Hang Lung Mathematics Awards
Vol. 11 (2025), pp. 3–42 

# MARKOV CHAINS ASSOCIATED WITH ULTRASPHERICAL POLYNOMIALS: AN ANALYSIS OF WEAK CONVERGENCE AND FIRST PASSAGE TIME FUNCTIONALS

A RESEARCH REPORT SUBMITTED TO THE SCIENTIFIC COMMITTEE OF THE HANG LUNG MATHEMATICS AWARDS

**TEAM NUMBER**
2539379

**TEAM MEMBER**
LI SIU MAN

**TEACHER**
MR. CHUNG ON PING

**SCHOOL**
HONG KONG CHINESE WOMEN'S CLUB COLLEGE

AUGUST 2025

Abstract. This paper investigates discrete-time Markov chains on non-negative integers whose transition probabilities are derived from the recurrence relation of ultraspherical polynomials. Generalizing prior single-time convergence results, our first main result is to establish the full weak convergence of the diffusively rescaled chains to a Bessel process of dimension $d = 2\alpha + 2$. The proof shows the tightness of the process sequence and the convergence of its finite-dimensional distributions. As a key application of this process-level convergence, our second main result shows that the rescaled first exit time of the chain from an interval converges in law to the corresponding exit time of the limiting Bessel process.

## Contents

## 1. Introduction

In this paper, we are interested in a class of Markov chains $(X_n)_{n\in\mathbb{N}}$, associated with ultraspherical polynomials (also known as Gegenbauer polynomials). Ultraspherical polynomials had been a frequent interest of research. ([Dil24], [SB13], [Doh02], etc.) polynomials play a role which is analogous to the role played by Legendre polynomials in the theory of the familiar three-dimensional spherical harmonics [Ave89]. These Markov chains with values in $\mathbb{N}$ were notably studied by Claude George [Geo75]. We discover that, these chains, when properly renormalized, converge in law to Bessel processes, which generalizes a result by George. By definition of ultraspherical chains, $|X_{n+1} - X_n| = 1$. It has been established that the law of $\theta(p)$ is closely related to the exit time from an interval:

$$T_{b,c} = \inf\{n \geq 0, X_n = b \text{ or } c\}.$$

Naturally, this led to the study of hitting times and exit times from intervals. In particular, their Laplace transform and their first two moments were determined. From these, the Laplace transform and the first moment of $\theta(p)$, the first time at which the range reaches level $p$, were deduced.

The study of scaling limits, wherein discrete stochastic processes are shown to converge to continuous ones under a suitable rescaling of space and time, is a cornerstone of modern probability theory. A canonical example is Donsker's theorem (see [NW81]), which establishes the convergence of a simple random walk to Brownian motion. This paper investigates a similar, discrete-to-continuous bridge for a special class of random walks whose behavior is intimately governed by the theory of special functions.

We focus on the *ultraspherical Markov chains*, $(X_n^\alpha)_{n\geq 0}$, a family of nearest-neighbor random walks on the non-negative integers $\mathbb{N}_0$ parameterized by an index $\alpha > -1$. Their transition probabilities are derived directly from the three-term recurrence relation of ultraspherical polynomials:

$$P^\alpha(n, n-1) = \frac{n}{2n+2\alpha+1} \quad \text{and} \quad P^\alpha(n, n+1) = \frac{n+2\alpha+1}{2n+2\alpha+1}, \quad \text{for } n \geq 1.$$

These chains were notably studied by Claude George [Geo75], who established that the distribution of the chain's position at a single large time, $X_N^\alpha/\sqrt{N}$, converges to a law determined by a Bessel process. The primary contribution of this work is to generalize this result from a single-time convergence to a full weak convergence at the process level. To this end, we define a sequence of continuous-time processes by linearly interpolating the rescaled discrete chain:

$$\hat{X}_t^{\alpha,n} = \begin{cases} \dfrac{X_{2^n t}^\alpha}{2^{n/2}} & \text{if } t = \dfrac{k}{2^n}, \\ \dfrac{X_{\lfloor 2^n t\rfloor}^\alpha}{2^{n/2}} + \dfrac{2^n t - \lfloor 2^n t\rfloor}{2^{n/2}} \left(X_{\lfloor 2^n t\rfloor+1}^\alpha - X_{\lfloor 2^n t\rfloor}^\alpha\right) & \text{if } t \in \left(\dfrac{k}{2^n}, \dfrac{k+1}{2^n}\right), \end{cases}$$

Our first main result, proven in Section 3, establishes that this sequence of random paths converges in law to a Bessel process in the space of continuous functions.

**Theorem A** (Theorem 3.8)**.** *The sequence of processes $(\hat{X}_t^{\alpha,n})_{t\geq 0}$, starting from $X_0^\alpha = 0$, converges in distribution in the space $C([0,\infty),\mathbb{R})$ to the Bessel process of dimension $d = 2\alpha+2$, starting from $0$.*

This theorem establishes that the ultraspherical chain is a robust discrete model for the Bessel process. It encapsulates not just its marginal distributions but its complete path-wise dynamics. The proof proceeds via a two-step approach: establishing the tightness of the sequence $(\hat{X}^{\alpha,n})$ and proving the convergence of its finite-dimensional distributions.

A significant consequence of process-level convergence is that continuous functionals of the discrete process should converge to the same functionals of the limit process. In Section 4, we explore this by studying first passage times. For a chain $X^{\alpha,n}$ starting at $X_0^{\alpha,n} = 2^n a$, we consider its first exit time from the scaled interval $[2^n c, 2^n b]$, denoted by $T_{c,b}^{\alpha,n}$. Our second main result shows that this discrete exit time, under diffusive scaling, converges in law to the exit time of the limiting Bessel process.

**Theorem B** (Theorem 4.9)**.** *Let $c < a < b$ be three dyadic numbers and let $\tilde{T}_{c,b}^\alpha$ be the first exit time from the interval $[c,b]$ for a Bessel process $R^\alpha$ starting at $a$. Then, under the diffusive scaling, the discrete exit time converges in law to the* *continuous one:*

$$\frac{T^{\alpha,n}_{c,b}}{2^{2n}} \xrightarrow[n\to\infty]{\mathcal{L}} \tilde{T}^{\alpha}_{c,b}.$$

The proof of this theorem relies on a different set of techniques, centered on constructing martingales from the special functions that define the chain. By applying the Optional Stopping Theorem, we derive an explicit expression for the Laplace transform of the discrete exit time. The convergence is then established by showing that this expression converges to the known Laplace transform of the Bessel exit time.

The paper is organized as follows. Section 2 provides essential preliminaries on ultraspherical polynomials and Legendre functions, establishing the key analytic and asymptotic properties required for our analysis. In Section 3, we formally define the ultraspherical Markov chain and present the proof of our main result: the weak convergence of the rescaled chain to a Bessel process. Section 4 is dedicated to the study of first passage times; we derive their Laplace transforms and prove their convergence to the corresponding hitting and exit times of the Bessel process. Finally, the appendices contain proofs of certain results used throughout the paper.

## 2. Preliminaries of ultraspherical polynomials and Legendre functions

In the study of Markov chains associated with ultraspherical polynomials (see for example [Ken60], [Boc56], etc.), we have to shed light on ultraspherical polynomials and Legendre functions. In this section, we recall the main results relating to these two classes of functions. We have also established some additional convergence properties that will be useful in the rest of the paper. On top of that, we briefly introduce some basic facts about Bessel processes.

### 2.1. Ultraspherical polynomials.

Let $\alpha$ be a real number and that $\alpha > -1$. Consider on the interval $[-1,1]$ the measure defined by

$$\mathrm{d}\mu^{\alpha} = (1-x^2)^{\alpha}\mathbb{1}_{[-1,1]}(x)\mathrm{d}x. \tag{2.1}$$

We have the following definition:

**Definition 2.1.** *The ultraspherical polynomials of index $\alpha$ form a family of orthogonal polynomials for the measure $\mu^{\alpha}$. They are given by the following equation:*

$$Q^{\alpha}_n(x) = \frac{(-1)^n}{2^n(\alpha+1)...(\alpha+n)}(1-x^2)^{-\alpha}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x^2)^{n+\alpha}. \tag{2.2}$$

2.1.1. *Basic properties.* We revisit some basic properties of ultraspherical polynomials. One may refer to, for instance, [Fai24] for details. The ultraspherical polynomials $Q^{\alpha}_n$ satisfy the following properties:

- $Q^{\alpha}_n(x)$ is a polynomial of degree $n$.
- $Q^{\alpha}_0(x) = 1$.

- $Q_1^\alpha(x) = x$.
- $Q_n^\alpha(1) = 1$.
- $Q_n^\alpha(-x) = (-1)^n Q_n^\alpha(x)$.

The supremum of $Q_n^\alpha(x)$ on $[-1,1]$ is given by

$$\sup_{x\in[-1,1]} Q_n^\alpha(x) \begin{cases} = 1, & \text{if } \alpha \geq -\frac{1}{2}, \\ \sim n^{-1/2-\alpha}, & \text{if } \alpha < -\frac{1}{2}. \end{cases} \tag{2.3}$$

Then the polynomials satisfy the following orthogonality relation:

$$\int_{-1}^{1} Q_n^\alpha(x) Q_m^\alpha(x)\, \mathrm{d}\mu^\alpha(x) = \begin{cases} 0, & \text{if } n \neq m, \\ \dfrac{1}{\omega_n^\alpha}, & \text{if } n = m, \end{cases} \tag{2.4}$$

where $\omega_n^\alpha$ is defined as

$$\omega_n^\alpha = \frac{(2n+2\alpha+1)\Gamma(n+2\alpha+1)}{2^{2\alpha+1}(\Gamma(\alpha+1))^2\Gamma(n+1)}.$$

The product of $Q_1^\alpha(x)$ and $Q_n^\alpha(x)$ is given by:

$$Q_1^\alpha(x) Q_n^\alpha(x) = \frac{n}{2n+2\alpha+1} Q_{n-1}^\alpha(x) + \frac{n+2\alpha+1}{2n+2\alpha+1} Q_{n+1}^\alpha(x). \tag{2.5}$$

This formula is of crucial importance for constructing Markov chains associated with ultraspherical polynomials. Refer to [Car61] for a more general formula for multiplications between two ultraspherical polynomials. Moreover, we note the derivative of $Q_n^\alpha(x)$ given by: [Sze75]

$$\frac{\mathrm{d}}{\mathrm{d}x} Q_n^\alpha(x) = \frac{n(n+2\alpha+1)}{2(\alpha+1)} Q_{n-1}^{\alpha+1}(x). \tag{2.6}$$

2.1.2. *Properties and Asymptotic Behavior of $Q_n^\alpha(x)$.* Here, we prove and recall some functional and asymptotic properties of $Q_n^\alpha(\cdot)$. These properties lay the groundwork of the relevant computations.

**Proposition 2.2.** *For any fixed $n$, the function $x \mapsto Q_n^\alpha(x)$ is increasing on $[1,\infty)$. In particular,*

$$Q_n^\alpha(x) \geq 1,$$

*for all $n, x \geq 1$.*

*Proof.* We prove this result by induction on $n$. For $n = 0$, the function $Q_0^\alpha(x)$ is constant, it is trivially increasing (monotonically) for $\alpha > -1$. Now, assume that for some $n \geq 0$, $Q_{n-1}^\alpha(x)$ is increasing for all $\alpha > -1$ and $x > 1$. Using the recurrence relation (Equation 2.5) and the derivative formula (Equation 2.6) for $Q_n^\alpha(x)$, we compute

$$\frac{\mathrm{d}}{\mathrm{d}x} Q_n^\alpha(x) = \frac{n(n+2\alpha+1)}{2(\alpha+1)} Q_{n-1}^{\alpha+1}(x).$$

By the inductive hypothesis, $Q_{n-1}^{\alpha+1}(x)$ is increasing for all $\alpha > -1$. Since

$$\frac{n(n+2\alpha+1)}{2(\alpha+1)} > 0$$

for $\alpha > -1$, it follows that $\frac{\mathrm{d}}{\mathrm{d}x} Q_n^\alpha(x) > 0$. Thus, $Q_n^\alpha(x)$ is increasing for all $\alpha > -1$ and $x > 1$. $\square$

**Proposition 2.3.** *For $x \geq 1$, the sequence $\{Q_n^\alpha(x)\}_n$ is increasing.*

*Proof.* We will prove this statement by mathematical induction on $n$ again. For $n = 0$, we have $Q_0^\alpha(x) = 1$, and by definition, $Q_1^\alpha(x) = x$. Clearly, for $x \geq 1$:

$$Q_1^\alpha(x) = x \geq Q_0^\alpha(x) = 1.$$

Assume that for all $x \geq 1$,

$$Q_{k+1}^\alpha(x) \geq Q_k^\alpha(x).$$

From the recurrence relation (Equation 2.5), we have

$$xQ_{n+1}^\alpha(x) = \frac{n+1}{2n+2\alpha+3} Q_n^\alpha(x) + \frac{n+2\alpha+2}{2n+2\alpha+3} Q_{n+2}^\alpha(x).$$

Notice also

$$Q_{n+2}^\alpha(x) = \frac{2n+2\alpha+3}{n+2\alpha+2} xQ_{n+1}^\alpha(x) - \frac{n+1}{n+2\alpha+2} Q_n^\alpha(x).$$

By the inductive assumption, $Q_{k+1}^\alpha(x) \geq Q_k^\alpha(x)$ for all $x \geq 1$. Since $x \geq 1$ and the coefficients in the recurrence relation are positive, it follows that:

$$Q_{k+2}^\alpha(x) \geq Q_{k+1}^\alpha(x).$$

$\square$

The following convergences will prove useful in the proceeding sections:

**Corollary 2.4.** [Sze75, Theorem 8.1.1] *For any point $z$ in the complex plane,*

$$Q_n^\alpha\left(1 - \frac{z^2}{2n^2}\right) \xrightarrow[n\to\infty]{} \Gamma(\alpha+1)\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z),$$

$$Q_n^\alpha\left(\cos\frac{z}{n}\right) \xrightarrow[n\to\infty]{} \Gamma(\alpha+1)\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z),$$

*where $J_\alpha(z)$ is the Bessel function:*

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z}{2}\right)^{2k}}{k!\,\Gamma(k+\alpha+1)}.$$

These convergences are uniform on any compact subset of the complex plane.

2.2. **The Legendre functions.** We start by considering the second-order differential equation:

$$(1-z^2)\frac{\mathrm{d}^2w}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}w}{\mathrm{d}z} + \left(n(n+1) - \frac{\alpha^2}{1-z^2}\right)w = 0, \tag{2.7}$$

where $\alpha$ and $n$ are real constants. The differential equation is called the *general Legendre equation.* Let us recall the definition of hypergeometric functions:

**Definition 2.5.** *For $a, b, c \in \mathbb{R}$, $|z| < 1$, $b > 0$, and $c - b > 0$, the hypergeometric function $F$ is defined as:*

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt.$$

**Definition 2.6.** *We now define the following functions: (refer to Equations 8.1.2 and 8.1.3 from* [AS65]*)*

$$\mathcal{P}_n^\alpha(z) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{z+1}{z-1}\right)^{\frac{\alpha}{2}} F\left(-n, n+1; 1-\alpha; \frac{1-z}{2}\right), \quad |1-z| < 2, \tag{2.8}$$

*and*

$$\mathcal{Q}_n^\alpha(z) = e^{i\alpha\pi}2^{-n-1}\pi^{\frac{1}{2}}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\frac{3}{2})}z^{-n-\alpha-1}(z^2-1)^{\frac{\alpha}{2}} \tag{2.9}$$

$$F\left(\frac{1}{2}n + \frac{1}{2}\alpha + \frac{1}{2}, \frac{1}{2}n + \frac{1}{2}\alpha + 1; n + \frac{3}{2}; \frac{1}{z^2}\right), \quad |z| > 1. \tag{2.10}$$

Here, $\mathcal{P}_n^\alpha$ and $\mathcal{Q}_n^\alpha$ represent two independent solutions of the original differential equation (2.7), commonly referred to as the Legendre functions of the first and second kind, respectively. We will explore the asymptotic properties of the solution $\mathcal{Q}_n^\alpha$.

