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GEOMETRIC CONSTRUCTION – AREA TRISECTION OF A CIRCLE

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Abstract. When dividing a cake of circle shape into equal parts, it is quite easy to divide it from the centre. However, if we need to divide it from its edge, how can we accomplish this task accurately?

This report aims to find a method to divide the area of a circle into 3 equal parts with two straight lines by Euclidean construction, i.e. the construction with compass and straightedge only. However, we were aware that it is difficult, if not impossible, to find the exact method of construction. Therefore, we try to find some methods to divide the area of circle approximately into three equal parts. In this report, we have three analytic approaches: by Lagrange Interpolating Polynomial, by infinite series of sine function and by method of bisection. Then, we will discuss three methods of construction, which are, inscribing a regular polygon with a large number of sides, inscribing a regular polygon with a small number of sides and bisecting the slope. At last, we will give a comparison of these three methods.

1. Introduction

In this report, we focus on the area trisection of a given circle under the rules of Euclidean construction.

We will talk about the background of Euclidean construction, the three famous problems of antiquity, and also the way to determine whether a line segment is constructible before we really start discussing the way to trisect the area. In this report, we will simplify the case, and consider area trisection by two straight lines. Indeed, we assume that the lines intersect at

[1]This work is done under the supervision of the authors' teacher, Mr. Wai-Hung Ho

a point B on the circumference. By doing so, we can reflect one line about the diameter passing through B to obtain another line.

We will start the discussion with the way of area trisection of a circle in an analytic method. We will approximate the angle between the two straight lines, which trisect area of the circle. Then we will give three different methods of approximation of the lines and also describe how to construct such lines.

The first method of construction is inscribing a regular polygon with a large number of sides into the circle. Our idea comes from the ancient method to find pi in China. As the number of sides increases, a regular polygon becomes more similar to a circle. We will not discuss the practical method of that part because the method is really complicated and we are not going to approximate the lines in that way.

The second method is inscribing a regular polygon with a small number of sides into the circle. We do not regard the polygon as a figure similar to a circle, but we regard this as an aid to locate the lines, which trisect the area of the circle. We will discuss how to locate such lines practically.

In the third method, we bisect the slope of a line at angle with a diameter repeatedly. Then we can obtain a line which is approximately the same with the line required. We will also discuss the actual method to construct the line.

Besides, we will compare these methods to find a suitable one for approximation.

2. Euclidean Constructions

In the book *Elements*, Euclid introduced five basic postulates in Euclidean geometry. They are:

1. A straight line can be drawn from any point to any point.
2. A finite straight line can be produced continuously in a straight line.
3. A circle may be described with any centre and distance.
4. All right angles are equal.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

"First two postulates can be done by an unmarked straightedge. The third one can be done with a Euclidean compass, which is different from a modern one. Euclidean compass collapses if either leg is lifted from the paper.

However, it was proved that a Euclidean compass could carry out all the construction performable with a modern compass." [2]

Then a game called compass-and-straightedge construction, also known as Euclidean construction, was developed.

In ancient Greece, the mathematicians were interested by 3 problems in Euclidean construction. They are: duplicating a cube, trisecting a general angle and squaring a circle.

In the following parts we assume the construction starts with a line segment with unit length.

These 3 problems of antiquity remained unsolved for over 2000 years.

"In 1801, Gauss asserted that a regular polygon of n sides is constructible if and only if n has the form $2^k p_1 p_2 \cdots p_i$, where the p's are distinct Fermat primes and $k \geqslant 0$.

This result eliminated the problem of trisecting a general angle because the ability to trisect a 60° angle enables one to construct a regular 9-gon, where $9 = 3^2$ which cannot be expressed in the form $2^k p_1 p_2 \cdots p_i$. Thus, trisecting a general angle is impossible.

In 1837, Wantzel proved that it was not possible to double the cube. The problem of the squaring of a circle resisted all attempts until 1882, when Ferdinand Lindemann proved that π is transcendental." [3]

Due to the rise of field theory in the last two centuries, the possibility of constructing a line segment with length a by Euclidean construction can be determined now.

"A real number a is constructible if a line segment with length a is constructible. Also, the set of constructible real numbers is a field and a real number a is constructible if and only if there is a sequence $L_0 \subseteq L_1 \subseteq \cdots \subseteq L_k$ of subfields of $\mathbb{R}$ such that $L_0 = \mathbb{Q}$, $[L_i : L_{i-1}] = 2$ for $1 \leqslant i \leqslant k$, and $a \in L_k$." [1]

"Also note that $L_{i+1} = L_i(\sqrt{\alpha_i})$, where $\alpha_i \in L_i$" [3]

Then we can observe that a is a constructible number if it satisfy any condition below.

1. a is rational.
2. $a = a_1 * a_2$, where $*$ is one of the four basic arithmetic operators and a_1, a_2 are known constructible numbers.
3. a is a square root of a known constructible number.

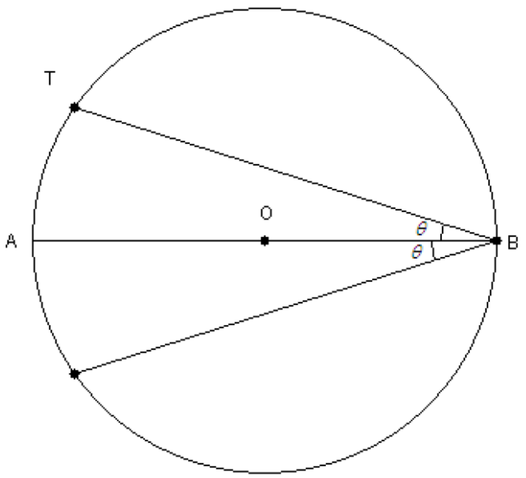
Conversely, a must satisfy at least one condition if it is constructible.

"Then two theorems directly follow.

1. From a given unit length, it is impossible to construct, with Euclidean tools, a line segment the magnitude of whose length is a root of a cubic equation with rational coefficients but with no rational root.
2. The magnitude of any length constructible with Euclidean tools from a given unit length is an algebraic number." [2]

These reject the possibility of solving the 3 problems of antiquity.

