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ILLUMINATION PROBLEM

TEAM MEMBERS
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Abstract. This article is a study of the best arrangements of bulbs in a room such that the minimum light intensity in the room attains maximum. The cases of circular and regular rooms are considered. [2]

1. Introduction

In the article, we aim at finding the best arrangement of bulbs in a room in order to attain maximum light intensity. Suppose there are n bulbs and we want to put them in a room. For any arrangement of the bulbs, there will be some points in the room, which have the minimum light intensity. We want to find the best way to put the bulbs so that this minimum intensity in the room is maximized.

We know that the intensity I is inversely proportional to square of the distance d from a bulb. We may assume that the variation constant is 1, i.e. $I = \dfrac{1}{d^2}$. We will consider rooms of circular and rectangular shape. We will also use the following two assumptions as a direction of finding the best arrangement:

1st assumption: The arrangement of bulbs is symmetric.
2nd assumption: More minimum intensity points will give a better arrangement.

[1]This work is done under the supervision of the authors' teacher, Mr. Chi-Hin Lau.
[2]The abstract is added by the editor.

The first assumption is expected since the domains we considered are symmetric. We made the second assumption since we believe that the light intensity should be more evenly distributed for the best arrangements. (We will see in section 2.4 that the second assumption is in general not correct.) The results in this article are based on these two assumptions. We cannot prove that these are the best arrangements of bulbs except for the simplest cases. These are only the best solutions we can find for the problem.

Now we set up the mathematical model for the problem. A room corresponds to a set $\Omega \subset \mathbb{R}^2$. For simplicity we will assume that the height of the room is zero. Suppose we have n bulbs of equal brightness putting at $\alpha_k = (x_{k_1}, x_{k_2}) \in \Omega$, $k = 1, 2, \ldots, n$. The light intensity of a point $p = (p_1, p_2) \in \Omega$ after normalization is given by

$$I(p) = \sum_{k=1}^{n} \frac{1}{(x_{k_1} - p_1)^2 + (x_{k_2} - p_2)^2}.$$

Now the light intensity of the minimum intensity point of the room will be

$$I_{\min}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \min_{p \in \Omega} \sum_{k=1}^{n} \frac{1}{(x_{k_1} - p_1)^2 + (x_{k_2} - p_2)^2}.$$

Our objective is to find $\alpha_k \in \Omega$, $k = 1, 2, \ldots, n$, such that $I_{\min}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is maximum for a given Ω.

Remarks: All numerical results given in this report are corrected to 5 decimal places.

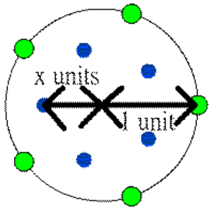
2. Circular Domain

In this chapter, we aim at how to put the bulbs in a circular room. We may assume that the circular room is of unit radius. Through out this chapter, let $\Omega = \{(x_1, x_2) : x_1^2 + x_2^2 \leqslant 1\} \subset \mathbb{R}^2$ be a circular room of unit radius, $\alpha_k = (x_{k_1}, x_{k_2}) \in \Omega$, with $1 \leqslant k \leqslant n$, be n bulbs in the room.

With the assumption that the bulbs are arranged in a symmetric way, we obtain a formula for the minimum intensity $I_{\min}$ in the circular room with n bulbs. We will show that the bulbs should be placed at the center of the room for $n = 1$, 2. Then we will find the best arrangement for $n = 3$, 4, 5 by simple differentiation. Finally we will see that the bulbs should be arranged in two triangles of different sizes for $n = 6$.

2.1. Formula for regular polygon

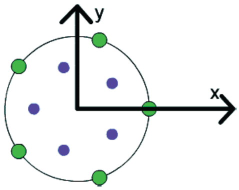
Here, we are going to derive the formula for regular polygon arrangement of bulbs. Assumed that the bulbs are arranged in a regular polygon with distance x units, $0 \leqslant x \leqslant 1$, from the centre of the room. We expect that the minimum intensity points would lie on the wall of the room. The following figure shows the case for $n = 5$.



where blue(the points inside) and green(the points on the circle) points represent bulbs and minimum intensity points respectively. Rotate the regular polygon made by the bulbs such that one of the minimum intensity points lies at $(1,0) \in \Omega$. In this case,

$$\begin{aligned}\alpha_k &= (x_{k_1}, x_{k_2}) \\ &= \left(x\cos\left(\frac{2k-1}{n}\pi\right), x\sin\left(\frac{2k-1}{n}\pi\right)\right) \in \Omega, \quad k = 1, 2, \ldots, n,\end{aligned}$$

where x is the distance of the bulbs from the centre.



Then the intensity at $(1,0)$ which is one of the minimum intensity points is

$$I_n(1,0) = \sum_{k=1}^{n} \frac{1}{(x_{k_1}-1)^2 + x_{k_2}^2}.$$

We are going to show that

$$I_n(1,0) = \frac{n(1-x^n)}{(1-x^2)(1+x^n)}. \tag{2.1.1}$$

We derive the above formula by means of complex number. We take x-axis as the real axis and the y-axis as the imaginary axis. As the bulbs are arranged in a regular polygon, they all lie in a circle with radius x units. In the

Argand diagram, the position of the bulbs $\alpha_k = ke^{\frac{2k-1}{n}\pi i} \in \mathbb{C}$, $k = 1, 2, \ldots, n$ are the roots of equation $z^n + x^n = 0$.

Now $1 \in \mathbb{C}$ is a minimum intensity point, so the intensity at 1 is $I_n(1) = \sum_{k=1}^{n} \frac{1}{|\alpha_k - 1|^2}$.

Lemma 1. *Suppose β_k, $k = 1, 2, \ldots, n$ are roots of the equation $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$ and $|\beta_k - w| = r$ for some $w \in \mathbb{C}$, $r \in \mathbb{R}$, then*

$$\sum_{k=1}^{n} |\beta_k|^2 = nr^2 - n|w|^2 - \left(\frac{a_1}{a_0}\right)\overline{w} - \left(\frac{\overline{a_1}}{\overline{a_0}}\right) w.$$

Remark: The assumption says that all lies on a circle and w, r are the centre and the radius of the circle respectively.

Proof of Lemma.

$$\begin{aligned}
\sum_{k=1}^{n} |\beta_k|^2 &= \sum_{k=1}^{n} |\beta_k + w - w|^2 \\
&= \sum_{k=1}^{n} ((\beta_k - w)(\overline{\beta_k - w}) + (\beta_k - w)(\overline{w}) + (\overline{\beta_k - w})(w) + (w\overline{w})) \\
&= \sum_{k=1}^{n} |\beta_k - w|^2 + \sum_{k=1}^{n} (\beta_k \overline{w} - w\overline{w}) + \sum_{k=1}^{n} (\overline{\beta_k} w - \overline{w} w) + \sum_{k=1}^{n} |w|^2 \\
&= \sum_{k=1}^{n} r^2 + \sum_{k=1}^{n} |w|^2 + \sum_{k=1}^{n} (\beta_k \overline{w}) + \sum_{k=1}^{n} (\overline{\beta_k} w) \\
&= nr^2 - n|w|^2 - \left(\frac{a_1}{a_0}\right)\overline{w} - \left(\frac{\overline{a_1}}{\overline{a_0}} w\right).
\end{aligned}$$

Here we used $\sum_{k=1}^{n} \beta_k = -\frac{a_1}{a_0}$. □

To find $I_n(1) = \sum_{k=1}^{n} \frac{1}{|\alpha_k - 1|^2}$, we can apply the lemma for $\beta_k = \frac{1}{\alpha_k - 1}$.