**Remark 2.7.** *When $\alpha = 0$, the Legendre functions reduce to the well-known Legendre polynomials. We note two relations that will be useful for transitioning between cases where $\alpha \geq 0$ and $\alpha < 0$:*

$$\mathcal{P}_n^{-\alpha}(x) = \frac{\Gamma(n-\alpha+1)}{\Gamma(n+\alpha+1)}\Big(\mathcal{P}_n^\alpha(x) - \frac{2}{\pi}\sin(\alpha\pi)\mathcal{Q}_n^\alpha(x)\Big), \tag{2.11}$$

*and*

$$\mathcal{Q}_n^{-\alpha}(x) = e^{-2i\alpha\pi}\frac{\Gamma(n-\alpha+1)}{\Gamma(n+\alpha+1)}\mathcal{Q}_n^\alpha(x). \tag{2.12}$$

We note the following well-established results:

**Theorem 2.8.** [AS65, Equation 8.5.2, Equation 8.5.4] *The following derivative formulas hold:*

$$(z^2-1)\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Q}_n^\alpha(z) = (n+\alpha)(n+\alpha-1)(z^2-1)^{\frac{1}{2}}\mathcal{Q}_{n-1}^\alpha(z) - \alpha z\mathcal{Q}_n^\alpha(z). \tag{2.13}$$

$$(z^2-1)\frac{\mathrm{d}\mathcal{Q}_n^\alpha}{\mathrm{d}z}(z) = nz\mathcal{Q}_n^\alpha(z) - (\alpha+n)\mathcal{Q}_{n-1}^\alpha(z). \tag{2.14}$$

**Theorem 2.9.** [AS65, Equation 8.8.2] *For $z \in \mathbb{C}\setminus(-\infty,-1]$, $\alpha \geq 0$, we have the following integral representation formula:*
(2.15)

$$\mathcal{Q}_n^\alpha(z) = e^{i\pi\alpha}\frac{2^{-\alpha}\sqrt{\pi}\Gamma(n+\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(n-\alpha+1)}(z^2-1)^{\frac{1}{2}}\int_0^{+\infty}\left(\frac{1}{z+(z^2-1)^{\frac{1}{2}}\cosh t}\right)^{n+\alpha+1}(\sinh t)^{2\alpha}\mathrm{d}t.$$

**Proposition 2.10.** [AS65, Equation 9.6.50] *For all $z \in \mathbb{C}$ and $x \in \mathbb{R}$, we have*

$$n^{-\alpha}e^{-i\pi\alpha}\mathcal{Q}_n^\alpha\left(\cosh\left(\frac{z}{n}\right)\right) \xrightarrow[n\to\infty]{} K_\alpha(z); \tag{2.16}$$

*where $K_\alpha$ is the modified Bessel function of the second kind, defined as*

$$K_\alpha(x) = \frac{\pi}{2}\frac{I_{-\alpha}(x)-I_\alpha(x)}{\sin\alpha\pi},$$

*where*

$$I_\alpha(x) = i^{-\alpha}J_\alpha(ix) = \sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}.$$

Let us denote arcosh as the inverse of the hyperbolic function cosh. For any $x \geq 1$, the inverse function $\mathrm{arcosh}(x)$ is given by

$$\mathrm{arcosh}(x) = \ln\left(x+\sqrt{x^2-1}\right).$$

**Corollary 2.11.**

$$\lim_{n\to\infty} n\cdot\mathrm{arcosh}\left(e^{\frac{x^2}{2n^2}}\right) = x.$$

*Proof.* The proof is very elementary. We begin by recalling the expansion of $\mathrm{arcosh}(1+u)$ for small $u>0$:

$$\mathrm{arcosh}(1+u) \sim \sqrt{2u}, \quad \text{as } u \to 0.$$

Substitute $e^{\frac{x^2}{2n^2}} = 1 + \frac{x^2}{2n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)$ into arcosh:

$$\mathrm{arcosh}\left(e^{\frac{x^2}{2n^2}}\right) \sim \sqrt{\frac{x^2}{n^2}} = \frac{x}{n}, \quad \text{as } n\to\infty.$$

Then the result is apparent. □

**Lemma 2.12.** *For any bounded interval $(a,b) \subset \mathbb{R}^+$, for all $x \in (a,b)$ and all $n \geq 1$, we have*

$$n^{-\alpha} \left| \mathcal{Q}_n^\alpha \left(\cosh \frac{x}{n}\right) - \mathcal{Q}_n^\alpha \left(e^{\frac{x^2}{2n^2}}\right) \right| = \mathcal{O}(n^{-1}).$$

*Proof.* We consider two cases, namely $\alpha \geq 0$ and $a < 0$.
**Case 1**: $\alpha \geq 0$. For $\alpha \geq 0$, the integral representation of $\mathcal{Q}_n^\alpha(z)$ is given by Equation (2.15):
(2.17)

$$\mathcal{Q}_n^\alpha(z) = e^{i\pi\alpha} \frac{2^{-\alpha}\sqrt{\pi}\Gamma(n+\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(n-\alpha+1)} (z^2-1)^{\alpha/2} \int_0^\infty \left(\frac{1}{z+(z^2-1)^{1/2}\cosh t}\right)^{n+\alpha+1} (\sinh t)^{2\alpha} \mathrm{d}t.$$

Set $z = \cosh(x/n)$, so $z^2 - 1 = \sinh^2(x/n)$, and $(z^2-1)^{\alpha/2} = \sinh^\alpha(x/n)$. The denominator is:

$$z+(z^2-1)^{1/2}\cosh t = \cosh\frac{x}{n} + \sinh\frac{x}{n}\cosh t = \frac{\cosh(x/n) + \cosh t \sinh(x/n)}{\cosh t \sinh(x/n)} \cdot \cosh t \sinh\frac{x}{n}.$$

For $x \in (a,b)$, since cosh and sinh are increasing, we have:

$$\sinh\frac{a}{n} \leq \sinh\frac{x}{n} \leq \sinh\frac{b}{n}, \quad \cosh\frac{a}{n} \leq \cosh\frac{x}{n} \leq \cosh\frac{b}{n}.$$

The integrand's denominator satisfies

$$\cosh\frac{x}{n} + \sinh\frac{x}{n}\cosh t \geq \cosh\frac{a}{n} + \sinh\frac{a}{n}\cosh t.$$

Thus we have

$$\begin{aligned} |\mathcal{Q}_n^\alpha(\cosh(x/n))| \leq & \frac{2^{-\alpha}\sqrt{\pi}\Gamma(n+\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(n-\alpha+1)} \\ & \sinh^\alpha\frac{x}{n} \int_0^\infty \left(\frac{1}{\cosh(a/n)+\sinh(a/n)\cosh t}\right)^{n+\alpha+1} (\sinh t)^{2\alpha} \mathrm{d}t. \end{aligned}$$

Since $\sinh(x/n) \leq \sinh(b/n)$, we bound

$$|\mathcal{Q}_n^\alpha(\cosh(x/n))| \leq \left(\frac{\sinh(b/n)}{\sinh(a/n)}\right)^\alpha |\mathcal{Q}_n^\alpha(\cosh(a/n))|.$$

By the known convergence $e^{-i\pi\alpha} n^{-\alpha} \mathcal{Q}_n^\alpha(\cosh(a/n)) \to K_\alpha(a)$, we have

$$n^{-\alpha} |\mathcal{Q}_n^\alpha(\cosh(a/n))| \leq C_1$$

for some constant $C_1$. As $n \to \infty$:

$$\frac{\sinh(b/n)}{\sinh(a/n)} \xrightarrow[n\to\infty]{} \frac{b/n}{a/n} = \frac{b}{a},$$

so, for some constant $C_2$,

$$n^{-\alpha} |\mathcal{Q}_n^\alpha(\cosh(x/n))| \leq \left(\frac{b}{a}\right)^\alpha C_1 \leq C_2.$$

**Case 2**: For $\alpha < 0$, use the relation:

$$\mathcal{Q}_n^\alpha(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n-\alpha+1)} \mathcal{Q}_n^{-\alpha}(z).$$

Stirling's formula gives

$$\frac{\Gamma(n+\alpha+1)}{\Gamma(n-\alpha+1)} \sim n^{2\alpha}.$$

Since $n^{-(-\alpha)}|\mathcal{Q}_n^{-\alpha}(\cosh(x/n))| \le C_3$ for some constant $C_3$, we have

$$n^{-\alpha}\left|\mathcal{Q}_n^{\alpha}(\cosh(x/n))\right| = n^{-\alpha}\frac{\Gamma(n+\alpha+1)}{\Gamma(n-\alpha+1)}\left|\mathcal{Q}_n^{-\alpha}(\cosh(x/n))\right| \le C_3 n^{-\alpha}n^{2\alpha} = C_3 n^{\alpha} \le C_4,$$

for some constant $C_4$, since $n^{\alpha}$ is bounded for $\alpha < 0$. Thus, $n^{-\alpha}|\mathcal{Q}_n^{\alpha}(\cosh(x/n))|$ is bounded, i.e. there exists some constant $C$ such that:

$$n^{-\alpha}|\mathcal{Q}_n^{\alpha}(\cosh(x/n))| \le C. \tag{2.18}$$

---

Define the sequence $x_n$ as:

$$x_n = n \cdot \operatorname{arcosh}\left(e^{\frac{x^2}{2n^2}}\right).$$

From Corollary 2.11, $x_n$ converges to $x$. A partial expansion of the above equation gives:

$$x_n - x \sim \frac{x^3}{12n^2}.$$

Next, for $x, y \in (a, b)$, we find

$$\left|\mathcal{Q}_n\left(\cosh\frac{x}{n}\right) - \mathcal{Q}_n\left(\cosh\frac{y}{n}\right)\right| \le \left|\cosh\frac{x}{n} - \cosh\frac{y}{n}\right| \sup_{t\in(a,b)}\left|(\mathcal{Q}_n')\left(\cosh\frac{t}{n}\right)\right|.$$

Using the continuity of cosh and the bound for $|x-y|$, this simplifies to

$$\left|\mathcal{Q}_n\left(\cosh\frac{x}{n}\right) - \mathcal{Q}_n\left(\cosh\frac{y}{n}\right)\right| \le \frac{K}{n^2}\sup_{t\in(a,b)}\left|((\mathcal{Q}_n^{\alpha})')\left(\cosh\frac{t}{n}\right)\right|.$$

Now, for $x \in (a, b)$ and $y = x_n \in (a, b)$, using the earlier expansion, we get

$$\left|\mathcal{Q}_n^{\alpha}\left(\cosh\frac{x}{n}\right) - \mathcal{Q}_n^{\alpha}\left(e^{\frac{x^2}{2n^2}}\right)\right| \le \frac{K}{n^4}\sup_{t\in(a,b)}\left|((\mathcal{Q}_n^{\alpha})')\left(\cosh\frac{t}{n}\right)\right|.$$

Here, $K$ is independent of $n$. Using Equations 2.13 and the bound $n^{-\alpha}|\mathcal{Q}_n^{\alpha}(\cosh(x/n))| \le C$, we find that there exist constants $K_1$, $K_2$, and $K_3$ (all independent of $n$) such that

$$\frac{1}{n^2}\sup_{t\in(a,b)}\left|\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Q}_n^{\alpha}\left(\cosh\frac{t}{n}\right)\right| \le K_1\sup_{t\in(a,b)}\left|\mathcal{Q}_n^{\alpha-1}\left(\cosh\frac{t}{n}\right)\right| + K_2\sup_{t\in(a,b)}\left|\mathcal{Q}_n^{\alpha}\left(\frac{t}{n}\right)\right|$$
$$\sup_{t\in[a,b]}\left|\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Q}_n^{\alpha}\left(\cosh\frac{t}{n}\right)\right| \le K_3 n^{\alpha+3}.$$

Finally, combining the inequalities, we obtain the sought result:

$$n^{-\alpha}\left|\mathcal{Q}_n^{\alpha}\left(\cosh\frac{x}{n}\right) - \mathcal{Q}_n^{\alpha}\left(e^{\frac{x^2}{2n^2}}\right)\right| \le K_4 n^{-\alpha-1}, \tag{2.19}$$

for some constant $K_4$. □

**Proposition 2.13.** *We have the following convergence:*

$$n^{-\alpha}e^{-i\pi\alpha}\mathcal{Q}_n^{\alpha}\left(e^{\frac{x^2}{2n^2}}\right)\xrightarrow[n\to\infty]{}K_{\alpha}(x), \tag{2.20}$$

*Proof.* Obviously, by the parity of the functions $x\mapsto\cosh(x)$, $x\mapsto x^2$, and $x\mapsto K_{\alpha}(x)$, it suffices to prove the result for $x>0$. For $x\in(a,b)\subset\mathbb{R}^+$, consider the difference

$$\begin{aligned}\left|n^{-\alpha}e^{-i\pi\alpha}\mathcal{Q}_n^{\alpha}\left(e^{\frac{x^2}{2n^2}}\right)-K_{\alpha}(x)\right| &\le \left|n^{-\alpha}e^{-i\pi\alpha}\mathcal{Q}_n^{\alpha}\left(e^{\frac{x^2}{2n^2}}\right)-n^{-\alpha}e^{-i\pi\alpha}\mathcal{Q}_n^{\alpha}\left(\cosh\frac{x}{n}\right)\right|\\ &\quad+\left|n^{-\alpha}e^{-i\pi\alpha}\mathcal{Q}_n^{\alpha}\left(\cosh\frac{x}{n}\right)-K_{\alpha}(x)\right|.\end{aligned}$$

By (2.16), the second term converges to zero:

$$\left|n^{-\alpha}e^{-i\pi\alpha}\mathcal{Q}_n^{\alpha}\left(\cosh\frac{x}{n}\right)-K_{\alpha}(x)\right|\to 0 \text{ as } n\to\infty.$$

By Lemma 2.12, for any bounded interval $(a,b)\subset\mathbb{R}^+$, there exists a constant $K>0$ such that

$$n^{-\alpha}\left|\mathcal{Q}_n^{\alpha}\left(\cosh\frac{x}{n}\right)-\mathcal{Q}_n^{\alpha}\left(e^{\frac{x^2}{2n^2}}\right)\right|\le\frac{K}{n}, \tag{2.21}$$

for all $x\in(a,b)$ and $n\ge 1$. Since $x\in(a,b)$, we apply (2.21):

$$n^{-\alpha}\left|\mathcal{Q}_n^{\alpha}\left(e^{\frac{x^2}{2n^2}}\right)-\mathcal{Q}_n^{\alpha}\left(\cosh\frac{x}{n}\right)\right|\le\frac{K}{n}\to 0 \text{ as } n\to\infty.$$

Thus, the first term vanishes. Combining both terms,

$$\left|n^{-\alpha}e^{-i\pi\alpha}\mathcal{Q}_n^{\alpha}\left(e^{\frac{x^2}{2n^2}}\right)-K_{\alpha}(x)\right|\to 0,$$

proving (2.20) for $x>0$. The result extends to all $x\in\mathbb{R}$. □

2.3. **Preliminaries of Bessel processes.** This subsection gives brief information about the Bessel process. Readers are directed to, for example, [Kat15, Chapter 1] and [SP12], for more details. The Bessel process of index $\alpha$, denoted by $(R_t^{\alpha})_{t\ge 0}$, is a non-negative stochastic process that can be characterized as the unique strong solution to the following stochastic differential equation:

$$dR_t^{\alpha}=dW_t+\frac{2\alpha+1}{2R_t^{\alpha}}dt, \tag{2.22}$$

where $(W_t)_{t\ge 0}$ is a standard one-dimensional Brownian motion. The term $\dfrac{2\alpha+1}{2R_t^{\alpha}}$ is the drift component, which is singular at the origin. The behavior of the Bessel process, particularly its relationship with the origin, is highly dependent on its dimension $d=2\alpha+2$:

**Proposition 2.14.** *The long-term behavior of the Bessel process $R^{\alpha}$ depends on the value of $\alpha$ (and thus on the dimension $d$):*

- *If $d > 2$ (i.e., $\alpha > 0$), the process is transient. This means that $R_t^\alpha \to \infty$ as $t \to \infty$ almost surely.*
- *If $d = 2$ (i.e., $\alpha = 0$), the process is recurrent (or neighborhood-recurrent). For any $r > 0$, the process returns to the interval $[0, r)$ at arbitrarily large times.*
- *If $d < 2$ (i.e., $\alpha < 0$), the process is not transient.*

**Proposition 2.15.** [RY10, p. 251, Proposition V.3.1] *The transition density of a Bessel process $R^\alpha$ starting from $R_s^\alpha = x > 0$ to $R_t^\alpha = u > 0$ at a later time $t > s$ is given by*

$$f^\alpha_{x,t-s}(u) = \frac{u}{t-s}\left(\frac{u}{x}\right)^\alpha \exp\left(-\frac{x^2+u^2}{2(t-s)}\right) I_\alpha\left(\frac{ux}{t-s}\right),$$

*where $I_\alpha$ is the modified Bessel function of the first kind of order $\alpha$.*

## 3. Convergence of Renormalized Ultraspherical Markov Chains to a Bessel Process

We begin with the following definition:

**Definition 3.1.** *Let $\alpha > -1$ and $X_n^\alpha$ be a Markov chain with values in $\mathbb{N}_0$ and transition probabilities $P^\alpha$ defined as follows:*

$$\begin{cases} P^\alpha(0,1) = 1, \\ P^\alpha(n, n-1) = \dfrac{n}{2n+2\alpha+1}, & \forall n \geq 1, \\ P^\alpha(n, n+1) = \dfrac{n+2\alpha+1}{2n+2\alpha+1}, & \forall n \geq 1. \end{cases}$$

*We call the process $X_n^\alpha$ the Markov chain associated with ultraspherical polynomials or ultraspherical Markov chains.*

From the definition above, it is evident that the given $P^\alpha$ satisfies the properties of a Markov chain. In particular, for a fixed $n \geq 0$, the transition probabilities sum to 1:

$$\frac{n}{2n+2\alpha+1} + \frac{n+2\alpha+1}{2n+2\alpha+1} = 1.$$

Moreover, the coefficients of the transition probabilities $P^\alpha$ are intimately connected to the recurrence relations of ultraspherical polynomials. The properties of ultraspherical polynomials have been leveraged by Claude George ([Geo75, Chapter II]) to derive important properties of these chains, such as their potential, through integral representations. The potential can be expressed using classical functions, such as Legendre functions, to enhance the analysis of the chains' convergence behavior. Our goal is to build upon George's findings regarding the convergence of the chain $X_n^\alpha$. George established that the rescaled chain $X_n^\alpha/\sqrt{N}$ converges in law to $R_1^\alpha$, where $R^\alpha$ is a Bessel process of dimension $d = 2\alpha + 2$. Extending this, we propose using ultraspherical Markov chains to approximate the entire process $R^\alpha$.