3. Area Trisection of A Circle



If BT is one of the line segment which trisect the area of the circle with centre O and $\angle OBT$, then the area bounded by BT, AB and arc AT

$$= \frac{2\theta}{2}(OA)^2 + \frac{1}{2}(OA)^2 \sin(\pi - 2\theta) = \frac{1}{2}(OA)^2(2\theta + \sin 2\theta)$$

Since BT and its reflection about AB divide the area of the circle into 3 equal parts, the area bounded by BT, AB and arc AT equals one-sixth of the area of the circle $\frac{\pi}{6}(OA)^2$

Therefore,

$$\sin 2\theta + 2\theta = \frac{\pi}{3}$$

Trisecting the area of a circle with two lines is possible if and only if a line segment with length $\sin 2\theta$ is constructible. To achieve this, we need to express $\sin 2\theta$ as an expression with only square roots and four basic arithmetic operators with rational coefficients. This is a difficult job since we have insufficient information on θ.

3.1. Analytic method to approximate the angle θ

3.1.1. Method 1: By Lagrange Interpolating Polynomial

Let $x = 2\theta$ and $f(x) = \sin x + x - \frac{\pi}{3}$.

Consider $f(0) = -\frac{\pi}{3}$, $f(\frac{\pi}{6}) = \frac{1}{2} - \frac{\pi}{6}$, $f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} - \frac{\pi}{12}$, $f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $f(\frac{\pi}{2}) = 1 + \frac{\pi}{6}$.

We have a quartic polynomial

$$\begin{aligned} P(x) = &\left(-\frac{\pi}{3}\right) \frac{(x-\frac{\pi}{6})(x-\frac{\pi}{4})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{(-\frac{\pi}{6})(-\frac{\pi}{4})(-\frac{\pi}{3})(-\frac{\pi}{2})} \\ &+ \left(\frac{1}{2} - \frac{\pi}{6}\right) \frac{x(x-\frac{\pi}{4})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{(\frac{\pi}{6})(-\frac{\pi}{12})(-\frac{\pi}{6})(-\frac{\pi}{3})} \\ &+ \left(\frac{\sqrt{2}}{2} - \frac{\pi}{12}\right) \frac{x(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{(\frac{\pi}{4})(\frac{\pi}{12})(-\frac{\pi}{12})(-\frac{\pi}{4})} \\ &+ \left(\frac{\sqrt{3}}{2}\right) \frac{x(x-\frac{\pi}{6})(x-\frac{\pi}{4})(x-\frac{\pi}{2})}{(\frac{\pi}{3})(\frac{\pi}{6})(\frac{\pi}{12})(-\frac{\pi}{6})} \\ &+ \left(1 - \frac{\pi}{6}\right) \frac{x(x-\frac{\pi}{6})(x-\frac{\pi}{4})(x-\frac{\pi}{3})}{(\frac{\pi}{2})(\frac{\pi}{3})(\frac{\pi}{4})(\frac{\pi}{6})} \end{aligned}$$

$$= 0.02879711246x^4 - 0.20434069602x^3 + 0.02137300753x^2 + 1.99562618428x - 1.04719755119$$

Solving the equation $P(x) = 0$, we get the result

$$x = 4.70 \pm 1.35i \text{ or } x = 0.53626407876 \text{ or } x = -2.83795433628$$

Hence, $x = 0.53626407876$, $\theta = \frac{x}{2} = 0.26813203938$.

3.1.2. Method 2: By infinite series of sine function

Let

$$\sin 2x + 2x = a_0 + a_1 x + a_2 x^2 + \cdots \tag{1}$$

Substitute $x = 0$ into (1), $a_0 = 0$. Differentiate (1) with respect to x 1 time,

$$2\cos 2x + 2 = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Substitute $x = 0$, $a_1 = 4$. Generally, differentiate (1) with respect to x n times, $n \geqslant 2$

$$2^n \sin\left(2x + \frac{n\pi}{2}\right) = n!a_n + (n+1)!a_{n+1}x + \cdots$$

and then substitute $x = 0$, $a_n = \dfrac{2^n \sin \frac{n\pi}{2}}{n!}$.

Therefore, $\sin 2x + 2x = 4x - \dfrac{4}{3}x^3 + \dfrac{4}{15}x^5 - \dfrac{8}{315}x^7 + \cdots$

Take the first two term as estimation,

$$4x - \frac{4}{3}x^3 = \frac{\pi}{3}$$
$$4x^3 - 12x + \pi = 0$$

Solving the above equation

$$x = 0.268232369341,\ 1.5822866081 \text{ (rej.)},\ -1.85051897744 \text{ (rej.)}$$

3.1.3. Method 2: By method of bisection

Let $f(x) = \sin 2x + 2x$
$x = 0, f(x) = 0$
$x = 1, f(x) = 2.90929$
Therefore, if $f(x) = \frac{\pi}{3}(1.047197551)$,

$$
\begin{aligned}
&0 < x < 1, x = 0.5, f(x) = 1.84747\\
&0 < x < 0.5, x = 0.25, f(x) = 0.97943\\
&0.25 < x < 0.5, x = 0.375, f(x) = 1.43164\\
&0.25 < x < 0.375, x = 0.3125, f(x) = 1.21010\\
&0.25 < x < 0.3125, x = 0.28125, f(x) = 1.09589\\
&0.25 < x < 0.28125, x = 0.265625, f(x) = 1.093786\\
&0.265625 < x < 0.28125, x = 0.2734375, f(x) = 1.06690\\
&0.265625 < x < 0.2734375, x = \frac{69}{256}, f(x) = 1.05239\\
&\frac{17}{64} < x < \frac{69}{256}, x = \frac{137}{512}, f(x) = 1.04513\\
&\frac{137}{512} < x < \frac{69}{256}, x = \frac{275}{1024}, f(x) = 1.04876\\
&\frac{137}{512} < x < \frac{275}{1024}, x = \frac{549}{2048}, f(x) = 1.04695\\
&\frac{549}{2048} < x < \frac{275}{1024}, x = \frac{1099}{4096}, f(x) = 1.04786\\
&\frac{549}{2048} < x < \frac{1099}{4096}, x = \frac{2197}{8192}, f(x) = 1.04740\\
&\frac{549}{2048} < x < \frac{2197}{8192}, x = \frac{4393}{16384}, f(x) = 1.047175
\end{aligned}
$$

As the process goes on, it can be found that

$$\frac{294815889495}{2^{40}} < x < \frac{589631778991}{2^{41}}$$

Take $\theta = \frac{1179263557981}{2^{42}} \approx 0.268133489494403$

Obviously, the value obtained using method of bisection is the most accurate as the accuracy can be improved if we repeat checking the value of $f(x)$. Therefore method of bisection should be the best one among the three methods mentioned.

