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THE JOHNSON-LEADER-RUSSELL QUESTION ON SQUARE POSETS

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Abstract. We study the problem of finding the maximum number of maximal chains in a given size-k subset of a square poset $[n] \times [n]$. This was proposed by Johnson, Leader, and Russell but not yet solved. Kittipassorn had given a conjectural solution to the problem. We verify Kittipassorn's conjecture for $0 \leq k \leq 3n-2$ and solve a variant problem for the case $3n-1 \leq k \leq 4n-4$, which also supports the conjecture. For general k, we find that the optimal configuration is given by a 1-Lipschitz function. We also generalize the problem to rectangle posets and give a solution to one particular poset.

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1. Introduction

Consider the subsets of the power set of $\{1, 2, ..., n\}$. Johnson, Leader, and Russell [1] solved asymptotically the maximum number of maximal chains in a subset with a given proportion of size. At the end of the paper, they considered a variant of the problem in which the set is $\mathfrak{P} = \{1, 2, ..., n\}^2$ with a partially order. They asked the following question, with $M(T)$ denoting the number of maximal chains of $\mathfrak{P}$ contained in T, where $T \subseteq \mathfrak{P}$, i.e.

$$M(T) := \#\{\mathfrak{m} \subseteq T : \mathfrak{m} \text{ is a maximal chain in } \mathfrak{P}\}.$$

Question 1.1 ([1], Question 9). *Given an integer k with $0 \leq k \leq n^2$, what is*

$$\max_{\substack{T \subseteq \mathfrak{P} \\ |T|=k}} M(T)?$$

In other words, we choose k elements in the set $\mathfrak{P}$ to form a subset T aiming for the greatest number of maximal chains in T. We can represent the set $\mathfrak{P} = \{1, 2, ..., n\}^2$ by an $n \times n$ square grid.

An equivalent statement to the problem is as follows: given an $n \times n$ grid and an integer k, among all configurations of k points in the grid, what is the maximum number of paths going from the bottom to top (only going upwards) that only pass through the selected points?

For example, Figure 1 shows a configuration with $n = 5$. The poset $\mathfrak{P}$ is the grid with 25 grid points (not the 16 squares) and the configuration T is the subset of $\mathfrak{P}$ as shown with the 14 blue points. The red line shows one of the maximal chains of $\mathfrak{P}$ contained in T.

Kittipassorn [2] made a contribution to the problem by imposing more constraints to the original problem: given the number of elements in each level $r_1, r_2, \ldots, r_{2n-1}$ instead of the total number of elements k in the grid, he gave a construction of optimal configurations. Therefore, we only need to consider Kittipassorn's configurations for any n and k. Kittipassorn also conjectured the answer for the original problem for all n and k.

In our paper, in contrast to Johnson, Leader, and Russell's paper of finding the asymptotic solution, we make progress on the exact value of the maximum number of maximal chains. We verify Kittipassorn's conjecture for $1 \leq k \leq 3n - 2$ and consider a variant problem for the case $3n - 1 \leq k \leq 4n - 4$, which also supports the conjecture. For general k, we find that the optimal configuration is given by a 1-Lipschitz function.

In Section 2, notation in poset theory is given. We also review the progress made by Kittipassorn. In Section 3, we solve the problem for the case $2n - 1 \leq k \leq 3n - 2$ and investigate the problem of general k. In Section 4, we develop an algorithm to compute the maximum number of maximal chains in T with the proven results. At

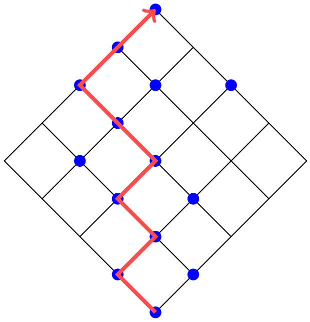

FIGURE 1. A maximal chain of $\mathfrak{P}$ contained in T

last, in Section 5, we consider a variant problem for the case $3n - 1 \leq k \leq 4n - 4$ and generalize the problem to rectangle posets.

2. Background

2.1. **Notations.** We first review some terminology and definitions from poset theory. For reference, we refer the reader to [4].

Let n be a positive integer. Denote $[n]$ denote the set $\{1, 2, ..., n\}$. Consider the partially ordered set, or in short, poset $(\mathfrak{P}, \succeq)$ where $\mathfrak{P} := [n]^2$, endowed with the partial order $\succeq$, defined by

$$(i, j) \succeq (i', j') \Leftrightarrow i \geq i' \text{ and } j \geq j'.$$

For simplicity, we will use $\mathfrak{P}$ instead of $(\mathfrak{P}, \succeq)$.

A **maximal chain** of $\mathfrak{P}$ is a chain in $\mathfrak{P}$ with $2n - 1$ elements.

Now we introduce some definitions that will be frequently used.

Definition 2.1. *We can partition the poset $\mathfrak{P}$ into* ***levels*** *by*

$$\mathfrak{P} = \bigsqcup_{d=1}^{2n-1} L_d,$$

where $L_d := \{(i, j) : i + j = d + 1\}$.

A maximal chain is thus a chain with exactly one element from each level L_d, where $d = 1, 2, \ldots, 2n - 1$.

Definition 2.2. *Deonte $P(\mathfrak{P})$ the power set of $\mathfrak{P}$. Define $M : P(\mathfrak{P}) \to \mathbb{Z}_{\geq 0}$ to be a function given by*

$$M(T) := \#\{\mathfrak{m} \subseteq T : \mathfrak{m} \textit{ is a maximal chain in } \mathfrak{P}\}.$$

In other words, M maps a subset T of the poset $\mathfrak{P}$ to the number of maximal chains of $\mathfrak{P}$ contained in T.