First of all we find the centre w and the radius r of the circle passes though all β_k. Since, α_k lies on a circle, $\alpha_k - 1$ lies on a circle with centre -1 and radius x with $0 < x < 1$. After the transformation $f(z) = \frac{1}{z}$, $\beta_k = \frac{1}{\alpha_k - 1}$

lies on a circle, its centre $w \in \mathbb{C}$ and radius $r \in \mathbb{R}$ are

$$w = \frac{-1}{1-x^2}, \quad r = \frac{x}{1-x^2}. \tag{2.1.2}$$

Secondly, since $\alpha_k = \dfrac{1}{\beta_k} + 1$ are the roots of the equation $z^n + x^n = 0$, β_k are roots of the equation $\left(\dfrac{1}{z} + 1\right)^n + x^n = 0$.

Simplify the equation we have

$$(1 + x^n)z^n + nz^{n-1} + \cdots + 1 = 0.$$

Therefore,

$$\frac{a_1}{a_0} = \frac{n}{1+x^n}. \tag{2.1.3}$$

Using (2.1.2), (2.1.3) and the lemma, we have

$$\begin{aligned}
I_n(1) &= \sum_{k=1}^{n} \frac{1}{|\alpha_k - 1|^2} \\
&= \sum_{k=1}^{n} |\beta_k|^2 \\
&= nr^2 - n|w|^2 - \left(\frac{a_1}{a_0}\right)\overline{w} - \left(\overline{\frac{a_1}{a_0}}\right) w \\
&= n\left(\frac{x}{1-x^2}\right)^2 - n\left(\frac{1}{1-x^2}\right)^2 - \frac{n}{1+x^n}\left(-\frac{1}{1-x^2}\right) \\
&\quad - \frac{n}{1+x^n}\left(-\frac{1}{1-x^2}\right) \\
&= \frac{n(1-x^n)}{(1-x^2)(1+x^n)}.
\end{aligned}$$

This completed the proof of (2.1.1).

2.2. One and Two Bulbs

When we have only one bulb, we have the following proposition, which tell us that we should put the bulb at the centre.

Proposition 1. *Let* $\Omega = \{(s,t) : s^2 + t^2 \leqslant 1\}$, *then*

$$\max_{(x,y)\in\Omega} \left(\min_{(s,t)\in\Omega} \frac{1}{(s-x)^2 + (t-y)^2} \right) = 1.$$

Remark: (x, y) is the position of the bulb and (s, t) is any point in the room.

Proof. For any $(x, y) \in \Omega$, there exists $(s, t) = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}\right)$ such that

$$
\begin{aligned}
\frac{1}{(s-x)^2+(t-y)^2} &= \frac{1}{\left(-\frac{x}{\sqrt{x^2+y^2}}-x\right)^2+\left(-\frac{y}{\sqrt{x^2+y^2}}-y\right)^2} \\
&= \frac{1}{\left(\sqrt{x^2+y^2}+1\right)^2} \\
&\leqslant 1.
\end{aligned}
$$

Therefore, $\min\limits_{(s,t)\in\Omega} \frac{1}{(s-x)^2+(t-y)^2} \leqslant 1$ and the equality holds if and only if $x = y = 0$. □

If there are two bulbs in a circular room, we are going to show that we should put both bulbs at the centre.

Proposition 2. *Let* $\Omega = \{(s, \alpha) : 0 \leqslant s \leqslant 1,\ \alpha \in [-\pi, \pi]\}$. *For any* (r_1, θ_1), $(r_2, \theta_2) \in \Omega$, *there exists* $(r, \theta) \in \Omega$ *such that*

$$
\frac{1}{r^2+r_1^2-2rr_1\cos(\theta-\theta_1)} + \frac{1}{r^2+r_2^2-2rr_2\cos(\theta-\theta_2)} \leqslant 2.
$$

Remark: (r_1, θ_1), (r_2, θ_2) are the positions of the bulbs in the unit circle and (r, θ) is any point in circular room.

Proof. For any (r_1, θ_1), $(r_2, \theta_1 + \alpha) \in \Omega$, $\alpha \in [0, \pi]$, there exists $(r, \theta) = \left(1, \theta_1 + \dfrac{\alpha}{2} + \pi\right)$ such that

$$\begin{aligned}
&\frac{1}{r^2 + r_1^2 - 2rr_1\cos(\theta - \theta_1)} + \frac{1}{r^2 + r_2^2 - 2rr_2\cos(\theta - \theta_2)} \\
&= \frac{1}{1 + r_1^2 - 2rr_1\cos\left(\theta_1 + \frac{\alpha}{2} + \pi - \theta_1\right)} \\
&\quad + \frac{1}{1 + r_2^2 - 2rr_2\cos\left(\theta_1 + \frac{\alpha}{2} + \pi - (\theta_1 + \alpha)\right)} \\
&= \frac{1}{1 + r_1^2 + 2rr_1\cos\left(\frac{\alpha}{2}\right)} + \frac{1}{1 + r_2^2 + 2rr_2\cos\left(\frac{\alpha}{2}\right)} \\
&\leqslant \frac{1}{1 + r_1^2} + \frac{1}{1 + r_2^2} \quad \left(\text{since } \frac{\alpha}{2} \in \left[0, \frac{\pi}{2}\right]\right) \\
&\leqslant 2
\end{aligned}$$

and the equality holds if and only if $r_1 = r_2 = 0$. $\square$

2.3. Three, four and five bulbs

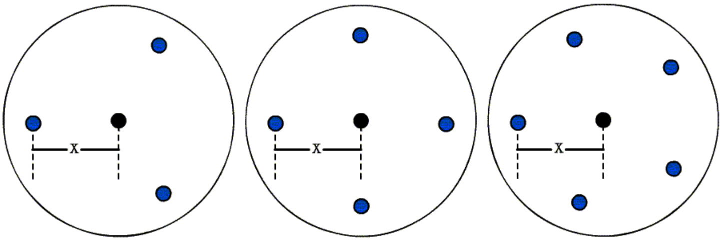
Let $I_{\min}(x)$ be the minimum intensity of the room when n bulbs are put in a shape of regular polygon, where x is the distant of the vertices from the centre. By the formula for the regular polygon (2.1.1),

$$I_{\min}(x) = \frac{n(1 - x^n)}{(1 - x^2)(1 + x^n)}.$$

In order to maximize $I_{\min}(x)$, we solve the equation $\dfrac{d}{dx}(I_{\min}) = 0$. The maximum values of $I_{\min}(x)$ and the corresponding values of x corrected to 5 decimal places are given in the following table.

n	$\frac{d}{dx}(I_{\min}) = 0$	x (5d.p.)	$I_{\min}$ (5d.p.)
3	$x^6 - 3x^3 + 3x - 1 = 0$ $(x-1)(x^2-1)(x^3 + x^2 + 2x - 1) = 0$	0.39265	3.14192
4	$x^9 - 4x^4 + 4x^2 - 1 = 0$ $(x^2-1)^2(x^4 + 2x^2 - 1) = 0$	0.64359	4.82843
5	$x^{10} - 5x^5 + 5x^3 - 1 = 0$ $(x+1)(x^2-1)(x^7 + x^6 + 2x^5 + 2x^4 + 3x^3 - 2x^2 - x - 1) = 0$	0.76763	7.04949

The following figures shows the position of the bulbs for $n = 3, 4, 5$.



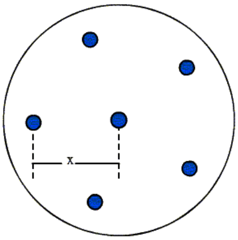
2.4. Six bulbs

The formula for regular polygon (2.1.1) can be used only to find the minimum intensity $I_{\min}$ on the boundary of the circular room. From Section 2.3, we can see that as n is increasing, x is increasing too. When $n \geqslant 6$, the intensity at the centre of the room would be smaller then $I_{\min}$. The following table compares the minimum intensity $I_{\min}$ on the boundary of the room with the intensity I_0 at the centre for the corresponding arrangement.

n	x	$I_{\min}$	I_0
3	0.392647	3.141916	19.45885
4	0.643594	4.828427	9.656854
5	0.767627	7.049489	8.485344
6	0.837018	9.784069	8.564108
7	0.879542	13.02489	9.048671
8	0.907418	16.76899	9.715722

It shows that we need to arrange the bulbs in another way.