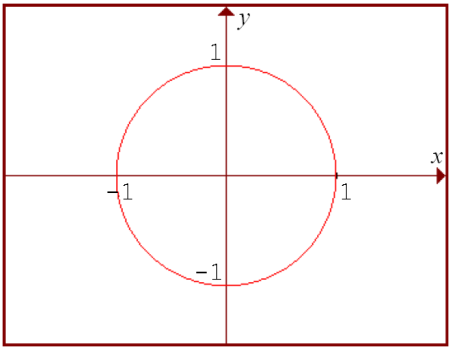
3.2. Construct the line to trisect the area of the circle

The error given is based on the area bound by both of the lines and the arc of the circle. If we consider the other two parts, the error should be halved.

3.2.1. Method 1: Inscribing a regular polygon with a large number of sides

As a circle is similar to a regular polygon with infinite number of sides, we would try to inscribe a regular polygon with a large number of sides into the circle. Then we divide the polygon into 3 parts with the same area from a vertex. Finally, we extend these 2 lines and find out the error for trisecting the area of the circumcircle.

Without loss of generality, assume the regular n-side polygon with centre O and is circumscribed by a unit circle, i.e. the distance between O and a vertex is 1. Let O be the origin, $V_0, V_1, V_2, \ldots, V_{n-1}$ be the vertexes of the regular n-side polygon and $V_0 = (0,1)$. Then $V_m = \left(\cos\frac{2m\pi}{n}, \sin\frac{2m\pi}{n}\right)$.



Denote $[A]$ be the area of polygon A.

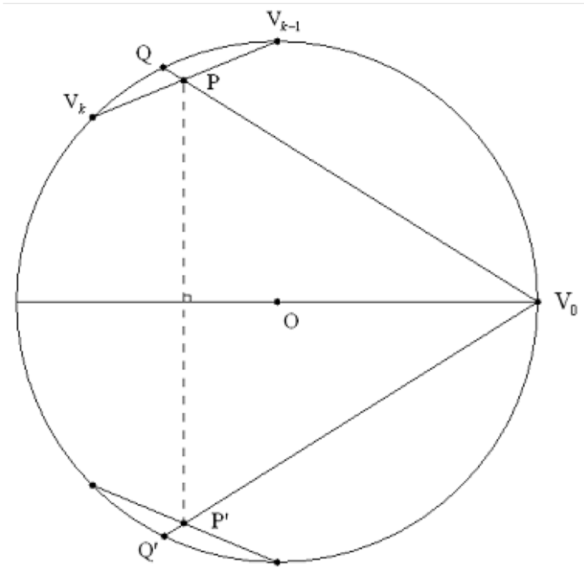
$$[V_0V_1V_2\cdots V_{n-1}] = \frac{n}{2}\sin\frac{2\pi}{n}$$

Let $[V_0V_1V_2\cdots V_{k-1}] \leqslant \frac{1}{3}[V_0V_1V_2\cdots V_{n-1}] \leqslant [V_0V_1V_2\cdots V_k]$

$$[V_0V_1V_2\cdots V_m] = \frac{m}{2}\sin\frac{2\pi}{n} - \frac{1}{2}\sin\frac{2m\pi}{n}$$

$$[V_0V_{k-1}V_k] = [V_0V_1V_2\cdots V_k] - [V_0V_1V_2\cdots V_{k-1}]$$

Let $P(x,y)$ be a point on $V_{k-1}V_k$ such that $[V_0V_1V_2\cdots V_{k-1}P] = \frac{1}{3}[V_0V_1V_2 \cdots V_{n-1}]$. Let $(x,-y)$ be P'. Extend V_0P and V_0P' and suppose they meet the circumference again at Q and Q' respectively.



Let $r = \frac{V_kP}{V_{k-1}V_k}$. Then

$$r = \frac{1}{[V_0V_1V_2\cdots V_k]}\{[V_0V_1V_2\cdots V_k] - \frac{1}{3}[V_0V_1V_2\cdots V_{n-1}]\}$$

$$\begin{aligned}
V_{k-1}V_k &= 2\cdot\left(1\cdot\sin\frac{\pi}{n}\right)\\
V_kP = r\cdot V_{k-1}V_k &= r\left\{2\left(1\cdot\sin\frac{\pi}{n}\right)\right\} = 2r\cdot\sin\frac{\pi}{n}\\
x &= \cos\frac{2k\pi}{n} + r\left\{\cos\frac{2(k-1)\pi}{n} - \cos\frac{2k\pi}{n}\right\},\\
y &= \sin\frac{2k\pi}{n} + r\left\{\sin\frac{2(k-1)\pi}{n} - \sin\frac{2k\pi}{n}\right\}
\end{aligned}$$

Absolute value of the slope of $V_0Q = \left|\dfrac{x-1}{y-0}\right| = \tan\angle QV_0O$.

$$\begin{aligned}
\angle QV_0O &= \tan^{-1}\left|\frac{x-1}{y}\right|\\
\angle QOV_{n/2} &= 2\times\angle QV_0O \quad \text{for even } n.
\end{aligned}$$

Area of the sector $OV_{n/2}Q = \frac{1}{2}\angle QOV_{n/2}$.

$$[OV_0Q] = \frac{1}{2}\sin(\pi - \angle QOV_{n/2})$$

Let $R =$ Area bounded by V_0Q, V_0Q' and arc QQ'. Then

$$R = 2 \times \{\,\text{Area of the sector } OV_{n/2}Q + [OV_0Q]\,\}$$

Computation for different n

$n = 4$
$[V_0V_1V_2V_3] = 2$

$k = 2$
$[V_0V_1] = 0$
$[V_0V_1V_2] = 1$
$r = 1/3$
$\angle QV_0O = 0.19739556$
$\angle QOV_2 = 0.39479112$
$[OV_0Q] = 0.192307692$
$R = 0.779406504$
$R/[V_0V_1V_2V_3] = 24.8092796\%$
Theoretically, percentage error $= -25.6\%$
Practical percentage error $= -25.6\%$ if we take $r = 1/3$

