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ON THE PRIME NUMBER THEOREM

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Abstract. It is possible to prove the Prime Number Theorem(PNT) by elementary methods. A. Selberg sketched his original elementary proof in a paper in 1949. This article is an attempt to complete the proof of the PNT by following the ideas in Selberg's paper.[2]

1. Introduction

This article is divided into 7 sections, namely:

1. Introduction
2. Background of the Prime Number Theorem
3. Basic Facts, Definitions and Theorems
4. Selberg's Original Proof
5. Results related to the Prime Number Theorem
6. Conjectures related to the Prime Number Theorem
7. Conclusion

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[1]This work is done under the supervision of the author's teacher, Mr. Wai-Man Ko.
[2]The abstract is added by the editor.

To explain these subtitles, first let me briefly mention my experiences in doing this project.

In fact, at the beginning I was not doing this topic. As I intended to work on the field of number theory, at first I either tried to find some interesting problems and thought about it, or tried to generalize some existing theorems. However, what I discovered in the first few months was that the problems I solved are too easy, or that the problems, though hard enough, are too particular and not so meaningful, or that the facts I obtained are actually some known results, or that the things I intended to work on are too difficult for me.

After a long period of choosing topics, I finally worked on problems concerning primes. During the time of research, I thought about a few problems. What I found was that the Prime Number Theorem(PNT) is actually very sharp. Many interesting results which are hard to prove directly are just consequences of the PNT. As I knew that the few existing proofs of PNT either use some advanced knowledge such as complex analysis or are elementary but quite long, I tried to prove the PNT on my own.

However, it is certainly too hard for me. Later when I read a paper written by Atle Selberg, I found that the historic 'completely' elementary proof of Selberg is indeed not his first proof of PNT. Hence instead of proving PNT on my own, I followed the sketched route mentioned by Selberg in the paper, and completed it.

To make this article self-contained, in sections 2 and 3 I will include the background of the PNT and some essential knowledge in analytic number theory. People who are familiar with them can jump to section 4, where Selberg's original proof will be completed. In sections 5 and 6 I will give my comments on the PNT and some results concerning primes.

2. Background of the Prime Number Theorem

The Prime Number Theorem is probably one of the most important theorems in mathematics. It was first conjectured by Gauss and Legendre independently in the late 18^{th} century, and was proved independently by Hadamard and Poussin in 1896. The simplest and most common form of the PNT is: If $\pi(x)$ is the number of primes not exceeding x, then

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1,$$

where log is the natural logarithm.

In other words,

$$\pi(x) \sim \frac{x}{\log x}.$$

The proofs of Hadamard and Poussin used methods from complex analysis. In that time, many mathematicians believed that the PNT is a 'deep' theorem and it could not be proved by using elementary methods.

In 1948, A. Selberg proved an asymptotic formula:

$$\sum_{p \leqslant x} (\log p)^2 + \sum_{pq \leqslant x} \log p \log q = 2x \log x + O(x), \tag{1}$$

where p, q run over all primes. This result is strong, though it is a consequence of the very famous Prime Number Theorem. However, what surprises us is that Selberg proved it with a completely elementary method! So we may use it to work out an elementary proof of some results obtained in analytic number theory, which previously looked inaccessible by elementary methods.

A few months later, P. Erdos successfully used this formula to prove a generalization of the Bertrand Postulate: For all real positive c, there exists real positive $\delta(c)$ such that

$$\pi((1+c)x) - \pi(x) > \delta(c)\frac{x}{\log x}, \tag{2}$$

(It is also a consequence of the Prime Number Theorem and we will prove it later). Erdos sent this proof to Selberg, who, two days later, successfully used (1) and (2) to prove the Prime Number Theorem! This proof is very important and meaningful, since Selberg and Erdos proved the theorem just using elementary techniques (i.e. without using complex analysis), which showed that the belief of mathematicians in the past is wrong! Later they jointly simplified the proof.

In 1949, Selberg published his proof in the Annals of Mathematics. However he chose to write his simplified proof ('simple' is in the sense that it avoids the concept of lower and upper limits) but just sketched his primary proof (Selberg commented that the two proofs are pretty different). As I think the original proof he sketched is much more direct and clear than the simplified one, so I decide to complete Selberg's original proof and my work is presented in this article.

3. Basic Facts and Theorems

In this section, some essential basic facts, definitions and theorems which will appear in Selberg's original proof are introduced. Readers who are familiar with number theory can skip this part.

Theorem 3.1. ***(Abel's identity)*** *For any arithmetical function $a(n)$, let*

$$A(x) = \begin{cases} \sum_{n \leqslant x} a(n) & \text{if } x \geqslant 1, \\ 0 & \text{if } 0 \leqslant x \leqslant 1. \end{cases}$$

If f is continously differentiable on the interval $[y, x]$ where $0 < y < x$, then

$$\sum_{y < n \leqslant x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) dt. \tag{3}$$

Proof. Using integration by parts we get

$$\begin{aligned} \sum_{y < n \leqslant x} a(n) f(n) &= \int_y^x f(t) dA(t) \\ &= [f(t) A(t)]_y^x - \int_y^x A(t) df(t) \\ &= A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) dt. \end{aligned}$$ □

Corollary 3.2. ***(Euler's Summation Formula)*** *If f is continuously differentiable on the interval $[y, x]$ where $0 < y < x$, then*

$$\sum_{y < n \leqslant x} f(n) = \int_y^x f(t) dt + \int_y^x (t - [t]) f'(t) dt + f(x)([x] - x) - f(y)([y] - y). \tag{4}$$

Proof. Let the $a(n)$ in (3) equal 1 for all $n \in \mathbb{N}$. Then

$$\sum_{y < n \leqslant x} f(n) = f(x)[x] - f(y)[y] - \int_y^x [t] f'(t) dt.$$

On the other hand, using integration by parts we have

$$\int_y^x t f'(t) dt = [t f(t)]_y^x - \int_y^x f(t) dt = x f(x) - y f(y) - \int_y^x f(t) dt.$$

Combining them we obtain the famous Euler's Summation Formula. □

Notice that both theorem 3.1 and corollary 3.2 may or may not work when doing approximations. Sometimes the terms $A(x)f(x) - A(y)f(y)$ estimates $\sum_{y<n\leqslant x} a(n)f(n)$ better, but sometimes the term $\int_y^x f(t)dt$ estimates $\sum_{y<n\leqslant x} f(n)$ better. Of course, sometimes both of them don't work and give a large error term.