One suggestion is to put one bulb at the centre. And the remaining 5 bulbs are arranged in a regular pentagon as in the following figure.



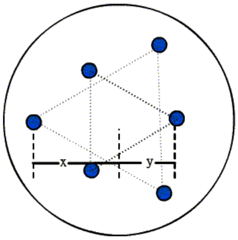
Using the table in Section 2.3, the minimum intensity is $I_{\min} = 7.04949 + 1 = 8.04949$.

The second suggestion is to make the hexagon smaller (x smaller) so that $I_{\min} = I_0$. Using the formula for regular polygon (2.1.1), we have

$$\frac{6(1-x^6)}{(1-x^2)(1+x^6)} = \frac{6}{x^2}. \tag{2.4.1}$$

We get $x = \sqrt{\frac{\sqrt{5}-1}{2}} = 0.78615$ and the minimum intensity is $I_{\min} = I_0 = 9.70820$. We see that this arrangement is better than the first suggestion.

The third suggestion is to arrange the bulbs in two triangles of different size as in the following figure.



The distances of the vertices of the two triangles from the centre are x and y. To find x, y so that $I_{\min}$ is maximum, we generalize the formula of regular polygon (2.1.1) to the intensity at any point in the room.

As before, suppose we have n bulbs of equal brightness putting at $\alpha_k \in \Omega$, $k = 1, 2, \ldots, n$, in a room $\Omega \subset \mathbb{C}$, where α_k are roots are the equation $z^n + x^n = 0$. The intensity $I(\omega)$ at any point $\omega \in \mathbb{C}$ in the room is

$$I(\omega) = \sum_{k=1}^{n} \frac{1}{|\omega - \alpha_k|^2}.$$

Proposition 3. *Let α_k be the roots of the equations $z^n + x^n = 0$, then*

$$I(\omega) = \sum_{k=1}^{n} \frac{1}{|\omega - \alpha_k|^2} = \frac{n}{|\omega|^2 - x^2} \mathrm{Re}\left(\frac{\omega^n - x^n}{\omega^n + x^n}\right). \tag{2.4.2}$$

Proof. To find $I(\omega) = \sum_{k=1}^{n} \frac{1}{|\alpha_k - \omega|^2}$, we can apply the lemma in Section 2.1 for $\beta_k = \frac{1}{\alpha_k - \omega}$. Since α_k lies on the circle when center 0 and radius x, β_k lies on the circle with centre $w \in \mathbb{C}$ and radius $r \in \mathbb{R}$ where

$$w = \frac{-\xi}{|\omega|^2 - x^2}, \quad r = \left|\frac{x}{|\omega|^2 - x^2}\right|. \tag{2.4.3}$$

Since $\alpha_k = \dfrac{1}{\beta_k} + \omega$ are the roots of the equation $z^n + x^n = 0$, β_k are roots of the equation $\left(\dfrac{1}{z} + \omega\right)^n + x^n = 0$. Simplify the equation we have

$$(\omega^n + x^n)z^n + n\omega^{n-1}z^{n-1} + \cdots + 1 = 0.$$

Therefore,

$$\frac{a_1}{a_0} = \frac{n\omega^{n-1}}{\omega^n + x^n}. \tag{2.4.4}$$

Using (2.4.3), (2.4.4) and the lemma in Section 2.1, we have

$$\begin{aligned}
I(\omega) &= \sum_{k=1}^{n} |\beta_k|^2 \\
&= nr^2 - n|w|^2 - 2\mathrm{Re}\left(\frac{a_1}{a_0}\overline{w}\right) \\
&= n\left|\frac{x}{|\omega|^2 - x^2}\right|^2 - n\left|\frac{-\overline{\omega}}{|\omega|^2 - x^2}\right|^2 - 2\mathrm{Re}\left(\frac{n\omega^{n-1}}{\omega^n + x^n}\right)\left(\frac{-\overline{\omega}}{\omega^n + x^n}\right) \\
&= \frac{n}{|\omega|^2 - x^2}\left(2\mathrm{Re}\left(\frac{\omega^n}{|\omega|^n + x^n}\right) - 1\right) \\
&= \frac{n}{|\omega|^2 - x^2}\mathrm{Re}\left(\frac{\omega^n - x^n}{\omega^n + x^n}\right).
\end{aligned}$$

□

Suppose that the 6 bulbs are arranged in two triangles. By (2.4.2), the intensity of a point $\omega \in \Omega$ in the room is

$$\mathrm{Re}\left(\frac{3(\omega^3 - x^3)}{(1-\omega^2)(\omega^3 + x^3)} + \frac{3(\omega^3 + y^3)}{(1-y^2)(\omega^3 - y^3)}\right)$$

where x and y are the distances of the bulbs from the centre. In order to maximize the minimum intensity, we compare it with the intensity at the centre. If the minimum intensity on the boundary of the room is equal to the centre, the equation

$$\mathrm{Re}\left(\frac{3(\omega^3 - x^3)}{(1-\omega^2)(\omega^3 + x^3)} + \frac{3(\omega^3 + y^3)}{(1-y^2)(\omega^3 - y^3)}\right) = \frac{3}{x^2} + \frac{3}{y^2} \tag{2.4.5}$$

should have 6 roots for $|\omega| = 1$. Consider

$$\frac{3(\omega^3 - x^3)}{(1-\omega^2)(\omega^3 + x^3)} + \frac{3(\omega^3 + y^3)}{(1-y^2)(\omega^3 - y^3)} - \left(\frac{3}{x^2} + \frac{3}{y^2}\right) = 0. \tag{2.4.6}$$

If (2.4.6) has 6 solutions for $|\omega| = 1$, then (2.4.5) will also have 6 solutions for $|\omega| = 1$. The equation (2.4.6) has 6 roots for $|\omega| = 1$ which are the 6 minimum intensity points on the boundary of the circular room. Rewrite (2.4.6) in a quadratic equation in ω^3.

$$\begin{aligned}&-x^5y^3+x^7y^3-x^3y^5+x^3y^7\\&+(x^5-x^7+x^3y^2-2x^5y^2+2x^7y^2-x^2y^3+x^4y^3\\&-x^3y^4-y^5+2x^2y^5+y^7-2x^2y^7)\omega^3\\&+(x^2-x^4+y^2-4x^2y^2+2x^4y^2-y^4+2x^2y^4)\omega^6=0. \qquad (2.4.7)\end{aligned}$$

So the roots in (2.4.7) should have modulus 1. This happens when

$$(-x^5y^3+x^7y^3-x^3y^5+x^3y^7)-(x^2-x^4+y^2-4x^2y^2+2x^4y^2-y^4+2x^2y^4)=0.$$

That is

$$\begin{aligned}&-x^2+x^4-y^2+4x^2y^2-2x^4y^2-x^5y^3\\&+x^7y^3+y^4-2x^2y^4-x^3y^5+x^3y^7=0. \qquad (2.4.8)\end{aligned}$$

We want to maximize $\dfrac{3}{x^2}+\dfrac{3}{y^2}$ subject to constrain (2.4.8).

There is a trivial critical point for $x=y$, which is a minimum and no other critical point was found. Thus the maximum value is obtained when 1 is a double zero of (2.4.7). In this case (2.4.6) will have only 3 zeros. Then the discriminant of (2.4.7) when it is considered as a quadratic equation in ω^3 is 0. Together with (2.4.8), this is equivalent to the equation

$$\begin{aligned}&(x^5-x^7+x^3y^2-2x^5y^2+2x^7y^2-x^2y^3+x^4y^3-x^3y^4-y^5+2x^2y^5\\&+y^7-2x^2y^7)+2(x^2-x^4+y^2-4x^2y^2+2x^4y^2-y^4+2x^2y^4)=0 \qquad (2.4.9)\end{aligned}$$

We used Mathematica to solve the system of equation (2.4.8), (2.4.9) and get the numerical values

$$x=0.84886,\ y=0.64321.$$

The intensity at 0 and 1, which are the minimum intensity points, is

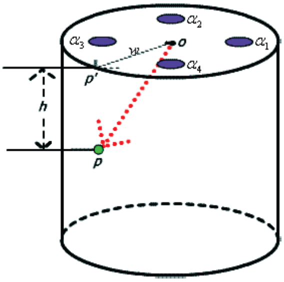
$$I=\frac{3(1-x^3)}{(1-x^2)(1+x^3)}+\frac{3(1-y^3)}{(1-y^2)(1+y^3)}=\frac{3}{x^2}+\frac{3}{y^2}=11.41470.$$

The minimum intensity when (2.4.1) is used is 9.70820. So we find that this is the best arrangement, among the three suggestions, for $n=6$.