Definition 2.3. *Define* $\mathrm{JLR} : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Z}_{\geq 0}$ *to be a function given by*

$$\mathrm{JLR}(n, k) := \max_{\substack{T \subseteq \mathfrak{P} \\ |T| = k}} M(T).$$

In other words, JLR maps the couple (n,k) to the maximum number of maximal chains of $\mathfrak{P}$ contained in T, with $|T|=k$.

For convenience, let us call a configuration $T \subseteq \mathfrak{P}$ **optimal** if

$$M(T) = \max_{S\,:\,|S|=|T|} M(S).$$

2.2. **Kittipassorn's configuration.** Teeradej Kittipassorn [2] considered a restricted case of the problem in which the number of elements in *each level* of T is also given, namely $r_1, r_2, ..., r_{2n-1}$. More precisely, we have the following variant of the Johnson-Leader-Russell question: given $r_1, r_2, \ldots, r_{2n-1}$, what is $\max M(T)$, where the maximization is over all configurations T such that $r_i = |T \cap L_i|$, for all $1 \le i \le 2n-1$? This has been solved by Kittipassorn. In the following, we will describe his solution to the problem.

Given $r_1, r_2, \ldots, r_{2n-1}$, Kittipassorn considered the following configuration $T^*(r_1, r_2, \ldots, r_{2n-1})$:

$$T^*(r_1, r_2, \ldots, r_{2n-1}) := \bigcup_{h=1}^{2n-1} \left\{ \left(\frac{h+1}{2} + t, \frac{h+1}{2} - t \right) : t = \alpha_h, \alpha_h + 1, ..., \beta_h \right\},$$

where for each $h = 1, 2, ..., 2n-1$, the numbers α_h and β_h are unique real numbers such that

$$\alpha_h + \beta_h \in \{0,1\}, \qquad \beta_h - \alpha_h + 1 = r_h, \qquad \text{and} \qquad h + 2\alpha_h \text{ is an odd integer.}$$

Another way to describe the configuration $T^*(r_1, r_2, \ldots, r_{2n-1})$ is that it is the unique configuration satisfying the following conditions:

(1) For each level, all the elements are condensed in the middle.
(2) If we have to break the left-right symmetry, all the extra elements are put on the right.

For "left-right symmetry", we mean that the number of elements in the right and in the left parts of $\mathfrak{P}$ in T is the same. In this paper, we call such a configuration *Kittipassorn's configuration*. We will use a shorthand notation $M(r_1, r_2, \ldots, r_{2n-1}) = M(T^*(r_1, r_2, \ldots, r_{2n-1}))$. The following example demonstrates how we form Kittipassorn's configurations.

Example 2.4. *Figure 2 shows the Kittipassorn's configuration $T^*(1,1,2,3,1,2,1)$ when $n = 4$ and $(r_1, r_2, r_3, r_4, r_5, r_6, r_7) = (1,1,2,3,1,2,1)$. The elements which break the left-right symmetry are coloured in red.*

Notice that the number of maximal chains in this configuration is 6, so

$$M(1,1,2,3,1,2,1) = 6.$$

Kittipassorn [2] proved that such configuration has the greatest number of maximal chains with given numbers of elements in each level $r_1, r_2, ..., r_{2n-1}$. We rephrase this in the following lemma. It solves the variant question by providing a construction of optimal configurations.

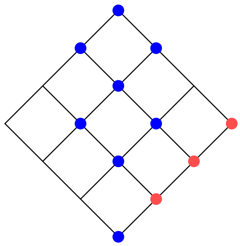

FIGURE 2. $T^*(1,1,2,3,1,2,1)$

Lemma 2.5 (Kittipassorn's lemma [2]). *Suppose that non-negative* $r_1, r_2, \ldots, r_{2n-1}$ *are given. Then*

$$\max_{T\,:\,\forall i,\,|T\cap L_i|=r_i} M(T) = M(r_1, r_2, \ldots, r_{2n-1}).$$

The proof of Lemma 2.5 is attached in Appendix A. With Kittipassorn's lemma, in order to compute

$$\max_{T\,:\,|T|=k} M(T),$$

it suffices to consider only the configurations which are Kittipassorn's configurations.

Moreover, Kittipassorn [2] proposed a conjectural solution to the original Johnson-Leader-Russell problem. Before introducing the conjecture, we need a new notation. Previously, we partition $\mathfrak{P}$ into $2n-1$ parts $L_1, L_2, ..., L_{2n-1}$ by the vertical positions of elements. Now we partition $\mathfrak{P}$ by the horizontal positions. For each $d = -(n-1), ..., n-1$, we partition $\mathfrak{P}$ into **columns** by

$$C_d := \{(i,j) \in \mathfrak{P} : j - i = d\}.$$

Conjecture 2.6 (Kittipassorn's conjecture [2]). *Let n be a positive integer. We define the sequence $T_1, T_2, \ldots, T_{n^2}$ of Kittipassorn's configurations by adding one element at a time so that each T_i has exactly i elements. To add elements from T_1 to T_{n^2}, we fill the columns in the following order:*

$$C_0, C_1, C_{-1}, C_2, C_{-2}, \ldots, C_n, C_{-n}.$$

In each column, we fill the points from bottom to top (see an example as in Figure 3).

Given $1 \leq k \leq n^2$, we have

$$\max_{T\ :\ |T|=k} M(T) = M(T_k).$$

For example, Figure 3 shows the order of adding the elements from T_1 to T_{16} according to Conjecture 2.6 when $n = 4$. For example, the configuration T_{11} contains the 11 points labeled 1 to 11.

Notice that the conjecture gives a construction of optimal configurations, which solves the original Johnson-Leader-Russell problem.

