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ON THE BASEL PROBLEM: GENERALIZATIONS TO OTHER POWER SERIES

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ABSTRACT. The Basel problem is about finding the sum of the reciprocals of all perfect squares. This problem is first posed by Pietro Mengoli in 1650 and was solved by Leonhard Euler in 1734. Euler proved that the sum of the series is $\frac{\pi^2}{6}$. In this report, inspired by an idea suggested by the YouTube channel 3blue1brown in 2018, we attempt to give a new proof to the Basel problem. After that, we discuss some possible generalizations of the Basel problem, by finding the sum of reciprocals of squares and cubes of the form $an+b$. Furthermore, we discuss how the sum of reciprocals of integral powers of $an+b$ can be computed, and the relation between $\zeta(3)$ and the results we have achieved.

KEYWORDS. Basel Problem, Riemann-Zeta Function, Power Series, Complex Numbers, Elementary Symmetric Polynomials, Vieta Formula, Limits, Maclaurin Series, Epsilon-Delta Definition of Limits

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1. Introduction and Main Results

The Basel problem is about finding the sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

This problem is first posed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734. Euler proved that the sum of the series is $\frac{\pi^2}{6}$.
In this report, inspired by an idea suggested by the YouTube channel 3blue1brown in 2018, we attempt to give a new proof to the Basel problem and discuss the generalizations of the Basel problem. With the idea in my new proof, we manage to prove the following results:

Theorem 1.1. *(The Basel problem)*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

Theorem 1.2. *($\zeta(4)$)*

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

Theorem 1.3. *For positive real numbers $a > b$,*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(an+b)^2} = \frac{\pi^2}{a^2 \sin^2\left(\frac{b}{a}\pi\right)}$$

Theorem 1.4. *For positive real numbers $a > b$,*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(an+b)^3} = \frac{\pi^3 \cos\left(\frac{b}{a}\pi\right)}{a^3 \sin^3\left(\frac{b}{a}\pi\right)}$$

Before proceeding to the main results, we provide an overview of the results that are used in this report. As these results are well known, proofs are omitted.

Proposition 1.5. *(Triangle inequality) For any complex numbers $c_1, c_2, \dots, c_n$,*

$$|c_1| + |c_2| + \dots + |c_n| \ge |c_1 + c_2 + \dots + c_n|$$

Proposition 1.6. *(Binomial theorem) For any nonnegative integer n, the polynomial $(x+y)^n$ can be expressed as*

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

Proposition 1.7. *(Vieta's formulas) For any polynomial*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

with roots $r_1, r_2, \dots, r_n$, we have

$$\begin{cases} r_1 + r_2 + \dots + r_{n-1} + r_n = -\dfrac{a_{n-1}}{a_n} \\ (r_1r_2 + r_1r_3 + \dots + r_1r_n) + (r_2r_3 + r_2r_4 + \dots + r_2r_n) + \dots + r_{n-1}r_n = \dfrac{a_{n-2}}{a_n} \\ \vdots \\ r_1r_2\dots r_n = (-1)^n \dfrac{a_0}{a_n} \end{cases}$$

Definition 1.8. *(Maclaurin Series) The Maclaurin series of a real or complex-valued function $f(x)$ that is infinitely differentiable at 0 is the power series*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

where $f^{(n)}(x)$ is the nth derivative of f.

Proposition 1.9. *(Cauchy–Schwarz inequality) For real numbers $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_n$,*

$$\left(a_1^2 + a_2^2 + \dots + a_n^2\right)\left(b_1^2 + b_2^2 + \dots + b_n^2\right) \geq (a_1b_1 + a_2b_2 + \dots + a_nb_n)^2$$

2. A Solution to the Basel Problem

Denote the sum of the infinite series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = C$$

Definition 2.1. *For any set S of several (can be infinitely many) nonzero complex numbers, define*

$$s_2(S) = \sum_{c \in S} \frac{1}{c^2}$$

if it can be calculated.

In particular, if A is the set of nonzero integers, then

$$s_2(A) = 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) = 2C$$

Therefore, to find C, we just need to find $s_2(A)$. Now, how to find $s_2(A)$? A video of 3blue1brown uses the concept of the brightness of lighthouses placed uniformly on a circle, which matches our definition of s_2 by the Inverse Square Law. By using the inverse Pythagors theorem, Grant Sanderson, the creator of this video, finds a family of sets such that the s_2 value of them are all equal to $\dfrac{\pi^2}{4}$, and the sets tends to the set of odd integers as the size of circle tends to infinity. Inspired by the video, we will consider a similar family of sets such that the s_2 value of them tends to $\dfrac{\pi^2}{6}$, and the sets tends to A as the size of circle tends to infinity. To be precise, we define

Definition 2.2. *For any positive integer $k \geq 2$, in the complex plane, let C_k be the circle centred at $\dfrac{ik}{2\pi}$ and passes through 0.*

Definition 2.3. *For any positive integer $k \geq 2$, consider the k-gon inscribed in C_k with vertices arranged counterclockwisely as $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ where the indices are considered* $\bmod k$ *and $\alpha_0 = 0$. Then let $A_k = \{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}$.*

