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Honorable Mention

The Erdős-Szekeres Conjecture
("Happy End Problem")

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THE ERDŐS-SZEKERES CONJECTURE ("HAPPY END PROBLEM")

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Abstract. The survey [1] conducted by W. Morris and V. Soltan mentioned that in 1935 Erdős-Szekeres proved that for any integer $n \geq 3$, there exists a smallest positive integer $g(n)$ points in general position in the plane containing n points that are the vertices of a convex n-gon. [See reviewer's comment (3)] They also conjectured that $g(n) = 2^{n-2} + 1$ for any integer $n \geq 3$. The conjecture is far from being solved for decades though many mathematicians had tried their very best on it. This paper is to investigate the Erdős-Szekeres conjecture by studying the greatest positive integer $f(n)$ points in general position in the plane which contains no convex n-gons. We successfully prove the cases when $n = 4, 5$ i.e. $f(4) = 4$ and $f(5) = 8$. For $n = 6$, we arrive at the conclusion that $f(6) \geq 16$ by creating an example of 16 points containing no convex hexagons. Moreover, we excitedly find an elegant proof for this example that one more point added to it will certainly give birth to a convex hexagon.

1. Background

The Erdős-Szekeres conjecture also known as the "Happy End Problem", has been investigated by mathematicians for several decades. The conjecture is not yet been proved, but some progress is made.

Define $g(n)$ the minimal possible M for a set of M points in general position, where no three points are collinear, must contain a convex n-gon. The conjecture states that $g(n) = 2^{n-2} + 1$.

It has been proved that the equality holds for $n = 3, 4, 5$ mathematically and $g(6) = 17$ has been proved by a computer search [2] in 2006.

Other researches [1] [See reviewer's commnet (7)] show that $g(n)$ is a finite number for all n, and $g(n)$ lies between a certain interval. The most recent interval of $g(n)$ obtained in [1] is

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-3} + 2.$$

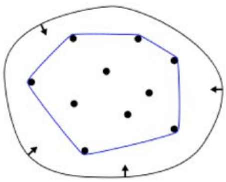
2. Introduction

Throughout this paper, all point set are assumed to be in general position, where no three points are collinear.

Through this investigation, we know that it is very hard to prove the conjecture by exhaustion since there are so many possibilities. Therefore, we try to find out some properties of the point sets forming no convex quadrilaterals, pentagons, hexagons and even heptagons. We carry out our investigation in three stages.

Definition 1. *A polygon $P_1P_2\ldots P_n$ is* convex *iff it contains all line segments P_iP_j connecting any two vertices i.e. for any distinct i, j, $P_iP_j \subset P_1P_2\ldots P_n$.*

Definition 2. *The* Convex Boundary $\Gamma(X)$ *of a point set X is the subset of X whose points are vertices of a convex polygon containing all the remaining points in X.*



By this concept, we can give the *Configuration* of any point set X by taking the convex boundary $\Gamma(X)$ of X, $\Gamma(\Gamma(X)), \ldots, \Gamma(\Gamma(\Gamma(\ldots(X))))$ and their corresponding numbers of elements in order. [See reviewer's comment (4)]
For example, the configuration of the point set on the right is $(6, 4)$. Also we can redefine the convex polygons in terms of convex boundary. Let $X = \{P_1, P_2, \ldots P_n\}$. A polygon $P_1P_2\ldots P_n$ is *convex* iff P_i lies outside the convex polygon formed by the points in $\Gamma(X \setminus P_i)$ for any $i = 1, 2, \ldots n$.

3. $g(4) = 5$

In this stage, we investigate the point sets without convex quadrilaterals. By definition, $g(3) = 3$ obviously. We are going to prove the Theorem 3.

Theorem 3. *Any five points must contain a convex quadrilateral, i.e. $g(4) = 5$.*

Proof. Since all the point sets of configuration $(3,1)$ form no convex quadrilateral, we consider point sets of configuration of $(3,2)$ i.e. with two points D, E inside a triangle ABC. DE must cut two of the sides of ABC. WLOG, assume that DE cuts AB and AC and then B, D, E and C forms a convex quadrilateral. Therefore $g(4) = 5$. □

4. $g(5) = 9$

In this stage, we start with a quadrilateral and use a software called Geometer's Sketchpad to add points one by one. We then shade off those regions giving a convex pentagon and find that no point can be further added to any eight points in general position. Moreover, we find an example of eight points without convex pentagons in [1]. So we are curious whether the example is unique and try out our own proof to it. Now we need some definitions and lemmas.

Definition 4. *The* ray $[A, B)$ *is defined as the set containing all points lying on the line segment AB or AB produced.*

Definition 5. *The* line (A, B) *is defined as the line joining the points A and B.*

Definition 6. *For any three points A, B, C,* beam $A : BC$ *denotes the set of all points in the interior of the region bounded by the segment BC, AB produced and AC produced.*

Definition 7. *For a convex quadrilateral ABCD,* beam $AB : CD$ *denotes the set of all points in the interior of the region bounded by the segment CD, AD produced and BC produced.*

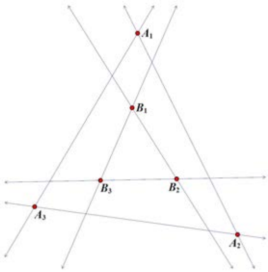
[See reviewer's commnet (5)]

Lemma 8. *Any eight points of* $(3, 3, 2)$ *must contain a convex pentagon.*

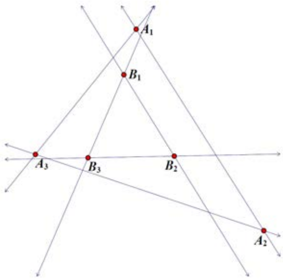
Proof. [See reviewer's commnet (6)] Let $X = \{A_1, A_2, A_3, B_1, B_2, B_3, D, E\}$ be a set of any 8 points, where $A_1, A_2, A_3 \in \Gamma(X)$, $B_1, B_2, B_3 \in \Gamma(\Gamma(X))$ and $D, E \in \Gamma(\Gamma(\Gamma(X)))$. The relative positions of A_1, A_2, A_3, B_1, B_2, B_3 can be divided in the 3 cases as shown below. We are now going to prove the lemma case by case.

Case 1 Since D and E are inside $\triangle B_1B_2B_3$, D, $E \notin A_iA_j : B_jB_i$ for any distinct i, j for otherwise D or E will form a convex pentagon with A_i, A_j, B_j and B_i. [See reviewer's comment (8)]
WLOG, let DE cuts B_1B_3 and B_1B_2.
By the similar argument in Theorem 3, D, E, B_2 and B_3 form a convex quadrilateral. Moreover, $A_2 \in DE : B_2B_3$ or $ED : B_2B_3$ and therefore D, E, B_2, A_2 and B_3 form a convex pentagon.

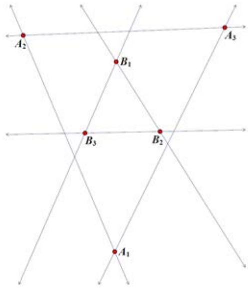
Case 2 [See reviewer's comment (9)] Since D and E are inside $\triangle B_1B_2B_3$, D, $E \notin A_iA_j : B_jB_i$, where (i, j) is neither $(2, 3)$ nor $(3, 2)$ for otherwise D or E will



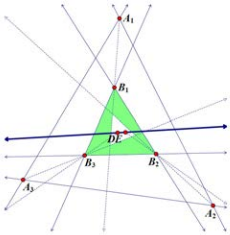
Case 1



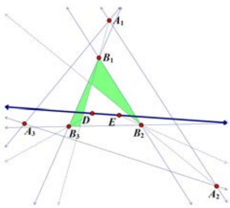
Case 2



Case 3



form a convex pentagon with A_i, A_j, B_j and B_i.
Assume that DE cuts B_1B_3 and B_1B_2.
By the argument in Theorem 3, D, E, B_2 and B_3 form a convex quadrilateral. Moreover, $A_2 \in DE : B_2B_3$ or $ED : B_2B_3$ and therefore D, E, B_2, A_2 and B_3 form a convex pentagon. The other cases can be proved similarly.

