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Honorable Mention

The Application of Graph Theory to Sudoku

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THE APPLICATION OF GRAPH THEORY TO SUDOKU

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Abstract. In this project, we establish the Sudoku graph by studying the relationship between Sudoku and graphs with the help of NEPS (Non-complete Extended P-Sum). The approach is to look for the chromatic polynomial of the Sudoku graph, so that we can find out the total number of possible solved Sudoku puzzles. Though the chromatic polynomial of the Sudoku graph is not presented in this research, we have found some properties of the polynomial that may provide inspirations for further researches.

1. Introduction

Sudoku, originally named Number Place, is a popular game which can be found in most newspapers. Researchers have been especially keen on finding the number of possible Sudoku puzzles, which was proved to be approximately 6.671×10^{21} by Felgenhauer and Jarvis [2] by writing a computer program.

We are greatly interested in this topic, and after reading several papers, it was discovered that the number of possible Sudoku puzzles can be found by another approach: relating Sudoku to graph theory. Therefore, we decided to work on the problem of finding the total number of valid Sudoku puzzles through this approach.

In Sanders paper, the relationship between Sudoku and graph was revealed [5], which we surprisingly found that Sudoku can be expressed as a graph, denoted by $\mathrm{Sud}(n)$. In Chapter 2, we aim to recall the findings of this relationship. [See reviewer's comment (2)]

After reading research about chromatic polynomial, which is defined to be the possible number of different proper colourings on a graph, we discovered that the chromatic polynomial of $\mathrm{Sud}(n)$ in fact refers to the number of possible Sudoku puzzles. Therefore, the number 6.671×10^{21} can be generated using the chromatic polynomial of Sud(n), which we intended to determine in Chapter 3, theoretically. However, due to the limitation of the power of computers, we are not able to calculate the chromatic polynomial. As a result, we plan to explore the relationship between larger graphs and smaller graphs, and find out the properties of chromatic polynomial, wishing to have a step closer.

2. The Relationship between Sudoku and Graph

2.1. Graph Theory

Definition 1. *A* graph $G = (V(G), E(G))$ *consists of a non-empty finite set* $V(G)$ *of elements called* vertices*, and a finite set* $E(G)$ *of unordered pairs of elements of* $V(G)$ *called* edges*. We call* $V(G)$ *the* vertex set *and* $E(G)$ *the* edge set *of* G*. An edge* $\{v, w\}$ *is said to join the vertices* v *and* w*, and is usually abbreviated to* vw*. For example, FIGURE 1 represents the graph* G *whose vertex set* $V(G)$ *is* $\{u, v, w, z\}$*, and whose edge set* $E(G)$ *consists of the edges* uv*,* uw*,* vw *and* wz*. The numbers of elements in* $V(G)$ *and* $E(G)$ *are denoted by* $|V(G)|$ *and* $|E(G)|$ *respectively*

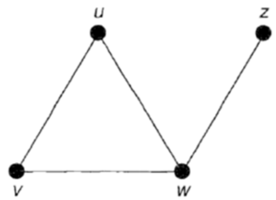


FIGURE 1

Definition 2. *A* subgraph *of a graph* G *is a graph, each of whose vertices belongs to* $V(G)$ *and each of whose edges belongs to* $E(G)$*. Thus the graph in FIGURE 2 is a subgraph of the graph in FIGURE 3.*

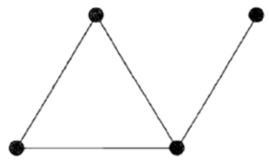


FIGURE 2

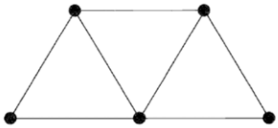
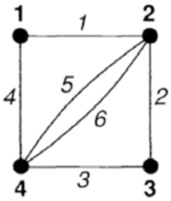


FIGURE 3

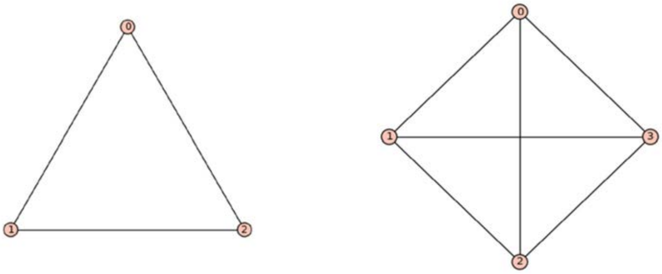
Definition 3. *If G is a graph with vertices labelled $\{1, 2, , n\}$, its* adjacency matrix *$A(G)$ is the $n \times n$ matrix whose ij-th entry is the number of edges joining vertex i and vertex j. If, in addition, the edges are labelled $\{1, 2, , m\}$, its* incidence matrix *$M(G)$ is the $n \times m$ matrix whose ij-th entry is 1 if vertex i is connected to edge j; and 0 otherwise. FIGURE 4 shows a labelled graph G with its adjacency matrix A and incidence matrix respectively.*



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

FIGURE 4

Definition 4. *A simple graph*[1] *in which each pair of distinct vertices are adjacent*[2] *is a* complete graph. *We denote the complete graph on n vertices by K_n. K_3 and K_4 are shown in FIGURE 5.*



K_3 K_4

FIGURE 5

[1] A *Simple graph* is a graph that has no loops and no multiple edges.
[2] *Adjacent*: Two vertices are adjacent if they are connected with one or more edges.

Definition 5. *Two graphs G_1 and G_2 are* isomorphic *(written as $G_1 \simeq G_2$) if there is a one-one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2. Let $\theta : V(G_1) \to V(G_2)$ be the isomorphism, then $x_1x_2 \in E(G_1)$ if and only if $\theta(x_1)\theta(x_2) \in E(G_2)$.*

2.2. Sudoku

The most common type of Sudoku puzzles consists of a 3×3 arrangement of square blocks, with 9 cells arranged in 3×3 in each square block. Each cell may have a number ranging from 1 to 9 or may be empty (see FIGURE 6). The objective is to fill all empty cells with numbers 1 to 9 such that each row, column and block consists of all 9 numbers from 1 to 9 (see FIGURE 7). [See reviewer's comment (3)]

9		5			7			
	3	8		1	9			4
2			4		5	9		8
3	1		8					
	8		7		1		6	
					4		2	1
5		7	9		3			2
1			5	7		6	9	
			1			7		5

FIGURE 6. A typical Sudoku puzzle

9	4	5	3	8	7	2	1	6
6	3	8	2	1	9	5	7	4
2	7	1	4	6	5	9	3	8
3	1	6	8	9	2	4	5	7
4	8	2	7	5	1	3	6	9
7	5	9	6	3	4	8	2	1
5	6	7	9	4	3	1	8	2
1	2	4	5	7	8	6	9	3
8	9	3	1	2	6	7	4	5

FIGURE 7. A solved Sudoku puzzle

Sudoku is closely related to graph theory as a Sudoku puzzle can be solved by considering it as a vertex colouring problem [5], which is the assignment of colours to the vertices of a graph in a way that no two adjacent vertices have the same colour. Given a Sudoku graph where all cells are empty, a Sudoku graph Sud(n), where n^4 is the number of cells, can be established by one-to-one mapping from cells to vertices, adding edges between two vertices if they are in the same row, column or block (see FIGURE 8). Solving a Sudoku puzzle is the same as colouring the whole graph with only n_2 colours, e.g. the 4×4 Sudoku puzzle has $n = 2$, and it can be solved by filling the puzzle with numbers 1 to 4, i.e. colouring the whole graph with $2^2 = 4$ colours.

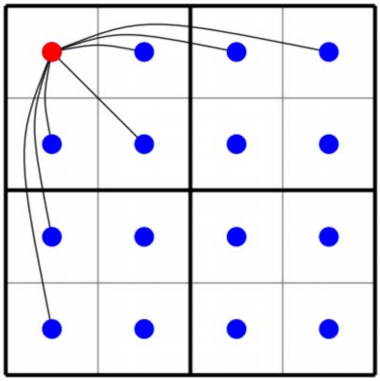


FIGURE 8. A partly mapped Sud(2) graph

2.3. NEPS

The NEPS (Non-complete Extended P-Sum) of graphs is a graph product operation, in which the vertex set of the resulting graph is the product of the vertex sets of starting graphs under a special basis.