Now we aim to show that as k tends to infinity, $s_2(A_k)$ tends to $s_2(A)$. But before that, we will establish the following results:

Lemma 2.4. *For $1 \leq j \leq k-1$,*

$$\sin(\arg(\alpha_j)) = \frac{|\alpha_j|\pi}{k}$$

Proof. Note that the antipode of point $O = 0$ with respect to C_k is $P = \dfrac{\pi i}{k}$ so the point $A_j = \alpha_j$ satisfies

$$\angle O A_j P = 90^\circ$$

As C_k is tangent to the real axis, we have

$$\sin(\arg(\alpha_j)) = \sin \angle A_j P O = \frac{A_j O}{OP} = \frac{|\alpha_j|\pi}{k}$$

□

Lemma 2.5. *For $k \in \mathbb{N}$,*

$$\sum_{j=1}^{k} \frac{1}{j^2} < 2$$

Proof. We have

$$\sum_{j=1}^{k} \frac{1}{j^2} \leq 1 + \sum_{j=2}^{k} \frac{1}{j(j-1)} = 1 + \sum_{j=2}^{k} \left(\frac{1}{j-1} - \frac{1}{j} \right) = 2 - \frac{1}{k} < 2$$

□

Lemma 2.6. *For $0 < x \leq \dfrac{\pi}{2}$,*

$$\frac{1}{\sin^2 x} - \frac{1}{x^2} \leq 1 - \frac{\pi^2}{4}$$

Proof. Let $f(x) = \dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}$ then $f\left(\dfrac{\pi}{2}\right) = 1 - \dfrac{4}{\pi^2}$ and

$$f'(x) = \frac{2}{x^3} - \frac{2\cos x}{\sin x} = \frac{2}{x^3 \tan x}\left(\tan x - x^3\right)$$

If f is monotonically increasing then we are done. To prove this, it suffices to prove that for $0 \leq x \leq \dfrac{\pi}{2}$,

$$\tan x - x^3 \geq 0$$

Recall that the Maclaurin Series of $\tan x$ is

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \text{terms involving higher powers}$$

So as all coefficients in the series are positive, when $0 \leq x \leq \dfrac{\pi}{2}$,

$$
\begin{aligned}
\tan x - x^3 &> x + \frac{1}{3}x^3 + \frac{2}{15}x^5 - x^3 \\
&= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 \\
&= \frac{2x}{15}\left(\left(x^2 - \frac{5}{2}\right)^2 + \frac{5}{4}\right) \\
&\geq 0
\end{aligned}
$$

as $x \geq 0$. So we are done.

□

Lemma 2.7. *For positive integers m, n and positive real number ϵ such that $m < \frac{n}{2}$ and $n \geq \frac{576\pi^2}{\epsilon^2}, \frac{6\pi^2 - 24}{\epsilon}$, we have*

$$\sum_{-m \leq j \leq m, j \neq 0} \left| \frac{1}{\alpha_j^2} - \frac{1}{j^2} \right| < \frac{\epsilon}{3}$$

Proof. The statement is equivalent to

$$\sum_{1 \leq j \leq m} \left| \frac{1}{\alpha_j^2} - \frac{1}{j^2} \right| < \frac{\epsilon}{6}$$

For any such j, by Cosine Rule and Lemma 2.4 we have

$$
\begin{aligned}
\left| \frac{1}{\alpha_j^2} - \frac{1}{j^2} \right|^2 &= \left| \frac{1}{\alpha_j^2} \right|^2 + \frac{1}{j^4} - 2\left| \frac{1}{\alpha_j^2} \right| \frac{1}{j^2} \cos\left(2 \arg \alpha_j\right) \\
&= \frac{1}{|\alpha_j|^4} + \frac{1}{j^4} - 2\frac{1}{|\alpha_j|^2}\frac{1}{j^2}\left(1 - 2\sin^2\left(\arg \alpha_j\right)\right) \\
&= \left(\frac{1}{|\alpha_j|^2} - \frac{1}{j^2}\right)^2 + 4\frac{1}{|\alpha_j|^2}\frac{1}{j^2}\left(\frac{|\alpha_j|\pi}{n}\right)^2 \\
&= \left(\frac{1}{|\alpha_j|^2} - \frac{1}{j^2}\right)^2 + \frac{4\pi^2}{n^2} \cdot \frac{1}{j^2}
\end{aligned}
$$

So by the Cauchy-Schwarz Inequality and Lemma 2.5,

$$\begin{aligned}\frac{1}{n}\left(\sum_{1\le j\le m}\left|\frac{1}{\alpha_j^2}-\frac{1}{j^2}\right|\right)^2 &< \frac{1}{m}\left(\sum_{1\le j\le m}\left|\frac{1}{\alpha_j^2}-\frac{1}{j^2}\right|\right)^2\\ &\le \sum_{1\le j\le m}\left|\frac{1}{\alpha_j^2}-\frac{1}{j^2}\right|^2\\ &= \sum_{1\le j\le m}\left(\frac{1}{|\alpha_j|^2}-\frac{1}{j^2}\right)^2+\frac{4\pi^2}{n^2}\sum_{1\le j\le m}\frac{1}{j^2}\\ &< \sum_{1\le j\le m}\left(\frac{1}{|\alpha_j|^2}-\frac{1}{j^2}\right)^2+\frac{8\pi^2}{n^2}\end{aligned}$$