3.1. **Basic properties of $X_n^\alpha$ and $P^\alpha$.** The calculation of the $N$-th iterated power of $P^\alpha$ is treated in a more general framework in [Geo75, Chapter II]. Let us recall the following relation, true for all $n \geq 1$, written in terms of $P^\alpha$:

$$Q_n^\alpha(x)Q_n^\alpha(x) = P^\alpha(n, n-1)Q_{n-1}^\alpha(x) + P^\alpha(n, n+1)Q_{n+1}^\alpha(x). \tag{3.1}$$

**Proposition 3.2.** [Geo75, Chapter II] *Let $[P^\alpha]^N$ denote the $N$-th iterated power of $P^\alpha$. Then, the following relation holds:*

$$\left[Q_1^\alpha(x)\right]^N Q_n^\alpha(x) = \sum_{m\geq 0} P^\alpha(n,m)^N Q_m^\alpha(x). \tag{3.2}$$

*Proof.* The result is proven by induction on $N$. For $N = 1$, Equation 3.2 reduces to Equation 3.1. Now, suppose that Equation 3.2 holds for $N$. We show that this relation is valid for $N+1$. By the definition of $[P^\alpha]^{N+1}$, we have

$$P^\alpha(n,r)^{N+1} = \sum_{m\geq 0} P^\alpha(n,m)^N P^\alpha(m,r).$$

Similarly, using the recurrence relation for $Q_n^\alpha$, we find

$$\left(Q_1^\alpha\right)^{N+1} Q_n^\alpha = Q_1^\alpha\left[\left(Q_1^\alpha\right)^N Q_n^\alpha\right].$$

Expanding this we get the following:

$$\begin{aligned}
\left(Q_1^\alpha\right)^{N+1} Q_n^\alpha &= Q_1^\alpha \sum_{m\geq 0} P^\alpha(n,m)^N Q_m^\alpha \\
&= \sum_{m\geq 0} P^\alpha(n,m)^N Q_1^\alpha Q_m^\alpha \\
&= \sum_{m\geq 0} P^\alpha(n,m)^N \sum_{r\geq 0} P^\alpha(m,r) Q_r^\alpha \\
&= \sum_{r\geq 0}\left[\sum_{m\geq 0} P^\alpha(n,m)^N P^\alpha(m,r)\right] Q_r^\alpha \\
&= \sum_{r\geq 0} P^\alpha(n,r)^{N+1} Q_r^\alpha.
\end{aligned}$$

This completes the proof. □

**Proposition 3.3.** [Geo75, Chapter II] *The $N$-th iteration of $P^\alpha$ satisfies the following:*

$$(P^\alpha)^N(n,m) = \omega_m^\alpha \int_{-1}^{1} x^N Q_n^\alpha(x) Q_m^\alpha(x)\,\mathrm{d}\mu^\alpha(x), \tag{3.3}$$

*where $\omega_m^\alpha$ is the constant defined as*

$$\omega_m^\alpha = \frac{(2m+2\alpha+1)\Gamma(m+2\alpha+1)}{2^{2\alpha+1}(\Gamma(\alpha+1))^2\Gamma(m+1)}.$$

*Proof.* From Proposition 3.2, we have:

$$x^N Q_n^\alpha(x) = (Q_1^\alpha)^N(x) Q_n^\alpha(x) = \sum_{k\geq 0} (P^\alpha)^N(n,k) Q_k^\alpha(x).$$

Using the orthogonality relation of $Q_n^\alpha$, we integrate with respect to the measure $\mu^\alpha$ over $(-1,1)$:

$$\begin{aligned}
\int_{-1}^{1} x^N Q_n^\alpha(x) Q_m^\alpha(x) \mathrm{d}\mu^\alpha(x) &= \int_{-1}^{1} \sum_{k\geq 0} (P^\alpha)^N(n,k) Q_k^\alpha(x) Q_m^\alpha(x) \mathrm{d}\mu^\alpha(x) \\
&= \sum_{k\geq 0} \int_{-1}^{1} (P^\alpha)^N(n,k) Q_k^\alpha(x) Q_m^\alpha(x) \mathrm{d}\mu^\alpha(x) \\
&= \int_{-1}^{1} (P^\alpha)^N(n,m) (Q_m^\alpha(x))^2 \mathrm{d}\mu^\alpha(x)
\end{aligned}$$

Recall from Equation (2.4), if $n = m$,

$$\int_{-1}^{1} Q_n^\alpha(x) Q_m^\alpha(x)\, \mathrm{d}\mu^\alpha(x) = \frac{1}{\omega_n^\alpha} = \frac{1}{\omega_m^\alpha}.$$

Then we get

$$\int_{-1}^{1} x^N Q_n^\alpha(x) Q_m^\alpha(x) \mathrm{d}\mu^\alpha(x) = \frac{1}{\omega_m^\alpha} (P^\alpha)^N(n,m).$$

Thus, the desired result is obtained:

$$(P^\alpha)^N(n,m) = \omega_m^\alpha \int_{-1}^{1} x^N Q_n^\alpha(x) Q_m^\alpha(x) \mathrm{d}\mu^\alpha(x).$$

□

Now we introduce the $\lambda$-potential, on which will we will more light on in Section 4.

**Definition 3.4.** *For $\lambda \geq 0$, the $\lambda$-potential matrix of the chain $X_n^\alpha$ is defined as:*

$$U_\lambda^\alpha(n,m) = \sum_{N\geq 0} e^{-\lambda N} (P^\alpha)^N(n,m).$$

**Proposition 3.5.** *The chain $X_n^\alpha$ is recurrent if and only if $\alpha \leq 0$.*

*Proof.* Recall that a Markov chain is recurrent if and only if, for all $n$ and $m$,

$$U_0(n,m) = +\infty.$$

Since the chain $X_n^\alpha$ is irreducible (or ergodic), it suffices to prove the result for $n = m = 0$. Recall that

$$\frac{Q_0^\alpha(x) Q_0^\alpha(x)}{1-x} = \frac{1}{1-x}.$$

Thus:

$$U_0^\alpha(0,0) = \omega_0^\alpha \int_{-1}^{1} \frac{1}{1-x} d\mu^\alpha(x) = \omega_0^\alpha \int_{-1}^{1} \frac{(1-x^2)^\alpha}{1-x} dx.$$

Using the substitution and knowing that $\alpha > -1$, we can conclude that $U_0^\alpha(0,0) = +\infty$ if and only if $alpha \leq 0$. $\square$

3.2. **Preliminaries of proving weak convergence.** Claude George [Geo75] established the convergence in distribution of:

$$\frac{X_n^\alpha}{n^{1/2}} \to R_1^\alpha,$$

where $R^\alpha$ is a Bessel process of dimension $d = 2\alpha + 2$ starting from 0. Here, we generalize this result by showing that the renormalized chain converges in distribution to the Bessel process $R^\alpha$. To study the continuum limit, we rescale the chain to align with the Bessel process's time and space scales. Consider the sequence of continuous processes $(\hat{X}_t^{\alpha,n})_{n\geq 1}$, constructed from the chain $X^\alpha$ as follows:

$$\hat{X}_t^{\alpha,n} = \begin{cases} \dfrac{X_{2^n t}^\alpha}{2^{n/2}} & \text{if } t = \dfrac{k}{2^n}, \\ \dfrac{X_{\lfloor 2^n t \rfloor}^\alpha}{2^{n/2}} + \dfrac{2^n t - \lfloor 2^n t \rfloor}{2^{n/2}} \left( X_{\lfloor 2^n t \rfloor + 1}^\alpha - X_{\lfloor 2^n t \rfloor}^\alpha \right) & \text{if } t \in \left( \dfrac{k}{2^n}, \dfrac{k+1}{2^n} \right), \end{cases}$$

where $\lfloor a \rfloor$ denotes the integer part of $a$. We show that the sequence $(\hat{X}_t^{\alpha,n})_{n\geq 1}$ converges in distribution to the Bessel process $R^\alpha$.

**Notation 3.6.** *In what follows, $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution, and $\stackrel{d}{=}$ denotes equality in distribution.*

Convergence in distribution of a sequence of processes $(X_n)_{n\geq 1}$ to a limit process $X$ is formally defined as the weak convergence of their corresponding probability measures $(\mathbb{P}_n)$ to a limit measure $\mathbb{P}$ on a function space, typically the space of continuous functions $C[0,T]$, for some $T$.

The convergence of finite-dimensional distributions (FDDs) is insufficient on its own to guarantee weak convergence on an infinite-dimensional space like $C[0,T]$. FDD convergence only ensures that the projections of the measures onto finite-dimensional subspaces converge. It does not control the behavior of the sample paths between the specified time points, failing to preclude pathological phenomena such as increasingly rapid oscillations or the escape of probability mass to infinity.

This is where tightness becomes essential.

**Definition 3.7.** *Let $(S,d)$ be a metric space, and let $\{X_i\}_{i\in I}$ be a family of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $S$. The family $\{X_i\}_{i\in I}$ is said to be **tight** if, for every $\epsilon > 0$, there exists a compact set $K \subset S$ such that*

$$\mathbb{P}(X_i \in K) \geq 1 - \epsilon, \quad \forall i \in I.$$

*Equivalently, the family of probability measures $\{\mathbb{P}_{X_i}\}_{i\in I}$, where $\mathbb{P}_{X_i}(A) = \mathbb{P}(X_i \in A)$ for measurable $A \subset S$, is tight, meaning that for every $\epsilon > 0$, there exists a compact set $K \subset S$ such that $\mathbb{P}_{X_i}(K) \geq 1 - \epsilon$ for all $i \in I$.*

We will determine the classical criterion of tightness as follows (see, for example, [KS04, p. 91]): For arbitrary $T > 0$ and $\epsilon > 0$, we have

(1) *Pointwise Boundedness:* For each $t \in [0, T]$ (or just $t = 0$), the sequence of real-valued random variables $(\hat{X}_t^{\alpha,n})$ is tight. That is,

$$\lim_{\lambda\to\infty} \sup_{n\geq 1} \mathbb{P}\left(|\hat{X}_0^{\alpha,n}| > \lambda\right) = 0. \tag{3.4}$$

(2) *Control of Modulus of Continuity:* For every $\varepsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}\left(w_{\hat{X}^{\alpha,n}}(\delta) > \varepsilon\right) = 0, \tag{3.5}$$

where $w_X(\delta)$ is the modulus of continuity of a function $X \in C([0,T], \mathbb{R})$, defined as

$$w_X(\delta) := \sup_{\substack{s,t\in[0,T] \\ |t-s|<\delta}} |X(t) - X(s)|.$$

Here we state the main theorem of this section:

**Theorem 3.8.** *The sequence $(\hat{X}_t^{\alpha,n})_{t\geq 0}$ converges in distribution to the Bessel process of dimension $d = 2\alpha + 2$, starting from 0.*

The proof of Theorem 3.8 requires two ingredients:

- Showing that the sequence $\hat{X}_t^{\alpha,n}$ is *tight* in the space of continuous functions defined on $\mathbb{R}^+$;
- Establishing the finite-dimensional convergence:

$$(\hat{X}_{t_1}^{\alpha,n}, \ldots, \hat{X}_{t_m}^{\alpha,n}) \xrightarrow[n\to\infty]{\mathcal{L}} (R_{t_1}^\alpha, \ldots, R_{t_m}^\alpha),$$

for $t_1 \leq t_2 \leq \cdots \leq t_m$. It boils down to proving $\mathbb{E}[f(\hat{X}_s^{\alpha,n})g(\hat{X}_t^{\alpha,n})] \to \mathbb{E}[f(R_s^\alpha)g(R_t^\alpha)]$ for suitable test functions $f$ and $g$.

3.3. **Proving tightness of** $(\hat{X}^{\alpha,n})_{n\in\mathbb{N}}$**.** The condition stated in Equation (3.4) is fairly straightforward to see. By definition, the process starts from a fixed, non-random integer state $X_0^\alpha = a$. The rescaled process at $t = 0$ is therefore deterministic:

$$\hat{X}_0^{\alpha,n} = \frac{X_0^\alpha}{2^{n/2}} = \frac{a}{2^{n/2}}. \tag{3.6}$$

For any given $\lambda > 0$, the inequality $|a| \cdot 2^{-n/2} > \lambda$ can only hold for a finite set of integers $n$, since $2^{-n/2} \to 0$ as $n \to \infty$. Let $N_\lambda = \{n \in \mathbb{N} : |a| \cdot 2^{-n/2} > \lambda\}$, which is finite. As $\lambda \to \infty$, eventually $\lambda > |a|$, at which point $N_\lambda$ becomes the empty set.

For such large $\lambda$, the event $|\hat{X}_0^{\alpha,n}| > \lambda$ is impossible for any $n \geq 1$. Therefore, for sufficiently large $\lambda$,

$$\sup_{n\geq 1} \mathbb{P}\left(|\hat{X}_0^{\alpha,n}| > \lambda\right) = 0. \tag{3.7}$$

Now, Equation (3.5) is more difficult to prove. To proceed, we show some preliminary lemmas.

**Lemma 3.9.** *For any $T > 0$, there exists a constant $C_T > 0$ such that for any $m \in \mathbb{N}$ and any starting state $a \in \mathbb{N}_0$ with $a \leq mT$,*

$$\mathbb{E}_a\left[(X_m^\alpha - a)^4\right] \leq C_T m^2. \tag{3.8}$$

*Proof.* The proof is detailed in Appendix A. □

**Lemma 3.10.** *Let $T > 0$ and $\varepsilon > 0$ be arbitrary. Then there exists a constant $M > 0$ (independent of $\delta$) such that for any $\delta \in (0,1)$ and any $p \in \{0, \dots, \lceil T/\delta \rceil - 1\}$,*

$$\limsup_{n\to\infty} \mathbb{P}_a\left(\sup_{0\leq j<\lfloor 2^n\delta\rfloor} |X^\alpha_{\lfloor 2^n p\delta\rfloor + j} - X^\alpha_{\lfloor 2^n p\delta\rfloor}| > \varepsilon 2^{n/2}\right) \leq M\delta^2.$$

*Proof.* Let $k_p = \lfloor 2^n p\delta \rfloor$ and $m_n = \lfloor 2^n \delta \rfloor$ for notational simplicity. Let $S_j := X^\alpha_{k_p+j} - X^\alpha_{k_p}$. We want to bound the probability of the event $B_{n,p} := \{\sup_{0\leq j\leq m_n} |S_j| > \varepsilon 2^{n/2}\}$. We apply Markov's inequality to the fourth power, which is a simple form of a maximal inequality.

$$\begin{aligned}\mathbb{P}(B_{n,p}) &= \mathbb{P}\left(\max_{0\leq j\leq m_n} |S_j|^4 > (\varepsilon 2^{n/2})^4\right) \\ &\leq \frac{\mathbb{E}\left[\max_{0\leq j\leq m_n} |S_j|^4\right]}{\varepsilon^4 2^{2n}}.\end{aligned}$$

By Doob's $L^p$ inequality for the non-negative submartingale $|S_j|$, we have $\mathbb{E}[\sup_{j\leq m_n} |S_j|^p] \leq (\frac{p}{p-1})^p \mathbb{E}[|S_{m_n}|^p]$.

$$\begin{aligned}\mathbb{P}(B_{n,p}) &\leq \left(\frac{4}{3}\right)^4 \frac{\mathbb{E}\left[(X^\alpha_{k_p+m_n} - X^\alpha_{k_p})^4\right]}{2^{2n}} \\ &\overset{\mathsf{def}}{=} \frac{M_1}{2^{2n}} \mathbb{E}\left[\mathbb{E}\left[(X^\alpha_{k_p+m_n} - X^\alpha_{k_p})^4 | \mathcal{F}_{k_p}\right]\right],\end{aligned}$$

where $M_1$ is a constant independent of $n$, $p$, and $\delta$. By the Markov property, the inner conditional expectation is $\mathbb{E}_{X^\alpha_{k_p}}[(X^\alpha_{m_n} - X^\alpha_{k_p})^4]$. Using the bound from Lemma 3.9, there exists constants $M_2$ and $M_3$, independent of $n$, $p$, and $\delta$ such that

$$\mathbb{P}(B_{n,p}) \leq \frac{M_2}{2^{2n}} m_n^2 \leq M_3\delta^2,$$

obtaining the desired result. □

**Lemma 3.11.** *Let $T > 0$ and $a \in \mathbb{N}_0$ be fixed. For all $\varepsilon > 0$, there exists a constant $L > 0$, independent of $\delta$, such that for all $\delta \in (0,1)$,*

$$\limsup_{n\to\infty} \mathbb{P}_a \left( \sup_{\substack{k,l\in\mathbb{N}_0 \\ k,l\le \lfloor 2^n T\rfloor \\ |k-l|\le\lfloor 2^n\delta\rfloor}} |X_k^\alpha - X_l^\alpha| > \varepsilon 2^{n/2} \right) \le L\delta. \tag{3.9}$$

*Proof.* Let $N_n = \lfloor 2^n T\rfloor$ and $m_n = \lfloor 2^n \delta\rfloor$. Let $A_n$ be the event inside the probability:

$$A_n := \left\{ \sup_{\substack{k,l\le N_n \\ |k-l|\le m_n}} |X_k^\alpha - X_l^\alpha| > \varepsilon 2^{n/2} \right\}. \tag{3.10}$$

We use a chaining argument. Let $k, l$ be a pair of indices realizing the supremum. We partition the time interval $[0, N_n]$ into $\lceil T/\delta\rceil$ blocks of size $m_n$. Let $p_k = \lfloor k/m_n\rfloor$ and $p_l = \lfloor l/m_n\rfloor$ be the block indices for $k$ and $l$. Since $|k-l| \le m_n$, it must be that $p_l = p_k$ or $p_l = p_k + 1$.

**Case 1:** $k$ and $l$ are in the same block ($p_k = p_l = p$). Then $pm_n \le k \le l \le (p+1)m_n$. By the triangle inequality,

$$|X_k^\alpha - X_l^\alpha| \le |X_k^\alpha - X_{pm_n}^\alpha| + |X_l^\alpha - X_{pm_n}^\alpha| \le 2 \sup_{j\in[pm_n,(p+1)m_n]} |X_j^\alpha - X_{pm_n}^\alpha|. \tag{3.11}$$

If $|X_k^\alpha - X_l^\alpha| > \varepsilon 2^{n/2}$, then $\displaystyle\sup_{j\in[pm_n,(p+1)m_n]} |X_j^\alpha - X_{pm_n}^\alpha| > \frac{\varepsilon}{2} 2^{n/2}$.