Experiences told us that it is hard to deal with the function $\pi(x)$. However some mathematicians later found a relation which links $\pi(x)$ and the so-called Chebyshev's ϑ-function together:

Definition 3.3. ***(Chebyshev's ϑ-function)*** *For $x > 0$, we define the Chebyshev's ϑ-function by*

$$\vartheta(x) = \sum_{p\leqslant x} \log p,$$

where p runs over all primes.

Theorem 3.4. *For $x \geqslant 2$ we have*

$$\vartheta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt \tag{5}$$

and

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t\log^2 t}dt. \tag{6}$$

Proof. Define $a(n)$ in theorem 3.1 as follows:

$$a(n) = \begin{cases} 1 & \text{if } n \text{ is a prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\vartheta(x) = \sum_{1<n\leqslant x} a(n)\log n = \pi(x)\log x - \pi(1)\log 1 - \int_1^x \frac{\pi(t)}{t}dt,$$

which proves (5) as $\pi(t) = 0$ for $t < 2$.

Next, define $b(n) = a(n)\log n$ and take a real k where $1 < k < 2$, then

$$\pi(x) = \sum_{k<n\leqslant x} \frac{b(n)}{\log n} = \frac{\vartheta(x)}{\log x} - \frac{\vartheta(k)}{\log k} + \int_k^x \frac{\vartheta(t)}{t\log^2 t}dt,$$

which proves (6) as $\vartheta(t) = 0$ for $t < 2$. $\square$

Now let's see the integral part of (5). What is its order of magnitude when assuming $\pi(x) \sim \dfrac{x}{\log x}$? The following lemma can prove that it is indeed $O(\dfrac{x}{\log x})$.

Lemma 3.5. *If $x \geqslant y > 1$, then*

$$\int_y^x \frac{dt}{\log^n t} = O(\frac{x}{\log^n x}). \tag{7}$$

In particular, $\displaystyle\int_y^x \frac{dt}{\log t} = O(\frac{x}{\log x})$.

Proof. Since

$$0 \leqslant \int_y^x \frac{dt}{\log^n t} = \int_y^{\sqrt{x}} \frac{dt}{\log^n t} + \int_{\sqrt{x}}^x \frac{dt}{\log^n t} \leqslant \frac{\sqrt{x}}{\log^n y} + \frac{x-\sqrt{x}}{\log^n \sqrt{x}} = O(\frac{x}{\log x}),$$

which proves (7). $\square$

It is then easy to see that $\vartheta(x) \sim x$ if $\pi(x) \sim \dfrac{x}{\log x}$. In fact the converse is also true.

Theorem 3.6. *The following relations are equivalent:*

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1 \tag{8}$$

$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1 \tag{9}$$

Proof. Divide both sides of (5) by x. If (8) is true,

$$\frac{\pi(t)}{t} = \frac{1}{t}O(\frac{t}{\log t}) = O(\frac{1}{\log t}),$$

so by lemma 3.5,

$$\frac{1}{x}\int_2^x \frac{\pi(t)}{t}dt = O(\frac{1}{x}\int_2^x \frac{dt}{\log t}) = O(\frac{1}{\log x}) = o(1).$$

Hence (8) implies (9).

Conversely, divide both sides of (6) by $\frac{x}{\log x}$. If (9) is true, $\vartheta(t) = O(t)$, so by lemma 3.5,

$$\frac{\log x}{x}\int_2^x \frac{\vartheta(t)}{t\log^2 t}dt = O(\frac{\log x}{x}\int_2^x \frac{1}{\log^2 t}dt) = O(\frac{1}{\log x}) = o(1).$$

Hence (9) implies (8). □

Now two more functions and a few theorems will be introduced. The easiest ones will be stated without proof. They will be used in section 4.

Definition 3.7. ***(Mangoldt function*** $\Lambda(n)$***)*** *For* $n \in \mathbb{N}$, $\Lambda(n)$ *is defined as follows:*

$$\Lambda(n) = \begin{cases} \log p & \textit{if } n = p^m \textit{ for some } m \geqslant 1 \\ 0 & \textit{otherwise} \end{cases}$$

Definition 3.8. ***(Chebyshev's*** ψ***-function)*** *For* $x > 0$, *we define the Chebyshev's* ψ*-fucntion by*

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n).$$

Theorem 3.9. *If* $F(x) = \sum_{n \leqslant x} f(n)$, *we have*

$$\sum_{n\leqslant x} f(n)\left[\frac{x}{n}\right] = \sum_{n\leqslant x} F(\frac{x}{n}). \tag{10}$$

Proof. Consider an array of numbers as follows:

$$\begin{array}{l} \underbrace{f(1), f(1), f(1), \ldots, f(1)}_{[x]} \\ \underbrace{f(2), f(2), \ldots, f(2)}_{[x/2]} \\ \underbrace{f(3), \ldots, f(3)}_{[x/3]} \\ \ldots \\ f([x]) \end{array}$$

Summing them row by row gives the L.H.S. of the formula and summing them column by column gives the R.H.S. of the formula. □

Theorem 3.10. *For $x > 0$, we have*

$$\sum_{n\leqslant x}\psi(\frac{x}{n}) = \sum_{n\leqslant x}\Lambda(n)\left[\frac{x}{n}\right] = \log[x]! = x\log x - x + O(\log x). \tag{11}$$

Proof. The first two equalities are trivial. The first one uses theorem 3.9 while the second one uses the unique factorization of integers. The last one uses Euler's summation formula as follows:

$$\begin{aligned}\log[x]! &= \sum_{n\leqslant x}\log n\\ &= \int_1^x \log t dt + \int_1^x \frac{t-[t]}{t}dt + \log x([x]-x)\\ &= x\log x - x + 1 + \int_1^x \frac{t-[t]}{t}dt + O(\log x)\\ &= x\log x - x + O(\int_1^x \frac{1}{t}dt) + O(\log x)\\ &= x\log x - x + O(\log x).\end{aligned}$$ □

Theorem 3.11.

$$\sum_{d|n}\mu(d) = \left[\frac{1}{n}\right] \tag{12}$$

where μ is the Möbius function.

Theorem 3.12.

$$\Lambda(n) = \sum_{d|n}\mu(d)\log\frac{n}{d}. \tag{13}$$

Proof. Use the Möbius inversion formula. □

Theorem 3.13. *If $s > 0$ and $s \neq 1$,*

$$\sum_{n\leqslant x}\frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}) \tag{14}$$

where ζ is the Riemann zeta function.