Now it suffices to have

$$\sum_{1\le j\le m}\left(\frac{1}{|\alpha_j|^2}-\frac{1}{j^2}\right)^2+\frac{8\pi^2}{n^2}\le\frac{\epsilon^2}{36n}$$

As $n\ge\dfrac{576\pi^2}{\epsilon^2}$, i.e.

$$\frac{8\pi^2}{n^2}\le\frac{\epsilon^2}{72n}$$

it suffices to have

$$\sum_{1\le j\le m}\left(\frac{1}{|\alpha_j|^2}-\frac{1}{j^2}\right)^2\le\frac{\epsilon^2}{72n}$$

As $m<\dfrac{n}{2}$, it suffices to have for all $1\le j\le m$,

$$\left(\frac{1}{|\alpha_j|^2}-\frac{1}{j^2}\right)^2\le\frac{\epsilon^2}{36n^2}$$

i.e. for all $1\le j<\dfrac{n}{2}$,

$$\frac{1}{j^2}-\frac{1}{|\alpha_j|^2}\le\frac{\epsilon}{6n}$$

Note that j is equal to the length of the arc from 0 to α_j along C_n, so zoom C_n back to an unit circle it suffices to have

$$\frac{1}{\sin^2 x}-\frac{1}{x^2}\le\frac{\epsilon}{6\pi^2}n$$

for $0<x\le\dfrac{\pi}{2}$ where $x=\arg(\alpha_j)$. But note that $1-\dfrac{4}{\pi^2}\le\dfrac{\epsilon}{6\pi^2}\cdot n$ so by Lemma 2.6 we are done. □

Lemma 2.8. *For positive integers m, n and positive real number ϵ such that $m < \frac{n}{2}$ and $m \geq \frac{\pi}{2}\sqrt{\frac{3n}{\epsilon}}$, we have*

$$\left|\sum_{m<j<n-m} \frac{1}{\alpha_j^2}\right| < \frac{\epsilon}{3}$$

Proof. First we claim that $\frac{x}{\sin x} \leq \frac{\pi}{2}$ for $0 < x < \frac{\pi}{2}$. If $f(x) = \frac{x}{\sin x}$, then

$$f'(x) = \frac{\tan x - x}{\sin^2 x \cos x}$$

If f is strictly increasing then we are done. To prove this, it suffices to prove that for $0 < x < \frac{\pi}{2}$,

$$\tan x > x$$

Indeed, note that $\tan 0 = 0$ and

$$\frac{d\tan(x)}{dx} = \sec^2 x > 1 = \frac{dx}{dx}$$

for $0 < x < \frac{\pi}{2}$ so $\tan x > x$ for $0 < x < \frac{\pi}{2}$ as desired. This implies that

$$|\alpha_m| = \frac{m}{\frac{\arg(\alpha_m)}{\sin(\arg(\alpha_m))}} \geq \frac{2m}{\pi}$$

Back to the lemma, we have

$$\left|\sum_{m<j<n-m} \frac{1}{\alpha_j^2}\right| \leq n\left|\frac{1}{\alpha_m^2}\right| < \frac{\pi^2 n}{4m^2} \leq \frac{\epsilon}{3}$$

□

Lemma 2.9. *For positive integers m, n and positive real number ϵ such that $m < \frac{n}{2}$ and $m \geq \frac{6}{\epsilon}$, we have*

$$\sum_{j\in\mathbb{Z}, |j|>m} \frac{1}{j^2} < \frac{\epsilon}{3}$$

Proof. It is equivalent to

$$\sum_{j=m+1}^{\infty} \frac{1}{j^2} < \frac{\epsilon}{6}$$

and we have

$$\sum_{j=m+1}^{\infty} \frac{1}{j^2} < \sum_{j=m+1}^{\infty} \frac{1}{j(j-1)} = \sum_{j=m+1}^{\infty} \left(\frac{1}{j-1} - \frac{1}{j}\right) = \frac{1}{m} \leq \frac{\epsilon}{6}$$

□

Lemma 2.10. *When k tends to positive infinity, the number $s_2(A_k)$ tends to $s_2(A)$.*

Proof. Consider any $\epsilon > 0$. It suffices to prove that there exists N such that for $n \geq N$,

$$|s_2(A_n) - s_2(A)| < \epsilon$$

By Lemma 2.7, 2.8, 2.9, when $N = \max\left(\frac{576\pi^2}{\epsilon^2}, \frac{12\pi^2}{\epsilon}, 4\right)$, for any $n \geq N$ there exists $m < \frac{n}{2}$ which satisfies

(i) $\displaystyle\sum_{-m\leq j\leq m, j\neq 0} \left|\frac{1}{\alpha_j^2} - \frac{1}{j^2}\right| < \frac{\epsilon}{3}$;

(ii) $\displaystyle\left|\sum_{m<j<n-m} \frac{1}{\alpha_j^2}\right| < \frac{\epsilon}{3}$;