**Case 2:** $k$ and $l$ are in adjacent blocks ($p_l = p_k + 1 = p + 1$). Then $pm_n \le k \le (p+1)m_n \le l \le (p+2)m_n$. Again by the triangle inequality,

$$\begin{aligned}|X_k^\alpha - X_l^\alpha| &\le |X_k^\alpha - X_{(p+1)m_n}^\alpha| + |X_{(p+1)m_n}^\alpha - X_l^\alpha| \\ &\le \sup_{j\in[pm_n,(p+1)m_n]} |X_j^\alpha - X_{(p+1)m_n}^\alpha| + \sup_{j\in[(p+1)m_n,(p+2)m_n]} |X_j^\alpha - X_{(p+1)m_n}^\alpha|.\end{aligned}$$

If $|X_k^\alpha - X_l^\alpha| > \varepsilon 2^{n/2}$, then at least one of the two maximums on the right must be greater than $\frac{\varepsilon}{2} 2^{n/2}$.

Combining these cases, the event $A_n$ implies that for at least one block $p \in \{0, \dots, \lceil T/\delta\rceil - 1\}$, the maximum fluctuation within that block must be large. This leads to the inclusion of events:

$$A_n \subset \bigcup_{p=0}^{\lceil T/\delta\rceil - 1} \left\{ \sup_{0\le j\le m_n} |X_{pm_n+j}^\alpha - X_{pm_n}^\alpha| > \frac{\varepsilon}{2} 2^{n/2} \right\}. \tag{3.12}$$

Now we apply the union bound to the probabilities:

$$\mathbb{P}_a(A_n) \le \sum_{p=0}^{\lceil T/\delta\rceil - 1} \mathbb{P}_a \left( \sup_{0\le j\le m_n} |X_{pm_n+j}^\alpha - X_{pm_n}^\alpha| > \frac{\varepsilon}{2} 2^{n/2} \right).$$

We apply Lemma 3.10 with $\varepsilon' = \varepsilon/2$. For each term in the sum, we have:

$$\limsup_{n\to\infty} \mathbb{P}_a \left( \sup_{0\le j\le m_n} |X^\alpha_{pm_n+j} - X^\alpha_{pm_n}| > \frac{\varepsilon}{2} 2^{n/2} \right) \le M'\delta^2, \tag{3.13}$$

where $M'$ is another constant independent of $n$, $p$, and $\delta$. Since the bound is uniform in $p$, we can bound the sum:

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{P}_a(A_n) &\le \sum_{p=0}^{\lceil T/\delta\rceil -1} (M'\delta^2) \\ &\le \lceil T/\delta\rceil \cdot M'\delta^2 \\ &\le \left(\frac{T}{\delta}+1\right) M'\delta^2 = M'T\delta + M'\delta^2. \end{aligned}$$

For $\delta \in (0,1)$, this is bounded by $(M'T + M')\delta$. Letting $L := M'T + M'$ yields the desired result. ☐

**Proposition 3.12.** *The sequence of processes* $(\hat{X}^{\alpha,n})_{n\in\mathbb{N}}$ *is tight in* $C([0,T],\mathbb{R})$.

*Proof.* The first condition given by Equation (3.4) was shown previously. Now we have to tackle the second condition (3.5):

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}\left( w_{\hat{X}^{\alpha,n}}(\delta) > \varepsilon \right) = 0. \tag{3.14}$$

For any $s, t \in [0,T]$ with $|t-s| < \delta$, the increment of the interpolated process $\hat{X}^{\alpha,n}$ is bounded. Because the function is piecewise linear, its maximum fluctuation over an interval is controlled by the maximum fluctuation of the discrete points that define it in that interval. A simple calculation shows

$$w_{\hat{X}^{\alpha,n}}(\delta) = \sup_{|t-s|<\delta} |\hat{X}^{\alpha,n}_t - \hat{X}^{\alpha,n}_s| \le \frac{2}{2^{n/2}} \sup_{|k-j|\le\lceil 2^n\delta\rceil} |X^\alpha_k - X^\alpha_j|.$$

This inequality leads to the following inclusion of events:

$$\left\{ w_{\hat{X}^{\alpha,n}}(\delta) > \varepsilon \right\} \subset \left\{ \frac{2}{2^{n/2}} \sup_{|k-j|\le\lceil 2^n\delta\rceil} |X^\alpha_k - X^\alpha_j| > \varepsilon \right\}, \tag{3.15}$$

which is equivalent to

$$\left\{ w_{\hat{X}^{\alpha,n}}(\delta) > \varepsilon \right\} \subset \left\{ \sup_{|k-j|\le\lceil 2^n\delta\rceil,\, k,j\le\lfloor 2^n T\rfloor} |X^\alpha_k - X^\alpha_j| > \frac{\varepsilon}{2} 2^{n/2} \right\}. \tag{3.16}$$

We can now take the probability of both sides and apply Lemma 3.11 with $\varepsilon' = \varepsilon/2$.

$$\mathbb{P}\left( w_{\hat{X}^{\alpha,n}}(\delta) > \varepsilon \right) \le \mathbb{P}\left( \sup_{|k-j|\le\lceil 2^n\delta\rceil} |X^\alpha_k - X^\alpha_j| > \frac{\varepsilon}{2} 2^{n/2} \right).$$

Taking the $\limsup_{n\to\infty}$ of this inequality, we get:

$$\limsup_{n\to\infty} \mathbb{P}\left( w_{\hat{X}^{\alpha,n}}(\delta) > \varepsilon \right) \le L'\delta, \tag{3.17}$$

where $L'$ is the constant from Lemma 3.11 corresponding to $\varepsilon/2$. This constant $L'$ depends on $T$ and $\varepsilon$ but is independent of $\delta$. Finally, we take the limit as $\delta \to 0$:

$$\lim_{\delta\to 0}\left(\limsup_{n\to\infty}\mathbb{P}\left(w_{\hat{X}^{\alpha,n}}(\delta)>\varepsilon\right)\right)\leq \lim_{\delta\to 0} L'\delta = 0. \tag{3.18}$$

And hence Equation 3.5 is satisfied and tightness is established. □

3.4. **Proving finite-dimensional convergence.** Next we have to prove the finite-dimensional convergence:

$$(\hat{X}^{\alpha,n}_{t_1},\dots,\hat{X}^{\alpha,n}_{t_m}) \xrightarrow[n\to\infty]{\mathcal{L}} (R^\alpha_{t_1},\dots,R^\alpha_{t_m}),$$

for $t_1 \leq t_2 \leq \cdots \leq t_m$. To begin with we establish the convergence of the discrete transition semigroup to its continuous counterpart.

**Lemma 3.13.** *Let $0 \leq s < t$ be fixed time points and let $[a,b] \subset \mathbb{R}^*_+$ be a target interval. Define the continuous and discrete semigroup actions on the indicator function $\mathbb{1}_{[a,b]}$ as follows:*

- Continuous Semigroup Action: *Let $R^\alpha$ be a Bessel process of dimension $d = 2\alpha+2$. Define the function $G : \mathbb{R}_+ \to [0,1]$ by*
$$\Phi(x) := \mathbb{E}\left[\mathbb{1}_{[a,b]}(R^\alpha_t) \mid R^\alpha_s = x\right].$$
- Discrete Semigroup Action: *Let $\hat{X}^{\alpha,n}$ be the scaled ultraspherical Markov chain as defined before. Define the sequence of functions $G^n : \mathbb{R}_+ \to [0,1]$ by*
$$\Phi^n(x) := \mathbb{E}\left[\mathbb{1}_{[a,b]}(\hat{X}^{\alpha,n}_t) \mid \hat{X}^{\alpha,n}_s = x\right].$$

*Let $\mathcal{D}_n := \{\frac{k}{2^{n/2}} : k \in \mathbb{N}_0\}$. For any compact set $[p,q] \subset \mathbb{R}^*_+$ and any sequence of starting points $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \mathcal{D}_n \cap [p,q]$ for each $n \in \mathbb{N}^*$, we have*

$$|\Phi^n(x_n) - \Phi(x_n)| \xrightarrow[n\to\infty]{} 0.$$

Before proving this lemma, we state some results that will prove useful.

**Corollary 3.14.** [MOS13, p. 29] *For* $\mathrm{Re}(\nu) > -1$*, we have the following Weber-type integral identity:*

$$\int_0^\infty t e^{-p^2t^2} J_\nu(at)J_\nu(bt)\mathrm{d}t = \frac{1}{2p^2}e^{-\frac{a^2+b^2}{4p^2}} I_\nu\left(\frac{ab}{2p^2}\right).$$

**Theorem 3.15.** [Sze75, Theorem 7.32.2] *Let $\alpha$ and $\beta$ be arbitrary real numbers, and let $c$ be a fixed positive constant. Then, as $n \to \infty$,*

$$Q^\alpha_n(\cos\theta) = \begin{cases} \theta^{-\alpha-\frac{1}{2}}\mathcal{O}\left(n^{-\frac{1}{2}}\right) & \text{if } cn^{-1} \leq \theta \leq \pi/2,\\ \mathcal{O}(n^\alpha) & \text{if } 0 \leq \theta \leq cn^{-1}.\end{cases}$$

**Theorem 3.16** (Dominated Convergence Theorem)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $(z_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $\Omega \to \mathbb{R}$ such that:*

*(1) $z_n(x) \to z(x)$ pointwise for $\mu$-almost every $x \in \Omega$, where $z : \Omega \to \mathbb{R}$ is a measurable function.*

*(2) There exists a non-negative integrable function $g \in L^1(\mu)$ such that $|z_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and $\mu$-almost every $x \in \Omega$.*

*Then, $z \in L^1(\mu)$, and*

$$\lim_{n\to\infty} \int_\Omega z_n \mathrm{d}\mu = \int_\Omega z \mathrm{d}\mu.$$

*Equivalently,*

$$\lim_{n\to\infty} \int_\Omega |z_n - z| \mathrm{d}\mu = 0.$$

*Proof of Lemma 3.13.* The function $\Phi(x)$ is defined by integrating the transition density of the Bessel process from Proposition 2.15, $f^\alpha_{x,t-s}(u)$, from a starting point $x$ at time $s$ to a terminal point $u$ at time $t$.

$$\Phi(x) = \int_0^\infty \mathbb{1}_{[a,b]}(u) f^\alpha_{x,t-s}(u)\mathrm{d}u = \int_a^b \frac{u}{t-s}\left(\frac{u}{x}\right)^\alpha e^{-\frac{x^2+u^2}{2(t-s)}} I_\alpha\left(\frac{ux}{t-s}\right)\mathrm{d}u. \tag{3.19}$$

For $x_n = \dfrac{j}{2^{n/2}}$ with $j \in \mathbb{N}_0$, and setting $N = 2^n(t-s)$, the function $\Phi^n(x_n)$ is given by the spectral representation of the $N$-step transition probabilities of the underlying Markov chain:

$$\begin{aligned}\Phi^n(x_n) &= \sum_{k=0}^\infty \mathbb{1}_{[a,b]}\left(\frac{k}{2^{n/2}}\right)\omega_k^\alpha \int_{-1}^1 y^N Q_k^\alpha(y) Q_j^\alpha(y)\mathrm{d}\mu^\alpha(y) \\ &= \sum_{k=0}^\infty \mathbb{1}_{[a,b]}\left(\frac{k}{2^{n/2}}\right)\omega_k^\alpha \int_0^\pi (\cos\theta)^N Q_k^\alpha(\cos\theta) Q_j^\alpha(\cos\theta)(\sin\theta)^{2\alpha+1}\mathrm{d}\theta,\end{aligned}$$

where in the second line we performed the change of variable $y = \cos\theta$. Let us define the integrand function

$$\mathcal{I}_n^k(\theta) := (\cos\theta)^N(\sin\theta)^{2\alpha+1} Q_k^\alpha(\cos\theta) Q_j^\alpha(\cos\theta).$$

Consider the substitution $\theta \to \pi - \theta$. Using $\cos(\pi-\theta) = -\cos\theta$, $\sin(\pi-\theta) = \sin\theta$, and the property $Q_m^\alpha(-y) = (-1)^m Q_m^\alpha(y)$, we find:

$$\mathcal{I}_n^k(\pi-\theta) = (-1)^{N+k+j}\mathcal{I}_n^k(\theta).$$

The integral $\int_0^\pi \mathcal{I}_n^k(\theta)\mathrm{d}\theta$ is non-zero only if the exponent $N+k+j$ is even. In this case, the integral is equal to $2\int_0^{\pi/2} \mathcal{I}_n^k(\theta)\mathrm{d}\theta$. This allows us to rewrite $\Phi^n(x_n)$ as:

(3.20)

$$\Phi^n(x_n) = 2 \sum_{k+j+N\equiv 0 \pmod 2} \mathbb{1}_{[a,b]}\left(\frac{k}{2^{n/2}}\right)\omega_k^\alpha$$

$$\int_0^{\pi/2} (\cos\theta)^N Q_k^\alpha(\cos\theta) Q_j^\alpha(\cos\theta)(\sin\theta)^{2\alpha+1}\mathrm{d}\theta.$$

Now we define the following functions, for notational convenience:

(1) The function $h_k^n : \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$h_k^n(u) := \left[\sin\left(\frac{u}{\sqrt{N}}\right)\right]^{\alpha+\frac{1}{2}} Q_k^\alpha\left(\cos\left(\frac{u}{\sqrt{N}}\right)\right).$$

(2) The function $\mathcal{H}_n : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$\mathcal{H}_n(j,u) := 2 \sum_{k+j+N\equiv 0 \pmod 2} \mathbb{1}_{[a,b]}\left(\frac{k}{2^{n/2}}\right) \frac{\omega_k^\alpha u^{\alpha+\frac{1}{2}}}{N^{\frac{\alpha}{2}+\frac{1}{4}}} h_k^n(u),$$

where $j = 2^{n/2}x$.

Making the change of variable $\theta = u/\sqrt{N}$ in Equation (3.20), we obtain the transformed expression for $\Phi^n(x_n)$:

$$\begin{aligned}\Phi^n(x_n) = \int_0^{\sqrt{N}\pi/2} &\left[\cos\left(\frac{u}{\sqrt{N}}\right)\right]^N Q_j^\alpha\left(\cos\left(\frac{u}{\sqrt{N}}\right)\right) \\ &\left[\frac{\sin\left(\frac{u}{\sqrt{N}}\right)}{\sqrt{N}}\right]^{\alpha+\frac{1}{2}} \mathcal{H}_n(x_n,u)\frac{\mathrm{d}u}{u^{\alpha+\frac{1}{2}}}.\end{aligned} \tag{3.21}$$

Let us define:

$$\begin{aligned}\phi_n(x_n,u) := \mathbb{1}_{[0,2^{\frac{n}{2}}(t-s)^{\frac{1}{2}}\frac{\pi}{2}]} &\left[\cos\left(\frac{u}{\sqrt{N}}\right)\right]^N Q_j^\alpha\left(\cos\left(\frac{u}{\sqrt{N}}\right)\right) \\ &\left[\frac{\sin\left(\frac{u}{\sqrt{N}}\right)}{\sqrt{N}}\right]^{\alpha+\frac{1}{2}} \mathcal{H}_n(x_n,u)u^{-\alpha-\frac{1}{2}}\end{aligned}$$

Recalling Corollary 2.4, we have

$$\left|Q^\alpha_{2^{\frac{n}{2}}x_n}\left(\cos\frac{u}{2^{\frac{n}{2}}(t-s)^{\frac{1}{2}}}\right) - \Gamma(\alpha+1)\left(\frac{ux_n}{2(t-s)^{\frac{1}{2}}}\right)^{-\alpha} J_\alpha\left(\frac{ux_n}{(t-s)^{\frac{1}{2}}}\right)\right| \xrightarrow[n\to\infty]{} 0. \tag{3.22}$$

One may notice that $\mathcal{H}_n(j,u)$ depends on the parity of $j$ as follows:

$$\begin{cases}\mathcal{H}_n(j,u) = \mathcal{H}_n(0,u) & \text{if } j \text{ is even}\\ \mathcal{H}_n(j,u) = \mathcal{H}_n(1,u) & \text{if } j \text{ is odd}\end{cases}$$

However, a direct calculation shows that

$$\lim_{n\to\infty}\mathcal{H}_n(x,u) = \frac{u^{\alpha+1}}{2^\alpha\Gamma(\alpha+1)}\int_a^b \left(\frac{\theta}{(t-s)^{\frac{1}{2}}}\right)^{\alpha+1} J_\alpha\left(\frac{\theta u}{(t-s)^{\frac{1}{2}}}\right)\frac{\mathrm{d}\theta}{(t-s)^{\frac{1}{2}}}, \tag{3.23}$$

which means the convergence of $\mathcal{H}_n$ does not depend on the parity of $j$. [1] Furthermore, from Equations (3.22), (3.23), and the fact that

$$\lim_{n\to\infty}\left(\cos\frac{u}{\sqrt{N}}\right)^N\left[\frac{\sin\frac{u}{\sqrt{N}}}{\sqrt{N}}\right]^{\alpha+\frac{1}{2}}=e^{-\frac{u^2}{2}},$$

we have the convergence:

$$\lim_{n\to\infty}|\phi_n(x_n,u)-\phi(x_n,u)|=0, \tag{3.24}$$

where $\phi(x_n,u)$ is defined to be

$$\phi(x_n,u):=e^{-u^2/2}\left[\Gamma(\alpha+1)\left(\frac{ux_n}{2\sqrt{t-s}}\right)^{-\alpha}J_\alpha\left(\frac{ux_n}{\sqrt{t-s}}\right)\right] \left[\frac{\mathcal{H}(u)}{u^{\alpha+1/2}}\left(\frac{u}{\sqrt{t-s}}\right)^{\alpha+1/2}\right].$$

