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ON THE IMAGES OF RATIONAL NUMBERS UNDER A CERTAIN TYPE OF FRACTAL INTERPOLATION FUNCTION

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Abstract. Fractal interpolation functions were first introduced by Barnsley [2]. In this paper, we investigate the fractal interpolation functions joining the points $(0,0),(x_1,y_1),(1,1)$ for rational x_1,y_1 in the closed unit interval, and show that, when $x_1 = 1/2$, the function will always map rationals to rationals, but will not map any rational to $1/2$ when y_1 has odd denominator. Moreover, we derive by similar considerations a result about a dynamical system on the closed unit interval.

Contents

1. Introduction

This paper aims to answer the following question:
Let f be the fractal interpolation function (introduced by Barnsley [2], and defined later in this paper) through $(0,0),(x_1,y_1),(1,1)$ for some rational numbers $x_1,y_1 \in [0,1]$, using only affine functions for L_j and F_j.

a Does f necessarily map rational numbers to rational numbers?
b If so, is every rational number in $[0,1]$ equal to f of some rational number?

We answer the first part with Corollary 3 in Section 4 (only if $a/b = 1/2$. If not, then no, it doesn't have to; if b is odd, and if $c/d = 1/2$, then $1/2$ does not get mapped to a rational number) and the second part with Corollary 2 in Section 3 (no, it isn't; $1/2$ is a rational number in $[0,1]$ which is a counterexample, if d is odd).
The other theorems are results of intrinsic interest uncovered while trying to answer this question.
In Section 2, we define fractal interpolation functions, and the special case we consider in this paper. We also outline some conventions in use in this paper.
In Section 3, we consider the special case where $a/b = 1/2$, and show that in this case, rationals are mapped to rationals, among other results. In section 4, we generalize and consider other values of a/b. Finally, after the conclusion, we acknowledge guidance received during our research. The appendix contains programs we used to calculate values of $f_{a/b,\,c/d}$, and plot the graph of $f_{1/2,\,1/3}$.

2. Preliminaries of Fractal Interpolation Functions

First, we will define fractal interpolation functions (or FIFs) in their most general form, and then restrict to the special case we consider in this paper. The following definition is quite long, but we will use some notation from it.
Let $N \in \mathbb{Z}^+$ be a positive integer, and let $x_0 < \cdots < x_N \in \mathbb{R}$ be points on the real line. For each $j \in \{0,\ldots,N-1\}$, let $L_j : \mathbb{R} \to \mathbb{R}$ be the affine map satisfying $L_j(x_0) = x_j, L_j(x_N) = x_{j+1}$. Let $y_0,\ldots,y_N \in \mathbb{R}$. Let $X = [x_0, x_n] \subset \mathbb{R}$.
For each $j \in \{0,\ldots,N-1\}$, let $\alpha_j \in [0,1)$, and let $F_j : X \times \mathbb{R} \to \mathbb{R}$ be functions satisfying the following conditions:

a For all $x \in X$, the function $F_j(x,-) : \mathbb{R} \to \mathbb{R}$ is a α_j-Lipschitz mapping on $\mathbb{R}$ (that is, it is a contraction). Explicitly, for all $x \in X, y_1, y_2 \in \mathbb{R}$, we have that $|F_j(x,y_1) - F_j(x,y_2)| \le \alpha_j |y_1 - y_2|$.
b $F_j(x_0,y_0) = y_j$, and $F_j(x_N,y_N) = y_{j+1}$.

Then we define the **fractal interpolation function (abbreviated FIF) associated with** $\{(L_j(x), F_j(x,y))\}_{j=0}^{N-1}$ to be the unique bounded function $f : X \to \mathbb{R}$ satisfying

$$\text{(A)} \qquad \forall x \in X, \forall j \in \{0,\ldots,N-1\}, f(L_j(x)) = F_j(x, f(x)).$$

It is known that there is only one bounded function satisfying the above, and it is continuous.[1]
Next, we will specialize to FIFs where $N = 2$, $(x_0,y_0) = (0,0)$, $(x_2,y_2) = (1,1)$, $(x_1,y_1) = (a/b, c/d) \in (0,1)$, where a,b and c,d are coprime pairs of integers with b,d nonzero, so $X = [0,1]$. (It is of course possible for x_1 and y_1 to be irrational,

but in this paper, we focus on the rational case.) Next, we choose α_1, α_2, F_0, and F_1 as follows:

$$\alpha_1 = \frac{c}{d},\ \alpha_2 = \frac{d-c}{d},\ F_0(x,y) = \frac{c}{d}\cdot y,\ F_1(x,y) = \frac{d-c}{d}\cdot y + \frac{c}{d}.$$

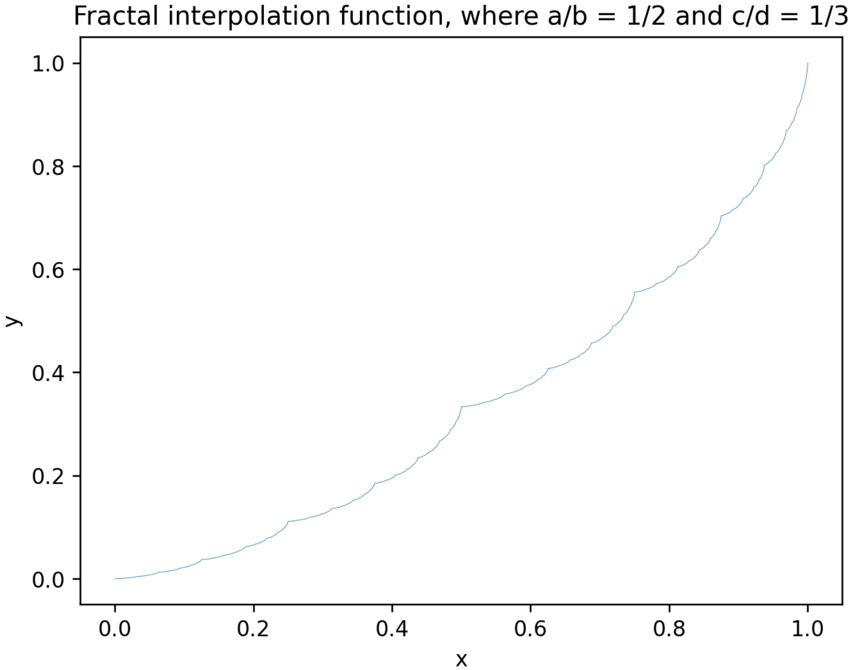
Since F_0 and F_1 do not depend on x, we will define $G_0(y) = F_0(x,y)$ and $G_1(y) = F_1(x,y)$. Note that we have

