Hang Lung Mathematics Awards 2012

Honorable Mention

How to Cut a Piece of Paper - Making Paper Cones with the Greatest Total Capacity

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HOW TO CUT A PIECE OF PAPER - MAKING PAPER CONES WITH THE GREATEST TOTAL CAPACITY

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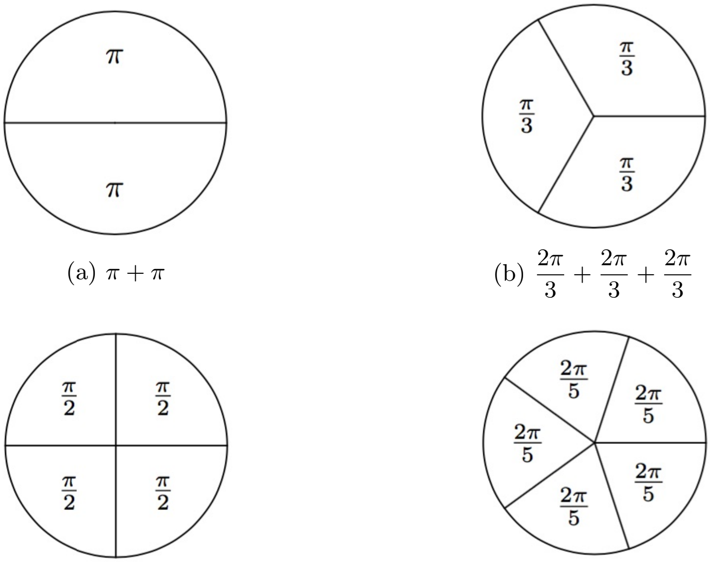
Abstract. Given a regular polygonal paper inscribed in a unit circle, the paper is cut along its radii and each division (consisting of one or more sub-divisions) is made into a cone. These cones are allowed to be slanted to obtain a greater capacity. The purpose of this study is to maximize the total capacity of cones made from the paper over all ways of divisions.

The methodology in this report is streamed into two parts - minimax strategy and bounds by inequalities. For triangular paper, the rims of cones are parameterized before their water depths are expressed explicitly. The capacities of cones are maximized over angles of slant. Different ways of division are compared to and out the optimal solution. Probing into general cases, various inequalities are set up analytically and exhaustively to bound the total capacities for comparisons.

To obtain the greatest capacities, cones made from one sub-division should be slanted but those from multiple sub-divisions should be held vertically. For a polygonal paper of six or more sides, it should be divided into two divisions, each comprising two or more sub-divisions with a central angle ratio of 0.648:1.352, approaching the way of division in circular paper.

1. Introduction

[See reviewer's comment (1)] The objective of our project consists of two problems. The first problem is to find the method to cut a paper circle into sectors and make cones with the maximum total capacity. The number of cones is not restricted and the cones are not necessarily identical. The capacity of a cone is easy to find but when there are more cones, the problem is not so simple. It can be quite a difficult Calculus problem as the number of variables (i.e. the number of cones) is also a variable. We used computer softwares to calculate the total capacities in many cases. It seems that the solution of the problem is a relatively simple case. For example, a unit circle can be cut into two sectors with each arc length π, three sectors each with $\frac{2\pi}{3}$ or four sectors each with arc length $\frac{\pi}{2}$. We can find the total capacity in each case to get some clues. We did not restrict our investigation just on evenly cut sectors but in general, it seems that the total capacity is the greatest when there are only two cones.



(a) $\pi + \pi$ (b) $\frac{2\pi}{3} + \frac{2\pi}{3} + \frac{2\pi}{3}$

(c) $\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}$ (d) $\frac{2\pi}{5} + \frac{2\pi}{5} + \frac{2\pi}{5} + \frac{2\pi}{5} + \frac{2\pi}{5}$

FIGURE 1. Ways of cutting the circle

We then switch our foci on two tasks. The first one is to find the best way to cut the circle into two sectors (to obtain the greatest total capacity) and we successfully solve it by using the symmetric property of the problem to reduce a polynomial equation in degree 7 to a cubic equation. The second task is to show that the total capacity of the cones will decrease if we cut a sector with central angle not exceeding $\frac{4\pi}{3}$ into two smaller sectors. This can assure us that the solution of the problem is to cut the circle into only two sectors. Our strategy to this second task is to consider the total capacity of two cones made with sectors such that the sum of the arc length does not exceed some constant $(\frac{2\sqrt{6}}{3}\pi)$. We use the convexity of the derivative of the total capacity and proved a useful inequality. Our strategy worked well as we managed to find a beautiful proof, much simpler than we expected, although we still need the 4th derivative of the volume of the cone.

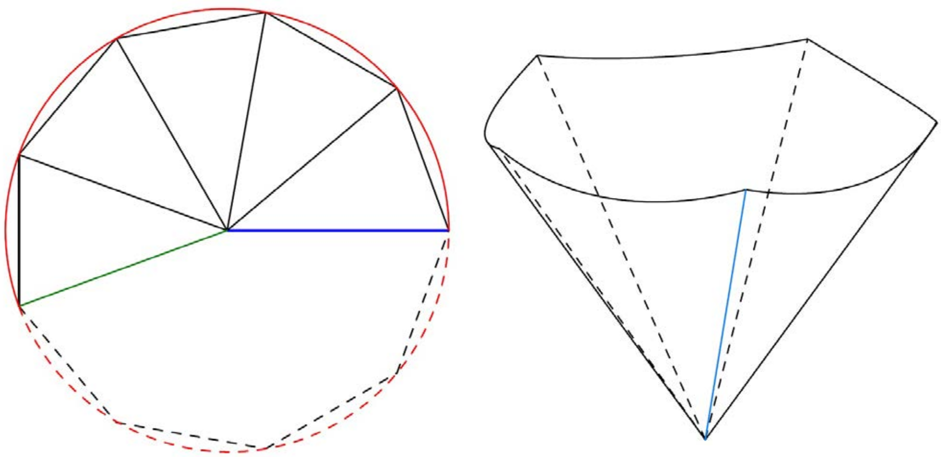
The second problem is a variation, or a generalization of the first problem. This time we are searching for the method to cut regular polygons along their radii (FIGURE

8a). The 'cones' (FIGURE 8b) are not really normal cones as they have zig-zag on the rim. We started with the seemingly simplest case: the 'sector' (actually is an isosceles triangle) is cut along two adjacent radii. The capacity is quite difficult to find, as we can slant the 'cone' in different ways and get a larger capacity than that when the 'cone' is held vertically. This part requires quite a bit of hard work and results a rather complicated trigonometric equation. We used the two kinds of Chebyshev Polynomials to help us to solve for some cases. In the due course, we managed to

find a general rule to factorize our polynomials (converted from the complicated trigonometric equations). We then tried to use these work to simplify our original trigonometric equation, without introducing Chebyshev Polynomials, and reduce it to a trigonometric equation as simple as

$$(n-1)\cos(n+1)\theta + (n+1)\cos(n-1)\theta = 0 \tag{1}$$

[See reviewer's comment (2)] which is not so difficult to solve numerically.



(a) A 5-portion paper in a paper nonagon

(b) 9,5-CONE

FIGURE 2

This seemingly simplest case turned out to be the most complicated case. We found that when the 'sector' consists of more 'isosceles triangles', the capacity of the 'cone' will be the greatest when it is held vertically. Although it took us some more hard work to prove this, it enable us to solve the problem with the results in the first problem, the circle problem. The polygon problem is still more complicated than the circle problem. Although (1) looks simple, it is not easy to find a solution in closed form and besides, the capacity of the 'simple' 'cone' is a complicated function of the solution of the equation. We avoided to find the actual value of this kind of 'cones' as we think it was almost impossible to do that. Instead, we found an upper bound for this capacity and do the final comparison work. The problems raised in the project are solved completely and the solution of the second problem is closely related to that of the first one, although they are exactly not the same.

2. How Much Water Can a Paper Circle Hold?

In many mathematics textbooks, we can find problems of calculating the volume (capacity) of a cone made with a given circular sector. We begin our project with a less commonly asked problem. When we cut a sector from a paper circle and make such a cone, there is always something left. The remained part is also a sector and can be used to make one more cone. So with a paper circle, we can make more than one cones. Actually, we can cut the paper circle into any number of sectors and make cones. Our first problem is to find the largest possible total capacity of these cones.

2.1. A Cone with the Greatest Capacity

As all circles are similar, we can assume the paper circle has unit radius. Let $V(x)$ be the capacity of the cone made with a sector of arc length x (where $0 \le x \le 2\pi$). The base radius of the cone is $\dfrac{x}{2\pi}$ and its height is $\sqrt{1-\left(\dfrac{x}{2\pi}\right)^2} = \dfrac{\sqrt{4\pi^2 - x^2}}{2\pi}$. Therefore, we have

$$\begin{aligned} V(x) &= \frac{\pi}{3}\left(\frac{x}{2\pi}\right)^2\left(\frac{\sqrt{4\pi^2 - x^2}}{2\pi}\right) \\ V(x) &= \frac{x^2\sqrt{4\pi^2 - x^2}}{24\pi^2} \end{aligned} \tag{2}$$

and its first derivative is

$$\begin{aligned} V'(x) &= \frac{1}{24\pi^2}\left(2x\sqrt{4\pi^2 - x^2} - \frac{x^3}{\sqrt{4\pi^2 - x^2}}\right) \\ &= \frac{8\pi^2 x - 3x^3}{24\pi^2\sqrt{4\pi^2 - x^2}}. \end{aligned}$$

V is continuous on $[0, 2\pi]$. Since $V'(x) > 0$ when $0 < x < \dfrac{2\sqrt{6}\pi}{3}$ and $V'(x) < 0$ when $\dfrac{2\sqrt{6}\pi}{3} < x < 2\pi$, V attains its greatest value when $x = \dfrac{2\sqrt{6}\pi}{3}$ (FIGURE 3). Thus the greatest possible capacity of such a cone is

$$\begin{aligned} V\left(\frac{2\sqrt{6}\pi}{3}\right) &= \frac{1}{24\pi^2}\left(\frac{2\sqrt{6}\pi}{3}\right)^2\sqrt{4\pi^2 - \left(\frac{2\sqrt{6}\pi}{3}\right)^2} \\ &= \frac{2\sqrt{3}\pi}{27} \\ &\approx 0.1283001\pi \end{aligned}$$

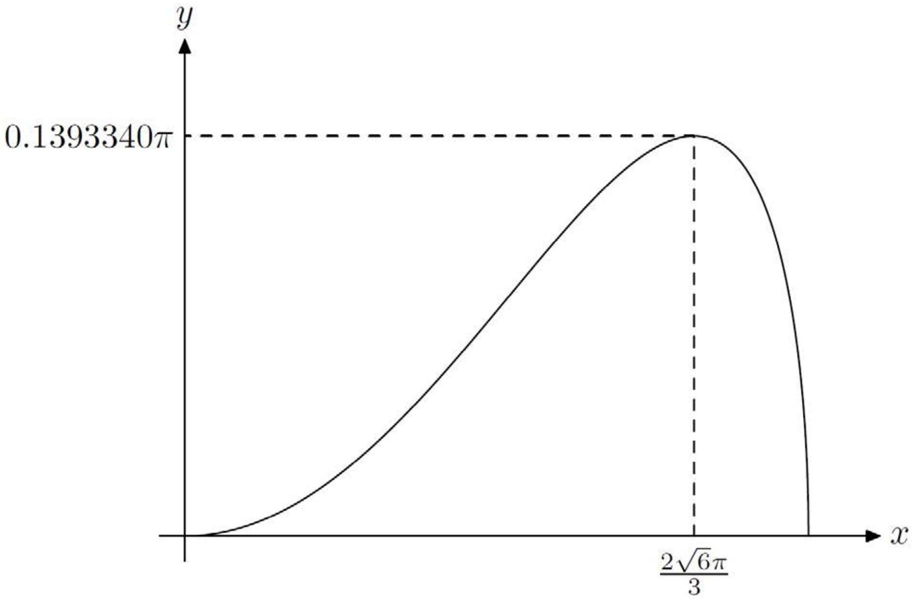


FIGURE 3. Graph of $V(x)$.

If the circular sector is cut from a paper unit circle, a sector with arc length $2\pi - \frac{2\sqrt{6}\pi}{3}$ will be left and when this sector is also used to make a cone, the total capacity of the cones is $V\left(\frac{2\sqrt{6}\pi}{3}\right) + V\left(2\pi - \frac{2\sqrt{6}\pi}{3}\right) \approx 0.1393340\pi$. However, this is not the way to make the greatest total capacity, as two identical semicircles will make two cones with total capacity $2V(\pi) = \frac{\sqrt{3}\pi}{12} \approx 0.1443376\pi$. But this is still not the greatest possible total capacity. FiIGUR 4 shows the graph of $V(x) + V(2\pi - x)$, which is the total capacity of the cones made when the paper circle is cut into two sectors. We can see from the graph that the total capacity of the cones made by two semi-circles is almost the greatest value of all possible total capacity, but ironically it is a local minimum there. We can see from the graph that we will have a larger total capacity if the paper circle is cut into two sectors with arc lengths approximately in the ratio 2 : 1. We will find the ratio in Section 2.4. This is actually the largest possible value of the total capacity, as we will show in the next two sections that we cannot increase the total capacity by cutting the sectors further.

2.2. Cutting a Sector

There are infinity many ways to cut the paper unit circle. Our problem of

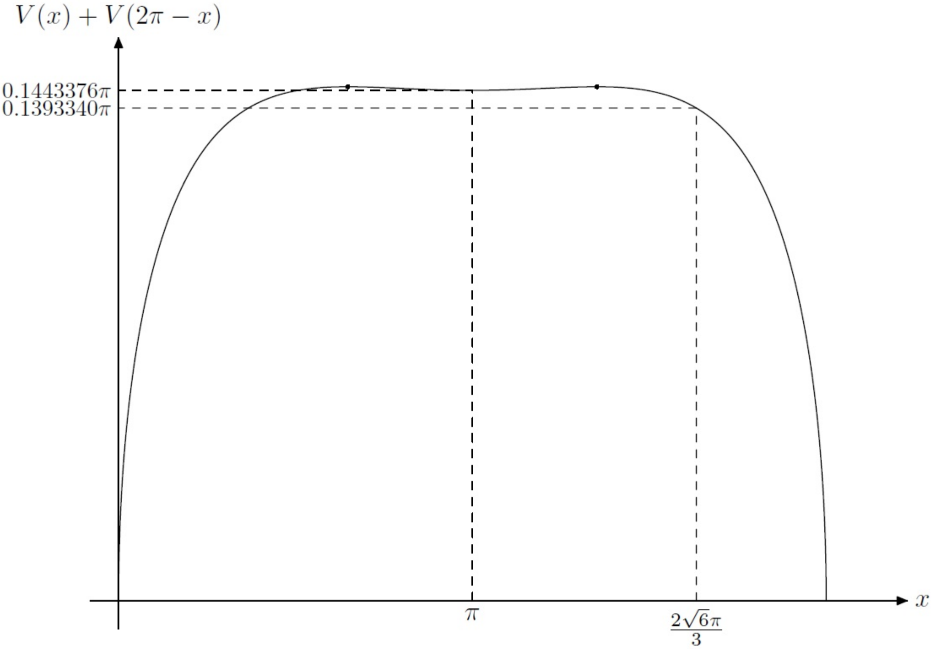


FIGURE 4. Graph of $V(x) + V(2\pi - x)$, showing that cutting the circle into 2 semi-circles does not give the greatest total capacity.

finding the greatest total capacity is equivalent to find

$$\max_{\substack{x_1+x_2+\ldots+x_n=2\pi \\ n\in\mathbb{N}\setminus\{1\}}} \sum_{j=1}^{n} V(x_j)$$

It seems to be very difficult to solve as the number of variables is not known. In TABLE 1, we find the total capacity of the cones when we cut a paper unit circle in different ways (and make cones). The table is not exhaustive, and it can never be. But with the help of these experiments, we make a conjecture that the total capacity is larger when the paper is cut into only two sectors.

As we don't know the number of variables, it is difficult to use Calculus of several variables. Instead, we will prove that when a sector is not too 'big' (which means that the arc length is not too 'long'), it is better not to divide the sector into small sectors. It means that the capacity of the cone made with the original sector is always larger than the total capacity of the cones made with those smaller sectors.

Suppose that we have a circular sector of radius 1 and arc length k, where $0 < k \leq 2\pi$. This sector can be cut into two (or more) smaller sectors. For $0 < x \leq k$, define a function f_k by

$$f_k(x) = V(x) + V(k - x)$$

Ways of cutting the paper circle	Total capacity of cones made
$\pi + \pi$	0.1443376π
$\left(\frac{2\sqrt{6}\pi}{3}\right) + \left(2\pi - \frac{2\sqrt{6}\pi}{3}\right)$	0.1393340π
$\frac{2\pi}{3} + \frac{2\pi}{3} + \frac{2\pi}{3}$	0.1047566π
$\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}$	0.0806871π
$\frac{2\pi}{5} + \frac{2\pi}{5} + \frac{2\pi}{5} + \frac{2\pi}{5} + \frac{2\pi}{5}$	0.0653197π
$0.6480277\pi + 1.3519723\pi$	0.1453532π

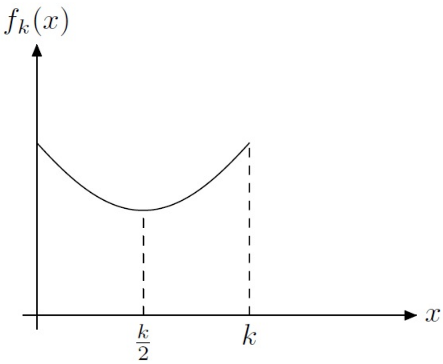
TABLE 1. Total capacity of cones made with different number of sectors.

which is the total capacity of the two cones made by the two smaller sectors with arc lengths x and $k - x$. We can prove that when k is not greater than $\frac{2\sqrt{6}\pi}{3}$, the total capacity of the cones made by those smaller sectors is less than the sector made by the original sector, i.e. $f_k(x) \leq f_k(0)$ for $0 \leq x \leq k$. (Theorem 1) [See reviewer's comment (3)]

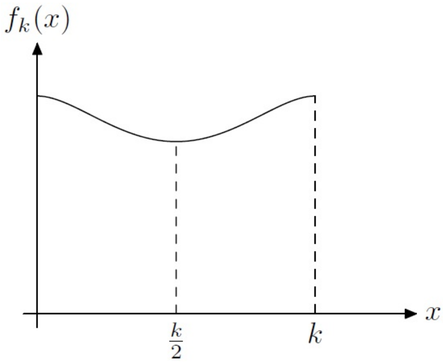
FIGURE 5a shows the graph of $f_k(x)$ for $0 \leq x \leq k$, when $k = \frac{5\pi}{4}$. [See reviewer's comment (4)] The graph is symmetric in the line $x = \frac{k}{2}$ for obvious reasons. f_k is decreasing on the left half of the graph and has the greatest value at $x = 0$ or $x = k$, meaning that we will have the greatest total capacity if we don't cut the sector into smaller sectors. FIGURE 5c show that when k is larger, we can obtain a greater total capacity if we cut the sector. It seems that there is an upper bound for the arc length of the sector that the total capacity will be the largest if we don't cut the sector. This upper bound is indeed $\frac{2\sqrt{6}\pi}{3}$, the value we found in Section 2.1 (See FIGURE 5b).

As mentioned above, as f_k is symmetric in the line $x = \frac{k}{2}$. If f_k is strictly decreasing on $\left[0, \frac{k}{2}\right]$ (assuming $k \leq \frac{2\sqrt{6}\pi}{3}$), then we can conclude that f_k attains its greatest value at $x = 0$ or k. This means that we cannot get a larger total capacity by cutting the sector to two smaller sectors. In order to do that, we proceed to find the derivatives of V up to the fourth order.