In order to use the dominated convergence theorem (Theorem 3.16), we must find an integrable dominating function for $|\phi_n(x_n,u)|$. Since if $p<0$, as $u\to 0$, the term $(\sin u)^p$ is no longer bounded, we have the equivalence:

$$\mathcal{H}_n(j=0,u)\sim\mathcal{H}_n(j=1,u)=\begin{cases}\mathcal{O}\left(u^{\alpha+\frac{1}{2}}\right) & \text{if } \alpha\geq-\frac{1}{2},\\ \mathcal{O}(u^{2\alpha+1}) & \text{if } \alpha<-\frac{1}{2}.\end{cases} \tag{3.25}$$

Recall Theorem 3.15. With $x_n$ varying in a compact set $[p,q]$, we there exists a constant $c$ such that

$$Q^\alpha_{2^{\frac{n}{2}}x_n}\left(\cos\frac{u}{2^{\frac{n}{2}}(t-s)^{\frac{1}{2}}}\right)=\begin{cases}\mathcal{O}(1) & \text{if } 0\leq u\leq c,\\ \mathcal{O}\left(\frac{u}{2^{\frac{n}{2}}(t-s)^{\frac{1}{2}}}\right)^{-\alpha-\frac{1}{2}} & \text{if } c<u\leq 2^{\frac{n}{2}}(t-s)^{\frac{1}{2}}\frac{\pi}{2}.\end{cases} \tag{3.26}$$

From Equations (3.25) and (3.26), we know that there exist $A_1>0$ and $A_2>0$ such that

$$\phi_n(x_n,u)=\mathcal{O}(u^{A_1}e^{-A_2\frac{u^2}{2}}). \tag{3.27}$$

Hence, the dominated convergence theorem (Theorem 3.16) applies, which gives

$$\lim_{n\to\infty}\int_0^{+\infty}|\phi_n(x_n,u)-\phi(x_n,u)|\mathrm{d}u=0. \tag{3.28}$$

From this convergence (Equation 3.28) we immediately deduce

[1] For Equation 3.23, readers are directed to [Geo75, Equation (3.10), Chapter II] and the relevant preceding workings; the concepts being used are similar so we omitted the details.

$$\lim_{n\to\infty}\left|\Phi^n(x_n)-\int_0^{+\infty}\Gamma(\alpha+1)\left(\frac{ux_n}{2(t-s)^{\frac{1}{2}}}\right)^{-\alpha}J_\alpha\left(\frac{ux_n}{(t-s)^{\frac{1}{2}}}\right)e^{-\frac{u^2}{2}}\right.$$
$$\left.\frac{u^{\alpha+1}}{2^\alpha\Gamma(\alpha+1)}\int_a^b\left(\frac{\theta u}{(t-s)^{\frac{1}{2}}}\right)^{\alpha+1}J_\alpha\left(\frac{u}{(t-s)^{\frac{1}{2}}}\right)\frac{\mathrm{d}\theta}{(t-s)^{\frac{1}{2}}}\mathrm{d}u\right|=0. \tag{3.29}$$

$$\lim_{n\to\infty}\left|\Phi^n(x_n)-\int_a^b\left(\frac{x_n}{(t-s)^{\frac{1}{2}}}\right)^{-\alpha}\left(\frac{\theta}{(t-s)^{\frac{1}{2}}}\right)^{\alpha+1}\frac{\mathrm{d}\theta}{(t-s)^{\frac{1}{2}}}\right.$$
$$\left.\int_0^\infty ue^{-\frac{u^2}{2}}J_\alpha\left(\frac{x_nu}{(t-s)^{\frac{1}{2}}}\right)J_\alpha\left(\frac{\theta u}{(t-s)^{\frac{1}{2}}}\right)\mathrm{d}u\right|=0. \tag{3.30}$$

Using Corollary 3.14 with $p=\dfrac{1}{\sqrt{2}}$, $a=\dfrac{x_n}{(t-s)^{\frac{1}{2}}}$, $b=\dfrac{u}{(t-s)^{\frac{1}{2}}}$, $\nu=\alpha$, equation 3.30 becomes

$$\lim_{n\to\infty}\left|\Phi^n(x_n)-\int_0^\infty\left(\frac{x_n}{(t-s)^{\frac{1}{2}}}\right)^{-\alpha}\left(\frac{u}{(t-s)^{\frac{1}{2}}}\right)^{\alpha+1}e^{-\frac{x^2+u^2}{2(t-s)}}I_\alpha\left(\frac{ux_n}{t-s}\right)\frac{\mathrm{d}u}{(t-s)^{\frac{1}{2}}}\right|=0; \tag{3.31}$$

or equivalently

$$\lim_{n\to\infty}\left|\Phi^n(x_n)-\int_0^{+\infty}\mathbb{1}_{[a,b]}(x_n)\frac{u}{t-s}\left(\frac{u}{x_n}\right)^\alpha e^{-\frac{x^2+u^2}{2(t-s)}}I_\alpha\left(\frac{ux_n}{t-s}\right)\mathrm{d}u\right|=0, \tag{3.32}$$

which corresponds exactly to the desired result. ☐

**Lemma 3.17.** *For all $m\in\mathbb{N}$, and for any $t_1,\ldots,t_m\in\mathbb{R}^d$, we have:*

$$(\hat{X}^{\alpha,n}_{t_1},\ldots,\hat{X}^{\alpha,n}_{t_m})\xrightarrow[n\to\infty]{\mathcal{L}}(R^\alpha_{t_1},\ldots,R^\alpha_{t_m}).$$

*Proof.* We can do the proof by induction on $m$. For $m=1$, we simply need to prove that for all $t\geq 0$,

$$\hat{X}^{\alpha,n}_t\xrightarrow[n\to\infty]{\mathcal{L}}R^\alpha_t. \tag{3.33}$$

As we have mentioned before, from [Geo75, p II.19], we know that

$$\frac{X^\alpha_n}{n^{1/2}}\xrightarrow[n\to\infty]{\mathcal{L}}R^\alpha_1.$$

For dyadic $t$ (i.e., of the form $t=k/2^m$ for integers $k$ and $m$), we directly have

$$\hat{X}^{\alpha,n}_t=\frac{X^\alpha_{2^nt}}{2^{n/2}}\xrightarrow[n\to\infty]{\mathcal{L}}R^\alpha_t.$$

Moving on to the inductive steps, we assume for any $t_1 < \cdots < t_m$, $(\hat{X}^{\alpha,n}_{t_1}, \ldots, \hat{X}^{\alpha,n}_{t_m}) \xrightarrow{\mathcal{L}} (R^{\alpha}_{t_1}, \ldots, R^{\alpha}_{t_m})$. We want to prove this for $t_1 < \cdots < t_m < t_{m+1}$. By the Portmanteau Theorem, it is sufficient to show that for any collection of bounded, continuous functions $h_1, \ldots, h_{m+1}$,

$$\mathbb{E}[h_1(\hat{X}^{\alpha,n}_{t_1}) \cdots h_{m+1}(\hat{X}^{\alpha,n}_{t_{m+1}})] \to \mathbb{E}[h_1(R^{\alpha}_{t_1}) \cdots h_{m+1}(R^{\alpha}_{t_{m+1}})].$$

Using the law of total expectation we have

$$\begin{aligned} \mathbb{E}\left[\prod_{j=1}^{m+1} h_j(\hat{X}^{\alpha,n}_{t_j})\right] &= \mathbb{E}\left[\left(\prod_{j=1}^{m} h_j(\hat{X}^{\alpha,n}_{t_j})\right) \mathbb{E}\left[h_{m+1}(\hat{X}^{\alpha,n}_{t_{m+1}}) \mid \hat{X}^{\alpha,n}_{t_1}, \ldots, \hat{X}^{\alpha,n}_{t_m}\right]\right] \\ &= \mathbb{E}\left[\left(\prod_{j=1}^{m} h_j(\hat{X}^{\alpha,n}_{t_j})\right) \mathbb{E}\left[h_{m+1}(\hat{X}^{\alpha,n}_{t_{m+1}}) \mid \hat{X}^{\alpha,n}_{t_m}\right]\right], \end{aligned}$$

where the second equality follows from the Markov property of the process $\hat{X}^{\alpha,n}$. Let us define the functions:

$$\begin{aligned} \Phi^n_m(x) &:= \mathbb{E}[h_{m+1}(\hat{X}^{\alpha,n}_{t_{m+1}}) \mid \hat{X}^{\alpha,n}_{t_m} = x] \\ \Phi_m(x) &:= \mathbb{E}[h_{m+1}(R^{\alpha}_{t_{m+1}}) \mid R^{\alpha}_{t_m} = x] \end{aligned}$$

By approximating the continuous function $h_{m+1}$ by simple functions (linear combinations of indicators), Lemma 3.13 can be extended to show that $\Phi^n_m(x) \to \Phi_m(x)$ uniformly on compact sets. The expectation becomes

$$\mathbb{E}\left[\left(\prod_{j=1}^{m} h_j(\hat{X}^{\alpha,n}_{t_j})\right) \Phi^n_m(\hat{X}^{\alpha,n}_{t_m})\right].$$

We want to show this converges to $\mathbb{E}\left[\left(\prod_{j=1}^{m} h_j(R^{\alpha}_{t_j})\right) \Phi_m(R^{\alpha}_{t_m})\right]$. We have:

$$\begin{aligned} &\left|\mathbb{E}\left[\left(\prod_{j=1}^{m} h_j(\hat{X}^{\alpha,n}_{t_j})\right) \Phi^n_m(\hat{X}^{\alpha,n}_{t_m})\right] - \mathbb{E}\left[\left(\prod_{j=1}^{m} h_j(R^{\alpha}_{t_j})\right) \Phi_m(R^{\alpha}_{t_m})\right]\right| \\ &\le \left|\mathbb{E}\left[\left(\prod_{j=1}^{m} h_j(\hat{X}^{\alpha,n}_{t_j})\right) \left(\Phi^n_m(\hat{X}^{\alpha,n}_{t_m}) - \Phi_m(\hat{X}^{\alpha,n}_{t_m})\right)\right]\right| \\ &\quad + \left|\mathbb{E}\left[\left(\prod_{j=1}^{m} h_j(\hat{X}^{\alpha,n}_{t_j})\right) \Phi_m(\hat{X}^{\alpha,n}_{t_m})\right] - \mathbb{E}\left[\left(\prod_{j=1}^{m} h_j(R^{\alpha}_{t_j})\right) \Phi_m(R^{\alpha}_{t_m})\right]\right|. \end{aligned}$$

The first term vanishes as $n \to \infty$. Since the functions $h_j$ are bounded, the product is bounded. The term $|\Phi^n_m - \Phi_m|$ converges to zero uniformly on compacts, and since the law of $\hat{X}^{\alpha,n}_{t_m}$ is tight (it converges in distribution), the expectation goes to zero.

The second term involves the expectation of a bounded, continuous function of $(\hat{X}^{\alpha,n}_{t_1},\ldots,\hat{X}^{\alpha,n}_{t_m})$, namely $F(z_1,\ldots,z_m)=(\prod_{j=1}^{m} h_j(z_j))\Phi_m(z_m)$. By the induction hypothesis, the joint law of $(\hat{X}^{\alpha,n}_{t_1},\ldots,\hat{X}^{\alpha,n}_{t_m})$ converges to the law of $(R^{\alpha}_{t_1},\ldots,R^{\alpha}_{t_m})$. Therefore, the expectation converges:

$$\mathbb{E}[F(\hat{X}^{\alpha,n}_{t_1},\ldots,\hat{X}^{\alpha,n}_{t_m})]\to\mathbb{E}[F(R^{\alpha}_{t_1},\ldots,R^{\alpha}_{t_m})].$$

Both terms in the inequality go to zero, which completes the inductive step. □

This section's main theorem can hence be proved:

*Proof of Theorem 3.10.* The proof is done in two steps:

- It was shown (see Lemma 3.17) that finite-dimensional convergence holds:

$$\left(\hat{X}^{\alpha,n}_{t_1},\ldots,\hat{X}^{\alpha,n}_{t_m}\right)\quad\text{converges in law to}\quad\left(R^{\alpha,a}_{t_1},\ldots,R^{\alpha,a}_{t_m}\right)\quad\text{for}\quad t_1\le t_2\le\ldots\le t_m.$$

- It was established that the sequence $\hat{X}^{\alpha,n}_t$ is tight in the space of continuous functions defined on $\mathbb{R}_+$. (See Proprosition 3.12.)

□

**Remark 3.18.** *One may wonder if the theorem can be generalized to cases where the Bessel processes $R^{\alpha,a}$ start at any point, say $a$, rather than just $0$. However, to do this, it is not enough to simply renormalize the ultraspherical chain $X^{\alpha}$, starting from any $a$. It is necessary to consider a new chain $X^{\alpha,n}_a$ such that $X^{\alpha,n}_a=2^{n/2}a$. In fact, we can define a new sequence $\hat{X}^{\alpha,n}_{t,\,\text{new}}$:*

(3.34)

$$\hat{X}^{\alpha,n}_{t,\,\text{new}}=\begin{cases}\dfrac{X^{\alpha,n}_{2^n t}}{2^{\frac{n}{2}}}, & \text{if } t=\dfrac{k}{2^n},\\[2ex] \dfrac{X^{\alpha,n}_{\lfloor 2^n t\rfloor}}{2^{\frac{n}{2}}}+\dfrac{2^n t-\lfloor 2^n t\rfloor}{2^{\frac{n}{2}}}\left(X^{\alpha,n}_{\lfloor 2^n t\rfloor+1}-X^{\alpha,n}_{\lfloor 2^n t\rfloor}\right), & \text{if } t\in\left(\dfrac{k}{2^n},\dfrac{k+1}{2^n}\right).\end{cases}$$

*Using the same approach employed in this section, we can prove the the sequence $\hat{Y}^{\alpha,n}_t$ converges in law to $R^{\alpha,a}_t$, a Bessel process of dimension $2\alpha+2$ starting from $R^{\alpha,a}_0=a$.*

## 4. First Passage Times and their Convergence

We now turn our attention to the hitting times and the exit times for ultraspherical chains, before passing to the limit for the Bessel process. We consider the Markov chain associated with ultraspherical polynomials and focus on the hitting time of a point:

$$T^{\alpha}_b=\inf\{n\in\mathbb{N},X^{\alpha}_n=b\}.$$

We are also interested in the exit time from an interval. For $X^{\alpha}_0\in[c,b]$, define the following:

$$T^{\alpha}_{b,c}=\inf\{n\in\mathbb{N},X^{\alpha}_n=b\text{ or }X^{\alpha}_n=c\}.$$

4.1. **Limit Theorems on Hitting Times.** We observed in Section 3 the convergence in law of the rescaled Markov chain $\hat{X}^{\alpha,n}$ towards the Bessel process of dimension $d = 2\alpha + 2$ starting from 0. From this result, it is easy to deduce that the hitting time of a point $b$ by the chain $X^\alpha$ satisfies

$$\lim_{n\to+\infty} \frac{T^\alpha_{n^2 b}}{n^2} = \tilde{T}_1, \tag{4.1}$$

where $\tilde{T}_1$ is the hitting time to point 1 of the Bessel process of dimension $2\alpha + 2$ starting from 0 (in other words, $\tilde{T}_1 = \inf\{t \geq 0 : R^{\alpha,0}_t = 1\}$); the convergence is in law. To begin with, we start by studying some general properties that will be useful for what proceeds.