Kittipassorn's conjecture implies the following formula for $2n-1 \leq k \leq 3n-2$:

$$\mathrm{JLR}(n,k) = 2^{k-2n+1}.$$

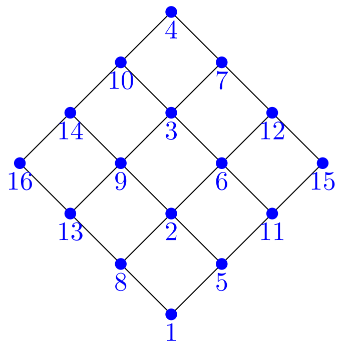

FIGURE 3. The order of elements added from T_1 to T_{16} when $n = 4$

In Subsection 3.1, we prove that this conjectural formula is correct.

The conjecture also implies the following formula for $3n - 1 \leq k \leq 4n - 4$:

$$\mathrm{JLR}(n, k) = 2^{4n-k-4} F_{2k-6n+7},$$

where F_i denotes the i-th Fibonacci number which are given by $F_0 = 0$, $F_1 = 1$, and for $i \geq 2$, we have $F_i := F_{i-1} + F_{i-2}$.

In our paper, we do not prove the above conjectural formula for $3n-1 \leq k \leq 4n-4$ but we show that this formula is true for a variant of the problem in Subsection 5.1.

It was observed by Tanya Khovanova [3] that for the case $k = 3n + c$ for a fixed integer $c \geq -1$, the number $\mathrm{JLR}(n, k)$ appears to double whenever n is increased by 1, for $n \geq c + 4$. While this has not been proved yet, the "doubling" phenomenon can be explained in view of Kittipassorn's conjecture. Thus we give the following remark.

Remark 2.7. *Let $c \geq -1$ be a fixed integer. Kittipassorn's conjecture implies the following conjectural formula: for all integers $n \geq c + 4$,*

$$\mathrm{JLR}(n, 3n + c) = 2^{n-c-4} F_{2c+7}.$$

If $n \geq c + 4$, then when n is increased by 1, we have that $\max M(T)$ is doubled.

3. MAIN RESULTS

3.1. The case $0 \leq k \leq 3n - 2$.

To begin our investigation, we start off by some small values of k. First consider the case when $0 \leq k \leq 2n - 2$.

Proposition 3.1. *If $0 \leq k \leq 2n - 2$, then*

$$\mathrm{JLR}(n, k) = 0.$$

Proof. Assume not, then there exists a configuration T with a maximal chain $\mathfrak{m} \subseteq T$. However, $|\mathfrak{m}| = 2n - 1$ but $|T| \leq 2n - 2$. This gives a contradiction. $\square$

Lemma 2.5 shows that given the number of elements in each level, we can give a unique construction of optimal configurations. Therefore, it remains to allocate the k elements to the $2n - 1$ levels. In the following lemma, we give an upper bound of $r_1 r_2 \cdots r_{2n-1}$ which helps to find the upper bound of $\mathrm{JLR}(n, k)$.

Lemma 3.2. *Given non-negative integers $r_1, r_2, ...r_{2n-1}$ such that $r_1 + r_2 + \cdots + r_{2n-1} = k$. Suppose $k = (2n-1)q + h$ where $q, h \in \mathbb{Z}_{\geq 0}$ and $0 \leq h < 2n-1$, we have*

$$r_1 r_2 \cdots r_{2n-1} \leq \lfloor \frac{k}{2n-1} \rfloor^{2n-h-1} \lceil \frac{k}{2n-1} \rceil^{h}.$$

The equality holds when there are $2n-h-1$ r_i's are equal to $\lfloor \frac{k}{2n-1} \rfloor$ and h r_i's are equal to $\lceil \frac{k}{2n-1} \rceil$.

Proof. We claim that $r_1 r_2 \cdots r_{2n-1}$ attains the maximum when all $r_i = \lfloor \frac{k}{2n-1} \rfloor$ or $\lceil \frac{k}{2n-1} \rceil$. If $2n-1 \mid k$, then the results follows by the AM-GM inequality. Now for the case that $2n-1 \nmid k$. Assume, for the sake of contradiction, that there exists $r_i > \lceil \frac{k}{2n-1} \rceil$ and $r_1 r_2 \cdots r_{2n-1}$ attains the maximum. As $r_1 + r_2 + \cdots + r_{2n-1} = k$, there exists $r_j \leq \lfloor \frac{k}{2n-1} \rfloor$. Thus $r_i > r_j + 1$. The product $(r_i - 1)(r_j + 1) = r_i r_j + r_i - r_j - 1 \geq r_i r_j$. This gives a contradiction.

Therefore, when $r_1 r_2 \cdots r_{2n-1}$ attains the maximum, it takes the form

$$\lfloor \frac{k}{2n-1} \rfloor^x \lceil \frac{k}{2n-1} \rceil^{2n-x-1},$$

where $0 \leq x \leq 2n-1$ is an integer. As $2n-1 \nmid k$, we have

$$x \lfloor \frac{k}{2n-1} \rfloor + (2n-x-1) \lceil \frac{k}{2n-1} \rceil = xq + (2n-x-1)(q+1) = k,$$

which gives $x = 2n-h-1$. The case where there exist $r_i < \lfloor \frac{k}{2n-1} \rfloor$ is similar. This completes the proof. □

The following proposition gives an explicit solution to the problem for the case $2n-1 \leq k \leq 3n-2$.

Proposition 3.3. *If $2n-1 \leq k \leq 3n-2$, then*

$$\mathrm{JLR}(n,k) = 2^{k-2n+1}.$$

Proof. Consider any configuration T with k elements. Let $r_i = |T \cap L_i|$, i.e. the number of elements in the i-th level, we have $r_1 + r_2 + \cdots + r_{2n-1} = k$. By Kittipassorn's lemma (Lemma 2.5), for the purpose of computing $\mathrm{JLR}(n,k)$, we may assume T is a Kittipassorn's configuration.