$n = 8$
$[V_0V_1V_2 \dots V_8] = 2\sqrt{2}$

$k = 4$
$[V_0V_1V_2V_3] = 1/\sqrt{2}$
$[V_0V_1V_2V_3V_4] = \sqrt{2}$
$r = 2/3$
$\angle QV_0O = 0.255495374$
$\angle QOV_4 = 0.510990747$
$[OV_0Q] = 0.244520838$
$R = 1.000032424$
$R/[V_0V_1V_2 \dots V_8] = 31.8320207\%$
Theoretically, percentage error $= -4.5\%$

Practical percentage error = −4.5% if we take $r = 2/3$

$n = 16$
$[V_0V_1V_2 \dots V_{16}] = 3.061467459$

$k = 7$
$[V_0V_1V_2 \dots V_6] = 0.794496907$
$[V_0V_1V_2 \dots V_7] = 1.148050297$
$r = 0.360797400$
$\angle QV_0O = 0.264420733$
$\angle QOV_8 = 0.528841465$
$[OV_0Q] = 0.252266705$
$R = 1.033374875$
$R/[V_0V_1V_2 \dots V_{16}] = 32.8933439\%$
Theoretically, percentage error = −1.32% Practical percentage error = −1.37% if we take $r = 9/25$

$n = 32$
$[V_0V_1V_2 \dots V_{32}] = 3.121445152$

$k = 14$
$[V_0V_1V_2 \dots V_{13}] = 0.990301977$
$[V_0V_1V_2 \dots V_{14}] = 1.174290538$
$r = 0.727267062$
$\angle QV_0O = 0.267280944$
$\angle QOV_{16} = 0.534561888$
$[OV_0Q] = 0.254732048$
$R = 1.044025984$
$R/[V_0V_1V_2 \dots V_{32}] = 33.2323792\%$
Theoretically, percentage error = −0.3% Practical percentage error = −0.3% if we take $r = 72/99$

$n = 64$
$[V_0V_1V_2 \dots V_{32}] = 3.136548491$

$k = 27$
$[V_0V_1V_2 \dots V_{26}] = 0.996437708$

$[V_0V_1V_2\dots V_{27}] = 1.087533026$
$r = 0.461240636$
$\angle QV_0O = 0.267911962$
$\angle QOV_{32} = 0.535823923$
$[OV_0Q] = 0.255274831$
$R = 1.046373585$
$R/[V_0V_1V_2\dots V_{32}] = 33.3071057\%$
Theoretically, percentage error $= -0.08\%$
Practical percentage error $= -0.073\%$ if we take $r = 6/13$

3.2.2. Method 2: Inscribing a regular polygon with a small number of sides

Instead of the method mentioned above, we try to locate the line of trisection by inscribing regular polygon with a small number of sides in order to reduce complexity. We will make use of the result in Section 3.1 and Appendices A & B.

Without loss of generality, assume the regular n-side polygon with centre O and is circumscribed by a unit circle, i.e. the distance between O and a vertex is 1. Let O be the origin, $V_0, V_1, V_2, \dots, V_{n-1}$ be the vertexes of the regular n-side polygon and $V_0 = (0, 1)$. Then $V_m = (\cos\frac{2m\pi}{n}, \sin\frac{2m\pi}{n})$.

3.2.2.1 Regular hexagon

Consider a regular hexagon. $\angle V_2V_0V_3 = \pi/6 > \theta$

Let R be a point on V_2V_3 such that $\angle V_3V_0R = \theta$
$\angle V_0V_2V_3 = \pi/3$
$\angle V_0RV_3 = 2\pi/3 - \theta$
$\sin\theta/V_3R = \sin(2\pi/3 - \theta)/V_0V_3$ (Sine Law)
$V_3R = (2 \times \sin\theta)/\sin(2\pi/3 - \theta)$
Take $\theta = 0.2681334894944403$
$V_3R = 0.547637285695908$
$V_3R/V_2V_3 = 0.547637285695908/1 = 0.547637285695908 \approx 11/20$
Practical percentage error $= 0.45\%$

To divide V_2V_3 into a ratio of $9:11$, we bisect the line twice. Then we divide the one of the segments into 5 equal parts, using the result of Appendix A.

3.2.2.2 Regular octagon

Consider a regular octagon. $\angle V_3V_0V_4 = \pi/8 > \theta$
Let R be a point on V_3V_4 such that $\angle V_4V_0R = \theta$
$\angle V_0V_3V_4 = 3\pi/8$
$\angle V_0RV_4 = 5\pi/8 - \theta$
$\sin\theta/V_4R = \sin(5\pi/8-\theta)/V_0V_4$ (Sine Law)
$V_4R = (2\times\sin\theta)/\sin(5\pi/8-\theta)$

Take $\theta = 0.268133489494403$
$V_4R = 0.534001757246977$
$V_4R/V_3V_4 = 0.534001757247056/[2\sin(\pi/8)] = 0.6977069191978 \approx 67/96$
Practical percentage error $= 0.03\%$

To divide V_3V_4 into a ratio of $29:67$, we bisect the line for five times. Then we trisect the one of the segments, using the result of Appendix A.

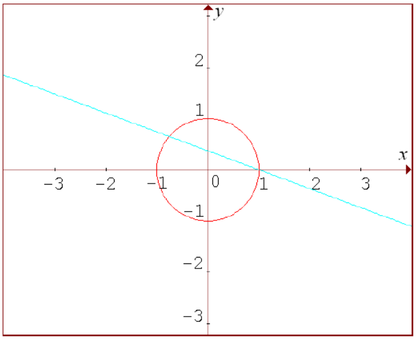
3.2.2.3 Simplified method

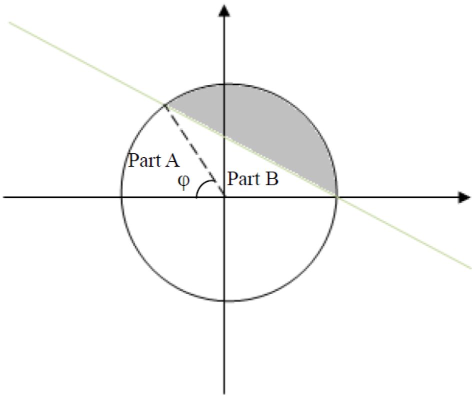
Consider a regular hexagon again. We bisect the inclined angle of the diameter and one of the diagonal once to obtain the line.
Practical percentage error $= -2.25\%$

3.2.3. Method 3: Bisecting the slope

Suppose there is a line with slope $m = \frac{y-0}{x-1}$ cut a unit circle with centre O.