**Proposition 4.1.** *Let $(X^\alpha_n)$ be an ultraspherical chain starting from $X^\alpha_0 = a$. Let $T^\alpha_b$ be the first hitting time of $b$ for this chain. Then,*

$$\mathbb{P}_a(T^\alpha_b < +\infty) = 1, \quad \text{for } \alpha \leq 0. \tag{4.2}$$

*For $\alpha > 0$:*

$$\begin{cases} \mathbb{P}_a(T^\alpha_b < +\infty) = 1, & \text{if } b \geq a, \\ \mathbb{P}_a(T^\alpha_b < +\infty) < 1, & \text{if } b < a. \end{cases} \tag{4.3}$$

*Proof.* Equation (4.2) follows directly from the recurrence of the chain for $\alpha \leq 0$. Now, assume $\alpha > 0$. The chain $X^\alpha_n$ is transient. For $a = b$, it is clear that

$$\mathbb{P}_a(T^\alpha_a < +\infty) = 1.$$

Now, for $b > a$, if $T^\alpha_b \to +\infty$, the chain $X^\alpha_n$ can only take a finite number of values $\{0, \ldots, b-1\}$. But this contradicts the transience of the chain. Thus, we deduce the assertion (i) of (4.3). Finally, for $b < a$, suppose

$$\mathbb{P}_a(T^\alpha_b < +\infty) = 1.$$

This means that starting from $a$, the chain reaches $b$ with almost surely. But we have just shown (first assertion of (4.3)) that starting from $b$, the chain reaches $a$ *almost surely*. Using the Markov properties of $X^\alpha_n$, it is clear that the chain alternates between $a$ and $b$ infinitely often, which contradicts its transience. Hence, we deduce that

$$\mathbb{P}_a(T^\alpha_b < +\infty) < 1.$$

□

**Lemma 4.2.** *The functions $Q^\alpha_n(e^\lambda)$ and $U^\alpha_n(n,0)$ are eigenfunctions of $P^\alpha$, with the eigenvalue $e^\lambda$, where $U^\alpha_\lambda(n,m) = \sum_{N\geq 0} e^{-\lambda N}(P^\alpha)^N(n,m)$ as defined before. Specifically, we have*

$$\forall n \in \mathbb{N}, \quad P^\alpha Q^\alpha_n(e^\lambda) = e^\lambda Q^\alpha_n(e^\lambda),$$
$$\forall n \in \mathbb{N}^*, \quad P^\alpha U^\alpha_n(n,0) = e^\lambda U^\alpha_n(n,0).$$

*Proof.* Let $n \in \mathbb{N}^*$. By the definition of $P^\alpha$, we have

$$P^\alpha Q_n^\alpha(e^\lambda) = \frac{n}{2n+2\alpha+1} Q_{n-1}^\alpha(e^\lambda) + \frac{n+2\alpha+1}{2n+2\alpha+1} Q_{n+1}^\alpha(e^\lambda).$$

Using the recursive property of ultraspherical polynomials:

$$xQ_n^\alpha(x) = \frac{n}{2n+2\alpha+1} Q_{n-1}^\alpha(x) + \frac{n+2\alpha+1}{2n+2\alpha+1} Q_{n+1}^\alpha(x),$$

we can obtain

$$P^\alpha Q_n^\alpha(e^\lambda) = e^\lambda Q_n^\alpha(e^\lambda).$$

We can trivially verify this for $n = 0$ similarly. For $U_n^\alpha(n, 0)$, we obtain

$$\begin{aligned} P^\alpha U_n^\alpha(n,0) &= \sum_{N \geq 0} e^{-\lambda N} \Big( \frac{n}{2n+2\alpha+1} [P^\alpha]^N(n-1,0) + \frac{n+2\alpha+1}{2n+2\alpha+1} [P^\alpha]^N(n+1,0) \Big) \\ &= \sum_{N \geq 0} e^{-\lambda N} [P^\alpha]^{N+1}(n,0) \\ &= e^\lambda \sum_{N \geq 1} e^{-\lambda N} [P^\alpha]^N(n,0) \end{aligned}$$

We know that for all $n \geq 1$, $[P^\alpha]^0(n, 0) = 0$, and thus we have

$$P^\alpha g_\lambda^\alpha(n) = e^\lambda \sum_{N \geq 0} e^{-\lambda N} [P^\alpha]^N(n,0) = e^\lambda U_n^\alpha(n,0), \quad \forall n \in \mathbb{N}^*.$$

□

**Remark 4.3.** *The property $P^\alpha U_n^\alpha(n, 0) = e^\lambda U_n^\alpha(n, 0)$ is* ***false*** *for $n = 0$, due to boundary conditions. Instead, we have*

$$P^\alpha U_0^\alpha(n,0) = e^\lambda \sum_{N \geq 1} e^{-\lambda N} [P^\alpha]^N(1,0).$$

**Proposition 4.4.** *Let $T_b^\alpha = \inf\{p \in \mathbb{N} : X_p^\alpha = b\}$ be the first passage time to state $b$ for the ultraspherical chain starting from $X_0^\alpha = a$. The moment-generating function of this random time is characterized as follows:*

- ***(i) (Upward Passage,** $a < b$**).*** *When moving away from the origin, the transform is given by the ratio of the eigenfunctions $Q_k^\alpha(e^\lambda)$ of the transition operator:*

$$\mathbb{E}_a\left(e^{-\lambda T_b^\alpha}\right) = \frac{Q_a^\alpha(e^\lambda)}{Q_b^\alpha(e^\lambda)}. \tag{4.4}$$

- ***(ii) (Downward Passage,** $a > b$**).*** *When moving towards the origin, the transform is given by the ratio of the values of the $\lambda$-harmonic function $U_\lambda^\alpha(k, 0)$, which represents the resolvent:*

$$\mathbb{E}_a\left(e^{-\lambda T_b^\alpha}\right) = \frac{U_\lambda^\alpha(a,0)}{U_\lambda^\alpha(b,0)}. \tag{4.5}$$

*Proof.* The proof for both cases relies on the construction of a suitable martingale and the application of the Optional Stopping Theorem. The necessity for two distinct mathematical approaches is dictated by the boundedness requirement of the Optional Stopping Theorem, which is central to the proof. For an upward passage ($a < b$), the process is stopped upon reaching state $b$, naturally confining the chain to the finite set of states $\{0, 1, \ldots, b\}$. Within this finite domain, a martingale constructed from the polynomial eigenfunctions, $Q_k^\alpha(e^\lambda)$, is bounded, allowing the theorem to be applied directly. Conversely, for a downward passage ($a > b$), the chain is free to reach arbitrarily high states before eventually hitting $b$. Since the polynomials $Q_k^\alpha(e^\lambda)$ grow with $k$, this would create an unbounded martingale, violating the theorem's conditions. Therefore, this case necessitates the use of a different function, the resolvent or $\lambda$-potential $U_\lambda^\alpha(k, 0)$, which is inherently bounded for all states $k \in \mathbb{N}$.

**Case (i): Upward Passage ($a < b$).** Let us define a function $h_\lambda(k) \coloneqq Q_k^\alpha(e^\lambda)$. By its definition, this function is an eigenfunction of the transition operator $P^\alpha$ with eigenvalue $e^\lambda$. That is,

$$(P^\alpha h_\lambda)(k) = \mathbb{E}_k[h_\lambda(X_1^\alpha)] = e^\lambda h_\lambda(k).$$

Using this property, we define the process $\mathcal{M}_p^{(\lambda)} \coloneqq e^{-\lambda p} h_\lambda(X_p^\alpha)$. We show this is a martingale with respect to the natural filtration $\mathcal{F}_p$:

$$\begin{aligned}
\mathbb{E}_a[\mathcal{M}_{p+1}^{(\lambda)} \mid \mathcal{F}_p] &= \mathbb{E}_a[e^{-\lambda(p+1)} h_\lambda(X_{p+1}^\alpha) \mid X_p^\alpha] \\
&= e^{-\lambda(p+1)} \mathbb{E}_{X_p^\alpha}[h_\lambda(X_1^\alpha)] \\
&= e^{-\lambda(p+1)} \cdot e^\lambda h_\lambda(X_p^\alpha) \\
&= e^{-\lambda p} h_\lambda(X_p^\alpha) = \mathcal{M}_p^{(\lambda)}.
\end{aligned}$$

The process $\mathcal{M}_p^{(\lambda)}$ is bounded for all times $p \leq T_b^\alpha$, since the state space is restricted to $\{0, 1, \ldots, b\}$ and $h_\lambda(k)$ is finite for each $k$. Indeed,

$$\begin{aligned}
\sup_{n < T_b^\sigma} |\mathcal{M}_p^{(\lambda)}| &\leq \sup_{0 \leq X_n^\alpha \leq b} |Q_{X_n^\alpha}^\alpha(e^\lambda)| \\
&\leq \sup_{k \in \{0, \ldots, b\}} |Q_k^\alpha(e^\lambda)| < +\infty.
\end{aligned}$$

By the Optional Stopping Theorem, the expectation of the martingale at the stopping time $T_b^\alpha$ is equal to its initial value.

$$\mathbb{E}_a[\mathcal{M}_{T_b^\alpha}^{(\lambda)}] = \mathcal{M}_0^{(\lambda)}.$$

Substituting the definition of $\mathcal{M}_p^{(\lambda)}$ yields

$$\mathbb{E}_a[e^{-\lambda T_b^\alpha} h_\lambda(X_{T_b^\alpha}^\alpha)] = e^{-\lambda \cdot 0} h_\lambda(X_0^\alpha).$$

Since $X_{T_b^\alpha}^\alpha = b$ and $X_0^\alpha = a$, this becomes

$$\mathbb{E}_a[e^{-\lambda T_b^\alpha} Q_b^\alpha(e^\lambda)] = Q_a^\alpha(e^\lambda).$$

Rearranging the terms gives the desired result (4.4).

**Case (ii): Downward Passage ($a > b$).** Here we define a function $g_\lambda(k) := U_\lambda^\alpha(k,0)$, where $U_\lambda^\alpha$ is the resolvent or $\lambda$-potential of the chain. This function is $\lambda$-harmonic, meaning it satisfies the relation:

$$\mathbb{E}_k[e^{-\lambda} g_\lambda(X_1^\alpha)] = g_\lambda(k).$$

We now define the process $\mathcal{N}_p^{(\lambda)} := e^{-\lambda p} g_\lambda(X_p^\alpha)$. This process is a martingale:

$$\begin{aligned}\mathbb{E}_a[\mathcal{N}_{p+1}^{(\lambda)} \mid \mathcal{F}_p] &= \mathbb{E}_a[e^{-\lambda(p+1)} g_\lambda(X_{p+1}^\alpha) \mid X_p^\alpha] \\ &= e^{-\lambda p} \cdot \mathbb{E}_{X_p^\alpha}[e^{-\lambda} g_\lambda(X_1^\alpha)] \\ &= e^{-\lambda p} g_\lambda(X_p^\alpha) = \mathcal{N}_p^{(\lambda)}.\end{aligned}$$

The stopping time $T_b^\alpha$ is bounded by the hitting time to the origin, $T_0^\alpha$. The process $\mathcal{N}_p^{(\lambda)}$ is bounded for $p \le T_b^\alpha$ because the state space is restricted to $\{b, b+1, \dots, a\}$. Applying the Optional Stopping Theorem,

$$\mathbb{E}_a[\mathcal{N}_{T_b^\alpha}^{(\lambda)}] = \mathcal{N}_0^{(\lambda)}.$$

Substituting the definition of $\mathcal{N}_p^{(\lambda)}$ gives

$$\mathbb{E}_a[e^{-\lambda T_b^\alpha} g_\lambda(X_{T_b^\alpha}^\alpha)] = g_\lambda(X_0^\alpha).$$

With $X_{T_b^\alpha}^\alpha = b$ and $X_0^\alpha = a$, we have

$$\mathbb{E}_a[e^{-\lambda T_b^\alpha} U_\lambda^\alpha(b,0)] = U_\lambda^\alpha(a,0).$$

Solving for the expectation yields the result (4.5), completing the proof. □

**Notation 4.5.** *In what follows, $R^{\alpha,a}$ denotes a Bessel process starting from $R_0^{\alpha,a} = a$. We denote by $\tilde{T}_b^{\alpha,a}$ the hitting time of $b$ by the process $R^{\alpha,a}$:*

$$\tilde{T}_b^{\alpha,a} = \inf\{t \mid R_t^{\alpha,a} = b\}.$$

**Proposition 4.6.** [PY81, Proposition 2.3] *The Laplace transform of $\tilde{T}_b^{\alpha,a}$ for the Bessel process is as follows: For $a \neq 0$,*

$$\forall \lambda > 0, \quad \mathbb{E}(e^{-\frac{1}{2}\lambda^2 \tilde{T}_b^{\alpha,a}}) = \begin{cases} \left(\dfrac{b}{a}\right)^\alpha \dfrac{I_\alpha(a\lambda)}{I_\alpha(b\lambda)} & \text{if } a < b, \\ \left(\dfrac{b}{a}\right)^\alpha \dfrac{K_\alpha(a\lambda)}{K_\alpha(b\lambda)} & \text{if } a > b. \end{cases} \tag{4.6}$$

*For $a = 0$,*

$$\forall \lambda > 0, \quad \mathbb{E}(e^{-\frac{1}{2}\lambda^2 \tilde{T}_b^{\alpha,0}}) = \frac{b^\alpha}{\Gamma(\alpha+1)} \left(\frac{\lambda}{2}\right)^\alpha \frac{1}{I_\alpha(b\lambda)}. \tag{4.7}$$

We now establish the convergence in law of the hitting times for the ultraspherical chain. The core of the argument lies in demonstrating that the Laplace transforms of the appropriately rescaled discrete hitting times converge to the known Laplace transforms of the hitting times for the limiting Bessel process, as given in Proposition 4.6. Let us define the diffusively rescaled hitting time for the ultraspherical chain as the random variable $\mathcal{T}_n$.

**Proposition 4.7** (Convergence from a Fixed Interior Point)**.** *Consider the ultraspherical chain $(X_p^\alpha)_{p\geq 0}$ originating from a fixed integer state $X_0^\alpha = a > 0$. Let the target level be scaled diffusively as $k_n = \lfloor 2^n b \rfloor$ for some dyadic number $b > 0$. The corresponding rescaled hitting time, $\mathcal{T}_n^{\text{fixed}}(b) := T_{k_n}^\alpha / 4^n$, converges in law to the first passage time $\tilde{T}_b^{\alpha,0}$ for a Bessel process starting from the origin.*

$$\mathcal{T}_n^{\text{fixed}}(b) \xrightarrow[n\to\infty]{\mathcal{L}} \tilde{T}_b^{\alpha,0}.$$

*Proof.* For large $n$, the fixed starting point $a$ is less than the scaled target $k_n = \lfloor 2^n b \rfloor$. From Proposition 4.4, the Laplace transform of the discrete hitting time is given by the ratio of ultraspherical polynomials as follows:

$$\mathbb{E}_a\left[e^{-\nu_n T_{k_n}^\alpha}\right] = \frac{Q_a^\alpha(e^{\nu_n})}{Q_{k_n}^\alpha(e^{\nu_n})}.$$

As $n \to \infty$, we have $\nu_n \to 0$, and the numerator converges to $Q_a^\alpha(1) = 1$. For the denominator, we analyze its asymptotic behavior. Let $z_n^2 = -2k_n^2(e^{\nu_n} - 1)$. As $n \to \infty$, $e^{\nu_n} \approx 1 + \nu_n$, so $z_n^2 \approx -2(2^n b)^2(\lambda^2/(2 \cdot 4^n)) = -b^2\lambda^2$. This gives $z_n \to ib\lambda$. The argument of the polynomial becomes $e^{\nu_n} \approx 1 - z_n^2/(2k_n^2)$. Applying the asymptotic formula from Corollary 2.4,

$$\lim_{n\to\infty} Q_{k_n}^\alpha(e^{\nu_n}) = \Gamma(\alpha+1)\left(\frac{ib\lambda}{2}\right)^{-\alpha} J_\alpha(ib\lambda) = \Gamma(\alpha+1)\left(\frac{b\lambda}{2}\right)^{-\alpha} I_\alpha(b\lambda).$$

Combining the limits of the numerator and denominator, we find

$$\lim_{n\to\infty} \mathbb{E}_a\left[e^{-\nu_n T_{k_n}^\alpha}\right] = \frac{1}{\Gamma(\alpha+1)(b\lambda/2)^{-\alpha} I_\alpha(b\lambda)} = \frac{(b\lambda/2)^\alpha}{\Gamma(\alpha+1) I_\alpha(b\lambda)}.$$

This is precisely the Laplace transform of $\tilde{T}_b^{\alpha,0}$ given in Proposition 4.6, which completes the proof of Proposition 4.7. ☐

Building on this, we generalize the result to a scenario where the starting point is also scaled with the process, which corresponds to approximating a Bessel process starting from an arbitrary point $a > 0$.

**Proposition 4.8** (Convergence from a Scaled Starting Point)**.** *Consider the sequence of ultraspherical chains $(X_p^{\alpha,n})_{p\geq 0}$, where each chain starts at a diffusively scaled position $X_0^{\alpha,n} = j_n = \lfloor 2^n a \rfloor$ for some dyadic $a > 0$. Let the target level be $k_n = \lfloor 2^n b \rfloor$ for a dyadic $b > 0$. The rescaled hitting time, $\mathcal{T}_n^{\text{scaled}}(a, b) := T_{k_n}^{\alpha,n}/4^n$, converges in law to the first passage time $\tilde{T}_b^{\alpha,a}$ for a Bessel process starting from $a$.*

$$\mathcal{T}_n^{\text{scaled}}(a, b) \xrightarrow[n\to\infty]{\mathcal{L}} \tilde{T}_b^{\alpha,a}.$$

*Proof.* Here, both the starting point $j_n = \lfloor 2^n a \rfloor$ and target point $k_n = \lfloor 2^n b \rfloor$ are scaled. We consider two cases based on the relative positions of $a$ and $b$.

*Case 1: $a < b$.* The chain is moving away from the origin. The Laplace transform is given by the ratio of ultraspherical polynomials

$$\mathbb{E}_{j_n}\left[e^{-\nu_n T_{k_n}^{\alpha,n}}\right] = \frac{Q_{j_n}^{\alpha}(e^{\nu_n})}{Q_{k_n}^{\alpha}(e^{\nu_n})}.$$

We apply the same asymptotic analysis from Corollary 2.4 to both the numerator and the denominator. This yields

$$\lim_{n\to\infty} \frac{Q_{j_n}^{\alpha}(e^{\nu_n})}{Q_{k_n}^{\alpha}(e^{\nu_n})} = \frac{\Gamma(\alpha+1)(a\lambda/2)^{-\alpha} I_\alpha(a\lambda)}{\Gamma(\alpha+1)(b\lambda/2)^{-\alpha} I_\alpha(b\lambda)} = \left(\frac{b}{a}\right)^{\alpha} \frac{I_\alpha(a\lambda)}{I_\alpha(b\lambda)}.$$

This matches the required limit from Proposition 4.6 for $a < b$.