Now we want to show that $M(T) \leq r_1 r_2 \cdots r_{2n-1}$. Let $\mathfrak{m} \in T$ be a maximal chain, then $\mathfrak{m}$ contains exactly one element in each level. For the i-th level, there are r_i choices of element that can be contained in $\mathfrak{m}$. Therefore, $M(T) \leq r_1 r_2 \cdots r_{2n-1}$.

As r_i are non-negative integers, $r_1 r_2 \cdots r_{2n-1}$ attains its maximum when there are exactly $k-2n+1$ 2's and $4n-k-2$ 1's by Lemma 3.2. Thus,

$$M(T) \leq 2^{k-2n+1} \cdot 1^{4n-k-2} = 2^{k-2n+1}.$$

For the construction for the equality case, an optimal configuration when $2n-1 \leq k \leq 3n-2$ has exactly one element in odd order of levels and at most two elements in even order of levels. For each level, the elements are condensed in the middle, i.e. it is a Kittipassorn's configuration. Notice that the total number of elements in such a configuration is at least $2n-1$ and at most $3n-2$. □

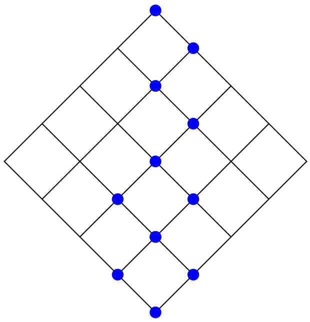

FIGURE 4. A configuration T such that $M(T) = 4$

Example 3.4. *Figure 4 gives a construction of T such that $M(T)$ attains the upper bound $2^{11-2\times5+1} = 4$ when $n = 5$ and $k = 11$.*

3.2. **Investigation on general k.** In this section, we investigate the problem of general values of k. The first proposition considers the upper bound of $M(T)$ for all $0 \leq k \leq n^2$.

Proposition 3.5. *For $0 \leq k \leq n^2$, we have*

$$\mathrm{JLR}(n, k) \leq 2^{k-2n+1}.$$

Proof. We have shown the case for $0 \leq k \leq 3n-2$ in Proposition 3.1 and 3.3. And as $M(T) \leq M(\mathfrak{P}) = \binom{2(n-1)}{n-1} \leq 2^{2n-2}$, the bound is trivial for $k \geq 4n-3$.

It now remains to show the inequality for the case $3n-1 \leq k \leq 4n-4$. Consider any configuration T with k elements. By Lemma 3.2 and similar to Proposition 3.3, we have

$$M(T) = r_1 r_2 \cdots r_{2n-1} \leq \lfloor \frac{k}{2n-1} \rfloor^{2n-h-1} \lceil \frac{k}{2n-1} \rceil^{h}.$$

On the other hand, as $3n-1 \leq k \leq 4n-4$, we have $q = 1$ and $h = k-2n+1$, and thus $\lfloor \frac{k}{2n-1} \rfloor = 1$ and $\lceil \frac{k}{2n-1} \rceil = 2$. This gives $M(T) \leq 1^{2n-h-1} \cdot 2^{k-2n+1} = 2^{k-2n+1}$. □

Remark 3.6. *The equality in Proposition 3.5 can only hold when $2n-1 \leq k \leq 3n-2$.*

This gives us the idea to consider the number of elements in consecutive levels. An intuitive idea is that no elements should be "wasted", i.e. it does not belong to any maximal chains. Observe that if there are more than $r_i + 1$ elements in the $(i+1)$-th level, then some elements in Kittipassorn's configuration are "wasted". Similarly, elements are "wasted" if there are less than $r_i - 1$ elements in the $(i-1)$-th level.

Therefore, removing the "wasted" elements will not affect the number of maximal chains in a configuration. On the other hand, as k is fixed, we can add new elements to the configuration. In the following lemma, we develop an algorithm to construct the new position of the removed elements such that the number of maximal chains is increased.

Lemma 3.7. *Let $T \subsetneq \mathfrak{P}$ be a configuration such that $M(T) > 0$. Then there exists $v \in \mathfrak{P} - T$ such that*

$$M(T) < M(T \cup \{v\}).$$

Proof. As $M(T) > 0$, there exists a maximal chain $\mathfrak{m} \subseteq T$. On the other hand, as $T \neq \mathfrak{P}$, there exists $v \in \mathfrak{P} - T$. As $v \notin \mathfrak{m}$, there are two cases: $\mathfrak{m}$ is to the left of v or $\mathfrak{m}$ is to the right of v.

First consider the case when $\mathfrak{m}$ is to the left of v (as in Figure 5).

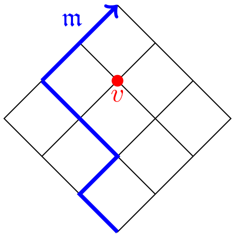

FIGURE 5. $\mathfrak{m}$ to the left of v

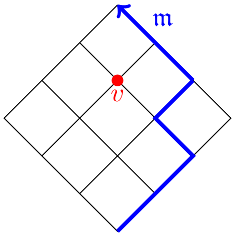

FIGURE 6. $\mathfrak{m}$ is to the right of v

Thus the set $U := \{\text{maximal chains } \mathfrak{m} \subseteq T : \mathfrak{m} \text{ is to the left of } v\}$ is non-empty.

Define a function $\mathbf{Area} : U \to \mathbb{Z}_{\geq 0}$ which maps a maximal chain to the number of elements in $\mathfrak{P}$ on its left. For example, $\mathbf{Area}(\mathfrak{m})$ in Figure 5 is 2. Since U is a non-empty finite set, the image $\mathbf{Area}(U)$ is a finite non-empty subset of $\mathbb{Z}_{\geq 0}$. Let $B := \max(\mathbf{Area}(U))$. Then there exists a maximal chain $\mathfrak{m}^* \in U$ such that $\mathbf{Area}(\mathfrak{m}^*) = B$.

