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DOING INDEFINITE INTEGRALS WITHOUT INTEGRATION

TEAM MEMBER
Chun SZETO

TEACHER
Mr. Alexander Kin Chit O

SCHOOL
G.T. (Ellen Yeung) College

Abstract. Residue theorem has been frequently used to tackle certain complicated definite integrals. However, it is never applied for indefinite integrals. Therefore, in this report, residue theorem and a some small tricks are applied to find antiderivatives.

The are mainly three interesting results:

1. Antiderivatives can be found without integration: antiderivatives can be represented by residues, while calculation of residues requires no knowledge of integration. For residues at poles, only differentiation is needed. For residues at essential singularities, Taylor series manipulation is required; still, it is just differentiation with algebra work. This allows fast computation of antiderivatives of rational functions, especially those with only simple poles, providing an alternative to partial fraction decomposition. This is also applicable for other functions. Moreover, this result implies that integration is not only the reverse of differentiation, integration is indeed equivalent to differentiation.
2. A universal functional form of antiderivative can be obtained: antiderivatives obtained by this method has a functional form that converges wherever it should converge. The functional form has the largest possible region of convergence on the complex plane.
3. As a weak tool for analytic continuation: since the universal functional form of antiderivative is obtained, differentiating yields a universal functional form of the integrand. If one knows the behaviour of f in the vicinity of every singularity of f, one can analytically continue f to its largest possible domain by the method presented in this report.

Residue theorem is the central tool to be used throughout this report. Certain simple inequalities such as triangle inequality and estimation lemma are occasionally applied

Introduction

This project is an attempt of expressing indefinite integrals in terms of residues, via residue theorem. The attempt is successful and the method produced results by means of residues that agree with well-known integration results. Surprisingly, doing integrals do not require integration at all.

As a generalization, integral of functions with infinitely many singularities are also handled. The resulting formula is less elegant, and requires knowledge of asymptotic behaviour of the function. Nevertheless, under certain circumstances, the asymptotic behaviour is easily predictable; the indefinite integral of that function, by the method presented in this report, also agrees with the expectation.

The report begins with some definitions to categorize singularities for easier discussion. Then, a case of finitely many singularities is handled. As a generalization, functions with 'singularities cluster' and 'singularities on branch' are also handled. Finally, universality of the functional form of the indefinite integral obtained is also briefly discussed.

1. Notations and Definitions

Suppose $f : \mathbb{C} \to \mathbb{C}$ has no branch points and natural boundaries.

Definition 1. *Let $\sigma(f) \subset \mathbb{C}$ such that f is holomorphic in $\mathbb{C}\backslash\sigma(f)$ and f has a non removable singularity at every $s \in \sigma(f)$. Also, $\sigma^*(f) := \sigma(f) \cup \{\infty\}$.*

In the following chapters, functions of the form $f(z)\ln(z-s)$ (s being a constant) are frequently encountered. Therefore, with respect to $f(z)$ and $\ln(z-s)$, the following sets and collections are defined for easier discussion:

Definition 2.

$$\begin{aligned}
&b = \{x \mid x \textit{ is on the branch cut of } \ln(z-s)\} \\
&\textit{(branch points are not part of branch cuts.)} \\
&A = \sigma(f)\backslash(B \cup C \cup C\mathrm{Nei}) \\
&B \textit{ (for branch)} = \sigma(f) \cap b \\
&C \textit{ (for cluster)} = \{x \mid x \textit{ is a cluster point of } \sigma(f)\} \\
&C\mathrm{Nei} = \bigcup_{k\in C} (\mathrm{Nei}_k \cap \sigma(f)), \textit{ where } \mathrm{Nei}_k \textit{ is defined below}
\end{aligned}$$

Definition 3. *For every $k \in C$, Nei_k: a sufficiently small punctured annulus centered at k, such that b and Nei_k are disjoint; moreover, for any $k_1, k_2 \in C$, Nei_{k_1} and Nei_{k_2} are disjoint. (*Nei *stands for neighbourhood.)*

For elements of $\sigma(f)$ in Nei_k, put the ones with farthest distance from k in the set $\mathrm{ring}_{k,1}$. Put the ones with second farthest distance from k in $\mathrm{ring}_{k,2}$. So on and so on.

Furthermore, define

$$\text{band}_{k,n} = \bigcup_{j=1}^{n} \text{ring}_{k,j}$$

$\text{ring}_{k,n}$ in general can be interpreted as the set of singularities nth farthest from k.

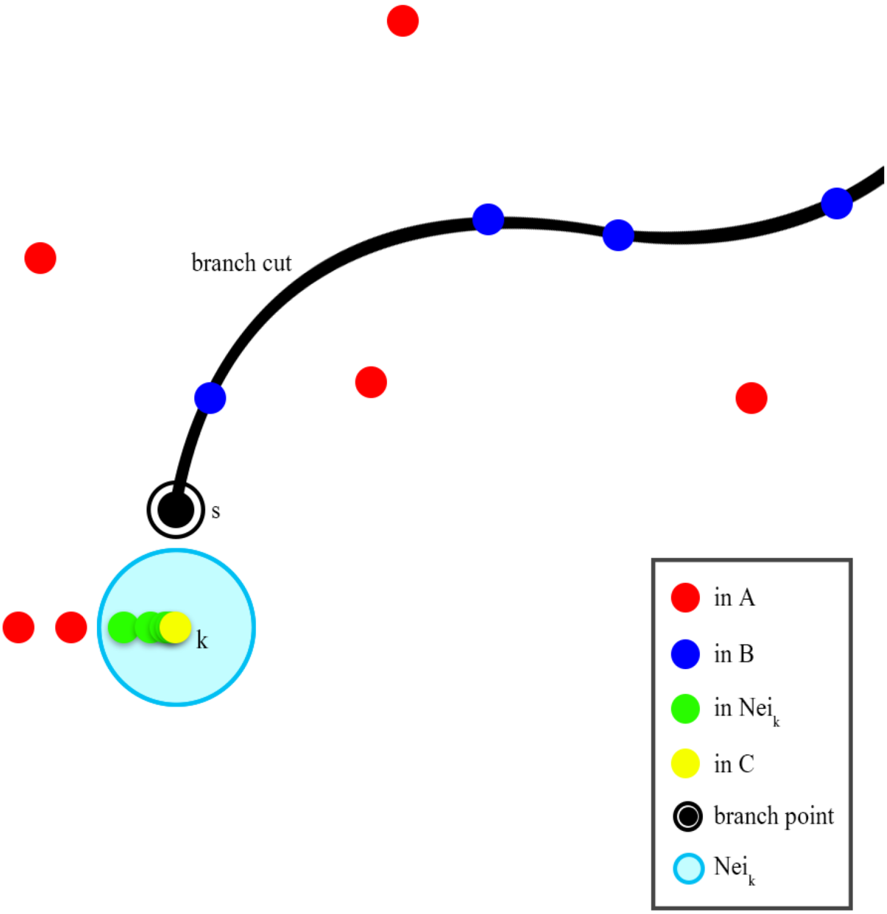
Equivalently, $\text{ring}_{k,n}$ is the set of singularities, lying on the boundary of the nth largest ring centered at k, where rings with no singularities on their boundaries are neglected.

For example, suppose the elements of $\sigma(f)$ in Nei_0 is $\{\dots, -\frac{1}{3}, -\frac{1}{2}, -1, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$. Then, $\text{ring}_{0,1} = \{1, -1\}$, $\text{ring}_{0,2} = \{\frac{1}{2}, -\frac{1}{2}\}$. In general, $\text{ring}_{0,n} = \{\frac{1}{n}, -\frac{1}{n}\}$.

Throughout this paper, there are some assumptions:

1. B and C are disjoint.
2. $C \subset \sigma^*(f)$
3. B, C are finite sets.
4. No elements of $\sigma(f)$ lie on the branch points of $\ln(z - s)$.

As a result, $A, B, C, C\text{Nei}$ are pairwise disjoint. Also, $A \cup B \cup C \cup C\text{Nei} = \sigma(f)$ if $\infty \notin C$, or $A \cup B \cup C \cup C\text{Nei} = \sigma^*(f)$ otherwise. (Note that a cluster point need not to be an element of the set.)



2. Representation of Indefinite Integrals in terms of Residues

2.1. Notations, definitions and several Lemmas

Definition 4.

$$\int_a^b f(x)dx$$

denotes the integral of $f(x)$ from a to b along a straight line on the complex plane.

Definition 5. $\mathrm{Log}_\theta(z)$ *denotes* $\ln z$ *with* $\arg(z) \in [\theta, \theta + 2\pi)$.

Definition 6. *For any* $z \in \mathbb{C}, \hat{z} := \frac{z}{|z|}, z^* := |z| \exp(-i \arg(z))$

Lemma 7.

$$|\ln z| \leq \sqrt{2}|\ln |z|| \quad \forall z : |\arg(z)| \leq |\ln |z||$$

Proof.

$$|\ln z| = \sqrt{\ln^2 |z| + \arg^2(z)} \leq \sqrt{2\ln^2 |z|} = \sqrt{2}|\ln |z||$$

Q.E.D. ☐

Lemma 8. *If* $|f(z)| \in O(|z|^{-(1+\epsilon)})$ *as* $|z| \to \infty$ *for any* $\epsilon > 0$*, then for any constant* s*,*

$$\lim_{R\to\infty} \int_{C(R)} f(z)\ln(z-s)dz = 0$$

where $C(R) = s + Re^{ia}, \alpha \leq a - \theta_0 \leq \beta$*, and* $\arg(z-s) \in [\theta_0, \theta_0 + 2\pi)$*.* $(\alpha, \beta$ *are constants and* $0 < \alpha < \beta < 2\pi$*.)*

Proof. As $R \to \infty$, $|z| = |s + Re^{ia}| \sim R \to \infty$ as well.