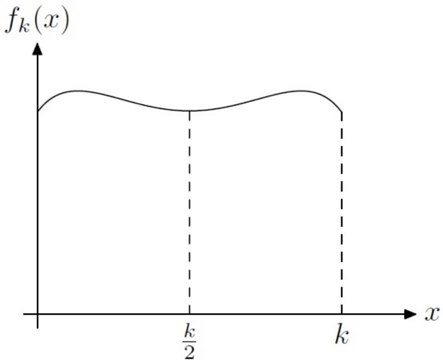
$$V''(x) = \frac{(4\pi^2 - x^2)(8\pi^2 - 9x^2) + x(8\pi^2 x - 3x^3)}{24\pi^2(\sqrt{4\pi^2 - x^2})^3}$$



(a) When $0 < k < \dfrac{2\sqrt{6}\pi}{3}$



(b) When $k = \dfrac{2\sqrt{6}\pi}{3}$



(c) When $\dfrac{2\sqrt{6}\pi}{3} < k < 2\pi$

FIGURE 5. Graph of $f_k(x)$ for different values of k.

$$= \frac{3x^4 - 18\pi^2 x^2 + 16\pi^4}{12\pi^2(\sqrt{4\pi^2 - x^2})^3}.$$

So,

$$\begin{aligned} V^{(3)}(x) &= \frac{1}{12\pi^2}\left[\frac{12x^3 - 36\pi^2 x}{(\sqrt{4\pi^2 - x^2})^3} + \frac{3x(3x^4 - 18\pi^2 x^2 + 16\pi^4)}{(\sqrt{4\pi^2 - x^2})^5}\right] \\ &= \frac{-x^5 + 10\pi^2 x^3 - 32\pi^4 x}{4\pi^2(\sqrt{4\pi^2 - x^2})^5} \end{aligned}$$

and

$$\begin{aligned} V^{(4)}(x) &= \frac{1}{4\pi^2}\left[\frac{-5x^4 + 30\pi^2 X^2 - 32\pi^4}{(\sqrt{4\pi^2 - x^2})^5} + \frac{5x(-x^5 + 10\pi^2 x^3 - 32\pi^4 x)}{(\sqrt{4\pi^2 - x^2})^7}\right] \\ &= \frac{-2\pi^2 x^2 - 32\pi^4}{(\sqrt{4\pi^2 - x^2})^7}. \end{aligned}$$

We are now ready to prove the following Theorem 1.

Theorem 1. *Let $k \leq \dfrac{2\sqrt{6}\pi}{3}$. When a sector with arc length k of a paper unit circle is cut into smaller sectors. The total capacity of the cones made with those smaller sectors is less than that of the cone made with the original sector.*

Proof. We will prove the theorem by showing that f_k is decreasing on $\left[0, \dfrac{k}{2}\right]$. Obviously, we have

$$f'_k(x) = V'(x) - V'(k-x).$$

So, we have

$$f'_k\left(\frac{k}{2}\right) = V'\left(\frac{k}{2}\right) - V'\left(\frac{k}{2}\right) = 0$$

and

$$\begin{aligned} f'_k(0) &= V'(0) - V'\left(\frac{k}{2}\right) \\ &= -V'(k). \end{aligned}$$

Here is where the upper bound $\dfrac{2\sqrt{6}\pi}{3}$ appears. When $k \leq \dfrac{2\sqrt{6}\pi}{3}$, $V'(k) \geq 0$ and hence

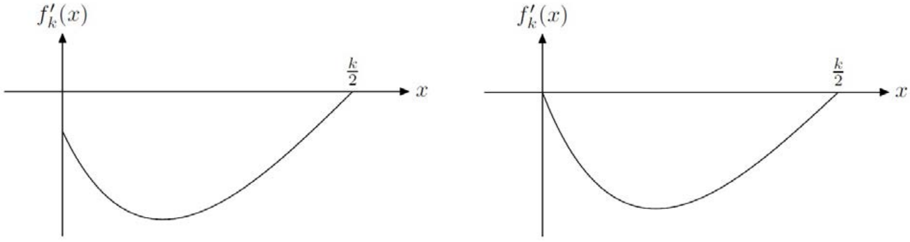
$$f'_k(0) \leq 0.$$

Now we have $f'_k(x) \leq 0$ at $x = 0$ and $x = \dfrac{k}{2}$. What we need is that $f'_k(x) \leq 0$ for all values in between, and it is true as f'_k is a convex function in $\left[0, \dfrac{k}{2}\right]$ (See FIGURE 6aand 6b). Since $v^{(4)}(x) = \dfrac{-2\pi^2 x^2 - 32\pi^4}{(\sqrt{4\pi^2 - x^2})^7}$ is always negative, $V^{(3)}$ is strictly decreasing. When $0 < x < \dfrac{k}{2}$, we have $k - x > \dfrac{k}{2} > x$ and hence $V^{(3)}(k-x) < V^{(3)}(x)$. Therefore,

$$\frac{d^2 f'_k(x)}{dx^2} = V^{(3)}(x) - V^{(3)}(k-x) > 0$$

for all $x \in \left(0, \dfrac{k}{2}\right)$. This proves that f'_k is strictly convex on $\left[0, \dfrac{k}{2}\right]$. So, for $x \in \left(0. \dfrac{k}{2}\right)$, we have

$$\begin{aligned} f'_k(x) &= f'\left(\left(1 - \frac{2x}{k}\right)(0) + \left(\frac{2x}{k}\right)\left(\frac{k}{2}\right)\right) \\ &< \left(1 - \frac{2x}{k}\right) f'(0) + \frac{2x}{k} f'\left(\frac{k}{2}\right) \\ &\leq 0. \end{aligned}$$



(a) When $k = \dfrac{2\sqrt{6}\pi}{3}$ (b) When $0 < k < \dfrac{2\sqrt{6}\pi}{3}$

FIGURE 6. $f_k'(x)$ is a convex function.

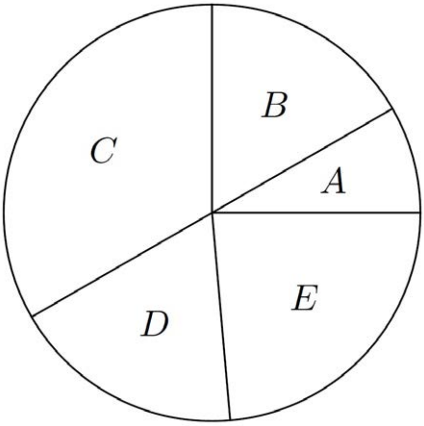
Therefore, f_k is strictly decreasing on $\left[0, \dfrac{k}{2}\right]$ and by symmetry, f_k attains its greatest values when $x = 0$ or k. So for $k \leq \dfrac{2\sqrt{6}\pi}{3}$, one cone is better than two cones, in the sense of obtaining the greatest total capacity. $\square$

2.3. Two Sectors Are the Best

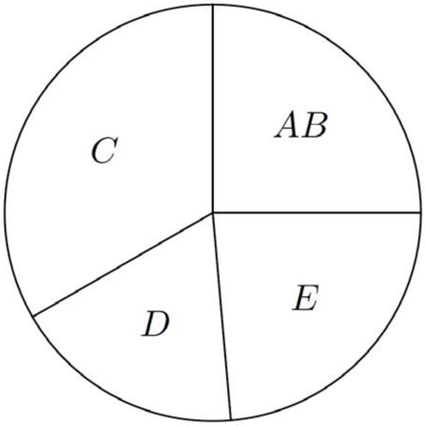
We can now use Theorem 1 to show that the greatest total capacity is obtained when the paper unit circle is cut into 2 sectors only.

If the circle is cut into 3 or more sectors, the smallest two sectors should have a total arc length not greater than $\dfrac{4\pi}{3}$, which is less than $\dfrac{2\sqrt{6}\pi}{3}$. By Theorem 1, the total capacity will increase if we assemble the 2 sectors into one. We can repeat this process as long as there are 3 or more sectors. So we can conclude that when the total capacity is the greatest, there must be only two sectors.

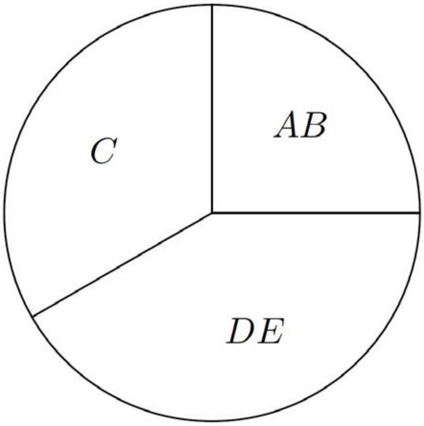
In FIGURE 7, a paper unit circle is cut into five sectors (and the sectors are used to made cones). We can increase the total capacity step by step, each time assembling the two smallest sectors. The process ends when there are only two sectors left and the only way we can further increase the total capacity is to vary the ratio of the arc lengths of the two cones.



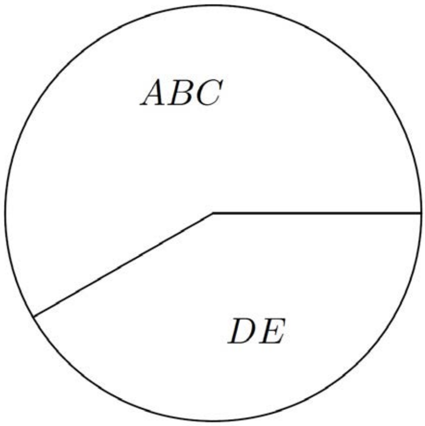
(a) A circle is divided into 5 sectors (Total capacity $= 0.0751010\pi$). A and B are the two smallest sectors and they have a sum of arc length smaller than $\frac{4\pi}{3}$.



(b) The capacity will increase if we combine sectors A and B (Total capacity $= 0.0838362\pi$). D and E become the two smallest sectors and they have a sum of arc length smaller than $\frac{4\pi}{3}$.



(c) The capacity will increase if we combine sectors D and E (Total capacity $= 0.107698\pi$). AB and C become the two smallest sectors and they have a sum of arc length smaller than $\frac{4\pi}{3}$.



(d) The capacity will increase if we combine sectors AB and C (Total capacity $= 0.1447359\pi$). There are only two sectors left, so the greatest total capacity is corresponding to a two-sectors case.

FIGURE 7. A paper unit circle is cut into five sectors. The corresponding total capacity increases as we assembling the two smallest sectors each time.

2.4. The Greatest Total Capacity

In the previous section, we have shown that the total capacity is the greatest when the unit circle is cut into exactly two sectors of some sizes. So the greatest total capacity is the greatest value of $f_{2\pi}$. We have

$$\begin{aligned} f'_{2\pi}(x) &= V'(x) - V'(2\pi - x) \\ &= \frac{8\pi^2 x - 3x^3}{24\pi^2\sqrt{4\pi^2 - x^2}} - \frac{8\pi^2(2\pi - x) - 3(2\pi - x)^3}{24\pi^2\sqrt{4\pi^2 - (2\pi - x)^2}}. \end{aligned}$$

Therefore, $f'_{2\pi}(x) = 0$ when

$$(8\pi^2 x - 3x^3)\sqrt{4\pi^2 - (2\pi - x)^2} = [8\pi^2(2\pi - x) - 3(2\pi - x)^3]\sqrt{4\pi^2 - x^2}$$

Squaring,

$$(8\pi^2 x - 3x^3)^2[4\pi^2 - (2\pi - x)^2] = [8\pi^2(2\pi - x) - 3(2\pi - x)^3]^2(4\pi^2 - x^2)$$

and results a polynomial equation of degree 7. This equation is symmetric about $x = \pi$. So, if we let $x = \pi - u$ (and hence $2\pi - x = \pi + u$). Then

$$(8\pi^2(\pi - u) - 3(\pi - u)^3)^2[4\pi^2 - (\pi + u)^2] = [8\pi^2(\pi + u) - 3((\pi + u))^3]^2(4\pi^2 - (\pi - u)^2)$$

and can be simplified as

$$8\pi u(-5\pi^6 + 51\pi^4 u^2 - 87\pi^2 u^4 + 9u^6) = 0$$

with roots 0, $\pm\sqrt{0.123884497}\pi$, $\pm\sqrt{0.495679566}\pi$, $\pm\sqrt{9.047102603}\pi$. (The factor in the brackets is cubic in u^2 and so this equation can be solved algebraically.)

By symmetry, we can consider those $x \in [0, \pi]$ only, i.e. $x = \pi$, $x \approx 0.295954855\pi$ of 0.648027704π.

Here $f'_{2\pi}(0.295954855\pi) \neq 0$, $f'_{2\pi}(0.0648027704\pi) = 0$.

$f''_{2\pi}(\pi) = \dfrac{1}{18\sqrt{3}\pi}$ and the $f_{2\pi}$ attains a minimum value when $x = \pi$.

$f''_{2\pi}(0.648027704\pi) = -0.021862974 < 0$ and the $f_{2\pi}$ attains its greatest value when $x = 0.648027704\pi$.

Theorem 2. [See reviewer's comment (5)] *The total capacity of cones made with all the sectors cut from a paper unit circle is the greatest when the circle is cut into exactly two sectors, with one of them with arc length approximately equal to 0.648027704π.*

The greatest possible total capacity is

$$f_{2\pi}(0.0648027704\pi) = 0.1453533215\pi$$

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3. Making CONEs with a Paper Triangle

In Section 2, we solved the problem of finding the way to cut a paper unit circle along its radii so that the resultant sectors can be used to make cones with a greatest total capacity. Circles are sometimes regarded as regular polygons of infinitely many sides. If we replace the paper circle with a paper regular polygon in the problem, it is natural to think that a similar result will hold when the number of sides of the polygon is large, as the 'cones' then made will be nearly circular cones. We can expect that in order to obtain the greatest total capacity, we should cut the paper (along the radii) into two parts, with ratio of central angles being approximately 0.648 : 1.352. But when the number of sides is not so large, it would not be so easy to make a conclusion. Is there any regular polygon that we should cut it into more than 2 pieces in order to obtain the greatest total capacity of the 'cones' formed?

3.1. Capacity Problem of Regular Polygons

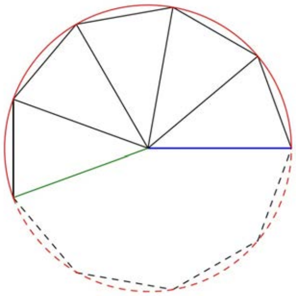
In our project, a polygon refers to a regular polygon which can be inscribed in a unit circle. In order to precisely define our problem, we need the following definitions.

Definition 3. *An n-sided regular polygon has n radii (the radius of regular polygon is the same as its circumradius* [4]*), which divide the polygon into n identical isosceles triangles. If a polygon is cut with some of the radii into several pieces, each piece is either an isosceles triangle or union of several isosceles triangles. We call such a piece a k-portion paper if it is the union of k isosceles triangles (when $k = 1$, it is merely an isosceles triangle.) (See FIGURE 8a and 8c)*

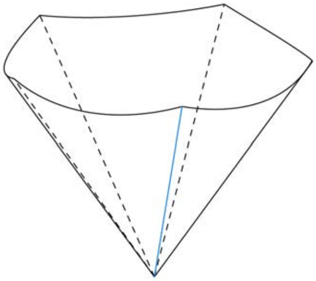
Definition 4. [See reviewer's comment (6)] *A k-portion paper ($1 \leq k \leq n$) can be folded into paper cups (without over- lapping). We call it a k-portion CONE, an n, k-CONE or simply a CONE if it is a part of an infinite right circular cone. The axis of a CONE is the axis of the corresponding right circular cone. (See FIGURE 8b and 8d)*

Definition 5. *We can fill an $n; k$-CONE (making with an n-sided polygon) with water, and the capacity of the CONE varies as the way we hold the CONE. The maximum capacity of the CONE is denoted by $V_{n,k}$.*

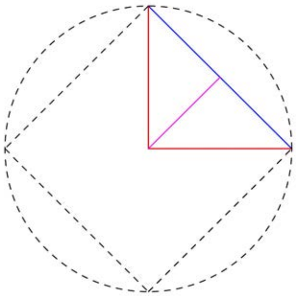
A polygon can be cut into several 1-portion or multi-portion papers and each CONE made with them can be held in some way so that its capacity is the greatest. This project is to find the way to maximize the sum of all the maximum capacities of the CONEs. For convenience, we call this the maximum (or greatest) total capacity of the CONEs.



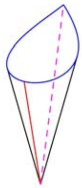
(a) A 5-portion paper in a paper nonagon



(b) 9,5-CONE



(c) A 1-portion paper in a paper square



(d) 4,1-CONE

FIGURE 8. Examples of k-portion papers and n, k-CONEs

3.2. A Paper Triangle

We start from a polygon with the least possible number of sides, a regular triangle. (FIGURE 2.2).

Obviously, we can cut the triangle into 1-portion paper and 2-portion paper, and there are totally 2 ways to make the CONEs with total capacity

(a) $3V_{3,1}$ (FIGURE 10a), and
(b) $V_{3,1} + V_{3,2}$ (FIGURE 10b).

3.3. Maximum Capacity of a $3, 2$-CONE

In order to find $V_{3,2}$, we have to first determine how to hold the 3,2-CONE so that it can be filled with the largest amount of water. As we will see in Theorem 13 [See reviewer's comment (7)], the CONE has the greatest capacity when it is held

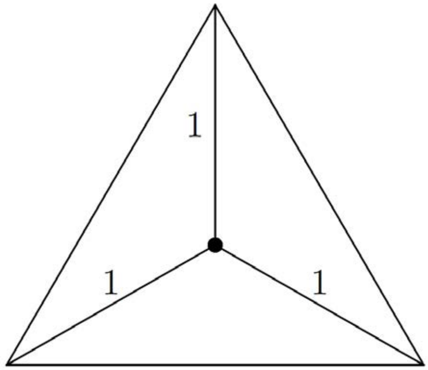
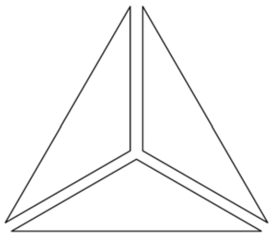
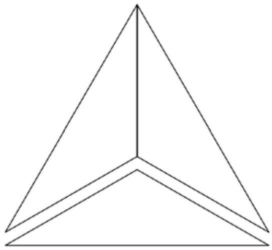


FIGURE 9. Regular triangle



(a) three 3-portion papers



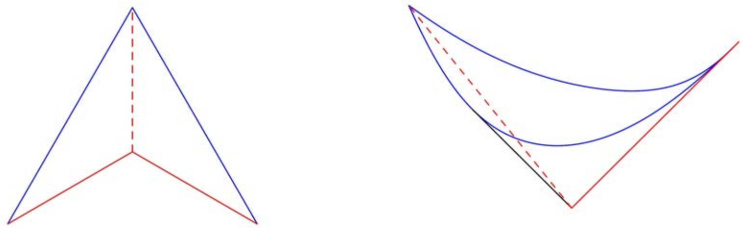
(b) one 1-portion paper and one 2-portion paper

FIGURE 10. Combination of portions of a triangular paper

vertically, which means that its axis is perpendicular to the horizontal. The shape of the water then is a right circular cone with slant height $\sin\dfrac{\pi}{6}$ (FIGURE 12a and FIGURE 12b).