$$L_0^{-1}(x) = \frac{b}{a}x, \text{ and } L_1^{-1}(x) = \frac{b}{b-a}x - \frac{a}{b-a},$$

which are the inverses of L_0 and L_1 respectively, as both are bijections.
So, after making these specializations, we find that the associated FIF, which for now we will write as $f_{a/b,\,c/d} : [0,1] \to \mathbb{R}$, satisfies the following functional equation:

$$\text{(1)} \quad \forall x \in [0,1], \quad f(x) = \begin{cases} \dfrac{c}{d} f\left(\dfrac{b}{a}x\right), & \text{for } 0 \le x \le \dfrac{a}{b} \\ \dfrac{d-c}{d} f\left(\dfrac{b}{b-a}x - \dfrac{a}{b-a}\right) + \dfrac{c}{d}, & \text{for } \dfrac{a}{b} \le x \le 1 \end{cases}$$

Note that the piecewise definition comes up in the above equation, because the images of L_0^{-1} and $L_1^{-1}(x)$ are $[0, a/b]$ and $[a/b, 1]$ respectively, and these are almost disjoint but for a/b.
See below for the graph of f when $a/b = 1/2$ and $c/d = 1/3$.



This special case is briefly discussed in section 5 of Barnsley and Harrington's paper [1], and some of them are the casino functions of Dubins and Savage [4]. (All of these are also examples of de Rham curves [3].) The resulting functions are continuous, and to each pair $(a/b, c/d)$ corresponds a unique such bounded function f. We will show that they are strictly increasing in Proposition 1 in Section 4, which does not depend on any results in Section 3.

A **dyadic rational** is defined as a rational number which gives an integer when multiplied by some nonnegative integer power of 2.

In this paper, we will frequently refer to "**rational numbers with odd denominators**". This is a bit of an abuse of terminology, since every rational number can have an even denominator, if it is not in lowest terms. So it should be understood that we mean "rational number with odd denominator when written in lowest terms" when we write "rational number with odd denominator". Also, we will use the notation $\mathbb{Z}_{(2)}$ for the set of such numbers. This is closed under addition and multiplication, which may be verified by hand.

For any French people reading this paper, $\mathbb{Z}^+$ denotes the set of all (strictly) positive integers, so it doesn't include 0. Instead, $\mathbb{Z}_{\geq 0}$ will be used to denote the set of nonnegative integers.

3. The case $a/b = 1/2$

First, consider the special case where $a/b = 1/2$. In this case, if $f_{1/2,\,c/d} : [0,1] \to \mathbb{R}$ is the associated FIF, then by the definition of an FIF, $f_{1/2,\,c/d}$ (which we will refer to as f for the rest of this section, leaving c/d implicit, since it doesn't matter very much) satisfies the following equation:

$$\text{(B)} \qquad \forall x \in [0,1], \quad f(x) = \begin{cases} \dfrac{c}{d} f(2x), & \text{for } 0 \leq x \leq \dfrac{1}{2} \\[2ex] \dfrac{d-c}{d} f(2x-1) + \dfrac{c}{d}, & \text{for } \dfrac{1}{2} \leq x \leq 1 \end{cases}$$

To show that rationals are mapped to rationals in this case, we can use the fact that, under the dyadic transformation (or doubling map, or bit shift map) from $[0,1)$ to $[0,1)$, all rational numbers are eventually periodic, with period equal to the period of the binary expansion of the rational number.

Before stating and proving the theorem, we'll note some key properties of the choice of a/b. Since $a/b = 1/2$, the affine maps L_j for $j \in \{0,1\}$ are simply expressed as $L_0(x) = 2x$ and $L_1(x) = 2x - 1$. If x is less than $1/2$, then L_0 shifts the binary expansion of x to the left, deleting the first digit, and otherwise, L_1 does this. Similarly, for all $x \in [0,1]$,

$$L_0^{-1}(x) = \frac{1}{2}x,$$

and

$$L_1^{-1}(x) = \frac{1}{2}x + \frac{1}{2},$$

where L_0^{-1} concatenates a zero on the left, and L_1^{-1} adjoins a one on the left, both shifting the binary expansion to the right. These properties enable the proof of the following theorem.

Theorem 1. *The image of any rational number in* $[0, 1]$ *under* f *is a rational number.*

Proof. First, let's deal with some special cases. By assumption, $f(0) = 0$, and $f(1) = 1$, so the theorem holds for 0 and 1.
Next, suppose that $p/q \in (0, 1)$ has an odd denominator. Then $\frac{p}{q}$ has exactly one binary expansion, which is repeating and non-terminating. So let $a_1, a_2, a_3, \dots \in \{0, 1\}$ be the digits of the binary expansion of $\frac{p}{q}$. Let $k \in \mathbb{Z}^+$ be the least positive integer such that for all $n \in \mathbb{Z}^+$, $a_n = a_{n \pmod k}$; that is, the binary expansion repeats with period k.
As noted above, L_0^{-1} and L_1^{-1} adjoin zeroes and ones respectively to the left of the binary expansion of x. Therefore, by the definition of $a_1, \dots a_k$,

$$\frac{p}{q} = L_{a_1}^{-1} L_{a_2}^{-1} \dots L_{a_k}^{-1} \left(\frac{p}{q}\right).$$

That is,

$$L_{a_k} \dots L_{a_2} L_{a_1} \left(\frac{p}{q}\right) = \frac{p}{q}.$$

Applying f to both sides, we have by equation (A) and the definitions of G_0, G_1,

$$f\left(\frac{p}{q}\right) = f\left(L_{a_k} \dots L_{a_2} L_{a_1} \left(\frac{p}{q}\right)\right) = G_{a_k} \dots G_{a_2} G_{a_1} f\left(\frac{p}{q}\right).$$