Definition 6. *Let $B \subseteq \{0,1\}^n \setminus \{0,\cdots,0\}$ be a set of binary n-tuples[3]. Given graphs $G_1,\cdots,G_n$ where $G_i = (V(G_i),E(G_i))$. The* NEPS *of these graphs with respect to the basis B is the graph G with vertex set $V(G) = V(G_1)\times\cdots\times V(G_n)$ in which two vertices, such as $(x_1,\cdots,x_n)$ and $(y_1,\cdots,y_n)$, are adjacent if and only if there exists an n-tuple $(\beta_1,\cdots,\beta_n) \in B$ such that when $\beta_i = 0$, $x_i = y_i$ and when $\beta_i = 1$, x_i is adjacent to y_i.*

For example, let graph G be the NEPS of G_1 and G_2 with respect to $B = \{(0,1),(1,0)\}$, where $V(G_1) = \{a_1,b_1,c_1\}$ and $V(G_2) = \{a_2,b_2\}$ (see FIGURE 9).

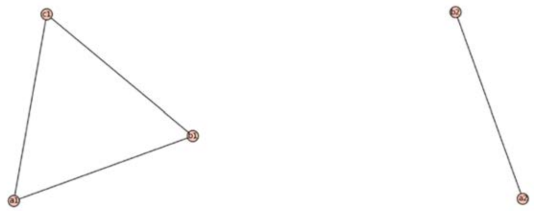


FIGURE 9

[3] *Tuple* is the number of numbers in each entry in the set, e.g. the set $\{(1,7,4),(6,7,3)\}$ has 3 tuples.

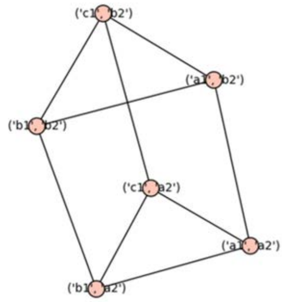


FIGURE 10

Then $V(G) = \{(a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2), (c_1, a_2), (c_1, b_2))\}$. With basis $B = \{(0, 1), (1, 0)\}$, for example, vertex (a_1, a_2) is adjacent to (a_1, b_2), (b_1, a_2), (c_1, a_2). Graph G is shown in FIGURE 10.

Definition 7. *For $n = 2$, commonly used products are the* direct sum *(or the Cartesian product) $G_1 + G_2$ with $B = \{(0, 1), (1, 0)\}$ and the* direct product *(or the tensor product) $G_1 \times G_2$ with $B = \{(1, 1)\}$.*

2.4. Sudoku and NEPS

It is proved that the Sudoku graph $\mathrm{Sud}(n)$ can be represented as NEPS. [1]

Lemma 8. *Let $n \in \mathbb{N}$ and $G_1 = K_n$, $G_2 = K_n$, $G_3 = K_n$, $G_4 = K_n$. If G is the NEPS of these graphs with basis $B = \{(0, 1, 0, 1), (1, 1, 0, 0), (0, 0, 1, 1), (1, 0, 0, 0),$ $(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$, then $G \simeq \mathrm{Sud}(n)$.*

Proof. Assume that $V(G_1) = V(G_2) = V(G_3) = V(G_4) = \{1, , n\}$. Construct a one-to-one mapping between the 4-tupules in $V(G) = \{1, , n\}^4$ and the cells of the Sudoku puzzle by the following method. For each vertex $v = (p, q, r, s) \in V(G)$, associate v with the cell Λ_v, where p, r index the horizontal and vertical block number respectively and q, s index the horizontal and vertical position in the block, i.e. the row number is $(p1)n + q$ and the column number is $(r1)n + s$. For example, let $n = 3$ and vertex $v = (2, 2, 2, 3)$. v then lies in the 2nd row of the 2nd horizontal block and the 3rd column of the 2nd vertical block (see FIGURE 11). The row number is thus 5 and the column number is 6.

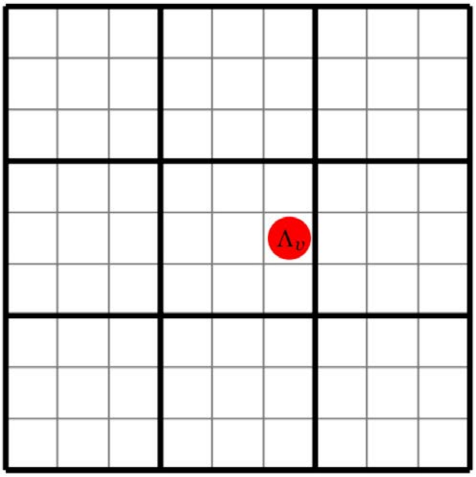


FIGURE 11

Let $B \subseteq \{0,1\}^n \setminus \{0,\cdots,0\}$. Fix a vertex v (say, $v = (2,2,2,3)$) and some $b \in B$. Now, consider how b selects the neighbours of v in G.
For $q = (1,1,0,0)$, when $v = (2,2,2,3)$, vertices $(1,1,2,3)$, $(1,3,2,3)$, $(3,1,2,3)$ and $(3,3,2,3)$ are selected, i.e. the cells that do not lie in the same horizontal block or the same relative horizontal position in the block as Λ_v but lie in the same column as Λ_v are selected (see FUGURE 12).

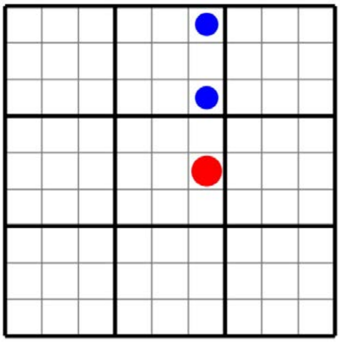


FIGURE 12

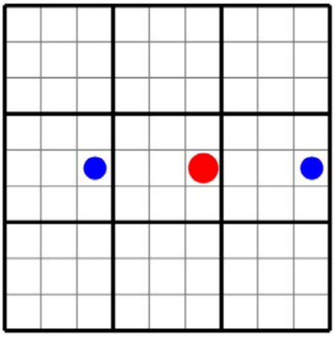


FIGURE 13

Similarly, for $q = (0,0,1,0)$, the cells that do not lie in the same vertical block but lie in the same relative vertical position in the block and the same row as Λ_v are selected (see FIGURE 13). It was found that subset $S_1 = \{(0,0,1,1),$
$(0,0,1,0),(0,0,0,1)\}$ selects the cells that lie in the same row as Λ_v; subset $S_2 =$
$\{(1,1,0,0),(1,0,0,0,),(0,1,0,0)\}$ selects the cells that lie in the same column as

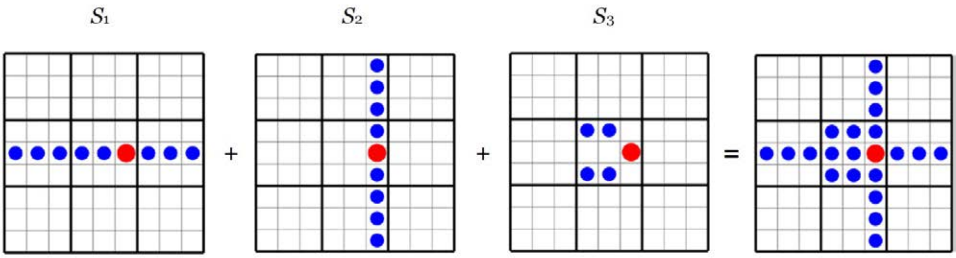


FIGURE 14

Λ_v and subset $S_3 = \{(0, 1, 0, 1)\}$ selects the cells that lie in the same block as Λ_v, excluding Λ_v itself and the cells selected by S_1 and S_2. It was observed that S_1, S_2 and S_3 are in fact the adjacencies of $\mathrm{Sud}(n)$ (see FIGURE 14). [See reviewer's comment (4)]

□

3. Chromatic Polynomial

3.1. Definition and Example

Definition 9. Proper colouring *refers to the vertex colouring of a graph G such that any two vertices connected by a common edge have different colours (see FIGURE 15)* [3]. *The* chromatic number*, denoted by $\chi(G)$, is the least number of colours with which we can achieve a proper colouring on G. For example, the chromatic number of complete graph K_n is n, i.e. $\chi(K_n) = n$. The* chromatic polynomial*, denoted by $\mathrm{chr}(G, k)$ where k represents the number of colours given, is a special function associated with each graph to count the number of possible different colourings on a graph with a given number of colours.*

For example, for complete graph K_4, the chromatic number is 4 and the chromatic polynomial is $k^4 6k^3 + 11k^2 6k$ (see FIGURE 15). [See reviewer's comment (5)]

It was proved in Section 2.4 that Sudoku graphs can be represented as the NEPS of certain graphs with a particular basis. Thus, it is observed that by finding the chromatic polynomial of $\mathrm{Sud}(n)$ and letting $k = n^2$ (as solving a Sudoku puzzle is the same as colouring the whole graph with only n^2 colours), the total number of solved Sudoku puzzles can be found.

