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FURTHER INVESTIGATION ON BUFFON'S NEEDLE PROBLEM

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Abstract. In this article, the well-known Buffon's Needle Problem is generalized. Instead of a needle, we consider dropping a triangle or a rhombus onto a plane with equally-spaced parallel lines and investigate the probability that the randomly-dropped figure intersects the lines.

1. Introduction

Assume that a "needle" (line segment) of length a is randomly dropped onto a plane with parallel lines which are separated by a distance d $(> a)$ from each other. The probability P that the needle intersects the lines is given by

$$P = \frac{2a}{\pi d}. \tag{1}$$

This is the answer to the Buffon's Needle Problem, which was first stated in 1777 and is one of the most famous problems in the field of geometrical probability [1]. In section 2 of this article, we investigate the modified problem in which a triangle is dropped instead of a needle. Finally, in section 3 of this article, we calculate the corresponding probability when a rhombus is dropped in place of a needle. The results are also verified by re-establishing (1) as a special case.

[1]This work is done under the supervision of the authors' teacher, Mr. Wai-Man Chu.

2. Dropping Triangles

Assume that a triangle ABC of sides a, b and c ($\leqslant a$) is randomly dropped onto a plane with parallel lines which are separated by a distance d ($> 2a$) from each other (see Figure 1). Let θ be the acute angle between side a and the gridline, and y be the distance between B and the next gridline in direction of $\overrightarrow{BC}$.

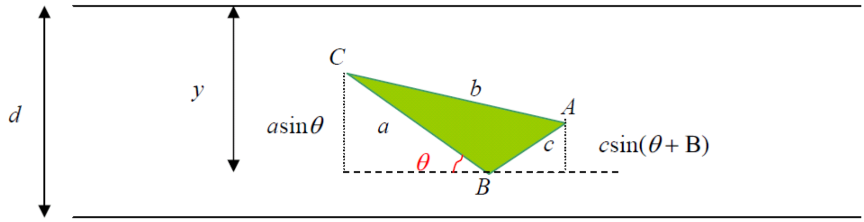

FIGURE 1

Since the triangle will intersect the gridlines if and only if at least two of its sides intersect the gridlines, we have

$$
\begin{aligned}
& P(\triangle ABC \text{ intersects the gridlines}) \\
=\ & P(a \text{ or } c \text{ intersects the gridlines}) \\
=\ & P(a \text{ intersects the gridlines}) + P(c \text{ intersects the gridlines}) \\
& - P(\text{both } a \text{ and } c \text{ intersects the gridlines}) \\
=\ & \frac{2a}{\pi d} + \frac{2c}{\pi d} - P(\text{both } a \text{ and } c \text{ intersects the gridlines}). \quad (\text{by (1)})
\end{aligned}
$$

Now, in order for both a and c to intersect the gridlines, either

(I) $c \sin(\theta + B) > a \sin\theta > y$ or
(II) $a \sin\theta > c \sin(\theta + B) > y$.

(Note that it is impossible for a and c to intersect two different gridlines as $a + c \leqslant 2a < d$.) We also observe that when $c \sin(\theta + B) = a \sin\theta$, AC is parallel to the gridlines and so $\theta = C$. Hence, $c \sin(\theta + B) > a \sin\theta$ if $\theta < C$ and $c \sin(\theta + B) < a \sin\theta$ if $\theta > C$.

Therefore, If B is an acute angle,

$$\begin{aligned}
&P(\text{both } a \text{ and } c \text{ intersects the gridlines})\\
=\ &\frac{\int_0^C a\sin\theta d\theta + \int_C^{\pi/2} c\sin(\theta+B)d\theta}{d\cdot\frac{\pi}{2}}\\
&(\text{Note that } y \text{ ranges from } 0 \text{ to } d \text{ and } \theta \text{ ranges from } 0 \text{ to } \frac{\pi}{2}.)\\
=\ &\frac{a[-\cos\theta]_0^C + c[-\cos(\theta+B)]_C^{\pi/2}}{\frac{\pi d}{2}}\\
=\ &\frac{a(1-\cos C) + c(\sin B - \cos A)}{\frac{\pi d}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&P(\triangle ABC \text{ intersects the gridlines})\\
=\ &\frac{2a}{\pi d} + \frac{2c}{\pi d} - \frac{2a(1-\cos C) + 2c(\sin B - \cos A)}{\pi d}\\
=\ &\frac{2a\cos C + 2c(1-\sin B+\cos A)}{\pi d}.
\end{aligned}$$