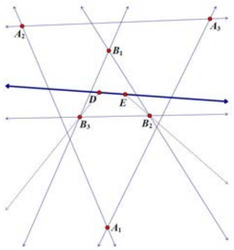


Case 3 WLOG, let DE cuts B_1B_3 and B_1B_2.
By the argument in Theorem 3, D, E, B_2 and B_3 form a convex quadrilateral. Moreover, $A_1 \in DE : B_2B_3$ or $ED : B_2B_3$ and therefore D, E, B_2, A_1 and B_3 form a convex pentagon.

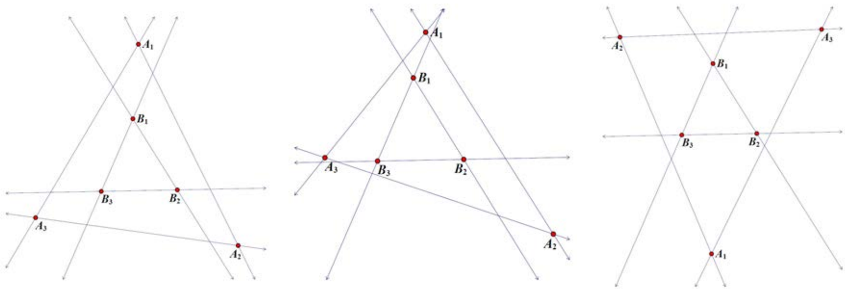
From the above deduction, we can conclude that the case $(3, 3, 2)$ must form at least one convex pentagon. □

Lemma 9. *Any eight points of* $(4, 3, 1)$ *must contain a convex pentagon.*

Proof. [See reviewer's comment (10)] For those point set X of $(4, 3, 1)$, we can imagine that one of the point A_4 are originally in $\Gamma(\Gamma(\Gamma(X)))$ and X becomes

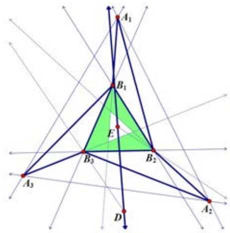


$(3,3,2)$. Similar to the proof of Lemma 8, let $X = \{A_1, A_2, A_3, B_1, B_2, B_3, D, E\}$ be a set of any 8 points, where D, A_1, A_2, $A_3 \in H(X)$, B_1, B_2, $B_3 \in \Gamma(\Gamma(X))$ and $E \in \Gamma(\Gamma(\Gamma(X)))$. The relative positions of A_1, A_2, A_3, B_1, B_2, B_3 can be divided in the 3 cases as shown below. We are now going to prove the lemma case by case.



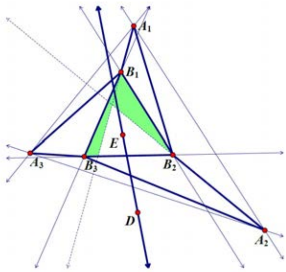
Case 1 Case 2 Case 3

Case 1 [See reviewer's comment (11)] Since E is inside $\triangle B_1B_2B_3$, $E \notin A_iA_j : B_jB_i$ for any distinct i, j for otherwise E will form a convex pentagon with A_i, A_j, B_j and B_i.
WLOG, let segment DE cuts segment B_2B_3. Thus, D, E, B_2 and B_3 form a convex quadrilateral. Moreover, $A_2 \in EB_3 : DB_2$ and therefore D, A_2, B_2, E and B_3 form a convex pentagon.

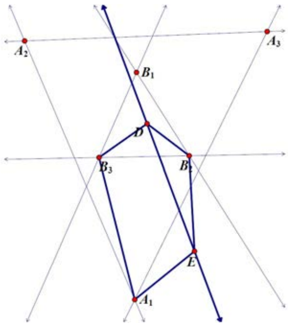


Case 2 [See reviewer's comment (12)] Since E is inside $\triangle B_1B_2B_3$, $E \notin A_iA_j : B_jB_i$, where (i,j) is neither $(2,3)$ nor $(3,2)$ for otherwise E will form a convex pentagon with A_i, A_j, B_j and B_i.
Assume that segment DE cuts segments B_2B_3 and A_2B_3. Thus, D, E, B_2 and

B_3 form a convex quadrilateral. Moreover, $A_2 \in EB_3 : DB_2$ and therefore D, A_2, B_2, E and B_3 form a convex pentagon. The other cases can be proved similarly.



Case 3 WLOG, let segment DE cuts segment B_2B_3.
Thus, D, E, B_2 and B_3 form a convex quadrilateral. Moreover, $A_1 \in DB_2 : EB_3$ and therefore D, B_2, E, A_1 and B_3 form a convex pentagon.

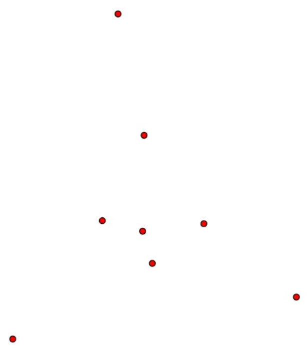


From the above deduction, we can conclude that the case $(4, 3, 1)$ must form at least one convex pentagon. ☐

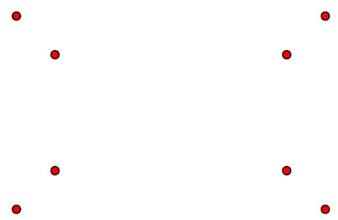
Now we shall apply these two lemmas to prove Theorem 10.

Theorem 10. *Any nine points must contain convex pentagons, i.e.* $g(5) = 9$.

Proof. By the concept of convex boundary, the configuration of any 8 points containing no convex pentagons is either of $(3, 3, 2)$, $(4, 3, 1)$, $(3, 4, 1)$ or $(4, 4)$. We now investigate them one by one. By Lemma 8 and 9, all point sets of $(3, 3, 2)$ and $(4, 3, 1)$ have at least one convex pentagon. The remaining point sets are of $(3, 4, 1)$ and $(4, 4)$. We have found two examples of $(3, 4, 1)$ and $(4, 4)$ without convex pentagons. [See reviewer's comment (13)]



$(3,4,1)$



$(4,4)$

By the examples, we get $f(5) \geq 8$.
Now if one more point is added to the point set of $(3,4,1)$, it becomes a point set of either $(3,4,2)$, $(3,5,1)$ or $(4,4,1)$ which contains convex pentagons by Lammas 8 and 9. Lastly, if one more point is added to $(4,4)$, it becomes either $(5,4)$, $(4,5)$ or $(4,4,1)$ and hence contains convex pentagons. Conclusively, any nine points must contain convex pentagons, i.e. $f(5) = 8$ and $g(5) = 9$. $\square$

5. $f(6) \geq 16$

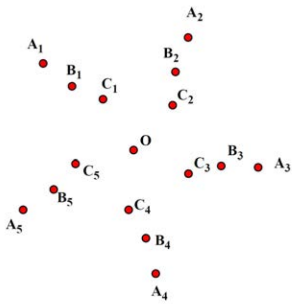
In this stage, we have study some papers in which we can't find any example of 16 points without convex hexagons in the literature, so we start to find one on our own by investigating the properties of convex polygon. We surprisingly find an example Y of configuration $(5,5,5,1)$ (Appendix 2). We then write a computer program (Appendix 1) for checking if there is any convex hexagon by exhaustion. Luckily, there is no convex hexagon in the point set. Finally, we give two mathematical proofs that Y contains no convex hexagon. That means we succeed to prove that $f(6) \geq 16$ and $g(6) \geq 17$ mathematically. For completeness, we further prove that if one point is added to the Y, at least one convex hexagon will be formed.

