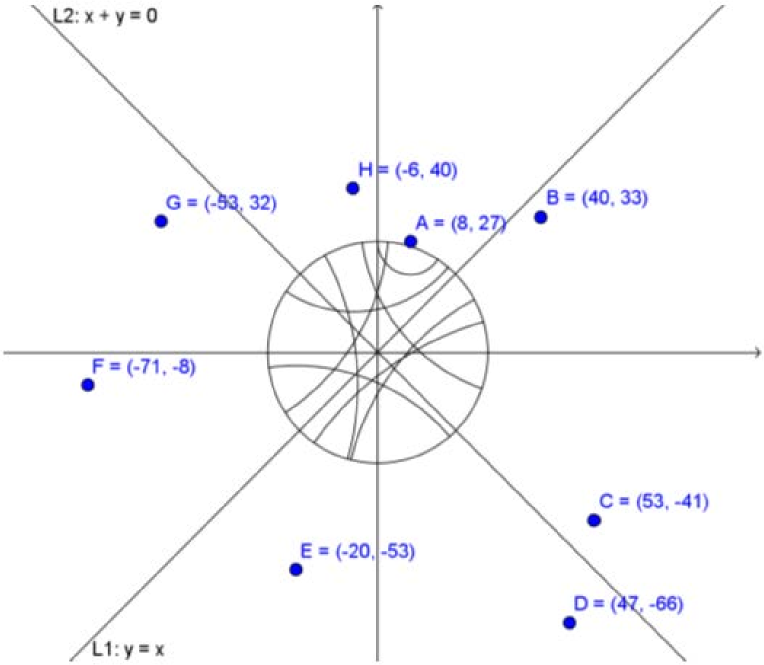
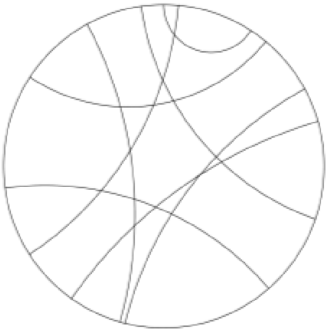


FIGURE 44

By erasing the excess lines and points, we now have the encoded data which is representing the sentence in 16 digits.



By applying the similar method, we can encode any data regardless the length in form of Poincaré Disk Model by GT method.

We regard this study as an open gate for studying Mathematics in a untraditional way we have encountered. With the base of this study, it gives us a new direction to study further though we have to stop here due to the deadline of submission. Be that as it may, our research will never cease. In the future, we will study the other GT method without applying the coordinate geometry. Also, we would like to figure out if the right kite number pattern does exist. If it does, the way to find out the least right kite number pattern, refer to Figure 45.

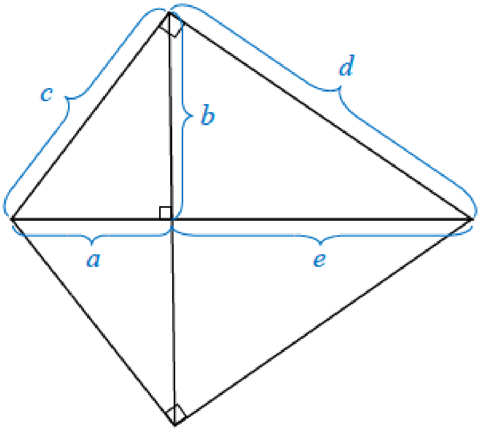


FIGURE 45

We are curious if there are any integers a, b, c, d, e can satisfy the above figure. If it does, we will name it as the right kite number patterns.

REFERENCES

[1] 莫海亮, 圓之吻——有趣的尺規作圖, 電子工業出版社, 2016.
[2] 沈淵源, 不可能的任務: 公鑰密碼傳奇, 三民書局, 2015.
[3] 丘成桐, 刘克峰, 杨乐, 季理真, 数学与人文第11辑：好的数学, 高等教育出版社, 2013.
[4] 欧几里得, 几何原本：建立空间秩序最久远最权威的逻辑推演语系, 江蘇人民出版社, 2011.
[5] Eli Maor, 畢氏定理四千年, 三民書局, 2015.
[6] Anderson, J.W., *Hyperbolic Geometry*, Springer-Verlag, New York, 1999.
[7] Goodman-Strauss, C., *Compass and Straightedge in the Poincaré Disk.*, Amer. Math. Monthly **108** (2001), 38-49.

Reviewer's Comments

The paper under review is a loose collection of topics about kites, which are quadrilaterals with one diagonal being the perpendicular bisector of another. The authors cite a lack of familiarity with kites among students as the reason why they write the paper on 'a synthetic approach in studying "kites" in a bid to let this beauty [sic] quadrilateral being brought [sic] to light'. The paper consists of two parts. The first one is about the geometry of kites, the construction of right kites, i.e. those kites with a pair of opposite angles being right angles, and its use in constructing Fibonacci sequence, spirals and some square roots. Those materials are in fact well-known to high school students. The second part, on the other hand, concerns the application of right kites and 'Poincaré disk model' in devising a way to encrypt and decrypt messages, which is elementary and not at all subtle.

One of the major issues about the paper is the proliferation of grammatical mistakes and incomprehensible sentence structure, which inevitably makes the paper unreadable. Sometimes the authors exhibit immature writing style which does not conform to the standard of academic writing. For instance, the questions in Section 5.2 on P.112 are utterly unnecessary. Both the abstract and the introduction are poorly written, as they do not give the readers a clear idea what specific problems they would like to address and how the paper is organized. Probably this is the result of the deficient and superficial mathematics content in the paper. Though there are a lot of illustrations showing some geometric constructions, the corresponding explanations in words are absent. See, for example, pages 108 and 109. Lastly, I find it a bit self-aggrandizing and misleading for them to mention the phrase 'Poincaré disk model' both in the title and in the second part of the paper, while the 'Poincaré disk' they use is no more than a disk in the Cartesian coordinate plane, and no hyperbolic geometry is involved. The following is a non-exhaustive list of grammatical mistakes and some specific issues I find in the paper.

(1) The reviewer has comments on the wordings, which have been amended in this paper.
(2) Title: it should be changed to 'A synthetic approach to studying the mysterious "right kites" and its applications to cryptography in relation to Poincaré Disk Model in view of Euclidean Geometry'.
(3) these questions are not necessary.
(4) change 'the elements' to 'any element'.
(5) change to 'The reason why there are 8 digits is owing to...'.
(6) add 'number of' after 'the'.