

INCENTRIC SUBDIVISIONS OF TRIANGLES AND ITERATED FUNCTION SYSTEMS

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ABSTRACT. In this paper, we have introduced and studied two new subdivision rules of triangles using incenter, namely, *inward* and *outward binary incentric subdivisions*, alongside the existing *senary* and *ternary incentric subdivisions*. To this end, we have used iterated function systems to establish approximation, convergence and denseness theorems in metric spaces as the main tools to study how diverse the shapes of the descendant triangles are upon repeated subdivisions of triangles using these old and new rules.

In the inward binary case, we have proved that the descendant triangles are dense in the space of triangles, similar to the senary case. We have also proved the distinctive behavior of the inward binary descendants that, in contrast to the senary case, they are spread over the space of triangle more and more evenly as subdivision keeps going, which is characterized by the weak convergence of their empirical measures to the uniform probability measure on the closure of the space of triangles.

In the outward binary case, we have proved that the descendant triangle branches all converge to a rather limited choice of triangles, similar to the ternary case. We have also followed up on connections between the ternary case and the Sierpiński triangle through formulating an old observation in precise terms and proved it.

We have also generalized both inward and outward binary incentric subdivisions to *quasi-incentric subdivisions*, and discovered that they preserve well the behaviors of the descendant triangles in their incentric counterparts.

KEYWORDS. Incenter, Incentric Subdivision, Triangle Subdivision, Iterated Function Systems, Empirical Measure, Uniform Measure, Banach Fixed-Point Theorem, Sierpiński Triangle

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1. INTRODUCTION

1.1. Background. Investigations into various aspects of successive subdivisions of triangles using triangle centers have attracted significant interest among mathematicians in recent years (see, for instance, [6], [7], [8], [11], [12] and [14], and [5] for a good reference list) since I. Bárány, A. F. Beardon and T. K. Carne have answered a question of V. Stakhovskii with the following theorem (see [1]):

Theorem 1. (Barycentric subdivision) The descendants of a triangle generated by successive barycentric subdivisions approximate arbitrarily closely any given triangle.

Starting with any initial triangle as the ancestor triangle, take the barycentric subdivision to obtain six daughter triangles (see Figure 1), then take the barycentric subdivision of each of these daughter triangles to obtain thirty-six granddaughter triangles and so on (see Figure 2). It is natural to ask how diverse the shapes of the descendant triangles are, up to similarity. The space of triangles used was constructed by representing each set of mutually similar triangles as at most six complex numbers in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Then through iterations of Möbius maps, the complex numbers representing the descendant triangles were generated, then were proved to approximate arbitrarily closely the complex numbers representing any triangle.

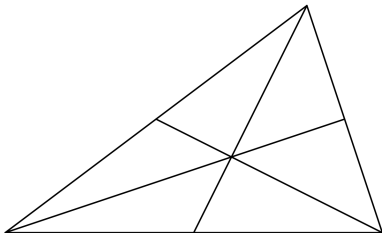


FIGURE 1.

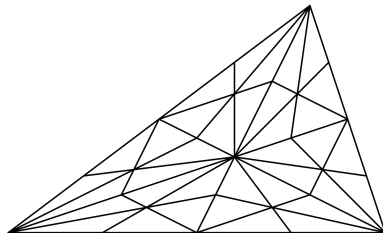


FIGURE 2.

It is then natural to try incenter in place of centroid, as S. Butler and R. Graham did with the following theorem (see [3] and [4]):

Theorem 2. (Senary incentric subdivision) The descendants of a triangle generated by successive senary incentric subdivisions approximate arbitrarily closely any given triangle.

and with the following observation (see [3]):

Theorem 3. (Ternary incentric subdivision) The descendants of a triangle generated by successive ternary incentric subdivisions approximate arbitrarily closely any given Sierpiński triangle, which means a triangle represented by a point in the Sierpiński triangle in the space of triangles.

The meanings of ‘senary’, ‘ternary’ and ‘incentric’ as we introduce here are literal: a *senary incentric subdivision* cuts a triangle into six daughter triangles by the

three angle bisectors (see Figures 3 and 4), and a *ternary incentric subdivision* cuts a triangle into three daughter triangles by the three angle bisectors truncated at the incenter (see Figures 5 and 6). The space of triangles used was the open equilateral triangle

$$\mathbb{P} = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = \pi, x > 0, y > 0, z > 0\}$$

sitting in the Euclidean space $\mathbb{R}^3$, in which each triangle with angles α, β, γ is represented as at most six points $(\alpha, \beta, \gamma), (\alpha, \gamma, \beta), (\beta, \alpha, \gamma), (\beta, \gamma, \alpha), (\gamma, \alpha, \beta), (\gamma, \beta, \alpha)$, and the descendant triangles were generated through iterations of linear maps on $\mathbb{R}^3$.

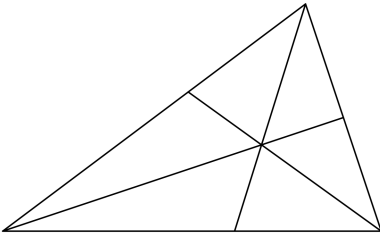


FIGURE 3.

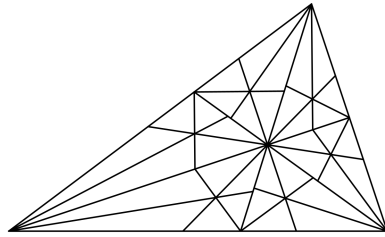


FIGURE 4.

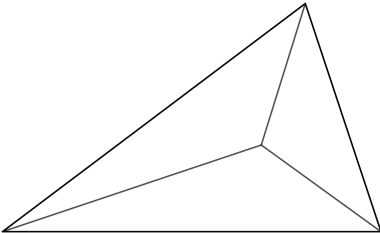


FIGURE 5.

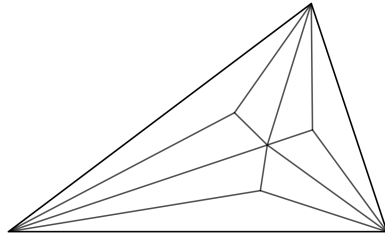


FIGURE 6.

1.2. Main Results. We now introduce two more incentric subdivision rules and their variants which, as far as we know, are new. We state our findings about them in conceptual forms first, and will re-state in precise terms in latter sections until we are ready to do so.

The *inward binary incentric subdivisions* begin with cutting an initial triangle into two daughter triangles by an angle bisector (see Figure 7). Each daughter triangle is cut into two granddaughter triangles by the angle bisector of its common angle shared with its mother triangle (see Figure 8), then each granddaughter triangle is cut into two great-granddaughter triangles by the angle bisector of its common angle shared with its mother triangle and so on (see Figures 9 and 10).

We discovered that, similar to senary incentric subdivision, the diversity of the shapes of the descendant triangles are extremely high:

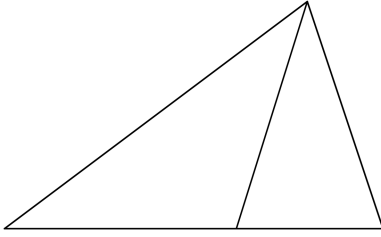


FIGURE 7.

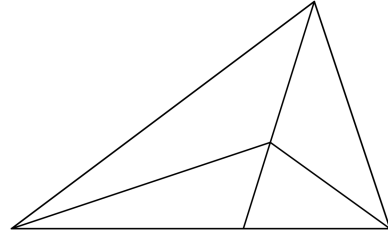


FIGURE 8.

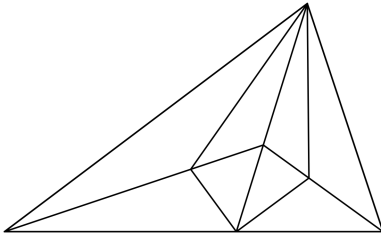


FIGURE 9.

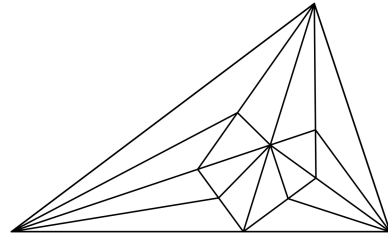


FIGURE 10.

Theorem 4. (Inward binary incentric subdivision) The descendants of a triangle generated by successive inward binary incentric subdivisions are dense in the space of triangles.

Moreover, we also discovered that as subdivision keeps going, the descendants are spread over the space of triangles more and more evenly. This creates a unique phenomenon that can be characterized in terms of weak convergence of measures:

Theorem 5. (Inward binary incentric subdivision) The sequence of empirical measures of the descendants of a triangle generated by successive inward binary incentric subdivisions converges weakly to the uniform probability measure on the closure of the space of triangles.

Theorems 4 and 5 will be re-stated in precise terms and proved in Section 3. Theorems 2 and 3 will also be re-stated in precise terms and re-proved using the approach similar to Theorem 4 in Sections 4 and 5.

The *outward binary incentric subdivisions* also begin with cutting an initial triangle into two daughter triangles by an angle bisector (see Figure 11). But this time, each daughter triangle is cut into two granddaughter triangles by the angle bisector of its angle opposite to its common side shared with its mother triangle (see Figure 12), then each granddaughter triangle is cut into two great-granddaughter triangles by the angle bisector of the angle opposite to its common angle shared with its mother triangle and so on (see Figures 13 and 14). We discovered that, similar to ternary incentric subdivision, the diversity of the shapes of the descendant triangles are extremely low (see Figures 47 to 56):

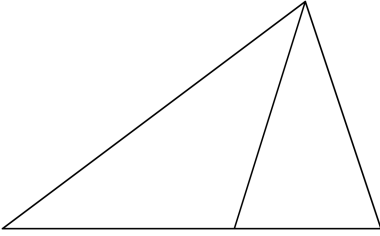


FIGURE 11.

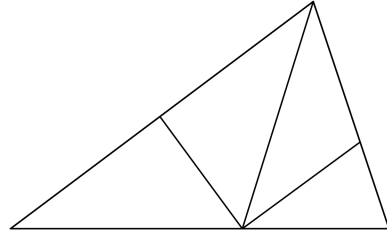


FIGURE 12.

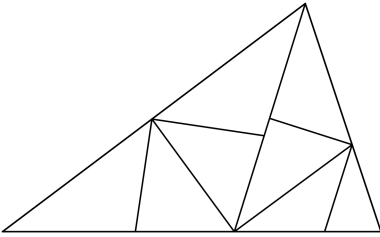


FIGURE 13.

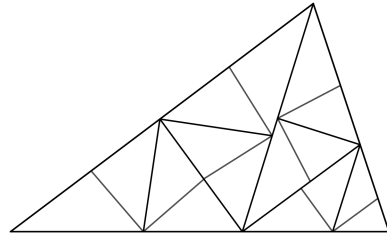


FIGURE 14.

Theorem 6. (Outward binary incentric subdivision) Every branch of a triangle generated by successive outward binary incentric subdivisions converges:

- (a) If the cuts is a zigzag, then among the three angles of the limit triangle, one angle is in common with the initial triangle, and one of the other two angles is double of the third angle.
- (b) If the cuts is eventually a zigzag, then among the three angles of the limit triangle, one angle is in common with the last daughter triangle before the zigzag started, and one of the other two angles is double of the third angle.
- (c) In all other cases, the limit triangle is the isosceles right-angled triangle.

Theorem 6 will be re-stated in precise terms and proved in Section 6.

The space of triangles to be used will be the open isosceles right-angled triangle

$$\mathbb{T} = \{(x, y) \in \mathbb{R}^2 : x + y < \pi, x > 0, y > 0\}$$

sitting in the Euclidean plane $\mathbb{R}^2$, in which each triangle with angles α, β, γ will be represented as at most six points $(\alpha, \beta), (\beta, \alpha), (\alpha, \gamma), (\gamma, \alpha), (\beta, \gamma), (\gamma, \beta)$, and the descendant triangles will be generated through iterations of affine maps on $\mathbb{R}^2$. Besides, we also generalized inward and outward binary incentric subdivision to *inward* (see Figures 15 to 18) and *outward* (see Figures 19 to 22) *binary r -quasi-incentric subdivisions* respectively, through replacing angle bisectors by cevians that divide angles into a fixed ‘directed’ ratio $r : s$, where $r + s = 1$ and $0 < r, s < 1$. We discovered that such generalizations preserve well the behaviors of descendant triangles as expressed in Theorems 4 and 6:

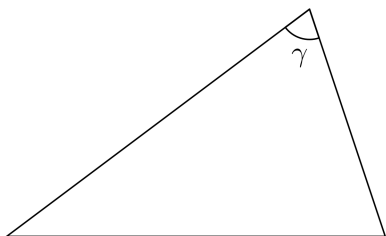


FIGURE 15.

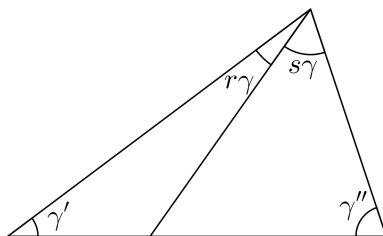


FIGURE 16.

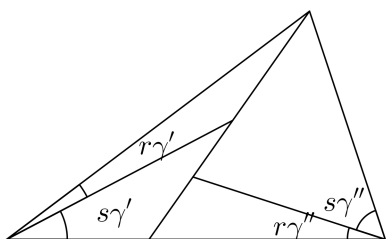


FIGURE 17.

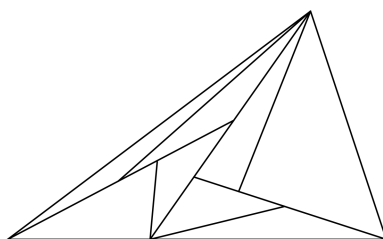


FIGURE 18.

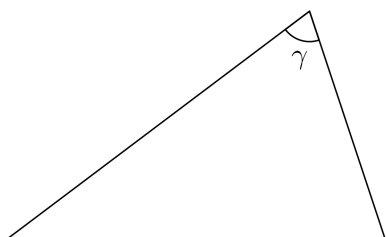


FIGURE 19.

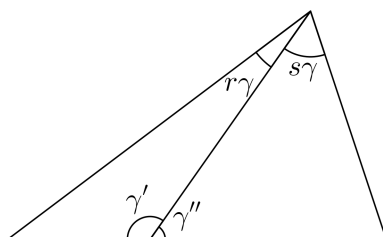


FIGURE 20.

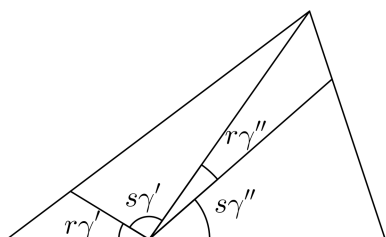


FIGURE 21.

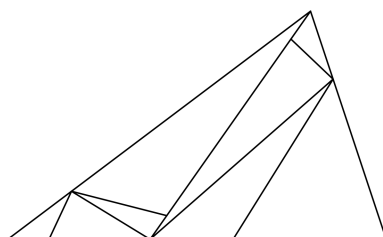


FIGURE 22.

Theorem 7. (Inward binary r -quasi-incentric subdivision) The descendants of a triangle generated by successive inward binary r -quasi-incentric subdivisions are dense in the space of triangles.

Theorem 8. (Outward binary r -quasi-incentric subdivision) Every branch of a triangle generated by successive outward binary r -quasi-incentric subdivisions converges:

- (a) If the cuts is a zigzag, then among the three angles of the limit triangle, one angle is in common with the initial triangle, and the other two angles are in the ratio $1 : r$ or $1 : s$.
- (b) If the cuts is eventually a zigzag, then among the three angles of the limit triangle, one angle is in common with the last daughter triangle before the zigzag started, and the other two angles are in the ratio $1 : r$ or $1 : s$.
- (c) If the cuts is not a zigzag nor eventually a zigzag, then the limit triangle is a right-angled triangle, and the other two angles are in the ratio $r : s$.