$$\begin{aligned}
\left|\lim_{R\to\infty} \int_{C(R)} f(z)\ln(z-s)dz\right| &= \lim_{R\to\infty} \left|\int_{\theta_0+\alpha}^{\theta_0+\beta} f(s+Re^{ia})\ln(Re^{ia})iRe^{ia}da\right| \\
&\leq \lim_{R\to\infty} \int_{\theta_0+\alpha}^{\theta_0+\beta} |f(s+Re^{ia})||\ln(Re^{ia})|Rda \\
&\leq \lim_{R\to\infty} \int_{\theta_0+\alpha}^{\theta_0+\beta} \frac{C}{R^{1+\epsilon}}|\ln(Re^{ia})|Rda \\
&\leq \lim_{R\to\infty} \int_{\theta_0+\alpha}^{\theta_0+\beta} \frac{C}{R^{\epsilon}} \cdot (\sqrt{2}|\ln R|)da \qquad \text{by Lemma 7} \\
&\leq \lim_{R\to\infty} \sqrt{2}(\beta-\alpha)C \cdot \frac{|\ln R|}{R^{\epsilon}} \\
&= 0
\end{aligned}$$

for some positive constant C. Q.E.D. ☐

Lemma 9. *For any constant* s*, if* $|f(z)| \in O(1)$ *as* $|z| \to s$*,*

$$\lim_{r\to 0^+} \int_{C(r)} f(z)\ln(z-s)dz = 0$$

where $C(r) = s + re^{ia}, \alpha \leq a - \theta_0 \leq \beta$*, and* $\arg(z-s) \in [\theta_0, \theta_0 + 2\pi)$*.* $(\alpha, \beta$ *are constants and* $0 < \alpha < \beta < 2\pi$*.)*

Proof.

$$
\begin{aligned}
\left|\lim_{r\to 0^+}\int_{C(r)} f(z)\ln(z-s)dz\right| &= \lim_{r\to 0^+}\left|\int_{\theta_0+\alpha}^{\theta_0+\beta} f(s+Re^{ia})\ln(re^{ia})ire^{ia}da\right| \\
&\le \lim_{r\to 0^+}\int_{\theta_0+\alpha}^{\theta_0+\beta} |f(s+re^{ia})||\ln(re^{ia})|rda \\
&\le \lim_{r\to 0^+}\int_{\theta_0+\alpha}^{\theta_0+\beta} C\cdot(\sqrt{2}|\ln r|)rda \qquad \text{by Lemma 7} \\
&\le \lim_{r\to 0^+}\sqrt{2}(\beta-\alpha)C\cdot r|\ln r| \\
&= 0
\end{aligned}
$$

for some positive constant C. Q.E.D. □

Lemma 10. *For a holomorphic function $f(z)$ on the straight line connecting a and b, $(\beta := b-a)$*

$$\int_a^b f(t)dt = \lim_{R\to\infty}\int_{\frac{1}{\beta}}^{R\hat{\beta}^*} \frac{f\left(\frac{1}{u}+a\right)}{u^2}du$$

if f is analytic at a.

Proof.

$$\int_a^b f(t)dt = \int_0^\beta f(t+a)dt = \int_{\frac{1}{R}\hat{\beta}}^{\beta} f(t+a)dt + \int_0^{\frac{1}{R}\hat{\beta}} f(t+a)dt \quad \forall R>0$$

Since

$$\left|\left(\int_a^b f(t)dt\right) - \left(\int_{\frac{1}{R}\hat{\beta}}^{\beta} f(t+a)dt + \int_0^{\frac{1}{R}\hat{\beta}} f(t+a)dt\right)\right| = 0 < \epsilon$$

for every $\epsilon > 0$ and for any $R > \frac{1}{\epsilon}$ (indeed for any $R > 0$), by the ϵ-δ definition of limit

$$\int_a^b f(t)dt = \lim_{R\to\infty}\int_{\frac{1}{R}\hat{\beta}}^{\beta} f(t+a)dt + \lim_{R\to\infty}\int_0^{\frac{1}{R}\hat{\beta}} f(t+a)dt \tag{1}$$

Enforcing $u = \frac{1}{t}$ on the first integral,

$$\int_{\frac{1}{R}\hat{\beta}}^{\beta} f(t+a)dt = \int_{\frac{1}{\beta}}^{R\hat{\beta}^*} f\left(\frac{1}{u}+a\right)\frac{du}{u^2}$$

Since f is analytic at a, near $z = a, |f(z)| \le M$ for some constant M. Then, for the second term, by estimation lemma

$$\left|\lim_{R\to\infty}\int_0^{\frac{1}{R}\hat{\beta}} f(t)dt\right| \le \lim_{R\to\infty} M\cdot\frac{1}{R} = 0$$

As a result, by (1),

$$\int_a^b f(t)dt = \lim_{R\to\infty} \int_{\frac{1}{\beta}}^{R\hat{\beta}^*} \frac{f\left(\frac{1}{u}+a\right)}{u^2} du$$

Q.E.D. □

Lemma 11. *Take* $\ln(z)$ *as the logarithm with* $\arg z \in [a, a+2\pi)$. *Let* $f(z)$ *be a function that is holomorphic on* $z = te^{i\theta}, t \in [0,\infty), a-k < \theta < a+k$ *for any arbitrarily small* k.

Let $\gamma_1(h) = s + he^{i(a+\delta)}, h \in [p,q]$. *Let* $\gamma_2(h) = s + he^{i(2\pi+a-\delta)}, h \in [q,p]$. *(*$p, q \in \mathbb{R}^+$*,* s *is a constant.) Then*

$$\lim_{\delta\to 0^+} \left(\int_{\gamma_1} + \int_{\gamma_2}\right) f(z)\ln(z-s)dz = -2\pi i \int_{pe^{ia}}^{qe^{ia}} f(s+\eta)d\eta$$

Proof.

$$\begin{aligned}
&\lim_{\delta\to 0^+} \int_{\gamma_1} f(z)\ln(z-s)dz \\
&= \lim_{\delta\to 0^+} \int_{\gamma_1} f(z)\ln|z-s|dz + \lim_{\delta\to 0^+} \int_{\gamma_1} f(z) i\arg(z-s)dz \\
&= \lim_{\delta\to 0^+} \int_{\gamma_1} f(z)\ln|z-s|dz + \lim_{\delta\to 0^+} i(a+\delta) \int_{\gamma_1} f(z)dz \\
&= \int_p^q \lim_{\delta\to 0^+} f(s+he^{i(a+\delta)})(\ln h)e^{i(a+\delta)}dh + ia \int_p^q \lim_{\delta\to 0^+} f(s+he^{i(a+\delta)})e^{i(a+\delta)}dh \\
&= \int_p^q f(s+he^{ia})(\ln h)e^{ia}dh + ia\int_p^q f(s+he^{ia})e^{ia}dh
\end{aligned}$$

The exchange of limit and the integral signs is justified by dominated convergence theorem: the assumed holomorphicity of f on the branch cut and its vicinity assures $|f(s+he^{i(a+\delta)})| < M$ for sufficiently small δ, for some positive constant M.

Similarly,

$$\begin{aligned}
&\lim_{\delta\to 0^+}\int_{\gamma_2} f(z)\ln(z-s)dz\\
&=\lim_{\delta\to 0^+}\int_{\gamma_2} f(z)\ln|z-s|dz+\lim_{\delta\to 0^+}\int_{\gamma_2} f(z)i\arg|z-s|dz\\
&=\lim_{\delta\to 0^+}\int_{\gamma_2} f(z)\ln|z-s|dz+\lim_{\delta\to 0^+} i(2\pi+a-\delta)\int_{\gamma_2} f(z)dz\\
&=\int_q^p \lim_{\delta\to 0^+} f(s+he^{i(2\pi+a-\delta)})(\ln h)e^{i(2\pi+a-\delta)}dh\\
&\quad+i(2\pi+a)\int_q^p \lim_{\delta\to 0^+} f(s+he^{i(2\pi+a-\delta)})e^{i(2\pi+a-\delta)}dh\\
&=-\int_p^q f(s+he^{ia})(\ln h)e^{ia}dh-i(2\pi+a)\int_p^q f(s+he^{ia})e^{ia}dh
\end{aligned}$$

Note that, from line 3 to line 4, it is assumed that $f(s+he^{i(2\pi+a)})=f(s+he^{ia})$ due to the presumed holomorphicity of f on branch cut.

Clearly,

$$\lim_{\delta\to 0^+}\left(\int_{\gamma_1}+\int_{\gamma_2}\right)f(z)\ln(z-s)dz=-2\pi i e^{ia}\int_p^q f(s+he^{ia})dh$$

With a substitution $\eta=he^{ia}$,

$$\lim_{\delta\to 0^+}\left(\int_{\gamma_1}+\int_{\gamma_2}\right)f(z)\ln(z-s)dz=-2\pi i\int_{pe^{ia}}^{qe^{ia}} f(s+\eta)d\eta$$

Q.E.D. □

2.2. Representation of indefinite integrals in terms of residues

Theorem 12. *For $f(z)\operatorname{Log}_\theta\left(z-\frac{1}{\zeta-c}\right)$, suppose its $B=C=\emptyset$. Then,*

$$\int_c^\zeta f(t)dt=-\sum_{s\in\sigma^*(f)}\operatorname{Res}_{z=1/(s-c)}\frac{f\left(\frac{1}{z}+c\right)\operatorname{Log}_\theta\left(z-\frac{1}{\zeta-c}\right)}{z^2}$$

where $\zeta\neq c,\theta:=-\arg(\zeta-c)$.

Proof. Let A of $f(x)$ be $\{s_k\}_{n\geq k\geq 1}$ ($s_1\equiv\infty$, see **Note**).

Let $J(\zeta)=\int_c^\zeta f(t)dt$, and $c,\zeta\in\mathbb{C}$.

Let $w=\zeta-c$. Let $\theta=\arg(\hat{\omega}^*)$.

By Lemma 10,

$$J(\zeta) = \lim_{R\to\infty} \int_{1/w}^{R\hat{\omega}^*} \frac{f\left(\frac{1}{u}+c\right)}{u^2} du$$

Consider the contour integral

$$\oint_{C(\epsilon,\delta,R)} \frac{f\left(\frac{1}{z}+c\right)\ln(z-\frac{1}{\omega})}{z^2} dz$$

where $C(\epsilon, \delta, R)$ is composed of four parts:

1. $\gamma_1(t) = 1/\omega + te^{i(\theta+\delta)}, t : [\epsilon, R]$
2. $\gamma_2(t) = 1/\omega + Re^{it}, t : [\theta+\delta, \theta+2\pi-\delta]$
3. $\gamma_3(t) = 1/\omega + te^{i(\theta+2\pi-\delta)}, t : [R, \epsilon]$
4. $\gamma_4(t) = 1/\omega + \epsilon e^{it}, t : [\theta+2\pi-\delta, \theta+\delta]$

Restrict $\arg(z-\frac{1}{\omega}) \in [\theta, \theta+2\pi)$.