Similarly, if B is an obtuse angle,

$$\begin{aligned}
&P(\text{both } a \text{ and } c \text{ intersects the gridlines})\\
=\ &\frac{\int_0^C a\sin\theta d\theta + \int_C^{\pi-B} c\sin(\theta+B)d\theta}{d\cdot\frac{\pi}{2}}\\
&(\text{Note that when } \theta > \pi - B, c\sin(\theta+B) \text{ becomes negative.})\\
=\ &\frac{a[-\cos\theta]_0^C + c[-\cos(\theta+B)]_C^{\pi-B}}{\frac{\pi d}{2}}\\
=\ &\frac{a(1-\cos C) + c(1 - \cos A)}{\frac{\pi d}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&P(\triangle ABC \text{ intersects the gridlines})\\
=\ &\frac{2a}{\pi d} + \frac{2c}{\pi d} - \frac{2a(1-\cos C) + 2c(1 - \cos A)}{\pi d}\\
=\ &\frac{2a\cos C + 2c\cos A}{\pi d}.
\end{aligned}$$

Combining the results, we have the following theorem:

Theorem 1. *If a triangle ABC of sides a, b and c ($\leqslant a$) is randomly dropped onto a plane with parallel lines which are separated by a distance d*

(> 2a) from each other, then the probability P that the triangle intersects the lines is given by

$$P = \begin{cases} \dfrac{2a\cos C + 2c(1 - \sin B + \cos A)}{\pi d}, & \textit{if } B \textit{ is acute,} \\ \dfrac{2a\cos C + 2c\cos A}{\pi d}, & \textit{if } B \textit{ is obtuse.} \end{cases} \quad (2)$$

Example 1: When $B = C = 0°$ and $A = 180°$, the triangle ABC becomes a "needle" of length a and by (2), $P = \dfrac{2a(1) + 2c(1 - 0 - 1)}{\pi d} = \dfrac{2a}{\pi d}$, which agrees with (1).

Example 2: When $A = C = 0°$ and $B = 180°$, the triangle ABC becomes a "needle" of length $a + c$ and by (2), $P = \dfrac{2a(1) + 2c(1)}{\pi d} = \dfrac{2(a + c)}{\pi d}$, which also agrees with (1).

3. Dropping Rhombuses

Assume that a rhombus $ABCD$ of diagonals $AC = a$ and $BD = b$ $(\leqslant a)$ is randomly dropped onto a plane with parallel lines which are separated by a distance d $(> a)$ from each other (see Figure 2). Let x be $\angle ACB$, θ be the acute angle between AC and the gridline, and y be the distance between C and the next gridline in the direction of $\overrightarrow{CA}$.

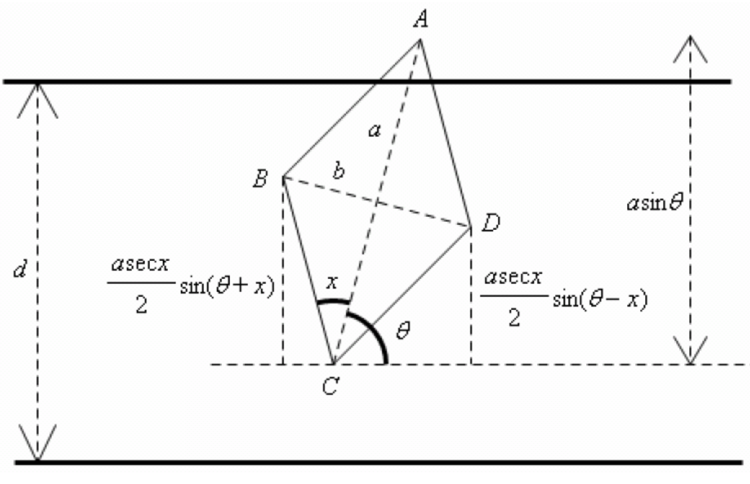

FIGURE 2

Since $\sec x = \dfrac{BC}{a/2}$, each side of the rhombus has length $\dfrac{a\sec x}{2}$. Hence, the "vertical" distances between C and the two ends of the diagonal BD are $\dfrac{a\sec x}{2}\sin(\theta+x)$ and $\dfrac{a\sec x}{2}\sin(\theta-x)$, as indicated in Figure 2.