Theorems 7 and 8 will be re-stated in precise terms and proved in Section 7.

1.3. Outline of Paper. In Section 2, we will build useful tools with contraction maps on metric spaces. In Sections 2.1 and 2.2, we will recall J. E. Hutchinson's work (see [9]) on invariant sets and invariant measures of iterated function systems. Taking inspirations from his work, we will build three tools in Theorem 10 about approximation, convergence and denseness in complete metric spaces. In Section 2.3, we will build another tool in Theorem 11 which can produce convergence results when the Banach fixed-point theorem might not fit. In Section 2.4, we will recall how contractivity of affine maps on Euclidean spaces is related to matrix norms. In Section 3, we will discuss the inward binary incentric subdivision. We will construct an iterated function system on the Euclidean space $\mathbb{R}^2$ that generate the descendants. Then, Theorem 4 will follow from Theorem 10 through showing that $\bar{\mathbb{T}}$ is the invariant set. After that, we will use the weak convergence theory in Section 2.2 to generate the empirical measures of the descendants and to prove Theorem 5 through showing that the uniform probability measure on $\bar{\mathbb{T}}$ is the invariant measure of the iterated function system.

In Sections 4 and 5, we will discuss the senary and the ternary incentric subdivisions. We will construct iterated function systems on the affine subspace containing $\mathbb{P}$ in the Euclidean space $\mathbb{R}^3$ that generate the descendants. Then, Theorems 2 and 3 will follow from Theorem 10 through showing that $\bar{\mathbb{P}}$ and the Sierpiński triangle are the invariant set of the respective iterated function system.

In Section 6, we will discuss the outward binary incentric subdivision. We will construct division maps on the Euclidean space $\mathbb{R}^2$ that generate the descendants. We will prove Theorem 6(a)–(b) using Theorem 9. After that, Proposition 12 will motivate Theorem 6(c), which will then be proved using Theorem 11.

In Section 7, we will generalize Theorems 4 and 6 to quasi-incentric subdivisions. We will prove Theorems 7 and 8 in similar ways as Theorems 4 and 6.

2. CONTRACTION MAPS ON METRIC SPACE

2.1. Invariant Set of Iterated Function System. Consider a complete metric space (X, d) with a finite set of contraction maps $S = \{S_1, \dots, S_m\}$ on it, i.e., there exist constants $R_1, \dots, R_m \in [0, 1)$ such that for $i = 1, \dots, m$,

$$(1) \quad d(S_i(x), S_i(y)) \leq R_i d(x, y), \quad x, y \in X.$$

The set S is called an *iterated function system* on the complete metric space (X, d) . If the system has just one contraction map, then we have (see [10]):

Theorem 9. (Banach fixed-point theorem) Suppose (X, d) is a complete metric space and T is a contraction map on X . Then, T has a unique fixed point x^* . Moreover, if $x_0 \in X$, then $d(T^n(x_0), x^*) \rightarrow 0$ as $n \rightarrow \infty$.

The Banach fixed-point theorem will be used to prove Theorems 6(a)–(b) and 8(a)–(b), and will now be used to establish powerful tools to prove many other results in this paper.

Hutchinson considered the set $\mathcal{B}(X)$ of all non-empty closed bounded subsets of the metric space X and endowed it with the Hausdorff metric $d_H : \mathcal{B}(X) \times \mathcal{B}(X) \rightarrow [0, \infty)$ defined by

$$(2) \quad d_H(A, B) = \max \left\{ \sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B) \right\},$$

where $d(x, U) = \inf_{u \in U} d(x, u)$ for $x \in X$ and $U \in \mathcal{B}(X)$. By showing that $(\mathcal{B}(X), d_H)$ is also a complete metric space and that $S : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$ defined by

$$(3) \quad S(A) = \bigcup_{i=1}^m S_i(A)$$

is a contraction map on $\mathcal{B}(X)$ with respect to the Hausdorff metric d_H , one can then apply Theorem 9 to obtain a unique fixed point $K \in \mathcal{B}(X)$ of the contraction map S , i.e.,

$$(4) \quad K = S(K) = \bigcup_{i=1}^m S_i(K),$$

and that for any $A \in \mathcal{B}(X)$,

$$(5) \quad d_H(S^n(A), K) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The compact subset $K \in \mathcal{B}(X)$ characterized by (4) is called the *invariant set* of the iterated function system S . We now apply Hutchinson's theory to build our first tool that will be used to prove Theorems 2, 3, 4 and 7:

Theorem 10. (Metric properties of invariant set) Suppose (X, d) is a complete metric space and $S = \{S_1, \dots, S_m\}$ is a finite set of contraction maps on X satisfying (1). Then, the compact subset $K \in \mathcal{B}(X)$ characterized by (4) possesses the following properties:

(a) **(Approximation)** Suppose $x_0 \in X$ and $z \in K$. Then, for any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \{1, \dots, m\}$ such that

$$(6) \quad d(z, S_{i_1} \cdots S_{i_n}(x_0)) < \varepsilon.$$

- (b) **(Backward iteration)** Suppose $x_0 \in X$ and $z \in K$. Then, there exists a $\{1, \dots, m\}$ sequence $(i_n)_{n=1}^\infty$ such that the backward iterates

$$(S_{i_1} \cdots S_{i_n}(x_0))_{n=1}^\infty$$

of x_0 converges to z :

$$d(S_{i_1} \cdots S_{i_n}(x_0), z) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

- (c) **(Denseness)** Suppose $x_0 \in K$. Then, the set

$$(7) \quad \{S_{i_1} \cdots S_{i_n}(x_0) : n \in \mathbb{N}, i_1, \dots, i_n \in \{1, \dots, m\}\}$$

consisting of all iterates of x_0 under S is a dense subset of K .

First proof of Theorem 10(a)

Proof. Let $R = \max\{R_1, \dots, R_m\} \in [0, 1)$ and $M = \sup_{y \in K} d(x_0, y)$, which is finite as K is bounded. Then, there exists $n \in \mathbb{N}$ such that $R^n M < \varepsilon$. Applying ‘ $K \subset S(K)$ ’ in (4) repeatedly, there exist $i_1, \dots, i_n \in \{1, \dots, m\}$ and $x_1, \dots, x_n \in K$ such that

$$(8) \quad z = S_{i_1}(x_1), x_1 = S_{i_2}(x_2), x_2 = S_{i_3}(x_3), \dots, x_{n-1} = S_{i_n}(x_n).$$

Then, applying (8) and then (1) repeatedly, we have

$$\begin{aligned} d(z, S_{i_1} \cdots S_{i_n}(x_0)) &= d(S_{i_1} \cdots S_{i_n}(x_n), S_{i_1} \cdots S_{i_n}(x_0)) \\ &\leq R_{i_1} d(S_{i_2} \cdots S_{i_n}(x_n), S_{i_2} \cdots S_{i_n}(x_0)) \\ &\quad \vdots \\ &\leq R_{i_1} \cdots R_{i_n} d(x_n, x_0) \\ &\leq R^n M < \varepsilon. \end{aligned}$$

■

Second proof of Theorem 10(a) by using (5)

Proof. By (5) with $A = \{x_0\}$, there exists $n \in \mathbb{N}$ such that

$$(9) \quad d_H(S^n(x_0), K) < \varepsilon.$$

By (2) with $A = S^n(x_0)$ and $B = K$, we also have

$$(10) \quad d(z, S^n(x_0)) \leq \sup_{x \in K} d(x, S^n(x_0)) \leq d_H(S^n(x_0), K).$$

Combining (9) and (10), we have

$$d(z, S^n(x_0)) < \varepsilon,$$

but then as $S^n(x_0) = \{S_{i_1} \cdots S_{i_n}(x_0) : i_1, \dots, i_n \in \{1, \dots, m\}\}$ is just a finite set, it follows that there exist $i_1, \dots, i_n \in \{1, \dots, m\}$ such that

$$d(z, S_{i_1} \cdots S_{i_n}(x_0)) < \varepsilon.$$

■

Proof of Theorem 10(b)

Proof. Let $R = \max \{R_1, \dots, R_m\} \in [0, 1)$ and

$$M = \sup (\{d(x_0, y) : y \in K\} \cup \{d(x_0, S_i(x_0)) : i = 1, \dots, m\}),$$

which is finite as K is bounded. The required sequence $(i_n)_{n=1}^\infty$ can be constructed by applying the method in the first proof of Theorem 10(a) repeatedly. First, there exist $n_1 \in \mathbb{N}$, $i_1, \dots, i_{n_1} \in \{1, \dots, m\}$ and $x_1 \in K$ such that $z = S_{i_1} \cdots S_{i_{n_1}}(x_1)$ and

$$d(z, S_{i_1} \cdots S_{i_{n_1}}(x_0)) \leq R^{n_1} M.$$

Next, there exist $n_2 \in \mathbb{N}$, $i_{n_1+1}, \dots, i_{n_2} \in \{1, \dots, m\}$ and $x_2 \in K$ such that $n_2 > n_1$, $x_1 = S_{i_{n_1+1}} \cdots S_{i_{n_2}}(x_2)$ and

$$d(z, S_{i_1} \cdots S_{i_{n_2}}(x_0)) \leq R^{n_2} M.$$

Inductively, for each $k \in \mathbb{N}$ with $k > 1$, there exist $n_k \in \mathbb{N}$, $i_{n_{k-1}+1}, \dots, i_{n_k} \in \{1, \dots, m\}$ and $x_k \in K$ such that $n_k > n_{k-1}$, $x_{k-1} = S_{i_{n_{k-1}+1}} \cdots S_{i_{n_k}}(x_k)$ and

$$d(z, S_{i_1} \cdots S_{i_{n_k}}(x_0)) \leq R^{n_k} M.$$

Now, for each $n \in \mathbb{N}$, pick the greatest n_k constructed above such that $n_k \leq n$, then we have

$$\begin{aligned} d(z, S_{i_1} \cdots S_{i_n}(x_0)) &\leq d(z, S_{i_1} \cdots S_{i_{n_k}}(x_0)) \\ &\quad + d(S_{i_1} \cdots S_{i_{n_k}}(x_0), S_{i_1} \cdots S_{i_{n_{k+1}}}(x_0)) \\ &\quad + \cdots + d(S_{i_1} \cdots S_{i_{n-1}}(x_0), S_{i_1} \cdots S_{i_n}(x_0)) \\ &\leq R^{n_k} M + R^{n_k} d(x_0, S_{i_{n_k+1}}(x_0)) \\ &\quad + \cdots + R^{n-1} d(x_0, S_{i_n}(x_0)) \\ &\leq R^{n_k} M + R^{n_k} M + \cdots + R^{n-1} M \\ &< R^{n_k-1} M + R^{n_k} M + \cdots \\ &= \frac{R^{n_k-1} M}{1-R} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad \blacksquare$$

Proof of Theorem 10(c)

Proof. Applying ' $K \supset S(K)$ ' in (4), we have

$$S_1(K), \dots, S_m(K) \subset \bigcup_{i=1}^m S_i(K) \subset K,$$

thus if $x_0 \in K$, then (7) is a subset of K , and then its denseness in K follows from Theorem 10(a). ■

2.2. Invariant Measure of Iterated Function System. Given an iterated function system $S = \{S_1, \dots, S_m\}$ on a complete metric space (X, d) , Hutchinson also considered the set $\mathcal{P}(X)$ of all Borel regular probability measures with compact support on the metric space X and endowed it with the Wasserstein-1 metric (see [13]) $W_1 : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty)$ defined by

$$(11) \quad W_1(\mu, \nu) = \sup \left\{ \mathbb{E}_\mu[f] - \mathbb{E}_\nu[f] : f \in \mathcal{C}(X), \|f\|_{\text{Lip}} \leq 1 \right\},$$

where $\mathcal{C}(X)$ is the set of all real-valued continuous functions on X that are bounded on bounded subsets of X and $\|f\|_{\text{Lip}}$ is the Lipschitz constant of f . By showing that $(\mathcal{P}(X), W_1)$ is also a complete metric space (see [2]) and that $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ defined by

$$(12) \quad \Phi(\mu) = \frac{1}{m} \sum_{i=1}^m \mu \circ S_i^{-1}$$

is a contraction map on $\mathcal{P}(X)$ with respect to the Wasserstein-1 metric W_1 , one can then apply Theorem 9 to obtain a unique fixed point $\kappa \in \mathcal{P}(X)$ of the contraction map Φ , i.e.,

$$(13) \quad \kappa = \Phi(\kappa) = \frac{1}{m} \sum_{i=1}^m \kappa \circ S_i^{-1},$$

and that for any $\mu \in \mathcal{P}(X)$,

$$(14) \quad W_1(\Phi^n(\mu), \kappa) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The probability measure $\kappa \in \mathcal{P}(X)$ characterized by (13) is called the *invariant measure* of the iterated function system S , and is supported on the invariant set K in (4). Moreover, W_1 metrizes the weak convergence

$$\mu_n \rightharpoonup \mu \text{ as } n \rightarrow \infty$$

of probability measures in the space $\mathcal{P}(X)$ (see [13]), which is, by the Portmanteau theorem (see [2]), equivalent to that

$$(15) \quad \liminf_{n \rightarrow \infty} \mu_n(U) \geq \mu(U)$$

for all open sets $U \subset X$.

Third proof of Theorem 10(a) by using (15)

Proof. Take $x \in X$ and $\mu = \delta_x$ in (12), and let T be a self-map of X and $A \subset X$ be any Borel set. By the definition of preimage, $x \in T^{-1}(A)$ if and only if $T(x) \in A$, thus

$$\delta_x \circ T^{-1} = \delta_{T(x)}.$$

Now, define $\mu_0 = \delta_{x_0}$ and $\mu_n = \Phi(\mu_{n-1})$ for $n \in \mathbb{N}$ using (12). By (12),

$$\mu_1 = \Phi(\mu_0) = \frac{1}{m} \sum_{i=1}^m \mu_0 \circ S_i^{-1} = \frac{1}{m} \sum_{i=1}^m \delta_{x_0} \circ S_i^{-1} = \frac{1}{m} \sum_{i=1}^m \delta_{S_i(x_0)}$$

and, inductively,

$$\begin{aligned}
 \mu_n &= \Phi(\mu_{n-1}) = \frac{1}{m} \sum_{i=1}^m \mu_{n-1} \circ S_i^{-1} \\
 &= \frac{1}{m} \sum_{i=1}^m \left(\frac{1}{m^{n-1}} \left(\sum_{i_1, \dots, i_{n-1} \in \{1, \dots, m\}} \delta_{S_{i_1} \dots S_{i_{n-1}}(x_0)} \right) \circ S_i^{-1} \right) \\
 &= \frac{1}{m^n} \sum_{i=1}^m \sum_{i_1, \dots, i_{n-1} \in \{1, \dots, m\}} \delta_{S_i S_{i_1} \dots S_{i_{n-1}}(x_0)} \\
 (16) \quad &= \frac{1}{m^n} \sum_{i_1, \dots, i_n \in \{1, \dots, m\}} \delta_{S_{i_1} \dots S_{i_n}(x_0)}.
 \end{aligned}$$

By (14), we have $\mu_n \rightharpoonup \kappa$ as $n \rightarrow \infty$. For each $\varepsilon > 0$, let $U = B(z, \varepsilon)$ which is the open ball with center z and radius ε . Since $z \in K$ on which κ is supported, $\kappa(U) > 0$ by the definition of support, then by (15), there exists $n \in \mathbb{N}$ such that

$$(17) \quad \mu_n(U) > \kappa(U) - \frac{\kappa(U)}{2} = \frac{\kappa(U)}{2} > 0,$$

which means that there exist $i_1, \dots, i_n \in \{1, \dots, m\}$ such that $\delta_{S_{i_1} \dots S_{i_n}(x_0)}(U) > 0$, i.e. (6). ■