Using the same notations as in Lemma 11, set $s = \frac{1}{\omega}, a = \theta, p = \epsilon$ and $q = R$;

$$\begin{aligned}
&\lim_{R\to\infty}\lim_{\epsilon\to 0^+}\lim_{\delta\to 0^+}\left(\int_{\gamma_1}+\int_{\gamma_2}\right)\frac{f(\frac{1}{z}+c)\ln(z-\frac{1}{\omega})}{z^2}dz \\
&= -2\pi i \lim_{R\to\infty}\lim_{\epsilon\to 0^+}\int_{\epsilon\hat{\omega}^*}^{R\hat{\omega}^*}\frac{f\left(\frac{1}{1/\omega+t}+c\right)}{(1/\omega+t)^2}dt \\
&= -2\pi i \lim_{R\to\infty}\lim_{\epsilon\to 0^+}\int_{\epsilon\hat{\omega}^*+1/\omega}^{R\hat{\omega}^*+1/\omega}\frac{f\left(\frac{1}{t}+c\right)}{t^2}dt \\
&= -2\pi i \lim_{R\to\infty}\int_{1/\omega}^{R\hat{\omega}^*+1/\omega}\frac{f\left(\frac{1}{t}+c\right)}{t^2}dt \\
&= -2\pi i \lim_{R\to\infty}\int_{1/\omega}^{R\hat{\omega}^*}\frac{f\left(\frac{1}{t}+c\right)}{t^2}dt \\
&= -2\pi i J(\zeta)
\end{aligned}$$

For the integral over γ_2, as $|z| \to \infty$,

$$|f(\frac{1}{z}+c)| \in O(1) \text{ and thus } \left|\frac{f(1/z+c)}{z^2}\right| \in O(|z|^{-2})$$

By Lemma 8, the integral over γ_2 vanishes as $R \to \infty$.

For the integral over γ_4, as assumed f is analytic at ζ, thus $f(\frac{1}{z}+c)$ is analytic at $z = \frac{1}{\omega}$; also, $\frac{1}{z^2}$ is analytic at $z = \frac{1}{w}$ because $\frac{1}{\omega} \neq 0$. As a result, $\left|\frac{f(1/z+c)}{z^2}\right|$ is bounded near $z = \frac{1}{\omega}$. By Lemma 9, the integral over γ_4 vanishes as $\epsilon \to 0^+$.

Thus,

$$\lim_{R\to\infty}\lim_{\epsilon\to 0^+}\lim_{\delta\to 0^+}\oint_{C(\epsilon,\delta,R)}\frac{f\left(\frac{1}{z}+c\right)\ln\left(z-\frac{1}{\omega}\right)}{z^2}dz=-2\pi i\cdot J(\zeta)$$

Note that as f has isolated singularities at s_k, thus $f(\frac{1}{z}+c)$ is not analytic at isolated points $z=\frac{1}{s_k-c}$. Under the limits, all singularities are included by the contour.

Therefore, by residue theorem, we

$$\lim_{R\to\infty}\lim_{\epsilon\to 0^+}\lim_{\delta\to 0^+}\oint_{C(\epsilon,\delta,R)}\frac{f\left(\frac{1}{z}+c\right)\ln\left(z-\frac{1}{\omega}\right)}{z^2}dz$$
$$=2\pi i\sum_{k=1}^{n}\mathrm{Res}_{z=1/(s_k-c)}\frac{f\left(\frac{1}{z}+c\right)\ln(z-\frac{1}{\omega})}{z^2}$$

As a result,

$$\int_c^\zeta f(t)dt=-\sum_{k=1}^{n}\mathrm{Res}_{z=1/(s_k-c)}\frac{f(\frac{1}{z}+c)\,\mathrm{Log}_\theta\left(z-\frac{1}{\zeta-c}\right)}{z^2}$$

Rewriting with a variation of notations,

$$\int_c^\zeta f(t)dt=-\sum_{s\in\sigma^*(f)}\mathrm{Res}_{z=1/(s-c)}\frac{f(\frac{1}{z}+c)\,\mathrm{Log}_\theta\left(z-\frac{1}{\zeta-c}\right)}{z^2}$$

Q.E.D. □

Note Although ∞ might not be a singularity of f, s_1 is still defined to be ∞. This is because 0 is likely to be a singularity of $\frac{f(1/z+c)}{z^2}$, so one should always evaluate $\mathrm{Res}_0\frac{f(1/z+c)}{z^2}$. For $\frac{1}{s_1-c}=0$, the corresponding value s_1 should be ∞.

Proposition 13. *Theorem 12 applies to indefinite integrals in a natural fashion. By Fundamental Theorem of Calculus,*

$$\int f(\zeta)d\zeta=-\sum_{s\in\sigma^*(f)}\mathrm{Res}_{z=1/(s-c)}\frac{f\left(\frac{1}{z}+c\right)\mathrm{Log}_\theta\left(z-\frac{1}{\zeta-c}\right)}{z^2}$$

with suitable assumption on f.

Not ideally, there is an explicit dependence on $\arg(\zeta-c)$. We shall try to remove it.

The principal logarithm (Log, or in this context, Log_0) can be related to Log_θ by

$$\mathrm{Log}\,z=\mathrm{Log}_\theta\,z-2\pi i\left\lfloor\frac{\Im\,\mathrm{Log}_\theta(z)}{2\pi}\right\rfloor$$

Therefore,

$$\int f(\zeta)d\zeta = -\sum_{s\in\sigma^*(f)} \mathrm{Res}_{z=1/(s-c)} \frac{f\left(\frac{1}{z}+c\right)\mathrm{Log}\left(z-\frac{1}{\zeta-c}\right)}{z^2} + C(\zeta)$$

where

$$C(\zeta) = -2\pi i \sum_{s\in\sigma^*(f)} \mathrm{Res}_{z=1/(s-c)} \frac{f\left(\frac{1}{z}+c\right)}{z^2} \left\lfloor \frac{\Im \mathrm{Log}_\theta(z-\frac{1}{\zeta-c})}{2\pi} \right\rfloor$$

However, $\left\lfloor \frac{\Im\,\mathrm{Log}_\theta(z-\frac{1}{\zeta-c})}{2\pi} \right\rfloor$ is a constant (derivative w.r.t. ζ is zero) almost everywhere. Thus, $C(\zeta)$ is a constant almost everywhere, and can be regarded as an integration constant, which can be omitted. As a result, a less rigorous formula is given

$$\int f(\zeta)d\zeta = -\sum_{s\in\sigma^*(f)} \mathrm{Res}_{z=1/(s-c)} \frac{f(\frac{1}{z}+c)\,\mathrm{Log}\left(z-\frac{1}{\zeta-c}\right)}{z^2}$$

2.3. Examples of application of Theorem 12 to find antiderivatives

1. $\int x^a dx$ where $a \neq 1$ In general, x^a cannot be meromorphically extended to the whole complex plane. Enforcing the substitution $x = e^y$ yields

$$\int x^a dx = \int e^{(a+1)y} dy$$

Choosing $c = 0$ and applying Theorem 12,

$$\int e^{(a+1)y} dy = \int_0^y e^{(a+1)t} dt = \mathrm{Res}_{z=0} \frac{e^{(a+1)/z} \cdot \ln\left(z - \frac{1}{y}\right)}{z^2}$$

By series expansion,

$$\frac{e^{(a+1)/z} \cdot \ln\left(z-\frac{1}{y}\right)}{z^2} = \frac{1}{z^2}\sum_{p=0}^{\infty} \frac{(a+1)^p}{p! z^p} \left(\ln\frac{-1}{y} - \sum_{q=1}^{\infty} \frac{y^q}{q} z^q\right)$$

Collecting the coefficient z^{-1} terms,

$$[z^{-1}] = -\sum_{p=0}^{\infty} \frac{(a+1)^p}{p!}\frac{y^{1+p}}{1+p} = -\frac{e^{(a+1)y}}{a+1} + 1 = -\frac{x^{(a+1)}}{a+1} + 1$$

Thus,

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C$$

2. $\int \tan\zeta d\zeta$

$\tan x$ has infinitely many poles on the complex plane, which is the case Theorem 12 cannot deal with. Enforcing $\zeta = \arctan x$ yields

$$\int \tan\zeta d\zeta = \int \frac{x}{1+x^2}dx$$

By Theorem 12 (choose $c = 0$),

$$\int \frac{x}{1+x^2}dx = -\sum \mathrm{Res}\frac{\frac{1/z}{1+z^{-2}}\ln(z-\frac{1}{x})}{z^2}$$

The poles are at $0, \pm i$.

$$\mathrm{Res}_{-i} = \lim_{z\to -i}(z+i)\frac{\ln(z-\frac{1}{x})}{z(z^2+1)} = -\frac{1}{2}\ln(-i-\frac{1}{x})$$

$$\mathrm{Res}_{i} = \lim_{z\to i}(z-i)\frac{\ln(z-\frac{1}{x})}{z(z^2+1)} = -\frac{1}{2}\ln(i-\frac{1}{x})$$

$$\mathrm{Res}_{0} = \lim_{z\to 0} z\cdot\frac{\ln(z-\frac{1}{x})}{z(z^2+1)} = \ln(-\frac{1}{x})$$

The sum of residues is

$$\begin{aligned}
&-\frac{1}{2}\ln(-i-\frac{1}{x}) - \frac{1}{2}\ln(i-\frac{1}{x}) + \ln(-\frac{1}{x}) \\
&= -\frac{1}{2}\ln(\frac{1}{x^2}+1) - \frac{1}{2}\ln(x^2) \\
&= -\frac{1}{2}\ln(1+x^2) \\
&= \ln\sqrt{1+x^2} \\
&= -\ln\sqrt{1+\tan^2\zeta} \\
&= -\ln\sec\zeta
\end{aligned}$$

Thus,

$$\int \tan\zeta d\zeta = \ln\sec\zeta + C$$

Note that the sloppy manipulation of logarithm is allowed, for the reasons as discussed in Proposition 13.

3. $\displaystyle\int \frac{d\zeta}{(\zeta-1)(\zeta-2)(\zeta-3)}$

Take $c = 0$. Let $g(z) = \frac{f(1/z)}{z^2}\ln\left(z-\frac{1}{\zeta}\right)$.