(iii) $\displaystyle\sum_{j\in\mathbb{Z}, |j|>m} \frac{1}{j^2} < \frac{\epsilon}{3}$;

Then by triangle inequality we have

$$\begin{aligned}|s_2(A_n) - s_2(A)| &\leq \sum_{-m\leq j\leq m, j\neq 0} \left|\frac{1}{\alpha_j^2} - \frac{1}{j^2}\right| + \left|\sum_{m<j<n-m} \frac{1}{\alpha_j^2}\right| + \sum_{j\in\mathbb{Z}, |j|>m} \frac{1}{j^2} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

□

Remark 2.11. *For* $N = \max\left(\frac{576\pi^2}{\epsilon^2}, \frac{6\pi^2 - 24}{\epsilon}, \frac{12\pi^2}{\epsilon}, \frac{24}{\epsilon}, 4\right)$, *the conditions of Lemma 2.7 is satisfied. Then take m to be any integer such that* $\frac{n}{4} \leq m < \frac{n}{2}$ *which exists as* $n \geq 4$. *The conditions of Lemma 2.8, 2.9 are satisfied as* $\frac{n}{4} \geq \frac{\pi}{2}\sqrt{\frac{3n}{\epsilon}}, \frac{6}{\epsilon}$.

Lemma 2.12. *For* $k \geq 2$,

$$s_2(A_k) = \frac{\pi^2(k-1)(k-5)}{3k^2}$$

Proof. For any $c \in A_k \cup \{0\}$, the number

$$\frac{i(c - \frac{ik}{2\pi})}{\frac{k}{2\pi}} = \frac{2\pi i}{k}c + 1$$

is a kth root of unity. So A_k is the set of all roots of the polynomial

$$P_k(x) = \frac{\left(\frac{2\pi i}{k}x + 1\right)^k - 1}{x}$$

Note that $P_k(x)$ have degree $k-1$, so let

$$P_k(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x + a_0$$

Let's find an expression to the coefficients of $P_k(x)$. By Binomial theorem, we have

$$a_t = \binom{k}{t+1}\left(\frac{2\pi i}{k}\right)^{t+1}$$

for $0 \le t \le k-1$. In particular, we have

$$a_0 = 2\pi i$$

$$a_1 = \frac{-2\pi^2(k-1)}{k}$$

$$a_2 = \frac{-4\pi^3 i(k-1)(k-2)}{3k^2}$$

Now the roots $r_1, r_2, \ldots, r_{k-1}$ of

$$Q_k(x) = a_0 x^{k-1} + a_1 x^{k-2} + \cdots + a_{k-2}x + a_{k-1}$$

are the reciprocals of the roots of $P_k(x)$, therefore

$$s_2(A_k) = r_1^2 + r_2^2 + \cdots + r_{k-1}^2$$

The last step is to calculate this expression, which we can use the Vieta's formulas. We have

$$\sum_{1\le t\le k-1} r_t = -\frac{a_1}{a_0} = -\frac{-2\pi^2(k-1)}{2\pi k i} = \frac{-\pi i(k-1)}{k}$$

$$\sum_{1\le t<u\le k-1} r_t r_u = \frac{a_2}{a_0} = \frac{-4\pi^3 i(k-1)(k-2)}{6k^2\pi i} = \frac{-2\pi^2(k-1)(k-2)}{3k^2}$$

So

$$\begin{aligned} s_2(A_k) &= \left(\sum_{t=1}^{k-1} r_t\right)^2 - 2\sum_{1\le t<u\le k-1} r_t r_u \\ &= \left(\frac{-\pi i(k-1)}{k}\right)^2 - 2\left(\frac{-2\pi^2(k-1)(k-2)}{3k^2}\right) \\ &= \frac{-\pi^2(k-1)^2}{k^2} + \frac{4\pi^2(k-1)(k-2)}{3k^2} \\ &= \frac{\pi^2(k-1)}{3k^2}\left(-3(k-1)+4(k-2)\right) \\ &= \frac{\pi^2(k-1)(k-5)}{3k^2} \end{aligned}$$

□

Finally we will prove Theorem 1.1. By Lemma 2.10 and 2.11,

$$s_2(A) = \lim_{k\to\infty} s_2(A_k) = \lim_{k\to\infty} \frac{\pi^2(k-1)(k-5)}{3k^2} = \lim_{k\to\infty} \frac{\pi^2}{3}\left(1-\frac{1}{k}\right)\left(1-\frac{5}{k}\right) = \frac{\pi^2}{3}$$

Therefore we have

$$C = \frac{\pi^2}{6}$$

which completes our proof of Theorem 1.1.

3. Extending to $\zeta(4)$ and Beyond

The Riemann zeta function ζ is defined as following: For any positive integer n,

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \dots$$

Then the Basel problem is equivalent to finding $\zeta(2)$. We have proved that $\zeta(2) = \dfrac{\pi^2}{6}$ in section 2, and in this section we will discuss other values of the Riemann zeta function.