In the introduction, we said that the best arrangement should be attained when we have maximum number of minimum intensity points. This example shows that this assumption is false. The number of minimum intensity points is 4 in this example and there are 7 if we use (2.4.1). This shows that the second assumption in the introduction is not true in general. Of course there is also a possibility that the result in this section is not the best arrangement.

2.5. 3-D generalization

In this section, we generalize (2.4.2) to 3 dimensional case. We assume that the bulbs are arranged on the top of a cylindrical room.



The intensity at point P vertically below P' is

$$I(x,P) = \frac{1}{|W-\alpha_1|^2+h^2} + \frac{1}{|W-\alpha_2|^2+h^2} + \cdots + \frac{1}{|W-\alpha_n|^2+h^2}, \quad (2.5.1)$$

where α_k, $k = 1, 2, \ldots, n$, are roots of $z^n + x^n = 0$ and W is the complex number represents P'. In order to apply the lemma in Section 2.1, we find A, W' in terms of x, h and W such that

$$|W-\alpha_k|^2 + h^2 = A|W'-\alpha_k|^2 \quad (2.5.2)$$

for $k = 1, 2, \ldots, n$. Note that $\alpha_k\overline{\alpha_k} = x^2$. Expand (2.5.2), we get

$$\begin{aligned} & W\overline{W} - (\overline{W}\alpha_k + W\overline{\alpha_k}) + \alpha_k\overline{\alpha_k} + h^2 \\ = {} & A[W'\overline{W'} - (\overline{W'}\alpha_k + W'\overline{\alpha_k}) + \alpha_k\overline{\alpha_k}] \end{aligned}$$

$$\begin{aligned} & (W\overline{W} + x^2 + h^2) - (\overline{W}\alpha_k + W'\overline{\alpha_k}) \\ = {} & A[W'\overline{W'} + x^2] - A(\overline{W'}\alpha_k + W'\overline{\alpha_k}). \end{aligned}$$

Compare coefficients of α_k, we have

$$\begin{cases} |W|^2 + x^2 + h^2 = A(|W'|^2 + x^2), \\ W = AW'. \end{cases}$$

From the second equation

$$W' = \frac{W}{A}.$$

Put it into the first equation

$$|W|^2 + x^2 + h^2 = \frac{|W|^2 + A^2x^2}{A}$$
$$A^2x^2 - A(|W|^2 + x^2 + h^2) + |W|^2 = 0.$$

Therefore we may take

$$A = \frac{|W|^2 + x^2 + h^2 \pm \sqrt{(|W|^2 + x^2 + h^2)^2 - 4x^2|W|^2}}{2x^2}$$
$$W' = \frac{W}{A} = \frac{2x^2W}{|W|^2 + x^2 + h^2 + \sqrt{(|W|^2 + x^2 + h^2)^2 - 4x^2|W|^2}}.$$

From (2.5.1), the intensity at P is

$$\begin{aligned} I(x,P) &= \frac{1}{A|W' - \alpha_1|^2} + \frac{1}{A|W' - \alpha_2|^2} + \cdots + \frac{1}{A|W' - \alpha_n|^2} \\ &= \frac{1}{A}\left(\frac{1}{|W' - \alpha_1|^2} + \frac{1}{|W' - \alpha_2|^2} + \cdots + \frac{1}{|W' - \alpha_n|^2}\right) \\ &= \frac{n}{A(|W'|^2 - x^2)}\mathrm{Re}\left(\frac{W'^n - x^n}{W'^n + A^nx^n}\right) \quad \text{by (2.4.2)} \\ &= \frac{nA}{|W|^2 - A^2x^2}\mathrm{Re}\left(\frac{W^n - A^nx^n}{W^n + A^nx^n}\right) \end{aligned}$$

where

$$A = \frac{|W|^2 + x^2 + h^2 \pm \sqrt{(|W|^2 + x^2 + h^2)^2 - 4x^2|W|^2}}{2x^2}.$$

3. Rectangular Domain

In this chapter, we consider rectangular rooms. Since the arrangement of bulbs depends only on the ratio of the length and width of the room, we may assume that the width of the room is two units. Throughout this chapter, let $\Omega = \{(x,y) : |x| \leqslant a, |y| \leqslant 1\} \subset \mathbb{R}^2$, $a \geqslant 1$, be a rectangular room and n be the number of bulbs in the room. We will make use of the two assumptions in the introduction to find the best arrangements although we have seen in section 2.3 that the second assumption is not true in general. We still think that this assumption is reasonable in our situation.

3.1. One Bulb

First of all we find the best arrangement for $n = 1$. When the bulb is put at the centre, the minimum intensity is $(1 + a^2)^{-1}$ which is attained at the four corners. We claim that this is the best arrangement. We have

Proposition 4. *Let* $\Omega = \{(x, y) : |x| \leqslant a, |y| \leqslant 1\}$, *then*

$$\max_{(x,y)\in\Omega}\left(\min_{(s,t)\in\Omega}\frac{1}{(s-x)^2+(t-y)^2}\right)=\frac{1}{1+a^2}.$$

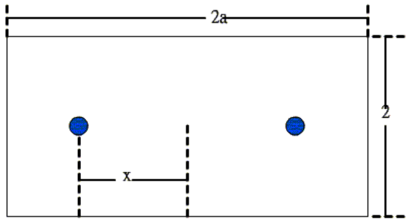
Proof. For all $(x, y) \in \Omega$, there exists $(s, t) = \left(-\frac{x}{|x|}a, -\frac{y}{|y|}\right)$ such that

$$\frac{1}{(s-x)^2+(t-y)^2}=\frac{1}{(a+|x|)^2+(1+|y|)^2}\leqslant\frac{1}{1+a^2}$$

and the equality holds if and only if $x = y = 0$. □

3.2. Two Bulbs

In this section, we consider two bulbs in a rectangular room. By our symmetric assumption, the two bulbs should be located at $(x, 0)$ and $(-x, 0)$.



We expect that the minimum intensity should be attained at the four corners and the mid-point of the long edge of the rectangular room, i.e. $(0, 1)$. There are 6 minimum intensity points. The intensity at $(0, 1)$ is $2(1+x^2)^{-1}$. Equate it with the intensity at the four corners we have

$$\frac{1}{1+(a-x)^2}+\frac{1}{1+(a+x)^2}-\frac{2}{1+x^2}=0$$

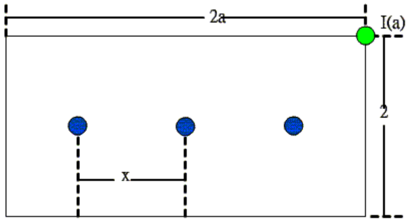
$$1 + a^2 - 3x^2 = 0.$$

Thus,

$$x=\frac{\sqrt{3(1+a^2)}}{3}.$$

3.3. Three Bulbs

When $n = 3$, we assume that the bulbs are located at $(-s, 0)$, $(0, 0)$ and $(s, 0)$.



We need to consider two cases depending on the size of a.