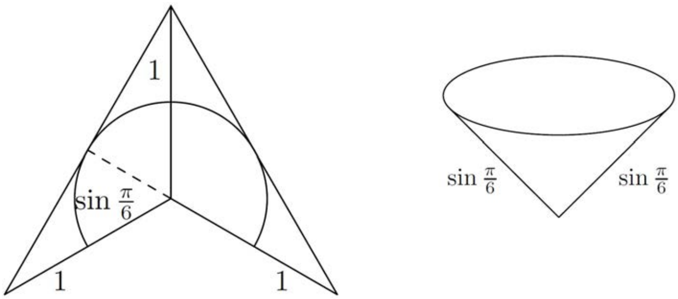
The base radius of the cone is $\dfrac{\frac{4\pi}{3}\sin\frac{\pi}{6}}{2\pi} = \dfrac{1}{3}$ and its height is $\sqrt{\left(\sin\dfrac{\pi}{6}\right)^2 - \left(\dfrac{1}{3}\right)^2} = \dfrac{\sqrt{5}}{6}$. Therefore,

$$V_{3,2} = \frac{1}{3}\pi\left(\frac{1}{3}\right)^2\left(\frac{\sqrt{5}}{6}\right) = \frac{\sqrt{5}\pi}{162} \tag{3}$$



(a) 2-portion paper (b) 3,2-CONE

FIGURE 11. The shape of a 3,2-CONE



(a) three 3-portion papers (b) sector of the inscribed circle of the paper triangle

FIGURE 12. Shape of water in a 3,2-CONE

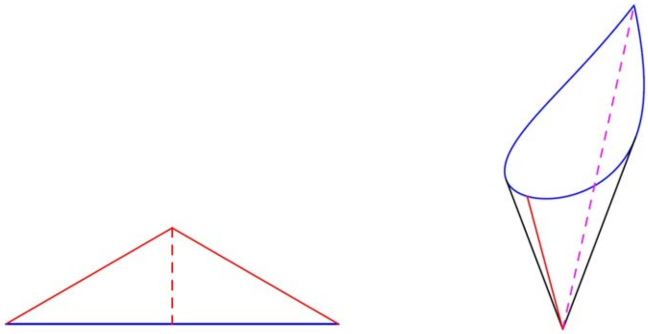
3.4. Maximum Capacity of a 3,1-CONE

The water inside a $3,2$-CONE has the shape of a right circular cone and its volume is easy to find. But a 3,1-CONE is much more complicated. When we fold the 1-portion paper into a CONE (FIGURE 13b), it is obvious that the CONE would not attain its maximum capacity if we hold it vertically.
Here we introduce a 3-dimensional Cartesian coordinate system into the picture so that the vertex of the CONE is the origin, the axis of the CONE is the positive z-axis and the shortest generatrix[1] projects on the positive x-axis. (FIGURE 14). By Theorem 9[2], the maximum capacity will be attained when the CONE is rotated about the y-axis by some positive angle ϕ.

We rotate the CONE about the y-axis through an angle ϕ and fill it with water. The depth of water h is the least value of z of the points on the rim of the CONE

[1]The perimeter of the base of a cone is called the 'directrix', and each of the line segments between the directrix and apex is a 'generatrix' of the lateral surface.[3]

[2]The statement here is not exactly the same as Theorem 9, but they are equivalent.



(a) 1-portion paper (b) 3,1-CONE

FIGURE 13. 3,1-CONE made from 1-portion paper

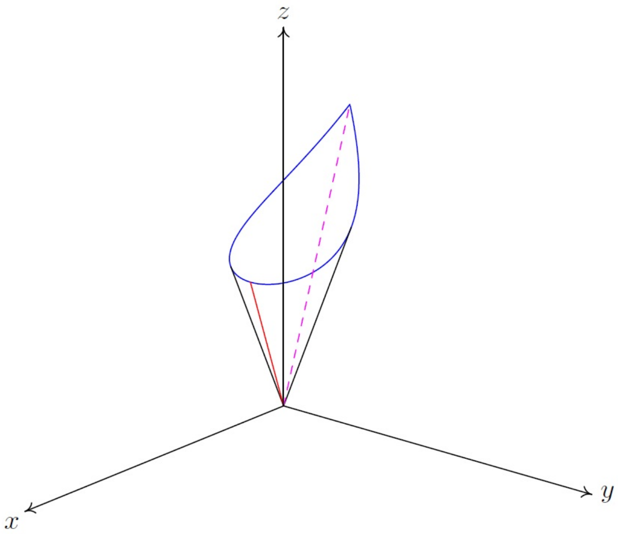


FIGURE 14. Introducing 3D Cartesian coordinate system

(FIGURE 15). The water has the shape of a right circular cone cut by an imaginary inclined plane (so that the water has an elliptical surface).

By Theorem 7, the capacity of the CONE is

$$V = \frac{\pi h^3 \sec^3 \phi \tan^2 \alpha}{3(\sqrt{1 - \tan^2 \phi \tan^2 \alpha})^3}$$

Here α is the semi-vertical angle of the CONE.

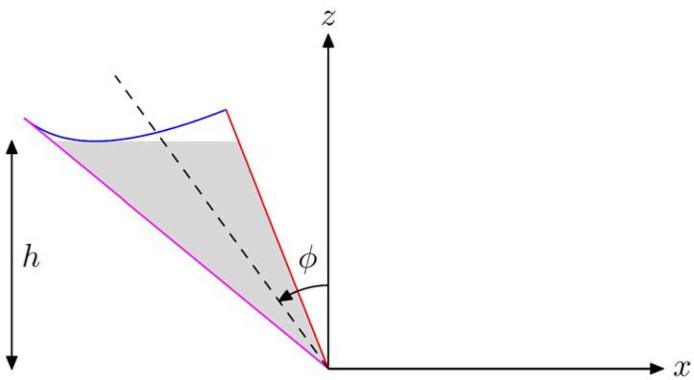


FIGURE 15. CONE rotated through an angle ϕ

3.4.1. Height of the CONE

In this subsection, we need to parametrize the rim of the CONE in order to find h. Suppose that the semi-vertical angle of the CONE is α, and the angle between the z-axis and the axis of the CONE is ϕ. α can be found using the following Theorem:

Theorem 6. *(a) If a sector of central angle x is folded to make a cone (without overlapping) and the semi-vertical angle of the cone is α, then*

$$\sin\alpha = \frac{x}{2\pi}.$$

(b) The semi-vertical angle of an n, k-CONE is given by

$$\sin\alpha = \frac{k}{n}.$$

Proof. (a) Without loss of generality, we may assume that the radius of the sector is 1. Then the arc length of the sector is x. This arc is folded to become the circumference of the base of the cone. So the base radius of the cone is $\dfrac{x}{2\pi}$. As the slant height of the cone is 1, we have $\sin\alpha = \dfrac{x}{2\pi}$.

(b) An n, k-CONE can be embedded in a right circular cone and this right circular cone can be made by a sector with a centre angle equal to the centre angle of the k-portion paper for making the CONE. So we have $x = \left(\dfrac{k}{n}\right)(2\pi)$ and hence

$$\sin\alpha = \frac{k}{n}.$$

□

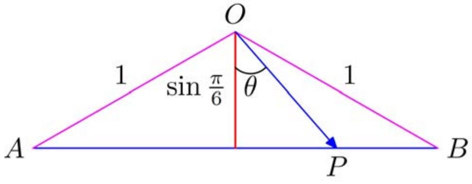


FIGURE 16. P is any point on AB [See reviewer's comment (8)]

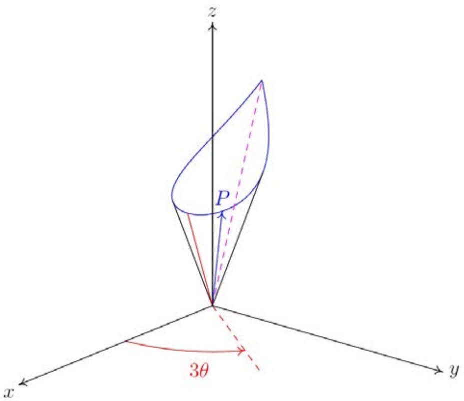


FIGURE 17. P is any point on the rim

For a 3,1-CONE, $\sin\alpha = \dfrac{1}{3}$ and hence $\tan\alpha = \dfrac{1}{\sqrt{8}}$.
In FIGURE 16, OAB is a 1-portion paper with altitude OD. For any point P on AB with such that $\angle POD = \theta$, where $\dfrac{-\pi}{3} \le \theta \le \dfrac{\pi}{3}$. ($\theta$ is positive when P lies between B and D and is negative when P lies between A and D.)

$$\cos\theta = \frac{\sin\dfrac{\pi}{6}}{OP}$$

$$OP = \frac{1}{2}\sec\theta$$

As the CONE is symmetric about the xz-plane, we can parameterize the half of the rim with positive y-coordinate only. The position vector of a point on this part of the rim of the CONE is

$$\frac{sec\theta}{2}\begin{pmatrix}\sin\alpha\cos 3\theta \\ \sin\alpha\sin 3\theta \\ \cos\alpha\end{pmatrix} = \frac{sec\theta}{6}\begin{pmatrix}\cos 3\theta \\ \sin 3\theta \\ \sqrt{8}\end{pmatrix}$$

where $0 \le \theta \le \dfrac{\pi}{3}$. [See reviewer's comment (9)]
If the CONE is rotated about the y-axis as described in the previous section, the above half-rim will be transformed to

$$\frac{sec\theta}{6}\begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}\begin{pmatrix} \cos 3\theta \\ \sin 3\theta \\ \sqrt{8} \end{pmatrix}$$

and hence any point on the rotated half-rim has the z-coordinate

$$z = \frac{\sec\theta}{6}(\sin\phi\cos 3\theta + \sqrt{8}\cos\phi) \tag{4}$$

The depth of water is the least value of z. So we have to find the derivative of z.

$$\begin{aligned}
\frac{dz}{d\theta} &= \frac{\sec\theta\tan\theta}{6}(\sin\phi\cos 3\theta + \sqrt{8}\cos\phi) + \frac{\sec\theta}{6}(-3\sin\phi\sin 3\theta)) \\
&= \frac{\sec\theta\tan\theta}{6}(\sin\phi\cos 3\theta + \sqrt{8}\cos\phi - 3\sin\phi\sin 3\theta\cot\theta) \\
&= \frac{\sec\theta\tan\theta\sin\phi}{6}[4\cos^3\theta - 3\cos\theta + \sqrt{8}\cos\phi - 3(3\sin\theta - 4\sin^3\theta)\cot\theta] \\
&= \frac{\sec\theta\tan\theta\sin\phi}{6}(4\cos^3\theta - 3\cos\theta + \sqrt{8}\cos\phi - 9\sin\theta + 12(1-\cos^2\theta)\cos\theta) \\
&= \frac{\sec\theta\tan\theta\sin\phi}{6}(-8\cos^3\theta + 8\cot\phi) \\
&= -\frac{4\sec\theta\tan\theta\sin\phi}{6}\left[\cos\theta - \frac{1}{\sqrt{2}}(\cot\phi)^{\frac{1}{3}}\right] \\
&\quad \cdot \left[\cos^2\theta + \frac{1}{\sqrt{2}}(\cot\phi)^{\frac{1}{3}}\cos\theta + \frac{1}{2}(\cot\phi)^{\frac{1}{3}}\right]
\end{aligned}$$

3.4.2. Capacity of the CONE for Small Angle of Slant from the Vertical

In order to find the maximum total capacity, we have to consider two cases:

1. when $0 \le \phi \le \alpha$;
2. when $\alpha < \phi < \dfrac{\pi}{2} - \alpha$.

If $0 \le \phi \le \alpha$, then $\cot\phi \ge \cot\alpha = \sqrt{8}$ and hence

$$\cos\theta - \frac{1}{\sqrt{2}}(\cot phi)^{\frac{1}{3}} \le 1\frac{1}{\sqrt{2}}(\sqrt{8})^{\frac{1}{3}} = 0.$$

So, $\frac{dz}{d\theta} \geq 0$ and hence Z is increasing on $\theta \in \left[0, \frac{\pi}{3}\right]$. The least value of z is $\frac{1}{6}(\sin\phi + \sqrt{8}\cos\phi)$ and thus the capacity of CONE is

$$
\begin{aligned}
V &= \frac{\pi\left[\frac{1}{6}(\sin\phi + \sqrt{8}\cos\phi)\right]^3 \sec^3\phi\left(\frac{1}{8}\right)}{3\left[\sqrt{1 - \tan^2\phi\left(\frac{1}{8}\right)}\right]^3} \\
&= \frac{\sqrt{8}\pi}{648}\left(\frac{\tan\phi + \sqrt{8}}{\sqrt{8 - \tan^2\phi}}\right)^3 \\
&= \frac{\sqrt{8}\pi}{648}\left(\frac{\sqrt{8} + \tan\phi}{\sqrt{8} - \tan\phi}\right)^{\frac{3}{2}} \\
&= \frac{\sqrt{8}\pi}{648}\left(-1 + \frac{2\sqrt{8}}{\sqrt{8} - \tan\phi}\right)^{\frac{3}{2}}.
\end{aligned}
$$

V is the greatest when $\tan\phi = \frac{1}{\sqrt{8}}$ and hence the greatest capacity of CONE for $0 \leq \phi \leq \alpha$ is

$$
\begin{aligned}
V &= \frac{\sqrt{8}\pi}{648}\left(-1 + \frac{2\sqrt{8}}{\sqrt{8} - \frac{1}{\sqrt{8}}}\right)^{\frac{3}{2}} \\
&= \frac{\sqrt{14}\pi}{588}.
\end{aligned}
$$

3.4.3. Capacity of the CONE for Large Angle of Slant from the Vertical

If $\alpha < \phi < \frac{\pi}{2} - \alpha$, $\frac{1}{\sqrt{8}} < \cot\phi < \sqrt{8}$ and hence

$$\frac{1}{2} < \frac{1}{\sqrt{2}}(\cot\phi)^{\frac{1}{3}} < 1.$$

So, $\frac{dx}{d\theta} \leq 0$ when $0 \leq \theta \leq \cos^{-1}\left[\frac{1}{\sqrt{2}}(\cot\phi)^{\frac{1}{3}}\right]$.
$\frac{dx}{d\theta} \geq 0$ when $\frac{\pi}{3} \geq \theta \geq \cos^{-1}\left[\frac{1}{\sqrt{2}}(\cot\phi)^{\frac{1}{3}}\right]$.
z is the least when

$$\cos\theta = \left[\frac{1}{\sqrt{2}}(\cot\phi)^{\frac{1}{3}}\right]$$

which implies that

$$\sec\theta = (\sqrt{8}\tan\phi)^{\frac{1}{3}}$$

and

$$\begin{aligned}\cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ &= 4\left(\frac{1}{\sqrt{8}}\cot\phi\right) - 3\left[\frac{1}{\sqrt{2}}(\cot\phi)^{\frac{1}{3}}\right] \\ &= \sqrt{2}\cot\phi - \frac{3}{\sqrt{2}}(\cot\phi)^{\frac{1}{3}}.\end{aligned}$$

Thus, the capacity of the CONE is

$$\begin{aligned}V &= \frac{\pi h^3\sec^3\phi\left(\frac{1}{8}\right)}{3\left(\sqrt{1-\tan^2\phi\left(\frac{1}{8}\right)}\right)^3} \\ &= \frac{\pi}{3}\left(\frac{\frac{1}{2}\sec\phi}{\sqrt{1-\frac{1}{8}\tan^2\phi}}\right)^3\left[\frac{\sec\theta}{6}(\sin\phi\cos 3\theta + \sqrt{8}\cos\phi)\right]^3 \\ &= \frac{\pi}{3}\left(\frac{\sqrt{2}\sec\phi(\sqrt{8}\tan\phi)^{\frac{1}{3}}}{6\sqrt{8-\tan^2\phi}}\right)^3\left[\sin\phi(\sqrt{2}\cot\phi - \frac{3}{\sqrt{2}}(\cot\phi)^{\frac{1}{3}}) + \sqrt{8}\cos\phi\right]^3 \\ &= \frac{\pi}{3}\left(\frac{(\sqrt{8}\tan\phi)^{\frac{1}{3}}}{6\sqrt{8-\tan^2\phi}}\right)^3[\tan\phi(2\cot\phi - 3(\cot\phi)^{\frac{1}{3}}) + 4]^3 \\ &= \frac{\pi}{3}\left(\frac{\sqrt{2}(\tan\phi)^{\frac{1}{3}}}{6\sqrt{8\cot^{\phi}-1}}\right)^3[6\cot\phi - 3(\cot\phi)^{\frac{1}{3}}]^3 \\ &= \frac{\pi}{6\sqrt{2}}\left[\frac{2(\cot\phi)^{\frac{2}{3}}-1}{\sqrt{8\cot^2\phi-1}}\right]^3.\end{aligned}$$

Let $f(u) = \dfrac{2u^{\frac{2}{3}}-1}{\sqrt{8u^2-1}}$ for $u > \dfrac{1}{\sqrt{8}}$, then

$$\begin{aligned}f'(u) &= \frac{(2)\sqrt{8u^2-1}\left(\frac{2}{3}\right)u^{\frac{-1}{3}} - \left(2u^{\frac{2}{3}}-1\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{8u^2-1}}\right)(16u)}{8u^1-1} \\ &= \frac{\frac{4}{3}\left(8u^{\frac{5}{3}} - u^{\frac{-1}{3}}\right) - 8\left(2u^{\frac{5}{3}} - u\right)}{(8u^2-1)^{\frac{3}{2}}} \\ &= \frac{-\frac{16}{3}u^{\frac{5}{3}} + 8u - \frac{4}{3}u^{\frac{-1}{3}}}{(8u^2-1)^{\frac{3}{2}}}\end{aligned}$$

$$= \frac{-\frac{4}{3}u^{\frac{-1}{3}}\left(4u^2 - 6u^{\frac{4}{3}} + 1\right)}{(8u^2-1)^{\frac{3}{2}}}$$

$$= \frac{-\frac{4}{3}u^{\frac{-1}{3}}\left[4\left(u^{\frac{2}{3}}\right)^3 - 6\left(u^{\frac{2}{3}}\right)^2 + 1\right]}{(8u^2-1)^{\frac{3}{2}}}$$

$$= \frac{-\frac{4}{3}u^{\frac{-1}{3}}\left[2\left(u^{\frac{2}{3}}\right) - 1\right]\left[2\left(u^{\frac{2}{3}}\right)^2 - 2\left(u^{\frac{2}{3}}\right) - 1\right]}{(8u^2-1)^{\frac{3}{2}}}$$

$$= \frac{-\frac{8}{3}u^{\frac{-1}{3}}\left(2u^{\frac{2}{3}} - 1\right)\left(u^{\frac{2}{3}} - \frac{1+\sqrt{3}}{2}\right)\left(u^{\frac{2}{3}} - \frac{1-\sqrt{3}}{2}\right)}{(8u^2-1)^{\frac{3}{2}}}.$$

So, $f'(u)\begin{cases} > 0 & \text{if } \frac{1}{2} < u < \frac{1+\sqrt{3}}{2} \\ < 0 & \text{if } \frac{1}{2\sqrt{2}} < u < \frac{1}{2} \text{ or } u > \frac{1+\sqrt{3}}{2} \end{cases}.$

f attains its greatest value when $u = \frac{1+\sqrt{3}}{2}$.

Therefore, V is the greatest when $(\cot\phi)^{\frac{2}{3}} = \frac{1+\sqrt{3}}{2}$.
The greatest capacity of the CONE is

$$V = \frac{\pi}{6\sqrt{2}}\left[\frac{2\left(\frac{1+\sqrt{3}}{2}\right) - 1}{\sqrt{8\left(\frac{1+\sqrt{3}}{2}\right)^3 - 1}}\right]^3$$

$$= \frac{\sqrt{2}\pi}{12}\left(\frac{\sqrt{3}}{\sqrt{(1+\sqrt{3})^3 - 1}}\right)^3$$

$$= \frac{\sqrt{2}\pi}{12(3+2\sqrt{3})^{\frac{3}{2}}}$$

$$= \frac{\sqrt{52\sqrt{3}-90}}{36}\pi.$$

As $\frac{\sqrt{14}}{588}\pi \le \frac{\sqrt{52\sqrt{3}-90}}{36}\pi$, the greatest capacity of 3,1-CONE is $\frac{\sqrt{52\sqrt{3}-90}}{36}\pi$ and the corresponding value of ϕ is given by

$$(\cot\phi)^{\frac{2}{3}} = \frac{1+\sqrt{3}}{2}$$

$$\tan\phi = \left(\frac{2}{1+\sqrt{3}}\right)^{\frac{3}{2}}$$
$$= \sqrt{6\sqrt{3}-10}.$$

3.5. Maximum Total Capacity of the CONEs

In the Sections 3.2, we know that we can divide the regular triangle into either three 1- portion papers, or one 1-portion and one 2-portion paper. The respectively total capacity are

(a) $3V_{3,1} = 3\left[\dfrac{\sqrt{52\sqrt{3}-90}}{36}\right] \approx 0.021512592\pi$, and

(b) $V_{3,1} + V_{3,2} = \dfrac{\sqrt{52\sqrt{3}-90}}{36} + \dfrac{\sqrt{5}\pi}{162} = 0.20973753\pi$.