2.3. Joint Contraction Maps with Common Fixed Point. Here we build our second tool that will be used to prove Theorems 6(c) and 8(c):

Theorem 11. (Convergence theorem for joint contraction maps with common fixed point) Suppose (X, d) is a metric space and T_1 and T_2 are two joint contraction maps on X , in the sense that there exists a constant $R \in [0, 1)$ such that for $(i, j) = (1, 2), (2, 1)$ and $k, l \in \mathbb{N}$,

$$(18) \quad d(T_i^k T_j^l(x), T_i^k T_j^l(y)) \leq R d(x, y), \quad x, y \in X.$$

Suppose $x_0 \in X$ and $I = (i_n)_{n=1}^\infty$ is a $\{1, 2\}$ sequence consisting of infinitely many 1s and infinitely many 2s. If T_1 and T_2 have a common fixed point x^* , then

$$(19) \quad d(T_{i_n} \dots T_{i_1}(x_0), x^*) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Consider the sequence $J = (j_n)_{n=1}^\infty$ defined by

$$j_n = \sum_{k=2}^n |i_k - i_{k-1}|,$$

which keeps track on the number of changes as one walks along I . Let $i = i_1$, $j \in \{1, 2\} \setminus \{i_1\}$ and $m \in \mathbb{N}$ such that $j_m = 0$ and $j_{m+1} = 1$, we have

$$T_{i_n} \dots T_{i_1}(x) = \begin{cases} T_i^m(x) & , \quad j_n = 0 \\ (T_j \dots T_j T_i \dots T_i)(x) & , \quad j_n = 1 \\ (T_i \dots T_i T_j \dots T_j)(T_i^m(x)) & , \quad j_n = 2 \\ (T_j \dots T_j T_i \dots T_i)(T_j \dots T_j T_i \dots T_i)(x) & , \quad j_n = 3 \\ (T_i \dots T_i T_j \dots T_j)(T_i \dots T_i T_j \dots T_j)(T_i^m(x)) & , \quad j_n = 4 \\ \vdots & \vdots \end{cases}$$

thus by that $T_1(x^*) = x^* = T_2(x^*)$ and applying (18) repeatedly, we have

$$\begin{aligned} d(T_{i_n} \cdots T_{i_1}(x_0), x^*) &= d(T_{i_n} \cdots T_{i_1}(x_0), T_{i_n} \cdots T_{i_1}(x^*)) \\ &\leq R^{\lfloor \frac{1}{2}(j_n+1) \rfloor} \max \{d(x_0, x^*), d(T_{i_n}^m(x_0), T_{i_n}^m(x^*))\} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad \blacksquare$$

2.4. Contractive Affine Maps and Matrix Norms. Consider a normed vector space $(\mathbb{R}^n, \|\cdot\|)$. The *matrix norm* $\|A\|$ of an $n \times n$ real matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

induced by the vector norm $\|\cdot\|$ is defined as

$$\|A\| = \sup \{ \|A\mathbf{u}\| : \mathbf{u} \in \mathbb{R}^n, \|\mathbf{u}\| = 1 \},$$

and that it is *compatible* with the vector norm that induced it, i.e.,

$$\|A\mathbf{x}\| \leq \|A\| \|\mathbf{x}\|, \quad \mathbf{x} \in \mathbb{R}^n.$$

Also, a matrix norm $\|\cdot\|$ is said to be *sub-multiplicative* if

$$\|AB\| \leq \|A\| \|B\|.$$

In this paper, we will use affine maps $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto T(\mathbf{x}) = \mathbf{b} + A\mathbf{x},$$

where $\mathbf{b} \in \mathbb{R}^n$, to represent subdivisions of triangles. We shall call $\mathbf{b}$ the *constant part* and A the *matrix part* of the affine map T . If $\|A\| < 1$, then by the aforementioned compatibility,

$$\begin{aligned} \|T(\mathbf{x}) - T(\mathbf{y})\| &= \|(\mathbf{b} + A\mathbf{x}) - (\mathbf{b} + A\mathbf{y})\| \\ &= \|A(\mathbf{x} - \mathbf{y})\| \\ &\leq \|A\| \|\mathbf{x} - \mathbf{y}\|, \end{aligned}$$

and the affine map T will be contractive. Matrix norms that we will use are the matrix 1-norm

$$\|A\|_1 = \max \left\{ \sum_{i=1}^n |a_{i,j}| : j = 1, 2, \dots, n \right\}$$

induced by the vector 1-norm $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$, and the matrix 2-norm

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$$

induced by the vector 2-norm $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$. Moreover, both matrix norms are sub-multiplicative.

3. INWARD BINARY INCENTRIC SUBDIVISION

To perform inward binary incentric subdivisions, construct division maps as the following invertible affine maps $C_1, C_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto C_i(\mathbf{x}) = \mathbf{c} + M_i \mathbf{x}, \quad i = 1, 2,$$

with constant and matrix parts

$$\mathbf{c} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}, \quad M_1 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad M_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

To understand their effects, note that the representation $(\angle CAB, \angle ABC) \in \mathbb{T}$ of any triangle $\triangle ABC$ induces an orientation $\angle CAB \rightarrow \angle ABC \rightarrow \angle BCA$ or $CA \rightarrow AB \rightarrow BC$ on its angles or sides. If $\mathbf{q} = (\alpha, \beta) \in \mathbb{T}$ represents a mother triangle with angles α, β, γ , then as we represent the first and the second daughter triangles as

$$C_1(\mathbf{q}) = \begin{bmatrix} \frac{\pi}{2} - \frac{\alpha}{2} - \frac{\beta}{2} \\ \frac{\pi}{2} - \frac{\alpha}{2} + \frac{\beta}{2} \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{2} \\ \frac{\gamma}{2} + \beta \end{bmatrix} \in \mathbb{T}, \quad C_2(\mathbf{q}) = \begin{bmatrix} \frac{\pi}{2} + \frac{\alpha}{2} - \frac{\beta}{2} \\ \frac{\pi}{2} - \frac{\alpha}{2} - \frac{\beta}{2} \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{2} + \alpha \\ \frac{\gamma}{2} \end{bmatrix} \in \mathbb{T},$$

respectively (see Figure 23),

- (i) the unmentioned angle γ in the representation $\mathbf{q}$ of the mother triangle will be the angle to be cut,
- (ii) both daughter triangles take the reversed orientation on angles or sides as the mother triangle,
- (iii) the daughter triangles $C_1(\mathbf{q}), C_2(\mathbf{q})$ will share the common angles α, β respectively with the mother triangle, and
- (iv) these two angles will be cut in the next subdivision, making it an inward subdivision.

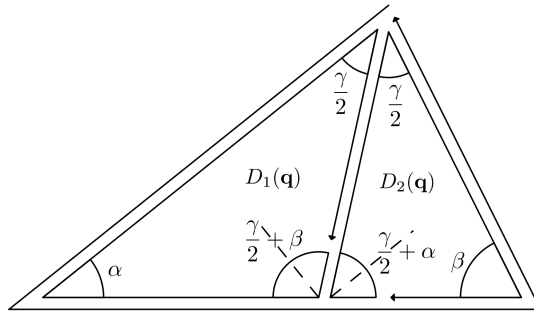


FIGURE 23.

To motivate Theorems 4 and 5 where the descendants are dense in $\mathbb{T}$ and their empirical measures approach the uniform measure on $\overline{\mathbb{T}}$, Figures 24 to 35 show some n -th generation descendants of the triangle $(\frac{9\pi}{20}, \frac{7\pi}{20}, \frac{\pi}{5})$ plotted in $\mathbb{P}$.

Here we re-state Theorem 4 in precise terms and prove it:

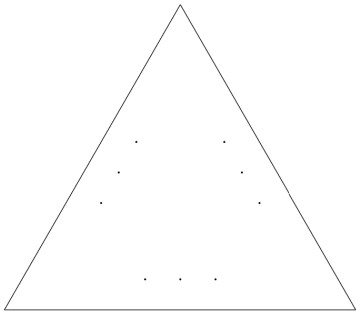


FIGURE 24. $n = 1$

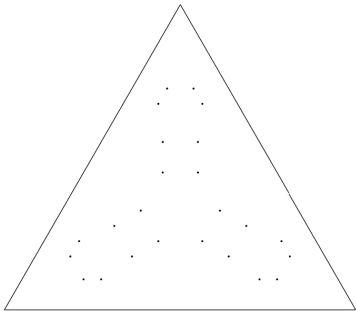


FIGURE 25. $n = 2$

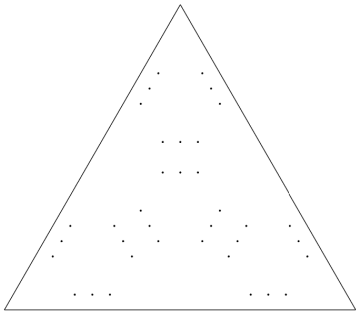


FIGURE 26. $n = 3$

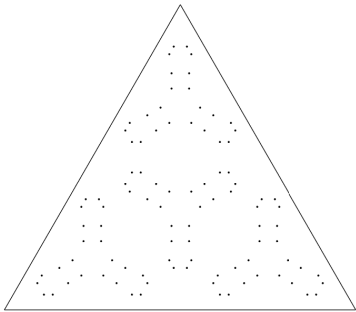


FIGURE 27. $n = 4$

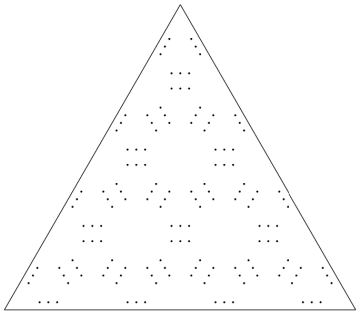


FIGURE 28. $n = 5$

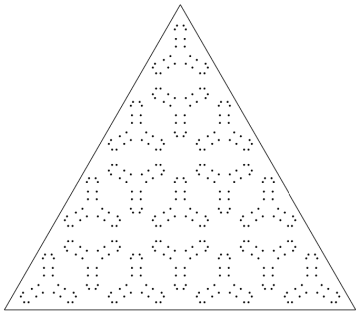


FIGURE 29. $n = 6$

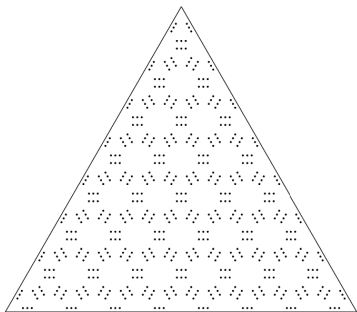


FIGURE 30. $n = 7$

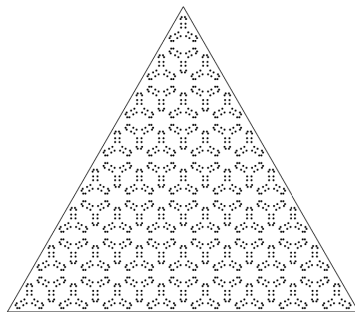


FIGURE 31. $n = 8$

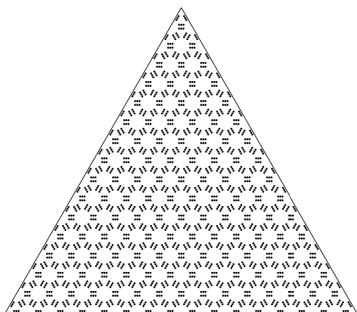


FIGURE 32. $n = 9$

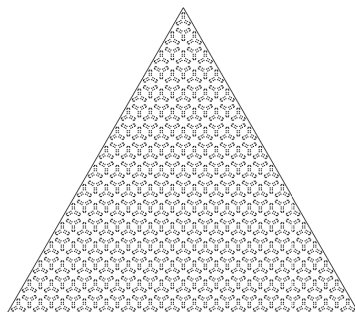


FIGURE 33. $n = 10$

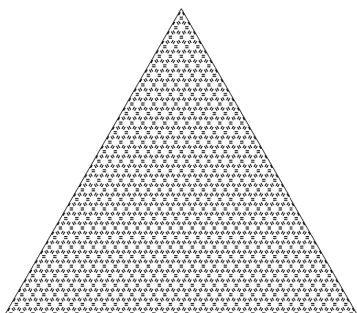


FIGURE 34. $n = 11$

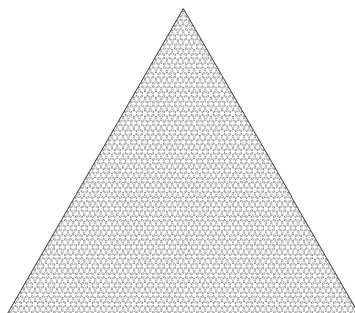


FIGURE 35. $n = 12$

Theorem 4. (Inward binary incentric subdivision) The descendants of a triangle generated by successive inward binary incentric subdivisions are dense in the space of triangles. More precisely, starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$, the set

$$(20) \quad \{C_{i_1} \cdots C_{i_n}(\mathbf{q}_0) : n \in \mathbb{N}, i_1, \dots, i_n \in \{1, 2\}\}$$

consisting of all iterates of $\mathbf{q}_0$ under

$$(21) \quad C = \{C_1, C_2\}$$

is a dense subset of $\mathbb{T}$.

Proof. First, we show that (21) is an iterated function system on $\mathbb{R}^2$. Indeed, by direct computations,

$$M_i^T M_i = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \|M_i\|_2^2 = \frac{1}{2}, \quad i = 1, 2,$$

thus, for $i \in \{1, 2\}$, since C_i is an affine map with matrix part M_i ,

$$(22) \quad \begin{aligned} d(C_i(\mathbf{x}), C_i(\mathbf{y})) &= \|M_i(\mathbf{x} - \mathbf{y})\|_2 \\ &\leq \|M_i\|_2 \|\mathbf{x} - \mathbf{y}\|_2 \\ &= \frac{1}{\sqrt{2}} d(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2. \end{aligned}$$

Next, we show that

$$(23) \quad C_1(\overline{\mathbb{T}}) \cup C_2(\overline{\mathbb{T}}) = \overline{\mathbb{T}}.$$

Indeed, since C_1, C_2 are affine maps and $\overline{\mathbb{T}}$ is a triangle, $C_1(\overline{\mathbb{T}}), C_2(\overline{\mathbb{T}})$ are the triangles with vertices

$$\begin{aligned} C_1\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}, \quad C_1\left(\begin{bmatrix} \pi \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_1\left(\begin{bmatrix} 0 \\ \pi \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \pi \end{bmatrix}, \\ C_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}, \quad C_2\left(\begin{bmatrix} \pi \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \quad C_2\left(\begin{bmatrix} 0 \\ \pi \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

respectively (see Figure 36).

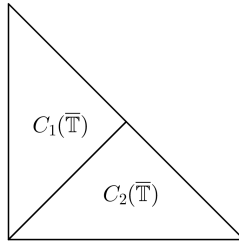


FIGURE 36.

By (22), (21) is an iterated function system on the complete metric space $\mathbb{R}^2$. Starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T} \subset \overline{\mathbb{T}}$, by (23) and Theorem 10(c), (20) is

a dense subset of $\overline{\mathbb{T}}$. Since $C_1, C_2 : \mathbb{T} \rightarrow \mathbb{T}$ and $\mathbf{q}_0 \in \mathbb{T}$, (20) is actually a dense subset of $\mathbb{T}$. $\blacksquare$

Here we re-state Theorem 5 in precise terms and prove it:

Theorem 5. (Inward binary incentric subdivision) The sequence of empirical measures of the descendants of a triangle generated by successive inward binary incentric subdivisions converges weakly to the uniform probability measure on the closure of the space of triangles. More precisely, starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$, the sequence $(\mu_n)_{n=1}^\infty$ of empirical measures of the n -th generation descendants defined by

$$(24) \quad \mu_n = \frac{1}{2^n} \sum_{i_1, \dots, i_n \in \{1, 2\}} \delta_{C_{i_1} \dots C_{i_n}(\mathbf{q}_0)}$$

converges weakly to the uniform probability measure μ^* on $\overline{\mathbb{T}}$.