Proof. Use Euler's summation formula. □

Theorem 3.14. *For $x > 0$, we have*

$$0 \leqslant \psi(x) - \vartheta(x) < \frac{\sqrt{x}\log^2 x}{2\log 2}.$$

Proof.

$$\begin{aligned}
0 \leqslant \psi(x) - \vartheta(x) &= \sum_{n \leqslant \log x/\log 2} \vartheta(x^{\frac{1}{n}}) - \vartheta(x) \\
&= \sum_{2 \leqslant n \leqslant \log x/\log 2} \vartheta(x^{\frac{1}{n}}) \\
&< \sum_{2 \leqslant n \leqslant \log x/\log 2} x^{\frac{1}{n}} \log x^{\frac{1}{n}} \\
&< \frac{\log x}{\log 2} \sqrt{x} \log \sqrt{x} \\
&= \frac{\sqrt{x} \log^2 x}{2 \log 2}
\end{aligned}$$

□

Finally, we come to a beautiful theorem which was proved by H.N. Shapiro in 1950. Its corollaries play an important role in the original proof of Selberg.

Theorem 3.15. ***(Shapiro's Tauberian Theorem)*** *If $a(n)$ is a nonnegative arithmetical function such that, for $x \geqslant 1$,*

$$\sum_{n \leqslant x} a(n) \left[\frac{x}{n}\right] = x \log x + O(x). \tag{15}$$

then

(a) *for $x \geqslant 1$, we have*

$$\sum_{n \leqslant x} \frac{a(n)}{n} = \log x + O(1); \tag{16}$$

(b) *there is a constant $M > 0$ such that*

$$\sum_{n \leqslant x} a(n) \leqslant Mx \text{ for all } x \geqslant 1;$$

(c) *there is a constant $m > 0$ and an $x_0 > 0$ such that*

$$\sum_{n \leqslant x} a(n) \geqslant mx \text{ for all } x \geqslant x_0.$$

Proof. Let

$$A(x) = \sum_{n \leqslant x} a(n), \ B(x) = \sum_{n \leqslant x} a(n) \left[\frac{x}{n}\right].$$

Then

$$\begin{aligned}
B(x)-2B(\frac{x}{2}) &= \sum_{n\leqslant x} a(n)\left[\frac{x}{n}\right] - 2\sum_{n\leqslant x/2} a(n)\left[\frac{x}{2n}\right] \\
&= \sum_{n\leqslant x/2} a(n)(\left[\frac{x}{n}\right]-2\left[\frac{x}{2n}\right]) + \sum_{x/2<n\leqslant x} a(n)\left[\frac{x}{n}\right] \\
&\geqslant \sum_{x/2<n\leqslant x} a(n)\left[\frac{x}{n}\right] \\
&= A(x)-A(\frac{x}{2}),
\end{aligned}$$

i.e.

$$A(x)-A(\frac{x}{2}) \leqslant B(x)-2B(\frac{x}{2}) \tag{17}$$

but from (15) we know that

$$B(x)-2B(\frac{x}{2}) = x\log x + O(x) - 2(\frac{x}{2}\log\frac{x}{2} + O(\frac{x}{2})) = O(x).$$

Hence from(17) we can find a constant M' such that

$$A(x)-A(\frac{x}{2}) \leqslant M'x \text{ for all } x \geqslant 1.$$

So

$$A(x) = \sum_{i=0}^{\infty}(A(\frac{x}{2^i}) - A(\frac{x}{2^{i+1}})) \leqslant \sum_{i=0}^{\infty} M'\frac{x}{2^i} = 2M'x.$$

(b) is then proved by letting $M = 2M'$.

Next, as $\left[\frac{x}{n}\right] = \frac{x}{n} + O(1)$ and $\sum_{n\leqslant x} a(n) = O(x)$ by (b), so

$$\begin{aligned}
B(x) &= \sum_{n\leqslant x} a(n)\left[\frac{x}{n}\right] \\
&= \sum_{n\leqslant x} a(n)(\frac{x}{n} + O(1)) \\
&= x\sum_{n\leqslant x}\frac{a(n)}{n} + O(\sum_{n\leqslant x} a(n)) \\
&= x\sum_{n\leqslant x}\frac{a(n)}{n} + O(x).
\end{aligned}$$

By (15) we get

$$\sum_{n\leqslant x}\frac{a(n)}{n} = \frac{1}{x}(B(x)+O(x)) = \frac{1}{x}(x\log x + O(x)) = \log x + O(1),$$

which proves (a).
Finally we wrtie

$$\sum_{n\leqslant x}\frac{a(n)}{n}=\log x+R(x).$$

(a) tells us that $|R(x)|<K$ for some $K>0$. If $x>\alpha x\geqslant 1$, then

$$\begin{aligned}\sum_{\alpha x<n\leqslant x}\frac{a(n)}{n}&=\sum_{n\leqslant x}\frac{a(n)}{n}-\sum_{n\leqslant \alpha x}\frac{a(n)}{n}\\&=\log x+R(x)-\log\alpha x-R(\alpha x)\\&\geqslant-\log\alpha-2K.\end{aligned}$$

Obviously we can choose a suitable α such that

$$\sum_{\alpha x<n\leqslant x}\frac{a(n)}{n}\leqslant\sum_{\alpha x<n\leqslant x}\frac{a(n)}{\alpha x}\leqslant\frac{1}{\alpha x}A(x).$$

Therefore $A(x)\leqslant mx$ for $m=\alpha$ and $x\geqslant x_0=\dfrac{1}{\alpha}$. □

Corollary 3.16. *For $x\geqslant 1$, we have*

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{n}=\log x+O(1) \tag{18}$$

and

$$\sum_{p\leqslant x}\frac{\log p}{p}=\log x+O(1). \tag{19}$$

Also there are positive constants M, m, M' and m' such that $\vartheta(x)\leqslant Mx$ for all $x\geqslant 1$ and $\vartheta(x)\geqslant mx$ for all sufficiently large x, and $\psi(x)\leqslant M'x$ for all $x\geqslant 1$ and $\psi(x)\geqslant m'x$ for all sufficiently large x.