Algorithm of our computer program

[See reviewers' comment (14)] We are now going to explain the algorithm of our computer program which is inspired by the definition of caps and cups in [1]. For any point set $\{P_i\}_{i=1}^n$ in a rectangular plane, we, WLOG, may assume that there are one leftmost and one rightmost point L and R. The line segment LR then may divides other P_i's into two groups: one above LR and one below LR. Rename those P_i's above (respectively, below) LR by $A_1, A_2, \ldots, A_k$ (respectively, $B_1, B_2, \ldots, B_{n-k-2}$) with their x-coordinates in ascending order for some k. Let $L = A_0$ and $R = A_{k+1} = B_{n-k-1}$. P_i's form a convex n-gon if and only if

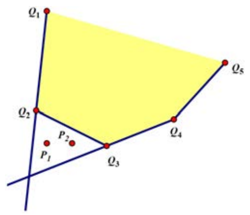
$\{A_0, A_1, A_2, \ldots, A_k, A_{k+1}\}$ form a $(k+2)$-cap and $\{A_0, B_1, B_2, \ldots, B_{n-k-2}, B_{n-k-1}\}$ form a $(n-k)$-cup or $\{P_i\}_{i=1}^{n}$ form an n-cap or n-cup, i.e. there exists $k \in \{1, 2, 3, ..., n\}$ such that the slopes of LA_i's are strictly decreasing and that of LB_i's are strictly increasing. By the algorithm we can pick any 6 points out of 16 points in general position for checking. There are totally 8008 trials.

Construction of 16 points without convex hexagons



Now we try to construct an example Y of 16 points of $(5,5,5,1)$ containing no convex hexagon. The figure on the right consists of 3 concentric regular pentagons with different sizes and their centre O(Appendix 2). We first plot the origin O and A_1. Then we plot B_1 and C_1 which are slightly above OA_1 such that $A_1B_1C_1$ form a 3-cup, A_1B_1O and A_1C_1O form 3-caps. Finally we rotate $OC_1B_1A_1$ clockwise by 72° each time to obtain $OC_iB_iA_i$, where $i = 2, 3, 4, 5$. So we have 16 points on the plane. We use our computer program for preliminary check and are glad to know that there is no convex hexagon at all. We then start to prove that the figure contains no convex hexagons. We find two different proofs, one by configuration and the other by union of beams (Appendix 3).

Lemma 11. *Let $Q_1Q_2 \ldots Q_m$ be a convex m-gon and $P_1, P_2 \in Q_1Q_4 : Q_3Q_2$. If the line (P_1, P_2) intersects the line segment Q_2Q_3, then $P_1, P_2, Q_1, Q_2, \ldots$ and Q_m can't form a convex $(m+2)$-gon.*



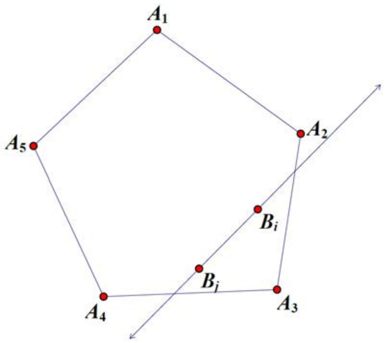
Proof. Since P_1P_2 is not parallel to Q_2Q_3, then one of them say P_2 is closer to Q_2Q_3, then P_2 lies inside the polygon $Q_1Q_2P_1 \ldots Q_m$ and hence P_1, P_2, Q_1, Q_2, $\ldots$ and Q_m can't form a convex $(m+2)$-gon. $\square$

By Lemma 11, we easily get Theorem 12 and Lemma 13.

Theorem 12. *Let $Q_1Q_2\dots Q_m$ be a convex m-gon and $\{P_1, P_2, \dots, P_n\} \subset Q_1Q_4 : Q_3Q_2$. If all the lines (P_i, P_j) intersect the line segment Q_2Q_3 for any distinct $i, j = 1, 2, ...n$, then $\{Q_1, Q_2, \dots, Q_m\}$ form no convex $(m+2)$-gon with any two P_i's.*

Lemma 13. *Any point set of $(5, 2)$ containing a convex hexagon if and only if the line joining the points inside the pentagon intersect two adjacent sides of the pentagon.*

Lemma 14. *Let $X = \{A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3\}$ be a point set of configuration $(5, 3)$, where $\Gamma(X) = \{A_1, A_2, A_3, A_4, A_5\}$ and $\Gamma(\Gamma(X)) = \{B_1, B_2, B_3\}$. If X contains a convex hexagon, then either at least one B_iB_j cuts two adjacent sides of the pentagon $\Gamma(X)$ or all the three lines (B_1, B_2), (B_3, B_2) and (B_1, B_3) cut the same pair of opposite sides of the pentagon $\Gamma(X)$. (equivalently, if one of the lines joining the vertices of the inner triangle neither cuts the two adjacent sides of the outer pentagon nor all lines cut the same pair of opposite sides of the pentagon, then X contains no convex hexagon.)*[See reviewer's comment (15)]

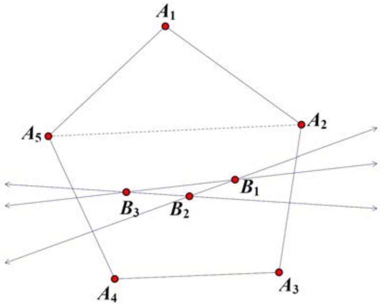


Proof. If X contains a convex hexagon, then its vertices of the hexagon are either from (i) $4A_i$'s and $2B_i$'s or (ii) $3A_i$'s and $3B_i$'s.

(i) $4A_i$'s and $2B_i$'s
It implies that B_i and B_j are contained in the beam $A_1A_5 : A_4A_2$ and hence B_iB_j cuts two adjacent sides A_2A_3 and A_4A_3.

(ii) $3A_i$'s and $3B_i$'s
WLOG., it implies that (B_1, B_2), (B_3, B_2) and (B_1, B_3) can't intersect A_5A_2 by Theorem 12 and hence can't intersect A_5A_1 and A_1A_2. So either all three lines (B_1, B_2), (B_3, B_2) and (B_1, B_3) intersect the opposite sides A_2A_3 and A_4A_5 or at least one of them intersect the adjacent sides A_2A_3 and A_4A_3 or A_4A_3 and A_4A_5.

$\square$



Definition 15. *Let $P_1P_2 \dots P_n$ be a convex n-gon.*
Define the beam union $U(P_1P_2 \dots P_n)$ *of the n-gon as below.*
For $n \geq 4$,

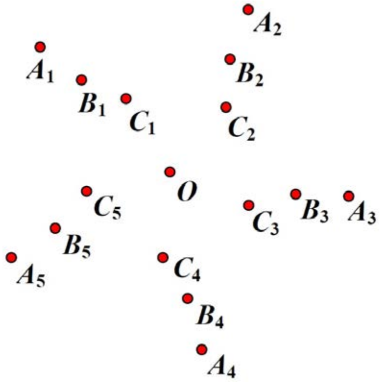
$$\begin{aligned} U(P_1P_2 \dots P_n) =& (P_nP_{n-3} : P_{n-2}P_{n-1}) \cup (P_{n-1}P_{n-4} : P_{n-3}P_{n-2}) \cup \dots \\ & \cup (P_2P_{n-1} : P_nP_1) \cup (P_1P_{n-2} : P_{n-1}P_n). \end{aligned}$$

For $n = 3$, $U(P_1P_2P_3) = (P_3 : P_2P_1) \cup (P_32 : P_1P_3) \cup (P_1 : P_3P_2)$.