Notice that $\mathfrak{m}^*$ cannot be the right boundary of $\mathfrak{P}$ because v is on its right. Hence, there exist $v_1, v_2, v_3 \in \mathfrak{m}^*$ in a certain configuration on Figure 7.

Now we want to show that $v_4 \notin T$. Suppose, for the sake of contradiction, that $v_4 \in T$, then $v_4 \neq v$ because $v \notin T$. This means that there exists a maximal chain $\mathfrak{m}^{**} = (\mathfrak{m}^* - \{v_2\}) \cup \{v_4\}$ which is also to the left of v. Thus $\mathfrak{m}^{**} \in U$. However, $\mathbf{Area}(\mathfrak{m}^{**}) = B + 1$, which contradicts the maximality of $\mathbf{Area}(\mathfrak{m}^*)$. Therefore, we have $v_4 \notin T$ and $M(T \cup \{v_4\}) > M(T)$ because $\mathfrak{m}^{**} \in T \cup \{v_4\}$ but $\mathfrak{m}^{**} \notin T$.

The other case where $\mathfrak{m}$ is to the right of v is proven similarly. And this completes the proof. $\square$

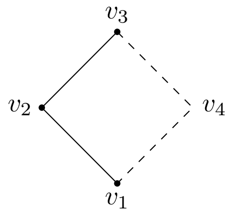

FIGURE 7. The existence of $v_1, v_2, v_3 \in \mathfrak{m}^*$

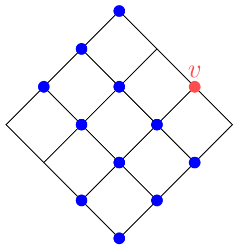

FIGURE 8. A configuration with a "wasted" element v

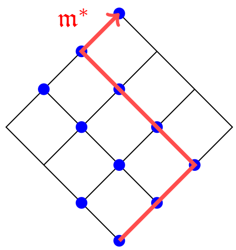

FIGURE 9. The maximal chain $\mathfrak{m}^*$ with the greatest **Area**

Example 3.8. *Given $n = 4$ and $k = 12$, consider a configuration as in Figure 8. Notice that the red point v is not in any of the maximal chains. Hence we can remove it without affecting the value of $M(T)$. Figure 9 shows the maximal chain $\mathfrak{m}^*$ with the greatest* ***Area****. By the algorithm in Lemma 3.7, we pick $u \notin T$. Notice that selecting u increases the number of maximal chains, as in Figure 10.*

Here we introduce some shorthand notation. For any $f : \{1, 2, \ldots, 2n-1\} \to \mathbb{Z}_{\geq 0}$ such that $0 \leq f(i) \leq |L_i|$, we define $T^*(f)$ to be

$$T^*(f) := T^*(f(1), f(2), ..., f(2n-1)).$$

We also define $M(f)$ to be

$$M(f) := M(T^*(f)).$$

Then we introduce the following theorem by considering r_i in consecutive levels. But before that, we give a definition of 1-Lipschitzness.

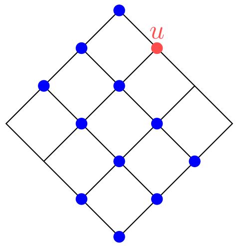

FIGURE 10. The improved configuration by adding a new element u

Definition 3.9. *A function $f : \{1, 2, ..., 2n-1\} \to \mathbb{R}$ is said to be **1-Lipschitz** if for all $x \in \{1, 2, ..., 2n-2\}$, we have*

$$|f(x+1) - f(x)| \leq 1.$$

The following lemma shows that no elements are "wasted" only if the configuration is given by a 1-Lipschitz function.

Lemma 3.10. *Let T^* be a Kittipassorn's configuration. For all $v \in T^*$, v is contained in at least one maximal chain only if T^* is given by a 1-Lipshchitz function.*

Proof. Without loss of generality, assume that $r_{h+1} > r_h$ for some $1 \leq h \leq 2n-1$. Then we pick the rightmost element $v = (i, j) \in L_{h+1} \cap T$. If $v \in \mathfrak{m}$ for some maximal chain $\mathfrak{m}$, then $(i-1, j) \in \mathfrak{m}$ or $(i, j-1) \in \mathfrak{m}$, which are elements in L_h. Now we want to find the number of elements in $L_h \cap T$ and $L_{h+1} \cap T$ respectively. In a Kittipassorn's configuration, we have

$$\frac{i+j-1}{2} + \beta_h = i - 1.$$

Thus $\beta_h = \frac{i-j-1}{2}$. As $\alpha_h + \beta_h \in \{0, 1\}$, $\alpha_h = -\frac{i-j-1}{2}$ or $-\frac{i-j-1}{2} + 1$. Thus $r_h = \beta_h - \alpha_h + 1 = i - j$ or $i - j + 1$. On the other hand, if $v \in L_{h+1} \cap T$ and is the rightmost element, $r_{h+1} = i - j + 1$ or $i - j + 2$ respectively. This gives $r_{h+1} - r_h = 0$ or 1. $\square$

Theorem 3.11. *Let n be a positive integer. For given k such that $2n - 1 \leq k \leq n^2$,*

$$\mathrm{JLR}(n, k) = \max M(f),$$

where the maximization of $M(f)$ is over all 1-Lipschitz functions $f : \{1, 2, ..., 2n-1\} \to \mathbb{Z}_{\geq 1}$ such that $\sum_{i=1}^{2n-1} f(i) = k$.