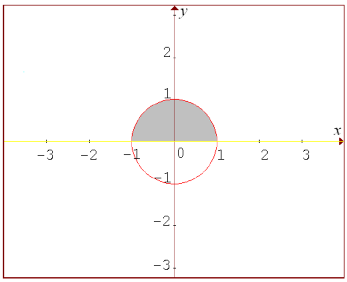
$\tan\frac{\varphi}{2} = |\,m\,|$.
Area of part A $= \frac{\varphi}{2} = \tan^{-1}|\,m\,|$.
Area of part B $= \frac{1}{2}\sin(\pi-\varphi) = \frac{1}{2}\sin(\pi - 2\tan^{-1}|\,m\,|)$.



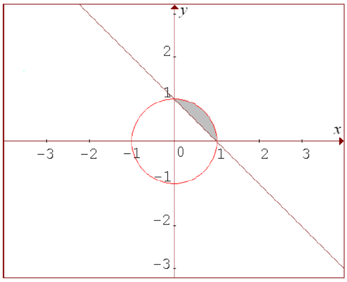


Shaded area = area of semicircle − Area of part A − Area of part B.

When the slope is 0, i.e. $m_1 = 0$, the shaded area is $\pi/2$, i.e. $\text{Area}_1 = \pi/2$.

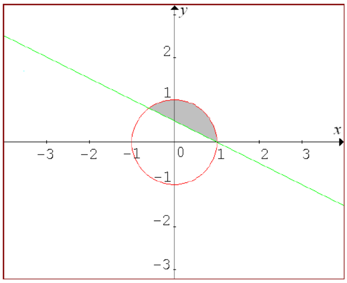


When the slope is −1, i.e. $m_2 = -1$, the shaded area is $\pi/4 - 1/2$, i.e. $\text{Area}_2 = \pi/4 - 1/2$. One-third of the area of the circle: $\pi(1)_2/3 = \pi/3 = 1.047197551$



So, Area$_2 < \pi/3 <$ Area$_1$. The slope of lines that divide the circle into three equal parts should be between -1 and 0, i.e. $-1 < m < 0$.

Take $m_3 = -1/2$, Shaded area



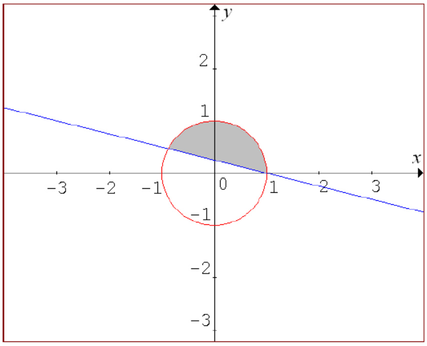
$$\begin{aligned}\text{Area}_3 &= \text{area of semicircle} - \text{Area of part A} - \text{Area of part B}\\ &= \pi/2 - \tan^{-1}|m_3| - 1/2 \times \sin(\pi - 2 \times \tan^{-1}|m_3|)\\ &= 0.707148717\end{aligned}$$

Since Area$_3 < \pi/3 <$ Area$_1$, the slope of lines that divide the circle into three equal parts should be between $-1/2$ and 0, i.e. $-1/2 < m < 0$.

Take $m_4 = -1/4$ Shaded area

$$\begin{aligned}\text{Area}_4 &= \text{area of semicircle} - \text{Area of part A} - \text{Area of part B}\\ &= \pi/2 - \tan^{-1}|m_4| - 1/2 \times \sin(\pi - 2 \times \tan^{-1}|m_4|)\\ &= 1.090523546\end{aligned}$$

Since Area$_3 < \pi/3 <$ Area$_4$, the slope of lines that divide the circle into three equal parts should be between $-1/2$ and $-1/4$, i.e. $-1 < m < -1/4$.



Take $m_5 = -3/8$

$$\begin{aligned}\text{Area}_5 &= \text{area of semicircle} - \text{Area of part A} - \text{Area of part B}\\ &= \pi/2 - \tan^{-1}|m_5| - 1/2 \times \sin(\pi - 2\times\tan^{-1}|m_5|)\\ &= 0.883258533\end{aligned}$$

Since $\text{Area}_5 < \pi/3 < \text{Area}_4$, the slope of lines that divide the circle into three equal parts should be between $-3/8$ and $-1/4$, i.e. $-3/8 < m < -1/4$.

Take $m_6 = -5/16$

$$\begin{aligned}\text{Area}_6 &= \text{area of semicircle} - \text{Area of part A} - \text{Area of part B}\\ &= \pi/2 - \tan^{-1}|m_6| - 1/2 \times \sin(\pi - 2\times\tan^{-1}|m_6|)\\ &= 0.983213949\end{aligned}$$

Since $\text{Area}_6 < \pi/3 < \text{Area}_4$, the slope of lines that divide the circle into three equal parts should be between $-5/16$ and $-1/4$, i.e. $-5/16 < m < -1/4$.

Take $m_7 = -9/32$

$$\begin{aligned}\text{Area}_7 &= \text{area of semicircle} - \text{Area of part A} - \text{Area of part B}\\ &= \pi/2 - \tan^{-1}|m_7| - 1/2 \times \sin(\pi - 2\times\tan^{-1}|m_7|)\\ &= 1.035995392\end{aligned}$$

Since $\text{Area}_7 < \pi/3 < \text{Area}_4$, the slope of lines that divide the circle into three equal parts should be between $-9/32$ and $-1/4$, i.e. $-9/32 < m < -1/4$.