Since both G_0 and G_1 are affine functions with rational coefficients, their composition $G_{a_k} \dots G_{a_2} G_{a_1}$ is too. Being a fixed point of an affine function with rational coefficients, $f(p/q)$ is therefore rational.
Next, consider the general case. In this case, the rational number can be expressed uniquely as some rational number r with an odd denominator divided by a nonnegative integer power of 2, so let $r \in \mathbb{Z}_{(2)}, n \in \mathbb{Z}^+$ be such that the rational number is equal to $2^{-n} r$.
Let's use strong induction on n. The base case $n = 0$, in which the rational number just has an odd denominator, is proved above.
In the induction case, let $m \in \mathbb{Z}^+$ be such that for all nonnegative integers $e \leq m$, for all $r \in \mathbb{Z}_{(2)}$ such that $2^{-e} r \in [0, 1]$, we have $f(2^{-e} r) \in \mathbb{Q}$.
Next, let $r \in \mathbb{Z}_{(2)}$ be such that $2^{-(m+1)} r \in [0, 1]$. We want to show that $f(2^{-(m+1)} r) \in \mathbb{Q}$. So split into cases depending on whether $2^{-(m+1)} r \leq \frac{1}{2}$.
If so, we have that $2^{-m} r \leq 1$, and $f(2^{-(m+1)} r) = c/d \cdot f(2^{-m} r)$ by equation (B). By our induction hypothesis, since $2^{-m} r \in [0, 1]$ and $2^m (2^{-m} r) \in \mathbb{Z}_{(2)}$, we know that $f(2^{-m} r)$ is rational, so $f(2^{-(m+1)} r)$ is also rational, and we are done in the first case where $f(2^{-(m+1)} r) \leq \frac{1}{2}$.

Otherwise, $2^{-(m+1)}r > \frac{1}{2}$, so we have that $2^{-m}r - 1 \in [0,1]$, and

$$f\left(\frac{r}{2^{m+1}}\right) = \frac{d-c}{d} f\left(\frac{r}{2^m} - 1\right) + \frac{c}{d},$$

which is in $\mathbb{Q}$ again by the induction hypothesis and because the sums and products of rationals are rational. $\square$

Program 4 follows the above proof to compute the value of the given FIF given a rational number in $[0,1]$ as input.
Over the next two results, we'll find a restriction on the denominator of the image of a rational number with odd denominator.

Lemma 1. *Let $k \in \mathbb{Z}_{\geq 0}$, and let $b_1, \ldots, b_k \in \{0,1\}$. Let $\varphi = |\{i \in \{1,\ldots,k\} : b_i = 0\}|$ be the number of zeros among $b_1, \ldots, b_k$. Then the composition $T := G_{b_k} \circ \cdots \circ G_{b_1}$ can be expressed as*

$$T(x) = \frac{c^{\varphi}(d-c)^{k-\varphi}x + B}{d^k}$$

for some integer B.

Proof. Use induction on k. If $k = 0$, then T is the identity on $\mathbb{R}$, and putting $B = 0$, we're done.
For the induction case, let $k+1 \in \mathbb{Z}^+$. Let $b_1, \ldots, b_{k+1} \in \{0,1\}$, φ be the number of zeros among $b_1, \ldots, b_k$, and let $S = G_{b_k} \circ \cdots \circ G_{b_1}$ and $T = G_{b_{k+1}} \circ G_{b_k} \cdots \circ G_{b_1} = G_{b_{k+1}} \circ S$. By our induction hypothesis, S has an expression $S(x) = \dfrac{c^{\varphi}(d-c)^{k-\varphi}x + B}{d^k}$ for some integer B.
Split into cases depending on whether b_{k+1} is 0 or 1. If it is 0, then for $x \in \mathbb{R}$,

$$T(x) = (G_{b_{k+1}} \circ S)(x) = \frac{c}{d}\left(\frac{c^{\varphi}(d-c)^{k-\varphi}x + B}{d^k}\right) = \frac{c^{\varphi+1}(d-c)^{k-\varphi}x + cB}{d^{k+1}},$$

so we are done in this case since there are indeed $\varphi + 1$ zeros among $b_1, \ldots, b_{k+1}$, and since $cB \in \mathbb{Z}$ by our induction hypothesis.
If $b_{k+1} = 1$, then for $x \in \mathbb{R}$,

$$\begin{aligned} T(x) &= (G_{b_{k+1}} \circ S)(x) \\ &= \frac{d-c}{d}\left(\frac{c^{\varphi}(d-c)^{k-\varphi}x + B}{d^k}\right) + \frac{c}{d} \qquad (2) \\ &= \frac{c^{\varphi}(d-c)^{k+1-\varphi}x + (d-c)B + c \cdot d^k}{d^{k+1}}, \end{aligned}$$

so we are also done in this case since there are indeed φ zeros among $b_1, \ldots, b_{k+1}$, and since $(d-c)B + c \cdot d^k \in \mathbb{Z}^+$ by our induction hypothesis. $\square$

Corollary 1. *Let $p, q \in \mathbb{Z}$ be such that $\gcd(p,q) = 1$, q is odd, and $(p/q) \in [0,1] \cap \mathbb{Q}$. Let $k \in \mathbb{Z}^+$ be the order of 2 in the group of units of $\mathbb{Z}/q\mathbb{Z}$. Let φ be the number of zeros in the repeating part of the binary expansion of p/q. Then, $(d^k - c^{\varphi}(d-c)^{k-\varphi}) \times f(p/q) \in \mathbb{Z}$.*

Proof. Following the notation in Theorem 1, let $a_1, a_2, a_3, \dots \in \{0,1\}$ be the digits of the binary expansion of p/q. By a result of number theory, the binary expansion repeats with period k, so, for all $n \in \mathbb{Z}^+$, $a_n = a_{n \pmod k}$. Thus, φ is the number of zeros among $a_1, \dots, a_k$.
Then we have that

$$f\left(\frac{p}{q}\right) = f\left(L_{a_k} \dots L_{a_2} L_{a_1}\left(\frac{p}{q}\right)\right) = G_{a_k} \dots G_{a_2} G_{a_1} f\left(\frac{p}{q}\right).$$

By the above Lemma, if we let $T := G_{a_k} \dots G_{a_2} G_{a_1}$, then for some integer $B \in \mathbb{Z}^+$,

$$f\left(\frac{p}{q}\right) = \frac{c^{\varphi}(d-c)^{k-\varphi} f\left(\frac{p}{q}\right) + B}{d^k}.$$

Rearranging, this implies that $(d^k - c^{\varphi}(d-c)^{k-\varphi}) \cdot f\left(\frac{p}{q}\right) = B \in \mathbb{Z}$ as desired. □

Since d and $c^{\varphi}(d-c)^{k-\varphi}$ are coprime by assumption, this corollary immediately implies, for example, that a rational number with odd denominator in $(0,1)$ can never get mapped to any rational number with denominator d.
We can derive an even stronger result with the following propositions.