We have $\sigma^*(f) = \{1, 2, 3, \infty\}$

By noting that for some arbitrary functions F (has simple pole at k) and G (analytic at k),

$$\mathrm{Res}_{z=k}\frac{F(1/z)}{z^2}G(z) = -G(k)\mathrm{Res}_{z=1/k}f(z)$$

effortlessly we have

$$\begin{aligned}
\mathrm{Res}_1 g(z) &= -\frac{1}{2}\ln\left(1-\frac{1}{\zeta}\right)\\
\mathrm{Res}_{1/2} g(z) &= \ln\left(\frac{1}{2}-\frac{1}{\zeta}\right)\\
\mathrm{Res}_{1/3} g(z) &= -\frac{1}{2}\ln\left(\frac{1}{3}-\frac{1}{\zeta}\right)\\
\mathrm{Res}_0 g(z) &= 0
\end{aligned}$$

As a result,

$$\begin{aligned}
\int \frac{d\zeta}{(\zeta-1)(\zeta-2)(\zeta-3)} &= -(\mathrm{Res}_1 g(z) + \mathrm{Res}_{1/2} g(z) + \mathrm{Res}_{1/3} g(z) + \mathrm{Res}_0 g(z))\\
&= \frac{1}{2}\ln\left(3-\frac{4}{3\zeta}+\frac{1}{\zeta^2}\right) - \frac{1}{2}\ln\left(4-\frac{4}{\zeta}+\frac{1}{\zeta^2}\right)\\
&= \frac{1}{2}\ln\frac{\frac{1}{3}(\zeta^2-4\zeta+3)}{\frac{1}{4}(\zeta^2-4\zeta+4)}\\
&= C+\frac{1}{2}\ln\left(1-\frac{1}{(\zeta-2)^2}\right)
\end{aligned}$$

which agrees with the integral calculator. This method avoids partial fraction decomposition.

From this chapter, we obtain a representation of indefinite integrals in terms of residues. This theorem has significant application: calculation of residues only requires differentiation (for poles) and, for the worst case, manipulation of Taylor series (for essential singularities); this theorem provides a method to find antiderivatives by residues, whose computation does not require any techniques of integration.

In Theorem 12, the assumptions on the distribution of singularities are strict. We shall try to relax these conditions and generalize the theorem.

3. Tools for $C \neq \emptyset$

In this chapter, we present a method to evaluate contour integrals around elements in C.

Let $f(z) := g(z)\ln(z-s)$ with $k \in C$.

3.1. Some related notations and definitions

Definition 14. *Let γ_n be a circular curve centered at k, contained in* Nei_k*, and enclosing* $\{ring_{k,n}, ring_{k,n+1}, \dots\}$ *only.*

Then, define

$$\lambda_n(f(z),k) = \frac{1}{2\pi i}\oint_{\gamma_n} f(z)dz$$
$$\lambda_\infty(f(z),k) = \lim_{n\to\infty}\frac{1}{2\pi i}\oint_{\gamma_n} f(z)dz$$

3.2. Existence of λ_∞

Intuitively, $\lambda_\infty(f(z),k)$ is an infinitely small circular contour integral of $f(z)$ around $z=k$, which may or may not exist.

Its existence shall be discussed.

Lemma 15. *First of all,*

$$\lambda_1(f(z),k) - \lambda_{N+1}(f(z),k) = \sum_{j\in\text{band}(k,N)} \text{Res}_{z=j} f(z)$$

for every $N \geq 1$.

Proof. This is an immediate consequence of residue theorem. □

Lemma 16.

$$\lim_{N\to\infty}\left(\lambda_{N+1}(f(z),k) + \sum_{j\in\text{band}(k,N)} \text{Res}_{z=j} f(z)\right)$$

always exists.

Proof. $\lambda_1(f(z),k)$ obviously always exists because it is an integral of a continuous function over a rectfiable curve.

Rearranging the equation in Lemma 15,

$$\lambda_1(f(z),k) = \lambda_{N+1}(f(z),k) + \sum_{j\in\text{band}(k,N)} \text{Res}_{z=j} f(z)$$

for every N, implying

$$\lambda_1(f(z),k) = \lim_{N\to\infty}\left(\lambda_{N+1}(f(z),k) + \sum_{j\in\text{band}(k,N)} \text{Res}_{z=j} f(z)\right)$$

Q.E.D. □

Lemma 17. $\lambda_\infty(f(z),k)$ *exists iff*

$$\lim_{N\to\infty}\sum_{j\in\text{band}(k,N)} \text{Res}_{z=j} f(z)$$

exists.

Proof. Taking limits on both sides of the equation in Lemma 15,

$$\lambda_1(f(z),k) - \lim_{N\to\infty} \lambda_{N+1}(f(z),k) = \lim_{N\to\infty} \sum_{j\in \mathrm{band}(k,N)} \mathrm{Res}_{z=j} f(z)$$
$$\lambda_1(f(z),k) - \lambda_\infty(f(z),k) = \lim_{N\to\infty} \sum_{j\in \mathrm{band}(k,N)} \mathrm{Res}_{z=j} f(z)$$

Obviously, the existence of $\lambda_\infty(f(z);k)$ is equivalent to the existence of

$$\lim_{N\to\infty} \sum_{j\in \mathrm{band}(k,N)} \mathrm{Res}_{z=j} f(z)$$

Q.E.D. □

3.3. Asymptotic formula for λ_∞

Definition 18. *Let $r \in \mathbb{R}$. If for any fixed $t \in [\theta_j, \theta_{j+1}]\backslash E$,*

$$\lim_{r\to 0^+} r \cdot (f(k+re^{it}) - D(r,t)) = 0 \tag{2}$$

where E is a finite set, then $D(r,t)$ is the angular divergence part of f about k in $[\theta_j, \theta_{j+1}]$, denoted as

$$f \sim D \quad \text{for } (k, \theta_j, \theta_{j+1})$$

Theorem 19. *If $f(z) \sim D(r,t)$ for $(k, \theta, \theta+2\pi)$, then*

$$\lambda_N(f(z),k) = o(1) + r_N e^{i\theta} \int_0^1 D(r_N, 2\pi u + \theta) i e^{2\pi i u} du \quad (N \to \infty)$$

where r_N is the radius of γ_N, which is the path of integration of $\lambda_N(f(z),k)$.

Proof. By parametrization,

$$\begin{aligned} &2\pi i \cdot \lambda_N(f(z),k) \\ &= \int_\theta^{\theta+2\pi} f(k+r_N e^{it}) i r_N e^{it} dt \\ &= \int_\theta^{\theta+2\pi} D(r_N,t) i r_N e^{it} dt + \int_\theta^{\theta+2\pi} (f(k+r_N e^{it}) - D(r_N,t)) i r_N e^{it} dt \end{aligned} \tag{3}$$

For the second integral, we have the estimation ($N \to \infty \implies r_N \to 0$)

$$\begin{aligned}
&\left|\int_{\theta}^{\theta+2\pi}(f(k+r_Ne^{it})-D(r_N,t))ir_Ne^{it}dt\right| \\
&\leq \int_{\theta}^{\theta+2\pi} r_N\left|f(k+r_Ne^{it})-D(r_N,t)\right|dt \\
&= \int r_N|f(k+r_Ne^{it})-D(r_N,t)|d\mu \\
&= \int o(1)d\mu \qquad (4)\\
&= o(1) \qquad (5)
\end{aligned}$$

(4): The integrand is of $o(1)$ except a set of measure zero, which does not make any difference to the integral.

(5): As permitted by dominated convergence theorem, since the integrand is bounded for small r.

Then, by (3) and the estimation

$$\lambda_N(f(z),k) = o(1) + \frac{1}{2\pi i}\int_{\theta}^{\theta+2\pi} D(r_N,t)ir_Ne^{it}dt \quad (N \to \infty)$$

With the substitution $u = \frac{t-\theta}{2\pi}$,

$$\lambda_N(f(z),k) = o(1) + r_Ne^{i\theta}\int_0^1 D(r_N, 2\pi u+\theta)ie^{2\pi iu}du \quad (N \to \infty)$$

Q.E.D. □

3.4. Examples of λ_∞ asymptotic formula

Indeed, Theorem 19 is quite trivial. The theorem basically gives the equivalence between the integral of a certain complicated function, and the integral of simple functions with similar diverging properties. For the purpose of illustration, an example is presented.

Example 20. *Consider* $f(z) = \dfrac{\cot\frac{1}{z}}{z^m}$ *for non-negative integer* $m \neq -1$.

It is commonly known that limits of trigonometric functions to infinity do not exist. However, from the perspective of complex plane, the 'real infinities' are the only ones that have no limit. Mathematically,

$$\lim_{a\to\infty}\cot(ae^{ib}) = -i \quad \forall b \in (0,\pi)$$
$$\lim_{a\to\infty}\cot(ae^{ib}) = i \quad \forall b \in (\pi,2\pi)$$

It can be shown that, due to the above two limits (derivation in appendix for 3.4):

$$f \sim D_0(r,t) := -i(re^{it})^{-m} \quad \textit{for } (0,\pi,2\pi)$$
$$f \sim D_1(r,t) := i(re^{it})^{-m} \quad \textit{for } (0,0,\pi)$$

Let

$$D(r,t) = D_0(r,t)\cdot H_{\pi,2\pi}(t) + D_1(r,t)\cdot H_{0,\pi}(t)$$

where $H_{x,y}(w)$ has value 1 when $y \geq w \geq x$ and 0 everywhere else.

By Theorem 19,

$$\begin{aligned}\lambda_N(f(z),0) &= o(1) + \int_0^{1/2} D_1(r_N, 2\pi t) r_N e^{2\pi i t} dt + \int_{1/2}^{1} D_0(r_N, 2\pi t) r_N e^{2\pi i t} dt \\ &= \frac{2r_N^{1-m}}{\pi(m-1)}(1-(-1)^{m-1}) + o(1)\end{aligned}$$

[See reviewer's comment (2a)]

Verification The correctness of this result can be verified by residue theorem. Notice that $2\pi i \lambda_N(f(z),0)$ is equivalent to a clockwise contour integral enclosing all the singularities outside the circle (consider the Riemann sphere). Thus,

$$2\pi i \lambda_N(f(z),0) = -2\pi i \mathrm{Res}_\infty - 2\pi i \sum_{j=-N, j\neq 0}^{N} \mathrm{Res}_{z=1/(j\pi)} f(z)$$

Also,

$$\mathrm{Res}_{z=1/(j\pi)} f(z) = -(j\pi)^{m-2}$$
$$\mathrm{Res}_\infty = \begin{cases} \frac{1}{3}, & \text{m=0} \\ -1, & \text{m=2} \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\lambda_N(f(z),0) = -\mathrm{Res}_\infty + \sum_{j=-N, j\neq 0}^{N} (j\pi)^{m-2}$$

For $m = 0$: Our example gives $\lambda_N(f(z),0) = \frac{1}{2\pi i}(o(1) - \frac{4i}{\pi} r_N) \to 0$.