Proof. We have already shown that (21) is an iterated function system on the complete metric space $\mathbb{R}^2$, thus the required weak convergence will follow from (14) through taking $x_0 = \mathbf{q}_0$ in (16) and checking that $\Phi(\mu^*) = \mu^*$. For any Borel set $A \subset \mathbb{R}^2$, by the definition of uniform probability measure and that μ^* is supported on $\overline{\mathbb{T}}$, we have

$$\mu^*(A) = \frac{\lambda(A \cap \overline{\mathbb{T}})}{\lambda(\overline{\mathbb{T}})},$$

and by noting that $\lambda((A \cap C_1(\overline{\mathbb{T}})) \cap (A \cap C_2(\overline{\mathbb{T}}))) \leq \lambda(C_1(\overline{\mathbb{T}}) \cap C_2(\overline{\mathbb{T}})) = 0$, we have

$$\begin{aligned} \lambda(A \cap C_1(\overline{\mathbb{T}})) + \lambda(A \cap C_2(\overline{\mathbb{T}})) &= \lambda((A \cap C_1(\overline{\mathbb{T}})) \cup (A \cap C_2(\overline{\mathbb{T}}))) \\ &= \lambda(A \cap (C_1(\overline{\mathbb{T}}) \cup C_2(\overline{\mathbb{T}}))) \\ &= \lambda(A \cap \overline{\mathbb{T}}). \end{aligned}$$

Also, as each C_i is an invertible affine map with $\det M_i = -\frac{1}{2}$, we have

$$C_i^{-1}(A) \cap \overline{\mathbb{T}} = C_i^{-1}(A) \cap C_i^{-1}(C_i(\overline{\mathbb{T}})) = C_i^{-1}(A \cap (C_i(\overline{\mathbb{T}}))).$$

Therefore,

$$\begin{aligned} \Phi(\mu^*)(A) &= \frac{1}{2} (\mu^*(C_1^{-1}(A)) + \mu^*(C_2^{-1}(A))) \\ &= \frac{1}{2} \left(\frac{\lambda(C_1^{-1}(A) \cap \overline{\mathbb{T}})}{\lambda(\overline{\mathbb{T}})} + \frac{\lambda(C_2^{-1}(A) \cap \overline{\mathbb{T}})}{\lambda(\overline{\mathbb{T}})} \right) \\ &= \frac{1}{2} \left(\frac{\lambda(C_1^{-1}(A \cap (C_1(\overline{\mathbb{T}}))))}{\lambda(\overline{\mathbb{T}})} + \frac{\lambda(C_2^{-1}(A \cap (C_2(\overline{\mathbb{T}}))))}{\lambda(\overline{\mathbb{T}})} \right) \\ &= \frac{1}{2} \left(\frac{|\det(M_1^{-1})| \lambda(A \cap (C_1(\overline{\mathbb{T}})))}{\lambda(\overline{\mathbb{T}})} + \frac{|\det(M_2^{-1})| \lambda(A \cap (C_2(\overline{\mathbb{T}})))}{\lambda(\overline{\mathbb{T}})} \right) \\ &= \frac{\lambda(A \cap \overline{\mathbb{T}})}{\lambda(\overline{\mathbb{T}})} = \mu^*(A), \end{aligned}$$

and μ^* is the fixed point of Φ as required. ■

4. SENARY INCENTRIC SUBDIVISION

To perform senary incentric subdivisions, construct division maps as the following invertible linear maps $A_1, \dots, A_6 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto A_i(\mathbf{x}) = K_i \mathbf{x}, \quad K_i = E_i K_0 E_i, \quad i = 1, \dots, 6,$$

where

$$K_0 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

To motivate Theorem 2 where the descendants are dense in $\mathbb{P}$, Figures 37 to 38 show some n -th generation descendants of the triangle $(\frac{9\pi}{20}, \frac{7\pi}{20}, \frac{\pi}{5})$ plotted in $\mathbb{P}$. In fact, Butler and Graham [3] mentioned that there exists a limiting distribution of the senary daughters, and produced histograms to approximate it. Figure 39 shows a similar histogram that we produced in higher resolution. Our histogram has been darkened on a log scale, to better illustrate the denseness of the descendants in $\mathbb{P}$.

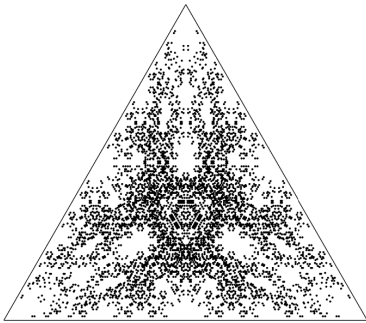


FIGURE 37. $n = 4$

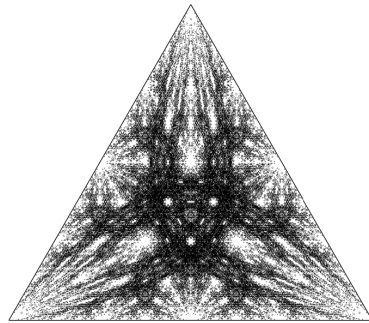
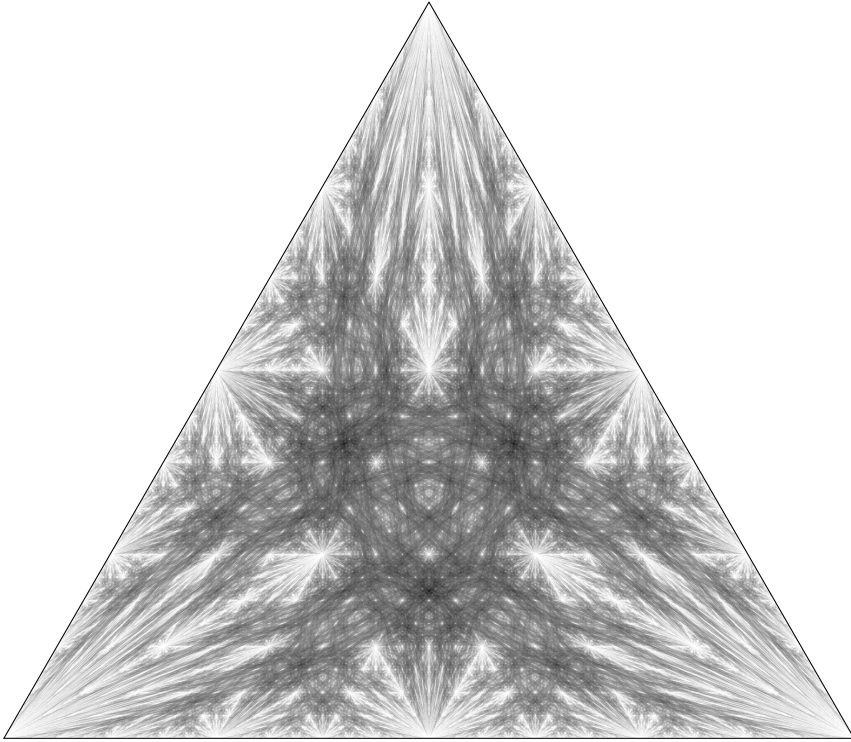


FIGURE 38. $n = 6$

Here we re-state Theorem 2 in precise terms and prove it:

Theorem 2. (Senary incentric subdivision) The descendants of a triangle generated by successive senary incentric subdivisions approximate arbitrarily closely any given triangle. Indeed, they are dense in the space of triangles. More precisely, starting with any initial triangle $\mathbf{q}_0 \in \mathbb{P}$, the set

$$(25) \quad \{A_{i_1} \cdots A_{i_n}(\mathbf{q}_0) : n \in \mathbb{N}, i_1, \dots, i_n \in \{1, \dots, 6\}\}$$


 FIGURE 39. $n = 11$ (histogram)

consisting of all iterates of $\mathbf{q}_0$ under

$$(26) \quad A = \{A_1, \dots, A_6\}$$

is a dense subset of $\mathbb{P}$.

Proof. First, we show that (26) is an iterated function system when restricted to the affine subspace

$$(27) \quad \mathbb{V} = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = \pi\}$$

of $\mathbb{R}^3$. Provide $\mathbb{V}$ with an $\mathbb{R}^2$ structure by taking $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ as the origin and by taking orthonormal basis

$$\begin{aligned} \mathbf{i}' &= \frac{-1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}, \\ \mathbf{j}' &= \frac{-1}{\sqrt{6}}\mathbf{i} + \frac{-1}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k} \end{aligned}$$

for the linear subspace

$$(28) \quad \mathbb{W} = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$$

of $\mathbb{R}^3$. Then, every $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathbb{W}$ admits a unique representation $\mathbf{a} + x'\mathbf{i}' + y'\mathbf{j}' \in \mathbb{V}$, with change of coordinates given by

$$(29) \quad \mathbf{x} = \mathbf{a} + Q^T \mathbf{x}',$$

$$(30) \quad \mathbf{x}' = Q(\mathbf{x} - \mathbf{a}),$$

where

$$(31) \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} \frac{\pi}{3} \\ \frac{\pi}{3} \\ \frac{\pi}{3} \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}, \quad QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

And then, in (x', y') coordinates, as

$$A_i(\mathbf{x}) = Q(A_i(\mathbf{a} + Q^T \mathbf{x}') - \mathbf{a}) = (QK_i \mathbf{a} - Q\mathbf{a}) + (QK_i Q^T) \mathbf{x}',$$

we can rewrite each $A_i|_{\mathbb{V}} : \mathbb{V} \rightarrow \mathbb{V}$ as

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \mapsto \mathbf{a}'_i + K'_i \mathbf{x}',$$

where

$$\mathbf{a}'_i = QK_i \mathbf{a} - Q\mathbf{a}, \quad K'_i = QK_i Q^T.$$

By direct computations,

$$K'_1 = \begin{bmatrix} 0 & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{2} & 0 \end{bmatrix}, \quad K'_3 = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{2} & 0 \end{bmatrix},$$

$$K'_2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{3} \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad K'_5 = \begin{bmatrix} \frac{1}{4} & \frac{5\sqrt{3}}{12} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix},$$

$$K'_4 = \begin{bmatrix} -\frac{1}{4} & \frac{5\sqrt{3}}{12} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} \end{bmatrix}, \quad K'_6 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{3} \\ 0 & \frac{1}{2} \end{bmatrix},$$

$$K'^T_1 K'_1 = K'^T_3 K'_3 = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{12} \end{bmatrix},$$

$$K'^T_2 K'_2 = K'^T_5 K'_5 = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{7}{12} \end{bmatrix},$$

$$K'^T_4 K'_4 = K'^T_6 K'_6 = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{7}{12} \end{bmatrix},$$

$$\|K'_i\|_2^2 = \max \left\{ \lambda : \det \left(K'^T_i K'_i - \lambda I \right) = 0 \right\} = \frac{3}{4}, \quad i = 1, \dots, 6.$$

As a result, for $i \in \{1, \dots, 6\}$,

$$\begin{aligned}
 d(A_i(\mathbf{x}), A_i(\mathbf{y})) &= \|K'_i(\mathbf{x} - \mathbf{y})\|_2 \\
 &\leq \|K'_i\|_2 \|\mathbf{x} - \mathbf{y}\|_2 \\
 (32) \qquad \qquad \qquad &= \frac{\sqrt{3}}{2} d(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{V}.
 \end{aligned}$$

Next, we show that

$$(33) \qquad A_1(\overline{\mathbb{P}}) \cup \dots \cup A_6(\overline{\mathbb{P}}) = \overline{\mathbb{P}}.$$

Indeed, since $A_1, \dots, A_6$ are linear maps and $\overline{\mathbb{P}}$ is a triangle, $A_1(\overline{\mathbb{P}}), \dots, A_6(\overline{\mathbb{P}})$ are the triangles with vertices

$$\begin{aligned}
 A_1 \left(\begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \\ 0 \end{bmatrix}, \quad A_1 \left(\begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix}, \quad A_1 \left(\begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}, \\
 A_2 \left(\begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ \frac{\pi}{2} \end{bmatrix}, \quad A_2 \left(\begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}, \quad A_2 \left(\begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix}, \\
 A_3 \left(\begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix}, \quad A_3 \left(\begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \\ 0 \end{bmatrix}, \quad A_3 \left(\begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix} \right) = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ \frac{\pi}{2} \end{bmatrix}, \\
 A_4 \left(\begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \\ 0 \end{bmatrix}, \quad A_4 \left(\begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ \frac{\pi}{2} \end{bmatrix}, \quad A_4 \left(\begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix}, \\
 A_5 \left(\begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix}, \quad A_5 \left(\begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ \frac{\pi}{2} \end{bmatrix}, \quad A_5 \left(\begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}, \\
 A_6 \left(\begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \\ 0 \end{bmatrix}, \quad A_6 \left(\begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix}, \quad A_6 \left(\begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}
 \end{aligned}$$

respectively (see Figure 40).

By (32), (26) is an iterated function system on the complete metric space $\mathbb{V}$. Starting with any initial triangle $\mathbf{q}_0 \in \mathbb{P} \subset \overline{\mathbb{P}}$, by (33) and Theorem 10(c), (25) is a dense subset of $\overline{\mathbb{P}}$. Since $A_1, \dots, A_6 : \mathbb{P} \rightarrow \mathbb{P}$ and $\mathbf{q}_0 \in \mathbb{P}$, (25) is actually a dense subset of $\mathbb{P}$. ■

Remark. Compared to Butler and Graham [4], we have shown an explicit computation of the Lipschitz constant $\frac{\sqrt{3}}{2}$ using the matrix 2-norm.

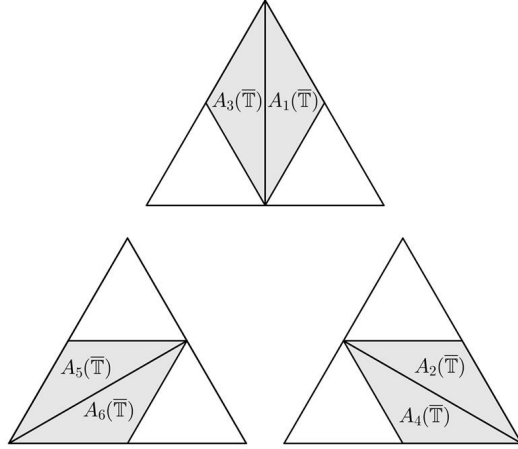


FIGURE 40.

5. TERNARY INCENTRIC SUBDIVISION

To perform ternary incentric subdivisions, construct division maps as the following invertible linear maps $B_1, B_2, B_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto B_i(\mathbf{x}) = L_i \mathbf{x}, \quad i = 1, 2, 3,$$

where

$$L_1 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad L_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad L_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

Let $\mathbb{S} \subset \overline{\mathbb{P}}$ denote the Sierpiński triangle in the closure of the space of triangles.

To motivate Theorem 3 where the descendants approximate Sierpińskian triangles arbitrarily closely, Figures 41 to 44 show some n -th generation descendants of the triangle $(\frac{9\pi}{20}, \frac{7\pi}{20}, \frac{\pi}{5})$.