Lemma 2. *For any $k \in \mathbb{Z}^+$, $\varphi \in \{1, \dots, k-1\}$, and coprime integers $c, d \in \mathbb{Z}^+$, $d^k - c^{\varphi}(d-c)^{k-\varphi}$ is odd.*

Proof. Split into cases depending on the parity of c and d. Since c and d are both coprime, they cannot both be even.
If d is odd, then since $\varphi, k-\varphi \geq 1$ by assumption, we know that $c^{\varphi}(d-c)^{k-\varphi}$ has both c and $d-c$ as factors. Since d is odd, (exactly) one of c and $d-c$ is even, so $c^{\varphi}(d-c)^{k-\varphi}$ is even. Since d^k is odd, this implies $d^k - c^{\varphi}(d-c)^{k-\varphi}$ is odd, being the difference of an odd number and an even number, so we're done.
If d is even, then c is odd, so $d-c$ is odd. Then d^k is even and $c^{\varphi}(d-c)^{k-\varphi}$ is odd, so $d^k - c^{\varphi}(d-c)^{k-\varphi}$ is odd. □

The above two results almost imply that every rational number with an odd denominator is always mapped to a rational with odd denominator, since the case of $\varphi = 0$, for example, is not covered by them. In fact, if d is odd, we can improve this to the result below.

Theorem 2. *If d is odd, then every rational number in $[0,1]$ gets mapped by f to a rational number with an odd denominator.*

Proof. Let $r \in [0,1] \cap \mathbb{Q}$ be any rational number in $[0,1]$. Let $p, q \in \mathbb{Z}$ be such that $r = \dfrac{p}{q}$ and $\gcd(p,q) = 1$. Let $n \in \mathbb{Z}_{\geq 0}$ be the largest nonnegative integer $m \in \mathbb{Z}_{\geq 0}$ such that $2^m | q$. Use induction on n.
Consider the base case, where $n = 0$. So q is odd. Let $k \in \mathbb{Z}^+$ be the order of 2 in the group of units of $\mathbb{Z}/q\mathbb{Z}$, and let φ be the number of zeros in the repeating part of the binary expansion of p/q. Then, by Corollary 1, $(d^k - c^{\varphi}(d-c)^{k-\varphi}) \times f(p/q) \in \mathbb{Z}$.

To show that $f(p/q)$ has an odd denominator, consider φ. By construction, $0 \leq \varphi \leq k$. Either $\varphi = 0$ or $k - \varphi = 0$, or $\varphi \in \{1, \ldots, k-1\}$. In the latter case, $\{1, \ldots, k-1\}$ is nonempty, so by Lemma 2, $d^k - c^\varphi(d-c)^{k-\varphi}$ is odd and so $f(p/q) \in \mathbb{Z}_{(2)}$.
Otherwise, $\varphi = 0$ or $k - \varphi = 0$, so by the definition of φ, the binary expansion of p/q must be all zeros or all ones, meaning that $p/q = 0$ or $p/q = 1$. In both cases, $f(p/q)$ has denominator 1, so $f(p/q) \in \mathbb{Z}_{(2)}$.
For the induction case, let $n \in \mathbb{Z}_{\geq 0}$ be such that for all rational numbers p/q (where $\gcd(p,q) = 1$) in $[0,1]$ such that n is the largest nonnegative integer such that $2^n | q$, we have that $f(p/q) \in \mathbb{Z}_{(2)}$. We want to show that for all rational numbers p/q (where $\gcd(p,q) = 1$) in $[0,1]$ such that $n+1$ is the largest nonnegative integer $m \in \mathbb{Z}_{\geq 0}$ such that $2^m | q$, we have that $f(p/q) \in \mathbb{Z}_{(2)}$.
To do this, let $p/q \in [0,1]$ (with $\gcd(p,q) = 1$) be such that $n+1$ is the largest nonnegative integer $m \in \mathbb{Z}_{\geq 0}$ such that $2^m | q$. (So q is even.) Split into cases depending on whether $p/q \leq 1/2$. If so, then $2(p/q) = p/(2^{-1}q) \in [0,1]$, and n is the largest nonnegative integer $m \in \mathbb{Z}_{\geq 0}$ such that $2^m | (2^{-1}q)$. Therefore $f(p/(2^{-1}q)) \in \mathbb{Z}_{(2)}$ by induction, so

$$f\left(\frac{p}{q}\right) = \frac{c}{d} \cdot f\left(\frac{p}{\left(\frac{q}{2}\right)}\right) \in \mathbb{Z}_{(2)},$$

because $\dfrac{c}{d} \in \mathbb{Z}_{(2)}$, and $\mathbb{Z}_{(2)}$ is closed under multiplication, so we are done in this case.
Otherwise, if $p/q > 1/2$, then $2(p/q) - 1 = (p - 2^{-1}q)/(2^{-1}q) \in [0,1]$, and n is the largest nonnegative integer $m \in \mathbb{Z}_{\geq 0}$ such that $2^m | (2^{-1}q)$. Therefore $f((p - 2^{-1}q)/(2^{-1}q)) \in \mathbb{Z}_{(2)}$, so

$$f\left(\frac{p}{q}\right) = \frac{d-c}{d} \cdot f\left(\frac{p - \frac{q}{2}}{\left(\frac{q}{2}\right)}\right) + \frac{c}{d} \in \mathbb{Z}_{(2)}$$

since $\dfrac{c}{d}, \dfrac{d-c}{d} \in \mathbb{Z}_{(2)}$, and $\mathbb{Z}_{(2)}$ is closed under multiplication and addition, so we are done in this case as well.
In all cases considered above, $f(p/q) \in \mathbb{Z}_{(2)}$, so it is a rational number with an odd denominator. □

Corollary 2. *If d is odd, then no rational number ρ in $[0,1]$ satisfies $f(\rho) = 1/2$.*

Proof. No odd multiple of $1/2$ is ever an integer, so $1/2 \notin \mathbb{Z}_{(2)}$, and the result follows by Theorem 2. □

4. The case $a/b \neq 1/2$

At the start of Section 3, we noted that because of the special choice of a/b, we were able to use the properties of binary expansions to obtain results. Naturally, one might ask whether rationals still get mapped to rationals if a/b is not equal to $1/2$, and whether we can use some analogue of binary expansion to prove this. Surprisingly, this property is not guaranteed once a/b is not $1/2$! This follows from Theorem 2, but we need to prove some more results to get there. We may continue to omit the subscripts on f_{x_1,y_1} when there is no confusion about what they are. Note that, in this section, a/b may not be $1/2$.