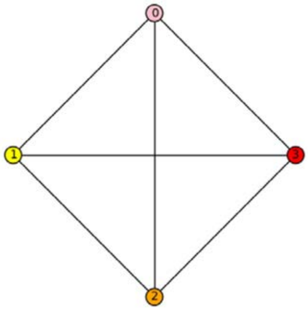


FIGURE 15. $\mathrm{chr}(K_4, k) = k^4 6k^3 + 11k^2 6k$

3.2. Deletion-contraction Property

The chromatic polynomial of a graph can be determined by the deletion-contraction property of a graph [4].

Definition 10. *For a graph $G = (V(G), E(G))$, where $i, j \in V(G)$ and $e = ij \in E(G)$, the action of unifying vertices i and j into one vertex is named* contraction, *denoted by G/e; and the action of deleting the edge $e = ij$ from G is named* deletion, *denoted by $G\backslash e$. (see FIGURE 16).*

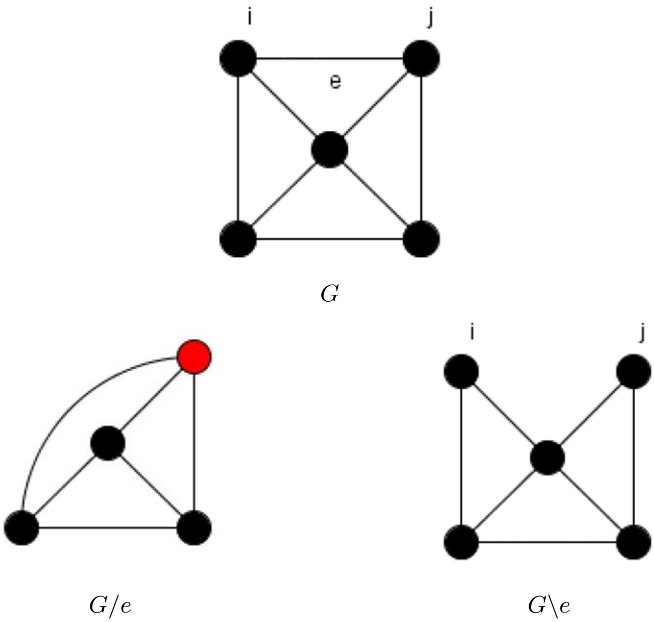


FIGURE 16

Theorem 11. *Let $G = (V(G), E(G))$ be a graph and e be one of its edge, we have* $\mathrm{chr}(G, k) = \mathrm{chr}(G\backslash e, k)$.

Proof. Consider vertices i and j from $V(G)$ with no edge between i and j. The colours of i and j would either be different or the same. When i and j have different colours, adding an edge $e = ij$ does not affect the colouring of all vertices. When i and j have the same colours, unifying i and j into one vertex, i.e. contracting, also does not change the colouring. [See reviewer's comment (6)] The following formula is thus produced:

$$\mathrm{chr}(G, k) = \mathrm{chr}(G + e, k) + \mathrm{chr}(G/e, k).$$

By substituting G with $G\backslash e$, which means deleting edge e from G, the formula becomes

$$\mathrm{chr}(G, k) = \mathrm{chr}(G\backslash e, k) - \mathrm{chr}(G/e, k).$$

□

Theorem 12. *Let n be a positive integer, we have the following results:*

(1) The chromatic polynomial of an empty graph with n vertices (i.e. null graph N_n): $\mathrm{chr}(N_n, k) = k^n$ *(see FIGURE 17 for N_4).*

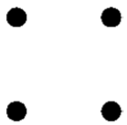


FIGURE 17. $\mathrm{chr}(N_4, k) = k^4$

Proof. Each vertex can be independently coloured by any of the k colours, giving k^n possibilities in total. □

(2) The chromatic polynomial of complete graph: $\mathrm{chr}(K_n, k) = k(k-1)(k-2)\cdots(k-n+1)$ *(see FIGURE 18 for K_4)*

Proof. All vertices must have different colours. When the first vertex is coloured by any of the k colours, the second vertex is coloured by any of the remaining $k-1$ colours, the third vertex is coloured by any of the remaining $k-2$ colours, etc., giving $k(k-1)(k-2)\cdots(k-n+1)$ possibilities in total. □

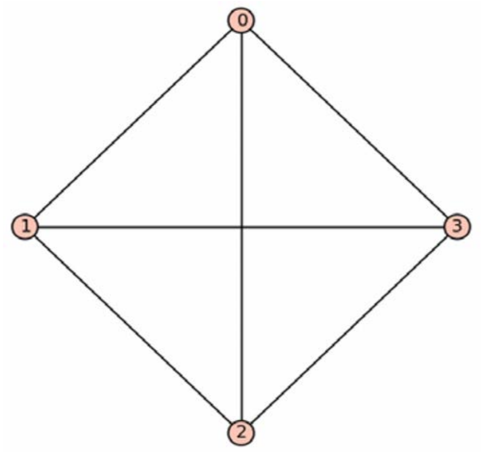


FIGURE 18. $\mathrm{chr}(K_4, k) = k(k-1)(k-2)(k-3)$

After using the deletion-contraction method for several times, any graph can be descended into null graphs or complete graphs. Thus, with Theorems 11 and 12, the chromatic polynomial of a graph can be determined.

For example, for the graph G in FIGURE 19, the chromatic polynomial can be determined by using deletion-contraction method repeatedly (see FIGURE 20).