Here we recall Theorem 3 and prove it:

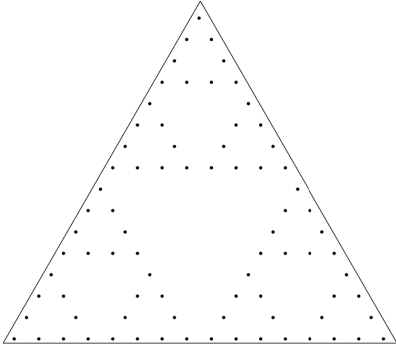
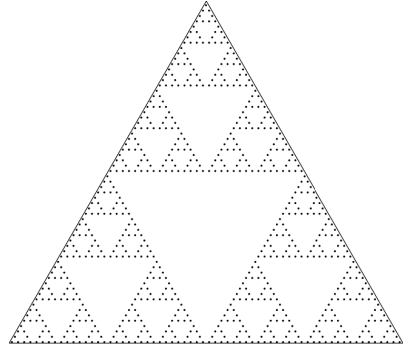
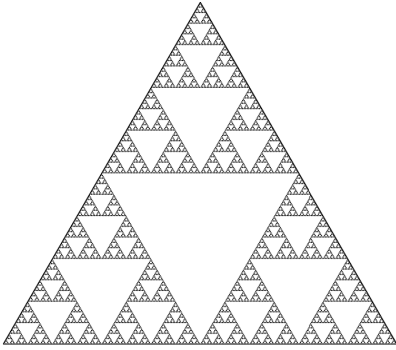
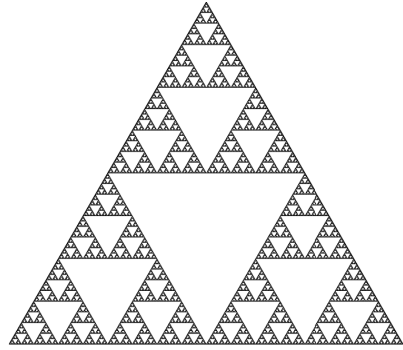
Theorem 3. (Ternary incentric subdivision) The descendants of a triangle generated by successive ternary incentric subdivisions approximate arbitrarily closely any Sierpińskian triangle, which means a triangle represented by a point in $\mathbb{S}$. Indeed, if the initial triangle is a Sierpińskian triangle, then the descendants are dense in the Sierpiński triangle. More precisely, starting with any initial triangle $\mathbf{q}_0 \in \mathbb{S} \cap \mathbb{T}$, the set

$$(34) \quad \{B_{i_1} \cdots B_{i_n}(\mathbf{q}_0) : n \in \mathbb{N}, i_1, \dots, i_n \in \{1, 2, 3\}\}$$

consisting of all iterates of $\mathbf{q}_0$ under

$$(35) \quad B = \{B_1, B_2, B_3\}$$

is dense in $\mathbb{S}$.


 FIGURE 41. $n = 4$

 FIGURE 42. $n = 6$

 FIGURE 43. $n = 8$

 FIGURE 44. $n = 10$

Proof. First, we show that (35) is an iterated function system when restricted to the affine subspace $\mathbb{V}$ in (27). Using the $\mathbb{R}^2$ structure on $\mathbb{V}$ and the change of coordinates given by (29)–(31), in (x', y') coordinates, as $B_i(\mathbf{x}) = Q(B_i(\mathbf{a} + Q^T \mathbf{a}') - \mathbf{a}) = (QL_i \mathbf{a} - Q\mathbf{a}) + (QL_i Q^T) \mathbf{x}'$, we can rewrite each $B_i|_{\mathbb{V}} : \mathbb{V} \rightarrow \mathbb{V}$ as

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \mapsto \mathbf{b}'_i + L'_i \mathbf{x}',$$

where

$$\mathbf{b}'_i = QL_i \mathbf{a} - Q\mathbf{a}, \quad L'_i = QL_i Q^T.$$

By direct computations,

$$L'_i = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \|L'_i\|_2^2 = \frac{1}{2}, \quad i = 1, 2, 3.$$

As a result, for $i \in \{1, 2, 3\}$,

$$\begin{aligned}
 d(B_i(\mathbf{x}), B_i(\mathbf{y})) &= \|L'_i(\mathbf{x} - \mathbf{y})\|_2 \\
 &\leq \|L'_i\|_2 \|\mathbf{x} - \mathbf{y}\|_2 \\
 (36) \qquad \qquad \qquad &= \frac{1}{\sqrt{2}} d(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{V}.
 \end{aligned}$$

Next, that $\mathbb{S}$ is the invariant set of (35), i.e.

$$(37) \qquad B_1(\mathbb{S}) \cup B_2(\mathbb{S}) \cup B_3(\mathbb{S}) = \mathbb{S},$$

simply follows from how B_1, B_2, B_3 correspond to the well-known self-similarity of $\mathbb{S}$ (see 45).

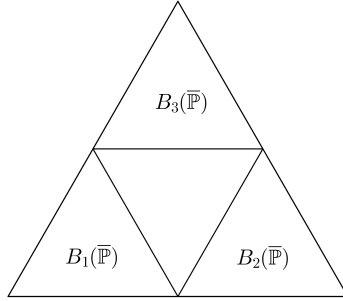


FIGURE 45.

By (36), (35) is an iterated function system on the complete metric space $\mathbb{V}$. Starting with any initial triangle $\mathbf{q}_0 \in \mathbb{S} \cap \mathbb{T}$, by (37) and Theorem 10(c), (34) is dense in $\mathbb{S}$. ■

Remark. We can explicitly find the sequence of descendants that converges to any given $\mathbf{s} \in \mathbb{S}$ from barycentric coordinates with respect to the vertices $U(\pi, 0, 0)$, $V(0, \pi, 0)$ and $W(0, 0, \pi)$ of $\mathbb{S}$:

$$\mathbf{s} = 0.u_1u_2u_3 \dots U + 0.v_1v_2v_3 \dots V + 0.w_1w_2w_3 \dots W,$$

where $u_n, v_n, w_n \in \{0, 1\}$ and $u_n + v_n + w_n = 1$ for all n . This is a well-known property of $\mathbb{S}$. Then, the backward iteration $(B_{i_1}B_{i_2} \dots B_{i_n}(\mathbf{q}_0))_{n=1}^\infty$ constructed by

$$i_k = \begin{cases} 1 & \text{if } u_k = 1 \\ 2 & \text{if } v_k = 1 \\ 3 & \text{if } w_k = 1 \end{cases}$$

converges to $\mathbf{s}$.

6. OUTWARD BINARY INCENTRIC SUBDIVISION

To perform outward binary incentric subdivisions, construct division maps as the following invertible affine maps $D_1, D_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto D_i(\mathbf{x}) = \mathbf{d}_i + N_i \mathbf{x}, \quad i = 1, 2,$$

with constant and matrix parts

$$\mathbf{d}_1 = \begin{bmatrix} 0 \\ \frac{\pi}{2} \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \quad N_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

To understand their effects, note that the representation $(\angle CAB, \angle ABC) \in \mathbb{T}$ of any triangle $\triangle ABC$ induces an orientation $\angle CAB \rightarrow \angle ABC \rightarrow \angle BCA$ or $CA \rightarrow AB \rightarrow BC$ on its angles or sides. If $\mathbf{q} = (\alpha, \beta) \in \mathbb{T}$ represents a mother triangle with angles α, β, γ , then as we represent the first and the second daughter triangles as

$$D_1(\mathbf{q}) = \begin{bmatrix} \frac{\pi}{2} - \frac{\alpha}{2} - \frac{\beta}{2} \\ \frac{\alpha}{2} \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{2} \\ \frac{\alpha}{2} \end{bmatrix} \in \mathbb{T}, \quad D_2(\mathbf{q}) = \begin{bmatrix} \frac{\pi}{2} - \frac{\alpha}{2} - \frac{\beta}{2} \\ \frac{\beta}{2} \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{2} \\ \frac{\beta}{2} \end{bmatrix} \in \mathbb{T},$$

respectively (see Figure 46),

- (i) the unmentioned angle γ in the representation $\mathbf{q}$ of the mother triangle will be the angle to be cut,
- (ii) both daughter triangles take the reversed orientation on angles or sides as the mother triangle,
- (iii) the daughter triangles $D_1(\mathbf{q}), D_2(\mathbf{q})$ will share the common angles α, β respectively with the mother triangle, and each will share a common side with the mother triangle, and
- (iv) the two angles opposite to these two common sides will be cut in the next subdivision, making it an outward subdivision.

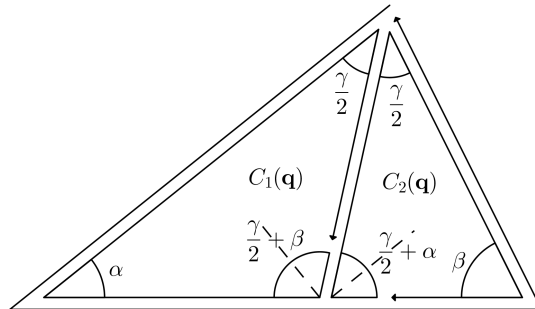


FIGURE 46.

To motivate Theorem 6(a)–(b) where the branches are converging to a limited choice of triangles, Figures 47 to 52 show the effects of D_1 and D_2 . The green and

red lines are the points in $\mathbb{T}$ of the forms

$$\begin{bmatrix} \alpha \\ \frac{\pi-\alpha}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{\pi-\beta}{3} \\ \beta \end{bmatrix}$$

respectively.

Here we re-state Theorem 6(a) in precise terms and prove it:

Theorem 6(a). (Outward binary incentric subdivision; Pure iterations)

Every branch of a triangle generated by successive outward binary incentric subdivisions converges. In particular:

- (a) If the cuts is a zigzag, then among the three angles of the limit triangle, one angle is in common with the initial triangle, and one of the other two angles is double of the third angle.

More precisely, starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$, the pure iterations converge with

$$D_1^n(\mathbf{q}_0) \rightarrow \begin{bmatrix} \alpha_0 \\ \frac{\pi-\alpha_0}{3} \end{bmatrix} \quad \text{as } n \rightarrow \infty, \quad D_2^n(\mathbf{q}_0) \rightarrow \begin{bmatrix} \frac{\pi-\beta_0}{3} \\ \beta_0 \end{bmatrix} \quad \text{as } n \rightarrow \infty.$$

Proof. Let $\mathbf{q}_{1,n} = \begin{bmatrix} \alpha_{1,n} \\ \beta_{1,n} \end{bmatrix}$ denote $D_1^n(\mathbf{q}_0)$. Since $D_1^n(\mathbf{q}_0) = D_1(D_1^{n-1}(\mathbf{q}_0))$, we have

$$(38) \quad \begin{bmatrix} \alpha_{1,n} \\ \beta_{1,n} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\pi}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \alpha_{1,n-1} \\ \beta_{1,n-1} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \frac{\pi-\alpha_0}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \alpha_{1,n-1} \\ \beta_{1,n-1} \end{bmatrix},$$

or simply $\mathbf{q}_{1,n} = D'_1(\mathbf{q}_{1,n-1})$, where $\mathbf{q}_{1,0} = \mathbf{q}_0$ and $D'_1(\mathbf{x}) = \mathbf{d}'_1 + N'_1\mathbf{x}$ is an affine map with constant and matrix parts

$$(39) \quad \mathbf{d}'_1 = \begin{bmatrix} \alpha_0 \\ \frac{\pi-\alpha_0}{2} \end{bmatrix}, \quad N'_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

Note that D'_1 is a contraction map with respect to $\|\cdot\|_1$ as

$$(40) \quad \begin{aligned} \|D'_1(\mathbf{x}) - D'_1(\mathbf{y})\|_1 &= \|N'_1(\mathbf{x} - \mathbf{y})\|_1 \\ &\leq \|N'_1\|_1 \|\mathbf{x} - \mathbf{y}\|_1 \\ &= \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|_1, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \end{aligned}$$

thus by Theorem 9, $\mathbf{q}_{1,n}$ converges to the fixed point $\mathbf{x}_1^*$ of D'_1 , which can be solved by

$$(41) \quad \mathbf{x}_1^* = D'_1(\mathbf{x}_1^*) = \mathbf{d}'_1 + N'_1\mathbf{x}_1^*$$

$$(42) \quad \mathbf{x}_1^* = (I - N'_1)^{-1} \mathbf{d}'_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \frac{\pi-\alpha_0}{2} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \frac{\pi-\alpha_0}{3} \end{bmatrix}.$$

Similarly, now let $\mathbf{q}_{2,n} = \begin{bmatrix} \alpha_{2,n} \\ \beta_{2,n} \end{bmatrix}$ denote $D_2^n(\mathbf{q}_0)$. Since $D_2^n(\mathbf{q}_0) = D_2(D_2^{n-1}(\mathbf{q}_0))$, we have

$$(43) \quad \begin{bmatrix} \alpha_{2,n} \\ \beta_{2,n} \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{2,n-1} \\ \beta_{2,n-1} \end{bmatrix} = \begin{bmatrix} \frac{\pi-\beta_0}{2} \\ \beta_0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{2,n-1} \\ \beta_{2,n-1} \end{bmatrix},$$

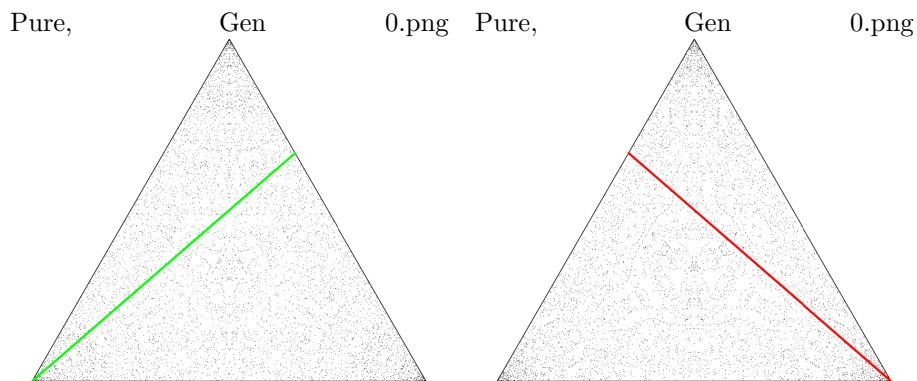


FIGURE 47. $D_1^0(\mathbb{T})$

FIGURE 48. $D_2^0(\mathbb{T})$

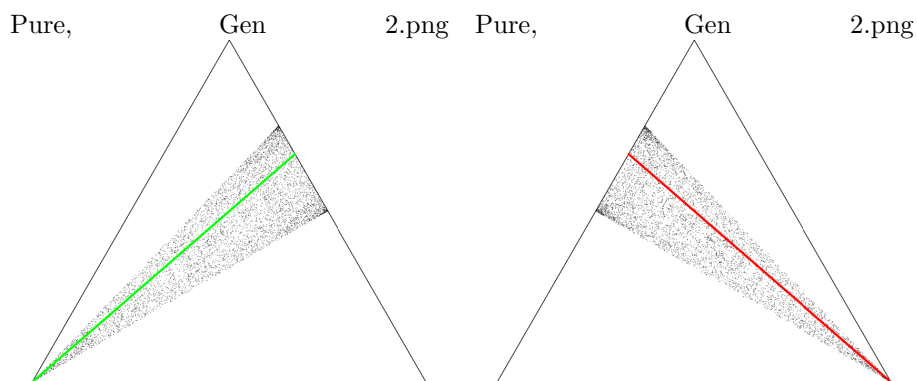


FIGURE 49. $D_1^2(\mathbb{T})$

FIGURE 50. $D_2^2(\mathbb{T})$

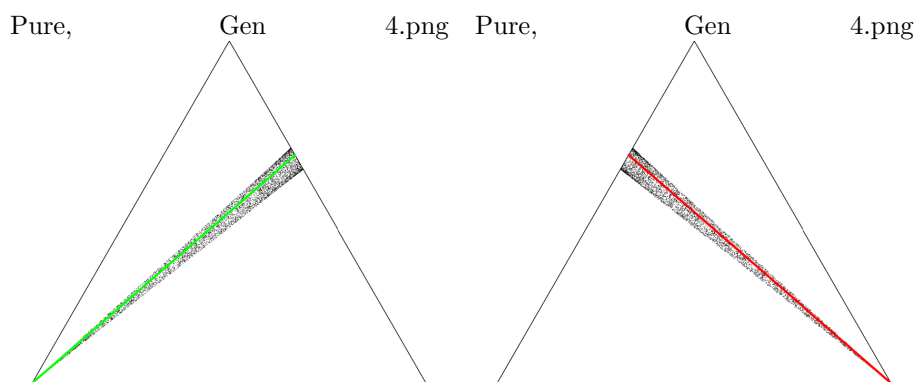


FIGURE 51. $D_1^4(\mathbb{T})$

FIGURE 52. $D_2^4(\mathbb{T})$

or simply $\mathbf{q}_{2,n} = D'_2(\mathbf{q}_{2,n-1})$, where $\mathbf{q}_{2,0} = \mathbf{q}_0$ and $D'_2(\mathbf{x}) = \mathbf{d}'_2 + N'_2\mathbf{x}$ is an affine map with constant and matrix parts

$$(44) \quad \mathbf{d}'_2 = \begin{bmatrix} \frac{\pi - \beta_0}{2} \\ \beta_0 \end{bmatrix}, \quad N'_2 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that D'_2 is a contraction map with respect to $\|\cdot\|_1$ as

$$(45) \quad \begin{aligned} \|D'_2(\mathbf{x}) - D'_2(\mathbf{y})\|_1 &= \|N'_2(\mathbf{x} - \mathbf{y})\|_1 \\ &\leq \|N'_2\|_1 \|\mathbf{x} - \mathbf{y}\|_1 \\ &= \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|_1, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \end{aligned}$$

thus by Theorem 9, $\mathbf{q}_{2,n}$ converges to the fixed point $\mathbf{x}_2^*$ of D'_2 , which can be solved by

$$(46) \quad \mathbf{x}_2^* = D'_2(\mathbf{x}_2^*) = \mathbf{d}'_2 + N'_2\mathbf{x}_2^*$$

$$(47) \quad \mathbf{x}_2^* = (I - N'_2)^{-1} \mathbf{d}'_2 = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\pi - \beta_0}{2} \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{\pi - \beta_0}{3} \\ \beta_0 \end{bmatrix}. \quad \blacksquare$$