For convenience, let $x_1 = a/b$ and $y_1 = c/d$. Then equation (1) in Section 2, slightly rearranged, states that $f_{x_1,y_1} : [0,1] \to \mathbb{R}$ is the unique bounded function (which is continuous [1]) satisfying

$$\forall x \in [0,1], f_{x_1,y_1}(L_0(x)) = G_0(f(x)) \text{ and } f_{x_1,y_1}(L_1(x)) = G_1(f(x)) \tag{3}$$

where $L_0, L_1, G_0, G_1 : \mathbb{R} \to \mathbb{R}$ are the affine functions which satisfy the following:

$$\begin{aligned} L_0(0) &= 0, L_0(1) = x_1, \\ L_1(0) &= x_1, L_1(1) = 1, \\ G_0(0) &= 0, G_0(1) = y_1, \\ G_1(0) &= y_1, G_1(1) = 1 \end{aligned} \tag{4}$$

In the following proposition, we will establish basic properties of the resulting function f_{x_1,y_1}.

Proposition 1. $f_{a/b,\,c/d} : [0,1] \to \mathbb{R}$ *has an image contained in* $[0,1]$*, and is strictly increasing, thus injective.*

Proof. This fact was mentioned briefly in Section 2, but we'll prove it again here. Restating (3), $f_{x_1,y_1} : [0,1] \to \mathbb{R}$ is the unique bounded function satisfying

$$\forall x \in [0,1], \quad f_{x_1,y_1}(x) = \begin{cases} \dfrac{c}{d} f_{x_1,y_1}(\dfrac{b}{a}x), & \text{for } 0 \le x \le \dfrac{a}{b} \\ \\ \dfrac{d-c}{d} f_{x_1,y_1}(\dfrac{b}{b-a}x - \dfrac{a}{b-a}) + \dfrac{c}{d}, & \text{for } \dfrac{a}{b} \le x \le 1, \end{cases}$$

and is continuous. Therefore, it suffices to first show that f_{x_1,y_1} is strictly increasing on a dense subset.

Let M be the submonoid of $\mathbb{R}^{\mathbb{R}}$ (with the monoid operation given by function composition) generated by $\{L_0, L_1\}$, and let $D := \{x \in [0,1] : \exists m \in M, \exists n \in \{0,1\}, m(n) = x\}$. Note that this implies $L_0(D) \subseteq D$, and $L_1(D) \subseteq D$ by definition. We want to show first that D is dense in $[0,1]$.

Subgoal: show that D is dense in $[0,1]$

Consider the distance function $\Delta : [0,1] \to [0,1]$ on $[0,1]$ defined by $\Delta(x) = \inf\limits_{d \in D}\{|d - x|\}$ for $x \in [0,1]$, which is the distance of x from D. By the definition of closure in metric spaces, to show that D is dense in $[0,1]$, it suffices to show that Δ is identically zero on $[0,1]$.

Since Δ is a point-to-set distance function, it is continuous by metric space theory, and therefore has a maximum point on $[0,1]$, since $[0,1]$ is compact. Let $(x, \Delta(x))$ be such a maximum point, and suppose for contradiction that $\Delta(x) > 0$. So $x \in [0, x_1] \cup [x_1, 1]$.

In the first case, where $x \in [0, x_1]$, we see that $x \in L_0([0,1])$. So let $p \in [0,1]$ be such that $L_0(p) = x$. Then note that

$$\begin{aligned} \Delta(x) &= \Delta(L_0(p)) \\ &= \inf_{d \in D}\{|d - L_0(p)|\} \end{aligned}$$

$$\begin{aligned}
&\leq \inf_{d \in L_0(D)} \{|d - L_0(p)|\} \\
&= \inf_{d \in D} \{|L_0(d) - L_0(p)|\} \\
&= \inf_{d \in D} \{x_1 |d - p|\} \\
&= x_1 \inf_{d \in D} \{|d - p|\} \\
&= x_1 \Delta(p) \\
&\leq x_1 \Delta(x)
\end{aligned}$$

where the equality between the fourth line and the fifth line is because L_0 is an affine function, which scales distances by x_1, and the last inequality is because $\Delta(x)$ is the maximum of Δ. Since $x_1 < 1$ by assumption, we must have that $\Delta(x) = 0$, a contradiction.

In the second case, where $x \in [x_1, 1]$, we see that $x \in L_1([0,1])$. So let $p \in [0,1]$ be such that $L_1(p) = x$. Then, similarly,

$$\begin{aligned}
\Delta(x) &= \Delta(L_1(p)) \\
&= \inf_{d \in D} \{|d - L_1(p)|\} \\
&\leq \inf_{d \in L_1(D)} \{|d - L_1(p)|\} \\
&= \inf_{d \in D} \{|L_1(d) - L_1(p)|\} \\
&= \inf_{d \in D} \{(1 - x_1)|d - p|\} \\
&= (1 - x_1) \inf_{d \in D} \{|d - p|\} \\
&= (1 - x_1)\Delta(p) \\
&\leq (1 - x_1)\Delta(x)
\end{aligned}$$

where the equality between the fourth line and the fifth line is because L_1 is an affine function, which scales distances by $1 - x_1$, and the last inequality is because $\Delta(x)$ is the maximum of Δ. Since $x_1 > 0$ by assumption, we must have that $\Delta(x) = 0$, a contradiction.
Therefore, we must have that $\Delta(x) = 0$, so D is indeed dense in $[0,1]$.
Subgoal: show that f is strictly increasing on D
Next, we want to show that f_{x_1,y_1} is strictly increasing on D. To do so, define $D_0 := \{0,1\}$ and, for each $n+1 \in \mathbb{Z}^+$, define $D_{n+1} := L_0(D_n) \cup L_1(D_n)$. By definition,

$$D = \bigcup_{k=0}^{\infty} D_k \subseteq [0,1].$$

To show that $D_0 \subseteq D_1 \subseteq D_2 \cdots$, let $n \in \mathbb{Z}_{\geq 0}$ be arbitrary. Then let $x \in D_n$. So for some $a_1, \ldots, a_n \in \{0,1\}$, and for some $\nu \in \{0,1\}$, $x = L_{a_n} \cdots L_{a_1}(\nu)$. Since $\nu = L_\nu(\nu)$ in both cases, we have that $\nu \in D_1$, so $x \in D_{n+1}$.
Therefore, it suffices to show that for all $n \in \mathbb{Z}_{\geq 0}$, f is strictly increasing on D_n. We will do this by induction.
For the base case $n = 0$, $f(0) = 0$ and $f(1) = 1$, so f is strictly increasing on D_0.