Take $m_8 = -17/64$

$$\begin{aligned}\text{Area}_8 &= \text{area of semicircle} - \text{Area of part A} - \text{Area of part B}\\ &= \pi/2 - \tan^{-1}|m_8| - 1/2 \times \sin(\pi - 2 \times \tan^{-1}|m_8|)\\ &= 1.063048111\end{aligned}$$

Since $\text{Area}_7 < \pi/3 < \text{Area}_8$, the slope of lines that divide the circle into three equal parts should be between $-9/32$ and $-17/64$, i.e. $-9/32 < m < -17/64$

Take $m_9 = -35/128$

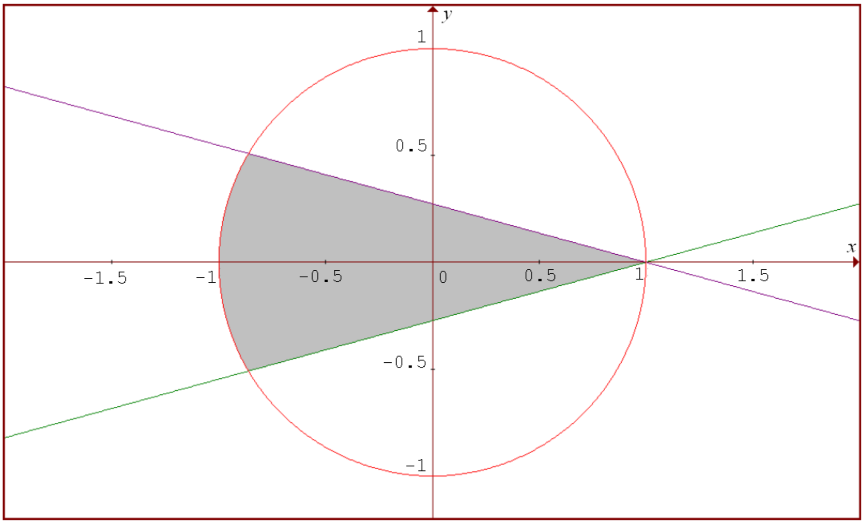
$$\begin{aligned}\text{Area}_9 &= \text{area of semicircle} - \text{Area of part A} - \text{Area of part B}\\ &= \pi/2 - \tan^{-1}|m9| - 1/2 \times \sin(\pi - 2 \times \tan^{-1}|m9|)\\ &= 1.049467983\end{aligned}$$

Since $\text{Area}_7 < \pi/3 < \text{Area}_9$, the slope of lines that divide the circle into three equal parts should be between $-9/32$ and $-35/128$, i.e. $-9/32 < m < -35/128$

and so on ...

The result will be more and more accurate if we continue to do so. The lines that divide the circle into three equal parts are approximately drawn with a slope of $\pm 35/128$.

Consider $m = \pm 35/128$.



Shaded area $= 2 \times (\pi/2 - \text{Area}_9) = 1.042656687$
Practical percentage error $= (1.042656687 - \pi/3)/(\pi/3) \times 100\% = -0.43\%$.

3.3. Comparison

A number of methods have been introduced in Section 3.3. In this part, we are going to compare these methods and investigate their own advantages and disadvantages.

3.3.1. Inscribing a regular polygon with a large number of sides

Firstly, this method can ensure an accurate result, as regular polygons with a large number of sides can be reasonably regarded as circle. For example, the result obtained from a 64-sided regular polygon has an error of 0.08% and the one with 32 sides has 0.3%.

However, to get a satisfactory result, we need to consider a regular polygon with a very large number of sides. The procedures are tedious. It is suggested to work with computers.

Besides, practically, it may not be possible to construct a regular polygon with a very large number of sides when we are asked to trisect a circle. It is because the circle given may have a radius of few centimeters.

3.3.2. Inscribing a regular polygon with a small number of sides

It is expected that the error of the result obtained from this method will be less accurate than inscribing a regular polygon with a very large number of sides since the error can be hardly reduced after we determine which regular polygon to be considered when complexity is also taken into consideration. However, this method is more convenient. The method may involve numerous steps for some cases (e.g. the method we mentioned in Section 3.2.2) due to the complicated method we introduce in Appendix A. Despite these, the error and the number of steps are acceptable when we consider a regular octagon.

Also, a simplified method is developed. Although the error is about 2%, it is enough for estimation.

3.3.3. Bisecting the slope

The accuracy can be controlled by repeating the procedures with a certain number of times.

However, although the degree of accuracy is guaranteed, the work is boring. To get an error of 0.43%, we have to repeat the procedures for 7 times, but this is acceptable compared with other methods mentioned.

4. Summary

We will end the report with the exact method to approximate the line of trisection. Since the method in Section 3.2.1 is too complicated, it is not included. For all of the methods, we should find the centre of the given circle (as stated in Appendix B.1) and label it as O first. Then draw a diameter of the circle. Label two ends of the diameter as A and B respectively.

Method 1

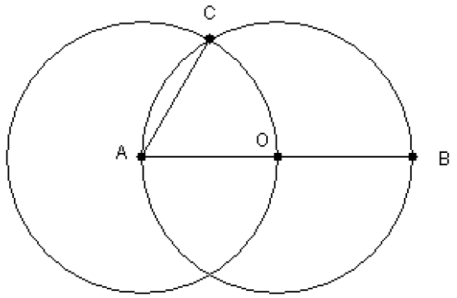
From the result of Section 3.2.2.1, we can construct one of the lines as the following.

1. Draw a circle with centre A and radius AO. Label one of the points of intersection of two circles as C.
2. Draw a line segment AC.
3. Bisect AC and label the mid-point as D.
4. Bisect CD and label the mid-point as E.
5. Divide DE into five equal segments such that $DF = DE/5$, where F lies on DE. Then $AF : FC = 11 : 9$.
6. Join B and F.
7. Extend BF until the line meets circumference of the circle again at X.

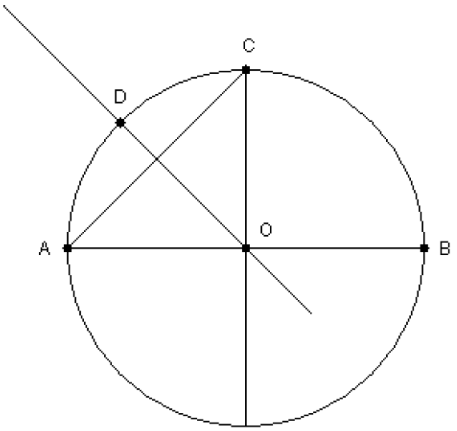
Method 2

From the result of Section 3.2.2.2, we can construct one of the lines as the following.

1. Draw a perpendicular bisector of AB. Label one of the intersecting points of the perpendicular bisector and the circle as C.