Proof. (18) is trivial in view of theorem 3.10 and theorem 3.15. For (19), we use Euler's summation formula:

$$\begin{aligned}
\sum_{n\leqslant x}\Lambda(n)\left[\frac{x}{n}\right] &= \sum_{p\leqslant x}\log p\sum_{m=1}^{\infty}\left[\frac{x}{p^m}\right] \\
&= \sum_{p\leqslant x}\log p\left[\frac{x}{p}\right] + \sum_{p\leqslant x}\log p\sum_{m=2}^{\infty}\left[\frac{x}{p^m}\right] \\
&\leqslant \sum_{p\leqslant x}\log p\left[\frac{x}{p}\right] + \sum_{p\leqslant x}\log p\sum_{m=2}^{\infty}\frac{x}{p^m} \\
&= \sum_{p\leqslant x}\log p\left[\frac{x}{p}\right] + x\sum_{p\leqslant x}\frac{\log p}{p(p-1)} \\
&\leqslant \sum_{p\leqslant x}\log p\left[\frac{x}{p}\right] + x\sum_{n=2}^{\infty}\frac{\log n}{n(n-1)}.
\end{aligned}$$

By the integral text, $\sum_{n=2}^{\infty}\frac{\log n}{n(n-1)}$ converges, so $x\sum_{n=2}^{\infty}\frac{\log n}{n(n-1)} = O(x)$. Hence

$$\sum_{n\leqslant x}\Lambda(n)\left[\frac{x}{n}\right] = \sum_{p\leqslant x}\log p\left[\frac{x}{p}\right] + O(x).$$

By theorem 3.10 we get

$$\sum_{p\leqslant x}\log p\left[\frac{x}{p}\right] = \sum_{n\leqslant x}\Lambda(n)\left[\frac{x}{n}\right] + O(x) = x\log x + O(x).$$

By letting the $a(n)$ in theorem 3.15 as follows:

$$a(n) = \begin{cases}\log n & \text{if } n \text{ is a prime,}\\ 0 & \text{otherwise,}\end{cases}$$

(19) follows immediately. □

4. Selberg's Original Proof

First of all, we prove an asymptotic formula (Theorem 4.2) discovered by A. Selberg in 1948 using a method (Lemma 4.1) given by Tatuzawa and Iseki in 1951. In fact, most of the existing elementary proofs of the Prime Number Theorem are based on this formula.

Lemma 4.1. *Let F be a real- or complex-valued function defined on $(0,+\infty)$, then*

$$F(x)\log x + \sum_{n\leqslant x} F(\frac{x}{n})\Lambda(n) = \sum_{d\leqslant x}\mu(d)G(\frac{x}{d}), \tag{20}$$

where

$$G(x) = \log x \sum_{n\leqslant x} F(\frac{x}{n}). \tag{21}$$

Proof. By theorem 3.11 and theorem 3.12, we have

$$F(x)\log x = \sum_{n\leqslant x} F(\frac{x}{n})\log\frac{x}{n}\left[\frac{1}{n}\right] = \sum_{n\leqslant x} F(\frac{x}{n})\log\frac{x}{n}\sum_{d|n}\mu(d)$$

and

$$\sum_{n\leqslant x} F(\frac{x}{n})\Lambda(n) = \sum_{n\leqslant x} F(\frac{x}{n})\sum_{d|n}\mu(d)\log\frac{n}{d}.$$

So

$$\begin{aligned} F(x)\log x + \sum_{n\leqslant x} F(\frac{x}{n})\Lambda(n) &= \sum_{n\leqslant x} F(\frac{x}{n})[\sum_{d|n}\mu(d)\log\frac{x}{n} + \sum_{d|n}\mu(d)\log\frac{n}{d}] \\ &= \sum_{n\leqslant x}\sum_{d|n} F(\frac{x}{n})\mu(d)\log\frac{x}{d} \\ &= \sum_{d\leqslant x}\mu(d)\log\frac{x}{d}\sum_{q\leqslant x/d} F(\frac{x/d}{q}) \\ &= \sum_{d\leqslant x}\mu(d)G(\frac{x}{d}) \end{aligned}$$

□

Theorem 4.2.

$$\vartheta(x)\log x + \sum_{p\leqslant x}\vartheta(\frac{x}{p})\log p = 2x\log x + O(x), \tag{22}$$

where p runs through all primes less than or equal to x.

Proof. By lemma 4.1 and theorem 3.10, when $F_1(x) = \psi(x)$ we have

$$\begin{aligned} G_1(x) &= \log x\sum_{n\leqslant x}\psi(\frac{x}{n}) \\ &= x\log^2 x - x\log x + O(\log^2 x). \end{aligned}$$

And when $F_2(x) = x - C - 1$ where C is the Euler's constant, we have

$$\begin{aligned} G_2(x) &= \log x \sum_{n \leqslant x} (\frac{x}{n} - C - 1) \\ &= x \log x \sum_{n \leqslant x} \frac{1}{n} - (C+1) \log x \sum_{n \leqslant x} 1 \\ &= x \log x(\log x + C + O(\frac{1}{x})) - (C+1) \log x(x + O(1)) \\ &= x \log^2 x - x \log x + O(\log x). \end{aligned}$$

Hence we have $G_1(x) - G_2(x) = O(\log^2 x)$, which implies $G_1(x) - G_2(x) = O(\sqrt{x})$.

By theorem 3.13,

$$\begin{aligned} \sum_{d \leqslant x} \mu(d) G_1(\frac{x}{d}) - \sum_{d \leqslant x} \mu(d) G_2(\frac{x}{d}) &= \sum_{d \leqslant x} \mu(d)[G_1(\frac{x}{d}) - G_2(\frac{x}{d})] \\ &= O(\sum_{d \leqslant x} \sqrt{\frac{x}{d}}) \\ &= O(\sqrt{x} \sum_{d \leqslant x} \frac{1}{\sqrt{d}}) \\ &= O(x), \end{aligned}$$

so we have

$$\sum_{d \leqslant x} \mu(d) G_1(\frac{x}{d}) = \sum_{d \leqslant x} \mu(d) G_2(\frac{x}{d}) + O(x).$$

Therefore, by the Shapiro's theorem and formula(18), we have

$$\begin{aligned} &\psi(x) \log x + \sum_{n \leqslant x} \psi(\frac{x}{n}) \Lambda(n) \\ &= (x - C - 1) \log x + \sum_{n \leqslant x} (\frac{x}{n} - C - 1) \Lambda(n) + O(x) \\ &= x \log x + x \sum_{n \leqslant x} \frac{\Lambda(n)}{n} - (C+1)\psi(x) + O(x) \\ &= x \log x + x(\log x + O(1)) + O(x) \\ &= 2x \log x + O(x). \end{aligned}$$

Finally define an arithmetical function $\Lambda_1(n)$ by

$$\Lambda_1(n) = \begin{cases} \log n & \text{if } n \text{ is a prime} \\ 0 & \text{otherwise} \end{cases}$$

and consider the expression

$$(\psi(x)-\vartheta(x))\log x+\sum_{n\leqslant x}\psi(\frac{x}{n})\Lambda(n)-\sum_{p\leqslant x}\vartheta(\frac{x}{p})\log p$$

$$=(\psi(x)-\vartheta(x))\log x+\sum_{n\leqslant x}\psi(\frac{x}{n})(\Lambda(n)-\Lambda_1(n))+\sum_{p\leqslant x}(\psi(\frac{x}{n})-\vartheta(\frac{x}{n}))\Lambda_1(n).$$