By using the same method of finding $\zeta(2)$, we can find $\zeta(4)$. We will use the notations in Definition 2.2, 2.3. Similar to section 2, we define

Definition 3.1. *For any set S of several (can be infinitely many) nonzero complex numbers, define*

$$s_4(S) = \sum_{c\in S} \frac{1}{c^4}$$

if it can be calculated.

In particular, if A is the set of nonzero integers, then

$$s_4(A) = 2\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right) = 2\zeta(4)$$

Therefore, to find $\zeta(4)$, we just need to find $s_4(A)$.

Remark 3.2. *This method doesn't work for finding odd values of ζ because $s_k(A) = 0$ if k is odd.*

Then using an argument similar to that in Lemma 2.4 to 2.10, we can deduce that

$$s_4(A_k) \to s_4(A) \text{ as } k \to +\infty$$

Lemma 3.3. *For $k \geq 2$,*

$$s_4(A_k) = \frac{\pi^4(k-1)(k^3+k^2-109k+251)}{45k^4}$$

Proof. Using the same argument and notations in Lemma 2.12,

$$s_4(A_k) = r_1^4 + r_2^4 + \dots + r_{k-1}^4$$

So we calculate a_0, a_1, a_2, a_3, a_4. Indeed, we have

$$\begin{aligned}
a_0 &= 2\pi i \\
a_1 &= \frac{-2\pi^2(k-1)}{k} \\
a_2 &= \frac{-4\pi^3 i(k-1)(k-2)}{3k^2} \\
a_3 &= \frac{2\pi^4(k-1)(k-2)(k-3)}{3k^3} \\
a_4 &= \frac{4\pi^5 i(k-1)(k-2)(k-3)(k-4)}{15k^4}
\end{aligned}$$

The last step is to calculate this expression, which can be done with the Vieta's formulas. We have

$$\begin{aligned}
\sum_{1\le t\le k-1} r_t &= -\frac{a_1}{a_0} = \frac{-\pi i(k-1)}{k} \\
\sum_{1\le t<u\le k-1} r_t r_u &= \frac{a_2}{a_0} = \frac{-2\pi^2(k-1)(k-2)}{3k^2}
\end{aligned}$$

$$\begin{aligned}
\sum_{1\le t<u<v\le k-1} r_t r_u r_v &= -\frac{a_3}{a_0} = -\frac{2\pi^4(k-1)(k-2)(k-3)}{6k^3\pi i} \\
&= \frac{\pi^3 i(k-1)(k-2)(k-3)}{3k^3} \\
\sum_{1\le t<u<v<w\le k-1} r_t r_u r_v r_w &= \frac{a_4}{a_0} = \frac{4\pi^5 i(k-1)(k-2)(k-3)(k-4)}{30k^4\pi i} \\
&= \frac{2\pi^4(k-1)(k-2)(k-3)(k-4)}{15k^4}
\end{aligned}$$

So

$$\begin{aligned}
s_4(A_k) &= \left(\sum_{t=1}^{k-1} r_t\right)^4 - 4\left(\sum_{t=1}^{k-1} r_t\right)^2 \left(\sum_{1\le t<u\le k-1} r_t r_u\right) \\
&+ 4\left(\sum_{t=1}^{k-1} r_t\right)\left(\sum_{1\le t<u<v\le k-1} r_t r_u r_v\right) + 2\left(\sum_{1\le t<u\le k-1} r_t r_u\right)^2 \\
&- 4 \sum_{1\le t<u<v<w\le k-1} r_t r_u r_v r_w \\
&= \left(\frac{-\pi i(k-1)}{k}\right)^4 - 4\left(\frac{-\pi i(k-1)}{k}\right)^2 \left(\frac{-2\pi^2(k-1)(k-2)}{3k^2}\right) \\
&+ 4\left(\frac{-\pi i(k-1)}{k}\right)\left(\frac{\pi^3 i(k-1)(k-2)(k-3)}{3k^3}\right) \\
&+ 2\left(\frac{-2\pi^2(k-1)(k-2)}{3k^2}\right)^2 - 4\left(\frac{2\pi^4(k-1)(k-2)(k-3)(k-4)}{15k^4}\right) \\
&= \frac{\pi^4(k-1)^4}{k^4} - \frac{8\pi^4(k-1)^3(k-2)}{3k^4} + \frac{4\pi^4(k-1)^2(k-2)(k-3)}{3k^4} \\
&+ \frac{8\pi^4(k-1)^2(k-2)^2}{9k^4} - \frac{8\pi^4(k-1)(k-2)(k-3)(k-4)}{15k^4} \\
&= \frac{\pi^4(k-1)}{45k^4}[45(k-1)^3 - 120(k-1)^2(k-2) + 60(k-1)(k-2)(k-3) \\
&+ 40(k-1)(k-2)^2 - 24(k-2)(k-3)(k-4)] \\
&= \frac{\pi^4(k-1)(k^3+k^2-109k+251)}{45k^4}
\end{aligned}$$

□

Finally we will prove Theorem 1.2. By Lemma 3.2,

$$\begin{aligned}
s_4(A) = \lim_{k\to\infty} s_4(A_k) &= \lim_{k\to\infty} \frac{\pi^4(k-1)(k^3+k^2-109k+251)}{45k^4} \\
&= \lim_{k\to\infty} \frac{\pi^4}{45}\left(1-\frac{1}{k}\right)\left(1+\frac{1}{k}-\frac{109}{k^2}+\frac{251}{k^3}\right) = \frac{\pi^4}{45}
\end{aligned}$$

Therefore we have

$$\zeta(4) = \frac{\pi^4}{90}$$

which completes our proof of Theorem 1.2.