Proof. Assume, for the sake of contradiction, that $\mathrm{JLR}(n, k) > \max M(f)$, i.e. there exists a Kittipassorn's configuration T which is optimal but is not given by a 1-Lipschitz function. Then by Lemma 3.10, there exists $v \in T$ that is not in any maximal chain. Hence we can remove it without changing $M(T)$, i.e.

$$M(T - \{v\}) = M(T).$$

After removing v, we have $T \subsetneq \mathfrak{P}$. By Lemma 3.7, there exists $u \in \mathfrak{P} - T$ such that $M((T - \{v\}) \cup \{u\}) > M(T)$. Also notice that $|(T - \{v\}) \cup \{u\}| = |T|$. This contradicts to the maximality of $M(T)$. $\square$

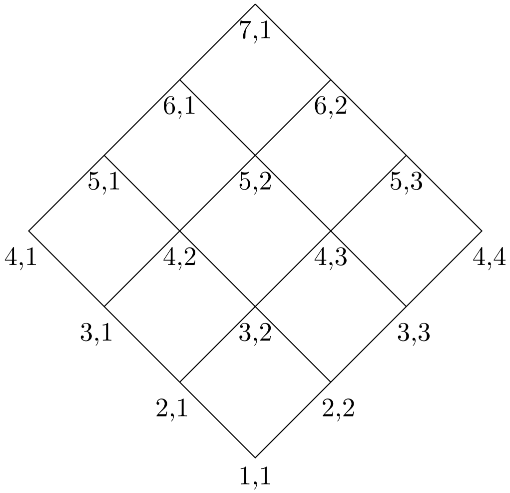

FIGURE 11. The indices of elements in a 4×4 poset

4. COMPUTATIONAL RESULTS

4.1. **Pseudo-code.** Using the results obtained in the previous section, we can develop a much more efficient computer program to compute $\mathrm{JLR}(n, k)$ than exhausting all the possibilities. By Lemma 2.5 and Theorem 3.11, we reduce the runtime of the program by considering the following three constraints:

(1) There is a total number of k elements in all levels, i.e. $r_1+r_2+\cdots+r_{2n-1} = k$.
(2) For all $1 \leq i \leq 2n-2, |r_{i+1} - r_i| \leq 1$.
(3) The configuration has to be a Kittipassorn's configuration.

The pseudo-code of the program is shown in Appendix C. From line 1 to 35, a function `chain` is defined by inputting n and $r_1, r_2, ..., r_{2n-1}$ and outputting $M(r_1, r_2, ..., r_n)$. Recall that $M(r_1, r_2, ..., r_n)$ is defined in Section 2.2. From line 36 to 51, we partition the given k into $2n-1$ parts: $r_1, r_2, \ldots, r_{2n-1}$ given by a 1-Lipschitz function, which is checked by the Boolean variable `bo`. It is true if and only if the difference of elements in any two levels is at most 1. Then we exhaust all possible Kittipassorn's configurations to find the optimal configuration.

Notice that each element is indexed by (i, j), where i is the order of level and j is the order counting from the left. For example, a 4×4 poset is indexed as:

Notice that this indexing is not the same as the normal definition, i.e. $\mathfrak{P} = [n] \times [n] = \{(i, j) : i, j \in [n]\}$. However, it is easier to develop the pseudo-code with the above indexing because it arranges the elements according to the level.

In the following example, we compare the efficiency of codes with and without our results.

Example 4.1. *We can use the following four different computer algorithms to compute the value of* $\mathrm{JLR}(7, 25)$*.*

(1) Running through all the subsets of 25 elements in the 7×7 grid: number of cases $= \binom{49}{25} \approx 6.32 \times 10^{13}$.
(2) Running through all the tuples ($1 \leq r_i \leq |L_i|$) and arrange them in Kittipassorn's configuration: number of cases $= 1 \times 2 \times \cdots \times 6 \times 7 \times 6 \times \cdots \times 1 \approx 3.62 \times 10^6$.
(3) Running through all tuples with $r_1 + r_2 + \cdots + r_{13} = 25$ and arrange them in Kittipassorn's configuration: number of cases $\approx 1.31 \times 10^5$.
(4) Running through all tuples with $r_1 + r_2 + \cdots + r_{13} = 25$ given by a 1-Lipschitz function and arrange them in Kittipassorn's configuration: number of cases $= 1100$.

Therefore, we can see that our code greatly improves the efficiency of computing $\mathrm{JLR}(n, k)$.

4.2. **Numerical results.** The computational results of $\mathrm{JLR}(n, k)$ when $2 \leq n \leq 6$ and $2n - 1 \leq k \leq n^2$ are shown in the following tables. However, this program does not follow the pseudo-code in Subsection 4.1. It is a brute-force program that does not use any result in this paper (Program 1 in Example 4.1). This is an inefficient code compared to the pseudo-code we developed with the mathematical results. However, as we did not use any of the results in this paper, we can use this code from scratch to check our results. Notice that the results of the program support the conjectural formulae in Conjecture 2.6 and Remark 2.7.

n	k	$\mathrm{JLR}(n,k)$
2	3	1
	4	2
3	5	1
	6	2
	7	4
	8	5
	9	6
4	7	1
	8	2
	9	4
	10	8
	11	10
	12	13
	13	15
	14	18

n	k	$\mathrm{JLR}(n,k)$
4	15	19
	16	20
5	9	1
	10	2
	11	4
	12	8
	13	16
	14	20
	15	26
	16	34
	17	39
	18	45
	19	54
	20	57
	21	61

n	k	$\mathrm{JLR}(n,k)$
5	22	64
	23	68
	24	69
	25	70
6	11	1
	12	2
	13	4
	14	8
	15	16
	16	32
	17	40
	18	52
	19	68
	20	89
	21	102

n	k	$\mathrm{JLR}(n,k)$
6	22	117
	23	135
	24	162
	25	171
	26	183
	27	197
	28	206
	29	218
	30	232
	31	236
	32	241
	33	245
	34	250
	35	251
	36	252

5. Variant problems

In this section, we will consider some variations of the original Johnson-Leader-Russell problem. The first variation restricts the values of $f(i)$ and the second one considers the number of maximal chains in rectangular posets (instead of square posets).