Small a

When a is small, consider the intensity at the corner,

$$I(s) = \frac{1}{1+a^2} + \frac{1}{1+(a-s)^2} + \frac{1}{1+(a+s)^2}$$

as a function of s. We want to maximize the intensity at the corners. Differentiate $I(s)$ with respect to s and factorize, we get

$$\frac{4s(-1+2a^2+3a^4-2s^2-2a^2s^2-s^4)}{(1+a^2-2as+s^2)^2(1+a^2-2as+s^2)^2} = 0.$$

Since $s \neq 0$,

$$(-1+2a^2+3a^4) - 2(1+a^2)s^2 - s^4 = 0. \tag{3.3.1}$$

We get

$$s = \sqrt{-1-a^2+2a\sqrt{1+a^2}}.$$

Large a

When a is large, we expect that the minimum intensity are attained at the corners and some points on the long edges. Fix s, let

$$I(v) = \frac{1}{1+v^2} + \frac{1}{1+(s-v)^2} + \frac{1}{1+(s+v)^2}$$

be the intensity at $(v, 1)$. We want to find the value of s such that the minimum value of $I(v)$ for $0 < v < a$ is equal to the intensity at the corner

$I(a)$. Consider the equation $I(v) - I(a) = 0$. Obviously $v^2 - a^2$ is a factor of $I(v) - I(a)$. The numerator of $\dfrac{I(v) - I(a)}{v^2 - a^2}$ is

$$\begin{aligned}&3 + 6a^2 + 3a^4 + 4a^4s^2 - 8a^2s^4 + a^4s^4 + 4s^6 - 2a^2s^6 + s^8\\&+ (6 + 12a^2 + 6a^4 - 8a^2s^2 - 8s^4 - 2a^2s^4 - 2s^6)v^2\\&+ (3 + 6a^2 + 3a^4 + 4s^2 + s^4)v^4\end{aligned}$$

which is a quadratic expression in v^2. If the minimum intensity along the upper edge of the room is equal to the intensity at the corners, then this expression should have unique zero for $0 < v < a$. Therefore the disriminant of this quadratic expression in v^2 is equal to zero. We have

$$\begin{aligned}&(6 + 12a^2 + 6a^4 - 8a^2s^2 - 8s^4 - 2a^2s^4 - 2s^6)^2\\&- 4(3 + 6a^2 + 3a^4 + 4s^2 + s^4)\\&(3 + 6a^2 + 3a^4 + 4a^4s^2 - 8a^2s^4 + a^4s^4 + 4s^6 - 2a^2s^6 + s^8) = 0.\end{aligned}$$

Factorizing this equation, we get

$$\begin{aligned}&s^2(4 + s^2)\\&(3 + 12a^2 + 18a^4 + 12a^6 + 3a^8 + 6s^2 + 6a^2s^2 - 6a^4s^2\\&- 6a^6s^2 + 3s^4 - 10a^2s^4 + 3a^4s^4 - 4a^2s^6) = 0.\end{aligned}$$

Since $s \neq 0, \pm 2i$, we have

$$\begin{aligned}&3 + 12a^2 + 18a^4 + 12a^6 + 3a^8 + (6 + 6a^2 - 6a^4 - 6a^6)s^2\\&+ (3 - 10a^2 + 3a^4)s^4 - 4a^2s^6 = 0\end{aligned} \tag{3.3.2}$$

which is a cubic equation in s^2.

Critical case

To find the critical case, eliminate s in (3.3.1) and (3.3.2)

$$\begin{cases}(-1 + 2a^2 + 3a^4) - 2(1 + a^2)s^2 - s^4 = 0\\ 3 + 12a^2 + 18a^4 + 12a^6 + 3a^8\\ +(6 + 6a^2 - 6a^4 - 6a^6)s^2 + (3 - 10a^2 + 3a^4)s^4 - 4a^2s^6 = 0.\end{cases}$$

We get

$$a^2(1 + a^2)(-9 - 20a^2 - 294a^4 + 12a^6 + 39a^8) = 0.$$

Since $a > 1$,

$$-9 - 20a^2 - 294a^4 + 12a^6 + 39a^8 = 0.$$

We have

$$a = 1.62394$$
$$s = 1.59905.$$

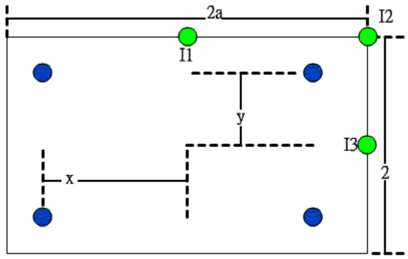
The conclusion is that if $a < 1.62394$, then we should solve (3.3.1) to find s. If $a > 1.62394$, then we should solve (3.3.2) to find s. If $a = 1.62394$, then both equations give the same value $s = 1.59905$.

3.4. Four Bulbs

When $n = 4$, there are 3 cases we need to consider.

Small a

When a is small, we expect that the four bulbs should be put at (x, y), $(-x, y)$, $(x, -y)$ and $(-x, -y)$.



Let

$$I_1 = 2\left(\frac{1}{x^2 + (1-y)^2} + \frac{1}{x^2 + (1+y)^2}\right)$$
$$I_2 = \frac{1}{(a-x)^2 + (1-y)^2} + \frac{1}{(a-x)^2 + (1+y)^2} + \frac{1}{(a+x)^2 + (1-y)^2} + \frac{1}{(a+x)^2 + (1+y)^2}$$
$$I_3 = 2\left(\frac{1}{(a-x)^2 + y^2} + \frac{1}{(a+x)^2 + y^2}\right)$$

be the intensities at $(0, 1)$, $(a, 1)$ and $(a, 0)$ respectively. With the second assumption in the introduction, we expect that $I_1 = I_2 = I_3$. To find x, y

in terms of a, consider $I_2 - I_3 = 0$, we have

$$\begin{aligned}
&a^2 + 3a^4 + 3a^6 + a^8 + x^2 + 10a^2x^2 + 13a^4x^2 + 4a^6x^2 + 3x^4 + 13a^2x^4 \\
&- 10a^4x^4 + 3x^6 + 4a^2x^6 + x^8 + y^2 - 2a^2y^2 - 3a^4y^2 - 2x^2y^2 \\
&- 18a^2x^2y^2 - 48a^4x^2y^2 - 3x^4y^2 - 48a^2x^4y^2 - 5y^4 + a^2y^4 - 6a^4y^4 \\
&+ x^2y^4 - 60a^2x^2y^4 - 6x^4y^4 + 7y^6 - 8a^2y^6 - 8x^2y^6 - 3y^8 = 0. \qquad (3.4.1)
\end{aligned}$$

From $I_1 - I_3 = 0$, we have

$$\begin{aligned}
&- a^2 + a^4 - x^2 - 4a^2x^2 + a^4x^2 - x^4 - 3a^2x^4 - y^2 + 4a^2y^2 \\
&+ a^4y^2 + 2x^2y^2 - 2a^2x^2y^2 + 3y^4 + a^2y^4 = 0. \qquad (3.4.2)
\end{aligned}$$

Eliminate y in (3.4.1) and (3.4.2) we have $p_1(x, a) =$

$$\begin{aligned}
&- 28 - 200a^2 + 609a^4 + 4676a^6 + 3630a^8 + 1272a^{10} + 261a^{12} + 20a^{14} \\
&+ 580x^2 + 772a^2x^2 - 7836a^4x^2 + 19220a^6x^2 + 14012a^8x^2 \\
&+ 3388a^{10}x^2 + 540a^{12}x^2 + 44a^{14}x^2 - 2144x^4 - 6784a^2x^4 \\
&- 92512a^4x^4 - 8064a^6x^4 + 13920a^8x^4 + 3328a^{10}x^4 + 96a^{12}x^4 \\
&+ 12608x^6 + 47040a^2x^6 - 130688a^4x^6 - 59264a^6x^6 - 3776a^8x^6 \\
&+ 960a^{10}x^6 + 19456x^8 + 38912a^2x^8 - 45056a^4x^8 - 30720a^6x^8 \\
&- 3072a^8x^8 + 4096x^{10} + 4096a^2x^{10} - 4096a^4x^{10} - 4096a^6x^{10} = 0.
\end{aligned}$$