By residue theorem we have

$$\lambda_N(f(z),0) = -\frac{1}{3} + \sum_{j=-N, j\neq 0}^{N} \frac{1}{(j\pi)^2} \to -\frac{1}{3} + \frac{1}{\pi^2}\cdot 2 \cdot \frac{\pi^2}{6} = 0$$

as expected.

For odd m: Our example gives $\lambda_N(f(z), \text{odd } m) = o(1) + 0 \to 0$.

By residue theorem we have

$$\lambda_N(f(z), \text{odd } m) = 0 + \sum_{j=-N, j\neq 0}^{N} (j\pi)^{m-2} = \sum_{j=1}^{N} (j\pi)^{m-2} + (-j\pi)^{m-2} = 0$$

as expected.

For even m: Our example gives $\lambda_N(f(z), \text{even } m) = o(1) + \frac{4/\pi}{m-1} r_N^{1-m} \to +\infty$ as $N \to \infty \implies r_N \to 0^+$.

By residue theorem we have

$$\lambda_N(f(z), \text{even } m) = -\text{Res}_\infty + \sum_{j=-N, j\neq 0}^{N} (j\pi)^{m-2} = -\text{Res}_\infty + 2\sum_{j=1}^{N} (j\pi)^{m-2} = 0$$

which clearly diverges to $+\infty$ as well.

Appendix for 3.4 Referring to (2),

$$\begin{aligned}
&\lim_{r\to 0^+} (f(re^{it}) - D_0(r,t)) \cdot r \\
&= \lim_{r\to 0^+} \frac{(re^{it})^{-m} \cot 1/(re^{it}) - (-i(re^{it})^{-m})}{1/r} \\
&= e^{-imt} \lim_{r\to 0^+} r^{1-m} (\cot 1/(re^{it})^{-1} + i) \\
&= e^{-imt} \lim_{r\to 0^+} ir^{1-m} \frac{\exp(2i/(re^{it}))}{-1 + \exp(2i/(re^{it}))} \\
&= 0
\end{aligned}$$

for $t \in (\pi, 2\pi)$, since $\lim_{r\to 0^+} \Re \dfrac{2i}{re^{it}} = -\infty$. Similarly for D_1.

Using the asymptotic formula for λ_∞, Theorem 12 in Chapter 2 can be generalized for $C \neq \emptyset$. The generalization will be presented in Chapter 5.

4. Tools for $B \neq \emptyset$

In this chapter, we present a method to evaluate an infinitely small circular contour integral of $f(z) \ln(z - s)$, around an element of its B.

Suppose $p \in B$, and in a punctured neighbourhood of p,

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - p)^j$$

4.1. Notations and definitions

Definition 21. *Define*

$$\kappa(f(z)\ln(z-s),p)=\frac{1}{2\pi i}\lim_{r\to 0^+}\mathrm{PV}\oint_{|z-p|=r}f(z)\ln(z-s)dz$$

An extended definition of residue, which is also defined for points on branch cuts, is adopted and will be utilized later:

Definition 22. *Suppose* $\ln(z-s)$ *is defined with* $\arg(z-s)\in[\theta,\theta+2\pi)$*. Then,*

$$\mathrm{Res}_{z=p}f(z)\ln(z-s):=a_{-1}(\ln|p-s|+i\theta)-\sum_{n=1}^{\infty}\frac{a_{-1-n}}{n(s-p)^n}$$

4.2. Evaluation of κ in closed form

In this section, we will derive the expression of $\kappa(f(z)\ln(z-s),p)$ in closed form.

Lemma 23. *If* $|f_j(x)|\leq g_j$ *and* $\sum_{j=-\infty}^{\infty}g_j$ *converges, then for a real* c*,*

$$\lim_{x\to c}\sum_{j=-\infty}^{\infty}f_j(x)=\sum_{j=-\infty}^{\infty}\lim_{x\to c}f_j(x)$$

Proof. Let μ be the counting measure on $\mathbb{Z}$.

$$S(x):=\sum_{j=-\infty}^{\infty}f_j(x)=\int f(x)d\mu$$

By the substitution $u=\frac{1}{x-c^-}$,

$$\lim_{x\to c}S(x)=\lim_{u\to\infty}S\left(\frac{1}{u}+c^-\right)=\lim_{u\to\infty}\int f\left(\frac{1}{u}+c^-\right)d\mu$$

By dominated convergence theorem,

$$\lim_{x\to c}S(x)=\lim_{u\to\infty}\int f\left(\frac{1}{u}+c^-\right)d\mu=\int\lim_{u\to\infty}f\left(\frac{1}{u}+c^-\right)d\mu$$

Further simplifying,

$$\int\lim_{u\to\infty}f\left(\frac{1}{u}+c^-\right)d\mu=\int\lim_{x\to c^-}f(x)d\mu=\int\lim_{x\to c}f(x)d\mu=\sum_{j=-\infty}^{\infty}\lim_{x\to c}f_j(x)$$

Q.E.D. □

Next, a useful, but not elegant, formula is presented as a corollary, with derivation as its proof.

Corollary 24. *Suppose* $\ln(z-s) = Log_\theta(z-s)$ *(refer to Definition 5). Let, for integer* k *and* $|p-s| > r > 0$*, define*

$$J(k,r) = i \lim_{\delta\to 0^+}\left(\int_{\theta+\delta}^{\theta+\pi-\delta} + \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)(re^{it})^{k+1}\ln(p+re^{it}-s)dt$$

Then,

1. *When* $k > -1, J(k,r) = 0$.
2. *When* $k = -1$, $\lim_{r\to 0^+} J(k,r) = 2\pi i \ln|p-s| - 2\pi\theta - 2\pi^2$.
3. *When* $k < -1$ *and* k *is odd,*
$$J(k,r) = \frac{2\pi i}{(k+1)(s-p)^{-1-k}}$$
4. *When* $k < -1$ *and* k *is even,*
$$J(k,r) = \frac{4\pi i(re^{i\theta})^{k+1}}{k+1} + \frac{2\pi i}{(k+1)(s-p)^{-1-k}}$$

Proof. For $k \neq -1$, integrating by parts,

$$\begin{aligned} &J(k,r)\\ &= i\lim_{\delta\to 0^+}\left(\frac{(re^{it})^{k+1}}{i(k+1)}\ln(p+re^{it}-s)\Big|_{\theta+\delta}^{\theta+\pi-\delta} + \frac{(re^{it})^{k+1}}{i(k+1)}\ln(p+re^{it}-s)\Big|_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)\\ &\quad - i\lim_{\delta\to 0^+}\left(\int_{\theta+\delta}^{\theta+\pi-\delta} + \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)\frac{(re^{it})^{k+1}}{i(k+1)}\frac{ire^{it}}{p+re^{it}-s}dt \end{aligned}$$

Let

1. $L_1 = \lim_{\delta\to 0^+}\ln(p+re^{i(\theta+\delta)}-s)$
2. $L_2 = \lim_{\delta\to 0^+}\ln(p+re^{i(\theta+\pi-\delta)}-s)$
3. $L_3 = \lim_{\delta\to 0^+}\ln(p+re^{i(\theta+\pi+\delta)}-s)$
4. $L_4 = \lim_{\delta\to 0^+}\ln(p+re^{i(\theta+2\pi-\delta)}-s)$

Then,

$$\begin{aligned} J(k,r) &= \frac{(re^{i\theta})^{k+1}}{k+1}((-1)^{k+1}L_2 - L_1 + L_4 - (-1)^{k+1}L_3)\\ &\quad - \frac{ir^{k+2}}{k+1}\int_\theta^{\theta+2\pi}\frac{e^{it(k+2)}}{p+re^{it}-s}dt \end{aligned}$$

Since $L_2 - L_3 = -2\pi i$ and $L_4 - L_1 = 2\pi i$, together with geometric series expansion,

$$J(k,r) = \frac{(re^{i\theta})^{k+1}}{k+1}\cdot 2\pi i(1+(-1)^k) - \frac{ir^{k+2}}{(k+1)(p-s)}\int_\theta^{\theta+2\pi}\sum_{j=0}^{\infty}\frac{e^{it(k+2)}r^j e^{itj}}{(s-p)^j}dt$$

By Fubini's theorem, the summation and integral can be interchanged, and thus it is simplified to

$$J(k,r) = \frac{(re^{i\theta})^{k+1}}{k+1} \cdot 2\pi i(1+(-1)^k) - \frac{ir^{k+2}}{k+1} \sum_{j=0}^{\infty} \frac{r^j}{(s-p)^{j+1}} \int_{\theta}^{\theta+2\pi} e^{it(k+2+j)} dt$$

Clearly, only with $k+2+j=0$ the integral would not be zero.

As a result, $J(k,r)=0$ for $k \geq 0$. For $k \leq -2$,

$$J(k,r) = \frac{(re^{i\theta})^{k+1}}{k+1} \cdot 2\pi i(1+(-1)^k) + \frac{ir^{k+2}}{k+1} \frac{r^{-2-k} \cdot 2\pi}{(s-p)^{-1-k}}$$

$$J(k,r) = \frac{(re^{i\theta})^{k+1}}{k+1} \cdot 2\pi i(1+(-1)^k) + \frac{2\pi i}{(k+1)(s-p)^{-1-k}}$$

Hence, the statements 1, 2 and 4 are proved.