Here we re-state Theorem 6(b) in precise terms, which follows from Theorem 6(a) as a corollary:

Theorem 6(b). (Outward binary incentric subdivision; Eventually pure iterations) Every branch of a triangle generated by successive outward binary incentric subdivisions converges. In particular:

- (b) If the cuts is eventually a zigzag, then among the three angles of the limit triangle, one angle is in common with the last daughter triangle before the zigzag started, and one of the other two angles is double of the third angle.

More precisely, suppose $I = (i_n)_{n=1}^\infty$ is a $\{1, 2\}$ sequence with finitely many 1 or finitely many 2. Starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$:

- (i) If i_k is the last 1 in I , then the eventually pure iteration

$$D_{i_n} \cdots D_{i_1}(\mathbf{q}_0) \rightarrow \begin{bmatrix} \frac{\pi - \beta_k}{3} \\ \beta_k \end{bmatrix} \quad \text{as } n \rightarrow \infty.$$

- (ii) If i_k is the last 2 in I , then the eventually pure iteration

$$D_{i_n} \cdots D_{i_1}(\mathbf{q}_0) \rightarrow \begin{bmatrix} \alpha_k \\ \frac{\pi - \alpha_k}{3} \end{bmatrix} \quad \text{as } n \rightarrow \infty.$$

Proposition 12. (Outward binary incentric subdivision; Alternating iterations) Starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$, for $(i, j) = (1, 2), (2, 1)$, the alternating iteration

$$(D_j D_i)^n(\mathbf{q}_0) \rightarrow \begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix} \quad \text{as } n \rightarrow \infty.$$

Proof. Since D_1 and D_2 are affine maps, so are $D_2 D_1$ and $D_1 D_2$, with matrix parts

$$N_2 N_1 = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \quad N_1 N_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

respectively. Note that for $(i, j) = (1, 2), (2, 1)$, $D_j D_i$ are contraction maps with respect to $\|\cdot\|_1$ as

$$\begin{aligned} \|D_j D_i(\mathbf{x}) - D_j D_i(\mathbf{y})\|_1 &= \|N_j N_i(\mathbf{x} - \mathbf{y})\|_1 \\ &\leq \|N_j N_i\|_1 \|\mathbf{x} - \mathbf{y}\|_1 \\ &= \frac{3}{4} \|\mathbf{x} - \mathbf{y}\|_1, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \end{aligned}$$

thus by Theorem 9, $(D_j D_i)^n(\mathbf{q}_0)$ converges to the fixed point $\mathbf{x}_{i,j}^*$ of $D_j D_i$, which can be solved by

$$\begin{aligned} \mathbf{x}_{1,2}^* &= D_2 D_1(\mathbf{x}_{1,2}^*) = \mathbf{d}_2 + N_2(\mathbf{d}_1 + N_1 \mathbf{x}_{1,2}^*) \\ \mathbf{x}_{1,2}^* &= (I - N_2 N_1)^{-1}(\mathbf{d}_2 + N_2 \mathbf{d}_1) = \begin{bmatrix} \frac{3}{4} & \frac{1}{8} \\ -\frac{1}{4} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix}, \\ \mathbf{x}_{2,1}^* &= D_1 D_2(\mathbf{x}_{2,1}^*) = \mathbf{d}_1 + N_1(\mathbf{d}_2 + N_2 \mathbf{x}_{2,1}^*) \\ \mathbf{x}_{2,1}^* &= (I - N_1 N_2)^{-1}(\mathbf{d}_1 + N_1 \mathbf{d}_2) = \begin{bmatrix} \frac{5}{8} & -\frac{1}{4} \\ \frac{1}{8} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix}. \quad \blacksquare \end{aligned}$$

To motivate Theorem 6(c) where the branches are all converging to the isosceles right-angled triangle, Figures 53 to 56 show the effect of $D_2 D_1$ which illustrate Proposition 12, and sheds some light on Theorem 6(c). The green and red lines are the same as in Figures 47 to 52. In particular, notice that the intersection point of the two lines is

$$\begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix},$$

showing that the isosceles right-angled triangle is invariant under both D_1 and D_2 .

Here we re-state Theorem 6(c) in precise terms and prove it:

Theorem 6(c). (Outward binary incentric subdivision; Mixed iterations)

Every branch of a triangle generated by successive outward binary incentric subdivisions converges. In particular:

- (c) If the cuts is not a zigzag nor eventually a zigzag, then the limit triangle is the isosceles right-angled triangle.

More precisely, suppose $I = (i_n)_{n=1}^\infty$ is a $\{1, 2\}$ sequence with infinitely many 1s and infinitely many 2s. Starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$, the mixed iteration

$$D_{i_n} \cdots D_{i_1}(\mathbf{q}_0) \rightarrow \begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix} \quad \text{as } n \rightarrow \infty.$$

Proof. First, we show that D_1 and D_2 have exactly one common fixed point

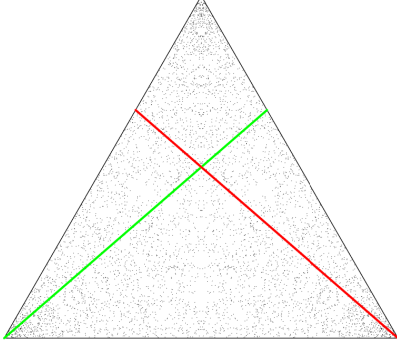
$$\begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix}.$$

Indeed, just note that the equation $D_1(\mathbf{x}) = \mathbf{x} = D_2(\mathbf{x})$ is equivalent to $N\mathbf{x} = \mathbf{r}$, where

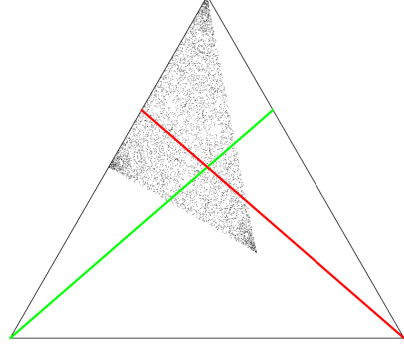
$$N = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}.$$

Gen

0.png

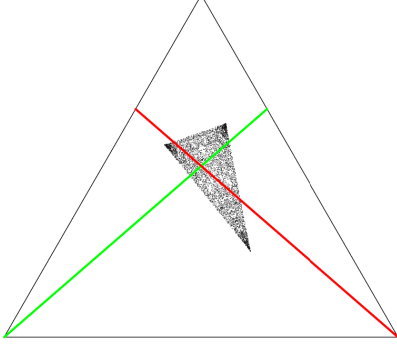
FIGURE 53. $(D_2D_1)^0(\mathbb{T})$

1.png

FIGURE 54. $(D_2D_1)^1(\mathbb{T})$

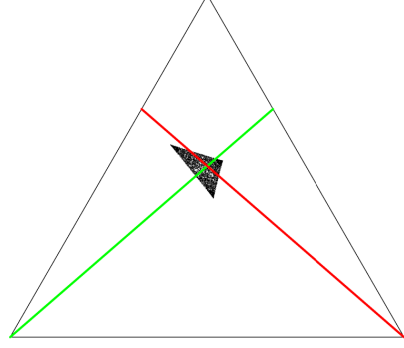
Gen

2.png

FIGURE 55. $(D_2D_1)^2(\mathbb{T})$

Gen

3.png

FIGURE 56. $(D_2D_1)^3(\mathbb{T})$

Solving it, we have

$$\mathbf{x} = N^{-1}\mathbf{r} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \end{bmatrix}.$$

Next, we show that D_1 and D_2 are joint contraction maps on the metric space $\mathbb{R}^2$. Since D_1 and D_2 are affine maps, so are $D_2^l D_1^k$ and $D_1^k D_2^l$, with matrix parts $N_2^l N_1^k$ and $N_1^k N_2^l$ respectively. Diagonalizing N_1 and N_2 ,

$$N_1 = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 1 \\ -\frac{1}{3} & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{bmatrix},$$

thus

$$N_1^k = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-\frac{1}{2})^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 1 \\ -\frac{1}{3} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} \left((-\frac{1}{2})^k - 1 \right) & (-\frac{1}{2})^k \end{bmatrix},$$

$$N_2^l = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (-\frac{1}{2})^l & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-\frac{1}{2})^l & \frac{1}{3} \left((-\frac{1}{2})^l - 1 \right) \\ 0 & 1 \end{bmatrix}.$$

First, we show that $\|N_2^l N_1^k\|_1 < 1$. To this end, write

$$N_2^l N_1^k = \begin{bmatrix} \frac{1}{9} \left((-\frac{1}{2})^{k+l} - (-\frac{1}{2})^k + 8(-\frac{1}{2})^l + 1 \right) & \frac{1}{3} \left((-\frac{1}{2})^{k+l} - (-\frac{1}{2})^k \right) \\ \frac{1}{3} \left((-\frac{1}{2})^k - 1 \right) & (-\frac{1}{2})^k \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then we have

$$|d| = \frac{1}{2^k} \leq \frac{1}{2},$$

$$|c| = \frac{1}{3} \left(1 - \frac{(-1)^k}{2^k} \right) \leq \frac{1}{3} \left(1 + \frac{1}{2} \right) = \frac{1}{2},$$

$$|b| = \frac{1}{3} \left(\frac{1}{2^k} - \frac{(-1)^l}{2^{k+l}} \right) \leq \frac{1}{3} \left(\frac{1}{2} + \frac{1}{4} \right) = \frac{1}{4},$$

$$|a| \leq \frac{1}{9} \left(\left(\frac{1}{2^k} - \frac{(-1)^l}{2^{k+l}} \right) + \left| 1 + \frac{(-1)^l 8}{2^l} \right| \right) \leq \frac{1}{9} \left(\frac{1}{2} + \frac{1}{4} + 3 \right) = \frac{5}{12}.$$

As a result,

$$\|N_2^l N_1^k\|_1 \leq \max \left\{ \frac{5}{12} + \frac{1}{2}, \frac{1}{4} + \frac{1}{2} \right\} \leq \frac{11}{12}.$$

As for $\|N_1^k N_2^l\|_1 < 1$, just note that $N_1 = EN_2E$, where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $E^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$$\|N_1^k N_2^l\|_1 = \|(EN_2^k E)(EN_1^l E)\|_1 \leq \|E\|_1 \|N_2^k N_1^l\|_1 \|E\|_1 \leq \frac{11}{12}.$$

Finally, the required result follows from Theorem 11. ■

7. INWARD AND OUTWARD BINARY QUASI-INCENTRIC SUBDIVISIONS

The *inward binary r -quasi-incentric subdivision*, as we introduce now, generalizes inward binary incentric subdivision through replacing angle bisectors by cevians that divide angles into a fixed ratio $r : s$, where $r, s \in (0, 1)$ and $r + s = 1$ (see Figures 15 to 18). To perform inward binary r -quasi-incentric subdivisions,

construct division maps as the following invertible affine maps $C_{r,1}, C_{r,2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto C_{r,i}(\mathbf{x}) = \mathbf{c}_r + M_{r,i}\mathbf{x}, \quad i = 1, 2,$$

with constant and matrix parts

$$\mathbf{c}_r = \begin{bmatrix} r\pi \\ s\pi \end{bmatrix}, \quad M_{r,1} = \begin{bmatrix} -r & -r \\ -s & r \end{bmatrix}, \quad M_{r,2} = \begin{bmatrix} s & -r \\ -s & -s \end{bmatrix}.$$

To understand their effects, note that the representation $(\angle CAB, \angle ABC) \in \mathbb{T}$ of any triangle $\triangle ABC$ induces an orientation $\angle CAB \rightarrow \angle ABC \rightarrow \angle BCA$ or $CA \rightarrow AB \rightarrow BC$ on its angles or sides. If $\mathbf{q} = (\alpha, \beta) \in \mathbb{T}$ represents a mother triangle with angles α, β, γ , then as we represent the first and the second daughter triangles as

$$C_{r,1}(\mathbf{q}) = \begin{bmatrix} r\pi - r\alpha - r\beta \\ s\pi - s\alpha + r\beta \end{bmatrix} = \begin{bmatrix} r\gamma \\ s\gamma + \beta \end{bmatrix} \in \mathbb{T},$$

$$C_{r,2}(\mathbf{q}) = \begin{bmatrix} r\pi + s\alpha - r\beta \\ s\pi - s\alpha - s\beta \end{bmatrix} = \begin{bmatrix} r\gamma + \alpha \\ s\gamma \end{bmatrix} \in \mathbb{T},$$

respectively (see Figure 57), we will have the same observations about the subdivisions as those in Section 3, just with (iii) replaced by

- (iii)' the daughter triangles $C_{r,1}(\mathbf{q}), C_{r,2}(\mathbf{q})$ will share the common angles α, β respectively with the mother triangle.

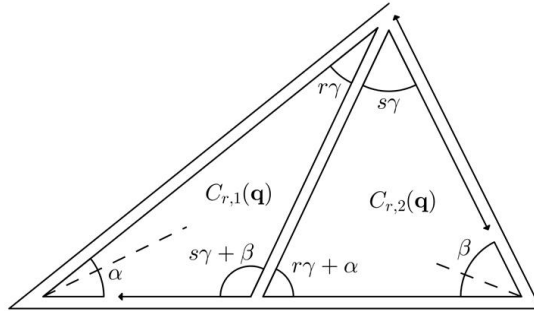


FIGURE 57.

To give intuition for Theorem 7, Figures 58 to 61 show the 12-th generation descendants for various values of r .