It can be easily seen that a point Q can join with a convex n-gon $P_1P_2 \dots P_n$ to form a convex $(n+1)$-gon iff $Q \in U(P_1P_2 \dots P_n)$.

Theorem 16. *The lower bound of $f(6)$ is 16 i.e. $f(6) \geq 16$.*

Proof. Let's define $G_i = \{A_i, B_i, C_i\}$, then Y is the union of G_i's and $\{O\}$. We shall prove Theorem 16 by showing that the point set Y does not contains a convex hexagon.
We started to choose any six points in Y. Among these six points, there are three cases: (1) three points from the same group, i.e. G_i for some i, (2) one point from each group or (3) two points from the same group.



Case 1: three from the same group

WLOG, we chose the point A_5, B_5 and C_5, point C_2 cannot be chosen as $U(A_5C_5C_2B_5)$ contains no point of Y, then A_1, B_1 and C_1 are the only candidates. However, A_5, B_5, C_5, C_1, B_1 and A_1 can't form a convex hexagon since B_1 is inside the polygon $A_5B_5C_5C_1A_1$. So we can't have point C_5. In other words, no convex hexagon will be formed by taking 3 points from the same group. [See reviewer's comment (16)]

Case 2: one from each group

In this case, O must be taken. However, point set of $(5, 1)$ forms no convex hexagon.

Case 3: two from the same group

WLOG, two points are chosen from G_5. The 3 cases are considered below. In these 3 cases, lines joining any two points from G_5 divide the other points into two parts. Since the lines joining any two points from both upper and lower parts do not intersect the segment A_5B_5, we can consider either the upper or lower part in each case. [See reviewer's comment (17)]

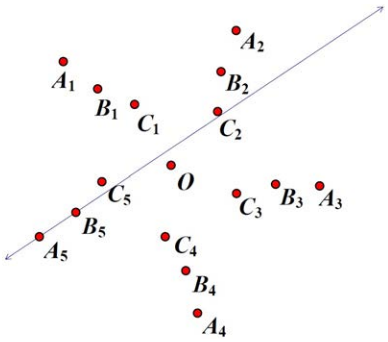
(i) A_5 and B_5 are included

Let's now consider the upper part first. By the result of Case 1, C_5 can't be chosen. Then the points in groups 1, 2 and 5 without C_5 form a point set of the configuration $(5, 3)$. None of the three lines (C_1, B_1), (B_1, B_2) and (C_1, B_2) intersect the adjacent side of the pentagon ${}_5B_5C_2A_2A_1$. On the other hand, (C_1, B_1) cuts opposite sides (B_5, C_2) and (A_5, A_1) which are different from the other two (B_1, B_2) and (C_1, B_2). By Lemma 14, they form no convex hexagon.

Next we consider the lower part. By case 1, we can't take 3 points from the same group.

For the lower part, we only need to consider the following 7 sets:
$\{A_5, B_5, O, A_3, A_4, B_3, B_4, C_4\}$, $\{A_5, B_5, O, A_3, A_4, B_3, B_4, C_3\}$, $\{A_5, B_5, O, A_3, A_4, B_3, C_4, C_3\}$, $\{A_5, B_5, O, A_3, A_4, B_4, C_4, C_3\}$, $\{A_5, B_5, O, B_3, A_4, B_4, C_4, C_3\}$, $\{A_5, B_5, O, A_3, B_4, B_3, C_4, C_3\}$ and $\{A_5, B_5, B_3, A_3, A_4, B_4, C_4, C_3\}$.

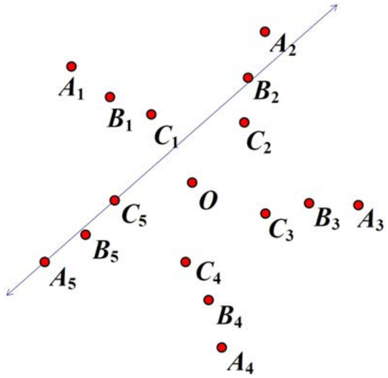
They are all of $(5, 3)$. Obviously, none of the lines joining the possible interior points B_3, B_4, C_3 and C_4 intersect two adjacent sides or all intersect the same pair of opposite sides of its corresponding pentagon. By Lemma 13, no convex hexagon can be formed.



(ii) A_5 and C_5 are included

We first consider the points in the upper part with A_5 and C_5. All these points form a point set of configuration $(5,2)$ and the line (B_1, C_1) does not intersect any two adjacent sides of $A_1A_2B_2C_5A_5$. By Lemma 13, they cannot form any convex hexagon.

Next we consider the lower part. Since the sole difference of this case from the lower part in Case 1 is that C_2 is added, it is sufficient to prove the situation when C_2 is chosen. In this situation, points O cannot be chosen as $U(B_5C_5C_2O)$ contains no point of the lower part. Thus we pick A_5, C_5, C_2 with two points from each group G_3 and G_4 to form a point set of $(5,2)$. Again, none of the lines joining the possible interior points B_3, B_4, C_3 and C_4 intersect two adjacent sides of its corresponding pentagon formed. By Lemma 13, no convex hexagon can be formed.

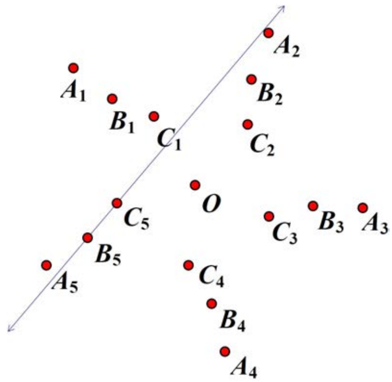


(iii) B_5 and C_5 are included

In this case, two more points A_2 and B_2 are further added to the (ii). By the results obtained in (i) and (ii), it is sufficient to prove the situation when both B_2 and C_2 are chosen.

As $U(B_5C_5B_2C_2) = \{O, C_4, A_4, B_4\}$ and the line joining any two points in the beam intersect B_5C_2, no convex hexagon can be formed by Theorem

12. Conclusively, there are no convex hexagons can be found in Y. Equivalently, we have proved that $f(6) \geq 16$ mathematically.



□

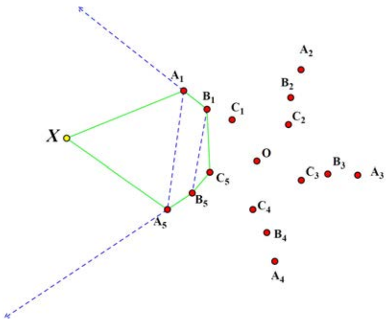
For completeness, we are going to prove Theorem 17.

Theorem 17. *If one more point is added to Y, a convex hexagon is formed.*

Proof. For any point X added, it must fall into one of these four zones.

1. Outside $A_1A_2A_3A_4A_5$
2. Inside $A_1A_2A_3A_4A_5$ but outside $B_1B_2B_3B_4B_5$
3. Inside $B_1B_2B_3B_4B_5$ but outside $C_1C_2C_3C_4C_5$
4. Inside $C_1C_2C_3C_4C_5$

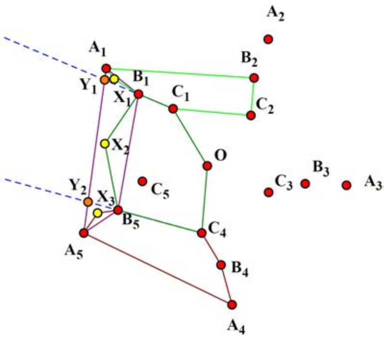
Case 1: When X is in zone 1
WLOG, assume X lies in beam $B_1B_5 : A_5A_1$. Then $XA_1B_1C_5B_5A_5$ is a convex hexagon.