For $k=-1$,

$$\begin{aligned}
J(k,r) &= i \lim_{\delta\to 0^+} \left(\int_{\theta+\delta}^{\theta+\pi-\delta} + \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta} \right) \ln(p+re^{it}-s)dt \\
&= i \lim_{\delta\to 0^+} \left(\int_{\theta+\delta}^{\theta+\pi-\delta} + \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta} \right) \ln|p+re^{it}-s|dt \\
&\quad - \lim_{\delta\to 0^+} \left(\int_{\theta+\delta}^{\theta+\pi-\delta} + \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta} \right) \arg|p+re^{it}-s|dt \\
&= i \int_{\theta}^{\theta+2\pi} \ln|p+re^{it}-s|dt \\
&\quad - \lim_{\delta\to 0^+} \left(\int_{\theta+\delta}^{\theta+\pi-\delta} + \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta} \right) \arg|p+re^{it}-s|dt
\end{aligned}$$

Then,

$$\begin{aligned}
\lim_{r\to 0^+} J(k,r) = i \lim_{r\to 0^+} &\int_{\theta}^{\theta+2\pi} \ln|p+re^{it}-s|dt \\
&- \lim_{r\to 0^+} \lim_{\delta\to 0^+} \int_{\theta+\delta}^{\theta+\pi-\delta} \arg(p+re^{it}-s)dt \\
&- \lim_{r\to 0^+} \lim_{\delta\to 0^+} \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta} \arg(p+re^{it}-s)dt
\end{aligned}$$

For the first integral on the right hand side, clearly the exchange of limit and integral is allowed by dominated convergence theorem,

$$\lim_{r\to 0^+} \int_{\theta}^{\theta+2\pi} \ln|p+re^{it}-s|dt = \int_{\theta}^{\theta+2\pi} \lim_{r\to 0^+} \ln|p+re^{it}-s|dt = 2\pi \ln|p-s|$$

For the second integral, by mean value theorem for integrals, there exists

$$\theta + \delta < c_1 < \theta + \pi - \delta$$

such that

$$\lim_{r\to 0^+} \lim_{\delta\to 0^+} \int_{\theta+\delta}^{\theta+\pi-\delta} \arg(p + re^{it} - s)dt = \lim_{r\to 0^+} \lim_{\delta\to 0^+} (\pi - 2\delta) \arg(p + re^{ic_1} - s) = \pi\theta$$

Similarly, for the third integral, there exists

$$\theta + \pi + \delta < c_2 < \theta + 2\pi - \delta$$

such that

$$\begin{aligned} \lim_{r\to 0^+} \lim_{\delta\to 0^+} \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta} \arg(p + re^{it} - s)dt &= \lim_{r\to 0^+} \lim_{\delta\to 0^+} (\pi - 2\delta) \arg(p + re^{ic_2} - s) \\ &= \pi(2\pi + \theta) \end{aligned}$$

Therefore,

$$\lim_{r\to 0^+} J(-1, r) = 2\pi i \ln|p - s| - 2\pi\theta - 2\pi^2$$

which proves the second statement as well.

Q.E.D. □

Theorem 25. *If $a_{-2j} = 0$ for every $j > 0$,*

$$\kappa(f(z)\ln(z - s), p) = \pi i \mathrm{Res}_{z=p} f(z) + \mathrm{Res}_{z=p} f(z)\ln(z - s)$$

Otherwise, $\kappa(f(z)\ln(z - s), p)$ does not exist.

[See reviewer's comment (2b)]

Proof. By parametrization and utilizing the Laurent series of $f(z)$ around $z = p$,

$$\begin{aligned} &2\pi i \cdot \kappa(f(z)\ln(z - s), p) \\ &= \lim_{r\to 0^+} \lim_{\delta\to 0^+} \left(\int_{\theta+\delta}^{\theta+\pi-\delta} + \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta} \right) \sum_{j=-\infty}^{\infty} a_j (re^{it})^{j+1} \ln(p + re^{it} - s) i dt \end{aligned}$$

Since

$$\begin{aligned}
&\left(\int_{\theta+\delta}^{\theta+\pi-\delta}+\int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)\sum_{j=-\infty}^{\infty}\left|a_j(re^{it})^{j+1}\ln(p+re^{it}-s)i\right|dt\\
&=\left(\int_{\theta+\delta}^{\theta+\pi-\delta}+\int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)\sum_{j=-\infty}^{\infty}|a_j|r^{j+1}\left|\ln(p+re^{it}-s)\right|dt\\
&=\sum_{j=-\infty}^{\infty}\left(\int_{\theta+\delta}^{\theta+\pi-\delta}+\int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)|a_j|r^{j+1}\left|\ln(p+re^{it}-s)\right|dt \qquad (6)\\
&=\sum_{j=-\infty}^{\infty}|a_j|\,r^{j+1}\left(\int_{\theta+\delta}^{\theta+\pi-\delta}+\int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)|\ln(p+re^{it}-s)|dt\\
&=\left(\sum_{j=-\infty}^{\infty}|a_j|\,r^{j+1}\right)\cdot\left(\int_{\theta+\delta}^{\theta+\pi-\delta}+\int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)|\ln(p+re^{it}-s)|dt
\end{aligned}$$

(6): The exchange of sum and integral is permitted by Tonelli's theorem.

The sum above obviously converges, due to the absolute convergence of Laurent series. The integral above converges as well: $|\ln(p+re^{it}-s)|$ might be discontinuous, nevertheless it is bounded, thus convergence is easily established by estimation lemma.

Therefore, by Fubini's theorem,

$$\begin{aligned}
&2\pi i\cdot\kappa(f(z)\ln(z-s),p)\\
&=\lim_{r\to0^+}\lim_{\delta\to0^+}\sum_{j=-\infty}^{\infty}\left(\int_{\theta+\delta}^{\theta+\pi-\delta}+\int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)a_j(re^{it})^{j+1}\ln(p+re^{it}-s)idt
\end{aligned}$$

Easily, we can obtain an inequality about the absolute value of the summand:

$$\begin{aligned}
&\left|\left(\int_{\theta+\delta}^{\theta+\pi-\delta}+\int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)a_j(re^{it})^{j+1}\ln(p+re^{it}-s)idt\right|\\
&\le\left(\int_{\theta+\delta}^{\theta+\pi-\delta}+\int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)|a_j(re^{it})^{j+1}\ln(p+re^{it}-s)i|dt\\
&=\left(\int_{\theta+\delta}^{\theta+\pi-\delta}+\int_{\theta+\pi+\delta}^{\theta+2\pi-\delta}\right)|a_j|r^{j+1}|\ln(p+re^{it}-s)|dt\\
&\le|a_j|r^{j+1}\cdot M
\end{aligned}$$

for some positive constant M.

Also, $\sum_{j=-\infty}^{\infty} |a_j| r^{j+1} \cdot M$ converges by the absolute convergence of Laurent series. By Lemma 23,

$$\begin{aligned}
&2\pi i \cdot \kappa(f(z)\ln(z-s), p) \\
&= \lim_{r\to 0^+} \sum_{j=-\infty}^{\infty} \lim_{\delta\to 0^+} \left(\int_{\theta+\delta}^{\theta+\pi-\delta} + \int_{\theta+\pi+\delta}^{\theta+2\pi-\delta} \right) a_j (re^{it})^{j+1} \ln(p + re^{it} - s) i dt \\
&= \lim_{r\to 0^+} \sum_{j=-\infty}^{\infty} a_j J(j,r)
\end{aligned}$$

(the same notations as in the statement of Corollary 24 is used.)

By Corollary 24, the sum can be rewritten into

$$\begin{aligned}
&2\pi i \cdot \kappa(f(z)\ln(z-s), p) \\
&= \lim_{r\to 0^+} \sum_{j=-\infty}^{-2} a_j J(j,r) + a_{-1} \lim_{r\to 0^+} J(-1,r) \\
&= \lim_{r\to 0^+} \left(\sum_{j=-\infty}^{-1} a_{2j} J(2j,r) + \sum_{j=-\infty}^{-1} a_{2j-1} J(2j-1,r) \right) + a_{-1} \lim_{r\to 0^+} J(-1,r) \\
&= \lim_{r\to 0^+} \left(\sum_{j=-\infty}^{-1} \frac{4\pi i a_{2j} (re^{i\theta})^{2j+1}}{2j+1} + \frac{2\pi i a_{2j}}{(2j+1)(s-p)^{-1-2j}} + \sum_{j=-\infty}^{-1} \frac{2\pi i a_{2j-1}}{(2j)(s-p)^{-2j}} \right) \\
&\quad + a_{-1}(2\pi i \ln|p-s| - 2\pi\theta - 2\pi^2) \\
&= \lim_{r\to 0^+} \left(4\pi i \sum_{j=-\infty}^{-1} \frac{a_{2j}(re^{i\theta})^{2j+1}}{2j+1} + 2\pi i \sum_{j=-\infty}^{-1} \frac{a_{2j}}{(2j+1)(s-p)^{-1-2j}} \right. \\
&\quad \left. + 2\pi i \sum_{j=-\infty}^{-1} \frac{a_{2j-1}}{(2j)(s-p)^{-2j}} \right) + a_{-1}(2\pi i \ln|p-s| - 2\pi\theta - 2\pi^2) \\
&= \lim_{r\to 0^+} \left(4\pi i \sum_{j=-\infty}^{-1} \frac{a_{2j}(re^{i\theta})^{2j+1}}{2j+1} + 2\pi i \sum_{j=-\infty}^{-2} \frac{a_j}{(j+1)(s-p)^{-1-j}} \right) \\
&\quad + a_{-1}(2\pi i \ln|p-s| - 2\pi\theta - 2\pi^2) \\
&= \lim_{r\to 0^+} \left(4\pi i \sum_{j=-\infty}^{-1} \frac{a_{2j}(re^{i\theta})^{2j+1}}{2j+1} \right) + 2\pi i \sum_{j=1}^{\infty} \frac{a_{-1-j}}{-j(s-p)^j} \\
&\quad + a_{-1}(2\pi i \ln|p-s| - 2\pi\theta - 2\pi^2)
\end{aligned}$$

The limit exists only when $a_{-2} = a_{-4} = a_{-6} = \cdots = 0$. ($f(z)$ is *locally odd* at p.)

If the limit exists, we have

$$\begin{aligned}\kappa(f(z)\ln(z-s),p) &= \pi i a_{-1} + \mathrm{Res}_{z=p} f(z)\ln(z-s)\\ &= \pi i \mathrm{Res}_{z=p} f(z) + \mathrm{Res}_{z=p} f(z)\ln(z-s)\end{aligned}$$

Note that the extended definition of residue (Definition 22) is adopted.

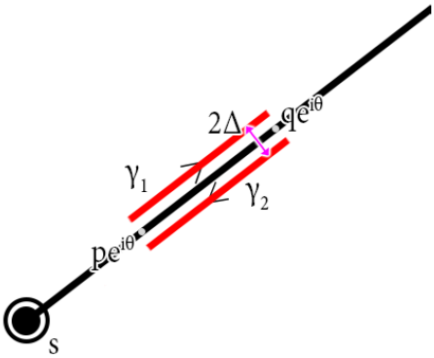
Q.E.D. □

Using Theorem 5, Theorem 9 in Chapter 2 can be generalized for $B \neq \emptyset$. The generalization will be presented in the next chapter.

5. Generalization of Representation of Indefinite Integrals in terms of Residues

With the tools in Chapter 3 and 4, Theorem 12 readily generalizes for functions with some anomalous distribution of singularities.