By theorem 3.14,

$$(\psi(x)-\vartheta(x))\log x=o(x). \tag{23}$$

By corollary 3.16,

$$\sum_{n\leqslant x}\psi(\frac{x}{n})(\Lambda(n)-\Lambda_1(n))=O(x\sum_{n\leqslant x}(\frac{\Lambda(n)}{n}-\frac{\Lambda_1(n)}{n}))=O(x). \tag{24}$$

By theorem 3.15 and theorem 1 of reference [4],

$$\sum_{p\leqslant x}(\psi(\frac{x}{n})-\vartheta(\frac{x}{n}))\Lambda_1(n)=O(\sqrt{x}\sum_{n\leqslant x}\frac{\Lambda_1(n)}{\sqrt{n}}).$$

As

$$\begin{aligned}\sum_{n\leqslant x}\frac{\Lambda_1(n)}{\sqrt{n}}&=\sum_{p\leqslant\sqrt{x}}\frac{\log p}{\sqrt{p}}+\sum_{\sqrt{x}<p\leqslant x}\frac{\log p}{\sqrt{p}}\\&\leqslant\sum_{p\leqslant\sqrt{x}}\log p+\frac{1}{\sqrt{x}}\sum_{\sqrt{x}<p\leqslant x}\log p\\&\leqslant\vartheta(\sqrt{x})+\frac{1}{\sqrt{x}}\vartheta(x)\\&=O(\sqrt{x}),\end{aligned}$$

we have

$$\sum_{p\leqslant x}(\psi(\frac{x}{n})-\vartheta(\frac{x}{n}))\Lambda_1(n)=O(\sqrt{x}\sum_{n\leqslant x}\frac{\Lambda_1(n)}{\sqrt{n}})=O(x). \tag{25}$$

Combining (23), (24) and (25), we get

$$(\psi(x)-\vartheta(x))\log x+\sum_{n\leqslant x}\psi(\frac{x}{n})\Lambda(n)-\sum_{p\leqslant x}\vartheta(\frac{x}{p})\log p=O(x),$$

or

$$\vartheta(x)\log x+\sum_{p\leqslant x}\vartheta(\frac{x}{p})\log p=2x\log x+O(x). \qquad\square$$

From this asymptotic formula, P. Erdös proved a result about the number of primes between two numbers (Theorem 4.3). Its proof(see reference [2]) is omitted here as it is quite lengthy. this result is found to be useful when giving an elementary proof of the PNT.

Theorem 4.3. *For an arbitrary positive real number δ, there exists a constant $K(\delta) > 0$ such that*

$$\pi(x+\delta x) - \pi(x) > K(\delta)\frac{x}{\log x}.$$

Now a series of lemmas will be proved. Using them we can prove an equivalent form of the Prime Number Theorem as stated in theorem 3.6.

Lemma 4.4. *Let* $\liminf_{x\to\infty} \dfrac{\vartheta(x)}{x} = a$ *and* $\limsup_{x\to\infty} \dfrac{\vartheta x}{x} = A$. *We have* $a + A = 2$.

Proof. Choose those $x \to \infty$ such that $\dfrac{\vartheta(x)}{x} \to a$. From (22) we get

$$\frac{1}{x\log x}\sum_{p\leqslant x}\vartheta(\frac{x}{p})\log p = 2 - a + o(1).$$

However by the definition of A, for arbitrary $\epsilon > 0$ we can find an $x_0 > 0$ such that

$$\frac{1}{x\log x}\sum_{p\leqslant x}\vartheta(\frac{x}{p})\log p < \frac{(A+\epsilon)x}{x\log x}\sum_{p\leqslant x}\frac{\log p}{p}$$

for $x > x_0$. By corollary 3.16,

$$\sum_{p\leqslant x}\frac{\log p}{p} = \log x + O(1),$$

hence

$$2 - a + o(1) = \frac{1}{x\log x}\sum_{p\leqslant x}\vartheta(\frac{x}{p})\log p < A + \epsilon + o(1),$$

which implies $2 - a \leqslant A + \epsilon$.

As ϵ is arbitrary, we get $2 - a \leqslant A$ or $2 \leqslant a + A$.

If we choose those $x \to \infty$ such that $\dfrac{\vartheta(x)}{x} \to A$ at the beginning, by similar arguments we can get $2 \geqslant a + A$. Therefore, $a + A = 2$. □

Lemma 4.5. *For arbitrary $\epsilon > 0$, we can choose a $x \to \infty$ (cf.* ***Reviewer's Comments*** *1) such that $\vartheta(x) \to ax$ and*

$$\vartheta(\frac{x}{p}) > (A-\epsilon)\frac{x}{p}$$

except for a set I satisfying

$$\sum_{p\in I}\frac{\log p}{p} = o(\log x).$$

Similarly, we can choose an $x' \to \infty$ *such that* $\vartheta(x') \to Ax'$ *and*

$$\vartheta(\frac{x'}{p}) > (a+\epsilon)\frac{x'}{p}$$

except for a set I' *satisfying*

$$\sum_{p\in I'} \frac{\log p}{p} = o(\log x).$$

Proof. For the first statement, if it is false, then there exists a set I such that $\sum_{p\in I} \frac{\log p}{p} > c_1 \log x$ and

$$\vartheta(\frac{x}{p}) \leqslant (A-\epsilon)\frac{x}{p} \text{ for all } p \in I.$$

However, by theorem 4.2 and corollary 3.16,

$$\begin{aligned}(2-a)x\log x + O(x) &= \sum_{p\leqslant x} \vartheta(\frac{x}{p})\log p \\ &= \sum_{p\in I} \vartheta(\frac{x}{p})\log p + \sum_{p\notin I} \vartheta(\frac{x}{p})\log p \\ &\leqslant \sum_{p\in I}(A-\epsilon)x\frac{\log p}{p} + \sum_{p\notin I}(A\frac{x}{p} + o(\frac{x}{p}))\log p \\ &= c_1(A-\epsilon)x\log x + A(1-c_1)x\log x + o(x\log x) \\ &= A - c_1\epsilon x\log x + o(x\log x)\end{aligned}$$

which contradicts that $a + A = 2$. Hence the first statement is proved. The second one can be prove by similar arguments. □

Using the notations in lemma 4.5, if we can find x, x', p and p' such that

$$\frac{x}{p} < \frac{x'}{p'} < (1+\epsilon)\frac{x}{p},$$

then we have

$$(A-\epsilon)\frac{x}{p} < \vartheta(\frac{x}{p}) \leqslant \vartheta(\frac{x'}{p'}) < (a+\epsilon)\frac{x'}{p'} < (a+\epsilon)(1+\epsilon)\frac{x}{p},$$

which implies

$$(A-\epsilon) < (a+\epsilon)(1+\epsilon).$$

As ϵ is arbitrary, we have $A \leqslant a$. Together with lemma 4.4, $\lim_{x\to 1} \frac{\vartheta(x)}{x} = 1$. Hence the Prime Number Theorem is proved.