For the induction case, let $n \in \mathbb{Z}_{\geq 0}$. Suppose that f is strictly increasing on D_n. We want to show that f is strictly increasing on D_{n+1}.
Let $x \in D_{n+1} = L_0(D_n) \cup L_1(D_n)$. Either $x < x_1$, or $x = x_1$, or $x > x_1$. If $x < x_1$, then $x \notin L_1(D_n)$, so $x \in L_0(D_n)$. Let $p \in D_n$ be such that $L_0(p) = x$. Then since $p < 1$ (because if $p = 1$, then we'd have $x = x_1$, which is false), by our induction hypothesis we have $f(p) < f(1) = 1$, so

$$f(x) = f(L_0(p)) = G_0(f(p)) < G_0(1) = y_1$$

Similarly, if $x > x_1$, then $x \notin L_0(D_n)$, so $x \in L_1(D_n)$. Let $p \in D_n$ be such that $L_1(p) = x$. Then since $p > 0$ (because if $p = 0$, then we'd have $x = x_1$, which is false), by our induction hypothesis we have $f(p) > f(0) = 0$, so

$$f(x) = f(L_1(p)) = G_1(f(p)) > G_1(0) = x_1.$$

Clearly, if $x = x_1$, then $f(x) = y_1$, by the definition of f.
Therefore, to show that f is strictly increasing on D_{n+1}, it suffices to show that it is strictly increasing on both $L_0(D_n)$ and $L_1(D_n)$. But this is equivalent (by the definition of FIF) to showing that both $G_0 \circ f$ and $G_1 \circ f$ are strictly increasing on D_n, which follows from the induction hypothesis and the fact that the composition of strictly increasing functions is strictly increasing.
Finally, we want to show that if f is strictly increasing on every D_n, then it is strictly increasing on D. So let $x, y \in D$, with $x < y$. Then there exist $m, n \in \mathbb{Z}_{\geq 0}$ such that $x \in D_m$ and $y \in D_n$. Without loss of generality, we may assume that $m \leq n$, otherwise, swap x and y. Then $D_m \subseteq D_n$, as shown above, so $x, y \in D_n$. Since f is strictly increasing on D_n, we have that $f(x) < f(y)$, as required.
Subgoal: show that the above implies the result
Since f_{x_1,y_1} is strictly increasing on D, a dense subset of $[0,1]$, and continuous, it is strictly increasing on all of $[0,1]$. Therefore, since $f(0) = 0$ and $f(1) = 1$, $f([0,1]) = [0,1]$ by the Intermediate Value Theorem, as required, and it is injective, being strictly increasing. □

Note the change in the subscripts in the statement and proof of the following theorem.

Theorem 3. *Considered as functions from* $[0,1]$ *to* $[0,1]$*,* f_{x_1,y_1} *and* f_{y_1,x_1} *are inverses.*

Proof. By Proposition 1, these two functions are both bijective. To show that they are inverses, define $L_0, L_1, G_0, G_1 : \mathbb{R} \to \mathbb{R}$ to be the affine functions satisfying the following:

$$\begin{aligned} L_0(0) = 0, L_0(1) = x_1, \\ L_1(0) = x_1, L_1(1) = 1, \\ G_0(0) = 0, G_0(1) = y_1, \\ G_1(0) = y_1, G_1(1) = 1 \end{aligned} \tag{5}$$

Then $f_{x_1,y_1} : [0,1] \to [0,1]$ is the unique bounded function satisfying

$$\forall x \in [0,1], f_{x_1,y_1}(L_0(x)) = G_0(f_{x_1,y_1}(x)) \text{ and } f_{x_1,y_1}(L_1(x)) = G_1(f_{x_1,y_1}(x))$$

while $f_{y_1,x_1} : [0,1] \to [0,1]$ is the unique bounded function satisfying

$$\forall x \in [0,1], f_{y_1,x_1}(G_0(x)) = L_0(f_{y_1,x_1}(x)) \text{ and } f_{x_1,y_1}(G_1(x)) = L_1(f_{y_1,x_1}(x)).$$

Let $g : [0,1] \to [0,1]$ be the inverse of f_{x_1,y_1}, which necessarily exists, since f_{x_1,y_1} is bijective. Then, for all $x \in [0,1]$, it follows by applying g on the outside and the inside that

$$g(f_{x_1,y_1}(L_0(g(x)))) = g(G_0(f(g(x)))) \text{ and } g(f_{x_1,y_1}(L_1(g(x)))) = g(G_1(f(g(x))))$$

so, by the construction of g,

$$L_0(g(x)) = g(G_0(x)) \text{ and } L_1(g(x)) = g(G_1(x)),$$

which is exactly the characterization of f_{y_1,x_1}. By uniqueness, then, $g = f_{y_1,x_1}$. □

With that out of the way, we can finally tackle the problem of whether $f_{a/b,\,c/d}$ necessarily maps rationals to rationals!

Corollary 3. *If b is odd, then if $c/d = 1/2$, then $f_{a/b,\,c/d}(1/2) \notin \mathbb{Q}$.*

Proof. Suppose $\kappa := f_{a/b,\,1/2}(1/2)$. By Theorem 3, applying $f_{a/b,\,1/2}$ to both sides, we have that $f_{1/2,\,a/b}(\kappa) = 1/2$. Since b is odd, by Corollary 2, κ must be irrational. □

In particular, this implies that any approach that uses a variant of binary expansion in attempt to show f maps rationals to rationals (even when $a/b \neq 1/2$) will fail. More rigorously, this can be stated as follows. Let $T : [0,1] \to [0,1]$ be the transformation defined by

$$T(x) = \begin{cases} L_0^{-1}(x) & \text{for } x \in \left[0, \dfrac{a}{b}\right] \\ L_1^{-1}(x) & \text{for } x \in \left[\dfrac{a}{b}, 1\right] \end{cases}$$

where $L_0, L_1 : \mathbb{R} \to \mathbb{R}$ are the affine functions satisfying

$$L_0(0) = 0, L_0(1) = \frac{a}{b}, \qquad (6)$$
$$L_1(0) = \frac{a}{b}, L_1(1) = 1.$$

Our result in Section 3 that rationals always got mapped to rationals relied on the fact that, when $a/b = 1/2$, the rationals were all eventually periodic under the transformation T. Program 4 in Appendix A also relies on this to compute $f_{1/2,\,c/d}$ of any rational number exactly.
Therefore, since rationals don't always get mapped to rationals when b is odd, we can deduce that some rationals, for example $1/2$, aren't eventually periodic under the transformation T when b is odd, by following the method of Theorem 1.