Now we let $\mathrm{JLR}(n, k; \mathfrak{U})$ to be the maximum number of maximal chains of $\mathfrak{P}$ in subset T with given size k in $\mathfrak{U} \subseteq \mathfrak{P}$, i.e.

$$\mathrm{JLR}(n, k; \mathfrak{U}) := \max_{T \in \mathfrak{U} : |T| = k} M(T).$$

5.1. **Conjectural solution for the case $3n-1 \le k \le 4n-4$.** In this section, we solve a variant problem of $\mathrm{JLR}(n,k)$ for $3n-1 \le k \le 4n-4$. First, we give the following conjecture:

Conjecture 5.1. *For $3n-1 \le k \le 4n-4$, we have*

$$\mathrm{JLR}(n,k) = \mathrm{JLR}(n,k;\mathfrak{U})$$

where $\mathfrak{U} = \{T \in \mathfrak{P} : \text{for } i \in 1,2,...,2n-1, |T \cap L_i| \le 2\}$.

Notice that Conjecture 5.1 is a special case of Kittipassorn's conjecture, Conjecture 2.6. For $3n-1 \le k \le 4n-4$, Kittipassorn's conjecture suggests that there exists an optimal configuration in which there are at most two elements in each level. Such an optimal configuration belongs to the set $\mathfrak{U}$. Therefore, to find $\mathrm{JLR}(n,k)$, it suffices to find the maximum number of maximal chains in the subset $\mathfrak{U}$.

We introduce two algebraic lemmas.

Lemma 5.2. *For $t \in \mathbb{Z}_{\ge 0}$ and $b_1, b_2, \dots, b_{t+2} \in \mathbb{Z}_{\ge 1}$, we have*

$$F_{b_1+2}F_{b_2+2}\cdots F_{b_t+2} \le 2^{t-1}F_{b_1+b_2+\cdots+b_t-t+3}.$$

The equality holds if and only if $t = 1$.

Proof. It suffices to show if $a, b \ge 1$, then $F_{a+2}F_{b+2} < 2F_{a+b+1}$.

For the base case, when $a = b = 1$, $F_3 \times F_3 = 4 \le 4 = 2 \times F_4$.

Apply strong induction on a. Assume $x \ge 1$ is a positive integer such that for all $1 \le i \le x$, $F_{i+2}F_{b+2} < 2F_{i+b+1}$. Then for $x+1$, as $F_{x+1}F_{b+2} < 2F_{x+b}$ and $F_{x+2}F_{b+2} < 2F_{x+b+1}$, we have $(F_{x+1} + F_{x+2})F_{b+2} < 2(F_{x+b+1} + F_{x+b})$, as required.

Notice that the above induction only works for $t \ge 2$. We can check that the equality holds when $t = 1$. $\square$

Lemma 5.3. *For $a, b \ge 2$, we have*

$$2^{a-1}F_b \le 2^a F_{b-1}.$$

The equality holds if and only if $b = 3$.

Proof. It follows from the fact that for all $b \ge 2$, $F_b = F_{b-1} + F_{b-2}$ and $F_{b-2} \le F_{b-1}$. The only case that this equality holds is when $b = 3$. $\square$

Proposition 5.4. *For given $3n-1 \le k \le 4n-4$,*

$$\mathrm{JLR}(n,k;\mathfrak{U}) = 2^{4n-k-4}F_{2k-6n+7}$$

where F_i denotes the i-th Fibonacci number.

Proof. For all $1 \le i \le 2n-1$, we have $f(i) = 1$ or 2 by the definition of $\mathfrak{U}$. Notice that $f(1) = 1$. Let t be the number of times when a sequence of consecutive 2 occurs, and the lengths of the sequences are $b_1, b_2, ..., b_t \ge 1$. Notice that we have $b_1 + b_2 + \cdots + b_t = k - 2n + 1$.

On the other hand, the number of 1's in the set of $f(i)$ is $4n-k-2$ and both of the first and last part are some sequences of 1's. We also know that sequences of 1's and 2's alternates. Thus we have $t \le 4n-k-3$.

Notice that $M(f) = F_{b_1+2}F_{b_2+2}\cdots F_{b_t+2}$. By Lemma 5.2, we have

$$F_{b_1+2}F_{b_2+2}\cdots F_{b_t+2} \le 2^{t-1}F_{b_1+b_2+\cdots b_t-t+3} = 2^{t-1}F_{k-2n+4-t}.$$

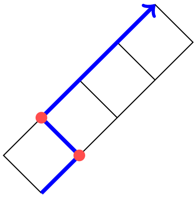

FIGURE 12. A maximal chain in $[n] \times [2]$

By Lemma 5.3, this is maximized when $t = 4n - k - 3$. The result follows. $\square$

5.2. **Generalization to rectangle posets.** In this section, we generalize the problem to rectangle posets. We consider the following poset

$$\mathfrak{P}_{n\times m} := [n] \times [m] = \{(i, j) : i \in [n] \text{ and } j \in [m]\}.$$

We consider the first non-trivial case, where $m = 2$ and n is any positive integer. We want to find

$$\mathrm{JLR}_{n\times 2}(n, k) := \max_{T:|T|=k} M(T)$$

where $T \subseteq \mathfrak{P}_{n\times 2}$.

We first define the following terminology in the poset $\mathfrak{P}_{n\times 2}$.

Definition 5.5. *Given a maximal chain $\mathfrak{m}$, an* ***overlap*** *is a set $\{(x, 1), (x, 2)\} \subseteq \mathfrak{m}$ for some $x \in [n]$.*

Example 5.6. *Figure 12 shows a maximal chain in $[n] \times [2]$ with an overlap (red points).*

With the above definition, we have the following lemma.

Lemma 5.7. *There is exactly one overlap in a maximal chain in $\mathfrak{P}_{n\times 2}$.*

Proof. In a maximal chain $\mathfrak{m} \in \mathfrak{P}_{n\times 2}$, let a be the number of elements in $\mathfrak{m}$ but not in any overlaps of $\mathfrak{m}$ and b be the number of elements in both $\mathfrak{m}$ and the overlaps. Hence we have $a + b = n + 1$, which is the total number of elements in a maximal chain.