Therefore, in order to get the maximum total capacity of CONEs made with the paper regular triangle, we should divide it into three 1-portion papers and incline the CONEs made with angle of slant $\tan^{-1}\sqrt{6\sqrt{3}-10}$ from the vertical.

4. Greatest Capacity of a CONE

4.1. Finding the Volume

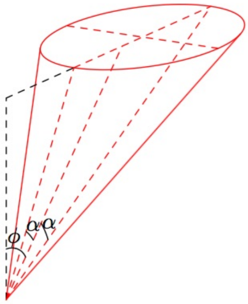
Our problem of finding the capacity of CONEs is actually about finding the volume of inclined cones, or equivalently right circular cones which are cut by inclined planes.

The volume of any cone with base area A and height h, no matter it is regular or non-regular, is given by $\frac{1}{3}Ah$; whereas the area of an ellipse of semi-major axis a and semi-minor axis b is given by πab.

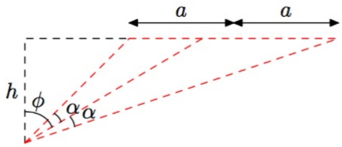
Therefore, in order to find the volume of the CONE with elliptical base, we need to find a and b, thus we can have the area of the elliptical water surface.

With the help of FIGURE 18, we can find a using trigonometry.

$$2a = h\tan(\phi+\alpha) - h\tan(\phi-\alpha)$$
$$= h\left(\frac{\tan\phi+\tan\alpha}{1-\tan\phi\tan\alpha} - \frac{\tan\phi-\tan\alpha}{1+\tan\phi\tan\alpha}\right)$$
$$= \frac{2h(\tan\alpha+\tan\alpha\tan^2\phi)}{1-\tan^2\tan^2\alpha}$$



(a) An inclined right circular cone



(b) The cones side view

FIGURE 18. An inclined right circular cone and its side view

$$a = \frac{h \sec^2 \phi \tan \alpha}{1 - \tan^2 \phi \tan^2 \alpha}. \tag{5}$$

FIGURE 18 is showing one of the possible cases. When the axis of the cone and the vertical make an angle less than the semi-vertical angle of the cone (See FIGURE 19), we have

$$2a = h \tan(\alpha + \phi) + h \tan(\alpha - \phi) \tag{6}$$

As $\tan(\alpha - \phi) = -\tan(\phi - \alpha)$, (6) will also result (5).

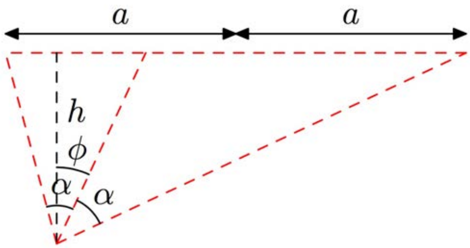


FIGURE 19. Side view of another cone

In order to find b, the trigonometric relation between ϕ, α and a is not enough for us to get the result, so instead of a mass calculation on the original cone, we introduce an imaginary right circular cone with the same vertex and axis as the given inclined cone, and its base passes through the minor axis of the base of the inclined cone (FIGURE 20).

If d is the distance between the centre of the elliptic water surface and the point where the axis of the imaginary cone meets the water surface.

For $\phi > \alpha$,

$$2a = h \tan(\phi + \alpha) - h \tan(\phi - \alpha)$$

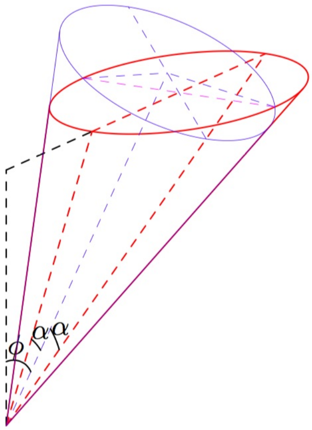
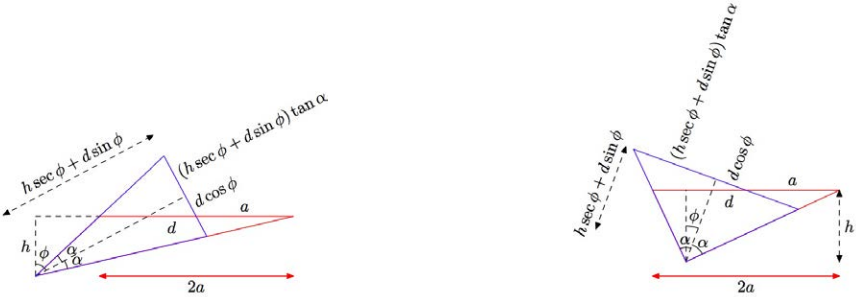


FIGURE 20. Introduce a circular cone to find b



(a) Sideview: if $\phi > \alpha$ (b) Sideview: if $\alpha > \phi$

FIGURE 21. Side view of the CONE and the right circular cone

$$a = d + h\tan\phi - h\tan(\phi - \alpha)$$

whereas, for $\alpha > \phi$,

$$\begin{aligned} 2a &= h\tan(\alpha + \phi) - h\tan(\alpha - \phi) \\ a &= d + h\tan\phi + h\tan(\alpha - \phi) \end{aligned}$$

As $\tan(\alpha - \phi) = -\tan(\phi - \alpha)$, the two cases are indeed equivalent, with a common solution of

$$\begin{aligned} d &= \frac{1}{2}[h\tan(\phi + \alpha) + h\tan(\phi - \alpha)] - h\tan\phi \\ &= \frac{h}{2}\left(\frac{\tan\phi + \tan\alpha}{1 - \tan\phi\tan\alpha} + \frac{\tan\phi - \tan\alpha}{1 + \tan\phi\tan\alpha}\right) - h\tan\phi \end{aligned}$$

$$
\begin{aligned}
&= \frac{h(\tan\phi + \tan\phi\tan^2\alpha)}{1-\tan^2\phi\tan^2\alpha} - h\tan\phi \\
&= \frac{h\tan\phi}{1-\tan^2\phi\tan^2\alpha}(1+\tan^2\alpha - 1 + \tan^2\phi\tan^2\alpha) \\
&= \frac{h\tan\phi\sec^2\phi\tan^2\alpha}{1-\tan^2\phi\tan^2\alpha}
\end{aligned}
$$

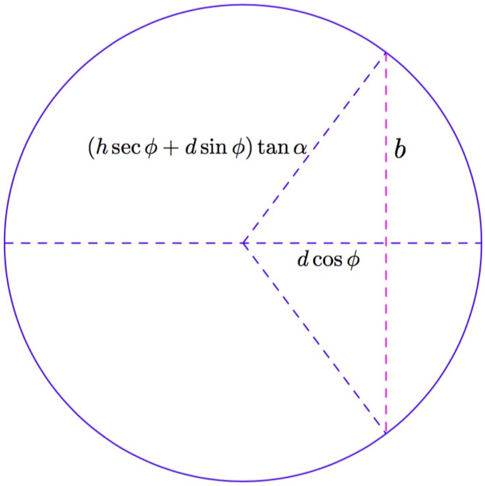


FIGURE 22. Top view of the cone

Now consider the base of the imaginary cone. The radius of the base is

$$
\begin{aligned}
(h\sec\phi + d\sin\phi)\tan\alpha &= \left(h\sec\phi + \frac{h\tan\phi\sec^2\phi\tan^2\alpha\sin\phi}{1-\tan^2\phi\tan^2\alpha}\right)\tan\alpha \\
&= \left(h\sec\phi + \frac{h\tan^2\phi\sec\phi\tan^2\alpha}{1-\tan^2\phi\tan^2\alpha}\right)\tan\alpha \\
&= h\sec\phi\left(1 + \frac{\tan^2\phi\tan^2\alpha}{1-\tan^2\phi\tan^2\alpha}\right)\tan\alpha \\
&= \frac{h\sec\phi\tan\alpha}{1-\tan^2\phi\tan^2\alpha}
\end{aligned}
$$

By Pythagoras' Theorem

$$
\begin{aligned}
b^2 &= \left(\frac{h\sec\phi\tan\alpha}{1-\tan^2\phi\tan^2\alpha}\right)^2 - (d\cos\phi)^2 \\
&= \left(\frac{h\sec\phi\tan\alpha}{1-\tan^2\phi\tan^2\alpha}\right)^2 - \left(\frac{h\tan\phi\sec\phi\tan^2\alpha}{1-\tan^2\phi\tan^2\alpha}\right)^2
\end{aligned}
$$

$$= \left(\frac{h\sec\phi\tan\alpha}{1-\tan^2\phi\tan^2\alpha}\right)^2(1-\tan^2\phi\tan^2\alpha)$$
$$= \frac{(h\sec\phi\tan\alpha)^2}{1-\tan^2\phi\tan^2\alpha}$$
$$b = \frac{h\sec\phi\tan\alpha}{\sqrt{1-\tan^2\phi\tan^2\alpha}}$$

We are now ready to prove the following theorem:

Theorem 7. *The volume of an inclined cone with semi-vertical angle α, angle of slant ϕ of its axis with the vertical and height h (the distance from the vertex to the base) is given by*

$$V = \frac{\pi h^3\sec^3\phi\tan^2\alpha}{2(\sqrt{1-\tan^2\phi\tan^2\alpha})^3}$$

Proof. Therefore, the volume of water is

$$V = \frac{\pi}{3}abh$$
$$= \frac{\pi h}{3}\left(\frac{h\sec^2\phi\tan\alpha}{1-\tan^2\phi\tan^2\alpha}\right)\left(\frac{h\sec\phi\tan\alpha}{\sqrt{1-\tan^2\phi\tan^2\alpha}}\right)$$
$$= \frac{\pi h^3\sec^3\phi\tan^2\alpha}{2(\sqrt{1-\tan^2\phi\tan^2\alpha})^3}$$

□

4.2. Parametrizing the CONE

order to find the depth of water inside a CONE, which is the least value of the z-coordinates of points on the rim, we need to introduce a parametrization of the rim. The rim of a k- portion CONE is the union of k congruent and inter-connecting portions. It is reasonable to consider the parametrization of one of the rims only. First we introduce a 3-dimensional Cartesian coordinate system to describe the CONE and parametrize one of k portions of the rims (FIGURE 23) such that

1. The positive z-axis is the axis of the CONE.
2. The origin is at the vertex of the CONE.
3. The mid-point of one of the portions of the rim is vertically above the positive x-axis. We call this portion of the rim the principal rim, although its selection is arbitrary.

As defined in Definition 2, the CONE is a part of an infinite right circular cone with same semi-vertical angle. It will be of much convenience to parametrize the extended cone first. (See Definition 8)

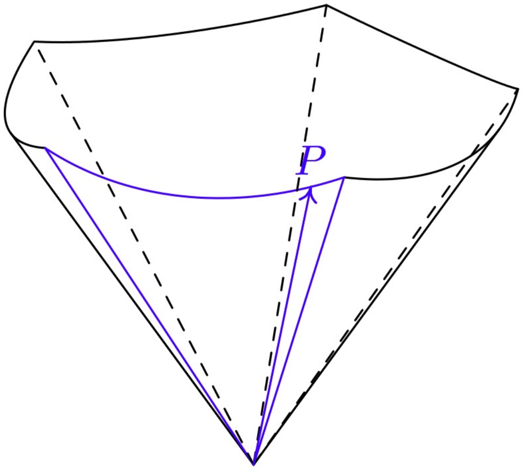


FIGURE 23. The principal rim (blue)

Definition 8. *A n,k-CONE is folded with a k-portion paper of a regular n-gon with radius 1. The k-portion paper is itself inscribed in a sector of the circumscribing unit circle, with centre angle $\dfrac{k}{n}(2\pi)$. If we fold the circular sector, we will obtain a right circular cone with slant height 1, and the original n,k-CONE can be embedded in this right circular cone. (FIGURE 24) We call this right circular cone the extended cone of the n,k-CONE.*

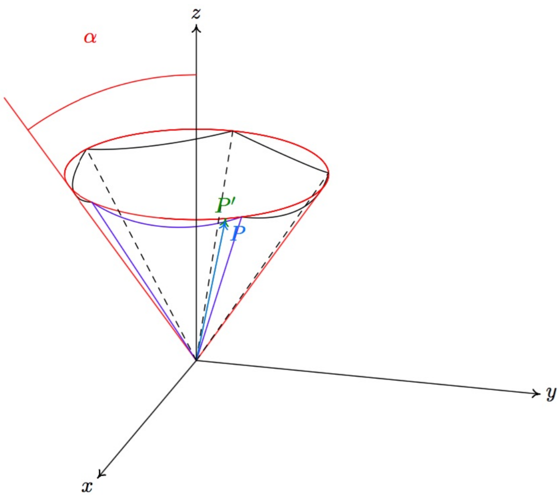


FIGURE 24. The principal rim (blue) and the extended cone (red) of a 9, 5-CONE

By Theorem 6, the semi-vertical angle of the CONE α is given by

$$\sin\alpha = \frac{k}{n}$$

In FIGURE 25, P is a point on a side P_0P_1 of the k-portion paper of a regular n-gon (which is going to be the principal rim of the CONE folded). OP (O is the centre of the corresponding regular polygon) is extended to meet the circumscribed circle at P. Suppose that OP makes an angle θ with OH $(-\pi < \theta < \pi)$), where $H$is the midpoint of P_0P_1. When the circumscribed sector is folded to make the extended cone, P will become a point of the circumference of the base of extended cone. The projection of OP on the xy-plane makes an angle ψ with the positive x axis (FIGURE 26) which is given by

$$(\sin\alpha)(\psi) = (1)(\theta)$$

$$\psi = \frac{n\theta}{k}$$

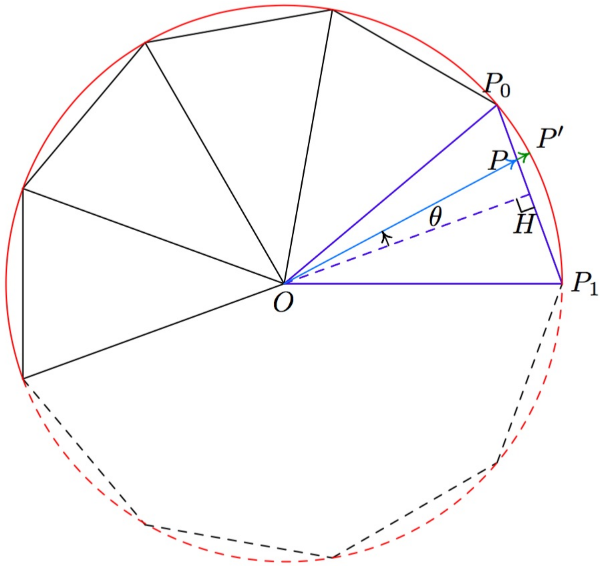


FIGURE 25. Parametrizing the extended cone

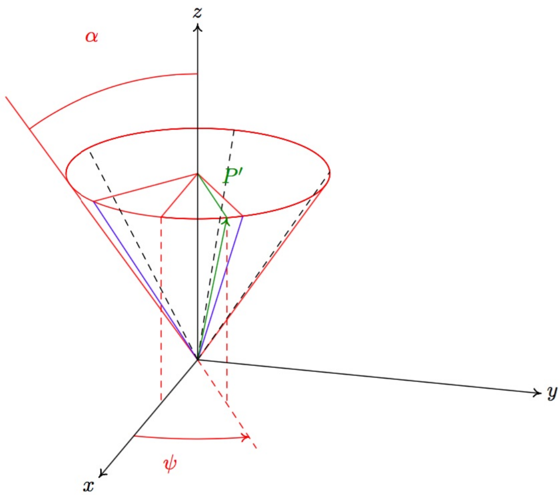


FIGURE 26. The polygonal paper

Hence, the circumference of the base of the extended cone is parametrized as:

$$\overrightarrow{OP'} = \begin{pmatrix} \sin\alpha\cos\psi \\ \sin\alpha\sin\psi \\ \cos\alpha \end{pmatrix} = \begin{pmatrix} \frac{k}{n}\cos\frac{n\theta}{k} \\ \frac{k}{n}\sin\frac{n\theta}{k} \\ \frac{1}{n}\sqrt{n^2-k^2} \end{pmatrix} = \frac{1}{n}\begin{pmatrix} k\cos\frac{n\theta}{k} \\ k\sin\frac{n\theta}{k} \\ \sqrt{n^2-k^2} \end{pmatrix}$$

We say that the extended cone and the respective CONE is at its standard position if it is parametrized as above. In general, we can rotate or incline the CONE (and the extended cone) in different ways. Sometimes, a CONE has its standard position does not guarantee its greatest capacity[1]. We can rotate the extended cone about the z-axis through an angle β. It makes no difference to the extended cone but one can see that the respective CONE look different (rotated), unless β is a multiple of $\frac{2\pi}{k}$. We can also slant the extended cone and the respective CONE in some direction. As the choice of the principal rim is arbitrary, we can describe the slanting as a rotation about the z-axis through an angle ϕ. The value of β can be restricted to be in the interval $\left[0, \frac{\pi}{k}\right]$ as the CONE is k-fold rotational symmetrical about its axis and is reflectional symmetrical. We can always choose a suitable principal rim so that ϕ is non-negative and as it doesnt make sense to have $\phi > \frac{\pi}{2} - \alpha$ (α is the semi-vertical angle of the CONE), we can assume that

[1]Later we will show that only 1-portion CONEs should be slanted in order to obtain the greatest capacity.

$\phi \in \left[0, \frac{\pi}{2} - \alpha\right]$. As we are going to parametrize the 'first rim' only, the parameter θ is in the interval $\left[-\frac{\pi}{n}, \frac{\pi}{n}\right]$.

When the CONE (and the extended cone) is rotated about the z-axis through β, the projection of the rim of CONE on the xy-plane is also rotated through β(FIGURE 27).