Theorem 4. *If b is odd, then $1/2$ is not an eventually periodic point under the transformation T, defined above.*

Proof. Suppose $1/2$ were an eventually periodic point under the transformation T. Then, for some $k \in \mathbb{Z}^+$, $T^k(1/2)$ is a fixed point of some power of T. That is, for some $\lambda \in \mathbb{Z}^+$, and some $a_1, \ldots, a_\lambda \in \{0,1\}$, we have that $(L_{a_1} \circ \cdots \circ L_{a_\lambda})(T^k(1/2)) = T^k(1/2)$, by the definition of T.

Applying $f_{a/b,\,1/2}$ (which will be abbreviated to f for the remainder of this proof) to both sides, and using its characteristic property, we have that

$$f\left(T^k\left(\frac{1}{2}\right)\right) = f\left((L_{a_1}\circ\cdots\circ L_{a_\lambda})\left(T^k\left(\frac{1}{2}\right)\right)\right) = (G_{a_1}\circ\cdots\circ G_{a_\lambda})\left(f\left(T^k\left(\frac{1}{2}\right)\right)\right).$$

For clarity, in this proof, $L_0, L_1, G_0, G_1 : \mathbb{R}\to\mathbb{R}$ are the affine functions satisfying the following:

$$\begin{aligned} L_0(0) &= 0, L_0(1) = \frac{a}{b}, \\ L_1(0) &= \frac{a}{b}, L_1(1) = 1, \\ G_0(0) &= 0, G_0(1) = \frac{1}{2}, \\ G_1(0) &= \frac{1}{2}, G_1(1) = 1 \end{aligned} \tag{7}$$

Since both G_0 and G_1 are affine functions with rational coefficients, and $f(T^k(1/2))$ is the fixed point of a composition of G_0's and G_1's, we must have that $f(T^k(1/2)) \in \mathbb{Q}$.

Similarly, by the definition of T, for some $b_1, \ldots, b_k \in \{0,1\}$, we know $(L_{b_1}\circ\cdots\circ L_{b_k})(T^k(1/2)) = 1/2$. Thus,

$$f\left(\frac{1}{2}\right) = f\left((L_{b_1}\circ\cdots\circ L_{b_k})\left(T^k\left(\frac{1}{2}\right)\right)\right) = (G_{b_1}\circ\cdots G_{b_k})\left(f\left(T^k\left(\frac{1}{2}\right)\right)\right).$$

So $f(1/2)$ is equal to the image of a rational number (which is $f(T^k(1/2))$) under a composition of affine functions with rational coefficients (which are G_0 and G_1), so it must be rational. However, this contradicts Corollary 3. Therefore, we must conclude that $1/2$ cannot be an eventually periodic point under the transformation T. □

By Theorem 4, not every rational number is an eventually periodic point under the transformation T. So we have obtained a proof of a result about dynamical systems just by considering FIFs!

5. Conclusion, extensions, and conjectures

In conclusion, $f_{a/b,\,c/d} : [0,1]\to[0,1]$ doesn't necessarily map rationals to rationals, for example when b is odd, but it must when $a/b = 1/2$. Even when $a/b = 1/2$, it is not guaranteed that every rational in $[0,1]$ is the image of some rational in $[0,1]$, which is seen with $1/2$ when d is odd. Because of this, we were able to derive a result about the dynamical transformation T (defined on the previous page, near the end of Section 4).

We believe that Theorem 3 can be generalized to the statement that for $x_1, y_1, z_1 \in [0,1]$, $f_{y_1,z_1}\circ f_{x_1,y_1} = f_{x_1,z_1}$, along with the fact that for any $x_1\in[0,1]$, f_{x_1,x_1} is the identity function on $[0,1]$. Due to insufficient time, we do not prove this statement in this paper. Using this, for any $a/b, c/d \in [0,1]$, we would have that

$$f_{a/b,\,c/d} = f_{1/2,\,c/d}\circ f_{a/b,\,1/2} = f_{1/2,\,c/d}\circ f^{-1}_{1/2,\,a/b},$$

allowing us to approximate $f_{a/b,\,c/d}$ knowing only how to compute $f_{1/2,\,c/d}$.

Furthermore, Theorem 1 used the idea of the dyadic transformation (defined on $[0,1)$ as $x \mapsto 2x \pmod 1$) to show that any $f_{1/2,\,c/d}$ mapped rationals to rationals. It should be possible to extend this process to any base, not just binary (base 2).

Conjecture 1. *Let $n \in \mathbb{Z}^+$ be such that $n \geq 2$. Let $0 < y_1 < \cdots < y_{n-1} < 1$ be rational numbers in $(0,1)$. Then the fractal interpolation function through $(0,0), (1/n, y_1), \ldots, ((n-1)/n, y_{n-1}), (1,1)$ maps rationals to rationals.*

Finally, we only dealt with the cases where b was odd, or where d was odd. By Theorem 1, we can't straightforwardly extend Corollary 3 to all cases where b is even, but can we still show it for most of these cases? We may ask the same question for Theorem 2 and Corollary 2.
Moreover, we found when running Program 4 that the denominator (in lowest terms) of the input to $f_{1/2,\,1/3}$ seemed always to be less than the denominator of the output (for example, $f_{1/2,\,1/3}(1/2) = 1/3$ by definition, and $2 < 3$, but less trivially, $f_{1/2,\,1/3}(1/4) = 1/9$, and $4 < 9$. We conjecture that this is true in general.)

Conjecture 2. *For any rational $c/d \in (0,1)$, for any $p/q \in (0,1)$, where p and q are nonnegative coprime integers, the denominator of $f_{1/2,\,c/d}(p/q)$ is strictly larger than q.*

6. Acknowledgements

In alphabetical order by surname, we would like to thank our team supervisor and teacher, Dr. Cheung Wai Shan for keeping our project on time and helping manage our meetings and assignment of work, our school's alumnus Isaac Li for thoroughly checking the typesetting of our paper, and Nicky Wong for searching the literature related to de Rham curves.