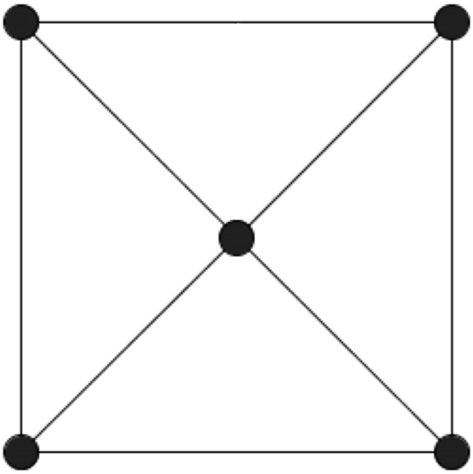


FIGURE 19

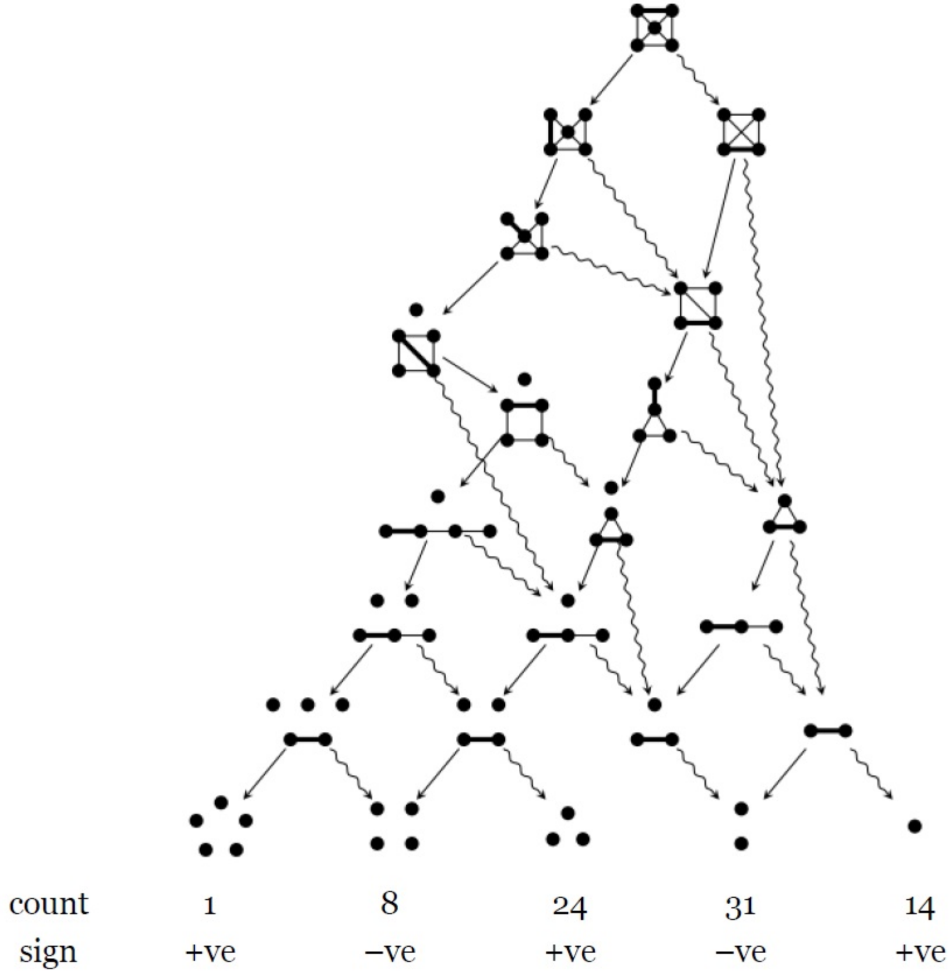
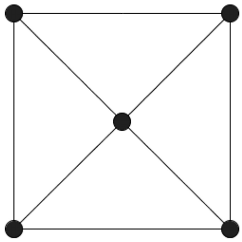


FIGURE 20

Note that $\rightarrow$ directs to graphs with deletion, while $\rightsquigarrow$ directs to graphs with contraction, e.g. FIGURE 22 refers to $G\backslash e$ and FIGURE 23 refers to G/e. By $\mathrm{chr}(G,k) = \mathrm{chr}(G\backslash e,k)\mathrm{chr}(G/e,k)$, the chromatic polynomial of G equals the difference between FIGURE 22 and FIGURE 23. After repeating this process, only null graphs N_5, N_4, N_3, N_2, N_1 are left. After counting the number of null graphs occurred and finding their signs, the chromatic polynomial of G can be calculated. The count of null graphs equals the number of paths G get to the null graph; and graphs of deletion have +ve sign while graphs of contraction have ve sign (as $\mathrm{chr}(G,k) = +\mathrm{chr}(G\backslash e,k)\mathrm{chr}(G/e,k)$).



G

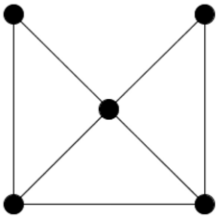


FIGURE 22

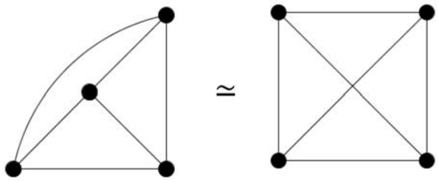


FIGURE 23

Therefore, the chromatic polynomial of G is $k^5 - 8k^4 + 24k^3 31k^2 + 14k$.

3.3. Examples

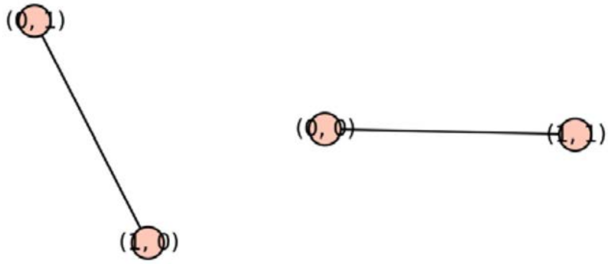
As the Sudoku graph $\mathrm{Sud}(n)$ is too large to start with, the chromatic polynomials of some smaller graphs are calculated.

(1) $G_1 = K_1 \times K_1$, with $B = \{(1, 1)\}$ (refer to Section 2.3 NEPS)

$\because G_1 \simeq N_1$
$\therefore \mathrm{chr}(G_1, k) = \underline{\underline{k}}$

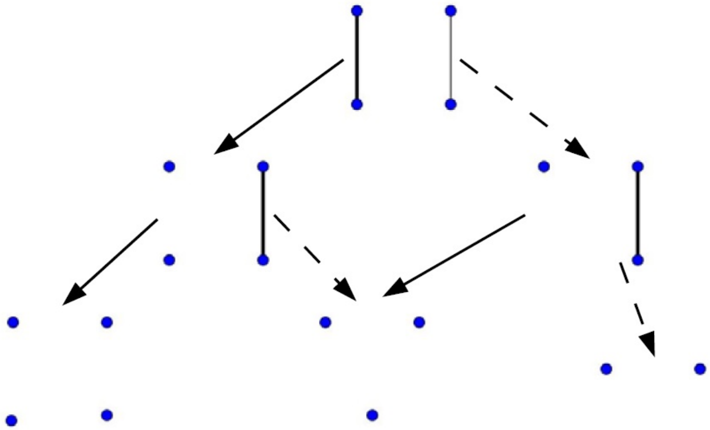
(2) $G_2 = K_2 \times K_2$, with $B = \{(1, 1)\}$

K_2



G_2

By using the deletion-contraction property (Note that $\rightarrow$ directs to graphs with deletion, while $\dashrightarrow$ directs to graphs with contraction):



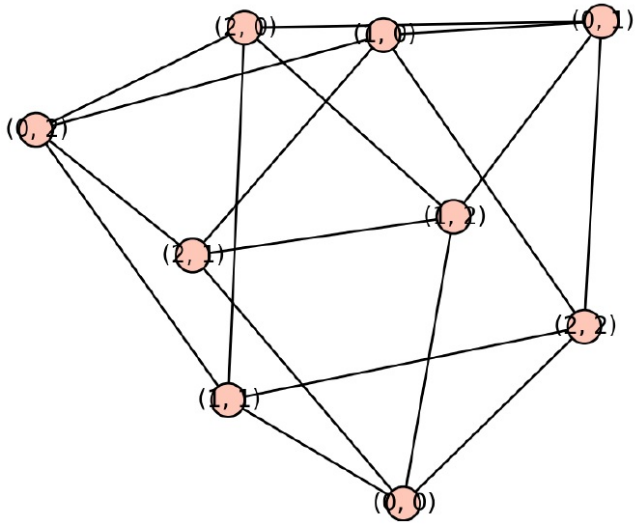
count	1	2	1
sign	+ve	−ve	+ve

$\therefore \operatorname{chr}(G_2, k) = \operatorname{chr}(N_4, k) - 2\operatorname{chr}(N_3, k) + \operatorname{chr}(N_2, k) = \underline{\underline{k^4 - 2k^3 + k^2}}$

(3) $G_3 = K_3 \times K_3$, with $B = \{(1, 1)\}$.
Besides using deletion-contraction method, we can also use Sage Virtual Machine to construct the graph G and calculate the chromatic polynomial of G.

$$\therefore \operatorname{chr}(G_3, k) = k^9 - 18k^8 + 147k^7 - 711k^6 + 220k^5 - 4545k^4 + 5878k^3 - 4308k^2 + 1336k$$

```
sage: G=(graphs.CompleteGraph(3)).tensor_product(graphs.CompleteGraph(3))
sage: G.show() # long time
```




```
sage: G.chromatic_polynomial()
x^9 - 18*x^8 + 147*x^7 - 711*x^6 + 2220*x^5 - 4545*x^4 + 5878*x^3 -
4308*x^2 + 1336*x
```

(4) $G_4 = K_4 \times K_4$, with $B = \{(1,1)\}$.
The Sage program requires extensive power to construct $K_4 \times K_4$. Without computers that are more powerful, $K_4 \times K_4$ cannot be constructed, not to mention calculating its chromatic polynomial.