Case 2: When X is in zone 2

WLOG, assume X lies in the quadrilateral $A_1B_1B_5A_5$. Let ray $[C_1, B_1)$ and ray $[C_4, B_5)$ intersect the segment A_1A_5 at Y_1 and Y_2 respectively. The quadrilateral $A_1B_1B_5A_5$ can be further divided into 3 three regions:

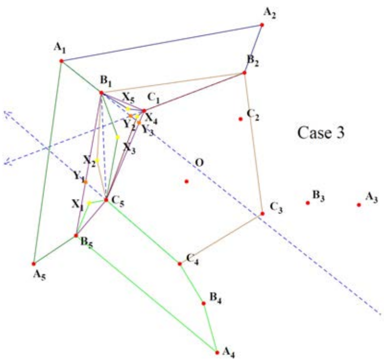
(1) $A_1B_1Y_1$, where $A_1X_1B_1C_1C_2B_2$ is a convex hexagon;
(2) $B_1B_5Y_2Y_1$, where $X_2B_1C_1OC_4B_5$ is convex hexagon;
(3) $A_5B_5Y_2$, where $X_3B_5C_4B_4A_4A_5$ is a convex hexagon.



Case 3: When X is in zone 3

WLOG, assume X lies in the quadrilateral $B_1C_1C_5B_5$. Let ray $[C_4, C_5)$ cuts segment B_1B_5 at Y_1, ray $[B_2, C_1)$ cuts ray $[A_1, B_1)$ at Y_2, and ray $[A_1, B_1)$ and segment C_1C_5 cut at Y_3. B_1C_1 C_5B_5 can be divided into 5 regions:

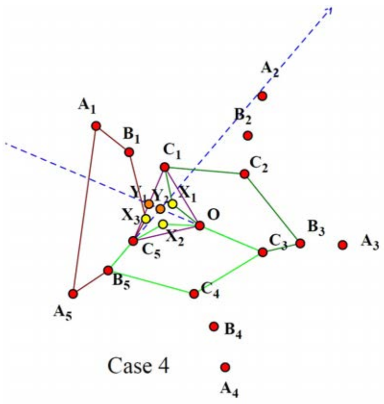
(1) $B_5C_5Y_1$, where $X_1C_5C_4B_4A_4B_5$ is convex;
(2) $B_1Y_1C_5$, where $X_2B_1B_2C_3C_4C_5$ is convex;
(3) $B_1C_5Y_3$, where $A_1B_1X_3C_5B_5A_5$ is convex;
(4) $C_1Y_2Y_3$, where $X_4C_1B_2C_3C_4C_5$ is convex;
(5) $B_1C_1Y_2$, where $A_1B_1X_5C_1B_2A_2$ is convex.



Case 4: When X is in the zone 4

WLOG, assume X lies in the triangle OC_1C_5. Let segment C_1C_5 and ray $[B_5, C_5)$ intersect the ray $[C_3, O)$ at Y_1 and Y_2 respectively. The triangle is divided into 3 regions:

(1) OY_1C_1, where $X_1C_1C_2B_3C_3O$ is a convex hexagon;
(2) OY_2C_5, where $X_2OC_3C_4B_5C_5$ is a convex hexagon;
(3) $Y_1Y_2C_5$, where $A_1B_1X_3C_5B_5A_5$ is a convex hexagon



Case 4

Conclusively, if one more point is added to this point set, a convex hexagon will be certainly formed. Hence the lower bound of $g(6)$ is 17 i.e. $g(6) \geq 17$. $\square$

6. Conclusion

We start our investigation on $g(5)$ by drawing diagrams using Geometer's Sketchpad, in the hope that we can figure out some patterns which may help our later investigation on $g(6)$. However we find that hundreds of diagrams are needed and decided to read the research papers written by pioneers. After reading Erdő's paper, we admire his beautiful and simple proof on $g(5) = 9$ without diagram and want to try on our own. Moreover, we accidentally discover an example which show that $g(5) > 8$ without words. It inspire us to construct a similar proof of $g(6) > 16$. Finally, an example of 16 points without any convex hexagon is successfully created and confirmed by our computer program. As we think that such confirmation is neither perfect nor elegant, we then try to prove the non-existence of convex hexagons mathematically.

We are not satisfied with what we have done because the results and techniques are confined to some particular positive integers and can't be generalized. We then turn our focus on the conjecture $g(n) = 2^{n-2} + 1$. After defining the greatest positive integer $f(n)$ points in general position in the plane contains no convex n-gons, i.e. $f(n) = g(n) - 1$, we can easily deduce that $f(n+1) = 2f(n)$ based on the conjecture. Then we try to construct a proof on $f(7) \geq 32$ by pasting two copies of our example demonstrating $f(6) \geq 16$. We start the experiment by placing two copies far away from each other with the slopes between any two points from two copies tend to infinity in order to avoid any formation of convex heptagons by taking union of points in both copies. Unfortunately, we fail and then try to paste these two copies in other ways such as compressing one set of points horizontally while another vertically. We keep trying since we still believe that the conjecture can be proved by building example of $2n$ points without convex $(n+2)$-gons by two blocks of 2^{n-1} points without convex $(n+1)$-gons.

This competition certainly broadens our horizon on Mathematics. Before the competition, none of us can imagine that geometrical problems can be solved in such way. By reading the articles written by Mathematicians, we experience the beauty and elegance of Mathematics. Also, the research challenges our patience. Unlike Mathematical problems we face in college, such open problem is so challenging that we do not have any confidence to solve the entire problem, so we have to learn how to do our best. We have to be patient even we may find nothing during the investigation. Though the paper is finished, we are not satisfied with our results and still want to tackle the conjecture. So we will continue with the hope that we may have some contributions one day.

REFERENCES

[1] W. Morris, and V. Soltan, *The Erdos-Szekeres problem on points in convex position-a survey*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 4, 437-458, DOI 10.1090/S0273-0979-00-00877-6. MR1779413

[2] G. Szekeres, and L. Peters, *Computer solution to the 17-point Erdős-Szekeres problem*, ANZIAM J. **48** (2006), no. 2, 151164, DOI 10.1017/S144618110000300X. MR2291511

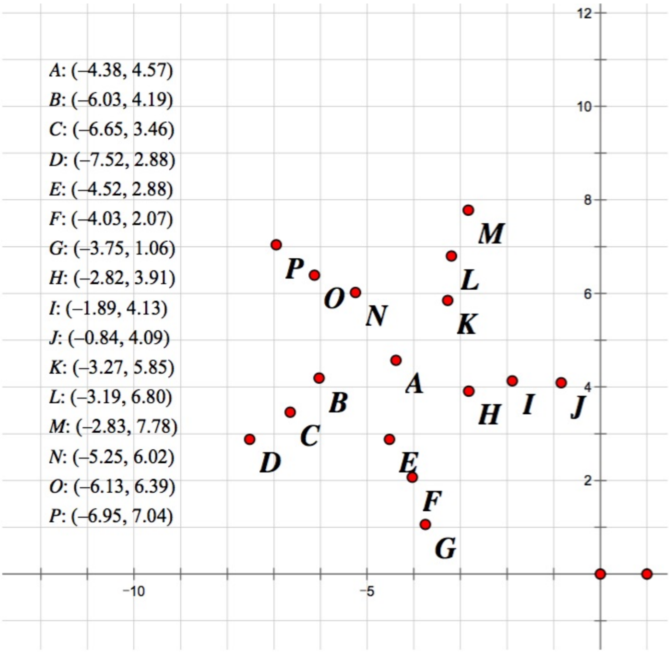
[3] Wikipedia, *Happy ending problem*, http://en.wikipedia.org/wiki/Happy_end_problem

[See reviewer's comment (18)]

7. Appendix

1 . **Computer Program**
http://dl.dropbox.com/u/9210037/CHECK%20g%286%29-new%20version.c
[See reviewer's comment (19)]

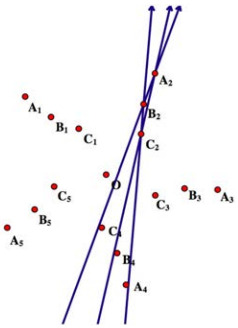
2 . **Example of 16 points of configuration (5, 5, 5, 1)**



3 . **Alternative Proof of** $g(6) > 16$
Now, we are going to prove that there exists a point set of 16 points without convex hexagons i.e. $f(6) > 16$.