Appendix A. Programs about $f_{1/2,\,c/d}$

In this appendix are some Python programs for computation with the above-mentioned FIFs.
First is Program 4, which computes the numerator and denominator of the output of the fractal interpolation function going through $(0,0), (1/2, c/d), (1,1)$ given a rational input, and also contains a function to approximate the inverse of the function.

```
from math import gcd

#parameters of FIF
c = 3
d = 4

def is_integer(n):
    if isinstance(n, float):
        return n.is_integer()
    elif isinstance(n, int):
        return True
    else:
```

```
        return False

def howEven(n):
    temp = n
    counter = -1
    while is_integer(temp):
        temp = temp/2
        counter = counter + 1
    return counter

def OddPartOf(n):
    temp = n
    counter = -1
    while is_integer(temp):
        temp = temp/2
        counter = counter + 1
    return int(temp*2)

def g(numerator, denominator):
    tempn = numerator
    coeff = 1
    const = 0
    provisDenom = 1
#(ax + b)/c
    while True:
        #print(tempn, denominator, coeff, const, provisDenom)
        if tempn == numerator and
        (coeff > 1 or const > 0 or provisDenom > 1):
            #print("1")
            return - const, coeff - provisDenom
        #end of loop (if periodic dyadic)
        elif tempn == 0:
            #print("2")
            return - const, coeff
        #end of loop (if periodic dyadic)
        elif tempn == denominator:
            #print("3")
            return provisDenom - const, coeff
        #end of loop (if periodic non-dyadic)
        else:
            if 2 * tempn - denominator < 0:
                #print("4a")
                coeff = d * coeff
                const = d * const
                provisDenom = c * provisDenom
                tempn = 2*tempn
```

```
            elif 2 * tempn - denominator >= 0:
                #print("4b")
                coeff = d * coeff
                const = d * const - c * provisDenom
                provisDenom = (d - c) * provisDenom
                tempn = 2*tempn - denominator

#Note that, in g, since the functions are actually
#applied *inside to outside*,
#I had to apply their inverses to f(x)
#instead of applying them directly to f(2x) or f(2x - 1).
#This results in them being applied in reverse order
#from the wrong end,
#so that it gets the right result.

#Update: I think this iterative approach worked well,
#so I rewrote the dyadic part to do that also

def function(numerator, denominator):
    if numerator == 0:
        #print("numerator == 0")
        return 0, 1
    elif numerator == denominator:
        #print("numerator == denominator")
        return 1, 1

    elif howEven(denominator) <= howEven(numerator)
    or OddPartOf(denominator) == 1:
        #print("howEven(denominator) <= howEven(numerator)
        or OddPartOf(denominator) == 1")
        array = g(numerator, denominator)
        return array[0], array[1]

    elif howEven(denominator) > howEven(numerator):
        if 2 * numerator - denominator < 0:
            array = function(2 * numerator, denominator)
            return c * array[0], d * array[1]
        if 2 * numerator - denominator > 0:
            array = function(2 * numerator - denominator,
            denominator)
            return (d - c) * array[0] + c * array[1],
            d * array[1]
        else:
            return "Error", "error"

def function2(numerator, denominator):
```

```
    array = function(numerator, denominator)
    divisor = gcd(array[0], array[1])
    return int(array[0]/divisor), int(array[1]/divisor)

#print(function(1, 7))

def generate_fractions(n):
    fractions = []
    for denominator in range(1, n + 1):
        for numerator in range(0, denominator + 1):
            if gcd(numerator, denominator) == 1:
                fractions.append((numerator, denominator))
    return fractions

#print(generate_fractions(20))

#for k in generate_fractions(30):
    #print(k, function2(k[0], k[1]))

def InverseApprox(n, d, j):
#n is numerator, d is denominator, j is
#how long you want the approximation to go on for
    temp = 0
    InverseApprox = [0, 1]
    for i in range(j):
        array2 = function2(InverseApprox[0], InverseApprox[1])
        temp = d * array2[0] - n * array2[1]
        if temp > 0:
            InverseApprox = [2 * InverseApprox[0] - 1,
            2 * InverseApprox[1]]
        elif temp < 0:
            InverseApprox = [2 * InverseApprox[0] + 1,
            2 * InverseApprox[1]]
    return InverseApprox

print(InverseApprox(1,2,80))
```

Next is the program we used to plot $f_{1/2,\,1/3}$, also written in Python.

```
import matplotlib.pyplot as plt
iterated=[]
x=[]
y=[]
diffquot=[]
initial = [[0, 1], [1, 0]] #f(0)=1 #f(1)=0
print("Doubling array: your initial array is [[0,1],[1,0]]")
```

```
def double(array):
    iterated=[]
    for index, pair in enumerate(array):
        iterated.append([2*pair[0],3*pair[1]])
        #appending the item in the main for
        #instead of inside the if-else statement
        if pair != array[-1]:
            pairnext = array[index + 1]
            middle = []
            middle.append(pair[0] + pairnext[0])
            middle.append(2*pair[1]+pairnext[1])
            iterated.append(middle)
    return iterated
while True:
    bool = input("Do you want to double the arrays?
    type y if so, and type yes if you want a graph: ")
    if bool[0] == "y":
        initial = double(initial)
        #print(initial)
        x=[]
        y=[]
        diffquot=[]
        for index, pair in enumerate(initial):
            x.append(pair[0])
            y.append(pair[1])
            if index == 0:
                continue
            else:
                diffquot.append(y[index-1] - y[index])
        #print(x)
        #print(y)
        realx = [element/x[-1] for element in x]
        realy = [1 - element/y[0] for element in y]
        #print(realx)
        #print(realy)
        print(f"Array now contains {len(x)} points")
        if bool == "yes":
            plt.figure(num=0,dpi=240)
            plt.plot(realx,realy, linewidth=0.25)
            plt.xlabel('x')
            plt.ylabel('y')
            plt.title("Fractal interpolation function,
            where a/b = 1/2 and c/d = 1/3")
            plt.savefig('fif.png', dpi=240, bbox_inches='tight')
            print("Graph saved as 'fif.png'")
```

```
        plt.show()
    #print(diffquot)
```

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EDITOR'S COMMENTS

This paper studies a family of Barnsley fractal interpolation functions through the points $(0,0),(x_1,y_1),(1,1)$ with rational data, and asks whether such a function sends rational numbers to rational numbers, and whether every rational value is reached. The team shows that when $x_1 = 1/2$ the function always sends rationals to rationals, but (when y_1 has an odd denominator) never sends any rational to $1/2$, while the general case $x_1 \neq 1/2$ behaves differently. These are well-posed questions about a specific family of functions linked to de Rham curves and the Dubins–Savage casino functions, and the key idea—connecting the eventual periodicity of binary expansions under the doubling map to whether the images stay rational—is a clean and clever observation.

The paper is honest and careful about its scope: it proves clear results in the manageable $x_1 = 1/2$ case (using number-theoretic structure such as the ring $\mathbb{Z}_{(2)}$ and odd-denominator rationals), and handles the harder general case partly, finishing with conjectures and possible extensions. Overall, it is a professionally written, technical paper on number theory with interesting contributions.