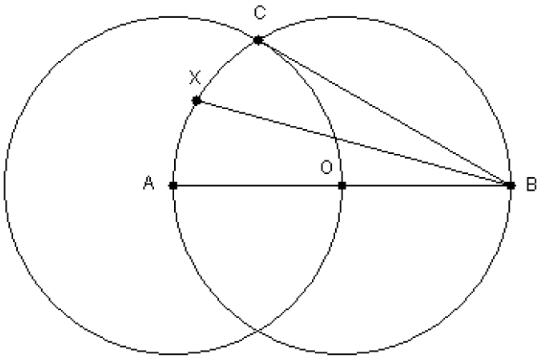
2. Draw a perpendicular bisector of AC. Label the intersecting points of the perpendicular bisector and the circle on the semi-circle ACB as D.
3. Bisect AD and label the mid-point as E.
4. Bisect DE and label the mid-point as F.
5. Bisect EF and label the mid-point as G.
6. Bisect GF and label the mid-point as H.
7. Bisect HF and label the mid-point as I.
8. Trisect HI such that $HJ = HI/3$, where J lies on DE. Then $AJ : JD = 67 : 29$.
9. Join B and J.
10. Extend BJ until the line meets circumference of the circle again at X.



Method 3

From the result of Section 3.2.2.3, we can construct one of the lines as the following.

1. Draw a circle with centre A and radius AO. Label one of the points of intersection of two circles as C.
2. Draw a line segment BC.
3. Draw a circle with centre B (the radius is not important). Label the points of intersection with AB and BC as D and E respectively.
4. Draw a circle C_1 with centre D and radius DE.
5. Draw a circle C_2 with centre E and radius DE. Label the points of intersection of C_1 and C_2 as F and G respectively.
6. Bisect GF and label the mid-point as H.
7. Join B and F.
8. Extend BF until the line meets circumference of the circle again at X. The line should also passes through G.



Method 4

From the result of Section 3.2.3, we can construct one of the lines as the following.

1. Draw a perpendicular bisector of AB. Label one of the intersecting points of the perpendicular bisector and the circle as C.
2. Bisect OC and label the mid-point as D.
3. Bisect OD and label the mid-point as E.
4. Bisect ED and label the mid-point as F.
5. Bisect EF and label the mid-point as G.
6. Bisect EG and label the mid-point as H.
7. Bisect EH and label the mid-point as I.
8. Bisect IH and label the mid-point as J. Then $OJ/OC = 35/128$.
9. Join B and J.
10. Extend BJ until the line meets circumference of the circle again at X.

Draw a circle with centre A and radius XA. Label another intersecting point of the circle and AB as Y. Join X and Y until the line meets the circle again. Label the point where XY meets the circle again as Z. Draw the line segment XZ. This is another line of trisection.

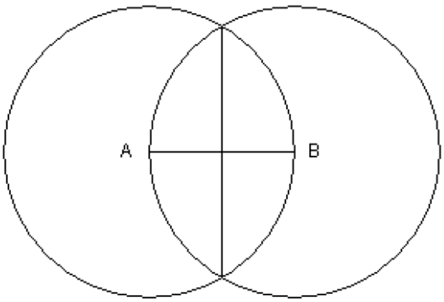
Appendix A. Dividing a line segment into n equal parts

A.1. Specific cases:

Bisecting a line segment

Suppose there is a line segment AB. To bisect it, the following method can be used.

1. Draw a circle with centre A and radius AB.
2. Draw a circle with centre B and radius AB.
3. Join the points of intersection of these two circles. The line is the perpendicular bisector of AB.



Trisecting a line

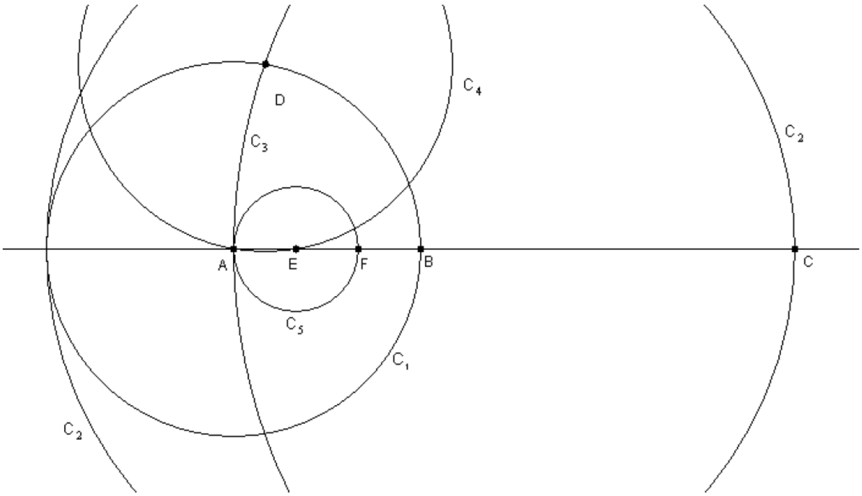
The method is given at

http://www41.homepage.villanova.edu/robert.styer/trisecting%20segment/

"Suppose there is a line segment AB. To trisect it, the following method can be used.

1. Extend AB into a long segment.

2. Draw a circle C_1 with centre A and radius AB
3. Draw a circle C_2 with centre B using the diameter of C_1 as the radius.
4. Set the compass on C, a point on the extended segment, where BC is the radius of C_2 and $AC = 3BC/2$. Draw a circle C_3 with radius AC.
5. Set the compass on one of the intersecting points of C_1 and C_3. Label it as D.
6. Draw a circle C_4 with radius AD. The point of intersection, E, of C_4 and AB divide AB into $1 : 2$.
7. Draw a circle C_5 with radius AE. Label the point of intersection of C_5 and AB as F. Then $AE : EF : FB = 1 : 1 : 1$."



A.1.1. General cases

A method is given at

http://www.mathopenref.com/constdividesegment.html

"Suppose there is a line segment AB. To divide a line segment into n equal parts, the following method can be used.

1. From point A, draw a line segment at an angle to the given line, and about the same length. The exact length is not important.
2. Set the compass on A, and set its width to a bit less than one fifth of the length of the new line.
3. Step the compass along the line, marking off 5 arcs. Label the last one C.