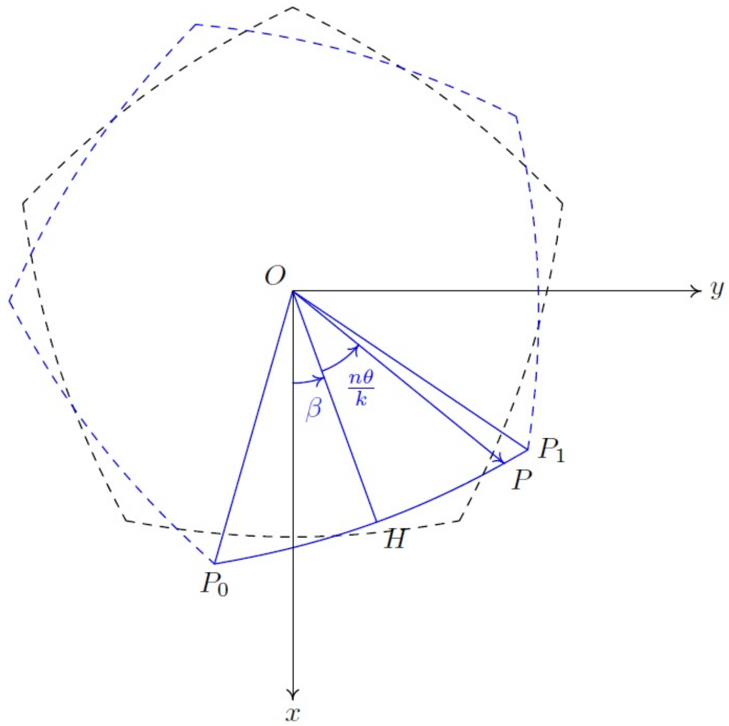
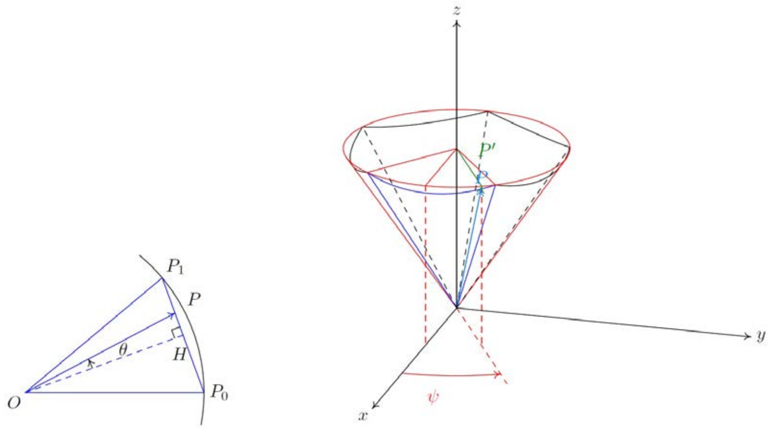


FIGURE 27. Projection of CONE rim onto xy-plane

From the figure, P_0, H, P, P_1 are all rotated about O such that OH makes an angle of β with the positive x-axis. Here the projection of the position vector $\overrightarrow{OP'}$ on the xy-plane makes an angle $\psi = \frac{n\theta}{k} + \beta$ with the positive x-axis. Hence, the circumference of the base of the rotated extended cone is given by

$$\frac{1}{n}\begin{pmatrix} k\cos\left(\frac{n\theta}{k} + \beta\right) \\ k\sin\left(\frac{n\theta}{k} + \beta\right) \\ \sqrt{n^2 - k^2} \end{pmatrix}.$$



(a) (b)

FIGURE 28. Reparametrizing the extended cone to CONE

Note that the 'principal rim' of the CONE can be obtained by scaling $\overrightarrow{OP}$ by the length of OP (FIGURE 28).

The height OH of the triangle OP_0P_1 is $\cos\frac{2\pi}{2n}=\cos\frac{\pi}{n}$ whereas the length of the OP is $\cos\frac{\pi}{n}\sec\theta$. Hence when $\phi=0$, the principal rim of the CONE is parametrized as:

$$\frac{1}{n}\cos\frac{\pi}{n}\sec\theta\begin{pmatrix} k\cos\left(\frac{n\theta}{k}+\beta\right)\\ k\sin\left(\frac{n\theta}{k}+\beta\right)\\ \sqrt{n^2-k^2}\end{pmatrix}.$$

Then we slant the CONE by rotating it through an angle $\phi\in\left[0.\frac{\pi}{2}-\alpha\right]$, suing the transformation matrix and results the position vector of a point on the principal rim

$$\frac{1}{n}\cos\frac{\pi}{n}\sec\theta\begin{pmatrix}\cos\phi & 0 & \sin\phi\\ 0 & 1 & 0\\ -\sin\phi & 0 & \cos\phi\end{pmatrix}\begin{pmatrix} k\cos\left(\frac{n\theta}{k}+\beta\right)\\ k\sin\left(\frac{n\theta}{k}+\beta\right)\\ \sqrt{n^2-k^2}\end{pmatrix}.$$

In particular, the z coordinates of a point on the principal rim is

$$z=\frac{1}{n}\cos\frac{\pi}{n}\sec\theta\left[\sqrt{n^2-k^2}\cos\phi-k\sin\phi\cos\left(\frac{n\theta}{k}+\beta\right)\right]. \tag{7}$$

4.3. To Rotate or Not to Rotate

4.3.1. z Is the Least for a Non-positive θ

When a 3-dimensional Cartesian coordinate system is introduced so that the vertex of the n,k-CONE is the origin. The rim of the a portion of the CONE has the z-coordinate

$$z=\frac{1}{n}\cos\frac{\pi}{n}\sec\theta\left[\sqrt{n^2-k^2}\cos\phi-k\sin\phi\cos\left(\frac{n\theta}{k}+\beta\right)\right],$$

where $\phi\in\left[0,\frac{\pi}{2}-\alpha\right]=\left[0,\tan^{-1}\frac{\sqrt{n^2-k^2}}{k}\right]$, $\beta\in\left[0,\frac{\pi}{k}\right]$ and $\theta\in\left[-\frac{\pi}{n},\frac{\pi}{n}\right]$.

z depends on three variables, namely ϕ, β and θ. ϕ and β decide the position of the CONE and θ is a parameter describing the rim. We write $z=z(\theta)$ to indicate that z is a function of θ. With ϕ and β fixed, the water depth h is the least value of z.

If z is the least when $\theta=\theta_0$, we can prove that $\theta_{)}\leq 0$. For instance, suppose that $\theta_0>0$. We will prove that $z(-\theta_0)\leq z(\theta_0)$. As $\sec\theta_0=\sec(-\theta_0)$, we only have to compare $\cos\left(\frac{n\theta_0}{k}+\beta\right)$ with $\cos\left(\frac{n(-\theta_0)}{k}+\beta\right)$. Since $\theta_0\in\left(0,\frac{\pi}{n}\right]$,

$$\pi\leq-\frac{\pi}{k}\leq\frac{n(-\theta_0)}{k}+\beta<\frac{n\theta_{)}}{k}+\beta\leq\frac{2\pi}{k}.$$

If $k\geq 2$,

$$\left|\frac{n(\theta_0)}{k}+\beta\right|\leq\left|\frac{n(\theta_0)}{k}\right|+\beta=\frac{n\theta_0}{k}+\beta\leq\pi.$$

As cosine is even and is strictly decreasing on $[0,\pi]$,
$\cos\left(\frac{n\theta_0}{k}+\beta\right)\leq\cos\left(\frac{n(-\theta_0)}{k}+\beta\right)$.
If $k=1$ and $\frac{n\theta_0}{k}+\beta\leq\pi$, the above argument still holds and

$$\cos\left(\frac{n\theta_0}{k}+\beta\right)\leq\cos\left(\frac{n(-\theta_0)}{k}+\beta\right).$$

If $k=1$ and $\frac{n\theta_0}{k}+\beta>\pi$, then $0\leq 2\pi-\beta-n\theta_0<\pi$. So $0\leq\frac{n(-\theta_0)}{k}+\beta<2\pi-\beta-n\theta_0\leq\pi$ if $\beta-n\theta_0\geq 0$ and $\left|\frac{n(-\theta_0)}{k}+\beta\right|=n\theta_0-\beta+2\pi-2n\theta_0=2\pi-\beta-n\theta_0\leq\pi$ if $\beta-n\theta_0<0$. Therefore, we have

$$\cos\left(\frac{n\theta_0}{k}+\beta\right)=\cos(2\pi-\beta-n\theta_0)\leq\cos\left(\frac{n(-\theta_0)}{k}+\beta\right).$$

So, in all cases, $\cos\left(\frac{n\theta_0}{k}+\beta\right) \leq \cos\left(\frac{n(-\theta_0)}{k}+\beta\right)$ and hence $z(-\theta_0) \leq (\theta_0)$. When z attains its least value, $\theta \leq 0$.

4.3.2. The capacity is the largest when $\beta = \frac{\pi}{k}$

The position of the CONE is determined by two parameters, β and ϕ. When ϕ is fixed, the capacity is proportional to the cube of the water depth. We will prove in this section that water depth is the greatest when $\beta = \frac{\pi}{k}$. Suppose that h is the least value of z when $\beta = \frac{\pi}{k}$. As shown in Section 4.3.1, $h = z(\theta_0)$ for some $\theta_0 \leq 0$. Although it is not easy to find the least value of $z(\theta)$ for each β (with ϕ fixed), we can prove that the water depth is the greatest when $\beta = \frac{\pi}{k}$ by proving that for any $\beta \in \left[0, \frac{\pi}{k}\right)$, there exists a θ such that $z(\theta) \leq h$.
Here we write $z = z(\beta, \theta)$ to indicate that z depends on both β and θ. For instance, $h = z\left(\frac{\pi}{k}, \theta_0\right)$.
If $\beta \in \left[\frac{n\theta_0}{k} + \frac{\pi}{k}, \frac{\pi}{k}\right)$, we can find a $\theta \in (\theta_0, 0]$ such that $\frac{n\theta}{k} + \beta = \frac{n\theta_0}{k} + \frac{\pi}{k}$. (See FIGURE 29) As secant is decreasing for $\theta \in (\theta_0, 0]$,

$$\sec\theta_0 \geq \sec\theta$$

and hence

$$\begin{aligned}
h &= \frac{1}{n}\cos\frac{\pi}{n}\sec\theta_0\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\left(\frac{n\theta_0}{k}+\frac{\pi}{k}\right)\right] \\
&= \frac{1}{n}\cos\frac{\pi}{n}\sec\theta_0\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\left(\frac{n\theta}{k}+\beta\right)\right] \\
&\geq \frac{1}{n}\cos\frac{\pi}{n}\sec\theta\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\left(\frac{n\theta}{k}+\beta\right)\right] \\
&= z(\beta, \theta)
\end{aligned}$$

and in particular, $h \geq z\left(\frac{n\theta_0}{k} + \frac{\pi}{k}, 0\right)$.

If $\beta \in \left[0, \frac{n\theta_0}{k} + \frac{\pi}{k}\right)$, then

$$\begin{aligned}
h &\geq z\left(\frac{n\theta_0}{k} + \frac{\pi}{k}, 0\right) \\
&= \frac{1}{n}\cos\frac{\pi}{n}\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\left(\frac{n\theta_0}{k}+\frac{\pi}{k}\right)\right] \\
&\geq \frac{1}{n}\cos\frac{\pi}{n}\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\beta\right]
\end{aligned}$$

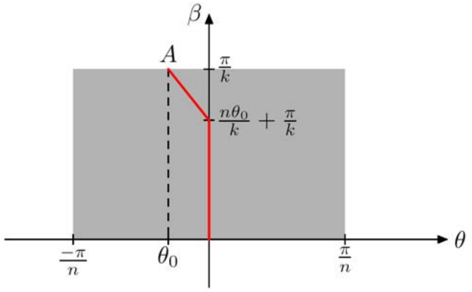


FIGURE 29. Along the red line segments, z is the largest at A.

$$= z(\beta, 0).$$

We can now conclude that

Theorem 9. *For fixed* $\phi \in \left[0, \frac{\pi}{2} - \alpha\right]$*, the water depth is the largest when* $\beta = \frac{\pi}{k}$*.*

From now on, we take $\beta = \frac{\pi}{k}$ and thus

$$z(\theta) = \frac{1}{n} \cos \frac{\pi}{n} \sec \theta \left[\sqrt{n^2 - k^2} \cos \phi - k \sin \phi \cos \left(\frac{n\theta}{k} + \frac{\pi}{k} \right) \right], \tag{8}$$

which depends on ϕ and θ only.

4.3.3. The Water Depth

By Theorem 7, $V = \dfrac{\pi h^3 \sec^3 \phi \tan^2 \alpha}{3(\sqrt{1 - \tan^2 \phi \tan^2 \alpha})^3}$. In our problem, h is the least value of $z(\theta)$ and depends on ϕ only. So V is a function of ϕ. However, it is very difficult to express V (or h) in terms of phi. Instead, we will express V in terms of θ, where θ is the value that makes z the largest for such a ϕ.

First we have to maximize z for a fixed ϕ.

$$\begin{aligned} \frac{dz}{d\theta} = \frac{1}{n} \cos \frac{\pi}{n} \sec \theta \Bigg[& \sqrt{n^2 - k^2} \cos \phi \tan \theta \\ & - k \sin \phi \tan \theta \cos \left(\frac{n\theta}{k} + \frac{\pi}{k} \right) + \frac{\pi}{k} + n \sin \phi \sin \left(\frac{n\theta}{k} + \frac{\pi}{k} \right) \Bigg]. \end{aligned} \tag{9}$$

Let

$$f(\theta) = \sqrt{n^2 - k^2} \cos \phi - k \sin \phi \cos \left(\frac{n\theta}{k} + \frac{\pi}{k} \right) + n \sin \phi \sin \left(\frac{n\theta}{k} + \frac{\pi}{k} \right) \cot \theta \tag{10}$$

for $\theta \in \left[-\frac{\pi}{n}, 0 \right)$. Then

$$\frac{dz}{d\theta} = \frac{1}{n} \cos \frac{\pi}{n} \sec \theta \tan \theta f(\theta) \tag{11}$$

when $\theta \neq 0$. Here we have

$$\begin{aligned} f'(\theta) &= \sin\phi \left[n\sin\left(\frac{n\theta+\pi}{k}\right) - n\sin\left(\frac{n\theta+\pi}{k}\right)\csc^2\theta + \frac{n^2}{k}\cos\left(\frac{n\theta+\pi}{k}\right)\cot\theta \right] \\ &= \frac{n\sin\phi}{k\sin^2\theta}\left[-k(1-\sin^2\theta)\sin\left(\frac{n\theta+\pi}{k}\right) + n\sin\theta\cos\theta\cos\left(\frac{n\theta+\pi}{k}\right)\right] \\ &= \frac{n\sin\phi\cos\theta}{k\sin^2\theta}\left[-k\cos\theta\sin\left(\frac{n\theta+\pi}{k}\right) + n\sin\theta\cos\left(\frac{n\theta+\pi}{k}\right)\right] \\ &= \frac{n\sin\phi\cos\theta}{2k\sin^2\theta}\left[(n-k)\sin\left(\frac{(n+k)\theta+\pi}{k}\right) - (n+k)\sin\left(\frac{(n-k)\theta+\pi}{k}\right)\right]. \end{aligned}$$

So, we have

$$f'(\theta) = \frac{n\sin\phi\cos\theta}{2k\sin^2\theta} g(\theta) \tag{12}$$

where

$$g(\theta) = (n-k)\sin\left(\frac{(n+k)\theta+\pi}{k}\right) - (n+k)\sin\left(\frac{(n-k)\theta+\pi}{k}\right). \tag{13}$$

We have

$$g'(\theta) = \frac{(n-k)(n+k)}{k}\left[\cos\left(\frac{(n+k)\theta+\pi}{k}\right) - \cos\left(\frac{(n-k)\theta+\pi}{k}\right)\right]. \tag{14}$$

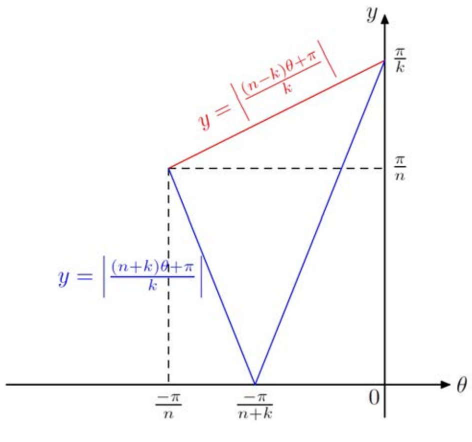


FIGURE 30. $\left|\frac{(n-k)\theta+\pi}{k}\right| > \left|\frac{(n+k)\theta+\pi}{k}\right|$ whenever $-\frac{\pi}{n} < \theta < 0$

As we can see in FIGURE 30, for $\theta \in \left(-\frac{\pi}{n}, 0\right)$,
$\pi > \left|\frac{(n-k)\theta+\pi}{k}\right| > \left|\frac{(n+k)\theta+\pi}{k}\right|$. So we have

$$\cos\left(\frac{(n+k)\theta+\pi}{k}\right) > \cos\left(\frac{(n-k)\theta+\pi}{k}\right)$$

and hence by (14), $g'(\theta) > 0$. So, g is strictly increasing.
As $g(0) = -2k\sin\frac{\pi}{k} \leq 0$,

$$g(\theta) < 0 \quad \text{for } \theta \in \left(-\frac{\pi}{n}, 0\right) \tag{15}$$

By (12),

$$f'(\theta) < 0$$

and thus f is strictly decreasing for $\theta \in \left[-\frac{\pi}{n}, 0\right)$.
From (10), when $k = 1$,

$$f(\theta) = \sqrt{n^2-1}\cos\phi + \sin\phi\cos n\theta - \frac{n\sin\phi\sin n\theta}{\tan\theta}.$$

By l'Hôpital's Rule,

$$\lim_{\theta\to 0^-}\frac{\sin n\theta}{tan\theta} = \lim_{\theta\to 0^-}\frac{n\cos n\theta}{\sec^2\theta} = n.$$

Therefore,

$$\lim_{\theta\to 0^-} f(\theta) = \sqrt{n^2-1}\cos\phi + \sin\phi - n^2\sin\phi$$
$$\lim_{\theta\to 0^-} f(\theta) = \sqrt{n^2-1}\cos\phi(1-\tan\phi\sqrt{n^2-1}) \tag{16}$$

We are now ready to prove the following theorems.

Theorem 10. *When $k = 1$ and $0 \leq \phi \leq \alpha$, z is the least if and only if $\theta = 0$.*

Proof. If $0 \leq \phi \leq \alpha$, $0 \leq \tan\phi \leq \tan\alpha = \frac{1}{\sqrt{n^2-1}}$. By (16), $\lim_{\theta\to 0^-} f(\theta) \geq 0$. Therefore, by (11), $\frac{dz}{d\theta} < 0$ for $\theta \in \left[-\frac{\pi}{n}, 0\right)$. So z attains its least value if and only if $\theta = 0$. □

Theorem 11. *When $k = 1$ and $\alpha \leq \phi \leq \frac{pi}{2} - \alpha$, z is the least for some $\theta < 0$ and this θ is unique.*

Proof. If $\alpha \leq \phi \leq \frac{pi}{2} - \alpha$, $\tan\phi > \tan\alpha = \frac{1}{\sqrt{n^2-1}}$ and $\tan\phi \leq \cot\alpha = \sqrt{n^2-1}$.
By (16), $\lim_{\theta\to 0^-} f(\theta) < 0$. We also have

$$f\left(-\frac{\pi}{n}\right) = \sqrt{n^2-1}\cos\phi - \sin\phi$$

$$= \sqrt{n^2-1}\cos\phi\left(1 - \frac{1}{sqrtn^2-1}\tan\phi\right)$$
$$\geq 0.$$

As f is strictly decreasing for $\theta \in \left[-\frac{\pi}{n}, 0\right)$, $f(\theta) = 0$ for a unique $\theta = \theta_0$. Moreover, we have

$$f(\theta)\begin{cases} > 0 & \text{if } -\frac{\pi}{n} \leq \theta < \theta_0 \\ < 0, & \text{if } \theta_0 < \theta < 0. \end{cases}.$$

By (11),

$$\frac{dz}{d\theta}\begin{cases} > 0 & \text{if } -\frac{\pi}{n} \leq \theta < \theta_0 \\ < 0, & \text{if } \theta_0 < \theta < 0 \end{cases}.$$

z attains its least value at $\theta = \theta_0$ □

When $k \geq 2$, note that

$$f\left(-\frac{\pi}{n}\right) = \sqrt{n^2-k^2}\cos\phi - k\sin\phi$$
$$= \begin{cases} \sqrt{n^2-k^2} > 0 & \text{if } \phi = 0 \\ \sqrt{n^2-k^2}\cos\phi\left(1 - \frac{k}{\sqrt{n^2-k^2}}\tan\phi\right) \geq 0 & \text{if } \phi > 0. \end{cases}$$

as $0 \leq \phi \leq \frac{\pi}{2} - \alpha$ implies that $\tan\phi \leq \cot\alpha = \frac{\sqrt{n^2-k^2}}{k}$.
Also, we have $\lim_{\theta\to 0^-} f(\theta) = -\infty$. As f is strictly decreasing, there exists a unique $\theta_0 \in \left[-\frac{\pi}{n}, 0\right)$ such that $f(\theta_0) = 0$. Moreover, we have

$$f(\theta)\begin{cases} < 0, & \text{if } \theta_0 < \theta < 0 \\ > 0 & \text{if } -\frac{\pi}{n} \leq \theta < \theta_0 \end{cases}. \tag{17}$$

For multi-portion CONE, we have the following theorems.