However I just found that there are flaws in my original proof of the existence of such x, x', p and p'(cf. **Reviewer's Comments** 2). A proof of this fact seems to be present in [2].

5. Results related to the Prime Number Theorem

This section contains some of my views on the Prime Number Theorem and results related to it. They may look discrete and have no obvious relations between each other. In fact most of them are problems I encountered when I was working on the proof of the PNT. It is possible that they have already been thought about by some people before.

First of all let us talk about the order of magnitude of the function $\pi(x)$. Someone may wonder how Gauss and Legendre could conjecture $\pi(x) \sim x/\log x$, as the distribution of primes is so irregular while $x/\log x$ is such a 'regular' function. Was this discovery just a coincidence with no reason behind? Or did Gauss and Legendre have extraordinary ability of observation when they were facing the table of primes? At first I wondered too, but what I found later is that it is indeed very natural to conjecture $\pi(x) \sim x/\log x$!

The sieve of Eratosthenes is probably the earliest efficient method for finding primes. Together with the inclusion-exclusion principle, it is easy to deduce that

$$\pi(x) - \pi(\sqrt{x}) + 1 = [x] - \sum_{p_1 \leqslant \sqrt{x}} \left[\frac{x}{p_1}\right] + \sum_{p_1 < p_2 \leqslant \sqrt{x}} \left[\frac{x}{p_1 p_2}\right] - \ldots + (-1)^n \left[\frac{x}{p_1 \ldots p_n}\right] \tag{26}$$

where $n = \pi(\sqrt{x}), p_i (i = 1, 2, \ldots, n)$ are primes not larger than x and $[x]$ is the floor function. For the L.H.S., due to the large difference between the orders of magnitude of x and $\sqrt{x}$, one may expect $\pi(x) - \pi(\sqrt{x}) + 1 \sim \pi(x)$ (It is actually true and is an easy consequence of the PNT). For the R.H.S., one may also expect it has the same order of magnitude as the expression with floor functions removed. (However this looks untrue from the calculations with a computer, which show that they differ by a constant factor close to

1.) Hence it is natural to guess

$$\begin{aligned}\pi(x) &\sim [x - \sum_{p_1 \leqslant \sqrt{x}} \frac{x}{p_1} + \sum_{p_1 < p_2 \leqslant \sqrt{x}} \frac{x}{p_1 p_2} - \ldots + (-1)^n \frac{x}{p_1 \ldots p_n}] \\ &= x \prod_{p \leqslant \sqrt{x}} (1 - \frac{1}{p}) \\ &= \frac{x}{\prod\limits_{p \leqslant \sqrt{x}} \frac{p}{p-1}} \\ &= \frac{x}{\prod\limits_{p \leqslant \sqrt{x}} (\sum\limits_{m=0}^{\infty} \frac{1}{p^m})}.\end{aligned}$$

Using the idea in Euler's proof of having infinitely many primes, one can expect

$$\pi(x) \sim \frac{x}{\prod\limits_{p \leqslant \sqrt{x}} (\sum\limits_{m=0}^{\infty} \frac{1}{p^m})} \sim \frac{x}{\sum\limits_{n=1}^{x} \frac{1}{n}} \sim \frac{x}{\log x}!$$

This deduction is of course not rigorous, and it is wrong actually! However it really suggests us that $\pi(x) \sim x/\log x$ is a good guess! At least there is a close relationship between the functions $\pi(x)$ and $x/\log x$. By direct calculation one may verify that it is indeed true.

After I got this idea, I spent some time on investigating them. Here are some observations:

Data 5.1. Let

$$f(x) = \frac{x}{\prod\limits_{p \leqslant \sqrt{x}} (\sum\limits_{m=0}^{\infty} \frac{1}{p^m})} \div \frac{x}{\log x} = \log x \prod_{p \leqslant x} (1 - \frac{1}{p}).$$

Values of $f(x)$:

x	$f(x)$	x	$f(x)$
10^2	1.05261	10^{10}	1.12258
10^4	1.10816	10^{12}	1.12288
10^6	1.11858	10^{14}	1.12291
10^8	1.12154		

It is natural to guess $\lim_{x\to\infty} \log x \prod_{p\leqslant x}(1-\frac{1}{p})$ exists and approximately equals 1.23. If it is true, it certainly implies

$$\lim_{x\to\infty} \frac{[x] - \sum\limits_{p_1\leqslant\sqrt{x}} \left[\frac{x}{p_1}\right] + \sum\limits_{p_1<p_2\leqslant\sqrt{x}} \left[\frac{x}{p_1p_2}\right] - \ldots + (-1)^n \left[\frac{x}{p_1\ldots p_n}\right]}{x \prod\limits_{p\leqslant\sqrt{x}} (1-\frac{1}{p})}$$
$$\approx 0.81.$$

In real situation, it is very hard to deal with the floor functions in (26). Using this formula I can only prove this very weak result:

Theorem 5.2. ***(Euclid's Second Theorem)*** *The number of primes is infinite.*

Proof. Apply the inequality $x-1<[x]\leqslant x$ to (26), we get

$$\begin{aligned}
&[x] - \sum_{p_1\leqslant\sqrt{x}} \left[\frac{x}{p_1}\right] + \sum_{p_1<p_2\leqslant\sqrt{x}} \left[\frac{x}{p_1p_2}\right] - \ldots + (-1)^n \left[\frac{x}{p_1\ldots p_n}\right] \\
&> x - 1 - \sum_{p_1\leqslant\sqrt{x}} \frac{x}{p_1} + \sum_{p_1<p_2\leqslant\sqrt{x}} \frac{x}{p_1p_2} - C_2^{\pi(\sqrt{x})} - \sum_{p_1<p_2<p_3\leqslant\sqrt{x}} \frac{x}{p_1p_2p_3} \cdots \\
&= x \prod_{p\leqslant\sqrt{x}} (1-\frac{1}{p}) - 2^{\pi(\sqrt{x})-1}
\end{aligned}$$

and

$$\begin{aligned}
&[x] - \sum_{p_1\leqslant\sqrt{x}} \left[\frac{x}{p_1}\right] + \sum_{p_1<p_2\leqslant\sqrt{x}} \left[\frac{x}{p_1p_2}\right] - \ldots + (-1)^n \left[\frac{x}{p_1\ldots p_n}\right] \\
&< x - \sum_{p_1\leqslant\sqrt{x}} \frac{x}{p_1} + C_1^{\pi(\sqrt{x})} + \sum_{p_1<p_2\leqslant\sqrt{x}} \frac{x}{p_1p_2} - \sum_{p_1<p_2<p_3\leqslant\sqrt{x}} \frac{x}{p_1p_2p_3} \\
&\quad + C_3^{\pi(\sqrt{x})} + \cdots \\
&= x \prod_{p\leqslant\sqrt{x}} (1-\frac{1}{p}) + 2^{\pi(\sqrt{x})-1}.
\end{aligned}$$