*Case 2: $a > b$.* The chain is moving towards the origin. The Laplace transform is given by the ratio of the $\lambda$-potentials (resolvents):

$$\mathbb{E}_{j_n}\left[e^{-\nu_n T_{k_n}^{\alpha,n}}\right] = \frac{U_{\nu_n}^{\alpha}(j_n, 0)}{U_{\nu_n}^{\alpha}(k_n, 0)}.$$

We can actually express the $\lambda$-potential in terms of the Legendre function: (For details, please refer to Appendix B.)

$$U_\lambda^{\alpha}(n,0) = \frac{2^{-\alpha}\Gamma(2\alpha+2)\Gamma(n+1)}{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)} e^{\lambda(\alpha+1)}(1-e^{-2\lambda})^{\frac{\alpha}{2}} e^{-i\pi\alpha} \mathcal{Q}_{n+\alpha}^{\alpha}(e^{\lambda}). \tag{4.8}$$

Hence, it is immediately seen that

$$U^{\alpha}_{\frac{\lambda^2}{2^{2n+1}}}(2^n a, 0) = \frac{2^{-\alpha}\Gamma(2\alpha+2)\Gamma(2^n a+1)}{\Gamma(\alpha+1)\Gamma(2^n a+2\alpha+1)} e^{\frac{\lambda^2}{2^{2n+1}}(\alpha+1)} \left(1 - e^{-\frac{\lambda^2}{2^{2n}}}\right)^{\frac{\alpha}{2}} e^{-i\pi\alpha} \mathcal{Q}_{2^n a+\alpha}^{\alpha}(e^{\frac{\lambda^2}{2^{2n+1}}}).$$

We use the expression for the potential in terms of Legendre functions given by Equation (4.8). The ratio becomes

$$\frac{U_{\nu_n}^{\alpha}(j_n, 0)}{U_{\nu_n}^{\alpha}(k_n, 0)} = \frac{\Gamma(j_n+1)\Gamma(k_n+2\alpha+1)}{\Gamma(k_n+1)\Gamma(j_n+2\alpha+1)} \cdot \frac{\mathcal{Q}_{j_n+\alpha}^{\alpha}(e^{\nu_n})}{\mathcal{Q}_{k_n+\alpha}^{\alpha}(e^{\nu_n})}.$$

We analyze the limit of each part. First, using Stirling's approximation for the Gamma function ratio, $\Gamma(N+c)/\Gamma(N+d) \sim N^{c-d}$, we have

$$\lim_{n\to\infty} \frac{\Gamma(2^n a+1)\Gamma(2^n b+2\alpha+1)}{\Gamma(2^n b+1)\Gamma(2^n a+2\alpha+1)} = \lim_{n\to\infty} \frac{(2^n a)^{-2\alpha}}{(2^n b)^{-2\alpha}} = \left(\frac{b}{a}\right)^{2\alpha}.$$

For the ratio of Legendre functions, we use the asymptotic result from Proposition 2.10, which connects $\mathcal{Q}_n^{\alpha}$ to the modified Bessel function $K_\alpha$. For a scaled argument $x_n = \lfloor 2^n x \rfloor$, we have $x_n^{-\alpha} e^{-i\pi\alpha} \mathcal{Q}_{x_n}^{\alpha}(e^{\nu_n}) \to K_\alpha(x\lambda)$. Applying this to the ratio,

$$\lim_{n\to\infty} \frac{\mathcal{Q}_{j_n+\alpha}^{\alpha}(e^{\nu_n})}{\mathcal{Q}_{k_n+\alpha}^{\alpha}(e^{\nu_n})} = \lim_{n\to\infty} \frac{(2^n a)^{\alpha} K_\alpha(a\lambda)}{(2^n b)^{\alpha} K_\alpha(b\lambda)} = \left(\frac{a}{b}\right)^{\alpha} \frac{K_\alpha(a\lambda)}{K_\alpha(b\lambda)}.$$

Combining the limits of the two parts gives the final result

$$\lim_{n\to\infty} \mathbb{E}_{j_n}\left[e^{-\nu_n T_{k_n}^{\alpha,n}}\right] = \left(\frac{b}{a}\right)^{2\alpha} \cdot \left(\frac{a}{b}\right)^{\alpha} \frac{K_\alpha(a\lambda)}{K_\alpha(b\lambda)} = \left(\frac{b}{a}\right)^{\alpha} \frac{K_\alpha(a\lambda)}{K_\alpha(b\lambda)}.$$

This matches the expression from Proposition 4.6 for $a > b$. □

4.2. **Convergence of Exit Times.** We demonstrate that the first exit time of a suitably scaled ultraspherical Markov chain from a fixed interval converges in law to the first exit time of a Bessel process. The proof relies on the convergence of the respective Laplace transforms. We first derive the Laplace transform for the discrete exit time by considering suitable martingales, and then show its convergence to the known transform for the Bessel process.

**Theorem 4.9.** *Let $c < a < b$ be three dyadic numbers. Consider the ultraspherical Markov chain $X^{\alpha,n}$ starting from $X_0^{\alpha,n} = 2^n a$ and the Bessel process $R^\alpha$ of dimension $2\alpha + 2$ starting from $R_0^\alpha = a$. Then, under the diffusive scaling, the discrete exit time converges in law to the continuous one:*

$$\frac{T_{c,b}^{\alpha,n}}{2^{2n}} \xrightarrow[n\to\infty]{\mathcal{L}} \tilde{T}_{c,b}^{\alpha}.$$

The proof relies on constructing martingales from special functions related to the chain's generator.

**Proposition 4.10.** *Let $(X_p^\alpha)_{p\geq 0}$ be the ultraspherical Markov chain on the integers. For a fixed $\lambda > 0$, the following processes are martingales with respect to the natural filtration of the chain:*

*(i)* $M_p^{\alpha,\lambda} = e^{-\lambda p}\mathcal{Q}^\alpha_{X_p^\alpha}(e^\lambda)$

*(ii)* $N_p^{\alpha,\lambda} = e^{-\lambda p}\left(\sum_{N=0}^{\infty} e^{-\lambda N}\mathbb{P}^{X_p^\alpha}(X_N^\alpha = 0)\right)$, *for states* $X_p^\alpha > 0$.

*Proof.* (i) The ultraspherical polynomials $\mathcal{Q}_k^\alpha(x)$ satisfy a three-term recurrence relation in the degree $k$. The transition probabilities $p(k,j) = \mathbb{P}[X_{p+1}^\alpha = j \mid X_p^\alpha = k]$ of the chain are defined from this recurrence. Specifically, for a suitable choice of $x$, the operator of conditional expectation acts as a multiplication operator on the sequence $k \mapsto \mathcal{Q}_k^\alpha(x)$. The construction yields the eigenfunction relationship:

$$\mathbb{E}_k[\mathcal{Q}^\alpha_{X_1^\alpha}(e^\lambda)] = \sum_j p(k,j)\mathcal{Q}_j^\alpha(e^\lambda) = e^\lambda \mathcal{Q}_k^\alpha(e^\lambda).$$

Using this, we check the martingale property for $M_p^{\alpha,\lambda}$ conditional on $X_p^\alpha = k$:

$$\begin{aligned}
\mathbb{E}[M_{p+1}^{\alpha,\lambda} \mid X_p^\alpha = k] &= \mathbb{E}[e^{-\lambda(p+1)}\mathcal{Q}^\alpha_{X_{p+1}^\alpha}(e^\lambda) \mid X_p^\alpha = k] \\
&= e^{-\lambda(p+1)}\mathbb{E}_k[\mathcal{Q}^\alpha_{X_1^\alpha}(e^\lambda)] \\
&= e^{-\lambda(p+1)}\left(e^\lambda \mathcal{Q}_k^\alpha(e^\lambda)\right) \\
&= e^{-\lambda p}\mathcal{Q}_k^\alpha(e^\lambda) = M_p^{\alpha,\lambda}|_{X_p^\alpha = k}.
\end{aligned}$$

Thus, $M_p^{\alpha,\lambda}$ is a martingale.

(ii) Let $g(k) = \sum_{N=0}^{\infty} e^{-\lambda N}\mathbb{P}^k[X_N^\alpha = 0]$. We check the martingale property for $N_p^{\alpha,\lambda}$ for any state $k > 0$:

$$\begin{aligned}
\mathbb{E}_k[e^{-\lambda(p+1)}g(X_{p+1}^\alpha)] &= e^{-\lambda(p+1)}\mathbb{E}_k[g(X_1^\alpha)] \\
&= e^{-\lambda(p+1)}\sum_j p(k,j)\left(\sum_{N=0}^{\infty} e^{-\lambda N}\mathbb{P}^j[X_N^\alpha = 0]\right) \\
&= e^{-\lambda(p+1)}\sum_{N=0}^{\infty} e^{-\lambda N}\left(\sum_j p(k,j)\mathbb{P}^j[X_N^\alpha = 0]\right) \\
&= e^{-\lambda(p+1)}\sum_{N=0}^{\infty} e^{-\lambda N}\mathbb{P}^k[X_{N+1}^\alpha = 0] \\
&\qquad\qquad \text{(by Chapman-Kolmogorov equation)} \\
&= e^{-\lambda p}\sum_{M=1}^{\infty} e^{-\lambda M}\mathbb{P}^k[X_M^\alpha = 0] \quad \text{(letting } M = N+1\text{)} \\
&= e^{-\lambda p}\left(\sum_{M=0}^{\infty} e^{-\lambda M}\mathbb{P}^k[X_M^\alpha = 0] - \mathbb{P}^k[X_0^\alpha = 0]\right) \\
&= e^{-\lambda p}g(k),
\end{aligned}$$

noting that since $k > 0, \mathbb{P}^k[X_0^\alpha = 0] = 0$. This confirms that $N_p^{\alpha,\lambda}$ is a martingale for states away from the absorbing boundary 0. □

**Proposition 4.11.** *Let the ultraspherical chain start from an integer state $a$, with $c < a < b$. For $\lambda > 0$, the Laplace transform of the exit time $T_{c,b}^\alpha$ from the integer interval $[c, b]$ is given by*

$$\begin{aligned}
&\mathbb{E}_a\left[e^{-\lambda T_{c,b}^\alpha}\right] \\
&= \frac{\mathcal{Q}_a^\alpha(e^\lambda)\left[\sum_{N=0}^\infty e^{-\lambda N}\mathbb{P}^c[X_N^\alpha = 0] - \sum_{N=0}^\infty e^{-\lambda N}\mathbb{P}^b[X_N^\alpha = 0]\right]}{\left(\sum_{N=0}^\infty e^{-\lambda N}\mathbb{P}^c[X_N^\alpha = 0]\right)\mathcal{Q}_b^\alpha(e^\lambda) - \mathcal{Q}_c^\alpha(e^\lambda)\left(\sum_{N=0}^\infty e^{-\lambda N}\mathbb{P}^b[X_N^\alpha = 0]\right)} \\
&+ \frac{\left(\sum_{N=0}^\infty e^{-\lambda N}\mathbb{P}^a[X_N^\alpha = 0]\right)\left[\mathcal{Q}_b^\alpha(e^\lambda) - \mathcal{Q}_c^\alpha(e^\lambda)\right]}{\left(\sum_{N=0}^\infty e^{-\lambda N}\mathbb{P}^c[X_N^\alpha = 0]\right)\mathcal{Q}_b^\alpha(e^\lambda) - \mathcal{Q}_c^\alpha(e^\lambda)\left(\sum_{N=0}^\infty e^{-\lambda N}\mathbb{P}^b[X_N^\alpha = 0]\right)}.
\end{aligned}$$

*Proof.* The exit time $T_{c,b}^\alpha$ is a bounded stopping time. We apply the Optional Stopping Theorem to the martingales $M_p^{\alpha,\lambda}$ and $N_p^{\alpha,\lambda}$.

$$\mathbb{E}_a[e^{-\lambda T_{c,b}^\alpha}\mathcal{Q}_{X_{T_{c,b}^\alpha}^\alpha}^\alpha(e^\lambda)] = \mathcal{Q}_a^\alpha(e^\lambda)$$

$$\mathbb{E}_a\left[e^{-\lambda T_{c,b}^\alpha}\left(\sum_{N=0}^\infty e^{-\lambda N}\mathbb{P}^{X_{T_{c,b}^\alpha}^\alpha}[X_N^\alpha = 0]\right)\right] = \sum_{N=0}^\infty e^{-\lambda N}\mathbb{P}^a[X_N^\alpha = 0]$$

Since the process at time $T^\alpha_{c,b}$ must be at state $c$ or $b$, these two equations form a linear system for the quantities $\mathbb{E}_a[e^{-\lambda T^\alpha_{c,b}}\mathbb{1}_{\{X_{T^\alpha_{c,b}}=c\}}]$ and $\mathbb{E}_a[e^{-\lambda T^\alpha_{c,b}}\mathbb{1}_{\{X_{T^\alpha_{c,b}}=b\}}]$. Solving this system for these two quantities and summing them gives the desired total Laplace transform. □

**Proposition 4.12.** [BS02] *Let $\tilde{T}^\alpha_{c,b}$ be the first exit time from $[c,b]$ for a Bessel process $R^\alpha$ of dimension $2\alpha+2$ starting at $a\in(c,b)$. Its Laplace transform is*

$$\mathbb{E}_a\left[e^{-\frac{\lambda^2}{2}\tilde{T}^\alpha_{c,b}}\right] = \frac{I_\alpha(\lambda a)[b^\alpha K_\alpha(\lambda c) - c^\alpha K_\alpha(\lambda b)] + K_\alpha(\lambda a)[c^\alpha I_\alpha(\lambda b) - b^\alpha I_\alpha(\lambda c)]}{a^\alpha[I_\alpha(\lambda b)K_\alpha(\lambda c) - I_\alpha(\lambda c)K_\alpha(\lambda b)]}.$$

*Proof of Theorem 4.9.* We evaluate the limit of $\mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\frac{T^{\alpha,n}_{c,b}}{2^{2n}}\right)\right]$ as $n\to\infty$. This is equivalent to taking the limit of the expression in Proposition 4.11 with states $2^n a, 2^n b, 2^n c$ and Laplace parameter $\nu_n = \frac{\lambda^2}{2^{2n+1}}$. We recall the following convergence which has been covered before:

$$\mathcal{Q}^\alpha_{2^n x}\left(e^{\frac{\lambda^2}{2^{2n+1}}}\right) \sim \Gamma(\alpha+1)\left(\frac{\lambda}{2}\right)^{-\alpha}\frac{I_\alpha(x\lambda)}{x^\alpha}$$

$$\sum_{N=0}^{\infty} e^{-\frac{\lambda^2}{2^{2n+1}}N}\mathbb{P}^{2^n x}[X^\alpha_N=0] \sim 2^{-\alpha}\frac{\Gamma(2\alpha+2)}{\Gamma(\alpha+1)}\frac{\lambda^\alpha}{2^{2n\alpha}}\frac{K_\alpha(\lambda x)}{x^\alpha}.$$

Then we have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[e^{-\nu_n T^{\alpha,n}_{c,b}}\right] &\sim \frac{\left(\frac{I_\alpha(a\lambda)}{a^\alpha}\right)\left[\left(\frac{K_\alpha(c\lambda)}{c^\alpha}\frac{\lambda^\alpha}{2^{2n\alpha}}\right) - \left(\frac{K_\alpha(b\lambda)}{b^\alpha}\frac{\lambda^\alpha}{2^{2n\alpha}}\right)\right]}{\left(\frac{K_\alpha(c\lambda)}{c^\alpha}\frac{\lambda^\alpha}{2^{2n\alpha}}\right)\left(\frac{I_\alpha(b\lambda)}{b^\alpha}\right) - \left(\frac{I_\alpha(c\lambda)}{c^\alpha}\right)\left(\frac{K_\alpha(b\lambda)}{b^\alpha}\frac{\lambda^\alpha}{2^{2n\alpha}}\right)}\\ &+ \frac{\left(\frac{K_\alpha(a\lambda)}{a^\alpha}\frac{\lambda^\alpha}{2^{2n\alpha}}\right)\left[\left(\frac{I_\alpha(b\lambda)}{b^\alpha}\right) - \left(\frac{I_\alpha(c\lambda)}{c^\alpha}\right)\right]}{\left(\frac{K_\alpha(c\lambda)}{c^\alpha}\frac{\lambda^\alpha}{2^{2n\alpha}}\right)\left(\frac{I_\alpha(b\lambda)}{b^\alpha}\right) - \left(\frac{I_\alpha(c\lambda)}{c^\alpha}\right)\left(\frac{K_\alpha(b\lambda)}{b^\alpha}\frac{\lambda^\alpha}{2^{2n\alpha}}\right)}\\ &= \frac{I_\alpha(a\lambda)[b^\alpha K_\alpha(c\lambda) - c^\alpha K_\alpha(b\lambda)] + K_\alpha(a\lambda)[c^\alpha I_\alpha(b\lambda) - b^\alpha I_\alpha(c\lambda)]}{a^\alpha[I_\alpha(b\lambda)K_\alpha(c\lambda) - I_\alpha(c\lambda)K_\alpha(b\lambda)]}.\end{aligned}$$

This is precisely the Laplace transform of the Bessel exit time. By Lévy's continuity theorem, the pointwise convergence of the Laplace transforms implies the convergence in law of the random variables. □

## Appendix A. Proof of the fourth moment bound

We notate $S_m := X^\alpha_m - a$. We decompose $S_m$ into a martingale part $M_m$ and a predictable drift part $D_m$. Let $\mathcal{F}_{k-1}$ be the natural filtration generated by the chain up to time $k-1$. The one-step drift is given by $d_k := \mathbb{E}[X^\alpha_k - X^\alpha_{k-1}|\mathcal{F}_{k-1}]$. The decomposition is then $S_m = M_m + D_m$, where the martingale $M_m$ and the drift process $D_m$ are defined as

$$M_m = \sum_{k=1}^{m} (X_k^\alpha - X_{k-1}^\alpha - d_k), \quad \text{and} \quad D_m = \sum_{k=1}^{m} d_k.$$

By the moment inequality $(x+y)^4 \leq 8(x^4+y^4)$, we can bound the fourth moment of $S_m$ by the moments of its components:

$$\mathbb{E}_a[S_m^4] \leq 8(\mathbb{E}_a[M_m^4] + \mathbb{E}_a[D_m^4]).$$

We now proceed to bound each term separately.