Using the Vieta's Formulas, we can compute elementary symmetric polynomials of the roots in terms of coefficients of $Q_k(x)$. Since we can express $\sum_{i=1}^{k-1} r_i^{2n}$ in terms of those elementary symmetric polynomials, we can actually use the similar analysis argument to compute $\zeta(2n)$ for any $n \in \mathbb{N}$. However, the complexity of such computation is extremely high, so we skipped this part and move on to other types of power series.

4. More Power Series

In section 2, we actually found the value of

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} = \frac{\pi^2}{3}$$

and in section 3, we discussed what will happen when the power changes. Now we see what will happen when the base changes. In this section, we will discuss how to determine

$$\sum_{n\in\mathbb{Z}} \frac{1}{(an+b)^m}$$

in closed form, for any $m \in \mathbb{N}$, and any positive real numbers $a > b$. To begin, we first consider the sum

$$\sum_{n\in\mathbb{Z}} \frac{1}{(an+b)^2}$$

Denote $B = a\mathbb{Z} + b$. We can compute this value in the same method of section 2, as shown below. We will use the same notation as Definition 2.1.

Definition 4.1. *For any positive integer k, in the complex plane, let D_k be the circle centred at $\frac{iak}{2\pi}$ and passes through 0.*

Definition 4.2. *For any positive integer k, consider the regular k-gon inscribed in D_k with vertices arranged counterclockwisely as $\beta_0, \beta_1, \ldots, \beta_{k-1}$ where the indices are considered $\bmod k$ and $\beta_0 = \frac{ak}{2\pi i}\left(\exp\left(\frac{2\pi ib}{ak}\right)\right)$. Then let $B_k = \{\beta_0, \beta_1, \ldots, \beta_{k-1}\}$.*

Then using an argument similar to that in Lemma 2.4 to 2.10, we can deduce that

$$s_2(B_k) \to s_2(B) \text{ as } k \to +\infty$$

Lemma 4.3. *For any k,*

$$s_2(B_k) = \frac{4\pi^2}{a^2}\left(\frac{1}{1-z}\frac{k-1}{k} - \frac{1}{(1-z)^2}\right)$$

where $z = \exp\left(\frac{2\pi ib}{a}\right)$.

Proof. For any $c \in B_k$, the number

$$\frac{i(c - \frac{iak}{2\pi})}{\frac{ak}{2\pi}} = \frac{2\pi i}{ak}c + 1$$

has its kth power equal to $z = \exp\left(\frac{2\pi ib}{a}\right)$. So B_k is the set of all roots of the complex polynomial

$$R_k(x) = \left(\frac{2\pi i}{ak}x + 1\right)^k - z$$

Note that $R_k(x)$ have degree k, so let

$$R_k(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$$

Let's find an expression to the coefficients of $R_k(x)$. By Binomial Theorem, we have

$$a_t = \binom{k}{t}\left(\frac{2\pi i}{ak}\right)^t$$

for $1 \le t \le k-1$ and $a_0 = 1 - z$. In particular, we have

$$\begin{aligned} a_0 &= 1 - z \\ a_1 &= \frac{2\pi i}{a} \\ a_2 &= \frac{-2\pi^2(k-1)}{a^2 k} \end{aligned}$$

Now the roots $r_1, r_2, \ldots, r_k$ of

$$S_k(x) = a_0 x^k + a_1 x^{k-1} + \cdots + a_{k-1}x + a_k$$

are the reciprocals of the roots of $P_k(x)$, therefore

$$s_2(B_k) = r_1^2 + r_2^2 + \cdots + r_k^2$$

The last step is to calculate this expression, which we can use the Vieta's formulas. We have

$$\begin{aligned} \sum_{1\le t\le k} r_t &= -\frac{a_1}{a_0} = \frac{2\pi i}{a(1-z)} \\ \sum_{1\le t<u\le k} r_t r_u &= \frac{a_2}{a_0} = \frac{-2\pi^2(k-1)}{a^2 k(1-z)} \end{aligned}$$

So

$$\begin{aligned} s_2(B_k) &= \left(\sum_{t=1}^{k} r_t\right)^2 - 2\sum_{1\le t<u\le k} r_t r_u \\ &= \left(\frac{2\pi i}{a(1-z)}\right)^2 - 2\left(\frac{-2\pi^2(k-1)}{a^2 k(1-z)}\right) \\ &= \frac{4\pi^2}{a^2}\left(\frac{1}{1-z}\frac{k-1}{k} - \frac{1}{(1-z)^2}\right) \end{aligned}$$