Sudoku graph $\mathrm{Sud}(n)$ is a much larger graph than $G_4 = K_4 \times K_4$, thus it is believed that the chromatic polynomial of0 $\mathrm{Sud}(n)$ cannot be calculated by computer. Therefore, this research will shift focus to exploring the properties of $\mathrm{chr}(\mathrm{Sud}(n), k)$, but not determining the chromatic polynomial of $\mathrm{Sud}(n)$ itself.

3.4. Determining chromatic polynomial from lower order to higher order

Theorem 13. *Let n be a positive integer, then $K_n + K_n$ is a subgraph of $K_{n+1} + K_{n+1}$, and $K_n \times K_n$ is a subgraph of $K_{n+1} \times K_{n+1}$, that is*

(1) $K_n + K_n \subseteq K_{n+1} + K_{n+1}$ and
(2) $K_n \times K_n \subseteq K_{n+1} \times K_{n+1}$.

Proof.

(1) Let $V(K_n) = \{1, 2, \cdots, n\}$, then $V(K_n + K_n) = \{(1,1), (1,2), (1,3), \cdots, (n,n)\}$. The vertices are adjacent to each other under the basis $\{(1,0), (0,1)\}$, e.g. vertex $(1,2)$ is adjacent to $(2,2)$, $(3,2)$, $\cdots$, $(n,2)$, $(1,1)$, $(1,3)$, $(1,4)$,$\cdots$, $(1,n)$, forming $2(n1)$ edges per vertex. By calculation, the total number of edges is
$$\frac{n}{2}2(n-1) = n(n-1).$$
Let $V(K_{n+1}) = \{1, 2, \cdots, n+1\}$, then $V(K_{n+1}+K_{n+1}) = \{(1,1), (1,2), (1,3), \cdots, (n+1, n+1)\}$. Consider only the vertices $\{1, 2, \cdots, n\}$ in K_{n+1}, these vertices would form vertices $\{(1,1), (1,2), (1,3), \cdots, (n,n)\}$ in $K_{n+1} + K_{n+1}$ and edges under $\{(1,0), (0,1)\}$. Thus the $n(n1)$ edges in $K_n + K_n$ also appears in $K_{n+1} + K_{n+1}$, showing that $V(K_n + K_n) \subseteq V(K_{n+1} + K_{n+1})$, and $E(K_n + K_n) \subseteq E(K_{n+1} + K_{n+1})$. This proves that $K_n + K_n \subseteq K_{n+1} + K_{n+1}$.

(2) Let $V(K_n) = \{1, 2, \cdots, n\}$, then $V(K_n \times K_n) = \{(1,1), (1,2), (1,3), \cdots, (n,n)\}$. The vertices are adjacent to each other under basis $\{(1,1)\}$, e.g. vertex $(1,2)$ is adjacent to $(2,1)$, $(2,3)$, $(2,4)$, $\cdots$, $(2,n)$, $(3,1)$, $(3,3)$, $(3,4)$, $\cdots$, $(3,n)$, $\cdots$ By calculation, the total number of edges is ${}_{n^2}C_2 - n(n-1) - 1$. Let $V(K_{n+1}) = \{1, 2, \cdots, n+1\}$, then $V(K_{n+1} \times K_{n+1}) = \{(1,1), (1,2), (1,3), \cdots, (n+1, n+1)\}$. Consider only the vertices $\{1, 2, \cdots, n\}$ in K_{n+1}, these vertices would form vertices $\{(1,1), (1,2), (1,3), \cdots, (n,n)\}$ in $K_{n+1} \times K_{n+1}$ and edges under $\{(1,1)\}$. Thus the ${}_{n^2}C_2 - n(n-1) - 1$ edges in $K_n \times K_n$ also appears in $K_{n+1} \times K_{n+1}$, showing that $V(K_n \times K_n) \subseteq V(K_{n+1} \times K_{n+1})$, and $E(K_n \times K_n) \subseteq E(K_{n+1} \times K_{n+1})$. This proves that $K_n \times K_n \subseteq K_{n+1} \times K_{n+1}$.

$\square$

With Theorem 13, it is deduced that the chromatic polynomials of the direct product of large complete graphs can be obtained by finding the chromatic polynomials of direct sum and direct product of small complete graphs first, then using the deletion-contraction property.

For example,

To find $\text{chr}(K_3 \times K_3, k)$, $\text{chr}(K_2 \times K_2, k)$ is first calculated. From Section 3.3 (2), $\text{chr}(K_2 \times K_2, k) = k^4 2k^3 + k^2$. Then, use the deletion-contraction method until only $K2 \times K_2$ and null graphs are left (see FIGURE 30).

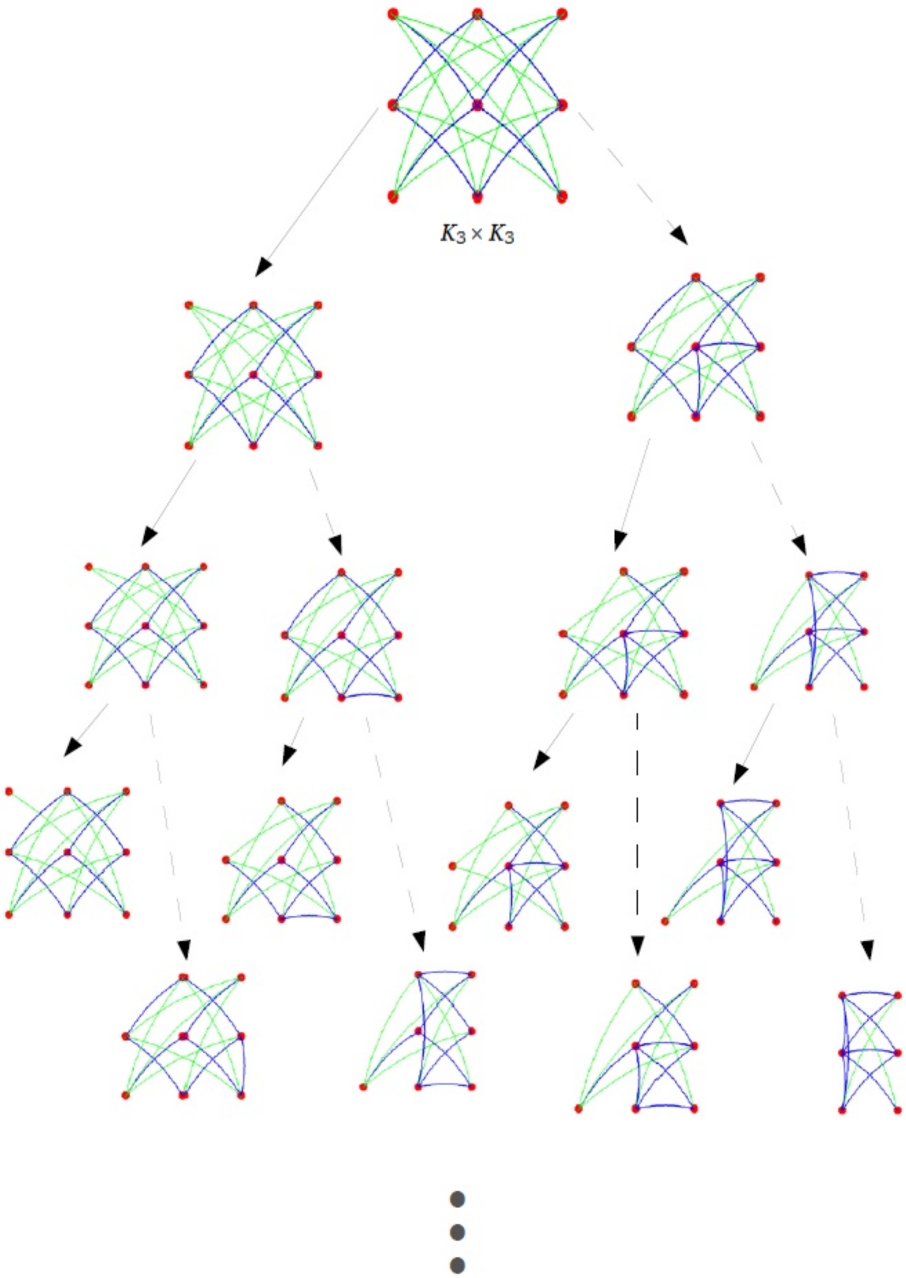


FIGURE 30

As the total number of graphs is too huge (about 40000) to be drawn by hands, the result is obtained by computer programming. The process in Fig. 3.4.1 is presented by a table:

Level	Sign	No. of graphs	Min. no. of vertices	Max. no. of vertices	Min. no. of edges	Max. no. of edges
1	1	1	9	9	18	18
2	-1	1	8	8	16	16
2	1	1	9	9	17	17
3	-1	2	8	8	15	15
3	1	2	7	9	14	16
4	-1	4	6	8	11	15
4	1	4	7	9	13	15

Level refers to the number of deletion and contraction used, where $K_3 \times K_3$ is in Level 1. FIGURE 30 has gone through 3 times of deletion-contraction, i.e. the graph is split to $23 = 8$ daughter graphs, and thus it is in Level 4.
Graphs of deletion have $+1$ sign while graphs of contraction have 1 sign.

The following is the description of the program:
First, populate vertices and edges to construct the $K_3 \times K_3$ graph (see FIGURE 31). The first level is thus formed:

Level	Sign	No. of graphs	Min. no. of vertices	Max. no. of vertices	Min. no. of edges	Max. no. of edges
1	1	1	9	9	18	18

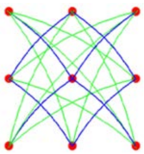


FIGURE 31

The action of contracting an edge (merging two vertices) is done by

1. removing one vertex, e.g. i, and the first edge, e.g. ij, then

2. replacing one vertex by another, e.g. replacing i by j, for all edges connected to i. In FIGURE 32, i is replaced by j, so edges ik, im, in are deleted, and edges jk, jm, jn are constructed (see FIGURE 32).

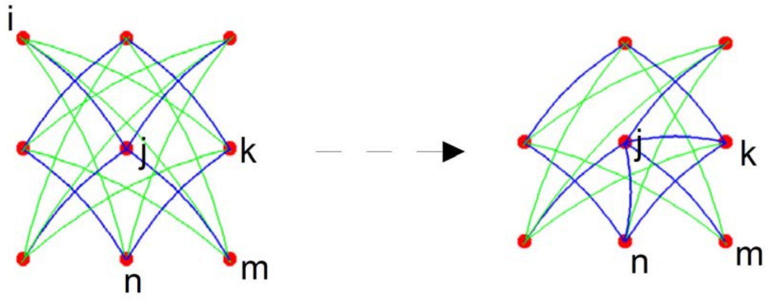


FIGURE 32

The sign of the contracted graph is the opposite of the graph of the previous level.

Level	Sign	No. of graphs	Min. no. of vertices	Max. no. of vertices	Min. no. of edges	Max. no. of edges
1	1	1	9	9	18	18
2	-1	1	8	8	16	16
2	1	1	9	9	17	17

The action of deleting an edge is done by removing the first edge from the graph of the previous level (see FIGURE 33).

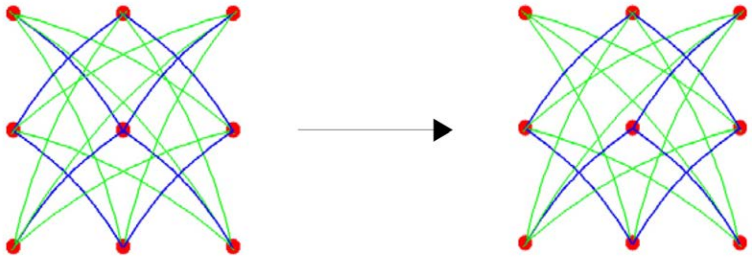


FIGURE 33

The sign of the deleted graph is the same as the graph of the previous level.

Level	Sign	No. of graphs	Min. no. of vertices	Max. no. of vertices	Min. no. of edges	Max. no. of edges
1	1	1	9	9	18	18
2	-1	1	8	8	16	16

The program repeats the two actions of contracting and deleting until only $K_2 \times K_2$ and null graphs are left:

Level	Sign	No. of graphs	Min. no. of vertices	Max. no of vertices	Min. no. of edges	Max. no. of edges
1	1	1	9	9	18	18
2	-1	1	8	8	16	16
2	1	1	9	9	17	17
3	-1	2	8	8	15	15
3	1	2	7	9	14	16
4	-1	4	6	8	11	15
4	1	4	7	9	13	15
5	-1	8	6	8	10	14
5	1	8	5	9	8	14
6	-1	16	4	8	5	13
6	1	16	5	9	7	13
7	-1	32	4	8	4	12
7	1	32	3	9	3	12
8	-1	64	2	8	1	11
8	1	64	3	9	2	11
9	-1	128	2	8	0	10
9	1	128	1	9	0	10
10	-1	254	2	8	0	9
10	1	254	1	9	0	9
11	-1	488	2	8	0	8
11	1	488	1	9	0	8
12	-1	894	2	8	0	7

12	1	894	1	9	0	7
13	-1	1516	2	8	0	7
13	1	1516	1	9	0	6
14	-1	2324	2	8	0	5
14	1	2324	1	9	0	5
15	-1	3148	2	8	0	4
15	1	3148	1	9	0	4
16	-1	3606	2	8	0	3
16	1	3606	1	9	0	3
17	-1	2486	2	8	0	2
17	1	3486	1	9	0	2
18	-1	2455	2	8	0	1
18	1	2455	1	9	0	1
19	-1	734	2	8	0	0
19	1	734	1	9	0	0

The $K_2 \times K_2$ graph appears in Level 17, being a graph of contraction.
The total number of null graphs is then counted (the first column refers to N_1, N_2, $\cdots$, N_9):

Null Graph	Sign	Count
1	1	1336
2	-1	4307
3	1	5876
4	-1	4544
5	1	2220
6	-1	711
7	1	147
8	-1	18
9	1	1

The chromatic polynomial of $K_3 \times K_3$ is therefore

$$\begin{aligned}&(k^9 - 18k^8 + 147k^7 - 711k^6 + 2220k^5 4544k^4 + 5876k^3 - 4307k^2 + 1336k)\\&\quad - (k^4 - 2k^3 + k^2)\\&= k^9 - 18k^8 + 147k^7 - 711k^6 + 2220k^5 - 4545k^4 + 5878k^3 - 4308k^2 + 1336k.\end{aligned}$$

Thus, it is believed that $\mathrm{chr}(K_n \times K_n)$ can be deduced from $\mathrm{chr}(K_{n-1} \times K_{n-1})$ for any integers $n \geq 1$ theoretically. This finding greatly reduces the work of deletion-contraction method.

3.5. Chromatic Polynomials and Chromatic Numbers of Product Graph

Lemma 14. *Let H, which is isomorphic to a complete graph K_n, be a subgraph of G where n is a positive integer. Then the chromatic polynomial of G is divisible by the chromatic polynomial of H, i.e.* $\mathrm{chr}(H,k)|\mathrm{chr}(G,k)$.

