Hang Lung Mathematics Awards 2012

Honorable Mention

Trajectories in Regular Pentagon

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TRAJECTORIES IN REGULAR PENTAGON

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Abstract. It is known that a light ray must obey the law of reflection when it is reflected by a plane mirror. In this report, we are going to find out whether a light ray in a regular pentagon[1] formed by 5 congruent plane mirrors can go back to the starting position and what the possible emitting angles are. Also, we will investigate the looping of the light trajectory after finite reflection.

First, we make an observation on some special cases. Then, we will consider the general cases and try to classify the looping trajectories. Properties of looping trajectories will be studied. Lastly, another approach, vectors, will be used to investigate this problem.

1. Introduction

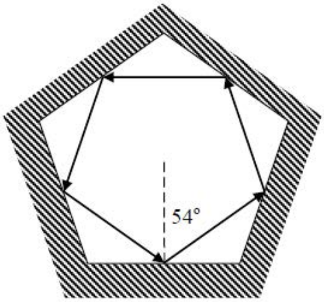
It is known that the law of reflection on a plane mirror is that "Angle of incidence equals to the angle of reflection." Before our investigation, the cases of hexagon, square and equilateral triangle were well studied by mathematicians. The results of them are significant for finding the angles that will return to their original position. [See reviewer's comment (1)]

Remark 1. *The angle of incidence of a light ray at a vertex is the angle made by the incident ray and the straight line passing through the vertex which is normal to the opposite side.*

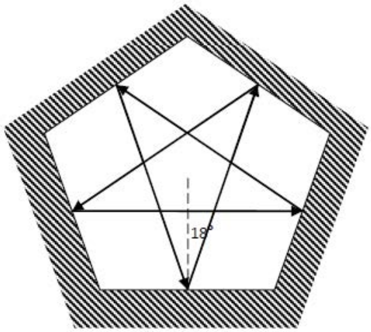
First, let us make an observation on some special cases.

Case 2. *A trajectory starting at the mid-point of the edge of the pentagon with emitting angle* $54°$

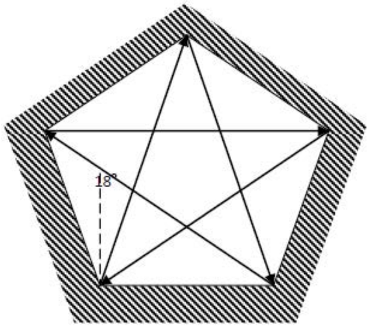
[1] In this paper, all pentagons we talk about are of unit side length.



Case 3. *A trajectory starting at the mid-point of the edge of the pentagon with emitting angle* 18°



Case 4. *A trajectory starting at the corner of the pentagon with emitting angle* 18°

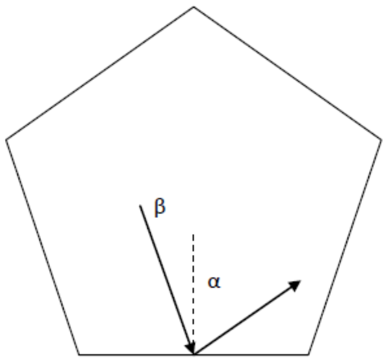


In the two special cases Case 3 and Case 4, it seems that the starting position of the light ray does not affect the result of whether it will return to their original position. [See reviewer's comment (2)] We will prove this phenomenon in chapter 7.

2. Emitting Angle, Returning Angle and Plane Development

We first give a precise definition of emitting angles and returning angles of a trajectory.

Definition 5. *Considering the starting position of a trajectory, the* ***emitting angle (α)*** *is the angle made by the light ray emitted and the normal to the side of the pentagon at the starting position. The* ***returning angle (β)*** *is the angle made by the light ray returned to the starting position and the normal to the side of the pentagon at the starting position.*

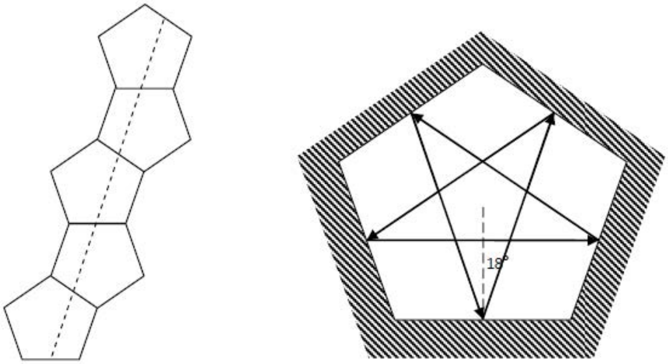


[See reviewer's comment (3)]

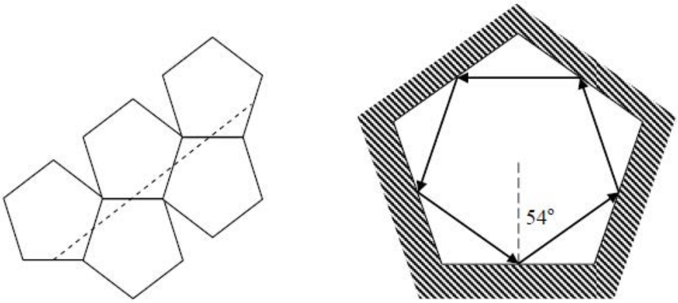
Now, we introduce the concept of plane development, which is very significant in our study.

Definition 6. *A* ***plane development*** *is a process of successively reflecting the pentagons along one of the sides to a new position, forming a series of pentagons.*

Example 7. *The following figure shows one plane development with a trajectory starting at the mid-point of the bottom edge of the pentagon and the emitting angle is* $18°$.



Example 8. *The following figure shows one plane development with a trajectory starting at the mid-point of the edge of the pentagon and the emitting angle is* 54°.



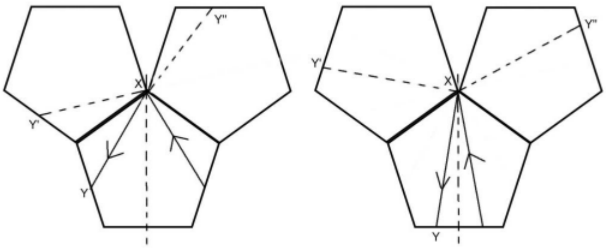
Proposition 9. *A straight line on a plane development is a valid trajectory if and only if it lies within the series of pentagons and it does not pass through any vertices.*

[See reviewer's comment (4)]

Proof. If a straight line lies within the series of pentagons and it does not pass through any vertices, we can obtain the corresponding trajectory in the original regular pentagon by simply restricting the path in the first pentagon (i.e. treat the sides of the first pentagon as 5 mirrors).

On the other hand, as a valid trajectory lies inside the regular pentagon, the reflected image of the trajectory in plane development should also lie inside the pentagons.

Besides, if the line trajectory is reflecting at a vertex, the reflected ray is symmetric to the incident ray along the line joining the vertex and the mid-point of the opposite side.



When plane development continues, the trajectory will not form a straight line.

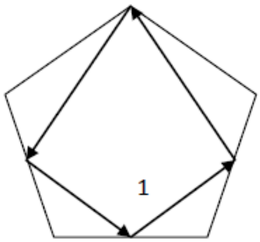
Hence a straight line on a plane development which is a valid trajectory must lie within the series of pentagons and does not pass through any vertices. □

With the plane development, we can analyze the trajectories in regular pentagon in a more systematic way and calculate the distance of the trajectory and the angle of incidence more effectively. Moreover, we can generalize our result using the method of plane development since it can be used not only for pentagons but also for other polygons. This will be further discussed in Chapter 8.

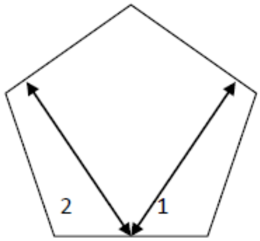
3. Looping of the Trajectories

If light ray can form loop in the regular pentagon, special patterns can be seen.