Eliminate x in (3.4.1) and (3.4.2) we have $p_2(y, a) =$

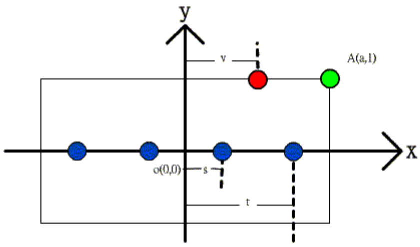
$$\begin{aligned}
&- 20a^2 - 261a^4 - 1272a^6 - 3630a^8 - 4676a^{10} - 609a^{12} + 200a^{14} \\
&+ 28a^16 - 44y^2 - 540a^2y^2 - 3388a^4y^2 - 14012a^6y^2 - 19220a^8y^2 \\
&+ 7836a^{10}y^2 - 772a^{12}y^2 - 580a^{14}y^2 - 96y^4 - 3328a^2y^4 - 13920a^4y^4 \\
&+ 8064a^6y^4 + 92512a^8y^4 + 6784a^{10}y^4 + 2144a^{12}y^4 - 960y^6 \\
&+ 3776a^2y^6 + 59264a^4y^6 + 130688a^6y^6 - 47040a^8y^6 - 12608a^{10}y^6 \\
&+ 3072y^8 + 30720a^2y^8 + 45056a^4y^8 - 38912a^6y^8 - 19456a^8y^8 \\
&+ 4096y^{10} + 4096a^2y^{10} - 4096a^4y^{10} - 4096a^6y^{10} = 0.
\end{aligned}$$

$p_1(x, a) = 0$ and $p_2(y, a) = 0$ are the equations that x and y should satisfy.

Medium a

When a is larger, we should put the bulbs along the x-axis at $(-t, 0)$, $(-s, 0)$,

$(s, 0)$ and $(t, 0)$ with $s < t$.



Let $I(v) = \dfrac{1}{1+(s-v)^2} + \dfrac{1}{1+(t-v)^2} + \dfrac{1}{1+(s+v)^2} + \dfrac{1}{1+(t+v)^2}$ be the intensity at $(v, 1)$. When the length a is not large, we want to maximize the intensities at $(0, 1)$ and $(a, 1)$ under constrain that their intensities are the same $I(0) = I(a)$. Now $I(0) = I(a)$ gives

$$\begin{aligned}&2 + 6a^2 + 6a^4 + 2a^6 - 3a^2s^2 - 2a^4s^2 + a^6s^2 + 3s^4 + a^2s^4 - 2a^4s^4 + s^6\\ &+ a^2s^6 - 3a^2t^2 - 2a^4t^2 + a^6t^2 - 18s^2t^2 - 6a^4s^2t^2 - 9s^4t^2 + 6a^2s^4t^2\\ &- 3s^6t^2 + 3t^4 + a^2t^4 - 2a^4t^4 - 9s^2t^4 + 6a^2s^2t^4 + t^6 + a^2t^6 - 3s^2t^6 = 0.\end{aligned}$$

We need to maximize $I(0)$ subject to the above constraint. Using the substitution $S = s^2 + t^2$ and $P = s^2t^2$, the above equation becomes

$$\begin{aligned}&2 + 6a^2 + 6a^4 + 2a^6 - 3a^2S - 2a^4S + a^6S + 4S^2 + a^2S^2 - 2a^4S^2 + S^3\\ &+ a^2S^3 + (-24 - 2a^2 - 2a^4 - 12S + 3a^2S - 3S^2)P + 6P^2 = 0. \qquad (3.4.3)\end{aligned}$$

Let $J = I(0) = \dfrac{2}{1+s^2} + \dfrac{2}{1+t^2} = \dfrac{2(2+S)}{1+S+P}$ be the intensity at $(0, 1)$. Now $P = \dfrac{2(2+S)}{J} - (1+S)$ and equation (3.4.3) becomes

$$\begin{aligned}&48 - 72J - 4a^2J - 4a^4J + 16J^2 + 4a^2J^2 + 4a^4J^2 + a^6J^2 + (24 - 48J\\ &+ 6a^2J + 16J^2 - 4a^2J^2 - 2a^4J^2)S + (-6J + 4J^2 + a^2J^2)S^2 = 0 \quad (3.4.4)\end{aligned}$$

which is a quadratic equation in S. We want to maximize $J = I(0)$ subject to (3.4.3).

Fix a positive real number J, equation (3.4.4) tells us whether there is an arrangement of bulbs so that the intensity at both $(0, 1)$ and $(a, 1)$ are J. In order words intensity J is attainable at $(0, 1)$ and $(a, 1)$ if (3.4.4) has real solution. Thus maximum intensity is obtained when the discriminant of (3.4.4) is 0. Therefore

$$(24-48J+6a^2J+16J^2-4a^2J^2-2a^4J^2)^2-4(-6J+4J^2+a^2J^2)$$
$$(48-72J-4a^2J-4a^4J+16J^2+4a^2J^2+4a^4J^2+a^6J^2)=0,$$

that is

$$\begin{aligned}&-144+288J-72a^2J-144J^2+264a^2J^2+39a^4J^2\\&-256a^2J^3-80a^4J^3-4a^6J^3+64a^2J^4+32a^4J^4+4a^6J^4=0.\end{aligned}\tag{3.4.5}$$

The corresponding value of S is

$$S=\frac{12-24J+3a^2J+8J^2-2a^2J^2-a^4J^2}{6J-4J^2-a^2J^2}.\tag{3.4.6}$$

Put $P=\dfrac{2(2+S)}{J}-(1+S)$ into (3.4.6), we get

$$P=\frac{24-36J+6a^2J+18J^2-11a^2J^2-2a^4J^2-4J^3+3a^2J^3+a^4J^3}{6J^2-4J^3-a^2J^3}$$

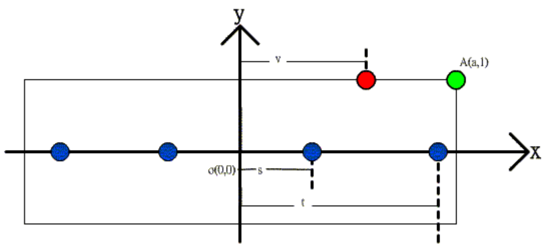
and s, t are the roots of the quadratic equation $x^4-Sx^2+P=0$, i.e.

$$\begin{aligned}&24-36J+6a^2J+18J^2-11a^2J^2-2a^4J^2-4J^3+3a^2J^3+a^4J^3\\&+(-12J+24J^2-3a^2J^2-8J^3+2a^2J^3+a^4J^3)x^2\\&+(6J^2-4J^3-a^2J^3)x^4=0.\end{aligned}\tag{3.4.7}$$

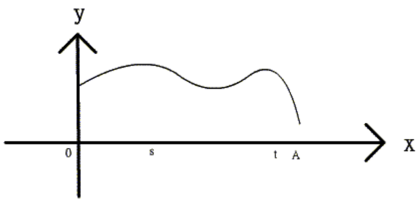
So, for a given a which is belong to the medium region (the range for medium a will be discussed later). We can find the values of s and t by solving (3.4.7).

Large a

When the length of the room is large, the bulbs should be put along the x-axis at $(-t,0)$, $(-s,0)$, $(s,0)$ and $(t,0)$ with $s<t$ as in the case of medium a.