Proof. Let j be any integer such that $0 \leq j < n$. As each vertex in K_n is adjacent to $n1$ vertices, the chromatic number of H is n and thus the chromatic number of G is not less than n. Therefore, when given only j colours, NO proper colouring on G can be achieved. Hence, when $k = j$, $\mathrm{chr}(G,k) = 0$ (as chromatic polynomial refers to the number of possible different proper colourings on a graph). It shows that j is a root of $\mathrm{chr}(G,k) = 0$ and thus $(k-j)$ is a factor of $\mathrm{chr}(G,k)$. Thus, k, $(k-1)$, $(k-2)$, $\cdots$, $(k-n+1)$ are factors of $\mathrm{chr}(G,k)$. As $\mathrm{chr}(K_n,k) = k(k-1)(k-2)\cdots(k-n+1)$, $\mathrm{chr}(H,k)|chr(G,k)$. □

Theorem 15. *Let n be a positive integer, we have the following results:*

(1) The chromatic polynomial of $K_n \times K_n$ is divisible by the chromatic polynomial of K_n, i.e. $\mathrm{chr}(K_n,k)|\mathrm{chr}(K_N \times K_n,k)$. *We define the quotient polynomial to be $f_n(k)$ such that* $\mathrm{chr}(K_n \times K_n,k) = \mathrm{chr}(K_n,k)f_n(k)$.
(2) The chromatic polynomial of $K_n + K_n$ is divisible by the chromatic polynomial of K_n, i.e. $\mathrm{chr}(K_n,k)|\mathrm{chr}(K_n + K_n,k)$. *We define the quotient polynomial to be $g_n(k)$ such that* $\mathrm{chr}(K_n + K_n,k) = \mathrm{chr}(K_n,k)g_n(k)$.

Proof. Let $G_n \simeq K_n$ and $V(G_1) = (x_1, \cdots, x_n)$, $G_2 \simeq K_n$ and $V(G_2) = (y_1, \cdots, y_n)$.

(1) With basis $B = \{(1,1)\}$, vertices (x_1,y_2), (x_2,y_3), $\cdots$, (x_n,y_1) are adjacent to one another. It is observed that these vertices forms K_n, proving that K_n is a subgraph of $K_n \times K_n$. Using Lemma 14, $\mathrm{chr}(K_n,k)|chr(K_n \times K_n,k)$ is proved.
(2) With basis $B = \{(0,1),(1,0)\}$, vertices (x_1,y_i), (x_2,y_i), $\cdots$, (x_n,y_i) are adjacent to one another. It is observed that these vertices forms K_n, proving that K_n is a subgraph of $K_n + K_n$. Using Lemma 14, $\mathrm{chr}(Kn,k)|\mathrm{chr}(K_n + K_n,k)$ is proved.

□

Definition 16. *A* Latin square *is an $n \times n$ matrix filled with n different numbers, each occurring exactly once in each row and once in each column, e.g. the Latin square constructed with 1, 2, 3, 4 is*

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

The general form of Latin square with $1,\ 2,\ \cdots,\ n$ *can be represented by:*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Note that $a_{11} = a_{22} = a_{33} = \cdots = a_{nn} = 1$; $a_{12} = a_{23} = a_{34} = \cdots = a_{(n-1)n} = a_{n1} = 2$; $a_{13} = a_{24} = a_{35} = \cdots = a_{(n-1)n} = a_{n2} = 3; \cdots ; a_{1n} = a_{21} = a_{32} = \cdots = a_{(n-1)(n-2)} = a_{n(n-1)} = n$ *is one of the valid Latin square:*

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n & 1 & 2 & \cdots & n-1 \\ n-1 & n & 1 & \cdots & n-2 \\ \vdots & & & & \vdots \\ 2 & 3 & 4 & \cdots & 1 \end{bmatrix}$$

Theorem 17. *The chromatic number of $K_n + K_n$ and $K_n \times K_n$ are both n where n is a positive integer, that is*

(1) $\chi(K_n + K_n) = n$
(2) $\chi(K_n \times K_n) = n$.

Proof. Let $G_1 \simeq K_n$ and $V(G_1) = (1, \cdots, n)$, $G_2 \simeq K_n$ and $V(G_2) = (n + 1, \cdots, 2n)$.

(1) Construct an $n \times n$ Latin square with positive integers $n+1$, $n+2$, $\cdots$, $2n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} n+1 & n+2 & n+3 & \cdots & 2n \\ 2n & n+1 & n+2 & \cdots & 2n-1 \\ 2n-1 & 2n & n+1 & \cdots & 2n-2 \\ \vdots & & & & \vdots \\ n+2 & n+3 & n+4 & \cdots & n+1 \end{bmatrix}$$

It is noted all entries represents different integers in each row and each column. Under basis $B = \{(1,0),(0,1)\}$, vertex $(1,a_{11})$ is NOT adjacent to $(2,a_{12})$, $(3,a_{13})$, $\cdots$, (n,a_{1n}).
Similarly, for $i = 1,2,\cdots,n$, vertices In general, vertices $(1,a_{i1})$ are not adjacent to $(2,a_{i2})$, $(3,a_{i3})$, $\cdots$, (n,a_{in}).
In general, The 1^{st} group of vertices $(1,a_{11}),(2,a_{12}),\cdots,(n,a_{1n})$ are shaded with colour C_1;
The 2^{nd} group of vertices $(1,a_{21}),(2,a_{22}),\cdots,(n,a_{2n})$ are shaded with colour C_2;

$$\vdots$$

The n^{th} group of vertices $(1,a_{n1}),(2,a_{n2}),\cdots,(n,a_{nn})$ are shaded with colour C_n.
Under this colouring scheme, all vertices are shaded with n colours $C_1,C_2,\cdots$, C_n.

For vertices that are adjacent under basis $\{(0,1)\}$, e.g. $(i,a_{1i}),(i,a_{2i}),\cdots$, (i,a_{ni}), where $i = 1,2,\cdots,n$, as (i,a_{1i}) is shaded with colour C_1, (i,a_{2i}) is shaded with colour $C_2,\cdots,(i,a_{ni})$ is shaded with colour C_n, the scheme ensures that these vertices must have different colours.
For vertices that are adjacent under basis $\{(1,0)\}$, e.g. $(1,a_{11}),(2,a_{22}),\cdots$, (n,a_{nn}) (Note that $a_{11} = a_{22} = a_{nn} = n+1$), as $(1,a_{11})$ is shaded with colour C_1, $(2,a_{22})$ is shaded with colour $C_2,\cdots,(n,a_{nn})$ is shaded with colour C_n, the scheme ensures that these vertices must have different colours.
Thus, if given n colours to colour the $K_n + K_n$ graph, a proper colouring can be achieved, proving $\chi(K_n + K_n) \leq n$.

Moreover, under the basis $\{(0,1)\}$, the vertices $(1,a_{11}),(1,a_{21}),\cdots,(1,a_{n1})$ are adjacent to one another, forming a K_n graph. As $\chi(K_n) = n$, $K_n + K_n$ cannot be coloured by less than n colours, this proves that $\chi(K_n + K_n) \geq n$. As $\chi(K_n + K_n) \leq n$ and $\chi(K_n + K_n) \geq n$, it is proved that $\chi(K_n + K_n) = n$.

For example, when $n = 4$, the Latin square is

$$\begin{bmatrix} 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 7 \\ 7 & 8 & 5 & 6 \\ 6 & 7 & 8 & 5 \end{bmatrix}$$

The 1^{st} group of vertices $(1,5),(2,6),(3,7),(4,8)$ are shaded with colour C_1;
The 2^{nd} group of vertices $(1,8),(2,5),(3,6),(4,7)$ are shaded with colour C_2;
The 3^{rd} group of vertices $(1,7),(2,8),(3,5),(4,6)$ are shaded with colour C_3;
The 4^{th} group of vertices $(1,6),(2,7),(3,8),(4,5)$ are shaded with colour C_4.
Thus, all vertices are shaded with 4 colours C_1,C_2,C_3,C_4.

For vertices that are adjacent under basis $\{(0,1)\}$, e.g. $(1,5)$, $(1,8)$, $(1,7)$, $(1,6)$, as $(1,5)$ is shaded with colour C_1, $(1,8)$ is shaded with colour C_2, $(1,7)$ is shaded with colour C_3, $(1,6)$ is shaded with colour C_4, the scheme ensures that these vertices must have different colours.
For vertices that are adjacent under basis $\{(1,0)\}$, e.g. $(1,5),(2,5),(3,5)$, $(4,5)$, as $(1,5)$ is shaded with colour C_1, $(2,5)$ is shaded with colour C_2, $(3,5)$ is shaded with colour C_3, $(4,5)$ is shaded with colour C_4, the scheme ensures that these vertices must have different colours.
Thus, if given 4 colours to colour the K_4+K_4 graph, a proper colouring can be achieved, proving $\chi(K_4+K_4)\le 4$.
Moreover, under the basis $\{(0,1)\}$, the vertices $(1,5),(1,8),(1,7)$ and $(1,6)$ are adjacent to one another, forming a K_4 graph. As $\chi(K_4)=4$, K_4+K_4 cannot be coloured by less than 4 colours, this proves that $\chi(K_4+K_4)\ge 4$.
As $\chi(K_4+K_4)\le 4$ and $\chi(K_4+K_4)\ge 4$, it is proved that $\chi(K_4+K_4)=4$.