Theorem 12. *When $k \geq 2$ and $\phi > 0$, z is the least for some $\theta < 0$ and this θ is unique.*

Proof. By (11 and (17), if θ_0 is the unique θ such that $f(\theta_0) = 0$, then $\frac{dz}{d\theta} < 0$ when $-\frac{\pi}{n} \leq \theta < \theta_0$ and $\frac{dz}{d\theta} > 0$ when $\theta_0 < \theta < 0$. z attains its least value at $\theta = \theta_0$. □

Theorem 13. *When $k \geq 2$ and $\phi = 0$, z is the least if and only if $\theta = 0$.*

Proof. By (8), when $\phi = 0$, $z(\theta) = \frac{1}{n}\cos\frac{\pi}{n}\sec\theta\sqrt{n^2-k^2}$. By the property of secant function, z is the least if and only if $\theta = 0$. □

4.3.4. Capacity of An 1-portion CONE When ϕ Is Small

By Theorem 10, when $k=1$ and $0 \le \phi \le \alpha$, the water depth is

$$h = \frac{1}{n}\cos\frac{\pi}{n}\left[\sqrt{n^2-1}\cos\phi + \sin\phi\right]$$

and by Theorem 7, the capacity of the $n,1$-CONE is given by

$$\begin{aligned}
V &= \frac{\pi h^3 \sec^3\phi \tan^2\alpha}{3(\sqrt{1-\tan^2\phi\tan^2\alpha})^3} \\
&= \frac{\pi h^3 \sec^3\phi\left(\dfrac{1}{n^2-1}\right)}{3\left(\sqrt{1-\dfrac{1}{n^2-1}\tan^2\phi}\right)^3}\left[\frac{1}{n}\cos\frac{\pi}{n}(\sin\phi+\sqrt{n^2-1}\cos\phi)\right]^3 \\
&= \frac{\pi\sqrt{n^2-1}\cos^3\dfrac{\pi}{n}}{3n^3\left(\sqrt{n^2-1}-\tan^2\phi\right)^3}(\tan\phi+\sqrt{n^2-1})^3 \\
&= \frac{\pi\sqrt{n^2-1}\cos^3\dfrac{\pi}{n}}{3n^3}\left[\sqrt{\frac{(\tan\phi+\sqrt{n^2-1})^2}{(\sqrt{n^2-1}-\tan\phi)(\sqrt{n^2-1}+\tan\phi)}}\right]^3 \\
&= \frac{\pi\sqrt{n^2-1}\cos^3\dfrac{\pi}{n}}{3n^3}\left(\sqrt{\frac{\sqrt{n^2-1}+\tan\phi}{\sqrt{n^2-1}-\tan\phi}}\right)^3
\end{aligned}$$

Theorem 14. *When $k=$ and $0 \le \phi \le \alpha$, the capacity of an $n,1$-CONE is*

$$V = \frac{\pi\sqrt{n^2-1}\cos^3\dfrac{\pi}{n}}{3n^3}\left(\sqrt{\frac{2\sqrt{n^2-1}}{\sqrt{n^2-1}-\tan\phi}-1}\right)^3$$

Since $0 \le \phi \le \alpha$, $\tan\alpha = \dfrac{1}{n^2-1}$. Therefore

$$\begin{aligned}
V &\le \frac{\pi\sqrt{n^2-1}\cos^3\dfrac{\pi}{n}}{3n^3}\left(\sqrt{\frac{2\sqrt{n^2-1}}{\sqrt{n^2-1}-\tan\alpha}-1}\right)^3 \\
&= \frac{\pi\sqrt{n^2-1}\cos^3\dfrac{\pi}{n}}{3n^3}\left(\sqrt{\frac{2\sqrt{n^2-1}}{\sqrt{n^2-1}-\dfrac{1}{n^2-1}}-1}\right)^3 \\
&= \frac{\pi\sqrt{n^2-1}\cos^3\dfrac{\pi}{n}}{3n^3}\left(\sqrt{\frac{n^2}{n^2-2}}\right)^3
\end{aligned}$$

and hence

$$V \le \frac{\pi\sqrt{n^2-1}\cos^3\dfrac{\pi}{n}}{3(\sqrt{n^2-2})^3} \tag{18}$$

The equality of (18) holds if and only if $\phi = \alpha$.

Theorem 15. *When $k = 1$ and $0 \le \phi \le \alpha$, the greatest capacity of an $n,1$-CONE is* $\dfrac{\pi\sqrt{n^2-1}\cos^3\dfrac{\pi}{n}}{3(\sqrt{n^2-2})^3}$

4.3.5. Capacity of a Multi-portion CONE

In Section 4.3.3, we proved that when $k \ge 2$, or when $k = 1$ and $\phi > \alpha$, z is the least at a unique $\theta = \theta_0$ and $\left.\dfrac{dz}{d\theta}\right|_{\theta=\theta_0} = 0$.

By (8) and (9), in such cases, the least value of z is given by

$$z = \frac{1}{n}\cos\frac{\pi}{n}\sec\theta_0\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\left(\frac{n\theta_0}{k}+\frac{\pi}{k}\right)\right]$$

where

$$\tan\theta_0\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\left(\frac{n\theta_0}{k}+\frac{\pi}{k}\right)\right] + n\sin\phi\sin\left(\frac{n\theta_0}{k}+\frac{\pi}{k}\right) = 0.$$

This least value of z is the water depth h. So we have

$$h\tan\theta_0 = -\cos\frac{\pi}{n}\sec\theta_0\sin\phi\sin\left(\frac{n\theta_0}{k}+\frac{\pi}{k}\right).$$

To simplify our notations, we simplify write ϑ for θ_0. So if $\vartheta \ne 0$,

$$h = -\frac{\cos\dfrac{\pi}{n}\sin\phi\sin\left(\dfrac{n\vartheta+\pi}{k}\right)}{\sin\vartheta} \tag{19}$$

where

$$\tan\vartheta\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\left(\frac{n\vartheta}{k}+\frac{\pi}{k}\right)\right] + n\sin\phi\sin\left(\frac{n\vartheta}{k}+\frac{\pi}{k}\right) = 0. \tag{20}$$

By Theorem 7,

$$\begin{aligned} V &= \frac{\pi h^3\sec^3\phi\tan^2\alpha}{3(\sqrt{1-\tan^2\phi\tan^2\alpha})^3} \\ &= \frac{\pi k^2}{3(n^2-k^2)}\left(\frac{h\sec\phi}{\sqrt{1-\dfrac{k^2\tan^2\phi}{n^2-k^2}}}\right)^3 \\ &= \frac{\pi k^2\sqrt{n^2-k^2}}{3}\left(\frac{h\sec\phi}{\sqrt{n^2-k^2-k^2\tan^2\phi}}\right)^3 \end{aligned}$$

$$= \frac{\pi k^2\sqrt{n^2-k^2}}{3}\left(-\frac{\cos\dfrac{\pi}{n}\sin\phi\sin\left(\dfrac{n\vartheta+\pi}{k}\right)\sec\phi}{\sin\vartheta\sqrt{n^2-k^2-k^2\tan^2\phi}}\right)^3$$

Therefore, we have

$$V^{\frac{2}{3}} = \left[\frac{\pi^2k^4(n^2-k^2)}{9}\right]^{\frac{1}{3}}\cos^2\frac{\pi}{n}\left[\frac{\tan^2\phi\sin^2\left(\dfrac{n\vartheta+\pi}{k}\right)}{\sin^2\vartheta(n^2-k^2-k^2\tan^2\phi)}\right]$$

$$V^{\frac{2}{3}} = \left[\frac{\pi^2k^4(n^2-k^2)}{9}\right]^{\frac{1}{3}}\cos^2\frac{\pi}{n}\left[\frac{\sin^2\left(\dfrac{n\vartheta+\pi}{k}\right)}{\sin^2\vartheta((n^2-k^2)\cot^2\phi-k^2)}\right] \tag{21}$$

As we can see in Section 4.3.3, for any given k and ϕ, there exists a unique ϑ such that $h = z(\vartheta)$. So it is possible to represent V in terms of ϑ. Here we should once again emphasize that ϑ is not the parameter θ describing the rim of the CONE as in Section 4.3.1. ϑ is the uniquely determined by ϕ.

Theorem 16. *Let $n, k \in \mathbb{Z}^+$, $\vartheta,\phi \in \mathbb{R}$ and $\alpha = \sin^{-1}\dfrac{k}{n}$. Suppose that one of the following conditions are satisfied:*

1. $n \geq 3$, $k \geq 2$, $\vartheta \in \left[-\dfrac{\pi}{n}, 0\right]$ *and* $\phi \in \left[0, \dfrac{\pi}{2}-\alpha\right]$
2. $n \geq 3$, $k \geq 1$, $\vartheta \in \left[-\dfrac{\pi}{n}, 0\right]$ *and* $\phi \in \left[\alpha, \dfrac{\pi}{2}-\alpha\right]$

If $\tan\vartheta\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\left(\dfrac{n\vartheta+\pi}{k}\right)\right] + n\sin\phi\sin\left(\dfrac{n\vartheta+\pi}{k}\right) = 0$, *then ϕ is a bijective function of ϑ.*

Proof. Assume that $n \geq 3$. By differentiation,

$$\begin{aligned}
0 &= \sec^2\vartheta\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\cos\left(\frac{n\vartheta+\pi}{k}\right)\right]\\
&\quad + \tan\vartheta\Bigg[-\sqrt{n^2-k^2}\sin\phi\frac{d\phi}{d\vartheta} - k\cos\phi\cos\left(\frac{n\vartheta+\pi}{k}\right)\frac{d\phi}{d\vartheta}\\
&\quad + n\sin\phi\sin\left(\frac{n\vartheta+\pi}{k}\right)\Bigg] + n\cos\phi\sin\left(\frac{n\vartheta+\pi}{k}\right)\frac{d\phi}{d\vartheta} + \frac{n^2}{k}\sin\phi\cos\left(\frac{n\vartheta+\pi}{k}\right)\\
&= \frac{-\sec^2\vartheta}{\tan\vartheta}\left[n\sin\phi\sin\left(\frac{n\vartheta}{k}\right)\right]\\
&\quad + \tan\vartheta\Bigg[-\sqrt{n^2-k^2}\sin\phi\frac{d\phi}{d\vartheta} - k\cos\phi\cos\left(\frac{n\vartheta+\pi}{k}\right)\frac{d\phi}{d\vartheta}
\end{aligned}$$

$$
\begin{aligned}
&+ n\sin\phi\sin\left(\frac{n\vartheta+\pi}{k}\right)\Bigg] + n\cos\phi\sin\left(\frac{n\vartheta+\pi}{k}\right)\frac{d\phi}{d\vartheta} + \frac{n^2}{k}\sin\phi\cos\left(\frac{n\vartheta+\pi}{k}\right)\\
&= -n\left(\frac{1}{\sin\vartheta\cos\vartheta} - \tan\vartheta\right)\sin\phi\sin\left(\frac{n\vartheta+\pi}{k}\right) - \sqrt{n^2-k^2}\sin\phi\tan\frac{d\phi}{d\vartheta}\\
&\quad - k\cos\phi\tan\vartheta\cos\left(\frac{n\vartheta+\pi}{k}\right)\frac{d\phi}{d\theta} + n\cos\phi\sin\left(\frac{n\vartheta+\pi}{k}\right)\frac{d\phi}{d\vartheta}\\
&\quad + \frac{n^2}{k}\sin\phi\cos\left(\frac{n\vartheta+\pi}{k}\right)\\
&= -n\left(\frac{cos\vartheta}{\sin\vartheta}\right)\sin\phi\sin\left(\frac{n\vartheta+\pi}{k}\right) + \frac{n^2}{k}\sin\phi\cos\left(\frac{n\vartheta+\pi}{k}\right)\\
&\quad + \Bigg[-\sqrt{n^2-k^2}\sin\phi\tan\vartheta - k\cos\phi\tan\vartheta\cos\left(\frac{n\vartheta+\pi}{k}\right)\\
&\quad + n\cos\phi\sin\left(\frac{n\vartheta+\pi}{k}\right)\Bigg]\frac{d\phi}{d\vartheta}\\
&= \frac{n\sin\phi}{k\sin\vartheta}\left[n\cos\left(\frac{n\vartheta+\pi}{k}\right)\sin\vartheta - k\sin\left(\frac{n\vartheta+\pi}{k}\right)\cos\vartheta\right]\\
&\quad - \sqrt{n^2-k^2}\sin\phi\tan\vartheta\frac{d\phi}{d\vartheta}\\
&\quad + \frac{\cos\phi}{\sin\phi}\left[-k\sin\phi\tan\vartheta\cos\left(\frac{n\vartheta+\pi}{k}\right) + n\sin\phi\sin\left(\frac{n\vartheta+\pi}{k}\right)\right]\frac{d\phi}{d\vartheta}\\
&= \frac{n\sin\phi}{2k\sin\vartheta}\left[(n-k)\sin\left(\frac{(n+k)\vartheta+\pi}{k}\right) - (n+k)\sin\left(\frac{(n-k)\vartheta+\pi}{k}\right)\right]\\
&\quad - \left[\sqrt{n^2-k^2}\sin\phi\tan\vartheta + \frac{\cos\phi}{\sin\phi}(\sqrt{n^2-k^2}\cos\phi\tan\vartheta)\right]\frac{d\phi}{d\vartheta}\\
&= \frac{n\sin\phi}{2k\sin\vartheta}g(\vartheta) - \frac{\sqrt{n^2-k^2}}{\sin\phi}\tan\vartheta\frac{d\phi}{d\vartheta}
\end{aligned}
$$

where $g(\vartheta)$ is defined in (13).
By (15),$g(\vartheta) < 0$ for $\vartheta \in \left(-\frac{\pi}{n}, 0\right)$.
Therefore, $\frac{d\phi}{d\vartheta} < 0$ for $\vartheta \in \left(-\frac{\pi}{n}, 0\right)$.
By continuity, $\phi = \phi(\vartheta)$ is a strictly decreasing and hence is a bijective function in ϑ on $\left[-\frac{\pi}{n}, 0\right]$.
When $\vartheta = -\frac{\pi}{n}$, we have

$$\tan\frac{-\pi}{n}\left[\sqrt{n^2-k^2}\cos\phi - k\sin\phi\right] = 0$$

$$\tan\phi = \frac{\sqrt{n^2-k^2}}{k}$$

$$\phi = \frac{\pi}{2} - \alpha.$$

1. Suppose that $k \geq 2$. When $\theta = 0$, we have

$$\begin{aligned} n \sin\phi \sin\frac{\pi}{k} &= 0 \\ \phi &= 0. \end{aligned}$$

When $k \geq 2$, $\phi : \left[-\frac{\pi}{n}, 0\right] \to \left[0, \frac{\pi}{2} - \alpha\right]$ is bijective. (See FIGURE 31)

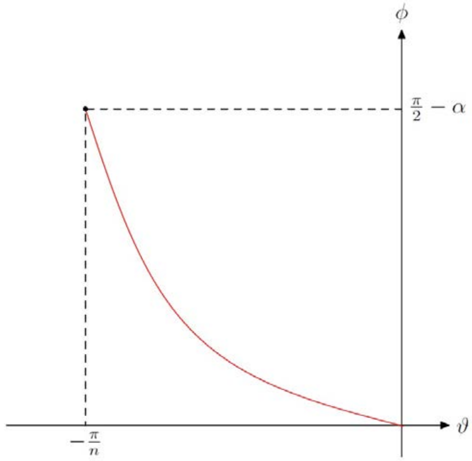


FIGURE 31. ϕ is a bijective function of ϑ when $n = 9$ and $k = 2$

2. Suppose that $k = 1$. Then we have

$$\tan\vartheta\left(\sqrt{n^2-1}\cos\phi + \sin\phi\cos n\vartheta\right) - n\sin\phi\sin n\vartheta = 0$$

By continuity, we have

$$\begin{aligned} \lim_{\vartheta\to 0^-}\left(\sqrt{n^2-1}\cos\phi + \sin\phi\cos n\vartheta\right) &= n\sin\phi \lim_{\vartheta\to 0^-}\frac{\sin n\vartheta}{\tan\vartheta} \\ \sqrt{n^2-1}\cos\phi + \sin\phi &= n\sin\phi \lim_{\vartheta\to 0^-}\frac{n\cos n\vartheta}{\sec^2\vartheta} \\ &= n^2\sin\phi \\ \tan\phi &= \frac{1}{\sqrt{n^2-1}} \\ \phi &= \alpha \end{aligned}$$

When $k = 1$, $\phi : \left[-\frac{\pi}{n}, 0\right] \to \left[\alpha, \frac{\pi}{2} - \alpha\right]$ is bijective. (See FIGURE 32)

$\square$

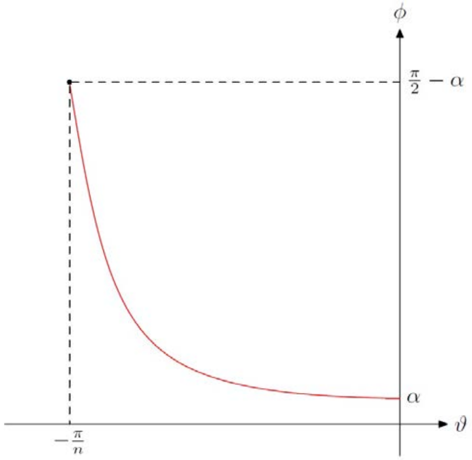


FIGURE 32. ϕ is a bijective function of ϑ when $n = 9$ and $k = 1$

As ϕ is a bijective function of ϑ, it is possible to express V in terms of ϑ. Let

$$\begin{aligned} s &= \frac{\sin^2\vartheta[(n^2-k^2)\cot^2\phi - k^2]}{\sin^2\left(\dfrac{n\vartheta+\pi}{k}\right)} \\ &= \frac{\cos^2\vartheta\tan^2\vartheta[(n^2-k^2)\cot^2\phi - k^2]}{\sin^2\left(\dfrac{n\vartheta+\pi}{k}\right)}. \end{aligned} \tag{22}$$

By (20),

$$\begin{aligned} \sqrt{n^2-k^2}\tan\vartheta\cos\phi &= k\sin\phi\tan\vartheta\cos\left(\frac{n\vartheta+\pi}{k}\right) - n\sin\phi\sin\left(\frac{n\vartheta+\pi}{k}\right) \\ \sqrt{n^2-k^2}\tan\vartheta\cot\phi &= k\tan\vartheta\cos\left(\frac{n\vartheta+\pi}{k}\right) - n\sin\left(\frac{n\vartheta+\pi}{k}\right) \\ \tan^2\vartheta[(n^2-k^2)\cot^2\phi - k^2] &= \left[k\tan\vartheta\cos\left(\frac{n\vartheta+\pi}{k}\right) - n\sin\left(\frac{n\vartheta+\pi}{k}\right)\right]^2 \\ &\quad - k^2\tan^2\vartheta \\ &= k^2\tan^2\vartheta\cos^2\left(\frac{n\vartheta+\pi}{k}\right) + n^2\sin^2\left(\frac{n\vartheta+\pi}{k}\right) \\ &\quad - 2nk\tan\vartheta\cos\left(\frac{n\vartheta+\pi}{k}\right)\sin\left(\frac{n\vartheta+\pi}{k}\right) \\ &\quad - k^2\tan^2\vartheta \\ &= -k^2\tan^2\vartheta\sin^2\left(\frac{n\vartheta+\pi}{k}\right) + n^2\sin^2\left(\frac{n\vartheta+\pi}{k}\right) \end{aligned}$$

$$- 2nk \tan\vartheta \cos\left(\frac{n\vartheta+\pi}{k}\right)\sin\left(\frac{n\vartheta+\pi}{k}\right)$$
$$= (n^2 - k^2\tan^2\vartheta)\sin^2\left(\frac{n\vartheta+\pi}{k}\right)$$
$$- 2nk \tan\vartheta \cos\left(\frac{n\vartheta+\pi}{k}\right)\sin\left(\frac{n\vartheta+\pi}{k}\right).$$