Hence

$$\prod_{p\leqslant\sqrt{x}} (1-\frac{1}{p}) + 2^{\pi(\sqrt{x})-1} > \pi(x) - \pi(\sqrt{x}) + 1 > \prod_{p\leqslant\sqrt{x}} (1-\frac{1}{p}) - 2^{\pi(\sqrt{x})-1}.$$

If there are only finitely many primes, $\pi(x) - \pi(\sqrt{x}) + 1 = 1$ for large x and $2^{\pi(\sqrt{x})-1}$ and $\prod_{p \leqslant \sqrt{x}} (1 - \frac{1}{p})$ are constants. However, $x \prod_{p \leqslant \sqrt{x}} (1 - \frac{1}{p}) - 2^{\pi(\sqrt{x})-1} \to \infty$ when $x \to \infty$, which leads to a contradiction! so the number of primes is infinite. □

In section 2 I have mentioned the Erdös proved a generalization of the Bertrand's Postulate using Selberg's asymptotic formula. Actually it is just an easy consequence of the powerful Prime Number Theorem.

Theorem 5.3. ***(Bertrand's Postulate)*** *For each $n \geqslant 2$, there is a prime p such that $n < p < 2n$.*

Proof. See reference [6] □

Theorem 5.4. ***(Generalization of Bertrand's Postulate)*** *Given arbitrary $c > 0$, there exists $K(c) > 0$ such that*

$$\pi((1+c)x) - \pi(x) \leqslant K(c)\frac{x}{\log x}$$

for sufficiently large x.

Proof. As $\frac{(1+c)x}{\log((1+c)x) = O(\frac{x}{\log x})}$, by the Prime Number Theorem,

$$\begin{aligned} &\pi((1+c)x) - \pi(x) \\ &= [\frac{(1+c)x}{\log((1+c)x)} + o(\frac{(1+c)x}{\log((1+c)x)})] - [\frac{x}{\log x} + o(\frac{x}{\log x})] \\ &= \frac{(1+c)x}{\log((1+c) + \log x} - \frac{x}{\log x} + o(\frac{x}{\log x}). \end{aligned}$$

Choose a c' such that $c > c' > 0$. For sufficiently large x,

$$\frac{(1+c)x}{\log((1+c) + \log x} > \frac{(1+c)x}{(1+c')\log x}.$$

Hence

$$\begin{aligned} \pi((1+c)x) - \pi(x) &> \frac{(1+c)x}{(1+c')\log x} - \frac{x}{\log x} + o(\frac{x}{\log x}) \\ &= c''\frac{x}{\log x} + o(\frac{x}{\log x}), \end{aligned}$$

where c'' is a constant. The theorem then follows. □

Many results can be proved by Bertrand's Postulate. For example, the following interesting theorem is its easy consequence.

Theorem 5.5. *If n is a natural number, then $n!$ is never a power of any interger.*

Proof. When $n = 1, 2, 3$ the theorem is obvious. When $n \geqslant 4$, by Bertrand's Postulate there is a prime p such that $[n/2] < p < 2[n/2]$. As $2p > n$, $p \| n!$. Hence $n!$ is never a power of any integer. □

For Selberg's original proof, I think his asymptotic formula is not necessary. Although it is very sharp, it came out very unnaturally. I guess there are other much more natural ways to deduce the PNT without using it.

6. Conjectures Related to the Prime Number Theorem

This section contains some 'phenomena' I observed when doing this project. They look true but I could not prove them. Some of them were mentioned in the last section, but I write them here once again. These conjectures are arranged in the order of increasing difficulty according to my opinions. Some of them might be conjectured by someone in the past.

1. Are the following two asymptotic formulae true?

For positive real n,

$$\sum_{p \leqslant x} \frac{\log^n p}{p} = \frac{\log^n}{n} + O(1).$$

For positive integer $n \geqslant 2$ (it may be true for positive real $n > 1$?),

$$\sum_{p \leqslant x} \frac{\log p}{\sqrt[n]{p}} \sim \frac{n}{n-1} x^{\frac{n-1}{n}}.$$

Comment: Let

$$f_n(x) = \sum_{p \leqslant x} \frac{\log^n p}{p} - \frac{\log^n x}{n}$$

and

$$g_n(x) = (\sum_{p \leqslant x} \frac{\log p}{\sqrt[n]{p}}) \div (\frac{n}{n-1} x^{\frac{n-1}{n}}).$$

Corollary 3.16 states that $f_1(x) = O(1)$. For other n, see the following data:

x	$f_2(x)$	x	$f_2(x)$	x	$f_2(x)$
10	-0.94941	10^4	-2.41264	10^7	-2.55145
10^2	-1.86573	10^5	-2.50292		
10^3	-2.24543	10^6	-2.54396		

x	$f_3(x)$	x	$f_3(x)$	x	$f_3(x)$
10	-1.57446	10^4	-8.65602	10^7	-10.1818
10^2	-5.04098	10^5	-9.54964		
10^3	-7.33604	10^6	-10.063		

x	$f_4(x)$	x	$f_4(x)$	x	$f_4(x)$
10	-3.03632	10^4	-40.6969	10^7	-57.905
10^2	-16.3872	10^5	-49.5742		
10^3	-30.2508	10^6	-56.0253		

x	$f_{0.1}(x)$	x	$f_{0.1}(x)$	x	$f_{0.1}(x)$
10	-9.68887	10^4	-9.78022	10^7	-9.78177
10^2	-9.76466	10^5	-9.78139		
10^3	-9.7769	10^6	-9.78174		

x	$f_{3/2}(x)$	x	$f_{3/2}(x)$	x	$f_{3/2}(x)$
10	-0.860821	10^4	-1.54845	10^7	-1.59068
10^2	-1.33465	10^5	-1.57718		
10^3	-1.48888	10^6	-1.5888		

x	$g_2(x)$	x	$g_2(x)$	x	$g_2(x)$
10	0.40788	10^4	0.961092	10^7	0.998071
10^2	0.731126	10^5	0.985598		
10^3	0.894516	10^6	0.994534		

x	$g_3(x)$	x	$g_3(x)$	x	$g_3(x)$
10	0.469714	10^4	0.979088	10^7	0.999215
10^2	0.789371	10^5	0.993294		
10^3	0.930852	10^6	0.997517		

x	$g_{1.1}(x)$	x	$g_{1.1}(x)$	x	$g_{1.1}(x)$
10	0.108999	10^4	0.504932	10^7	0.735068
10^2	0.256633	10^5	0.597177		
10^3	0.391766	10^6	0.673431		

I was inspired by Corollary 3.16 to conjecture this two formulae. They seem true but I did not work on them at all. I am not sure whether they are easy or not.