(2) Under basis $B=\{(1,1)\}$, vertex $(1,n+1)$ is NOT adjacent to $(1,n+2),(1,n+3),\cdots,(1,2n)$. Similarly, for $i=1,2,\cdots,n$, vertices $(i,n+1)$ are not adjacent to $(i,n+2),(i,n+3),\cdots,(i,2n)$.
In general,
The 1^{st} group of vertices $(1,n+1),(1,n+2),\cdots,(1,2n)$ are shaded with colour C_1;
The 2^{nd} group of vertices $(2,n+1),(2,n+2),\cdots,(2,2n)$ are shaded with colour C_2;

$$\vdots$$

The n^{th} group of vertices $(n,n+1),(n,n+2),\cdots,(n,2n)$ are shaded with colour C_n.
Thus, all vertices are shaded with n colours $C_1,C_2,\cdots,C_n$.

For vertices that are adjacent, e.g. $(1,n+i),(2,n+i),\cdots,(n,n+i)$, where $i=1,2,,n$, as $(1,n+i)$ is shaded with colour C_1, $(2,n+i)$ is shaded with colour C_2, $\cdots$, $(n,n+i)$ is shaded with colour C_n, the scheme ensures that these vertices must have different colours. Thus, if given n colours to colour the

$K_n \times K_n$ graph, a proper colouring can be achieved, proving $\chi(K_n \times K_n) \leq n$. [See reviewer's comment (7)]

Moreover, under the basis $\{(1,1)\}$, the vertices $(1, n+1), (2, n+2), \cdots, (n, n+i)$ are adjacent to one another, where $i = 1, 2, \cdots, n$, forming a K_n graph. As $\chi(K_n) = n$, $K_n \times K_n$ cannot be coloured by less than n colours, this proves that $\chi(K_n \times K_n) \geq n$.
As $\chi(K_n \times K_n) \leq n$ and $\chi(K_n \times K_n) \geq n$, it is proved that $\chi(K_n \times K_n) = n$.

For example, when $n = 4$,
The 1^{st} group of vertices $(1,5), (1,6), (1,7), (1,8)$ are shaded with colour C_1;
The 2^{nd} group of vertices $(2,5), (2,6), (2,7), (2,8)$ are shaded with colour C_2;
The 3^{rd} group of vertices $(3,5), (3,6), (3,7), (3,8)$ are shaded with colour C_3;
The 4^{th} group of vertices $(4,5), (4,6), (4,7), (4,8)$ are shaded with colour C_4.
Thus, all vertices are shaded with 4 colours C_1, C_2, C_3, C_4.

Consider any vertices that are adjacent, e.g. $(1,5), (2,6), (3,7)$ and $(4,8)$, as $(1,5)$ is shaded with colour C_1, $(2,6)$ is shaded with colour C_2, $(3,7)$ is shaded with colour C_3, $(4,8)$ is shaded with colour C_4, the scheme ensures that these vertices must have different colours. Thus, if given 4 colours to colour the $K_4 \times K_4$ graph, a proper colouring can be achieved, proving $\chi(K_4 \times K_4) \leq 4$. Moreover, under the basis $\{(1,1)\}$, the vertices $(1,5), (2,6), (3,7), (4,8)$ are adjacent to one another, forming a K_4 graph. As $\chi(K_4) = 4$, $K_4 \times K_4$ cannot be coloured by less than 4 colours, proving that $\chi(K_4 \times K_4) = 4$.

□

4. Conclusion

It is amusing that the relationship between Sudoku and graph theory helps to find the number of possible Sudoku puzzles. Although the chromatic polynomial of Sud(n) cannot be calculated due to the lack of a high power computer, the finding of determining chromatic polynomials from product graph of lower order to product graph of higher order is of great significance in calculating chr(Sud(n), k), as it reduces the scale of deletion-contraction method. Furthermore, the proof of $\chi(K_n \times K_n) = n$ gives us a brief picture of how the chromatic number of product graphs can be determined, thus helping to find the chromatic number of larger graphs, e.g. $K_n \times K_n \times K_n$.

For further research, the properties of $K_n \times K_n \times K_n \times K_n$ can be explored, as the NEPS of Sudoku is in fact $K_n \times K_n \times K_n \times K_n$ under a particular basis. With advanced technology in the future, the chromatic polynomial of$K_n \times K_n \times K_n \times K_n$ can be found with extensive power.

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[5] T. Sander, *Sudoku graphs are integral*, Electron. J. Combin., **16** (2009), N25. MR2529816

Reviewer's Comments

Sudoku has been a popular game in recent years and become a fixture in the puzzle section of many newspapers. The paper under review is motivated by the following question: how many solved Sudoku puzzles are there in total? This was already answered by Felgenhauer and Jarvis using computer programming. Nevertheless, in this paper the authors discuss a graph theoretic approach to this problem using such concepts as Sudoku graph and chromatic polynomials. They first review elements of graph theory and in particular an operation called non-complete extended P-sum (NEPS). Then they describe the construction of the Sudoku graph by applying suitable NEPSs to copies of complete graphs. Sudoku graph is so named because the ways of coloring its vertices suitably (called proper colorings) are in bijective correspondence with all the solved Sudoku puzzles. After showing this link, the authors discuss chromatic polynomials, which give the number of proper colorings of graphs in general, and an inductive algorithm of computing them using contraction and deletion of vertices. Though the authors do not make much headway in computing the chromatic polynomial of Sudoku graph due to overwhelming computational complexity and lack of computer power, they do manage to obtain some partial results which serve as stepping stones to a complete solution, e.g. an inductive algorithm of computing the chromatic polynomial of $K_n \times K_n$ and the least number of colors needed to properly color $K_n + K_n$ and $K_n \times K_n$.

In general the paper is very well-written and a pleasant reading for me. The authors have done an excellent job in the exposition of various constructions and algorithms with sufficient illustrations and examples (e.g. the proof of Lemma 8, the explanation on pp.331-333, 335-342 and the proof of Theorem 17). They also put in some effort in making the paper reader-friendly. For example, they highlight the terminology they define in bold for the convenience of the (cursory) readers. The following are some minor problems I found in the paper as well as some suggestions for improvement.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. It is better to say 'we were surprised to find that...'.
3. It is better to replace this part with a separate mathematical definition of Sudoku graph $\mathrm{Sud}(n)$.
4. It is better to replace 'are in fact' by 'in fact describe'. The sets $S_1, \cdots, S_3$ are not adjacencies of the graph, but describe the adjacencies.
5. It is better to mention that the polynomial is $k(k-1)(k-2)(k-3)$ because this factorized form appears again later, e.g. on p.330.
6. It is better to say '...does not affect the properness of the colouring of all the vertices'. The point is that adding an edge still gives a proper colouring. In the last sentence, again, it is better to say 'does not change the properness of the colouring'.

7. In fact $(1, n+i), (2, n+i), \cdots, (n, n+i)$ are NOT adjacent. The authors should rewrite the whole paragraph and say something along the line: If n vertices in this graph are adjacent, then they are from different rows and columns of the array

$$\begin{bmatrix} (1, n+1) & (1, n+2) & \cdots & (1, 2n) \\ (2, n+1) & (2, n+2) & \cdots & (2, 2n) \\ \vdots & \vdots & \ddots & \vdots \\ (n, n+1) & (n, n+2) & \cdots & (n, 2n) \end{bmatrix}.$$

According to the above colouring scheme, these vertices must have different colours.