So by (22),

$$s = \cos^2\vartheta(n^2 - k^2\tan^2\vartheta) - 2nk\sin\vartheta\cos\vartheta\cot\left(\frac{n\vartheta+\pi}{k}\right) \tag{23}$$

and by (21)

$$V^{\frac{2}{3}} = \left[\frac{\pi^2k^4(n^2-k^2)}{9}\right]^{\frac{1}{3}}\left(\cos^2\frac{\pi}{n}\right)\left(\frac{1}{s}\right). \tag{24}$$

4.3.6. Maximum Capacity of Multi-portion CONE

As in (23),

$$\begin{aligned} s &= \cos^2\vartheta(n^2 - k^2\tan^2\vartheta) - 2nk\sin\vartheta\cos\vartheta\cot\left(\frac{n\vartheta+\pi}{k}\right) \\ &= n^2\cos^2\vartheta - k^2\sin^2\vartheta - 2nk\sin\vartheta\cos\vartheta\cot\left(\frac{n\vartheta+\pi}{k}\right) \\ &= \frac{n^2-k^2}{2} + \frac{n^2+k^2}{2}\cos 2\vartheta - nk\sin 2\vartheta\cot\left(\frac{n\vartheta+\pi}{k}\right) \end{aligned}$$

Thus,

$$\begin{aligned} \frac{ds}{d\vartheta} &= -(n^2+k^2)\sin 2\vartheta - 2nk\cos 2\vartheta\cot\left(\frac{n\vartheta+\pi}{k}\right) + n^2\sin 2\vartheta\csc^2\left(\frac{n\vartheta+\pi}{k}\right) \\ &= \sin 2\vartheta\left[n^2\cot^2\left(\frac{n\vartheta+\pi}{k}\right) - 2nk\cot 2\vartheta\cot\left(\frac{n\vartheta+\pi}{k}\right) - k^2\right] \\ &= \sin 2\vartheta\left[\left(n\cot\left(\frac{n\vartheta+\pi}{k}\right) - k\cot 2\vartheta\right)^2 - k^2\csc^2\theta\right] \\ &= \sin 2\vartheta\left[n\cot\left(\frac{n\vartheta+\pi}{k}\right) - k\cot 2\vartheta - k\sec 2\vartheta\right] \\ &\quad \left[n\cot\left(\frac{n\vartheta+\pi}{k}\right) - k\cot 2\vartheta + k\sec 2\vartheta\right] \\ &= \frac{1}{\sin 2\vartheta\sin^2\left(\frac{n\vartheta+\pi}{k}\right)} \\ &\quad \left[n\cos\left(\frac{n\vartheta+\pi}{k}\right)\sin 2\vartheta - k\sin\left(\frac{n\vartheta+\pi}{k}\right)(1+\cos 2\vartheta)\right] \end{aligned}$$

$$\left[n\cos\left(\frac{n\vartheta+\pi}{k}\right)\sin 2\vartheta + k\sin\left(\frac{n\vartheta+\pi}{k}\right)(1-\cos 2\vartheta)\right]$$
$$= \frac{4\sin\vartheta\cos\vartheta}{\sin 2\vartheta\sin^2\left(\frac{n\vartheta+\pi}{k}\right)}\left[n\cos\left(\frac{n\vartheta+\pi}{k}\right)\sin\vartheta - k\sin\left(\frac{n\vartheta+\pi}{k}\right)\cos\vartheta\right]$$
$$\left[n\cos\left(\frac{n\vartheta+\pi}{k}\right)\cos\vartheta + k\sin\left(\frac{n\vartheta+\pi}{k}\right)\sin\vartheta\right]$$
$$= \frac{1}{2\sin^2\left(\frac{n\vartheta+\pi}{k}\right)}$$
$$\left[(n-k)\sin\left(\frac{(n+k)\vartheta+\pi}{k}\right) - (n+k)\sin\left(\frac{(n-k)\vartheta+\pi}{k}\right)\right]$$
$$\left[(n-k)\cos\left(\frac{(n+k)\vartheta+\pi}{k}\right) + (n+k)\cos\left(\frac{(n-k)\vartheta+\pi}{k}\right)\right]$$

Hence we have

$$\frac{ds}{d\vartheta} = \frac{g(\vartheta)\rho(\vartheta)}{2\sin^2\left(\frac{n\vartheta+\pi}{k}\right)} \tag{25}$$

where $g(\vartheta)$ is defined as (13) and

$$\rho(\vartheta) = (n-k)\cos\left(\frac{(n+k)\vartheta+\pi}{k}\right) + (n+k)\cos\left(\frac{(n-k)\vartheta+\pi}{k}\right). \tag{26}$$

By (15), $g(\vartheta) < 0$ when $\vartheta \in \left(-\frac{\pi}{n}, 0\right)$, $-\frac{\pi}{n} < \frac{(n+k)\vartheta+\pi}{k} < \frac{\pi}{k}$ and $\frac{\pi}{n} < \frac{(n-k)\vartheta+\pi}{k} < \frac{\pi}{k}$. So if $k \geq 2$, both $\cos\left(\frac{(n+k)\vartheta+\pi}{k}\right)$ and $\cos\left(\frac{(n-k)\vartheta+\pi}{k}\right)$ are positive. By (25), if $k \geq 2$

$$\frac{ds}{d\vartheta} < 0$$

for $\vartheta \in \left(-\frac{\pi}{n}, 0\right)$. Therefore, s is strictly decreasing on $\left(-\frac{\pi}{n}, 0\right)$. By (24), V is strictly increasing on $\left(-\frac{\pi}{n}, 0\right)$. By continuity, V is the greatest when $\vartheta = 0$ and by (20), V is the greatest when $\phi = 0$.

Theorem 17. *When $k \geq 2$, the capacity of n, k-CONE is the greatest when $\phi = 0$, and the greatest capacity is*

$$V_{n,k} = \frac{\pi}{3n^3}k^2\sqrt{n^2-k^2}\cos^3\frac{\pi}{n}.$$

4.3.7. Maximum Capacity of 1-portion CONE

To investigate the maximum capacity of an 1-portion CONE, we put $k = 1$ in (26),

$$\rho(\vartheta) = -(n-1)\cos(n+1)\vartheta - (n+1)\cos(n-1)\vartheta$$

So we have $\rho(0) = -2n < 0$ and $\rho\left(-\frac{\pi}{n}\right) = 2n\cos\frac{\pi}{n} > 0$.
As

$$\rho'(\vartheta) = (n-1)(n+1)[\sin(n+1)\vartheta + \sin(n-1)\vartheta].$$

If $\vartheta \in \left(-\frac{\pi}{n+1}, 0\right)$, we have $-\pi < (n+1)\vartheta < (n-1)\vartheta < 0$ and hence $\rho'(\vartheta) < 0$.
If $\vartheta \in \left(-\frac{\pi}{n}, -\frac{\pi}{n+1}\right)$, then $-\pi < 2-\pi-(n+1)\vartheta < (n-1)\vartheta \le -\frac{\pi}{2}$ and thus

$$\sin(n+1)\vartheta + \sin(n-1)\vartheta = \sin(n-1)\vartheta - \sin[-2\pi - (n+1)\vartheta] < 0$$

and hence $\rho'(\vartheta) < 0$. By continuity, $\rho$is strictly decreasing on $\left[-\frac{\pi}{n}, 0\right]$.
There exist a unique ϑ_1 such that $\rho(\vartheta_1) = 0$.
By (15) and (25), $\frac{ds}{d\vartheta} < 0$ for $\vartheta \in \left[-\frac{\pi}{n}, \vartheta_1\right)$ and $\frac{ds}{d\vartheta} > 0$ for $\vartheta \in (\vartheta_1, 0]$. Thus, s attains its least value (for $\alpha \le \phi \le \frac{\pi}{2} - \alpha$) if and only if $\vartheta = \vartheta_1$.
Therefore, V attains its greatest value (for $\alpha \le \phi \le \frac{\pi}{2} - \alpha$) if and only if $\vartheta = \vartheta_1$.

By Theorem 10, if $0 \le \phi \le \alpha$, the capacity of the 1-portion CONE is the greatest when $\vartheta = 0$, or equivalently, $\phi = \alpha$. Here we can show that the capacity will be even larger when $\vartheta = \vartheta_1$ (and $\phi > \alpha$).
By the proof of Theorem 16, we know that when $\alpha \le \phi \le \frac{\pi}{2} - \alpha$, $\phi = \alpha$ if and only if $\vartheta = 0$. we also have $\frac{d\phi}{d\vartheta} < 0$ for $\left(-\frac{\pi}{n}, 0\right)$.

Suppose that $\phi(\vartheta_1) = \phi_1$. Then for $\phi \in (\alpha, \phi_1)$, we have $\vartheta \in (\vartheta_1, 0)$. By (24) and the chain rule,

$$\begin{aligned}
\frac{dV^{\frac{2}{3}}}{d\phi} &= \left[\frac{\pi^2 k^4 (n^2 - k^2)}{9}\right]^{\frac{1}{3}} \left(\cos^2\frac{\pi}{n}\right) \frac{\frac{d(s^{-1})}{ds} \cdot \frac{ds}{d\vartheta}}{\frac{d\phi}{d\vartheta}} \\
&= \left[\frac{\pi^2 k^4 (n^2 - k^2)}{9}\right]^{\frac{1}{3}} \left(\cos^2\frac{\pi}{n}\right) \left(-\frac{1}{s^2}\right) \frac{\frac{ds}{d\vartheta}}{\frac{d\phi}{d\vartheta}} \\
&> 0
\end{aligned}$$

By continuity, V is strictly increasing for $\phi \in [\alpha, \phi_1]$. So we have the following theorem.

Theorem 18. *The capacity of an $n, 1$-CONE is greatest when $\vartheta = \vartheta_1$, where ϑ_1 is the only root of the equation $(n-1)\cos(n+1)\vartheta + (n+1)\cos(n-1)\vartheta = 0$ in the interval $\left(-\frac{\pi}{n}, 0\right)$. When the CONE has the greatest capacity, $\beta = \pi$ and $\tan\vartheta_1(\sqrt{n^2-1}\cos\phi + k\sin\phi\cos n\vartheta_1) - n\sin\phi\sin n\vartheta_1 = 0$.*

5. What Can We Do with a Paper Polygon?

Tedious as it seems, the calculation in Section 3 provokes us to find the maximal total capacity of CONEs made from a paper regular polygon.

5.1. Capacity of CONEs

Now we go on to consider the total capacity of a k-portion CONE made from a n-sided polygonal paper. For $k \geq 2$, by Theorem 17 the greatest capacity of an n, k-CONE is

$$V_{n,k} = \frac{\pi}{3n^3}k^2\sqrt{n^2-k^2}\cos^3\frac{\pi}{n}.$$

It is hard to find a general formula for $V_{n,1}$ in closed forms. Instead, we will find an upper bound B_n of $V_{n,1}$ to do the comparison.

A convenient way to choose B_n as the volume of some inclined circular cone which is an extension of the $n, 1$-CONE. We inclined the CONE (rotate it about the y-axis) in a way that the lowest and the highest points of the rim are of the same height (FIGURE 33). We can then extend the generatrices of the CONE to the same level as these two points, to form an inclined circular cone, which is clearly an extension of the original CONE. The volume of this inclined circular cone is denoted by B_n.

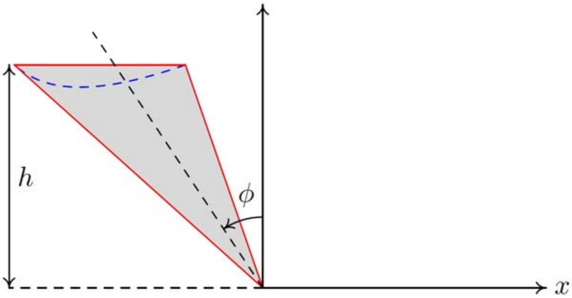


FIGURE 33. Upper bound(red) for a 1-portion cone's capacity

Here we have

$$\cos(\phi + \alpha) = \cos\frac{\pi}{n}\cos(\phi - \alpha)$$

$$\begin{aligned}
\cos\phi\cos\alpha - \sin\phi\sin\alpha &= \cos\frac{\pi}{n}(\cos\phi\cos\alpha + \sin\phi\sin\alpha) \\
\frac{sqrtn^2 - 1\cos\phi - \sin\phi}{n} &= \cos\frac{\pi}{n}\left(\frac{sqrtn^2 - 1\cos\phi + \sin\phi}{n}\right) \\
\sqrt{n^2-1} - \tan\phi &= \cos\frac{\pi}{n}\left(\sqrt{n^2-1} + \tan\phi\right) \\
\tan\phi &= \frac{\sqrt{n^2-1}\left(1-\cos\dfrac{\pi}{n}\right)}{1+\cos\dfrac{\pi}{n}}
\end{aligned}$$

which is equivalent to

$$\frac{\tan\phi}{\sqrt{n^2-1}} = \frac{1-\cos\dfrac{\pi}{n}}{1+\cos\dfrac{\pi}{n}}. \tag{27}$$

Note also that the water level h in this case is given by

$$h = \cos(\phi+\alpha).$$

By Theorem 7, the upper bound B_n is given by

$$\begin{aligned}
B_n = \frac{\pi h^3\sec^3\phi\tan^2\alpha}{3(\sqrt{1-\tan^2\phi\tan^2\alpha})^3} &= \frac{\pi\cos^3(\phi+\alpha)\sec^3\phi\tan^2\alpha}{3(\sqrt{1-\tan^2\phi\tan^2\alpha})^3} \\
&= \frac{\pi(\cos\alpha - \sin\alpha\tan\phi)^3\tan^2\alpha}{3(\sqrt{1-\tan^2\phi\tan^2\alpha})^3} \\
&= \frac{\frac{\pi}{n}\left(\sqrt{n^2-1}-\tan\phi\right)^3\frac{1}{n^2-1}}{3\left(\sqrt{1-\frac{\tan^2\phi}{n^2-1}}\right)^3}.
\end{aligned}$$

By (27), we have

$$\begin{aligned}
B_n &= \frac{\dfrac{\pi}{n}\left(\sqrt{n^2-1}\right)^3\left(1-\dfrac{1-\cos\frac{\pi}{n}}{1+\cos\frac{\pi}{n}}\right)^3\frac{1}{n^2-1}}{3\left[\sqrt{1-\left(\dfrac{1-\cos\frac{\pi}{n}}{1+\cos\frac{\pi}{n}}\right)^2}\right]^3} \\
&= \frac{\frac{\pi}{n^3}\left(\sqrt{n^2-1}\right)\left(2\cos\frac{\pi}{n}\right)^3}{3\left(\sqrt{4\cos\frac{\pi}{n}}\right)^3} \\
&= \frac{\pi}{3n^3}\sqrt{n^2-1}\left(\cos\frac{\pi}{n}\right)^{\frac{3}{2}}.
\end{aligned}$$

Hence we have the following theorem

Theorem 19. *An upper bound B_n of the maximum capacity $V_{n,1}$ of an $n, 1$-CONE is given by*

$$B_n = \frac{\pi}{3n^3}\sqrt{n^2-1}\left(\cos\frac{\pi}{n}\right)^{\frac{3}{2}}.$$

5.2. Are Two CONEs Still the Best?

In Section 2, we can see that maximum total capacity of the cones made with sectors of a paper circle is attained at 2-cone case. But in Section 3, we can see that the maximum total capacity of the CONEs made with portions of a regular triangle is the 3-pieces case. If we regard a circle as a polygon of infinitely many sides, these results suggest that when the number of sides of the polygon is large enough, the greatest total capacity will be attained by making two CONEs only. But when there are fewer sides, we may need more CONEs to maximize the total capacity.

By Theorem 17, $k \geq 2$, the capacity of an n, k-CONE is the greatest when it is held vertically, and the water inside is in the shape of a right circular cone. If we have two multi-portion papers cut from a regular n-sided polygon, as long as the sum of the central angles does not exceed $\frac{2\sqrt{6}}{3}\pi$, we can increase the total capacity of the CONEs made by assembling them to only one multi-portion paper (Theorem 1). If k and l are integers such that $2lk2$ and $k < \frac{\sqrt{6}}{3}$, then

$$V_{n,l} + V_{n,k-1} < V_{n,k} \tag{28}$$

In the paper circle problem, if the central angle is too large (larger than $\frac{2\sqrt{6}}{3}\pi$), we will not cut the sector into smaller sectors as this will reduce the total capacity of the cones. But in the paper polygon problem, it is possible that the total capacity will increase when we cut a multi-portion paper. This is because, by Theorem 18, the capacity of a 1-portion CONE in a slant position can be larger than that when it is in the standard position. Therefore, we have to compare

1. $2V_{n,1}$ with $V_{n,2}$, and
2. $V_{n,1} + V_{n,k-1}$ with $V_{n,k}$, when $k \geq 3$.

As it is difficult to find $V_{n,1}$, we will use B_n instead in the following comparisons:

1. What we want to have is that

$$V_{n,2} > 2B_n, \tag{29}$$

   or equivalent,

$$\frac{4\pi}{3n^3}\sqrt{n^2-4}\cos^3\frac{\pi}{n} > \frac{2\pi}{3n^3}\sqrt{n^2-1}\left(\cos\frac{\pi}{n}\right)^{\frac{3}{2}}$$

$$2\sqrt{n^2-4} > \sqrt{n^2-1}\left(\sec\frac{\pi}{n}\right)^{\frac{3}{2}}$$

This is true for $n \geq 4$ as we have

$$\left(\sec\frac{\pi}{n}\right)^{\frac{3}{2}} \leq \left(\sec\frac{\pi}{4}\right)^{\frac{3}{2}} = \sqrt[4]{8}.$$

and

$$2\sqrt{\frac{n^2-4}{n^2-1}} = 2\sqrt{1-\frac{3}{n^2-1}} \geq 2\sqrt{1-\frac{3}{(4)^2-1}} = 2\sqrt{\frac{4}{5}} = \sqrt[4]{\frac{256}{25}} > \sqrt[4]{8}$$

Since $B_n > V_{n,1}$, we have

$$2V_{n,1} \leq V_{n,2} \tag{30}$$

for $n \geq 4$

2. Here we need (when $3 \leq k \leq \left|\frac{2n}{3}\right|$)

$$V_{n,k} > B_n + V_{n,k-1}$$

which is equivalent to

$$k^2\sqrt{n^2-k^2} - (k^2-1)\sqrt{n^2-(k-1)^2} > \sqrt{n^2-1}\left(\sec\frac{\pi}{n}\right)^{\frac{3}{2}} \tag{31}$$

Let $f(x) = x^2\sqrt{n^2-x^2}$ for $0 < x < n$. Then

$$f'(x) = 2x\sqrt{n^2-x^2} - \frac{x^2}{\sqrt{n^2-x^2}} = \frac{2n^2x-3x^3}{\sqrt{n^2-x^2}}$$

By Largrange's Mean Value Theorem, there exist at least a point c in $(k-1,k)$, such that

$$f(k) - f(k-1) = f'(c)$$

$$\begin{aligned} k^2\sqrt{n^2-k^2} - (k-1)^2\sqrt{n^2-(k-1)^2} &= \frac{2n^2c-3c^3}{\sqrt{n^2-c^2}} \\ &> \frac{2n^2c-3c^3}{\sqrt{n^2-1}} \end{aligned}$$

So it suffices to prove that

$$\frac{2n^2c-3c^3}{n^2-1} > \left(\sec\frac{\pi}{n}\right)^{\frac{3}{2}}. \tag{32}$$

It is easy to prove that 32 holds when $n \geq 6$.
First we have

$$\left(\sec\frac{\pi}{n}\right)^{\frac{3}{2}} \leq \left(\sec\frac{\pi}{6}\right)^{\frac{3}{2}} = \left(\frac{2}{\sqrt{3}}\right)^{\frac{3}{2}} = \left(\frac{64}{27}\right)^{\frac{1}{4}} < \left(\frac{625}{256}\right)^{\frac{1}{4}} = \frac{5}{4}.$$

For $2 < k-1 < x < k$, let

$$g(x) = \frac{2n^2x-3x^3}{n^2-1} - \frac{5}{4}.$$

Then we have

$$g(2) = \frac{4n^2 - 24}{n^2 - 1} - \frac{5}{4} = \frac{11}{4} - \frac{20}{n^2 - 1} \geq \frac{11}{4} - \frac{20}{6^2 - 1} = \frac{61}{28} > 0$$

and

$$\begin{aligned} g\left(\frac{3n}{4}\right) &= \frac{2n^2\left(\frac{3n}{4}\right) - 3\left(\frac{3n}{4}\right)^3}{n^2 - 1} - \frac{5}{4} \\ &= \frac{15n^3}{64(n^2 - 1)} - \frac{5}{4} \\ &> \frac{15n^3}{64n^2} - \frac{5}{4} \\ &\geq \frac{15}{64}(6) - \frac{5}{4} \\ &> 0 \end{aligned}$$

Since $g''(x) = -\dfrac{18x}{n^2 - 1} < 0$ for all real x, g is a concave function and hence $g(x) > 0$ for all $0 < x < \dfrac{3n}{4}$. (FIGURE 34) Since $\left\lceil \dfrac{2n}{3} \right\rceil \leq \dfrac{3n}{4}$ for all[1] integers

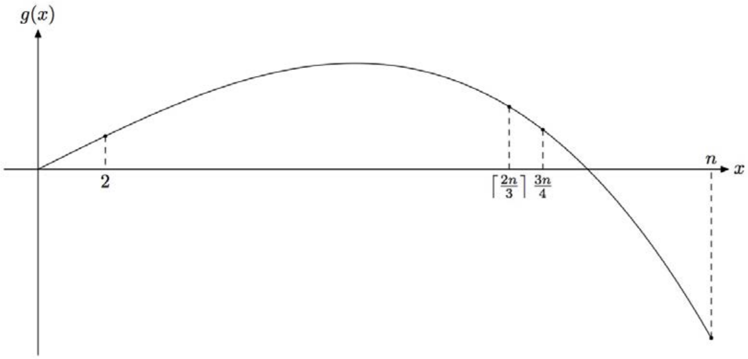


FIGURE 34. The graph of $g(x)$

$n \geq 6$, $g(x) > 0$ for $0 \leq x \leq \left\lceil \dfrac{2n}{3} \right\rceil$. (32) holds for $n \geq 6$.