Definition 10. *We now define a* ***single-looped trajectory*** *to be a trajectory which will repeat after going back to the starting point once and a* ***double-looped trajectory*** *to be the trajectory which will repeat after going back to the starting point twice but not once.*



Single - looped trajectory



Double - looped trajectory

Theorem 11. *All looping trajectories are either single-looped trajectories or double-looped trajectories.*

Proof. Let the emitting angle and the returning angle of a looping trajectory be α and β respectively. (Fig. 1)

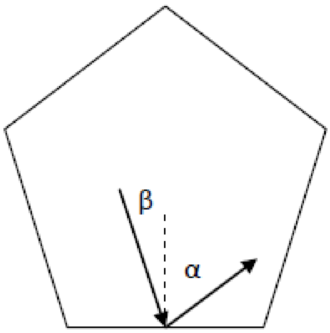


FIGURE 1

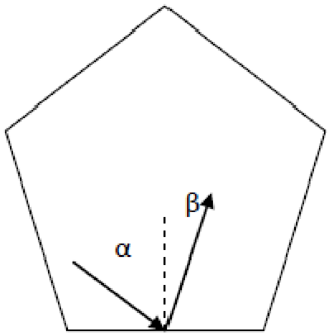


FIGURE 2

As the direction of trajectories is reversible, there exists another trajectory with the emitting angle β and the returning angle α. (Fig. 2) If $\alpha \neq \beta$, the light ray will follow the trajectory in fig. 1 and fig. 2 successively. Therefore, these two trajectories form a double-looped trajectory.

If $\alpha = \beta$, the trajectory will repeat after going back to the starting point according to the law of reflection. □

[See reviewer's comment (5)]

In the following two chapters, we are going to discuss single-looped trajectories and double-looped trajectories separately.

4. Single-looped Trajectories

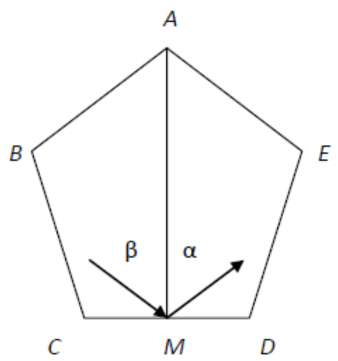
Now we are going to find out what the single-looped trajectories are. In this chapter, the starting point of the trajectories is assumed to be the mid-point of an edge.

First, we have the following characterization of single-looped trajectories starting from the mid-point of one edge:

Proposition 12. *A looping trajectory starting from the mid-point of one edge is a single-looped trajectory if and only if either one of the followings holds:*

i It contains a line which is parallel to the starting side;
ii It passes through the vertex opposite to the starting side.

Proof. For single-looped trajectories, by the law of reflection, the emitting angle is equal to the returning angle of the single-looped trajectories, i.e. $\alpha = \beta$.



With the notation in the above figure, the trajectory should have reflectional symmetry about AM. Therefore, the trajectory will either

i contain a line which is parallel to CD, or
ii pass through vertex A. [See reviewer's comment (6)]

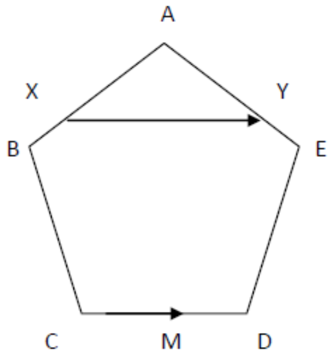
On the other hand, if any one of the two conditions holds, by law of reflection, the trajectory must have reflectional symmetry about AM. Hence the emitting angle is equal to the returning angle. Therefore it is a single-looped trajectory. □

Then we investigate the nature of the single-looped trajectories in these two conditions.

(i) The single-looped trajectory contains a line which is parallel to CD

For the trajectory which contains a line parallel to CD, the line may be in the interior or boundary of ΔABE or in the interior of trapezium $BCDE$. This gives two cases.

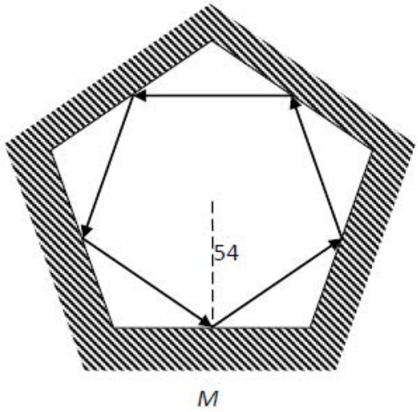
Case 1 The trajectory contains a line which is parallel to CD in ΔABE.



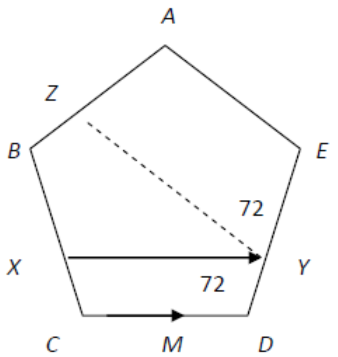
Let the line be XY. We have $XY//CD$. Now,

$$AB = AE$$
$$\angle ABE = \angle AEB = 36^\circ$$
$$\angle AXY = \angle AYX = 36^\circ$$

Continuing the reflections, many $108^\circ - 36^\circ - 36^\circ$ isosceles triangles will be formed. As the trajectory should return to point M, this gives a trajectory shown in the following figure. [See reviewer's comment (7)]



Case 2 The trajectory contains a line which is parallel to CD in trapezium $BCDE$.



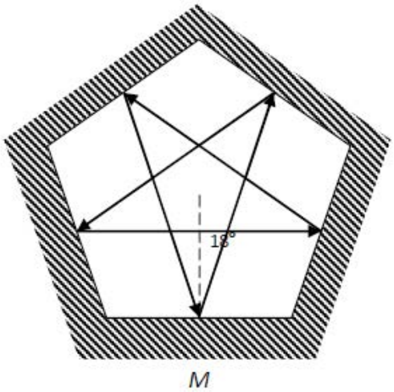
Let the line be XY. We have $XY//CD$. Now,

$$\angle XYD = 180^\circ - 108^\circ = 72^\circ$$

Reflect the light ray XY along the normal of DE to form YZ, we have

$$\angle ZYE = 72^\circ$$
$$\therefore ZY//AE$$

Similarly, many lines which are parallel to the sides of the pentagon will be formed. As the trajectory should return to point M, this gives a trajectory shown in the following figure.



(ii) The single-looped trajectory passes though vertex A.

For the looping trajectories which pass through vertex A, we find a series of the trajectories using plane development.

Remark 13. *From now on, we extend the method of plane development to trajectories which passes through vertex A (the vertex opposite to the starting side) but not other vertices, by the following way: As the trajectory is symmetric about AM, we can study the trajectory by only drawing the half trajectory from M to A in the plane development. Then it is a straight line and we treat it as the corresponding trajectory of the original path in the plane development.*

Consider the following pattern of plane development. Denote it as ***Pattern 1a***.

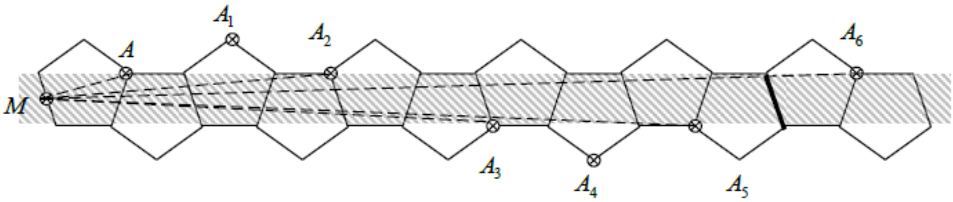


Figure of Pattern 1a

With the notation in the figure of Pattern 1a, M is the starting point of the light rays. A_n is the reflected point of A. The trajectories passing through vertex A are formed by joining M and A_n. If the line lies in the shaded region, the trajectory exists and has a unique emitting angle. Otherwise, the trajectory does not exist. From the figure of pattern 1a, the position of the points corresponding to A is periodic and of period 10 (the position repeats for every 10 pentagons).

For convenience, we make the following definition:

Definition 14. *A plane development of a trajectory is said to be of* ***of k pentagons*** *if the trajectory lies on the interior of exactly k pentagons on the plane development.*

Proposition 15. *Single-looped trajectories in Pattern 1a plane development passing through vertex A must be of* $10n+1, 10n+4, 10n+6, 10n+9$ *pentagons, where* $n = 0, 1, 2, \dots$.

Proof. From the figure of Pattern 1a, only the vertices corresponding to A in the $1^{\text{st}}, 4^{\text{th}}, 6^{\text{th}}$ and 9^{th} pentagons lie within the shaded region (for common vertices of two pentagons, we let them belong to the former pentagon). Only for these cases, the straight line joining M and that vertex corresponding to A lies within the series of pentagons. By Proposition 9 and 12, the result follows. [See reviewer's comment (8)] □

These plane developments represent trajectories with unique emitting angles. The value of the angles will be discussed in the later chapter.