The intensity along the upper edge of the room is as below:



Let $I(v) = \dfrac{1}{1+(s-v)^2} + \dfrac{1}{1+(t-v)^2} + \dfrac{1}{1+(s+v)^2} + \dfrac{1}{1+(t+v)^2}$ be the intensity at $(v, 1)$, $S = s^2 + t^2$, $P = s^2t^2$ and $J = \dfrac{2}{1+s^2} + \dfrac{2}{1+t^2} = \dfrac{2(2+S)}{1+S+P}$ be the intensity at $(0, 1)$ as before. We expect that the local minimum intensity between s and t should be equal to the intensity at $(0, 1)$ and $(a, 1)$, i.e. $I(0) = I(a)$ and $I(v) = I(0)$ should have unique solution for $0 < v < a$. Consider the equation $I(v) - J =$

$$\frac{1}{1+(s-v)^2} + \frac{1}{1+(t-v)^2} + \frac{1}{1+(s+v)^2} + \frac{1}{1+(t+v)^2} - J = 0.$$

We have

$$\begin{aligned}
&(1+P+S)(-4+J+JP-2S+JS)\\
&+(-12+4J+12P-12JP-4S+2JS-2JPS-2S^2+2JS^2)v^2\\
&+(-12+6J+2JP+2S-2JS+JS^2)v^4\\
&+(-4+4J-2JS)v^6+Jv^8=0.
\end{aligned}$$

The constant term is 0 since $I(0) = J$ by definition. Removing the solution $v = 0$, the equation becomes

$$\begin{aligned}
&-12+4J+12P-12JP-4S+2JS-2JPS-2S^2+2JS^2\\
&+(-12+6J+2JP+2S-2JS+JS^2)v^2\\
&+(-4+4J-2JS)v^4+Jv^6=0.
\end{aligned}$$

Using $P = \dfrac{2(2+S)}{J} - 1 - S$, we get

$$\begin{aligned} &48 - 72J + 16J^2 + 24S - 48JS + 16J^2S - 6JS^2 + 4J^2S^2 \\ &+ (-4J + 4J^2 + 6JS - 4J^2S + J^2S^2)v^2 \\ &+ (-4J + 4J^2 - 2J^2S)v^4 + J^2v^6 = 0. \end{aligned} \tag{3.4.8}$$

Now $I(a) = J$ gives

$$\begin{aligned} &48 - 72J - 4a^2J - 4a^4J + 16J^2 + 4a^2J^2 + 4a^4J^2 + a^6J^2 + 24S - 48JS \\ &+ 6a^2JS + 16J^2S - 4a^2J^2S - 2a^4J^2S - 6JS^2 + 4J^2S^2 + a^2J^2S^2 = 0 \end{aligned} \tag{3.4.9}$$

which is just the same as (3.4.4). Let $f(v)$ be the left hand side of equation (3.4.8). Now $f(v) - f(a)$ is a cubic polynomial in v^2 which has a factor $v^2 - a^2$. Since (3.4.8) has a repeat root, $f(v) - f(a) =$

$$\begin{aligned} &- J(a - v)(a + v) \\ &(-4 - 4a^2 + 4J + 4a^2J + a^4J + 6S - 4JS - 2a^2JS + JS^2) \\ &+ (-4 + 4J + a^2J - 2JS)v^2 + Jv^4 = 0 \end{aligned}$$

has an unique solution for $0 < v < a$. Thus the discriminant of the last factor is 0. So

$$\begin{aligned} &(-4 + 4J + a^2J - 2JS)^2 \\ &- 4J(-4 - 4a^2 + 4J + 4a^2J + a^4J + 6S - 4JS - 2a^2JS + JS^2) = 0 \end{aligned}$$

that is

$$16 - 16J + 8a^2J - 8a^2J^2 - 3a^4J^2 - 8JS + 4a^2J^2S = 0. \tag{3.4.10}$$

Eliminating S in (3.4.9) and (3.4.10), we get

$$\begin{aligned} &(-12 + 16J + a^2J) \\ &(-128 + 32a^2J + 128a^2J^2 + 120a^4J^2 - 208a^4J^3 - 34a^6J^3 \\ &+ 64a^4J^4 + 20a^6J^4 + a^8J^4) = 0. \end{aligned}$$

This equation tells us how to find the minimum intensity directly without knowing the arrangement of bulbs. The minimum intensity is the largest root of the second factor

$$\begin{aligned} &- 128 + 32a^2J + 128a^2J^2 + 120a^4J^2 - 208a^4J^3 - 34a^6J^3 \\ &+ 64a^4J^4 + 20a^6J^4 + a^8J^4 = 0. \end{aligned} \tag{3.4.11}$$

To find s and t, from (3.4.9) we have

$$S = \frac{16 - 16J + 8a^2J - 8a^2J^2 - 3a^4J^2}{8J - 4a^2J^2}$$

and $P = \dfrac{2(2+S)}{J} - 1 - S$, thus

$$P = \frac{32 - 16J + 16a^2J + 8J^2 - 40a^2J^2 - 6a^4J^2 + 12a^2J^3 + 3a^4J^3}{8J^2 - 4a^2J^3}$$

and s, t are positive roots of $x^4 - Sx^2 + P = 0$, i.e.

$$\begin{aligned}
&32 - 16J + 16a^2J + 8J^2 - 40a^2J^2 - 6a^4J^2 + 12a^2J^3 + 3a^4J^3 \\
&+ (-16J + 16J^2 - 8a^2J^2 + 8a^2J^3 + 3a^4J^3)x^2 \\
&+ (8J^2 - 4a^2J^3)x^4 = 0.
\end{aligned}$$

3.5. Critical Cases

We discussed the cases for $n = 4$. Now, we find the values of a that distinguish different situations. For the critical values of a, there are two bulbs arrangements such that the minimum intensities are equal. There are two critical cases for $n = 4$.

<u>Part A: Between Small a and Medium a</u>

For small a, the minimum intensity is

$$J = 2\left(\frac{1}{x^2 + (1-y)^2} + \frac{1}{x^2 + (1+y)^2}\right).$$

Put it into the equation (3.4.5) that J satisfy for medium a, we get

$$
\begin{aligned}
&81 - 246a^2 - 231a^4 - 48a^6 + 432x^2 - 434a^2x^2 - 682a^4x^2 - 176a^6x^2 \\
&+ 900x^4 + 514a^2x^4 - 457a^4x^4 - 224a^6x^4 + 864x^6 + 1494a^2x^6 + 372a^4x^6 \\
&- 96a^6x^6 + 270x^8 + 766a^2x^8 + 503a^4x^8 + 16a^6x^8 - 144x^{10} - 182a^2x^{10} \\
&+ 86a^4x^{10} + 16a^6x^{10} - 108x^{12} - 138a^2x^{12} - 39a^4x^{12} + 18a^2x^{14} + 9x^{16} \\
&- 2634a^2y^2 - 1650a^4y^2 - 240a^6y^2 + 648x^2y^2 - 3892a^2x^2y^2 - 3506a^4x^2y^2 \\
&- 640a^6x^2y^2 + 2160x^4y^2 + 2706a^2x^4y^2 - 852a^4x^4y^2 - 480a^6x^4y^2 \\
&+ 2232x^6y^2 + 5256a^2x^6y^2 + 1980a^4x^6y^2 + 288x^8y^2 + 122a^2x^8y^2 \\
&+ 742a^4x^8y^2 + 80a^6x^8y^2 - 648x^{10}y^2 - 1044a^2x^{10}y^2 - 234a^4x^{10}y^2 \\
&- 144x^{12}y^2 + 126a^2x^{12}y^2 + 72x^{14}y^2 - 540y^4 - 7766a^2y^4 - 3673a^4y^4 \\
&- 416a^6y^4 - 576x^2y^4 - 4926a^2x^2y^4 - 4068a^4x^2y^4 - 672a^6x^2y^4 \\
&+ 2196x^4y^4 + 9540a^2x^4y^4 + 2298a^4x^4y^4 - 96a^6x^4y^4 + 2592x^6y^4 \\
&+ 3172a^2x^6y^4 + 2108a^4x^6y^4 + 160a^6x^6y^4 - 756x^8y^4 - 3150a^2x^8y^4 \\
&- 585a^4x^8y^4 - 864x^{10}y^4 + 378a^2x^{10}y^4 + 252x^{12}y^4 + 432y^6 - 7290a^2y^6 \\
&- 2844a^4y^6 - 288a^6y^6 - 1224x^2y^6 + 5224a^2x^2y^6 + 668a^4x^2y^6 \\
&- 128a^6x^2y^6 + 2304x^4y^6 + 6964a^2x^4y^6 + 2732a^4x^4y^6 + 160a^6x^4y^6 \\
&+ 1296x^6y^6 - 4920a^2x^6y^6 - 780a^4x^6y^6 - 2160x^8y^6 + 630a^2x^8y^6 \\
&+ 504x^{10}y^6 + 846y^8 + 174a^2y^8 - 153a^4y^8 - 48a^6y^8 - 1296x^2y^8 \\
&+ 5810a^2x^2y^8 + 1678a^4x^2y^8 + 80a^6x^2y^8 + 3564x^4y^8 - 4230a^2x^4y^8 \\
&- 585a^4x^4y^8 - 2880x^6y^8 + 630a^2x^6y^8 + 630x^8y^8 - 1440y^{10} + 1714a^2y^{10} \\
&+ 398a^4y^{10} + 16a^6y^{10} + 2808x^2y^{10} - 1908a^2x^2y^{10} - 234a^4x^2y^{10} \\
&- 2160x^4y^{10} + 378a^2x^4y^{10} + 504x^6y^{10} + 756y^{12} - 354a^2y^{12} - 39a^4y^{12} \\
&- 864x^2y^{12} + 126a^2x^2y^{12} + 252x^4y^{12} - 144y^{14} + 18a^2y^{14} + 72x^2y^{14} \\
&+ 9y^{16} = 0.
\end{aligned}
$$