5.1. The generalization



Lemma 26.

$$\lim_{\Delta\to 0^+}\left(\int_{\gamma_1}+\int_{\gamma_2}\right) f(z)\ln(z-s)dz = -2\pi i\int_{pe^{i\theta}}^{qe^{i\theta}} f(t)dt$$

[See reviewer's comment (2c)]

Proof. Let $\hat{k} = i\hat{s}$.

$$
\begin{aligned}
&\lim_{\Delta\to 0^+}\left(\int_{\gamma_1}+\int_{\gamma_2}\right) f(z)\ln(z-s)dz \\
&= \lim_{\Delta\to 0^+}\left(\int_{\gamma_1}+\int_{\gamma_2}\right) f(z)\ln|z-s|dz + i\lim_{\Delta\to 0^+}\left(\int_{\gamma_1}+\int_{\gamma_2}\right) f(z)\arg(z-s)dz \\
&= \left(\int_{pe^{i\theta}}^{qe^{i\theta}}+\int_{qe^{i\theta}}^{pe^{i\theta}}\right) f(z)\ln|z-s|dz \\
&\quad + i\lim_{\Delta\to 0^+}\int_{pe^{i\theta}+\Delta\hat{k}}^{qe^{i\theta}+\Delta\hat{k}} f(z)\arg(z-s)dz + i\lim_{\Delta\to 0^+}\int_{pe^{i\theta}-\Delta\hat{k}}^{qe^{i\theta}-\Delta\hat{k}} f(z)\arg(z-s)dz
\end{aligned}
$$

Obviously the first term is zero.

For the second term, by the substitution $z = ue^{i\theta} + \Delta\hat{k}$,

$$
\begin{aligned}
& i\lim_{\Delta\to 0^+}\int_{pe^{i\theta}+\Delta\hat{k}}^{qe^{i\theta}+\Delta\hat{k}} f(z)\arg(z-s)dz \\
&= i\lim_{\Delta\to 0^+}\int_p^q f(ue^{i\theta}+\Delta\hat{k})\arg(ue^{i\theta}+\Delta\hat{k}-s)e^{i\theta}du \\
&= i\int_p^q f(ue^{i\theta})\theta e^{i\theta}du \\
&= i\theta\int_{pe^{i\theta}}^{qe^{i\theta}} f(t)dt
\end{aligned}
$$

From the second line to the third line, dominated convergence theorem is applied to exchange limit and integral, and $\lim_{\Delta\to 0^+}\arg(ue^{i\theta}+\Delta\hat{k}-s)=\theta$ is used.

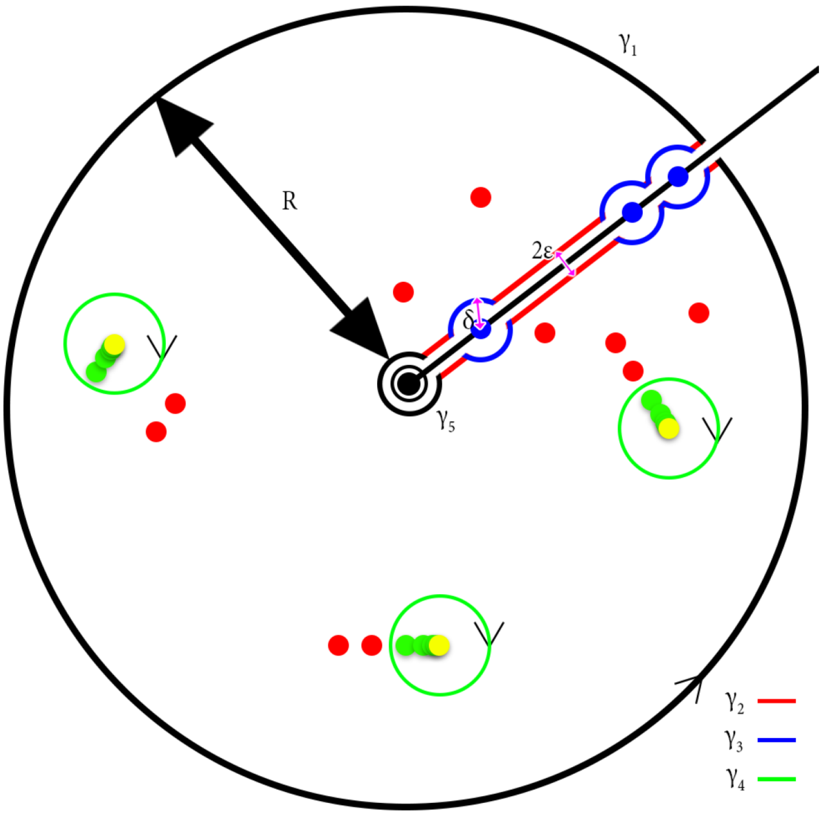
For the third term, by the substitution $z = ue^{i\theta} - \Delta\hat{k}$

$$
\begin{aligned}
& i\lim_{\Delta\to 0^+}\int_{pe^{i\theta}-\Delta\hat{k}}^{qe^{i\theta}-\Delta\hat{k}} f(z)\arg(z-s)dz \\
&= i\lim_{\Delta\to 0^+}\int_p^q f(ue^{i\theta}-\Delta\hat{k})\arg(ue^{i\theta}-\Delta\hat{k}-s)e^{i\theta}du \\
&= -i\int_p^q f(ue^{i\theta})(2\pi+\theta)e^{i\theta}du \\
&= -i(2\pi+\theta)\int_{pe^{i\theta}}^{qe^{i\theta}} f(t)dt
\end{aligned}
$$

Similarly, $\lim_{\Delta\to 0^+}\arg(ue^{i\theta}-\Delta\hat{k}-s)=2\pi+\theta$ is used. As a result,

$$\begin{aligned}&\lim_{\Delta\to 0^+}\left(\int_{\gamma_1}+\int_{\gamma_2}\right)f(z)\ln(z-s)dz\\&=0+i\theta\int_{pe^{i\theta}}^{qe^{i\theta}}f(t)dt-i(2\pi+\theta)\int_{pe^{i\theta}}^{qe^{i\theta}}f(t)dt\\&=-2\pi i\int_{pe^{i\theta}}^{qe^{i\theta}}f(t)dt\end{aligned}$$

Q.E.D. □



Theorem 27. *Let* $g(z):=\dfrac{f(\frac{1}{z}+c)\ln(z-\frac{1}{\omega})}{z^2}$, *where* $\arg(z-\frac{1}{\omega})\in[\arg\hat{\omega}^*,\arg\hat{\omega}^*+2\pi)$.

Suppose $f(z)$ *is locally odd at every element in its* B.

For every $s \in C$, let $D_s(x,y) \sim g(z)$ for $(s, 0, 2\pi)$. Then,

$$\begin{aligned}
&-\mathrm{PV}\int_c^\zeta f(t)dt \\
&= \sum_{s\in A\cup B} \mathrm{Res}_{z=s} g(z) + \pi i \sum_{s\in B} \mathrm{Res}_{z=s} \frac{f(\frac{1}{z}+c)}{z^2} \\
&\quad + \sum_{s\in C} \lim_{N\to\infty} \left(r_N e^{i\theta} \int_0^1 D(r_N, 2\pi u + \theta) i e^{2\pi i u} du + \sum_{j\in \mathrm{band}(s,N)} \mathrm{Res}_{z=j} g(z) \right)
\end{aligned}$$

where $\zeta \neq c$.

[See reviewer's comment (2d)]

Proof. For $f(t)$ with finitely many singularities on the straight line connecting ζ and c, $(\omega := \zeta - c)$

$$J(\zeta) := \mathrm{PV}\int_c^\zeta f(t)dt = \mathrm{PV}\lim_{R\to\infty}\int_{\frac{1}{\omega}}^{R\hat{\omega}^*} \frac{f(\frac{1}{u}+c)}{u^2} du$$

Let $\mathbb{D}(c,R) = \{z \mid |z-c| \leq R\}$.

Let $\mathbb{L}(\epsilon, R)$ be the rectangle with vertices $\frac{1}{\omega} \pm i\epsilon\hat{\omega}^*$, $\frac{1}{\omega} + (R \pm i\epsilon)\hat{\omega}^*$.

Let

$$\mathbb{U}(\epsilon, R) = \mathbb{D}\left(\frac{1}{\omega}, R\right) \backslash \mathbb{L}(\epsilon, R) \backslash \bigcup_{s\in B} \mathbb{D}(s,\delta) \backslash \bigcup_{s\in C} \mathrm{Nei}_s \backslash \mathbb{D}\left(\frac{1}{\omega}, \delta\right)$$

Let $\Gamma(\epsilon, \delta, R) = \partial\mathbb{U}(\epsilon, R)$. (To visualize the contour, refer to the image above.)

By residue theorem,

$$\lim_{\delta\to 0^+}\lim_{\epsilon\to 0^+}\lim_{R\to\infty}\oint_\Gamma g(z)dz = 2\pi i \sum_{s\in A} \mathrm{Res}_{z=s} g(z) \tag{7}$$

Also,

$$\oint_\Gamma = \int_{\gamma_1(R)} + \int_{\gamma_2(\epsilon,\delta,R)} + \int_{\gamma_3(\delta)} + \int_{\gamma_4} + \int_{\gamma_5(\delta)}$$

For the first integral,

$$\lim_{R\to\infty}\int_{\gamma_1(R)} g(z)dz = 0$$

by Lemma 8.