Then we introduce another pattern of plane development which is slightly different from the pattern 1a, denoted as ***Pattern 1b***. Note that the pentagon is first reflected along the adjacent side of the starting position.

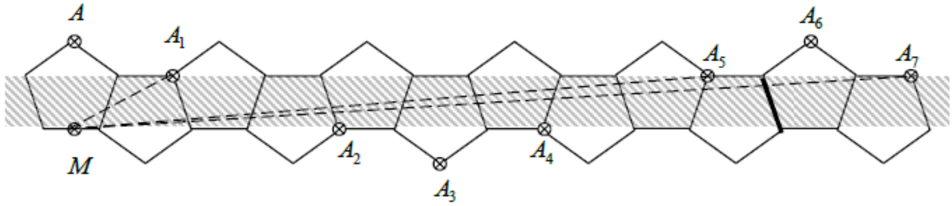


Figure of Pattern 1b

Proposition 16. *Single-looped trajectories in Pattern 1b plane development passing through vertex A must be of* $10n+2, 10n+9$ *pentagons, where* $n = 0, 1, 2, \dots$.

Proof. From the figure of Pattern 1b, only the vertices corresponding to A in the 2^{nd} and 9^{th} pentagons lie within the shaded region and valid trajectories can be drawn. Only for these cases, the straight line joining M and that vertex corresponding to A lies within the series of pentagons. By Proposition 9 and 12, the result follows. □

The corresponding trajectories will also have unique emitting angles.

We introduce another pattern of plane development denoted as ***Pattern 2***.

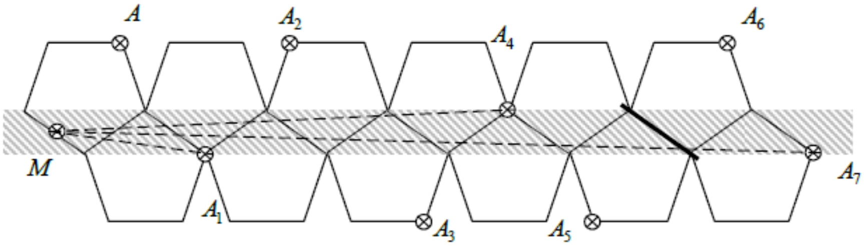


Figure of Pattern 2

Proposition 17. *Single-looped trajectories in Pattern 1b plane development passing through vertex A must be of* $10n + 2, 10n + 7$ *pentagons, where* $n = 0, 1, 2, \ldots$.

Proof. From the figure of Pattern 2, only the vertices corresponding to A in the 2^{nd} and 7^{th} pentagons lie within the shaded region. Only for these cases, the straight line joining M and that vertex corresponding to A lies within the series of pentagons. By Proposition 9 and 12, the result follows. □

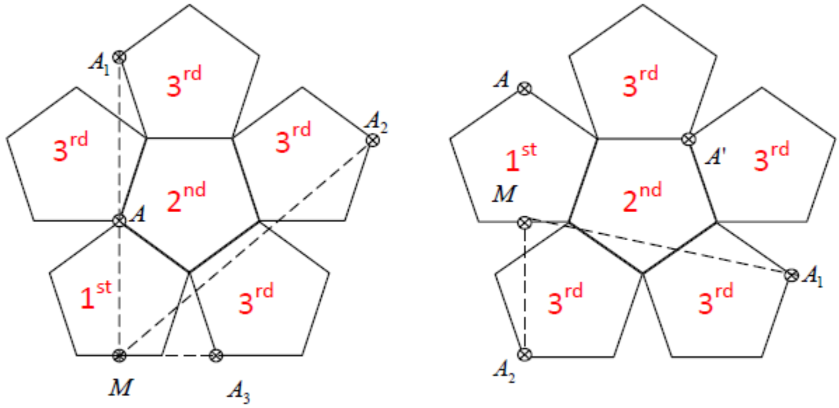
From Proposition 15 - 17, as n is arbitrary, we have the following two corollaries.

Corollary 18. *There exist infinitely many single-looped trajectories.*

Corollary 19. *The set of path lengths of all single-looped trajectories has no upper bound.*

Proposition 20. *There is no single-looped trajectory of 3 pentagons passing through vertex A.*

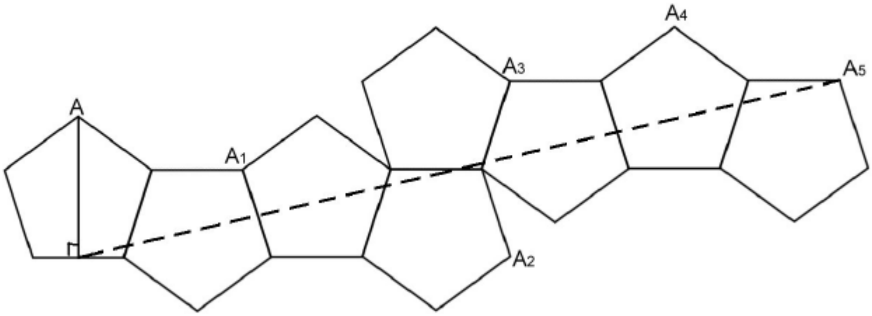
Proof. The figures below show all possible plane developments of 3 pentagons. All straight lines formed do not lie inside the pentagons and hence there is no trajectory of 3 pentagons.



□

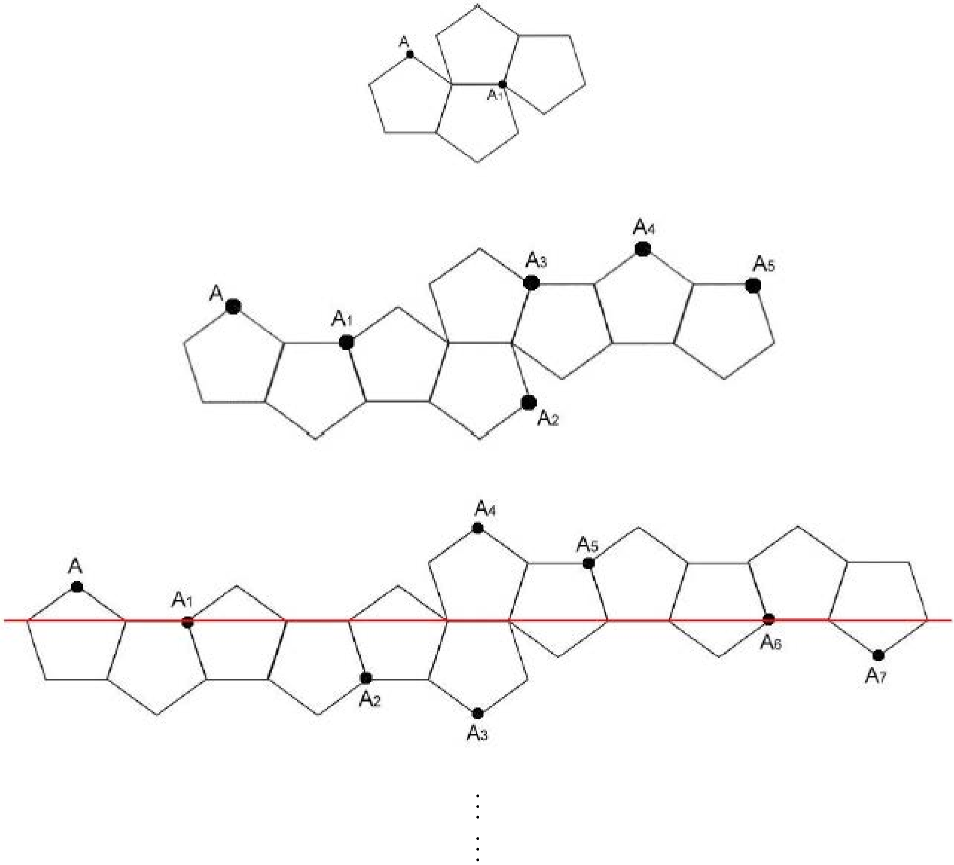
We than ask a question: Do all trajectories passing through vertex A lie in pattern 1a, Pattern 1b, or Pattern 2 plane development?

The answer is no because of the following example:



The above plane development neither belong to Pattern 1a, Pattern 1b, nor Pattern 2, while a looping trajectory is found (the dotted line).