Let group $G_i = \{A_i, B_i, C_i\}$, where $i = 1, 2, 3, 4, 5$ and $Y = \bigcup_{i=1}^{5} G_i \cup \{0\}$.
We plan to prove that Y contains no convex hexagons by considering three cases:
Case 1: 3 points from the same group,
Case 2 : 2 points from the same group and
Case 3 : 1 point from each group.

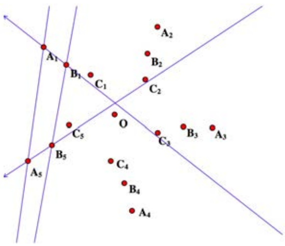
Case 1 3 points from a group
WLOG, let's take Group 2 as an example. Then $U(A_2B_2C_2) \cap Y = \{A_3, B_3, C_3, C_4\}$. So the only candidates are A_3, B_3C_3. However, B_3 lies inside $A_2B_2C_2C_3A_3$, no convex hexagon can be formed.



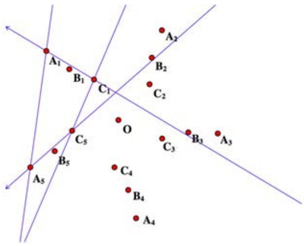
Case 2 2 points from a group

Case 2.1 2 points from a group and 1 point from all other groups.
The point set form is of (5, 1) and hence contains no convex hexagon.

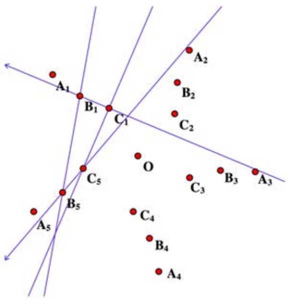
Case 2.2 2 points from each of two adjacent groups.
WLOG, let's take group 1 and group 5 as an example.



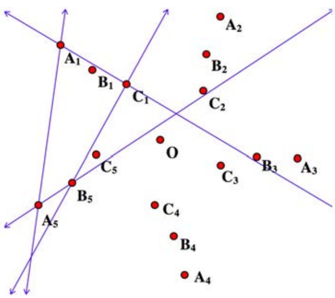
(i) $A_1A_5B_5B_1$,
$U(A_1A_5B_5B_1) \cap Y = \{C_5\}$



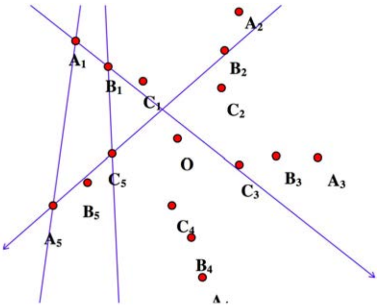
(ii) $A_1A_5C_5C_1$,
$U(A_1A_5C_5C_1) \cap Y = \{B_5\}$



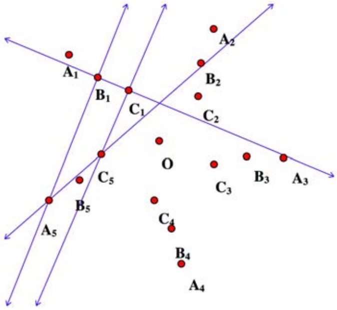
(iii) $B_1B_5C_5C_1$,
$U(B_1B_5C_5C_1) \cap Y = \{A_5\}$



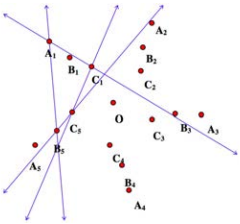
(iv) $A_1A_5B_5C_1$,
$U(A_1A_5B_5C_1) \cap Y = \{C_5\}$



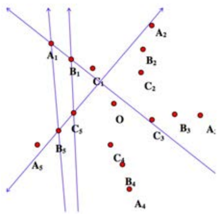
(v) $A_5A_1B_1C_5$,
$U(A_5A_1B_1C_5) \cap Y = \{B_5\}$



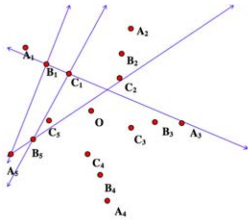
(vi) $A_5B_1C_1C_5$,
$U(A_5B_1C_1C_5) \cap Y = \{B_5\}$



(vii) $A_1B_5C_5C_1$,
$U(A_1B_5C_5C_1) \cap Y = \{A_5\}$



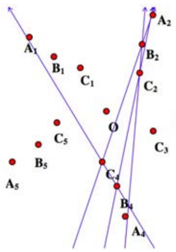
(viii) $A_1B_1C_5B_5$,
$U(A_1B_1C_5B_5) \cap Y = \{A_5\}$



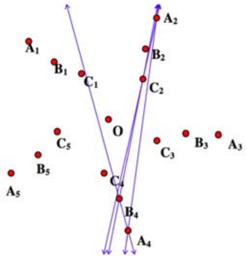
(ix) $A_5B_1C_1B_5$,
$U(A_5B_1C_1B_5) \cap Y = \{C_5\}$

Since the beam union of the quadrilaterals formed in (i)-(ix) by any two points from each of two adjacent groups G_1 and G_5 consists of only one element, therefore no convex hexagon can be formed.

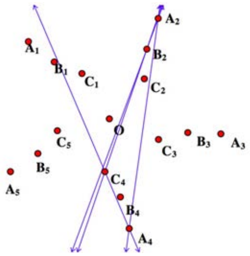
Case 2.3 2 points from each of two opposite groups
WLOG, let's take group 2 and group 4 as example.



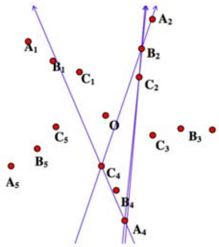
(i) $B_2C_4B_4C_2$.
$U(B_2C_4B_4C_2) \cap Y = \{A_1, B_1, C_1, O\}$. By case 2.2, O must be taken. But O lies inside $B_2C_2B_4C_4A_1$, $B_2C_2B_4C_4B_1$ and $B_2C_2B_4C_4C_1$. Therefore no convex hexagon is formed.



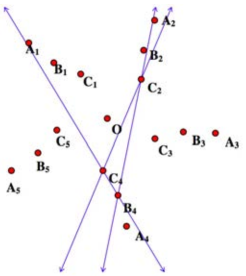
(ii) $A_2A_4B_4C_2$.
$U(A_2A_4B_4C_2) \cap Y = \{A_3, B_3, C_3\}$.
By case 2.2, only one of $\{A_3, B_3, C_3\}$ can be taken and hence no convex hexagon is formed.



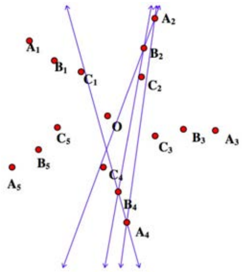
(iii) $A_2A_4C_4B_2$.
$U(A_2A_4C_4B_2) \cap Y = \{A_3, B_3, C_3\}$ which is the same as (ii) and hence no convex hexagon is formed.