On the other hand, we have $a+b/2 = n$. Solving the two equations simultaneously, we have $a = n - 1$ and $b = 2$. Thus there is exactly one overlap in a maximal chain. $\square$

Proposition 5.8. *Let $T \subseteq \mathfrak{P}_{n\times 2}$. If $1 \leq k \leq 2n$ then*

$$\mathrm{JLR}_{n\times 2}(n, k) = k - n.$$

Proof. Let T be any configuration with k elements in $\mathfrak{P}_{n\times 2}$. Suppose, for the sake of contradiction, that $M(T) \geq k-n+1$. However, by Lemma 5.7, we know that there is a overlap in each of the maximal chains. Hence if there are $k-n+1$ maximal chains, there are at least $k - n + 1 + n = k + 1$ elements. This gives a contradiction. $\square$

6. Acknowledgements

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Appendix

A. **Proof of Kittipassorn's lemma.** The proof below of Kittipassorn's lemma (Lemma 2.5) is from Kittipassorn's unpublished manuscript [2].

To prove this lemma, Kittipassorn first define the ordering of the poset elements in each level. Let $v_j \in L_i \cap T$. The ordering of v_j's is defined to be: the middle element is named v_1, add one element at a time going outward columns from the middle. If we have to break the left-right symmetry, add a element to the right first.

Define $a(v)$ and $b(v)$ to be the number of chains containing $(1,1)$ and $v=(i,j)$ with $i-j+1$ elements in a configuration T and a Kittipassorn's configuration respectively. For convention, if $v \notin T$, then $a(v)=0$; if $v \notin T^*$, then $b(v)=0$.

Instead of proving his original lemma, Kittipassorn proves a stronger proposition:

Proposition 6.1. *Let $u_1, u_2, ..., u_{r_i}$ in any configuration T and $v_1, v_2, ..., v_{r_i}$ in a Kittipassorn's configuration T^*. For all $i = 1, 2, ..., 2n-1$, we have*

$$\begin{gathered} a(u_1) \le b(v_1) \\ a(u_1) + a(u_2) \le b(v_1) + b(v_2) \\ \vdots \\ a(u_1) + a(u_2) + ... + a(u_{r_i}) \le b(v_1) + b(v_2) + ... + b(v_{r_i}). \end{gathered}$$

Proof. Let $\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_{r_{i+1}}$ and $\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_{r_{i+1}}$ be elements in $L_{i+1} \cap T$ and $L_{i+1} \cap T^*$ respectively. Apply induction on i, we have

$$\begin{aligned} b(\tilde{v}_1) + b(\tilde{v}_2) + ... + b(\tilde{v}_{r_{i+1}}) &= 2b(v_1) + ... + 2b(v_{k_{i-1}}) + b(v_i) + b(v_{i+1}) \\ &\geq 2a(u_1) + ... + 2a(u_{k_{i-1}}) + a(u_i) + a(u_{i+1}) \\ &\geq a(\tilde{u}_1) + a(\tilde{u}_2) + ... + a(\tilde{u}_{r_{i+1}}), \end{aligned}$$

as required. □

B. **Pseudo-code for computing** $\mathrm{JLR}(n, k)$.

```
FUNCTION chain (positive integer n, vector (r[1], r[2],..., r[2n-1]))
    a: 2D array [0..(2n-1),0..n] of non-negative integers
    b: 2D array [0..(2n-1),0..n] of boolean
    x, z: positive integer
    for i from 1 to 2n-1 do
        if r[i] != 0 then
                if i <= n then x := ceil(i/2)
                else x := ceil((2n-i)/2
                if (i mod 2 = 0) then x := x+1
                if (r[i] mod 2 = 0) and (i mod 2 = 0) then
                    z := ceil(r[i]/2)
                else z := floor(r[i]/2)+1
                y:=x-1
                for j from 1 to z do
                        b[i][x] := TRUE
                        x := x+1
            for j from 1 to (r[i]-z) do
                    b[i][y] := TRUE
                        y := y-1
        else b[i][j] := FALSE
    a[1][1] := 1
    for i from 1 to 2n-1 do
        if i <= n then x := i
        else x := 2n-i
        for j from 1 to x do
        if b[i][j] then
            if i<=n then
                   a[i][j] := a[i-1][j-1] +  a[i-1][j]
                else a[i][j] := a[i-1][j] + a[i-1][j+1]
            else a[i][j] := 0
    RETURN a[2n-1][1]
MAIN
    INPUT positive integers n, k
    max, count: non-negative integer
    bo: boolean
    bo := TRUE
    r[1] := 1
    r[2n-1] := 1
    Partition k-2 into 2n-3 parts: r[2], r[3], ..., r[2n-2]
    for each partition do
        for each i from 2 to 2n-1,
            if abs(r[i]-r[i-1])>1 then
               bo := FALSE
       if bo then
           count := chain(n,r[1], r[2], ..., r[2n-1])
       if max < count then max := count
    OUTPUT max
```

References

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REVIEWERS' COMMENTS

The author of this paper studied the Johnson-Leader-Russell question on square posets about finding the subset of given size k in the square poset $[n] \times [n]$ that contains the largest number of $(2n-1)$-chains. Progress toward the problem was made by Kittipassorn. The author builds on Kittipassorn's work to settle the question in a small but nontrivial range of values of k, and in a slightly larger range under a reasonable-seeming hypothesis about where the maximum could be obtained. The paper also introduced a very natural and interesting extension to the case of rectangular posets, and some codes that can efficiently compute exact answers in a large family of cases.

All three reviewers were impressed by the excellent exposition and high quality results of the paper, and compare it with strong undergraduate theses.