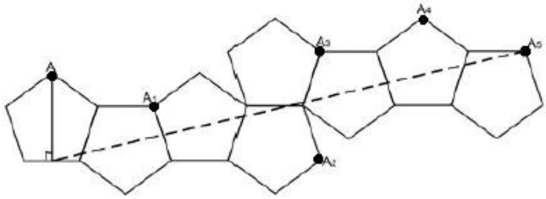
Motivated by the above example, we can show that there are actually infinitely many patterns of plane development which contain looping trajectories.
Consider the plane developments below.



We say that they consist of two rows of pentagons, the upper row and the lower row. And the line formed by extending the side of pentagons joining two rows of pentagons (the red line in the above figure) is called ***middle horizontal line.*** Suppose the trajectory starts from the mid-point of the lowest side of the left most pentagon.

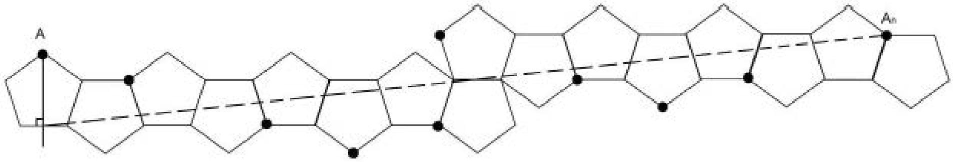
Proposition 21. *Single-looped trajectories in plane development arranged as the above figures must be of* $10n + 4$ *pentagons on each row, or of* $10n + 8$ *pentagons on the lower row with* $10n + 7$ *pentagons on the upper row, where* $n = 0, 1, 2, \dots$.

Proof.
Case 1



If the last corresponding vertex A_n is at the position as shown above, the vertical distance from starting position to the middle horizontal line is equal to that from the middle horizontal line to vertex A_n. The intersection point of the trajectory and the middle horizontal line is the mid-point of the trajectory. Therefore the intersection point must be on the side of the pentagons which joining the two lines of pentagons. The trajectory must lie inside the series of pentagons and does not pass through any vertices. By proposition 9 and 12, it is a single-looped trajectory.

Case 2



Similarly, if the last corresponding vertex A_n is at the position as shown above, the intersection point of the trajectory and the middle horizontal line must be on the side of pentagons which joining two lines of pentagons. Therefore, the trajectory must lie inside the series of pentagons and does not pass through any vertices. By Proposition 9 and 12, it is a single-looped trajectory.

Case 3

If the last vertex A_n is at the other 3 positions, the trajectory cannot lie inside the series of pentagons obviously.

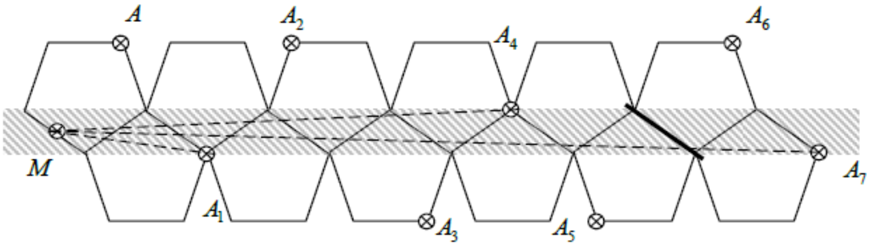
By observation, when the number of pentagons on each line increases by 2, the position of the last corresponding vertex A_n will be rotated by $\frac{4\pi}{5}$ in clockwise direction. The position of last corresponding vertex A_n is periodic of 10 (when the number of pentagons on each row increases by 10, the position of the last corresponding vertex A_n will go back to the original position).

When number of pentagons is $10n + 4$ on each row, the position of the last corresponding vertex A_n is the same as the figure of case 1, single-looped trajectories can be formed.

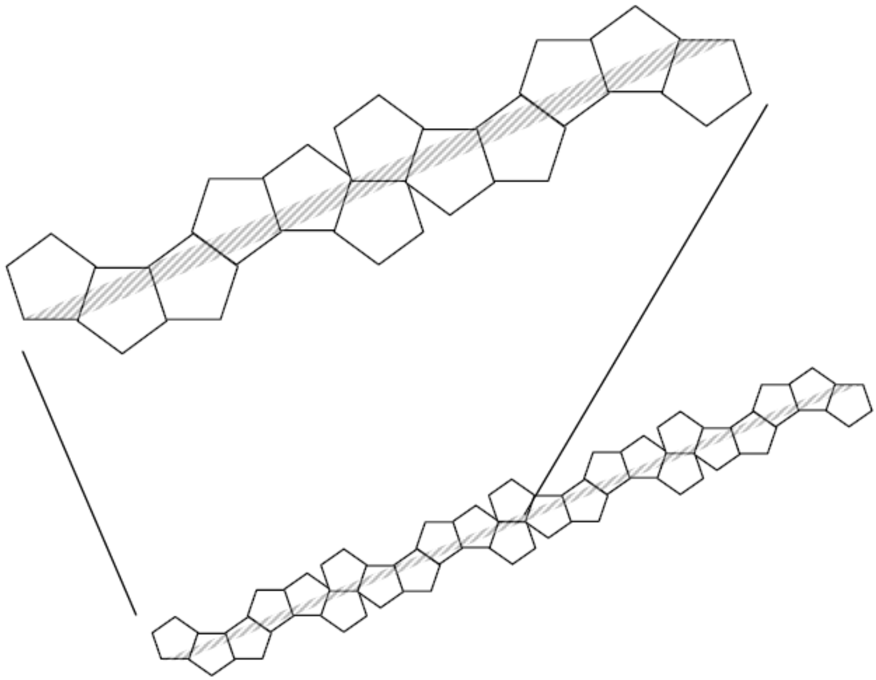
When number of pentagons is $10n+8$ on each row, the position of the last corresponding vertex A_n is the same as the figure of case 2, single-looped trajectories can be formed. But the trajectory will not cover the last pentagon, so it is on plane development of $10n+8$ pentagons on the lower row with $10n+7$ pentagons on the upper row. □

One can also produce infinitely many new patterns which contain looping trajectories by replacing the each pentagon in Pattern 1a by with n pentagons, where $n = 2, 3, 4, 5, 6, 7, 8, 9 \ldots$. [See reviewer's comment (9)] Some examples are illustrated below.

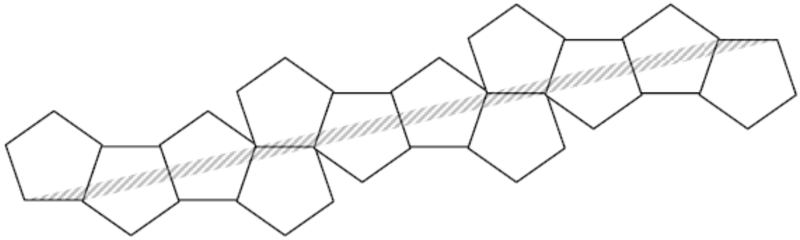
Replace each pentagon in Pattern 1a with 2 pentagons (pattern 2 is formed):



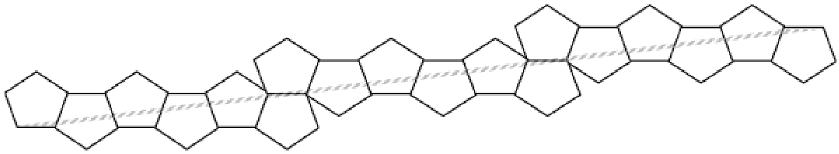
Replace each pentagon in Pattern 1a with 3 pentagons:



Replace each pentagon in Pattern 1a with 4 pentagons:

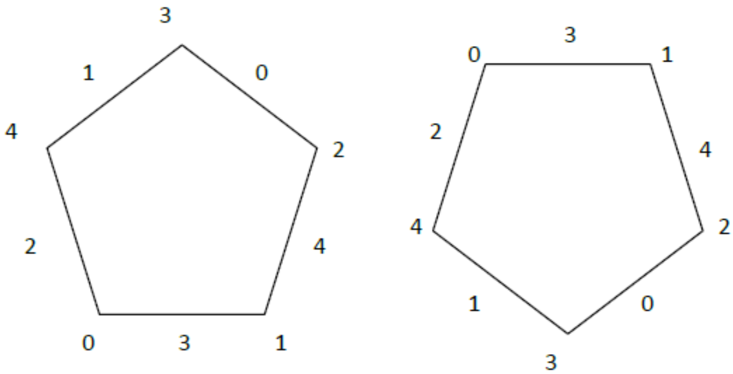


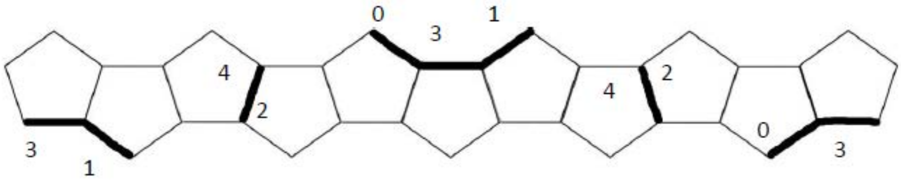
Replace each pentagon in Pattern 1a with 6 pentagons:



Proposition 22. *All patterns which contain looping trajectories generated by the above replacing method must be periodic.*

Proof. With the notation of the figure, the positions of the five vertices and their opposite sides are named 0, 1, 2, 3 and 4 respectively.