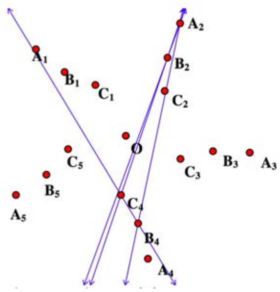
(iv) $A_4C_4B_2C_2$.
$U(A_4C_4B_2C_2) \cap Y = \{C_1, O\}$. But $\{A_4, C_4, B_2, C_2, C_1, O\}$ is of (5,1). Hence no convex hexagon is formed.



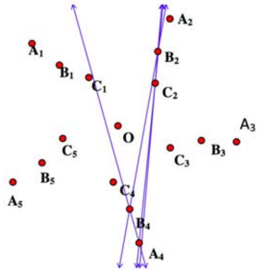
(v) $A_2B_4C_2C_4$.
As $A_2 \notin U(B_4C_2C_4)$
therefore $\{A_2, B_4, C_2, C_4\}$ cannot form a convex quadrilateral. By case 2.2, O must be chosen. However point set of (5,1) contains no convex hexagons.



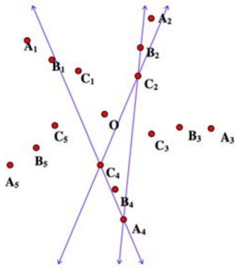
(vi) $A_2B_2C_4B_4$,
$U(A_2B_2C_4B_4) \cap Y = \{A_3, B_3, C_3\}$ as same as (ii). Hence no convex hexagon is formed.



(vii) $A_2B_2C_4B_4$.
As $U(A_2B_2C_4B_4) \cap Y = O$ and there are only 5 points, no convex hexagon is formed.



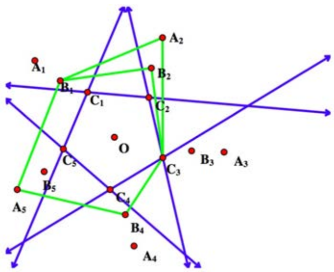
(viii) $A_2B_4B_2C_2$.
As $U(A_2B_4B_2C_2) \cap Y = O$ and there are only 5 points, no convex hexagon is formed.



(ix) $A_2A_4C_2C_4$.
As $A_2 \notin U(B_4C_2C_4)$, therefore points $\{A_2, A_4, C_2, C_4\}$ cannot form a convex quadrilateral. By case 2.2, O must be chosen. However point set of (5,1) contains no convex hexagons.

Since the beam union of the quadrilaterals formed in (i)-(ix) by any two points from each of two opposite groups G_2 and G_4 consists of only one element, therefore no convex hexagon can be formed.

Case 2.4 2 points from a group with or without O.



By cases 2.2 and 2.3, the remaining 3 points come from 3 or 4 other groups. Obviously, the point sets are of (5,1) and hence there is no convex hexagon can be formed.

Case 3 1 point from each group

In this case, O must be included and form a point set of (5,1) which contains no convex hexagon. Conclusively, this point set of 16 points consists no convex hexagon and hence $f(6) \geq 16$ and $g(6) \geq 17$.

Reviewer's Comments

This paper investigates the Erdős-Szekeres'conjecture, which gives a relationship between the number of points in a general-position point set and its largest convex polygon. More precisely, the conjecture states that the smallest number $g(n)$ of points for which any general position arrangement contains a convex subset of n points is $2^{n-2}+1$. The general case remains unproven. The first non-trivial case starts from $n = 4$, and it is now known that the conjecture is true up to $n = 6$.

This paper gives the proof of the fact that $g(4) = 5$ and $g(5) = 9$, i.e. the fact that a set of 5 (respectively 9) points in general position guarantees the existence of a convex quadrilateral (respectively pentagon) whose vertices are from this set. They also showed that $g(6) \geq 17$ by giving a 16-point configuration which does not contain any convex hexagon. Finally, they showed that in their example, a convex hexagon will be created by adding one more point (which is of course true because $g(6) = 17$).

They make extensive use of the (x, y, z) configuration to classify the distribution of points. E.g. a $(4, 3, 1)$ configuration consists of 8 points, in which 4 points form a convex quadratical, whose interior contains a triangle formed by 3 of the remaining points, and the remaining one point is contained in the interior of this triangle.

Here are my comments and suggestions about both the mathematics and the style of this paper.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. Overall, the reviewer feels that some of their arguments rely purely on geometric intuition or a particular way in which they draw a figure and may not be rigorous (or general) enough.

   For example, in Case 1 of the proof of Lemma 9, they claimed that "Moreover, $A_2 \in EB_3 : DB_2$" (without proof). Actually, renaming $B_2 \leftrightarrow B_3$ and $A_2 \leftrightarrow A_3$ will not affect their assumptions but now $A_2 \notin EB_3 : DB_2$ but $A_3 \in EB_3 : DB_2$. So at least their consideration seems to be incomplete at times. There are quite a number of such instances and it is not easy to know how many cases they have missed.
3. The description below is problematic:

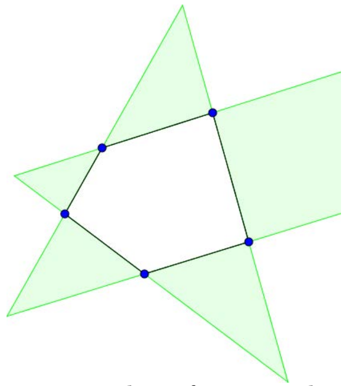
   "... for any integer $n \geq 3$, there exists a smallest positive integer $g(n)$ points in general position in the plane containing n points that are the vertices of a convex n-gon"

   It can be changed to, for example:

   "... for any integer $n \geq 3$, there exists a smallest number $g(n)$ such that for any $g(n)$ points in general position on the plane, there is a subset of n points that form the vertices of a convex polygon."

4. They first define $\Gamma(X)$ as the convex boundary of a set X of points in the plane. The definition is okay. However, they go on to talk about $\Gamma(X)$, $\Gamma(\Gamma(X))$ and so on, and then claim that the "configuration" of X is defined as the number of $\Gamma(X)$, $\Gamma(\Gamma(X))$, and so on. This is not the correct way to define "configuration". In fact, it is obvious that $\Gamma(X) = \Gamma(\Gamma(X)) = \Gamma(\Gamma(\Gamma(X))) = \cdots$.

   If they insist on using Γ to define configuration, then it can be defined like this:
   Take $X_1 = X$, and let $y_1 = |\Gamma(X_1)|$ (the number of points in $\Gamma(X_1)$). Now inductively define $X_{i+1} = X_i \setminus \Gamma(X_i)$ and take $y_{i+1} = |\Gamma(X_{i+1})|$. It is clear that y_i must terminate (i.e. become 0 eventually). We then say X is of $(y_1, y_2, \cdots)$-configuration.
5. It is suggested that they illustrate the concepts in Definition 4, 5, 6, 7 by some figures. E.g. Definition 7 can be illustrated as follows:



   It is even better to mention that, for example, adding a point at any position inside $AB : CD$ will produce a convex pentagon whose vertices are A, B, C, D and the added point.
6. As mentioned before, their notions of $\Gamma(X)$, $\Gamma(\Gamma(X))$ do not make sense. So their statement "$B_1, B_2, B_2 \in \Gamma(\Gamma(X))$, $D, E \in \Gamma(\Gamma(\Gamma(X)))$ " is wrong. The first sentence should be deleted. They can say something like:
   "Suppose D, E is enclosed in $\Delta B_1B_2B_3$ and $\Delta B_1B_2B_3$ is enclosed in $\Delta A_1A_2A_3$."
7. The reviewer would suggest citing the original paper:
   "Erdős, P.; Szekeres, G. (1961), "On some extremum problems in elementary geometry", Ann. Univ. Sci. Budapest. Eőtvős Sect. Math., 34: 5362. Reprinted in: Erdős, P."
   rather than citing a survey paper for this result.
8. p. 275, Case 1. The way they wrote the argument is confusing, especially the first paragraph (the sentence $D, E \notin A_iA_j : B_jB_i$ is very confusing, as it is not always true). This case should be rewritten, for example, as follows:

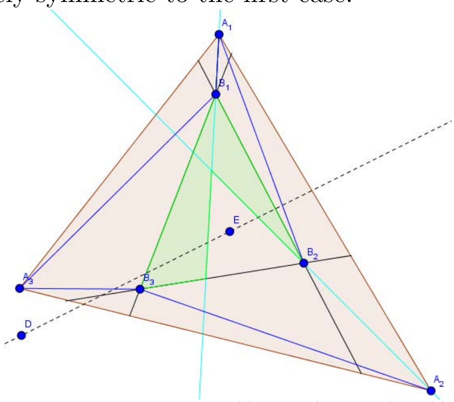
   "If either D or E lies in $A_iA_j : B_jB_i$, for some distinct i, j then it's done as D or E will then form a convex pentagon with A_i, A_j, B_j and B_i.