Combine it with (3.4.1)

$$
\begin{aligned}
&1 + 3a^2 + 3a^4 + a^6 - 3a^2x^2 - 2a^4x^2 + a^6x^2 - 6x^4 + a^2x^4 - 5a^4x^4 \\
&- 8x^6 + 7a^2x^6 - 3x^8 + 4y^2 + 13a^2y^2 + 10a^4y^2 + a^6y^2 - 48x^2y^2 \\
&- 18a^2x^2y^2 - 2a^4x^2y^2 - 60x^4y^2 + a^2x^4y^2 - 8x^6y^2 - 10y^4 + 13a^2y^4 \\
&+ 3a^4y^4 - 48x^2y^4 - 3a^2x^2y^4 - 6x^4y^4 + 4y^6 + 3a^2y^6 + y^8 = 0
\end{aligned}
$$

and (3.4.2)

$$\begin{aligned}&-a^2+a^4-x^2-4a^2x^2+a^4x^2-x^4-3a^2x^4-y^2+4a^2y^2+a^4y^2\\&+2x^2y^2-2a^2x^2y^2+3y^4+a^2y^4=0.\end{aligned}$$

We used Mathematica to solve the above system of equations and get

$$a = 2.28660,$$

$$x = 1.35022,\ y = 0.79582.$$

Using (3.4.5) and (3.4.7), we have

$$s = 0.86998,\ t = 24836,$$

and the minimum intensity is

$$J = 1.46869.$$

Remarks: This is the first example we see that the best arrangement is not unique.

Part B: Between Medium a and Large a

The minimum intensity for the medium and large a are given by (3.4.5) and (3.4.11). Eliminating J in these two equations, we get

$$\begin{aligned}&(-3872+3394a^2-558a^4-218a^6+37a^8)^2\\&(-1183744+539136a^2+3312a^4-23352a^6+2151a^8)^2=0.\end{aligned}$$

(3.4.5) and (3.4.11) have no common solution for J when the second factor is 0. Therefore,

$$-3872+3394a^2-558a^4-218a^6+37a^8=0.$$

We used Mathematica to solve this equation and get

$$a = 2.53167.$$

Using (3.4.7), we have

$$s = 0.89294,\ t = 2.50741,$$

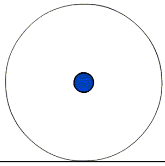
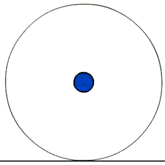
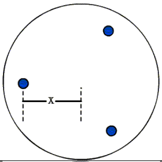
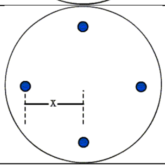
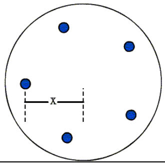
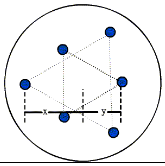
(3.4.5) or (3.4.11) gives the minimum intensity

$$J = 1.38721.$$

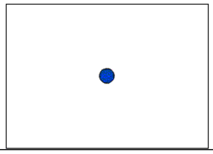
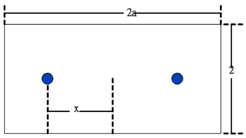
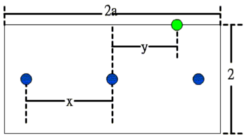
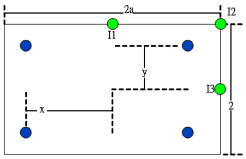
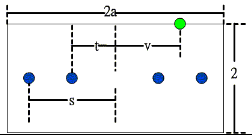
4. Summary and Conclusion

Our results are summarized in the following two tables

Circular domain

n	Figure	Arrangement	Minimum intensity
1		The bulb should be placed at the centre.	1
2		The two bulbs should be placed at the centre.	2
3		Three bulbs form an equilateral triangle and the distance of the bulbs from the centre is $x = 0.39265$.	3.14192
4		Four bulbs form a square and the distance of the bulbs from the centre is $x = 0.64359$.	4.82843
5		Five bulbs form a regular pentagon and the distance of the bulbs from the centre is $x = 0.76763$.	7.04949
6		Six bulbs form two equilateral triangles and the distance of the bulbs from the centre are $x = 0.84886$ and $y = 0.64321$.	11.41470

Rectangular domain

n	Figure	Remarks
1		The bulb is placed place at the centre of the room.
2		The bulbs are placed at $(x, 0)$ and $(-x, 0)$ where $x = \dfrac{\sqrt{3(1+a^2)}}{3}$.
3		The bulbs are placed at $(x, 0)$, $(0, 0)$ and $(-x, 0)$ where $x = \sqrt{-1 - a^2 + 2a\sqrt{1+a^2}}$ for $1 < a \leqslant 1.62394$ and $3 + 12a^2 + 18a^4 + 12a^6 + 3a^8 + (6 + 6a^2 - 6a^4 - 6a^6)x^2 + (3 - 10a^2 + 3a^4)x^4 - 4a^2x^6 = 0$ for $a > 1.62394$.
4		Where $a \leqslant 2.2866$, the bulbs are placed at (x, y), $(-x, y)$, $(x, -y)$ and $(-x, -y)$ where x, y satisfy (3.4.1) and (3.4.2).
		Where $a \geqslant 2.28660$, the bulbs are placed at $(s, 0)$, $(-s, 0)$, $(t, 0)$ and $(-t, 0)$ when s, t satisfy (3.4.7) for $2.2866 < a \leqslant 2.53167$ and s, t satisfy (3.4.11) for $a \geqslant 2.53167$.

The objective of this project is to find the arrangements of a given number of bulbs in a circular or rectangular room to maximize the minimum intensity. For simplicity we assume that the rooms are 2 dimensional. The main tool of solving the problem on circular domain is the intensity formula for regular polygon (2.4.2)

$$I(\omega) = \frac{n}{|\omega|^2 - x^2} \mathrm{Re} \left(\frac{\omega^n - x^n}{\omega^n + x^n} \right).$$

This formula can be proved using complex numbers and Vieta's theorem. We found that the bulbs should be arranged in two triangles of different size when there are six bulbs. We also found a 3-D generalization of this formula.

In the rectangular case, we found that the solution of the problem is not unique in general. Two different arrangements may give the same minimum intensity when there are four bulbs.

Except for some simple cases, we cannot prove that the results in this report are the best arrangements. Even for the case when there are 3 bulbs in a circular domain, we have to assume that the bulbs are arranged in a symmetric way. The results in this report are just the best arrangements we can find.

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