Consider the bottom side of the pentagon. As shown in the figure, the position of corresponding sides of it in pattern 2 is periodic. The positions of the corresponding sides are $3, 1, 4, 2, 0, 3, 1, 4, 2, 0 \ldots$.

As a result, the position of the corresponding sides in the n^{th} pentagons $\equiv 3n \pmod 5$.

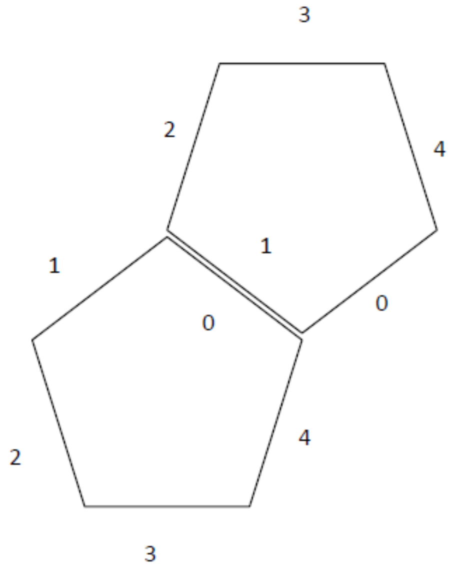
Also, by definition, the numbers of the vertex and its opposite sides have no change when reflected about a horizontal line. When the pentagon is reflected about a horizontal line, the position of the corresponding side in the upper pentagon
$\equiv 3n' + 3 - 3 \pmod 5$
$\equiv 3(n' + 1) - 3 \pmod 5$
where $3n' \pmod 5$ is the position of the corresponding side in the lower pentagon.

For patterns formed by replacing 1 pentagon with m pentagons (m is even) to be periodic, the position of the side in 1^{st} pentagon of $(k+1)^{\text{th}}$ row$\equiv 3 \pmod 5$

$$3km + 3 - 3k \equiv 3 \pmod 5$$
$$3km - 3k \equiv 0 \pmod 5$$
$$3k(m-1) \equiv 0 \pmod 5$$

If $m \equiv 1 \pmod 5$, the equation holds for all k. Therefore, the patterns will have a period of m.

If $m \equiv 0, 2, 3, 4 \pmod 5$, the equation holds for $k \equiv 0 \pmod 5$ the patterns will have a period of $5m$.



If the reflection of the pentagon is same as the figure, the position of the corresponding side in the upper pentagon
$\equiv 3n' + 3 - 2 \pmod 5$
$\equiv 3(n' + 1) - 2 \pmod 5$
where $3n' \pmod 5$ is the position of the corresponding side in the lower pentagon.

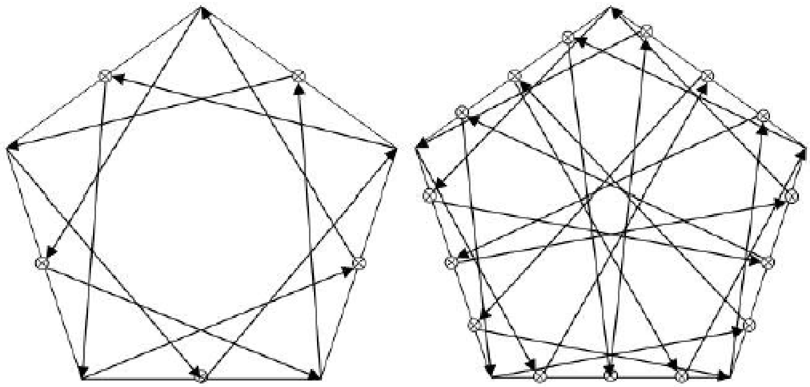
For patterns formed by replacing 1 pentagon with l pentagons (l is odd) to be periodic, the position of the sides of 1^{st} pentagons of $(2k+1)^{\text{th}}$ row $\equiv 3 \pmod 5$

$$3(2kl) + 3 - (2+3)k \equiv 3 \pmod 5$$
$$6kl - 5k \equiv 0 \pmod 5$$
$$6kl \equiv 0 \pmod 5$$

If $l \equiv 0 \pmod 5$, the equation holds for all k. Therefore, the patterns will have a period of $2l$.

If $l \equiv 1, 2, 3, 4 \pmod 5$, the equation holds for $k \equiv 0 \pmod 5$ the patterns will have a period of $10l$. $\square$

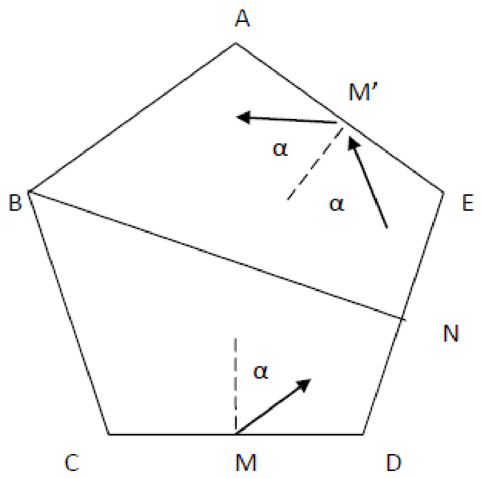
Now, for trajectories passing through more than 1 vertex (i.e. at least one vertex other than vertex A), the method of plane development is not applicable. Two examples are shown below.



Denote the other 4 vertices of the regular pentagon as B, C, D, E. We obtain the following result.

Proposition 23. *Any trajectory passing through any one of B, C, D, E must be a single-looped trajectory passing through all 5 vertices.*

Proof. Consider a trajectory starting at M passing through vertex B.



As it passes through B, it should have reflectional symmetry about BN, where N is the mid-point of DE. Therefore, the trajectory should pass through M' which is the mid-point of AE. At M', the angle of incidence is equal to the emitting angle

α. According to the law of reflection, the angle of reflection is equal to the angle of incidence α. Consequently, the trajectory will pass through all of the 5 vertices and all of the 5 mid-points of the side of the pentagons. Finally, it will reach point M with the returning angle α. A single-looped trajectory passes through vertex A is given. Similarly, cases with trajectories starting at M passing through C, D and E respectively will result in some single-looped trajectories passing through vertex A.

As a result, trajectories passing through any one of B, C, D, E must be a single-looped trajectory passing through all 5 vertices. □

Corollary 24. *Any looping trajectory passing through any vertex of the regular pentagon must be a single-looped trajectory.*

Proof. The result follows immediately from Proposition 12 and Proposition 23. □

5. Double-looped Trajectories

First we give a characterization of double-looped trajectories starting from the mid-point of one edge:

Proposition 25. *A looping trajectory starting from the mid-point of one edge is a double-looped trajectory if and only if it does not contain a line which is parallel to the starting side and does not pass through the vertex opposite to the starting side.*

Proof. By Theorem 11, every trajectory is either a single-looped trajectory or a double-looped trajectory. [See reviewer's comment (10)] Hence by Proposition 12, the result follows. □

To study double-looped trajectories, we introduce two patterns of plane development below. In this chapter, the starting point of the trajectories is assumed to be the mid-point of an edge.