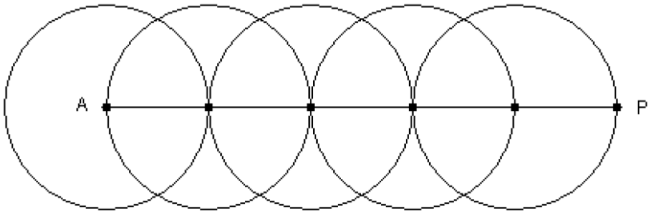
4. With the compass width set to CB, draw an arc from A just below it.
5. With the compass width set to AC, draw an arc from B crossing the one drawn in step 4. This intersection is point D.
6. Draw a line from D to B.
7. Using the same compass width as used to step along AC, step the compass from D along DB making 4 new arcs across the line.
8. Draw lines between the corresponding points along AC and DB.
9. Done. The lines divide the given line segment AB into 5 congruent parts."

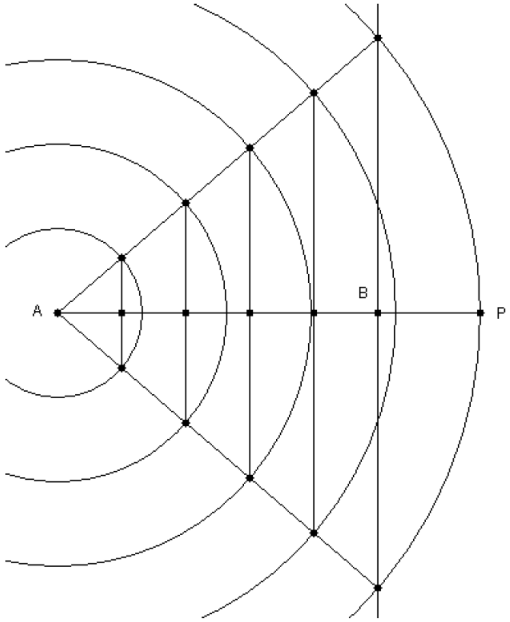
Due to the use of modern compass in the above method, it has been modified. Suppose there is a line segment AB. To divide it into n equal parts, we can use the following method.

1. Extend AB to a point P such that AP is long enough.
2. Draw a circle with centre A. The radius of this circle must be larger than AB/n (it can equals $AB/2^m, 2^m \leqslant n$, for convenient). Let the intersecting point of AP and the circle be A_1.
3. Draw a circle with centre A_1 and radius AA_1. Let the intersecting point of AP and the circle be A_2.
4. Repeat the step until a circle with centre A_{n-1} and radius $A_{n-1}A_n$ is drawn. Repeat the step until a circle with centre A_{n-1} and radius $A_{n-1}A_n$ is drawn.
5. Draw a circle C_i with centre A and radius AA_i for every $0 < i < n$.
6. Draw a circle with centre B (the radius is not important). Label the intersecting points of the circle and AP as B_1 and B_2 respectively.
7. Draw a perpendicular bisector of B_1B_2. Label the intersecting points of the circle and the perpendicular bisector as Q and R respectively.
8. Draw line segments AQ and AR.
9. Label the intersecting points of C_i and AQ as Q_i and the intersecting points of C_i and AR as R_i.
10. Join Q_iR_i with a straight line for every $0 < i < n$. These lines cut AB into n equal parts.

To divide a line segment into a ratio $a : b$, we only construct some essential lines instead of all of them.

Here comes an example for $n = 5$. Dividing a line into 5 equal parts

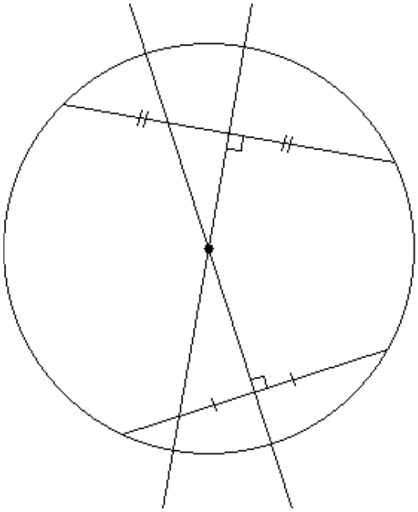




Appendix B. Constructing a 2^m-side polygon or a hexagon

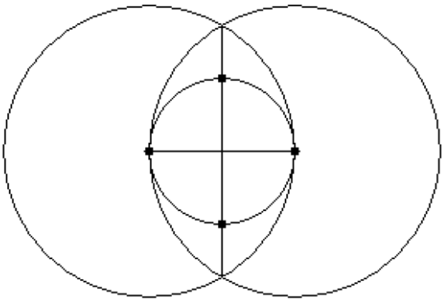
B.1. Finding the centre of a given circle

We will discuss the method to find the centre of a given circle first. To achieve that, we may draw two line segments in the circle. After that, draw the perpendicular bisectors of these two segments. The intersection of these two perpendicular bisectors is the centre.



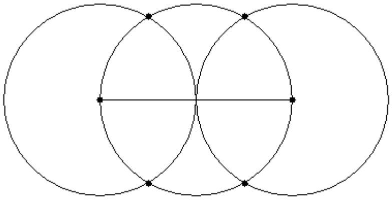
B.2. Constructing a 2^m-side polygon

To construct a 2^m-side polygon, we can first draw a line pass through the centre, i.e. the diameter. Then, draw a perpendicular bisector to the diameter. The 4 intersecting points of these two diameters and the circle are the vertices of a square. Draw a perpendicular bisector to each side of the 2^m-side polygon. The intersecting points of these perpendicular bisectors and the circle, as well as the original vertices of 2m-side polygon, are the vertices of a 2^{m+1}-side polygon. Therefore, we can construct any 2^m-side polygon for any positive integer $m \geqslant 2$.



B.3. Constructing a hexagon

To construct a hexagon, we can draw a diameter first. Then, draw a circle using one end as centre with the same radius as the original circle. Repeat the step using another end. The four points of intersection, as well as two ends of the diameter, are the vertices of a hexagon.



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[4] http://www41.homepage.villanova.edu/robert.styer/trisecting%20segment/
[5] http://www.mathopenref.com/constdividesegment.html

The content in quotation marks and reference code represent the content we have quoted, which may be slightly amended for better presentation.

Reviewer's Comments

After reading the submitted paper, the reviewer has only comments on the wordings, which have been amended in this paper.