Now, since the rhombus will intersect the gridlines if and only if at least one of its diagonals intersects the gridlines, we have

$$\begin{aligned}
&P(\text{rhombus}ABCD \text{ intersects the gridlines})\\
=\ &P(AC \text{ or } BD \text{ intersects the gridlines})\\
=\ &P(AC \text{ intersects the gridlines}) + P(BD \text{ intersects the gridlines})\\
&- P(\text{both } AC \text{ and } BD \text{ intersects the gridlines})\\
=\ &\frac{2a}{\pi d} + \frac{2b}{\pi d} - P(\text{both } AC \text{ and } BD \text{ intersects the gridlines}). \quad \text{(by (1))}
\end{aligned}$$

In order for both AC and BD to intersect the gridlines, $a\sin\theta > y$ and $\dfrac{a\sec x}{2}\sin(\theta+x) > y > \dfrac{a\sec x}{2}\sin(\theta-x)$. (Note that it is impossible for AC and BD to intersect two different gridlines as $b \leqslant a < d$.) Hence, either

(I) $\dfrac{a\sec x}{2}\sin(\theta+x) > a\sin\theta > y > \dfrac{a\sec x}{2}\sin(\theta-x)$ or

(II) $a\sin\theta > \dfrac{a\sec x}{2}\sin(\theta+x) > y > \dfrac{a\sec x}{2}\sin(\theta-x)$.

Therefore,

$$\begin{aligned}
&P(\text{both } AC \text{ and } BD \text{ intersects the gridlines})\\
=\ &\frac{\int_0^x a\sin\theta d\theta + \int_x^{\pi/2}\{\frac{a\sec x}{2}\sin(\theta+x) - \frac{a\sec x}{2}\sin(\theta-x)\}d\theta}{d\cdot\frac{\pi}{2}}\\
&(\text{Note that } y \text{ ranges from 0 to } d \text{ and } \theta \text{ ranges from 0 to } \frac{\pi}{2}.)\\
=\ &\frac{a[-\cos\theta]_0^x + \int_x^{\pi/2}\frac{a\sec x}{2}\cdot 2\cos\theta\sin x d\theta}{\frac{\pi d}{2}}\\
=\ &\frac{a(1-\cos x) + \int_x^{\pi/2} a\cos\theta\tan x d\theta}{\frac{\pi d}{2}}.\\
=\ &\frac{a(1-\cos x) + a\tan x[\sin\theta]_x^{\pi/2}}{\frac{\pi d}{2}}\\
=\ &\frac{2a(1-\cos x) + 2a\tan x(1-\sin x)}{\pi d}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& P(\text{rhombus}ABCD \text{ intersects the gridlines}) \\
= & \frac{2a}{\pi d} + \frac{2b}{\pi d} - \frac{2a(1-\cos x) + 2a\tan x(1-\sin x)}{\pi d} \\
= & \frac{2a + 2(a\tan x) - 2a + 2a\cos x - 2a\tan x + 2a\tan x\sin x}{\pi d} \\
= & \frac{2a\cos x + 2a\frac{\sin x}{\cos x}\sin x}{\pi d} \\
= & \frac{2a\sec x}{\pi d}.
\end{aligned}$$

Finally, since each side of the rhombus has length $\frac{a\sec x}{2}$, we have the following theorem:

Theorem 2. *If a rhombus $ABCD$ is randomly dropped onto a plane with parallel lines separated by a distance d, which is longer than the diagonals of $ABCD$, then the probability P that the rhombus intersects the lines is given by*

$$P = \frac{\textit{Perimeter of the rhombus}}{\pi d}. \tag{3}$$

Example 3: When $AB = BC = CD = DA = L$ and $\angle BCD = 0°$, the rhombus $ABCD$ becomes a "needle" of length $2L$ and by (3), $P = \frac{4L}{\pi d} = \frac{2(2L)}{\pi d}$, which also agrees with (1).

4. Acknowledgment

Our teacher, Mr. Chu, has provided great help to us, for example, he asked us challenging questions to stimulate our thinking. We would like to thank him very much.

We also want to thank our school, Munsang College (Hong Kong Island), for giving us this chance to participate in this research project, from which we have learned a lot!

REFERENCES

[1] Beese, G., *Buffon's Needle - An Analysis and Simulation*, retrieved from University of Illinois at Urbana-Champaign Web site:
http://www.mste.uiuc.edu/reese/buffon/buffon.html