So we assume otherwise. Then we can without loss of generality assume that DE cuts B_1B_2 and B_1B_3. Then by the argument in Theorem 1, we can assume (by renaming $D \leftrightarrow E$ is necessary) DEB_2B_3 is a convex quadrilateral. Then we have either $A_2 \in DE : B_2B_3$ of $A_3 \in DE : B_2B_3$. i.e. either $DEB_2A_2B_3$ or $DEB_2A_3B_3$ is a convex pentagon."

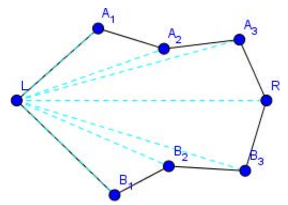
9. The problem is the same as Case 1. At least the first two paragraphs should be completely rewritten, for example, as follows:

   "We can assume $D, E \notin A_iA_j : B_jB_i$ for $(i,j) = (2,3)$ or $(i,j) = (3,2)$ for otherwise D or E will then form a convex pentagon with A_i, A_j, B_j and B_i.

   So suppose otherwise. Assume first that DE cuts B_1B_3 and B_1B_2."
10. Same problem as before. Their notions $\Gamma(\Gamma(X))$ and $\Gamma(\Gamma(\Gamma(X)))$ do not make sense. The first two sentence should be deleted (and the reviewer don't know what the first sentence means anyway).
11. The same problem as Case 1 and Case 2 of Lemma 8: $E \notin A_iA_j : B_jB_i$ is in general not true and so the first paragraph in these two cases should be rewritten (see (8), (9)).
12. They only proved the case where DE cuts B_2B_3 and A_2A_3, and then claimed that the other cases can be proved similarly. The reviewer don't think the other cases are "similar". Take for example, in the following figure, DE cuts B_1B_3 (and also A_3B_3 and A_1B_2), they did not indicate how to find the convex pentagon (although it can be done: $EB_1A_3DB_3$ is convex). The remaining cases are not entirely symmetric to the first case.



13. There is no proof that the given figures ((3, 4, 1) and (4, 4)) contains no convex pentagon. It is very hard to verify their claim by inspection as there are 56 possible pentagons.
14. The algorithm as described in the later part cannot be used to detect a convex n-gon. In fact, the slope of LA_i being strictly decreasing and the slope of LB_i being strictly increasing is never sufficient to prove the n-gon is convex, as can be seen in the following figure:



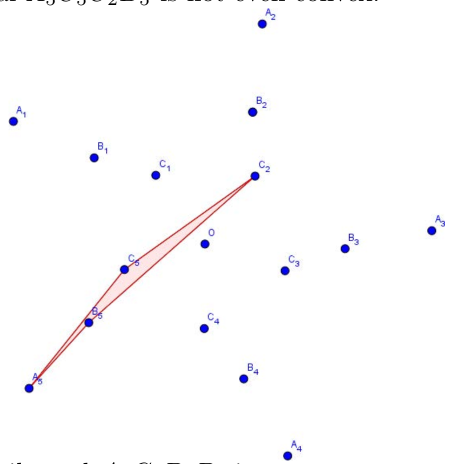
The correct way is to verify that the slope $p_i = A_{i-1}A_i$ being decreasing and the slope $q_i = B_{i-1}B_i$ being increasing. As the reviewer does not have access to their computer program, the reviewer cannot determine whether their implementation is correct or not.

15. Again $\Gamma(\Gamma(X)) = \{B_1, B_2, B_3\}$ does not make sense. Delete this sentence and replace it by: $A_1A_2A_3A_4A_5$ forms a convex pentagon which encloses the triangle $\Delta B_1B_2B_3$.

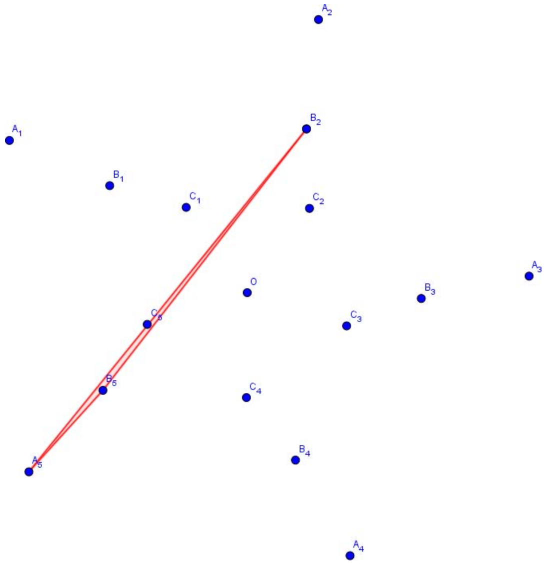
    Also, the sentence inside the last bracket is not the contrapositive statement of Lemma 14. Replace it by:
    "Equivalently, if none of the lines B_iB_j cuts two adjacent sides of the outer pentagon and not all of the lines B_iB_j cut the same pair of opposite sides of the outer pentagon, then X contains on convex hexagon."

16. The reviewer thinks C_2 should actually be B_2. Since the reviewer could not download their program, the reviewer plotted it himself (using their description on p. 277 and 278) as shown (the coordinates on p. 287 are not exact anyway).

    The quadrilateral $A_5C_5C_2B_5$ is not even convex.



However, the quadrilateral $A_5C_5B_2B_5$ is convex.



So any C_2 here should be changed to B_2. They also did not explain why they only consider "C_2" (which should be B_2), A_1, B_1 and C_1. In fact, the reviewer also found that $A_2 \in U(A_5B_5C_5)$ but they seemed to have ignored this point.

17. The reviewer does not understand the following sentence:
"Since the lines joining any two points from both upper and lower parts do not intersect the segment A_5B_5, we can consider either the upper or lower part in each case."
In fact, it seems that in each of the subcases ((i), (ii) and (iii)), the "upper part" and the "lower part" are different. Perhaps what they mean is:
"In each of the following subcases, by Theorem 12 or Lemma 13, we only consider the possibility of forming a convex hexagon by adjoining either only points in the upper part or points in the lower part, and the upper and lower part depend on each subcase."
18. The reference [3], which is the Wikipedia article about the "Happy Ending problem", is not cited in the paper, although it can be relevant. The reviewer thinks it can be cited, for example, in the introduction.
Also, the reviewer think the References should be moved to the end of the paper, after the appendix.
19. The link of the computer program is dead.

Overall, the reviewer thinks their results are correct in general (after all, they want to find alternative proofs of some known results). However, some of their arguments seem to rely purely on geometric intuition or a particular way in which they draw a figure, and seem to be a bit ad hoc. The reviewer think some of their arguments are not very complete or rigorous. The reviewer also have doubts about how their computer program is written (see (14)). Since the reviewer cannot download their program, the reviewer cannot verify the correctness.