For the second integral, we can repeatedly apply Lemma 11 for every pair of segments right next to each other and separated by the branch cut. Clearly

$$\lim_{\delta\to0^+}\lim_{\epsilon\to0^+}\lim_{R\to\infty}\int_{\gamma_2(\epsilon,\delta,R)} g(z)dz = -2\pi i\,\mathrm{PV}\,J(\zeta)$$

For the third integral, by definition

$$\lim_{\delta\to0^+}\int_{\gamma_3(\delta)} g(z)dz = -2\pi i\sum_{s\in B}\kappa(g(z),s)$$

By Theorem 25,

$$\lim_{\delta\to0^+}\int_{\gamma_3(\delta)} g(z)dz = -2\pi i\sum_{s\in B}\left(\pi i\mathrm{Res}_{z=s}\frac{f(\frac{1}{z}+c)}{z^2}+\mathrm{Res}_{z=s}g(z)\right)$$

For the fourth integral,by Lemma 15,

$$\int_{\gamma_4} g(z)dz = -2\pi i\sum_{s\in C}\lim_{N\to\infty}\left(\lambda_{N+1}(g(z),s)+\sum_{j\in\mathrm{band}(s,N)}\mathrm{Res}_{z=j}g(z)\right)$$

By Theorem 19,

$$\begin{aligned}&\int_{\gamma_4} g(z)dz\\ &= -2\pi i\sum_{s\in C}\lim_{N\to\infty}\left(r_Ne^{i\theta}\int_0^1 D(r_N,2\pi u+\theta)ie^{2\pi iu}du+\sum_{j\in\mathrm{band}(s,N)}\mathrm{Res}_{z=j}g(z)\right)\end{aligned}$$

For the fifth integral, by Lemma 9,

$$\lim_{\delta\to0^+}\int_{\gamma_5(\delta)} g(z)dz = 0$$

Assembling everything into (7),

$$\begin{aligned}&-2\pi i\,\mathrm{PV}\,J(\zeta)-2\pi i\sum_{s\in B}\left(\pi i\mathrm{Res}_{z=s}\frac{f(\frac{1}{z}+c)}{z^2}+\mathrm{Res}_{z=s}g(z)\right)\\ &-2\pi i\sum_{s\in C}\lim_{N\to\infty}\left(r_Ne^{i\theta}\int_0^1 D(r_N,2\pi u+\theta)ie^{2\pi iu}du+\sum_{j\in\mathrm{band}(s,N)}\mathrm{Res}_{z=j}g(z)\right)\\ &=2\pi i\sum_{s\in A}\mathrm{Res}_{z=s}g(z)\end{aligned}$$

Further simplifying,

$$\begin{aligned}
&- \mathrm{PV}\, J(\zeta) \\
&= \sum_{s\in A\cup B} \mathrm{Res}_{z=s} g(z) + \pi i \sum_{s\in B} \mathrm{Res}_{z=s} \frac{f(\frac{1}{z}+c)}{z^2} \\
&\quad + \sum_{s\in C} \lim_{N\to\infty} \left(r_N e^{i\theta} \int_0^1 D(r_N, 2\pi u+\theta) i e^{2\pi i u} du + \sum_{j\in \mathrm{band}(s,N)} \mathrm{Res}_{z=j} g(z) \right)
\end{aligned}$$

When $B = C = \emptyset$, this equation reduces to Theorem 12.

Q.E.D. $\square$

5.2. Example

For the purpose of illustration, an example is given.

Example 28.

$$\int \csc\zeta d\zeta$$

Take $c = -\frac{\pi}{2}$. Let

$$g(z) = \frac{\csc(\frac{1}{z} - \frac{\pi}{2})}{z^2} \ln\left(z - \frac{1}{\zeta + \frac{\pi}{2}}\right)$$

From Theorem 27, we can infer that it is not important to determine what elements are in B; what is important is the elements in $A \cup B$.

Firstly,

$$\sigma(g) = \left\{ \frac{\pm 1}{n\pi + \frac{\pi}{2}} \right\}_{n\geq 0}$$

Clearly, $C = \{0\}$. Take $\mathrm{Nei}_0 = \{x \mid |x| \leq r\}$, where $r < \frac{1}{|\zeta - c|}$ is a constant.

Suppose

$$\mathrm{Nei}_0 \cap \sigma(g) = \left\{ \frac{\pm 1}{n\pi + \frac{\pi}{2}} \right\}_{n\geq n_0}$$

Then,

$$\begin{aligned}
\mathrm{ring}_{0,1} &= \left\{ \frac{1}{n_0\pi + \frac{\pi}{2}}, -\frac{1}{n_0\pi + \frac{\pi}{2}} \right\} \\
\mathrm{ring}_{0,2} &= \left\{ \frac{1}{(n_0+1)\pi + \frac{\pi}{2}}, -\frac{1}{(n_0+1)\pi + \frac{\pi}{2}} \right\} \\
&\dots
\end{aligned}$$

and by definition

$$A \cup B = \left\{ \frac{\pm 1}{n\pi + \frac{\pi}{2}} \right\}_{0 \le n \le n_0 - 1}$$

Moreover, surprisingly, $D(r,t) := 0 \sim g$ for $(0,0,2\pi)$.

Therefore, by Theorem 27,

$$-\int \csc\zeta d\zeta = \sum_{s \in A \cup B} \mathrm{Res}_{z=s} g(z) + \lim_{N\to\infty} \sum_{j \in \mathrm{band}(0,N)} \mathrm{Res}_{z=j} g(z)$$

Obviously, this can be rewritten to

$$-\int \csc\zeta d\zeta = \lim_{N\to\infty} \sum_{n=0}^{N-1} \mathrm{Res}\left(g(z), \frac{1}{n\pi + \frac{\pi}{2}}\right) + \mathrm{Res}\left(g(z), \frac{-1}{n\pi + \frac{\pi}{2}}\right)$$

Since

$$\begin{aligned} &\mathrm{Res}\left(g(z), \frac{1}{n\pi + \frac{\pi}{2}}\right) + \mathrm{Res}\left(g(z), \frac{-1}{n\pi + \frac{\pi}{2}}\right) \\ &= (-1)^{n+1} \ln \frac{n\pi - \zeta}{n\pi + \pi + \zeta} \end{aligned}$$

Without loss of generality, let $N = 2k+1$ be an odd number. By induction,

$$-\int \csc\zeta d\zeta = -\lim_{N\to\infty} \ln \frac{-\zeta \prod_{n=1}^{k}((2n\pi)^2 - \zeta^2)}{(N\pi + \zeta)\prod_{n=1}^{k}((2n-1)^2\pi^2 - \zeta^2)}$$

$$\begin{aligned} \int \csc\zeta d\zeta &= \lim_{N\to\infty} \ln -\frac{\left(\prod_{n=1}^{k} \frac{2n}{2n-1}\right)^2}{N\pi + \zeta} \frac{\zeta \prod_{n=1}^{k}\left(1 - \left(\frac{\zeta}{2n\pi}\right)^2\right)}{\prod_{n=1}^{k}\left(1 - \left(\frac{\zeta}{(2n-1)\pi}\right)^2\right)} \\ &= \lim_{N\to\infty} \ln h_N(\zeta) + \ln \frac{\sin\zeta/2}{\cos\zeta/2} \\ &= \ln\tan\frac{\zeta}{2} + \lim_{N\to\infty} \ln h_N(\zeta) \end{aligned}$$

where

$$h_N(\zeta) := -\frac{\left(\prod_{n=1}^{k} \frac{2n}{2n-1}\right)^2}{N\pi + \zeta}$$

It can be shown that

$$\frac{d}{d\zeta} \lim_{N\to\infty} h_N(\zeta) = 0$$

for all ζ, and thus $h_\infty(\zeta)$ can be regarded as a constant.

Thus,

$$\int \csc\zeta d\zeta = \ln\tan\frac{\zeta}{2} + C$$

as expected.

6. Universality

Since the only assumption made when deriving Theorem 27 is $\zeta \neq c$, certainly the left hand side of the equation should converge for every ζ except c, unless ζ is a singularity of f. As mentioned in the abstract, Theorem 27 provides a universal functional form of $\int f(\zeta)d\zeta$.

By partially differentiating both sides of the equation in Theorem 27, with respect to ζ, one obtains the universal functional form of $f(\zeta)$. This gives the analytic continuation of f to the largest possible domain.

Of course, doing such requires knowledge of behaviour of f around every singularity. Thus, this method of analytic continuation is quite lame. For instance, even knowing the integral form of Gamma function on the right half plane, one cannot analytically continue it to the left half plane by Theorem 27, because in no way the behaviour of f can be directly observed from the integral form of Gamma function.

[See reviewer's comment (2e)]

Postface

The tricks used in this report might be a bit out of the standard syllabus of secondary schools. I have been continuously acquiring such knowledge from internet resources about complex analysis, for more than a year. Residue theorem happens to be the most interesting one among the other theorems. As I drilled down, I was surprised that none of the internet resources show an application of residue theorem to indefinite integrals. It turns out that there are some curious, non-trivial results.

A friend asked me after skimming the draft of this report, "Did *you* write it?"

"Yes, of course, *I* wrote it."

REFERENCES

[1] Wikipedia, *Residue theorem*, https://en.wikipedia.org/wiki/Residue_theorem

[See reviewer's comment (2f)]

Reviewer's Comments

This article tried to express indefinite integrals in terms of residues, via residue theorem. Here are the reviewer's comments on the paper:

1. Novelty and methodology: The author seems to have a quite solid background on the topic of complex analysis. The motivation of the paper is interesting, yet the reviewer believes it is not a new idea in finding indefinite integrals using residues. In fact, in Chapter 2, the author first considered definite integral using residues (which is a classical application of residue theorem) and then generalised to the case of indefinite integrals. However, the author was not "truly" computing indefinite integral; he was in fact finding *definite integral* from c to ζ for some specific values of c. More precisely, in Chapter 2.2 Proposition 13, the reviewer feels very puzzled on the following lines: "By Fundamental Theorem of Calculus
$$\int f(\zeta)d\zeta = -\sum_{s\in\sigma^*(f)} \operatorname{Res}_{z=1/(s-c)} \frac{f(\frac{1}{z}+c)\log_\theta(z-\frac{1}{\zeta-c})}{z^2}$$
with suitable assumption on f." What is c in the above formula? It appears to the reviewer that the integral on the left side is still a definite integral (just hiding ζ and c on the left side of the formula). Later in many applications, the author just took $c = 0$ and computed the right side explicitly. Yet what the author did was just computing the definite integral $\int_0^\zeta f(\xi)d\xi$ (the reviewer changed the dummy variable to ξ). A similar thing happened in Chapter 5 when the author gave a formula in terms of c and ζ (refer to Theorem 27 page 75) and then applied the formula on $\int \csc(\zeta)d\zeta$ by choosing $c = -\frac{\pi}{2}$.
2. Organisation: The organisation of the paper is not good, and there are many missing definitions and details in stating theorems and lemmas. The author omitted some details in proving theorems and lemmas as well. Here are some examples:
   (a) When defining $f(z)$, what are the definitions of $g(z)$ and s? And the reviewer cannot find $k \in C$ from the definition of f.
   (b) What are the definitions of those a_j's? is there any condition imposed on them?
   (c) What are the definition and conditions for s?
   (d) What are the definition and conditions for ω? And what is the meaning for $f(z)$ being "locally odd at every element in its B"? And what does it mean for "$(s, 0, 2\pi)$"?
   (e) The author mentioned that "By partially differentiating both sides of the equation in Theorem 27, with respect to ζ, one obtains the universal functional form of $f(\zeta)$. This gives the analytic continuation of f to the largest possible domain."Why and how does it give the analytic continuation of f to the largest possible domain?

(f) The listed references are not sufficient. Is there any other reference rather than "wikipedia" (for example Ahlfors [1])?

REFERENCES

[1] Ahlfors, L.: Complex Analysis, Third Edition. New York: McGraw Hill, 1979.