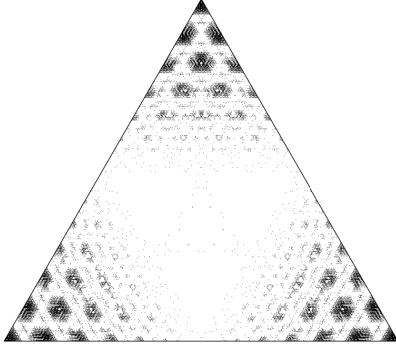
Here we re-state Theorem 7 in precise terms and prove it:

Theorem 7. (Inward binary r -quasi-incentric subdivision) The descendants of a triangle generated by successive inward binary r -quasi-incentric subdivisions are dense in the space of triangles. More precisely, starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$ and any fixed $r \in (0, 1)$, the set

$$(48) \quad \{C_{r,i_1} \cdots C_{r,i_n}(\mathbf{q}_0) : n \in \mathbb{N}, i_1, \dots, i_n \in \{1, 2\}\}$$

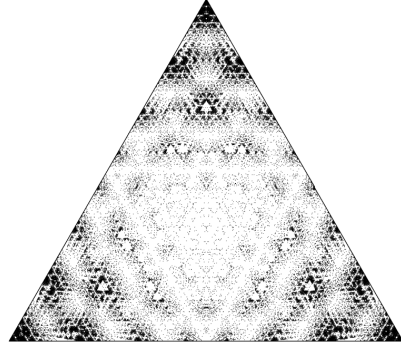
Generation,

r=0.10.png


 FIGURE 58. $r = 0.10$

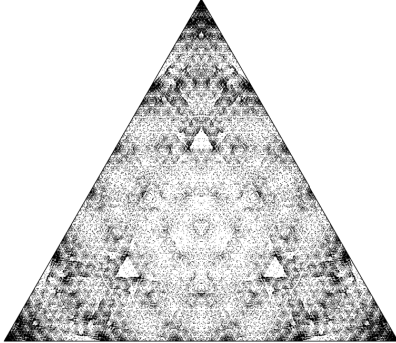
Generation,

r=0.18.png


 FIGURE 59. $r = 0.18$

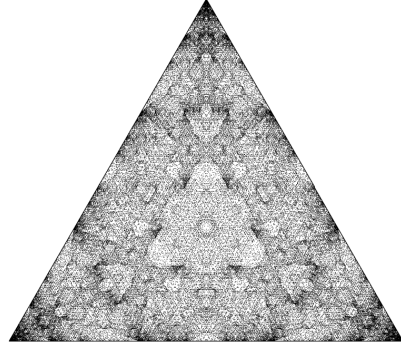
Generation,

r=0.25.png


 FIGURE 60. $r = 0.25$

Generation,

r=0.33.png


 FIGURE 61. $r = 0.33$

consisting of all iterates of $\mathbf{q}_0$ under

$$(49) \quad C = \{C_{r,1}, C_{r,2}\}$$

is a dense subset of $\mathbb{T}$.

Proof. First, we show that

$$(50) \quad C' = \{C_{r,j}C_{r,i} : i, j \in \{1, 2\}\}$$

is an iterated function system on $\mathbb{R}^2$. Indeed, by direct computations, writing $s = 1 - r$,

$$\begin{aligned} (M_{r,1}M_{r,1})^T (M_{r,1}M_{r,1}) &= \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}, & \|M_{r,1}M_{r,1}\|_2 &= r^2 < 1, \\ (M_{r,2}M_{r,2})^T (M_{r,2}M_{r,2}) &= \begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix}, & \|M_{r,2}M_{r,2}\|_2 &= s^2 < 1, \\ (M_{r,2}M_{r,1})^T (M_{r,2}M_{r,1}) &= \begin{bmatrix} s^2 & 0 \\ 0 & r^2 \end{bmatrix}, & \|M_{r,2}M_{r,1}\|_2 &= \max\{r^2, s^2\} < 1, \\ (M_{r,1}M_{r,2})^T (M_{r,1}M_{r,2}) &= \begin{bmatrix} s^2 & 0 \\ 0 & r^2 \end{bmatrix}, & \|M_{r,1}M_{r,2}\|_2 &= \max\{r^2, s^2\} < 1, \end{aligned}$$

thus, for $i, j \in \{1, 2\}$, since $C_{r,i}, C_{r,j}$ are affine maps with matrix parts $M_{r,i}, M_{r,j}$, so is $C_j C_i$, with matrix part $M_{r,j} M_{r,i}$,

$$\begin{aligned} d(C_{r,j}C_{r,i}(\mathbf{x}), C_{r,j}C_{r,i}(\mathbf{y})) &= \|M_{r,j}M_{r,i}(\mathbf{x} - \mathbf{y})\|_2 \\ &\leq \|M_{r,j}M_{r,i}\|_2 \|\mathbf{x} - \mathbf{y}\|_2 \\ (51) \quad &\leq \max\{r^2, s^2\} d(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2. \end{aligned}$$

Next, we show that

$$C_{r,1}(\overline{\mathbb{T}}) \cup C_{r,2}(\overline{\mathbb{T}}) = \overline{\mathbb{T}},$$

so that

$$(52) \quad \bigcup_{i,j \in \{1,2\}} C_{r,j}C_{r,i}(\overline{\mathbb{T}}) = \bigcup_{j \in \{1,2\}} C_{r,j} \left(\bigcup_{i \in \{1,2\}} C_{r,i}(\overline{\mathbb{T}}) \right) = \bigcup_{j \in \{1,2\}} C_{r,j}(\overline{\mathbb{T}}) = \overline{\mathbb{T}}.$$

Indeed, since $C_{r,1}, C_{r,2}$ are affine maps and $\overline{\mathbb{T}}$ is a triangle, $C_{r,1}(\overline{\mathbb{T}}), C_{r,2}(\overline{\mathbb{T}})$ are the triangles with vertices

$$\begin{aligned} C_{r,1} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} r\pi \\ s\pi \end{bmatrix}, \quad C_{r,1} \left(\begin{bmatrix} \pi \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_{r,1} \left(\begin{bmatrix} 0 \\ \pi \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \pi \end{bmatrix}, \\ C_{r,2} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} r\pi \\ s\pi \end{bmatrix}, \quad C_{r,2} \left(\begin{bmatrix} \pi \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \quad C_{r,2} \left(\begin{bmatrix} 0 \\ \pi \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

respectively (see Figure 62).

By (51), (50) is an iterated function system on the complete metric space $\mathbb{R}^2$. Starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T} \subset \overline{\mathbb{T}}$, by (52) and Theorem 10(c),

$$(53) \quad \{C_{r,i_1} \cdots C_{r,i_{2n}}(\mathbf{q}_0) : n \in \mathbb{N}, i_1, \dots, i_n \in \{1, 2\}\}$$

is a dense subset of $\overline{\mathbb{T}}$. Since $C_{r,1}, C_{r,2} : \mathbb{T} \rightarrow \mathbb{T}$ and $\mathbf{q}_0 \in \mathbb{T}$, (53) is actually a dense subset of $\mathbb{T}$. Note that (53) is a subset of (48), thus (48) is also a dense subset of $\mathbb{T}$. ■

The *outward binary r -quasi-incentric subdivision*, as we introduce now, generalizes outward binary incentric subdivision through, just like the inward binary r -quasi-incentric subdivision, replacing angle bisectors by cevians that divide angles into a fixed ratio ‘directed’ $r : s$, where $r, s \in (0, 1)$ and $r + s = 1$ (see Figures 19 to 22).

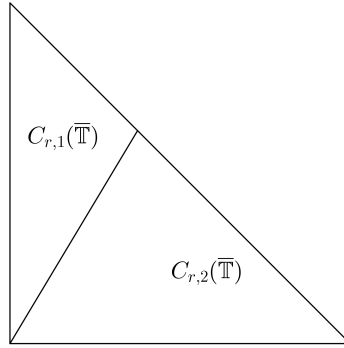


FIGURE 62.

To perform outward binary r -quasi-incentric subdivisions, construct division maps as the following invertible affine maps $D_{r,1}, D_{r,2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto D_{r,i}(\mathbf{x}) = \mathbf{d}_{r,i} + N_{r,i}, \quad i = 1, 2,$$

with constant and matrix parts

$$\mathbf{d}_{r,1} = \begin{bmatrix} 0 \\ r\pi \end{bmatrix}, \quad \mathbf{d}_{r,2} = \begin{bmatrix} s\pi \\ 0 \end{bmatrix}, \quad N_{r,1} = \begin{bmatrix} 1 & 0 \\ -r & -r \end{bmatrix}, \quad N_{r,2} = \begin{bmatrix} -s & -s \\ 0 & 1 \end{bmatrix}.$$

To understand their effects, note that the representation $(\angle CAB, \angle ABC) \in \mathbb{T}$ of any triangle $\triangle ABC$ induces an orientation $\angle CAB \rightarrow \angle ABC \rightarrow \angle BCA$ or $CA \rightarrow AB \rightarrow BC$ on its angles or sides. If $\mathbf{q} = (\alpha, \beta) \in \mathbb{T}$ represents a mother triangle with angles α, β, γ , then as we represent the first and the second daughter triangles as

$$D_{r,1}(\mathbf{q}) = \begin{bmatrix} \alpha \\ r\pi - r\alpha - r\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ r\gamma \end{bmatrix} \in \mathbb{T},$$

$$D_{r,2}(\mathbf{q}) = \begin{bmatrix} s\pi - s\alpha - s\beta \\ \beta \end{bmatrix} = \begin{bmatrix} s\gamma \\ \beta \end{bmatrix} \in \mathbb{T},$$

respectively (see Figure 63), we will have the same observations about the subdivisions as those in Section 6, just with (iii) replaced by

- (iii)' the daughter triangles $D_{r,1}(\mathbf{q}), D_{r,2}(\mathbf{q})$ will share the common angles α, β respectively with the mother triangle, and each will share a common side with the mother triangle.

To motivate Theorem 8(a)–(b) where the branches are converging to a limited choice of triangles, Figures 64 to 69 with $r = 0.25$ show the effects of $D_{r,1}$ or $D_{r,2}$. The green and red lines are the points in $\mathbb{T}$ of the forms

$$\begin{bmatrix} \alpha \\ \frac{r}{r+1}(\pi - \alpha) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{s}{s+1}(\pi - \beta) \\ \beta \end{bmatrix}$$

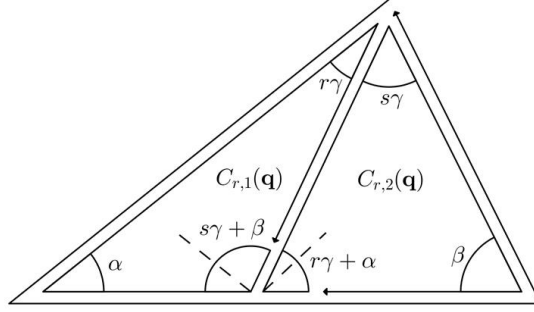


FIGURE 63.

respectively.

Here we re-state Theorem 8(a) in precise terms and prove it:

Theorem 8(a). (Outward binary r -quasi-incentric subdivision; Pure iterations) Every branch of a triangle generated by successive outward binary r -quasi-incentric subdivisions converges. In particular:

- (a) If the cuts is a zigzag, then among the three angles of the limit triangle, one angle is in common with the initial triangle, and the other two angles are in the ratio $1 : r$ or $1 : s$.

More precisely, starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$, the pure iterations converge with

$$D_{r,1}^n(\mathbf{q}_0) \rightarrow \begin{bmatrix} \alpha_0 \\ \frac{r}{r+1}(\pi - \alpha_0) \end{bmatrix} \text{ as } n \rightarrow \infty, \quad D_{r,2}^n(\mathbf{q}_0) \rightarrow \begin{bmatrix} \frac{s}{s+1}(\pi - \beta_0) \\ \beta_0 \end{bmatrix} \text{ as } n \rightarrow \infty.$$

Proof. Let $\mathbf{q}_{r,1,n} = \begin{bmatrix} \alpha_{r,1,n} \\ \beta_{r,1,n} \end{bmatrix}$ denote $D_{r,1}^n(\mathbf{q}_0)$. Since $D_{r,1}^n(\mathbf{q}_0) = D_{r,1}(D_{r,1}^{n-1}(\mathbf{q}_0))$, we have

$$(54) \quad \begin{bmatrix} \alpha_{r,1,n} \\ \beta_{r,1,n} \end{bmatrix} = \begin{bmatrix} 0 \\ r\pi \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -r & -r \end{bmatrix} \begin{bmatrix} \alpha_{r,1,n-1} \\ \beta_{r,1,n-1} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ r\pi - r\alpha_0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} \begin{bmatrix} \alpha_{r,1,n-1} \\ \beta_{r,1,n-1} \end{bmatrix},$$

or simply $\mathbf{q}_{r,1,n} = D'_{r,1}(\mathbf{q}_{r,1,n-1})$, where $\mathbf{q}_{r,1,0} = \mathbf{q}_0$ and $D'_{r,1}(\mathbf{x}) = \mathbf{d}'_{r,1} + N'_{r,1}\mathbf{x}$ is an affine map with constant and matrix parts

$$(55) \quad \mathbf{d}'_{r,1} = \begin{bmatrix} \alpha_0 \\ r\pi - r\alpha_0 \end{bmatrix}, \quad N'_{r,1} = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix}.$$

Note that $D'_{r,1}$ is a contraction map with respect to $\|\cdot\|_1$ as

$$(56) \quad \begin{aligned} \|D'_{r,1}(\mathbf{x}) - D'_{r,1}(\mathbf{y})\|_1 &= \|N'_{r,1}(\mathbf{x} - \mathbf{y})\|_1 \\ &\leq \|N'_{r,1}\|_1 \|\mathbf{x} - \mathbf{y}\|_1 \\ &= r \|\mathbf{x} - \mathbf{y}\|_1, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \end{aligned}$$