2. Is $\vartheta(x) < x + O(x)$ for all $x \geqslant 1$?

Comment: By using a computer we can see that $\vartheta(x) < x$ for $1 \leqslant x \leqslant 1,000,000$. Moreover, the value of $\vartheta(x) - x$ seems unbounded below. As the performances of $\vartheta(x)$ and $\psi(x)$ are similar[3] , I also tested whether $\psi(x)$ has similar performance as $\vartheta(x)$. However it turned out that this is wrong. Certainly, if this is true, then it immediately gives another proof of the PNT, as $\liminf\limits_{x\to\infty} \dfrac{\vartheta(x)}{x} + \limsup\limits_{x\to\infty} \dfrac{\vartheta(x)}{x} = 2$ and $\limsup\limits_{x\to\infty} \dfrac{\vartheta(x)}{x} \leqslant 1$ imply

$$\liminf_{x\to\infty} \frac{\vartheta(x)}{x} = \limsup_{x\to\infty} \frac{\vartheta(x)}{x} = \lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1.$$

Also this inequality can be used to prove a problem stated in conjecture 4.

3. Is the following generalization of Shapiro's Tauberian Theorem true?

Let $a(n)$ be a nonnegative arithmetical function such that

$$\sum_{n\leqslant x} a(n) \left[\frac{x}{n}\right] = x\log x + Kx + o(x)$$

for all $x \geqslant 1$, where K is a constant. Then for $x \geqslant 1$ there exists a constant K' such that

$$\sum_{n\leqslant x} \frac{a(n)}{n} = \log x + K' + o(1).$$

Comment: This generalization seems true when $a(n) = \log n$ or $a(n) = \Lambda(n)$, and I guess it is true for general $a(n)$. However, no matter whether it is true for general $a(n)$ or it is true for $a(n) = \Lambda(n)$ or $a(n) = \log n$, the PNT will be an easy consequence.

4. Does $\pi(x + c_1x^{c_2}) - \pi(x)$ tend to infinity for all $c_1, c_2 > 0$?

Comment: Tested with a computer, it seems to be true (I tested it for $1 \leqslant x \leqslant 1,000,000$). The case $c_1 > 0, c_2 \geqslant 1$ is an easy consequence of the Prime Number Theorem. If it is true for some c_2 such that $c_2 < 0.5$, it implies the unsolved problem: for $n > 1$, there is a prime p such that $n^2 < p(n+1)^2$. It is surely an extremely difficult problem.

Another interesting problem is to find some functions $f(x)$ such that $\pi(x + f(x)) - \pi(x)$ is bounded. I tried $f(x) = \log^n x$. Tested with a computer, it seems to be bounded for small n. However, when I tested it with n equal to 2.xxx, it seems to grow very very slow! An even harder problem is: what n

[3]It is well-known that both of the relations $\vartheta(x) \sim x$ and $\psi(x) \sim x$ are equivalent to the PNT. However $\psi(x)/x$ converges faster to 1 than $\vartheta(x)/x$ when $x \to \infty$.

will make $\pi(x+\log^n x)-\pi(x)$ unbounded? The following are some related data:

x	$\pi(x+\log x)-\pi(x)$	x	$\pi(x+\log x)-\pi(x)$
10	1	10^6	1
10^2	2	10^7	0
10^3	0	10^8	1
10^4	2	10^9	2
10^5	1	10^{10}	1

x	$\pi(x+\log^2 x)-\pi(x)$	x	$\pi(x+\log^2 x)-\pi(x)$
10	2	10^8	17
10^2	5	10^9	23
10^3	7	10^{10}	19
10^4	8	10^{11}	34
10^5	9	10^{12}	27
10^6	14	10^{13}	30
10^7	12		

x	$\pi(x+\log^3 x)-\pi(x)$	x	$\pi(x+\log^3 x)-\pi(x)$
10	4	10^7	254
10^2	20	10^8	360
10^3	49	10^9	431
10^4	84	10^{10}	513
10^5	130	10^{11}	633
10^6	202	10^{12}	758

The case $n=1$ is relatively easy. it is easy to prove that it does not tend to infinity by establishing a bounded subsequence: Let $x=\prod_{i=1}^{m} p_i - p_{m+1}$, where p_i are primes. By the Prime Number Theorem, $\vartheta(p_m)<2p_m$ for large m. So

$$\pi(x+\log x)-\pi(x)\leqslant \pi(x+\vartheta(p_m))-\pi(x)\leqslant \pi(x+2p_{m+1}-1)-\pi(x)\leqslant 2.$$

Proving $\pi(x+\log x)-\pi(x)$ is bounded for all x is much more difficult. If conjecture 2 is true, we can prove that $\pi(x+\log x)-\pi(x)=0$ for infinitely many x by choosing p_m such that $p_{m+1}-p_m$ is larger than the upper bound of $\vartheta(p_m)-p_m$ and using similar arguments as above.

7. Conclusion

The real time I spent on this report is very little. It was mainly because I used most of the time to prepare my AL public examination and the IMO2006 until the end of July. Most of my work came out in August.

After reading this article, one may feel that it looks like a report rather than a research, as the topic I did is not new and I did not obtain any meaningful result. The topics and problems concerning primes are still to hard for me now. My present knowledge does not allow me to get any nice things in this stage.

My completion of Selberg's proof may look trivial and easy to mathematicians. However, to me it is not easy at all. I did spend a long time to work out the details of the proof on my own. No matter how people evaluate this article, I think I have gained a lot in the process, and I love number theory more now! In this article I only used the most fundamental knowledge in analytic number theory. In the future I will certainly learn deeper in this field, and also in algebraic number theory.

In section 6, I stated a few 'conjectures'. In fact many conjectures came out of my mind when I was working on this topic. Conjecturing something is easy but proving something is hard! In the future I will continue to work on conjectures 1,2 and 3. They are meaningful and they look solvable. I wish I can enter the final, and I will have obtained some concrete results for sharing in December!

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Reviewer's Comments

1. In the statement of lemma 4.5, it is not standard in Mathematics to say "choose a $x \to \infty$" and "choose a $x' \to \infty$".

2. On page 15, last paragraph of section 4, the author should clarify whether such proof exists.