Since $B_n > V_{n,1}$, we have

$$V_{n,1} + V_{n,k-1} \leq V_{n,k} \tag{33}$$

[1]For $n \geq 12$, $\dfrac{3n}{4} - \left\lceil \dfrac{2n}{3} \right\rceil > \dfrac{3n}{4} - \left(\dfrac{2n}{3} + 1\right) = \dfrac{n}{12} - 1 \geq 0$. Cases for $6 \leq n \leq 11$, the claim can be verified numerically.

n	k	$k^2\sqrt{n^2-k^2}-(k^2-1)\sqrt{n^2-(k-1)^2}-\sqrt{n^2-1}\left(\sec\frac{\pi}{n}\right)^{\frac{3}{2}}$
4	3	$9\sqrt{7}-8\sqrt{3}-\sqrt[4]{1800}\approx 3.44179971$
5	3	$36-4\sqrt{21}-2\sqrt{6}\left(\sec\frac{\pi}{5}\right)^{\frac{3}{2}}\approx 10.93730896$
5	4	$12-2\sqrt{6}\left(\sec\frac{\pi}{5}\right)^{\frac{3}{2}}\approx 5.267611740$

TABLE 2. Cases for $4\le n\le 5$

for $n\ge 6$ and $3\le k\le\left\lceil\frac{2n}{3}\right\rceil$. For $4\le n\le 5$, we exhaust all cases to check the validity of (31) (TABLE 2). Note that in each case, the difference is greater than 0. So (33) also holds for $4\le n\le 5$.

Now we have the following result.

Theorem 20. *For a paper polygon with 4 or more sides, the maximum total capacity of CONEs as made with portions of this paper is attained in a case of 2 CONEs.*

Proof. As we have

$$V_{n,l}+V_{n,k-l}<V_{n,k} \tag{34}$$

for any positive integers k and l such that $l<k\le\left\lceil\frac{2n}{3}\right\rceil$, we can always assemble the two smallest pieces of portion papers to increase the total capacity, as long as there are 3 CONEs or more. The maximum total capacity must be attained in a case of 2 CONEs □

5.3. How to Cut the Paper?

In the previous section, we know that the greatest total capacity can be obtained in a two- CONE cases if the polygon has 4 sides or more. But in what way should the paper be divided?

Theorem 21. *For a paper polygon of 6 or more sides, the solution of the greatest total capacity problem does not involve any 1-portion CONEs.*

Proof. It suffices to show that

$$V_{n,2}+Vn,n-2>B_n+V_{n,n-1} \tag{35}$$

as this implies that $V_{n,2}+V_{n,n-2}>V_{n,1}+V_{n,n-1}$. Recall (29) in Section 5.2

$$V_{n,2}>2B_n$$

for $n \geq 4$. It is sufficient to verify that

$$B_n + V_{n,n-2} > V_{n,n-1}$$

i.e.

$$\sqrt{n^2-1}\left(\sec\frac{\pi}{n}\right)^{\frac{3}{2}} + (n-2)^2\sqrt{n^2-(n-2)^2} \geq (n-1)^2\sqrt{n^2-(n-1)^2}$$

Note that, for $n \geq 8$, we have

$$\begin{aligned}
\frac{(n-2)^2\sqrt{n^2-(n-2)^2}}{(n-1)^2\sqrt{n^2-(n-1)^2}} &= \left(\frac{n-2}{n-1}\right)^2\sqrt{\frac{4n-4}{2n-1}} \\
&= \left(1-\frac{1}{n-1}\right)^2\sqrt{1-\frac{2}{2n-1}} \\
&\geq \left(1-\frac{1}{8-1}\right)^2\sqrt{1-\frac{2}{2(8)-1}} \\
&= \sqrt{\frac{1728}{1715}} \\
&> 1
\end{aligned}$$

and we are done.

n	$V_{n,2} + V_{n,n-2} - B_n - V_{n,n-1}$
6	0.012360483
7	0.025468439

TABLE 3. Cases for $6 \leq n \leq 7$

For $6 \leq n \leq 7$, we adopt a brute-force method and check case-by-case[1] (TABLE 3). As the difference in each case is greater than 0, it can concluded that the inequality holds for every $n \geq 6$. □

5.4. Marginal Case - Square Paper

In the case of a paper square, we know from the Theorem 20 that the greatest total capacity is attained at one of the 2-CONE cases. But whether a 1-portion CONE should be included is not known. Hence, we first need to find the maximum capacity of a 1-portion CONE before making comparisons between different combinations of CONEs.

[1]The expressions are too complicated and trivial so we only list out numerical values.

5.4.1. Capacity of 1-portion CONE

Bt Theorem 18, the maximum capacity of $V_{4,1}$ corresponds to

$$3\cos 5\theta + 5\cos 3\theta = 0$$

This can be solved by using Multiple Angle Formulae, or we can use Chebyshev Polynomials[1] to simplify the calculations. Let $u = \cos\theta$ and $T_m = \cos m\theta$ (for $m \in \mathbb{Z}^+$). We have (see Appendix A)

$$\begin{aligned} 0 &= 3T_5 + 5T_3 \\ &= 3(16u^5 - 20u^3 + 5u) + 5(4u^3 - 3u) \\ &= 8u^3(6u^2 - 5) \end{aligned}$$

The only solution of this equation such that $-\dfrac{\pi}{4} \le \theta \le 0$ is $u = \sqrt{\dfrac{5}{6}}$.

Hence $V_{4,1} = \dfrac{\sqrt{5}\pi}{243} \approx 0.0092019258\pi$.

5.4.2. Comparing the Total Capacity

The capacities of multi-portion CONEs are calculated as follows:

$$V_{4,2} = \frac{\pi(2^2)\sqrt{4^2 - 2^2}}{3 \cdot 4^3} \cos^3 \frac{\pi}{4} \approx 0.025515518\pi$$
$$V_{4,3} = \frac{\pi(3^2)\sqrt{4^2 - 3^2}}{3 \cdot 4^3} \cos^3 \frac{\pi}{4} \approx 0.043847548\pi.$$

The total capacity of combinations are listed in TABLE 4: In conclusion, the maximum total capacity in the case of pentagonal paper is approximately 0.053049473π, attained at a combination of 1-portion and 3-portion CONEs.

Combination	Total Capacity
$V_{4,1} + V_{4,3}$	0.053049473π
$V_{4,2} + V_{4,2}$	0.051031036π

Table 4. Comparing total capacity of different combinations of CONEs made from square paper

5.5. Marginal Case - Pentagonal Paper

Previously in Section 5.2, we know that a larger capacity can be obtained in cases of 2 CONEs than those of more divisions. However, the inequalities in that section fail to predict whether a 1-portion CONE should be involved. For this reason, we first need to find the maximum capacity of an 1-portion CONE prior to the comparisons.

5.5.1. 1-portion CONE Capacity

By Theorem 18, the maximum capacity of $V_{5,1}$ corresponds to (see Appendix A)

$$\begin{aligned}0 &= 4T_6 + 6T_4 \\ &= 4(32u^6 - 48u^4 + 18u^2 - 1) + 6(8u^4 - 8u^2 + 1) \\ &= 2(64u^6 - 72u^4 + 12u^2 + 1)\end{aligned}$$

The solution of this equation such that $-\frac{\pi}{5} \le \theta \le 0$ is $u \approx 0.946772794$. Hence $V_{5,1} \approx 0.08094773\pi$.

5.5.2. Comparing the Total Capacity

The capacities of multi-portion CONEs are calculated as follows:

$$\begin{aligned}V_{5,2} &= \frac{\pi(2^2)\sqrt{5^2-2^2}}{3\cdot 5^3}\cos^3\frac{\pi}{5} \approx 0.025882803\pi \\ V_{5,3} &= \frac{\pi(3^2)\sqrt{5^2-3^2}}{3\cdot 5^3}\cos^3\frac{\pi}{5} \approx 0.050832816\pi \\ V_{5,4} &= \frac{\pi(4^2)\sqrt{5^2-4^2}}{3\cdot 5^3}\cos^3\frac{\pi}{5} \approx 0.06777088\pi\end{aligned}$$

The total capacity of combinations are listed in TABLE 5: In conclusion, the maximum total capacity in the case of pentagonal paper is approximately 0.076715619π, attained at a combination of a 2-portion and a 3-portion CONEs.

Combination	Total Capacity
$V_{5,1} + V_{5,4}$	0.075871860π
$V_{5,2} + V_{5,3}$	0.076715619π

TABLE 5. Comparing total capacity of different combinations of CONEs made from pentagonal paper

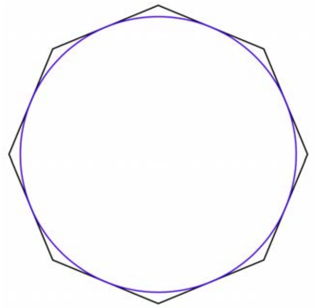


FIGURE 35. The octagonal paper and its inscribed circle

5.6. The Complete Solution for Polygonal Paper of More Sides

With the results in previous sections, we will illustrate our solution to a particular case of n-sided polygonal paper where $n \geq 6$. Here, we take a piece of octagonal paper as an example.

By Theorem 20 and Theorem 21, the maximum total capacity must be attained in a certain two-CONE case with multi-portion CONEs only.

Next, as only multiple-portion cones are within consideration, we can directly go into considering the inscribed circle of the paper (FIGURE 35). Recall that in Section 2, we have the maximum total capacity attained at central angle $\theta = 0.648027704\pi$. Then the corresponding peak (FIGURE 36)[1] for this octagon is given by

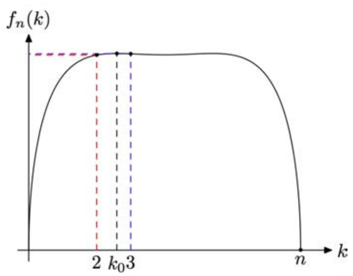
$$\frac{k_0}{8} \approx \frac{0.648027704\pi}{2\pi}$$
$$k_0 \approx 2.592110816$$

But whether the maximum total capacity lies on the central angle ratio of $2:6$ or $3:5$ needs checking as listed in TABLE 6:

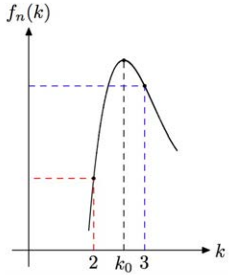
Combination	Total Capacity
$V_{8,2} + V_{8,6}$	0.1137065124π
$V_{8,3} + V_{8,5}$	0.1144215614π

TABLE 6. Comparing total capacity of different combinations of CONEs made from paper octagon

[1]Here we take $k : n = \theta : 2\pi$



(a)



(b)

FIGURE 36. The graph of $f_n(k)$ and its magnified part(500X)

Hence the maximum total capacity in the case of paper octagon is approximately 0.1144215614π, attained at a combination of 3-portion and 5-portion CONEs.

6. Conclusion

The objective of this project is to find the method to cut a paper regular polygon along its radii to make cone-like containers (named CONEs here) that together can fill with the largest amount of water. We managed to solve the problem completely.

Although there are many different ways to cut the paper, the best way (in the sense of holding more water) is always to cut the polygon to two unequal pieces, with one piece with area about twice as that of the other one, with the exception of the case of regular triangle. The problem and the solution has a limiting case, which is a circle. The circle problem is the foundation of the general problem, as the general problem can be reduced to the circle problem in all but a few cases. These exceptional few cases are interesting as it is difficult to give a closed form of the solution. Fortunately, we can prove that the solution is related to the root of a simple (simple to state, but not so simple to solve) trigonometric equation. This enable us to solve many cases using computing softwares.

The following table shows the ways to cut a regular n-sided polygon so that the total capacity of the CONEs made is the greatest.

n	Combination of portions	Total Capacity	Ratio of central angle
3	$1, 1, 1$	0.0215125920π	/
4	$1, 3$	0.0530494730π	$1:3$
5	$2, 3$	0.0767156190π	$1:1.5$
6	$2, 4$	0.0944023744π	$1:2$
7	$2, 5$	0.1061194595π	$1:2.5$
8	$3, 5$	0.1144215614π	$1:1.666666667$
9	$3, 6$	0.1206003233π	$1:2$
10	$3, 7$	0.1249595700π	$1:2.333333333$
10^2	$32, 68$	0.1451358859π	$1:2.125$
10^4	$3240, 6760$	0.1453531930π	$1:2.086419753$
10^6	$324014, 675986$	0.1453532145π	$1:2.086286395$
10^8	$32401385, 67598615$	0.1453532145π	$1:2.086287824$
10^10	$3240138518, 6759861482$	0.1453532145π	$1:2.086287807$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞(circle)	/	0.145353214π	$1:2.086287806$

TABLE 7

Appendix A. Chebyshev Polynomials of the First Kind

The Chebyshev polynomials[2] are named after the Russian mathematician Pafnuty Lvovich Chebyshev. There are two kinds of Chebyshev polynomials, the first kind (TABLE 8) which are denoted by T_n and the second kind which are denoted by U_n. The Chebyshev polynomials of the first kind is defined by

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \quad \text{for } n \in \mathbb{Z}^+. \end{aligned}$$

If we let $x = \cos\theta$ for some angle θ, then for any positive integer n, $T_n(x) = \cos n\theta$. By Theorem 18, $V_{n,1} = \left[\frac{\pi^2 k^4 (n^2 - k^2)}{9}\right]^{\frac{1}{2}} \left(\cos^3 \frac{\pi}{n}\right) \left(\frac{1}{s}\right)^{\frac{3}{2}}$, where $s = s(\theta)$ is defined by (23) and θ is the root of

$$(n-1)T_{n+1} + (n+1)T_{n-1} = 0$$

in $\left[-\frac{\pi}{n}, 0\right]$.

n	T_n
3	$4u^3 - 3u$
4	$8u^4 - 8u^2 + 1$
5	$16u^5 - 20u^3 + 5u$
6	$32u^6 - 48u^4 + 18u^2 - 1$
7	$64u^7 - 112u^5 + 56u^3 - 7u$
8	$128u^8 - 256u^6 + 160u^4 - 32u^2 + 1$
9	$256u^9 - 576u^7 + 432u^5 - 120u^3 + 9u$
10	$521u^10 - 1280u^8 + 1120u^6 - 400u^4 + 50u^2 - 1$

TABLE 8. Chebyshev Polynomials T_n

Listed in the following are some numerical values.

n	u	s	$V_{n,1}$
3	0.82644583	6.4641016	0.007170864π
4	0.91287093	13.5	0.009201926π
5	0.94677279	22.513089	0.008094773π
6	0.96391327	33.519507	0.006600217π
7	0.97385800	46.523171	0.005322611π
8	0.98016416	61.525472	0.004323273π
9	0.98442252	78.527017	0.003555106π
10	0.98743676	97.528105	0.002962243π
11	0.98965020	118.52890	0.002499530π
12	0.99132426	141.52950	0.002133574π
13	0.99262149	166.52997	0.001840240π
14	0.99364735	193.53033	0.001602101π
15	0.99447274	222.53063	0.001406473π
16	0.99514679	253.53087	0.001244012π
17	0.99570444	286.53107	0.001107751π

18	0.99617105	321.53124	0.000992429π
19	0.99656546	358.53138	0.000894020π
20	0.99690184	397.53150	0.000809408π
21	0.99719105	438.53160	0.000736156π
22	0.99744154	481.53169	0.000672335π
23	0.99765992	526.53177	0.000616406π
24	0.99785146	573.53184	0.000567129π
25	0.99802039	622.53190	0.000523497π
26	0.99817014	673.53195	0.000484684π
27	0.99830350	726.53200	0.000450011π
28	0.99842279	781.53204	0.000418911π
29	0.99852992	838.53208	0.000390912π
30	0.99862648	897.53211	0.000365617π

TABLE 9. Numerical values of $V_{n,1}$

REFERENCES

[1] S. Hollos, and R. Hollos, *Chebyshev polynomials*, Tutorial articles available on the Exstrom Labs website (2006), http://www.exstrom.com/journal/sigproc/chebident.pdf
[2] Wikipedia, *Chebyshev polynomials*, http://en.wikipedia.org/wiki/Chebyshev_polynomials
[3] Wikipedia, *Cone*, http://en.wikipedia.org/wiki/Cone
[4] Wikipedia, *Radius*, http://en.wikipedia.org/wiki/Radius

Reviewer's Comments

The reviewer has some comments about the presentation of this paper and the typos.

1. The grammatical tense is always changing and not consistent in "Introduction".
2. Equation (1), it may be better to roughly mention the meaning of n and θ.
3. "less than the sector" should be "less than the capacity of the cone".
4. There is no specialty for $k = \dfrac{5\pi}{4}$, how about replacing it with $0 < k < \dfrac{2\sqrt{6}}{3}\pi$ for consistency?
5. Although it does not affect the conclusion, it is better to mention $f_{2\pi}(0) = f_{2\pi}(2\pi) = 0$ and as a consequence the maximum of $f_{2\pi}(x)$ is attained at $x = 0.648027704\pi$.
6. The definition of "n, k-CONE" is not clear. How about moving there the Definition 8 on page 355, which constructively defines the "n,k-CONE" and the corresponding extended cone.
7. It should be Theorem 13, not Theorem 9, that implies the greatest capacity for $3, 2$-CONE.
8. The point "D" is missing.
9. It may be more readable to mention the geometrical calculation of the angle 3θ.