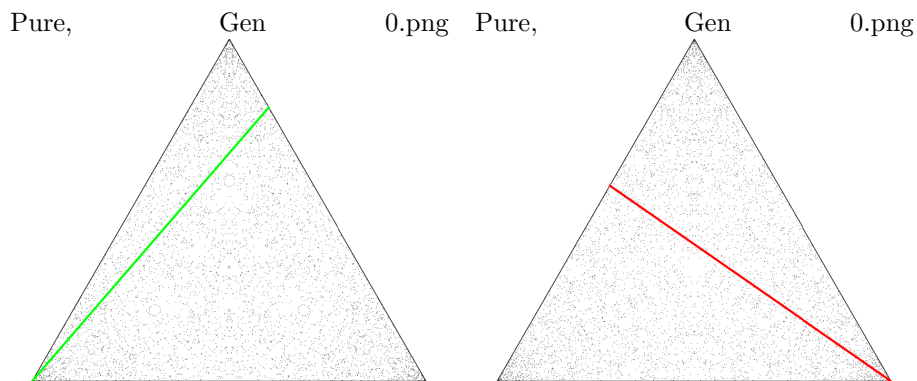

 FIGURE 64. $D_{r,1}^0(\mathbb{T})$

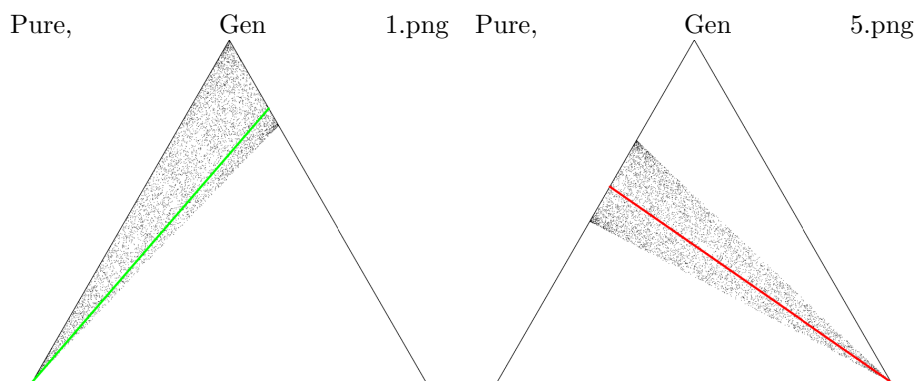
 FIGURE 65. $D_{r,2}^0(\mathbb{T})$

 FIGURE 66. $D_{r,1}^1(\mathbb{T})$

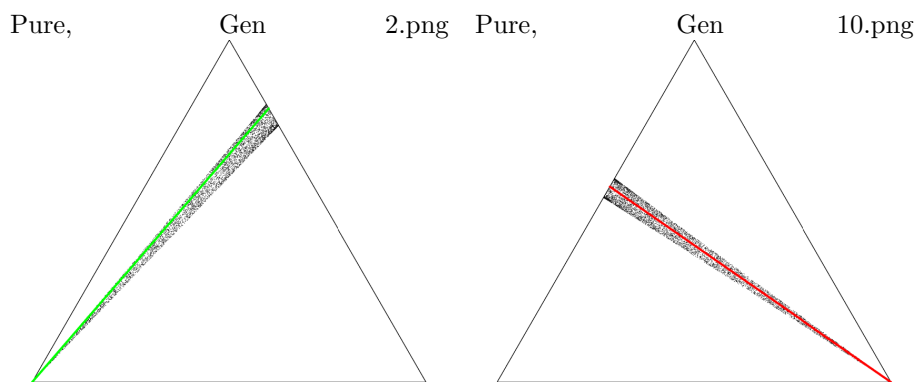
 FIGURE 67. $D_{r,2}^5(\mathbb{T})$

 FIGURE 68. $D_{r,1}^2(\mathbb{T})$

 FIGURE 69. $D_{r,2}^{10}(\mathbb{T})$

thus by Theorem 9, $\mathbf{q}_{r,1,n}$ converges to the fixed point $\mathbf{x}_{r,1}^*$ of $D'_{r,1}$, which can be solved by

$$(57) \quad \mathbf{x}_{r,1}^* = D'_{r,1}(\mathbf{x}_{r,1}^*) = \mathbf{d}'_{r,1} + N'_{r,1}\mathbf{x}_{r,1}^*$$

$$(58) \quad \mathbf{x}_{r,1}^* = (I - N'_{r,1})^{-1} \mathbf{d}'_{r,1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r+1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ r\pi - r\alpha_0 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \frac{r}{r+1}(\pi - \alpha_0) \end{bmatrix}.$$

Similarly, now let $\mathbf{q}_{r,2,n} = \begin{bmatrix} \alpha_{r,2,n} \\ \beta_{r,2,n} \end{bmatrix}$ denote $D_{r,2}^n(\mathbf{q}_0)$. Since

$$D_{r,2}^n(\mathbf{q}_0) = D_{r,2}(D_{r,2}^{n-1}(\mathbf{q}_0)),$$

we have

$$(59) \quad \begin{bmatrix} \alpha_{r,2,n} \\ \beta_{r,2,n} \end{bmatrix} = \begin{bmatrix} s\pi \\ 0 \end{bmatrix} + \begin{bmatrix} -s & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{r,2,n-1} \\ \beta_{r,2,n-1} \end{bmatrix} = \begin{bmatrix} s\pi - s\beta_0 \\ \beta_0 \end{bmatrix} + \begin{bmatrix} -s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{r,2,n-1} \\ \beta_{r,2,n-1} \end{bmatrix},$$

or simply $\mathbf{q}_{r,2,n} = D'_{r,2}(\mathbf{q}_{r,2,n-1})$, where $\mathbf{q}_{r,2,0} = \mathbf{q}_0$ and $D'_{r,2}(\mathbf{x}) = \mathbf{d}'_{r,2} + N'_{r,2}\mathbf{x}$ is an affine map with constant and matrix parts

$$(60) \quad \mathbf{d}'_{r,2} = \begin{bmatrix} s\pi - s\beta_0 \\ \beta_0 \end{bmatrix}, \quad N'_{r,2} = \begin{bmatrix} -s & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $D'_{r,2}$ is a contraction map with respect to $\|\cdot\|_1$ as

$$(61) \quad \begin{aligned} \|D'_{r,2}(\mathbf{x}) - D'_{r,2}(\mathbf{y})\|_1 &= \|N'_{r,2}(\mathbf{x} - \mathbf{y})\|_1 \\ &\leq \|N'_{r,2}\|_1 \|\mathbf{x} - \mathbf{y}\|_1 \\ &= s \|\mathbf{x} - \mathbf{y}\|_1, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \end{aligned}$$

thus by Theorem 9, $\mathbf{q}_{r,2,n}$ converges to the fixed point $\mathbf{x}_{r,2}^*$ of $D'_{r,2}$, which can be solved by

$$(62) \quad \mathbf{x}_{r,2}^* = D'_{r,2}(\mathbf{x}_{r,2}^*) = \mathbf{d}'_{r,2} + N'_{r,2}\mathbf{x}_{r,2}^*$$

$$(63) \quad \mathbf{x}_{r,2}^* = (I - N'_{r,2})^{-1} \mathbf{d}'_{r,2} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s\pi - s\beta_0 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{s}{s+1}(\pi - \beta_0) \\ \beta_0 \end{bmatrix}. \quad \blacksquare$$

Here we re-state Theorem 8(b) in precise terms, which follows from Theorem 8(a) as a corollary:

Theorem 8(b). (Outward binary r -quasi-incentric subdivision; Eventually pure iterations) Every branch of a triangle generated by successive outward binary r -quasi-incentric subdivisions converges. In particular:

- (b) If the cuts is eventually a zigzag, then among the three angles of the limit triangle, one angle is in common with the last daughter triangle before the zigzag started, and the other two angles are in the ratio $1 : r$ or $1 : s$.

More precisely, suppose $I = (i_n)_{n=1}^\infty$ is a $\{1, 2\}$ sequence with finitely many 1s or finitely many 2s. Starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$:

- (i) If i_k is the last 1 in I , then the eventually pure iteration

$$D_{i_n} \cdots D_{i_1}(\mathbf{q}_0) \rightarrow \begin{bmatrix} \frac{s}{s+1}(\pi - \beta_k) \\ \beta_k \end{bmatrix} \quad \text{as } n \rightarrow \infty.$$

(ii) If i_k is the last 2 in I , then the eventually pure iteration

$$D_{i_n} \cdots D_{i_1}(\mathbf{q}_0) \rightarrow \left[\frac{r}{r+1} (\pi - \alpha_k) \right] \text{ as } n \rightarrow \infty.$$

To motivate Theorem 8(c) where the branches are all converging to a specific triangle, Figures 70 to 73 show the effect of $D_{r,2}D_{r,1}$. The green and red lines are the same as in Figures 64 to 69. In particular, notice that the intersection point of the two lines is

$$\left[\frac{s\pi}{2}, \frac{r\pi}{2} \right],$$

showing that the limit triangle is invariant under both $D_{r,1}$ and $D_{r,2}$.

Gen

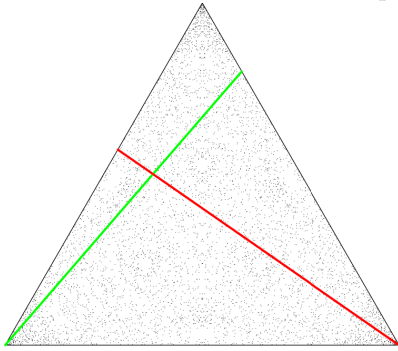


FIGURE 70. $(D_{r,2}D_{r,1})^0(\mathbb{T})$

0.png Gen

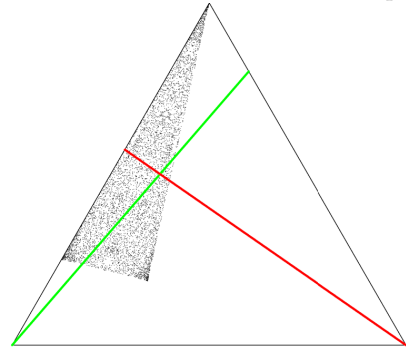


FIGURE 71. $(D_{r,2}D_{r,1})^1(\mathbb{T})$

1.png

Gen

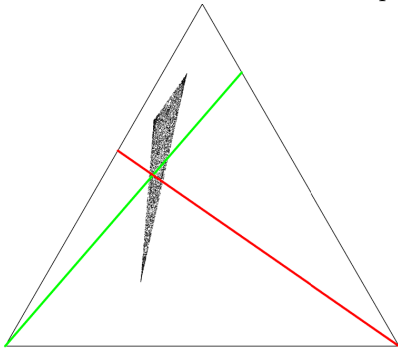


FIGURE 72. $(D_{r,2}D_{r,1})^2(\mathbb{T})$

2.png Gen

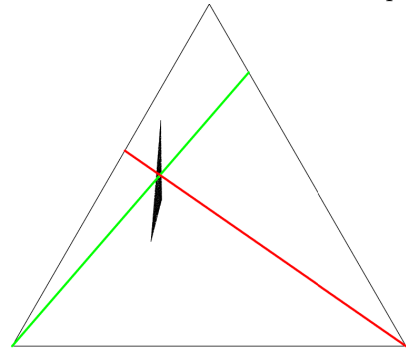


FIGURE 73. $(D_{r,2}D_{r,1})^3(\mathbb{T})$

3.png

Here we re-state Theorem 8(c) in precise terms and prove it:

Theorem 8(c). (Outward binary r -quasi-incentric subdivision; Mixed iterations) Every branch of a triangle generated by successive outward binary r -quasi-incentric subdivisions converges. In particular:

- (c) If the cuts is not a zigzag nor eventually a zigzag, then the limit triangle is a right-angled triangle, and the other two angles are in the ratio $r : s$.

More precisely, suppose $I = (i_n)_{n=1}^\infty$ is a $\{1, 2\}$ sequence with infinitely many 1s and infinitely many 2s. Starting with any initial triangle $\mathbf{q}_0 \in \mathbb{T}$, the mixed iteration

$$D_{r,i_n} \cdots D_{r,i_1}(\mathbf{q}_0) \rightarrow \left[\begin{array}{c} \frac{s\pi}{2} \\ \frac{r\pi}{2} \end{array} \right] \text{ as } n \rightarrow \infty.$$

Proof. First, we show that $D_{r,1}$ and $D_{r,2}$ have exactly one common fixed point

$$\left[\begin{array}{c} \frac{s\pi}{2} \\ \frac{r\pi}{2} \end{array} \right].$$

Indeed, just note that the equation $D_{r,1}(\mathbf{x}) = \mathbf{x} = D_{r,2}(\mathbf{x})$ is equivalent to $N\mathbf{x} = \mathbf{r}$, where

$$N = \begin{bmatrix} r & r+1 \\ s+1 & s \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} r\pi \\ s\pi \end{bmatrix}.$$

Solving it, we have

$$\mathbf{x} = N^{-1}\mathbf{r} = \begin{bmatrix} -\frac{s}{2} & \frac{r+1}{2} \\ \frac{s+1}{2} & -\frac{r}{2} \end{bmatrix} \begin{bmatrix} r\pi \\ s\pi \end{bmatrix} = \begin{bmatrix} \frac{s\pi}{2} \\ \frac{r\pi}{2} \end{bmatrix}.$$

Next, we show that $D_{r,1}$ and $D_{r,2}$ are joint contraction maps on the metric space $\mathbb{R}^2$. Since $D_{r,1}$ and $D_{r,2}$ are affine maps, so are $D_{r,2}^l D_{r,1}^k$ and $D_{r,1}^k D_{r,2}^l$, with matrix parts $N_{r,2}^l N_{r,1}^k$ and $N_{r,1}^k N_{r,2}^l$ respectively. Diagonalizing $N_{r,1}$ and $N_{r,2}$,

$$N_{r,1} = \begin{bmatrix} -\frac{r+1}{r} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix} \begin{bmatrix} -\frac{r}{r+1} & 0 \\ \frac{r}{r+1} & 1 \end{bmatrix}, \quad N_{r,2} = \begin{bmatrix} -\frac{s}{s+1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \frac{s}{s+1} \end{bmatrix},$$

thus

$$\begin{aligned} N_{r,1}^k &= \begin{bmatrix} -\frac{r+1}{r} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-r)^k \end{bmatrix} \begin{bmatrix} -\frac{r}{r+1} & 0 \\ \frac{r}{r+1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{r+(-r)^k}{r+1} & (-r)^k \end{bmatrix}, \\ N_{r,2}^l &= \begin{bmatrix} -\frac{s}{s+1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-s)^l \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \frac{s}{s+1} \end{bmatrix} = \begin{bmatrix} (-s)^l & \frac{-s+s(-s)^l}{s+1} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

First, we show that $\|N_{r,2}^l N_{r,1}^k\|_1 < 1$. To this end, write

$$\begin{aligned} & N_{r,2}^l N_{r,1}^k \\ &= \begin{bmatrix} \frac{rs}{(r+1)(s+1)} - \frac{rs(-r)^k}{(r+1)(s+1)} + \frac{2(-s)^l}{(r+1)(s+1)} + \frac{rs(-r)^k(-s)^l}{(r+1)(s+1)} & -\frac{s(-r)^k}{s+1} + \frac{s(-r)^k(-s)^l}{s+1} \\ \frac{-r}{r+1} + \frac{r(-r)^k}{r+1} & (-r)^k \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \end{aligned}$$

then we have

$$\begin{aligned}
 |d| &= (-r)^k \leq r, \\
 |c| &= \frac{r(1 - (-r)^k)}{r+1} \leq \frac{r(r+1)}{r+1} = r, \\
 |b| &= \frac{sr^k(1 - (-s)^l)}{s+1} \leq \frac{sr(s+1)}{s+1} = r - r^2, \\
 |a| &\leq \frac{rsr^k(1 - (-s)^l)}{(r+1)(s+1)} + \frac{|rs + 2(-s)^l|}{(r+1)(s+1)} \\
 &\leq \frac{rsr(s+1)}{(r+1)(s+1)} + \frac{s(s+1)}{(r+1)(s+1)} \\
 &= \frac{(1-r)(r^2+1)}{r+1}.
 \end{aligned}$$

As a result,

$$\begin{aligned}
 \|N_{r,2}^l N_{r,1}^k\|_1 &\leq \max \left\{ \frac{(1-r)(r^2+1)}{r+1} + r, r - r^2 + r \right\} \\
 &= \max \left\{ \frac{r(1 - (1-r)^2) + 1}{r+1}, 1 - (1-r)^2 \right\} < 1.
 \end{aligned}$$

As for $\|N_{r,1}^k N_{r,2}^l\|_1 < 1$, just note that $N_{r,1} = EN_{1-r,2}E$, where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and

$$E^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ then}$$

$$\|N_{r,1}^k N_{r,2}^l\|_1 = \|(EN_{1-r,2}^k E)(EN_{1-r,1}^l E)\|_1 \leq \|E\|_1 \|N_{1-r,2}^k N_{1-r,1}^l\|_1 \|E\|_1 < 1.$$

Finally, the required result follows from Theorem 11. ■

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EDITOR'S COMMENTS

Building on earlier work on barycentric and incentric triangle subdivisions, this paper introduces two new subdivision rules—inward and outward binary incentric subdivisions—along with further generalizations, and uses iterated function systems to study how the resulting triangles spread out across the space of triangle shapes. One particularly nice result is that the inward-binary descendants spread out evenly, in the sense that their empirical measures converge to the uniform measure on the space of triangle shapes. This is a stronger and more interesting statement than simply saying the triangles fill the space, and it sets this case apart from the earlier senary case. The team also takes an observation about the connection between the ternary case and the Sierpiński triangle and makes it precise with a proof. It is a strong paper in the interface of geometry and analysis.