Denote the following pattern as ***Pattern 1***

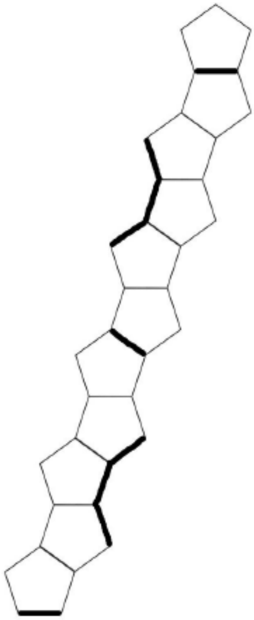


Figure of Pattern 1

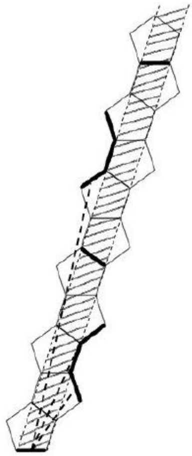
Suppose the light trajectory starts from the mid-point of the lowest side of the lowest pentagon. The thickened sides of the pentagons are the corresponding sides of the starting side. Therefore, when the light trajectory goes to the mid-point of the thickened sides of the pentagons in this plane development, the light trajectory goes back to the original starting position in the regular pentagon formed by plane mirrors.

If the straight light trajectory goes outside the plane development of pentagons, possible light trajectory in the regular pentagon formed by plane mirror cannot be formed, because the part of trajectory outside the plane development is not the correct reflected ray.

Also, we can see that Pattern 1 plane development is periodic and of period 10 (The position of the thickened side repeats for every 10 pentagons).

Proposition 26. *Except the looping trajectories on plane development of 2 pentagons, looping trajectories in Pattern 1 plane development must be of* $10n, 10n+3, 10n+5, 10n+8$ *pentagons, where* $n = 0, 1, 2, \ldots$.

Proof.

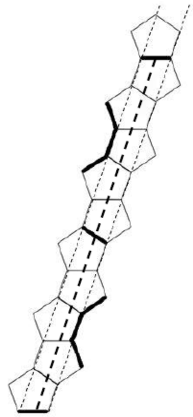


From the above figure of Pattern 1, trajectories of 2 pentagons are found. Although they do not lie in the shaded region, they lie in the plane development. Except these cases, only the vertices corresponding to A in the $3^{\text{rd}}, 5^{\text{th}}, 8^{\text{th}}$ and 10^{th} pentagons lie within the shaded region. Only for these cases, the straight line joining M and that vertex corresponding to A lies within the series of pentagons. By Proposition 9, the result follows. □

Then we try to classify the trajectories in Proposition 26.

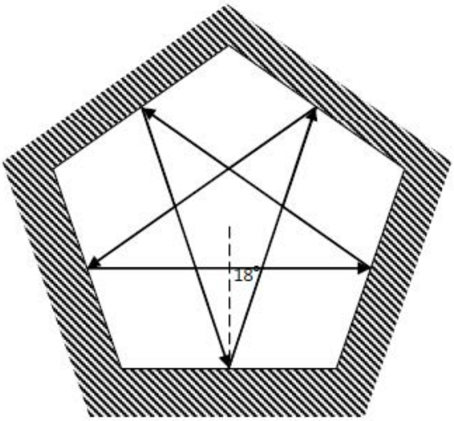
Proposition 27. *Looping trajectories in Pattern 1 plane development of* $10n + 5, 10n$ *pentagons, where* $n = 0, 1, 2, \ldots,$ *are single-looped trajectories.*

Proof.



As the trajectory is parallel to one of the sides of the pentagon, by Proposition 12, it is a single-looped trajectory.

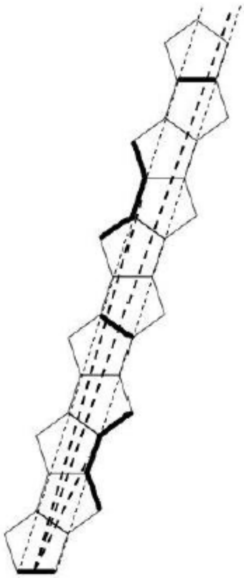
Converting it back to trajectory in regular pentagon, we also observe that it is actually the same as one case shown on Ch.1 (the below figure)



□

Proposition 28. *Looping trajectories in Pattern 1 plane development of* $10n+3, 10n+8$ *pentagons, where* $n = 0, 1, 2, \ldots,$ *are double-looped trajectory.*

Proof.



It is obvious that the light trajectory will not be parallel to any thickened sides of the pentagons and will not pass through any vertices of the pentagons. By Proposition 25, they are double-looped trajectories. □

Now consider another pattern of plane development. This time the light trajectory starts from the mid-point of the lowest side of the left most pentagon, as shown in the following figure. Denote this pattern as ***Pattern 2.***

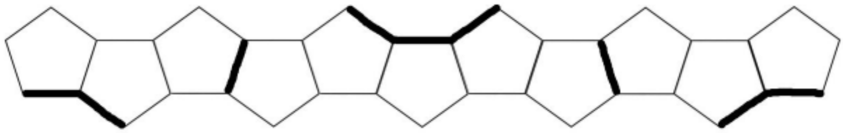
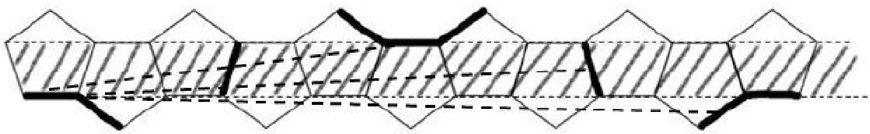


Figure of Pattern 2

Proposition 29. *Looping trajectories in Pattern 2 plane development must be of* $10n+3, 10n+6, 10n+8$ *pentagons, where* $n = 0, 1, 2, \dots$ *, and they must be double-looped trajectories.*

Proof.



Only the thickened sides of the $3^{\text{rd}}, 6^{\text{th}}$, and 8^{th} pentagons are within the shaded region. This plane development is also periodic with the period of 10 pentagons. Therefore, when plane development of $10n + 3, 10n + 6, 10n + 8$ pentagons are formed, where $n = 0, 1, 2, 3 \dots$, and the light trajectory goes to the mid-point of the corresponding side of the last pentagon to the starting point, the light trajectory goes back to the starting position in the regular pentagon formed by mirrors.

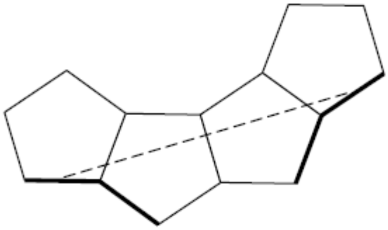
It is obvious that the light trajectory will not be parallel to any thickened sides of the pentagons and will not pass through any vertices of the pentagons. By Proposition 25, they are double-looped trajectories. □

From Proposition 28 and 29, as n is arbitrary, we have the following corollaries.

Corollary 30. *There exist infinitely many double-looped trajectories.*

Corollary 31. *The set of path lengths of all double-looped trajectories has no upper bound.*

It is natural to ask whether pattern 1 and 2 contain all types of trajectories. Unfortunately the answer is no, due to the following example of a trajectory of 4 pentagons.



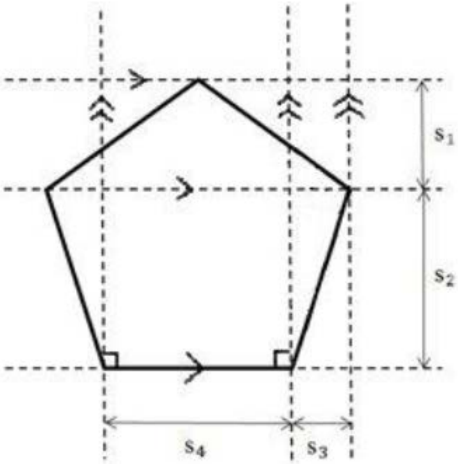
6. Method of Finding the Emitting Angles

After considering the cases of looping trajectories in regular pentagon, we are interested in finding an expression or an exact value of the emitting angles in the cases for which the trajectories will go back to the original position. We develop a method to find the emitting angles below.