□

Finally we will prove Theorem 1.3. By Lemma 4.3,

$$\begin{aligned} s_2(B) = \lim_{k\to\infty} s_2(B_k) &= \lim_{k\to\infty} \frac{4\pi^2}{a^2}\left(\frac{1}{1-z}\frac{k-1}{k} - \frac{1}{(1-z)^2}\right) \\ &= \frac{4\pi^2}{a^2}\left(\frac{1}{1-z} - \frac{1}{(1-z)^2}\right) \\ &= \frac{4\pi^2}{a^2}\frac{z}{(1-z)^2} \end{aligned}$$

To further rewrite $s_2(B)$ in terms of a and b, notice that as $s_2(B)$ is a positive real number, and $|z| = 1$, we have

$$\begin{aligned} s_2(B) = \left|\frac{4\pi^2}{a^2}\frac{z}{(1-z)^2}\right| &= \frac{4\pi^2}{a^2}\frac{1}{|1-z|^2} \\ &= \frac{4\pi^2}{a^2}\frac{1}{4\sin^2\left(\frac{1}{2}\arg(z)\right)} \\ &= \frac{\pi^2}{a^2\sin^2\left(\frac{b}{a}\pi\right)} \end{aligned}$$

which completes our proof of Theorem 1.3.

After settling the case of squares, we turn to compute

$$\sum_{n\in\mathbb{Z}} \frac{1}{(an+b)^3}$$

for positive real numbers $a > b$. First, define s_3 similarly. We will use the same notation as Definition 4.1, 4.2. Then using an argument similar to that in Lemma 2.4 to 2.10, we can deduce that

$$s_3(B_k) \to s_3(B) \text{ as } k \to +\infty$$

Lemma 4.4. *For any k,*

$$s_3(B_k) = \frac{4\pi^3 i}{a^3}\left(\frac{2}{(1-z)^3} - \frac{3}{(1-z)^2}\frac{k-1}{k} + \frac{1}{1-z}\frac{(k-1)(k-2)}{k^2}\right)$$

where $z = \exp\left(\dfrac{2\pi i b}{a}\right)$.

Proof. For any $c \in B_k$, the number

$$\frac{i(c - \frac{iak}{2\pi})}{\frac{ak}{2\pi}} = \frac{2\pi i}{ak}c + 1$$

has its kth power equal to $z = \exp\left(\dfrac{2\pi i b}{a}\right)$. So B_k is the set of all roots of the complex polynomial

$$R_k(x) = \left(\frac{2\pi i}{ak}x + 1\right)^k - z$$

Note that $R_k(x)$ have degree k, so let

$$R_k(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$$

Let's find an expression to the coefficients of $R_k(x)$. By Binomial theorem, we have

$$a_t = \binom{k}{t}\left(\frac{2\pi i}{ak}\right)^t$$

for $1 \le t \le k-1$ and $a_0 = 1 - z$. In particular, we have

$$\begin{aligned}
a_0 &= 1 - z \\
a_1 &= \frac{2\pi i}{a} \\
a_2 &= \frac{-2\pi^2(k-1)}{a^2 k} \\
a_3 &= \frac{-4\pi^3(k-1)(k-2)i}{3a^3k^2}
\end{aligned}$$

Now the roots $r_1, r_2, \ldots, r_k$ of

$$S_k(x) = a_0 x^k + a_1 x^{k-1} + \cdots + a_{k-1}x + a_k$$

are the reciprocals of the roots of $P_k(x)$, therefore

$$s_3(B_k) = r_1^3 + r_2^3 + \cdots + r_k^3$$

The last step is to calculate this expression, which we can use the Vieta's formulas. We have

$$\begin{aligned}
\sum_{1\le t\le k} r_t &= -\frac{a_1}{a_0} = \frac{-2\pi i}{a(1-z)} \\
\sum_{1\le t<u\le k} r_t r_u &= \frac{a_2}{a_0} = \frac{-2\pi^2(k-1)}{a^2k(1-z)} \\
\sum_{1\le t<u<v\le k-1} r_t r_u r_v &= -\frac{a_3}{a_0} = \frac{4\pi^3(k-1)(k-2)i}{3a^3k^2(1-z)}
\end{aligned}$$

So

$$\begin{aligned}
s_3(B_k) &= \left(\sum_{t=1}^{k-1} r_t\right)^3 - 3\left(\sum_{t=1}^{k-1} r_t\right)\left(\sum_{1\le t<u\le k-1} r_t r_u\right) + 3\sum_{1\le t<u<v\le k-1} r_t r_u r_v \\
&= \left(\frac{-2\pi i}{a(1-z)}\right)^3 - 3\left(\frac{-2\pi i}{a(1-z)}\right)\left(\frac{-2\pi^2(k-1)}{a^2k(1-z)}\right) + 3\left(\frac{4\pi^3(k-1)(k-2)i}{3a^3k^2(1-z)}\right) \\
&= \frac{4\pi^3 i}{a^3}\left(\frac{2}{(1-z)^3} - \frac{3}{(1-z)^2}\frac{k-1}{k} + \frac{1}{1-z}\frac{(k-1)(k-2)}{k^2}\right)
\end{aligned}$$