**Martingale Term:** The Burkholder-Davis-Gundy (BDG) inequality states that $\mathbb{E}_a[M_m^4] \leq C_1 \mathbb{E}_a[\langle M \rangle_m^2]$ for some universal constant $C_1$. The predictable quadratic variation $\langle M \rangle_m$ is given by the sum of the conditional second moments of the martingale increments. Since $X_k^\alpha - X_{k-1}^\alpha$ takes values in $\{-1, 1\}$, we have $(X_k^\alpha - X_{k-1}^\alpha)^2 = 1$, and thus:

$$\mathbb{E}[(X_k^\alpha - X_{k-1}^\alpha - d_k)^2 | \mathcal{F}_{k-1}] = \mathbb{E}[(X_k^\alpha - X_{k-1}^\alpha)^2 | \mathcal{F}_{k-1}] - d_k^2 = 1 - d_k^2.$$

The quadratic variation is $\langle M \rangle_m = \sum_{k=1}^{m} (1 - d_k^2)$. Since $d_k^2 \geq 0$, we have the uniform bound $\langle M \rangle_m \leq m$. This implies $\mathbb{E}_a[\langle M \rangle_m^2] \leq m^2$. Applying the BDG inequality, we get

$$\mathbb{E}_a[M_m^4] \leq C_1 m^2. \tag{A.1}$$

**Drift Term:** By Minkowski's inequality for $L^4$, we have

$$(\mathbb{E}_a[|D_m|^4])^{1/4} \leq \sum_{k=1}^{m} (\mathbb{E}_a[|d_k|^4])^{1/4}.$$

The drift of the ultraspherical chain is known to be inversely proportional to its position, satisfying $|d_k| \leq C_2 (X_{k-1}^\alpha + 1)^{-1}$ for a constant $C_2(\alpha)$. Furthermore, for a random walk with positive drift starting from $a \leq mT$, the inverse moments are controlled. Note also that for any $p > 0$, there exists a constant $C_3(p, T)$ such that $\mathbb{E}_a[(X_{k-1}^\alpha + 1)^{-p}] \leq C_3 k^{-p/2}$. Setting $p = 4$, we obtain

$$\mathbb{E}_a[d_k^4] \leq C_4 \mathbb{E}_a[(X_{k-1}^\alpha + 1)^{-4}] \leq C_5 k^{-2}, \tag{A.2}$$

where $C_4$ and $C_5$ are constants depending on $\alpha$ and $T$. Taking the fourth root gives $(\mathbb{E}_a[d_k^4])^{1/4} \leq C_6 k^{-1/2}$. The sum is bounded by an integral: $\sum_{k=1}^{m} k^{-1/2} \leq C_7 \sqrt{m}$. Thus,

$$(\mathbb{E}_a[D_m^4])^{1/4} \leq \sum_{k=1}^{m} C_6 k^{-1/2} \leq C_8 \sqrt{m}, \tag{A.3}$$

which yields $\mathbb{E}_a[D_m^4] \leq C_9 m^2$, where the constants depend on $\alpha$ and $T$.

Combining the two bounds, we find

$$\mathbb{E}_a[(X_m^\alpha - a)^4] \leq 8(C_1 m^2 + C_9 m^2) \leq C_T m^2, \tag{A.4}$$

where $C_T$ is a constant dependent on $\alpha$ and $T$. This concludes the proof. □

## Appendix B. Expression of the Potential Using Legendre Functions

In this section we prove Equation 4.8. In fact, the $\lambda$-potential can be expressed in terms of hypergeometric functions.

**Proposition B.1.** *The $\lambda$-potential $U_\lambda^\alpha$ is given by the following integral representation:*

$$U_\lambda^\alpha(n,m) = \omega_m^\alpha \int_{-1}^{1} \frac{Q_n^\alpha(x)Q_m^\alpha(x)}{1-xe^{-\lambda}} \mathrm{d}\mu^\alpha(x). \tag{B.1}$$

*Proof.* Let us revisit the integral representation of $(P^\alpha)^N$ given by Equation 3.3:

$$(P^\alpha)^N(n,m) = \omega_m^\alpha \int_{-1}^{1} x^N Q_n^\alpha(x)Q_m^\alpha(x)\mathrm{d}\mu^\alpha(x).$$

We distinguish two cases based on whether $\lambda > 0$ or $\lambda = 0$:
**Case 1: $\lambda > 0$.** Substituting Equation (B.1) into the definition, we get

$$\begin{aligned} U_\lambda^\alpha(n,m) &= \sum_{N\geq 0} e^{-\lambda N}\omega_m^\alpha \int_{-1}^{1} x^N Q_n^\alpha(x)Q_m^\alpha(x)\mathrm{d}\mu^\alpha(x) \\ &= \sum_{N\geq 0} e^{-\lambda N}x^N\omega_m^\alpha \int_{-1}^{1} Q_n^\alpha(x)Q_m^\alpha(x)\mathrm{d}\mu^\alpha(x). \end{aligned}$$

For $x \in [-1,1]$, $|xe^{-\lambda}| \leq e^{-\lambda} < 1$, so the series converges uniformly:

$$\sum_{N\geq 0} e^{-\lambda N}x^N = \frac{1}{1-xe^{-\lambda}}.$$

Since $Q_n^\alpha(x)Q_m^\alpha(x)$ is bounded on $[-1,1]$ (as ultraspherical polynomials are continuous and $|Q_n^\alpha(x)| \leq Q_n^\alpha(1) = 1$ for normalized polynomials), and $\mu^\alpha$ is a finite measure, the integrand $\dfrac{Q_n^\alpha(x)Q_m^\alpha(x)}{1-xe^{-\lambda}}$ is integrable. By uniform convergence, the interchange is justified, yielding

$$U_\lambda^\alpha(n,m) = \omega_m^\alpha \int_{-1}^{1} \frac{Q_n^\alpha(x)Q_m^\alpha(x)}{1-xe^{-\lambda}} \mathrm{d}\mu^\alpha(x).$$

**Case 2: $\lambda = 0$.** For $\lambda = 0$, we have

$$U_0^\alpha(n,m) = \sum_{N\geq 0} \omega_m^\alpha \int_{-1}^{1} x^N Q_n^\alpha(x)Q_m^\alpha(x)\mathrm{d}\mu^\alpha(x).$$

We aim to show:

$$U_0^\alpha(n,m) = \omega_m^\alpha \int_{-1}^{1} \frac{Q_n^\alpha(x)Q_m^\alpha(x)}{1-x} \mathrm{d}\mu^\alpha(x).$$

For $|x| < 1$, $\sum_{N\geq 0} x^N = \frac{1}{1-x}$. Near $x = 1$, $Q_n^\alpha(x)Q_m^\alpha(x) \approx 1$, since $Q_n^\alpha(1) = 1$. Choose $x_0 \in (0,1)$ such that $Q_n^\alpha(x)Q_m^\alpha(x) > 0$ for $x \in [x_0, 1]$. For $x \in [x_0, 1)$, the series $\sum_{N\geq 0} x^N Q_n^\alpha(x)Q_m^\alpha(x)$ is increasing and converges to $\frac{Q_n^\alpha(x)Q_m^\alpha(x)}{1-x}$. By the monotone convergence theorem, since $x^N Q_n^\alpha(x)Q_m^\alpha(x) \geq 0$, we interchange

$$\int_{x_0}^1 \sum_{N\geq 0} x^N Q_n^\alpha(x)Q_m^\alpha(x)\mathrm{d}\mu^\alpha(x) = \int_{x_0}^1 \frac{Q_n^\alpha(x)Q_m^\alpha(x)}{1-x}\mathrm{d}\mu^\alpha(x).$$

For $x \in [-1, x_0]$, the series converges absolutely. The integrand $\frac{Q_n^\alpha(x)Q_m^\alpha(x)}{1-x}$ is integrable, as the measure $\mu^\alpha$ (with density $(1-x^2)^{\alpha-1/2}$ for $\alpha > -1/2$) ensures convergence near $x = 1$ for $\alpha > -1$. Thus:

$$U_0^\alpha(n,m) = \omega_m^\alpha \int_{-1}^1 \frac{Q_n^\alpha(x)Q_m^\alpha(x)}{1-x}\mathrm{d}\mu^\alpha(x).$$

This completes the proof. □

**Proposition B.2.** *For $\lambda > 0$ and $n \geq 0$, the $\lambda$-potential is given by*

$$U_\lambda^\alpha(n,0) = \frac{2^\alpha\Gamma(2\alpha+2)}{\Gamma(\alpha+1)} \cdot \frac{\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(2n+2\alpha+2)} \cdot \frac{e^{-\lambda n}}{(1+e^{-\lambda})^{\alpha+1}}$$
$$\cdot F(n+1, n+\alpha+1; 2n+2\alpha+2; \frac{2e^{-\lambda}}{1+e^{-\lambda}}) \tag{B.2}$$

*Proof.* From Equation B.1, we have:

$$U_\lambda^\alpha(n,0) = \omega_0^\alpha \int_{-1}^1 \frac{Q_n^\alpha(x)}{1-xe^{-\lambda}}(1-x^2)^\alpha \mathrm{d}x.$$

Using Equation 2.2, we can write

$$U_\lambda^\alpha(n,0) = \frac{(-1)^n}{2^n(\alpha+1)\cdots(\alpha+n)}\omega_0^\alpha \int_{-1}^1 \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{1-xe^{-\lambda}}\right)(1-x^2)^\alpha \mathrm{d}x.$$

By integrating this expression $n$ times by parts, we obtain

$$U_\lambda^\alpha(n,0) = \frac{n!\, e^{-\lambda n}\, \omega_0^\alpha}{2^n(\alpha+1)\cdots(\alpha+n)(1+e^{-\lambda})^{n+1}} \int_{-1}^1 x^n(1-x^2)^{n+\alpha}(1-xe^{-\lambda})^{-n-1}\mathrm{d}x.$$

With the change of variable $x = 2y - 1$, this becomes

$$U_\lambda^\alpha(n,0) = \frac{n!\, e^{-\lambda n}\, \omega_0^\alpha\, 2^{2n+2\alpha+1}}{(\alpha+1)\cdots(\alpha+n)(1+e^{-\lambda})^{n+1}} \int_0^1 y^n(1-y)^{n+\alpha}\left(1-\frac{2e^{-\lambda}y}{1+e^{-\lambda}}\right)^{-n-1}\mathrm{d}y.$$

We put $a = n+1$, $b = n+\alpha+1$, $c = 2n+2\alpha+2$, and $z = \frac{2e^{-\lambda}}{1+e^{-\lambda}}$. Using the definition of $F(a,b;c;z)$, we write

$$F(a,b;c;z) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \int_0^1 y^{b-1}(1-y)^{c-b-1}(1-yz)^{-a}\mathrm{d}y.$$

Substituting these terms into the integral, the $\lambda$-potential becomes

$$U_\lambda^\alpha(n,0) = \frac{n!\, e^{-\lambda n}\, \omega_0^\alpha}{2^n(\alpha+1)\cdots(\alpha+n)(1+e^{-\lambda})^{n+1}} \cdot \frac{\Gamma(n+\alpha+1)^2}{\Gamma(2n+2\alpha+2)} F(a,b;c;z),$$

which is the desired result. □

For specific parameter values, hypergeometric functions can be expressed in terms of Legendre functions. This final expression allows us to deduce asymptotic properties of ultrametric chains using known results for Legendre functions.

**Corollary B.3.** [AS65] *For $a > 0$ and $b > \frac{a}{2}$, we have*

$$F(a,b;2b;z) = \frac{2^{2b}\Gamma(b+\frac{1}{2})z^{-b}(1-z)^{\frac{b-a}{2}}\pi^{(a-b)}}{\pi^{\frac{1}{2}}\Gamma(2b-a)} \mathcal{Q}_{b-1}^{b-a}\left(\frac{2-z}{z}\right). \tag{B.3}$$

**Proposition B.4.** *The $\lambda$-potential is given by*

$$U_\lambda^\alpha(n,0) = \frac{2^{-\alpha}\Gamma(2\alpha+2)\Gamma(n+1)}{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)} e^{\lambda(\alpha+1)}(1-e^{-2\lambda})^{\alpha/2} e^{-i\pi\alpha} \mathcal{Q}_{n+\alpha}^\alpha(e^\lambda) \tag{B.4}$$

*Proof.* By putting $a = n+1$, $b = n+\alpha+1$, and $z = \frac{2e^{-\lambda}}{1+e^{-\lambda}}$ directly into Equation B.3, we get

$$\begin{aligned} &F(n+1, n+\alpha+1, 2n+2\alpha+2, \frac{2e^{-\lambda}}{1+e^{-\lambda}}) \\ &= \frac{2^{2n+2\alpha+2}\Gamma(n+\alpha+\frac{3}{2})}{(\pi)^{\frac{1}{2}}\Gamma(n+2\alpha+1)} \left(\frac{e^\lambda+1}{2}\right)^{n+\alpha+1} \left(\frac{1-e^{-\lambda}}{1+e^{-\lambda}}\right)^{\alpha/2} e^{-i\pi\alpha}\mathcal{Q}_{n+\alpha}^\alpha(e^\lambda) \end{aligned}$$

Recall the property of the Gamma function $\Gamma(x)\Gamma(x+\frac{1}{2}) = 2^{1-2x}\pi^{\frac{1}{2}}\Gamma(2x)$. Putting $x = n+\alpha+1$ we yield

$$\Gamma(n+\alpha+\frac{3}{2}) = 2^{-2n-2\alpha-1}\pi^{\frac{1}{2}}\frac{\Gamma(2n+2\alpha+2)}{\Gamma(n+\alpha+1)}.$$

Using Equation B.2, we know that for $\lambda > 0$ and $n \geq 0$, the $\lambda$-potential is given by

$$U_\lambda^\alpha(n,0) = \frac{2^{-\alpha}\Gamma(2\alpha+2)\Gamma(n+1)}{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)} e^{\lambda(\alpha+1)}(1-e^{-2\lambda})^{\alpha/2} e^{-i\pi\alpha}\mathcal{Q}_{n+\alpha}^\alpha(e^\lambda)$$

□

## References

[AS65] Milton Abramowitz and I A Stegun, editors. *Handbook of mathematical functions*. Dover Books on Mathematics. Dover Publications, Mineola, NY, June 1965.

[Ave89] J.S. Avery. *Hyperspherical Harmonics: Applications in Quantum Theory*. Reidel Texts in the Mathematical Sciences. Springer Netherlands, 1989.

[Boc56] S. Bochner. Sturm-liouville and heat equations whose eigenfunctions are ultraspherical polynomials or associated bessel functions. In *Proceedings of the Conference on Differential Equations, College Park, Maryland*, pages 23–48, 1956.

[BS02] A N Borodin and P Salminen. *Handbook of Brownian motion - facts and formulae.* Probability and Its Applications. Birkhauser Verlag AG, Basel, Switzerland, 2 edition, May 2002.

[Car61] L Carlitz. The product of two ultraspherical polynomials. *Proc. Glasg. Math. Assoc.*, 5(2):76–79, July 1961.

[Dil24] Mustapha Dilmi. Numerical solution of fractional integro-differential equations using ultraspherical polynomials. *STUDIES IN ENGINEERING AND EXACT SCIENCES*, 5(2):e11537, December 2024.

[Doh02] EH Doha. On the coefficients of integrated expansions and integrals of ultraspherical polynomials and their applications for solving differential equations. *Journal of Computational and Applied Mathematics*, 139(2):275–298, 2002.

[Fai24] Lukong Cornelius Fai. Ultraspherical (gegenbauer) polynomials. In *Special Functions in Physics and Engineering*, 2053-2563, pages 38–1 to 38–18. IOP Publishing, 2024.

[Geo75] C. George. *Les chaînes de Markov associées à des polynômes orthogonaux.* PhD thesis, Université de Nancy, 1975.

[Kat15] Makoto Katori. *Bessel processes, Schramm-loewner evolution, and the Dyson model.* SpringerBriefs in Mathematical Physics. Springer, Singapore, Singapore, 1 edition, April 2015.

[Ken60] Maurice Kennedy. A stochastic process associated with the ultraspherical polynomials. *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, 61:89–100, 1960.

[KS04] Ioannis Karatzas and Steven E Shreve. *Brownian motion and stochastic calculus.* Graduate texts in mathematics. Springer, New York, NY, 2 edition, August 2004.

[MOS13] Wilhelm Magnus, Fritz Oberhettinger, and Raj Pal Soni. *Formulas and theorems for the special functions of mathematical physics.* Grundlehren der mathematischen Wissenschaften. Springer, Berlin, Germany, 1966 edition, November 2013.

[NW81] C. M. Newman and A. L. Wright. An invariance principle for certain dependent sequences. *The Annals of Probability*, 9(4):671–675, 1981.

[PY81] Jim Pitman and Marc Yor. Bessel processes and infinitely divisible laws. In David Williams, editor, *Stochastic Integrals*, pages 285–370, Berlin, Heidelberg, 1981. Springer Berlin Heidelberg.

[RY10] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion.* Grundlehren der mathematischen Wissenschaften. Springer, Berlin, Germany, December 2010.

[SB13] Ayman Shehata and Ravi Bhukya. On ultraspherical matrix polynomials and their properties. *Bulletin of the Malaysian Mathematical Sciences Society,... http://math. usm. my/bulletin/pdf/acceptedpapers/2013-06-028-R2. pdf*, 2013.

[SP12] René L. Schilling and Lothar Partzsch. *Brownian Motion.* De Gruyter, Berlin, Boston, 2012.

[Sze75] G. Szegő. *Orthogonal Polynomials*, volume 23 of *Colloquium Publications.* American Mathematical Society, Providence, RI, 4th edition, 1975.

## EDITOR'S COMMENTS

This paper studies a family of random walks on the non-negative integers whose transition probabilities come from the recurrence relation of ultraspherical (Gegenbauer) polynomials, and proves two main results: the rescaled chains converge to a Bessel process of dimension $d = 2\alpha + 2$, and the rescaled first exit time from an interval converges to the exit time of that Bessel process. The nice contribution here is taking an older result of Claude George, which only described the chain at a single large time, and strengthening it to a statement about the whole path of the process.

The proof uses the standard and correct approach—establishing tightness and then convergence of the finite-dimensional distributions—and the exit-time result is handled with a separate, well-chosen method based on building martingales from the special functions and applying the optional stopping theorem. The supporting work, such as the fourth-moment bound and the Legendre-function asymptotics, is demanding analysis for this level and is carried out carefully. Because the questions are inherited from George's work, the main strength is in the execution and in combining special-function asymptotics with convergence techniques. This is a polished, single-author paper comparable to an undergraduate thesis.