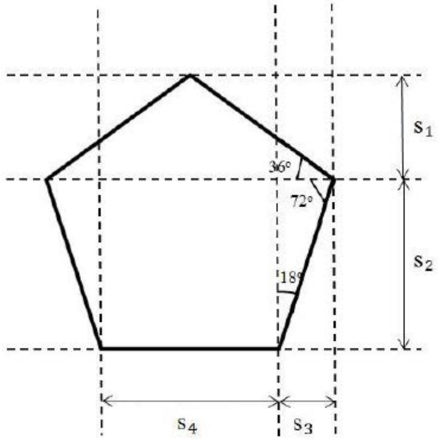
Proposition 32. *For any looping trajectory in the regular pentagon formed by plane mirrors which pass through non of the vertices, the emitting angle is given by*

$$\tan^{-1}\frac{k_3(\sqrt{5}-1)+4k_4}{k_1\sqrt{10-2\sqrt{5}}+k_2\sqrt{10+2\sqrt{5}}}$$

where k_1, k_2, k_3, k_4 are rational numbers such that the trajectory on the plane development has vertical distance $k_1s_1+k_2s_2$ and horizontal distance $k_3s_3+k_4s_4$, where s_1, s_2, s_3, s_4 denote the lengths indicated as the below figure.



Proof. Let s_1, s_2, s_3, s_4 be the indicated lengths in the figure below.

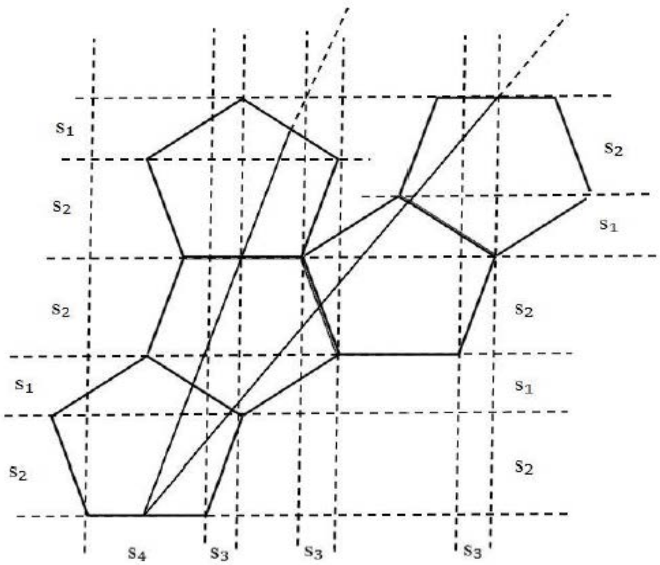


It is easy to obtain

$$s_1 = \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$$
$$s_2 = \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$$
$$s_3 = \sin 18^\circ = \frac{\sqrt{5}-1}{4}$$
$$s_4 = 1$$

As the trajectories only pass through at most 1 vertex, we can use the method of plane development.

By proposition 9, all the looping trajectories can form a straight line on a series of pentagons by their corresponding plane development.



When plane development continues, the pentagons repeat.

Therefore, the horizontal distance between the starting point and ending point must be in terms of s_3 and s_4. The vertical distance must be in terms of s_1 and s_2. The angle between the light trajectory and the side of pentagon is

$$\begin{aligned}
&\tan^{-1}\frac{\text{vertical distance}}{\text{horizontal distance}}\\
=&\tan^{-1}\frac{k_1s_1+k_2s_2}{k_3s_3+k_4s_4} \quad \text{where } k_1,k_2,k_3,k_4 \text{ are rational numbers}\\
=&\tan^{-1}\frac{k_1\frac{\sqrt{10-2\sqrt{5}}}{4}+k_2\frac{\sqrt{10+2\sqrt{5}}}{4}}{k_3\frac{\sqrt{5}-1}{4}+k_4(1)}\\
=&\tan^{-1}\frac{k_1\sqrt{10-2\sqrt{5}}+k_2\sqrt{10+2\sqrt{5}}}{k_3(\sqrt{5}-1)+4k_4}
\end{aligned}$$

Hence, the emitting angle is given by

$$\begin{aligned}
&90^\circ-\tan^{-1}\frac{k_1\sqrt{10-2\sqrt{5}}+k_2\sqrt{10+2\sqrt{5}}}{k_3(\sqrt{5}-1)+4k_4}\\
=&\tan^{-1}\frac{k_3(\sqrt{5}-1)+4k_4}{k_1\sqrt{10-2\sqrt{5}}+k_2\sqrt{10+2\sqrt{5}}}.
\end{aligned}$$

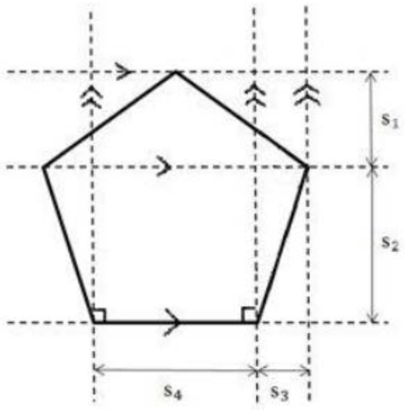
□

For looping trajectories which pass through more than 1 vertex, by Proposition 23, it must pass through all 5 vertices of the regular pentagon. We cannot directly apply Proposition 32. Fortunately, we can still find the emitting angles of them by modifying our previous method.

Proposition 33. *For all looping trajectories in the regular pentagon formed by plane mirrors which pass through 1 or more vertices, the emitting angle is given by*

$$\tan^{-1}\frac{k_3(\sqrt{5}-1)+4k_4}{k_1\sqrt{10-2\sqrt{5}}+k_2\sqrt{10+2\sqrt{5}}}$$

where k_1,k_2,k_3,k_4 are rational numbers such that the segment of trajectory from the starting point to the first vertex it meets on the plane development has vertical distance $k_1s_1+k_2s_2$ and horizontal distance $k_3s_3+k_4s_4$, where s_1,s_2,s_3,s_4 denote the lengths indicated as the below figure.

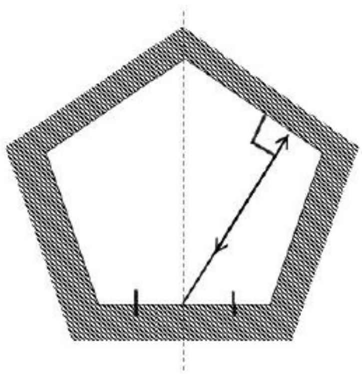


Proof. The segment of trajectory from the starting point to the first vertex it meets is a straight line on the plane development. By the same method in the proof of Proposition 32, the result follows. □

By Proposition 32 and 33, for any looping trajectory, we can find an exact expression of its emitting angle.

Then we give two examples to illustrate how to make use of Proposition 32.

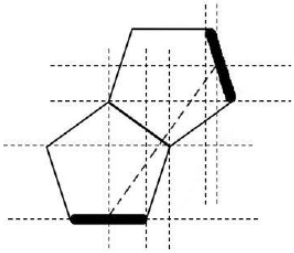
Example 34. *We start with the simplest case.*



The emitting angle

$$= 180^\circ - 90^\circ - \frac{108^\circ}{2} \quad (\textit{angle sum of } \Delta)$$
$$= 36^\circ$$

By using the plane development,

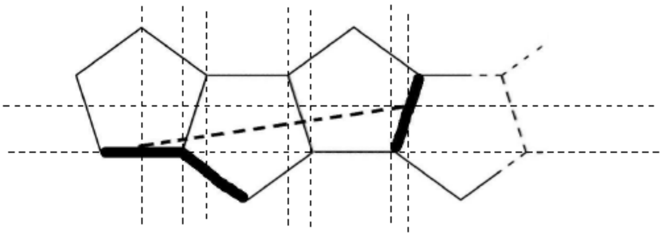


Vertical distance $= s_1 + 1.5s_2$
Horizontal distance $= 1.5s_3 + s_4$
The emitting angle

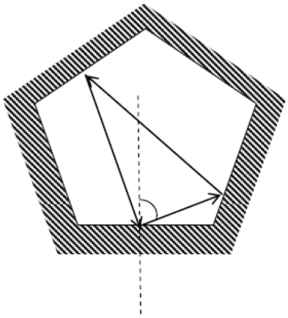
$$
\begin{aligned}
&= 90^\circ - \tan^{-1}\frac{\textit{vertical distance}}{\textit{horizontal distance}}\\
&= 90^\circ - \tan^{-1}\frac{s_1 + 1.5s_2}{1.5s_3 + s_4}\\
&= \tan^{-1}\frac{1.5(\sqrt{5}-1)+4}{\sqrt{10-2\sqrt{5}}+1.5\sqrt{10+2\sqrt{5}}}\\
&= 36^\circ
\end{aligned}
$$

The results are consistent.