$\square$

Finally we will prove Theorem 1.4. By Lemma 4.4,

$$\begin{aligned} s_3(B) &= \lim_{k\to\infty} s_3(B_k) \\ &= \lim_{k\to\infty} \frac{4\pi^3 i}{a^3}\left(\frac{2}{(1-z)^3} - \frac{3}{(1-z)^2}\frac{k-1}{k} + \frac{1}{1-z}\frac{(k-1)(k-2)}{k^2}\right) \\ &= \frac{4\pi^3 i}{a^3}\left(\frac{2}{(1-z)^3} - \frac{3}{(1-z)^2} + \frac{1}{1-z}\right) \\ &= \frac{4\pi^3 i}{a^3}\frac{z(z+1)}{(1-z)^3} \end{aligned}$$

To further rewrite $s_3(B)$ in terms of a and b, notice that as $s_3(B)$ is a positive real number, and $|z| = 1$, we have

$$\begin{aligned} s_3(B) &= \frac{4\pi^3 i}{a^3}\frac{z(z+1)}{(1-z)^3} \\ &= \left|\frac{4\pi^3 i}{a^3}\frac{z(z+1)}{(1-z)^3}\right| \\ &= \frac{4\pi^3}{a^3}\cdot\frac{|z+1|}{|1-z|^3} \\ &= \frac{4\pi^3}{a^3}\cdot\frac{2\cos\left(\frac{1}{2}\arg(z)\right)}{8\sin^3\left(\frac{1}{2}\arg(z)\right)} \\ &= \frac{\pi^3\cos\left(\frac{b}{a}\pi\right)}{a^3\sin^3\left(\frac{b}{a}\pi\right)} \end{aligned}$$

which completes our proof of Theorem 1.4.

Using the Vieta's Formulas, we can compute elementary symmetric polynomials of the roots in terms of coefficients of $S_k(x)$. Since we can express $\sum_{i=1}^{k-1} r_i^m$ in terms of those elementary symmetric polynomials, we can actually use the similar analysis argument to compute $\sum_{n\in\mathbb{Z}} \frac{1}{(an+b)^m}$ for any $m \in \mathbb{N}$. Different from the result in section 3, we can actually determine the sum in closed form for all powers m, rather than just even powers. This is because under our construction, there is no symmetry about the point 0, so the sum on odd powers will still be valid. However, the complexity of such computation is again extremely high, so this part is omitted as well. Nevertheless, we know such sums are computable with such method.

5. Discussion

In this section we are going to discuss about $\zeta(3)$. The value of $\zeta(3)$ is still unknown, only being proved to be irrational by Roger Apery in 1978. In fact, the result we have achieved towards the end of section 4 has a strong relation with $\zeta(3)$.

For example, if we take $a = 3, b = 1$ in Theorem 1.4, we have

$$\begin{aligned}\frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \ldots &= \sum_{n\in\mathbb{Z}} \frac{1}{(3n+1)^3} \\ &= \frac{\pi^3 \cos\left(\frac{\pi}{3}\right)}{3^3 \sin^3\left(\frac{\pi}{3}\right)} \\ &= \frac{4\sqrt{3}\pi^3}{243}\end{aligned}$$

Let's call this sum S. Then we have

$$\frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \cdots = \frac{S}{8}$$

Summing up these 2 series we have

$$\frac{1}{1^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \cdots = \frac{9S}{8}$$

which can also be computed by taking $a = 6, b = 1$ in Theorem 1.4. So summing up the second and the third series we have

$$\frac{1}{1^3} + \frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} + \cdots = \frac{5S}{4}$$

Also, note that

$$\begin{aligned}&\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{7^3} + \ldots \\ &= \left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3}\right) - \left(\frac{1}{3^3} + \frac{1}{6^3} + \frac{1}{9^3} + \frac{1}{12^3} + \frac{1}{15^3}\right) \\ &= \zeta(3) - \frac{1}{27}\zeta(3) \\ &= \frac{26}{27}\zeta(3)\end{aligned}$$

Then

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{13^3} + \cdots = \frac{1}{2}\left(\frac{5S}{4} + \frac{26}{27}\zeta(3)\right)$$

so actually if we can find the value of $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{13^3} + \ldots$ then we can find $\zeta(3)$.

However, the current mechanism only allows us to compute reciprocal of cubes with alternating signs (either repetitions of $++--$, $+-+-$, or their reverse), and does not allow us to compute such sums with all terms having positive signs. Another mechanism is required.

References

[1] Grant Sanderson (3blue1brown) (2018) *Why is pi here? And why is it squared? A geometric answer to the Basel problem* https://youtu.be/d-o3eB9sfls

REVIEWERS' COMMENTS

This paper studies the exact sums of infinite series of the form

$$\sum_{n\in A} \frac{1}{(an+b)^m}$$

for various choices of the set A (such as $\mathbb{N}$), integer $m \geq 2$, and real numbers a and b. The author was able to discover a geometric interpretation of the above infinite sum (motivated by a YouTube video), namely replacing it by the sum of equidistributed points over a large circle centered at $ik/(2\pi)$ for some large $k \geq 1$. The author was able to recover classical results such as the case $A = \mathbb{N}$, $a = 1$, $b = 0$ and m is even. Reviewers also suggested more proper acknowledgements is needed when citing references.