Example 35. *We use the example of double-looped trajectory again.*



Vertical distance $= 0.5s_2$
Horizontal distance $= 2.5s_3 + 2.5s_4$



The emitting angle

$$\begin{aligned}
&= 90^\circ - \tan^{-1}\frac{\textit{vertical distance}}{\textit{horizontal distance}}\\
&= 90^\circ - \tan^{-1}\frac{0.5s_2}{2.5s_3+2.5s_4}\\
&= \tan^{-1}\frac{2.5(\sqrt{5}-1)+2.5(4)}{0.5\sqrt{10+2\sqrt{5}}}\\
&= 81.7^\circ \quad , \textit{cor. to 3 sig. fig.}
\end{aligned}$$

Therefore, given any looping trajectory, we can use Proposition 32 or 33 to find an exact expression of its emitting angle.

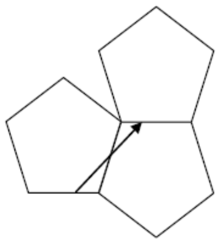
7. Another Approach, Vectors

In this chapter, we will use the techniques of vectors to show the independence of starting position of looping trajectories.

Proposition 36. *If a trajectory is looping, then after some parallel translation, it is still looping.*

[See reviewer's comment (11)]

Proof. First, we consider the plane development of a looped trajectory and treat it as a vector.



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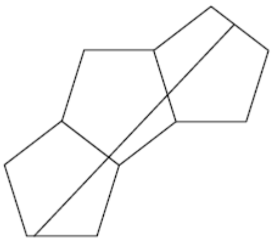
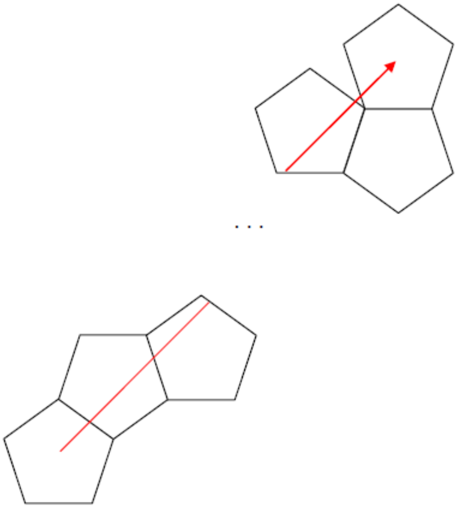


Since it is a vector, we can translate its starting point of it from the side of the pentagon to the center of the pentagon.



Case 1 The plane development of the trajectory is the same after translation.

In this case, it is obvious that it can go back to its original position because the route of the trajectory is the same.

Case 2 The plane development of the trajectory is different after translation.

If we consider the plane development on the regular dodecahedron, the looped trajectory means a loop on its surface and the parallel translation is the same on it. Then the statement is trivially correct, since the loop on the dodecahedron is just like an extensive rubber band, if we move it up or down, the rubber band will become smaller. □

Hence, the starting position of the trajectory does not affect whether it is looping or not. This implies that all the results in the chapter 4 to chapter 6 can be extended to arbitrary starting position.
As a remark, from the paper [1], the length of any trajectory that does not touch the vertices of the pentagon is in the form of $\sqrt{a+b\phi}$, where $\phi = \frac{1+\sqrt{5}}{2}, a, b \in \mathbb{Q}$.

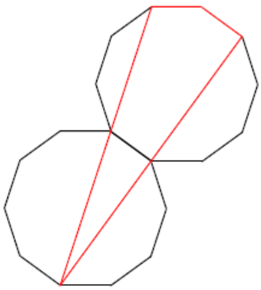
8. Further Discussion

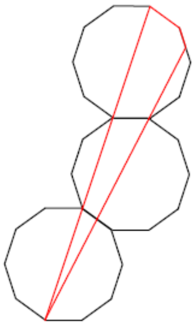
Using the method of plane development, we can extend out study to other regular polygons.

As we said before, the method of plane development and parallel translation of the trajectory can be used in the other polygons. Therefore, we hope that there is also a formula to find all emitting angles such that the trajectory will become a loop. In addition, from the previous results and our own result, it seems that the formula is related to the value of $\tan\alpha$, where α represents the angles of the regular polygon. For example, in the case of equilateral triangle, the required formula is $\theta = \tan^{-1}\frac{p\sqrt{3}}{q}, p, q \in \mathbb{Z}$ and $\tan 60° = \sqrt{3}$; in the case of pentagon (our result), the required formula is $\theta = \tan^{-1}\sqrt{a+b\phi}, a, b, \in \mathbb{Q}, \phi = \frac{1+\sqrt{5}}{2}$, and $\tan 108° = -\sqrt{5+2\sqrt{5}} = -\sqrt{1+4\phi}$. Therefore, we believe that the formulas of the other regular polygons are somehow related to the value of $\tan\alpha$.

Besides, by studying the plane development, we find that the looping trajectory is unstable with respect to the emitting angle in the cases of regular polygons. It means that if we adjust the emitting angle slightly, the plane development of the trajectory will change greatly.

When the number of regular polygons in a plane development increases, the possible range of emitting angles of looping trajectories decrease greatly. This phenomenon appears because the starting position and the path of the looping trajectory are fixed. For example, on a plane development of 2 decagons, the range of emitting angles of looping trajectories in plane development is around 22.5 degree. When the plane development is of 3 decagons, the range significantly decreases to around 10 degree.





9. Conclusion

In the previous chapters, we have successfully found some trajectories for which the light ray can go back to the starting position and their unique emitting angles. Also, we have proved that the looping trajectories are either single-looped or double-looped. Then we gave two characterizations of single-looped and double-looped trajectories. Besides, we discovered that there are infinitely many single-looped and double-looped trajectories. We also investigated some patterns of plane developments which contain looping trajectories.

In addition, using the method of plane development, we found an effective way for obtaining the exact value of the emitting angle of any trajectory.

More importantly, we proved that the looping property of trajectories is independent of the starting position of the trajectories. Hence the results of the previous chapters are also valid for looping trajectories with any starting position.

Finally, we made some further investigations on the trajectories in regular polygons. We believe that our method can be further generalized.

REFERENCES

[1] Dmitry Fuchs, *Geodesics on a regular dodecahedron*, http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_mathematik/2010/2009_91.pdf

Reviewer's Comments

We organize our comments in two parts: mathematical and expositional. In what follows ln means the n-th line from the top of the page, and l$-n$ means the n-th line from the bottom.

Comments on the mathematical content

1. Section 1 first paragraph. Cite some references for the known cases of hexagon, square and equilateral triangle. It would be nice to explain why this problem (trajectories of reflecting rays) is mathematically interesting.
2. Do you mean that a periodic path can start at any position in the pentagon?
3. Definition 5. Comment on the case where the light ray hits a corner of the pentagon.
4. Before Proposition 9. Point out that the plane development does not apply to trajectories that hit the corners of the pentagon (explained in the proof of Proposition 9).
5. Proof of Theorem 11. For this argument to work, it seems that the ray must start at the midpoint of the side. If so, please clarify (as in the next chapter).
6. Argue why the ray must pass through vertex A if it does not contain a line parallel to CD.
7. The argument is rather sloppy. Please supply the details (same for Case 2 below).
8. Proof of Proposition 15. Comment on the period 10.
9. Do these cases cover all looping trajectories?
10. Proof of Proposition 25. Every *looped* trajectory.
11. Proposition 36. Define parallel translation properly. (In particular, the translation is performed on the plane development.) The statement of the proposition claims that there exists a translation so that the trajectory is still looping (or for all?). It is not clear to me how the argument proves this.

Comments on exposition

As far as the logical development goes the paper is well structured. After considering some special cases the authors develop the concept of plane development and use it to investigate the properties of various looping trajectories. Most results are carefully stated and the proofs are clear.

On the other hand, the authors should try to provide more motivations about the problems and point out connections with previous results. Here are a few explicit questions the reviewer has in mind:

- What are the main known results achieved by mathematicians?
- What are the key ideas and methods used? How is the current case (pentagon) different from the previous ones?

- Connections with problems in, say, dynamical systems.
- Was the method of plane development used in similar problems?
- Some interesting open problems.

While it is understandable that the authors may not be able to answer all these questions satisfactorily (at this level), some discussion would be highly appreciated. The paper cites only one paper. It is not easy for a general reader to appreciate the 'bigger picture'.

There are some grammatical mistakes throughout the paper, but they are not serious.