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THE PRODUCT OF CONSECUTIVE INTEGERS IS NEVER A SQUARE

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ABSTRACT. The aim of our project was to prove our conjecture that the product of consecutive positive integers is never a square. In our investigation, we had developed three approaches to prove it.
In the first approach, we used the fact that a number lying between two consecutive squares is never a square to prove that the product of eight consecutive integers is never a square. Then we made use of relatively prime-ness of consecutive integers to prove the rest.
In the second approach, we had used Bertrand's Postulate Theorem to obtain a beautiful theorem that the product of consecutive positive integers is never a square if there is a prime number among them. Besides we had found some interesting results from this theorem.
When we started our project, we thought that our conjecture had not been proved. However, we found later in a website that our conjecture has already been proved by two famous mathematicians P. Erdos and J.L. Selfridge in 1939. Although our conjecture was proved, we didn't give up but tried our best to develop our third approach.
In the third approach, we had referred to an academic journal [1] written by P. Erdos and J.L. Selfridge and knew that the square-free parts of consecutive integers are distinct. By counting, we arrived at a necessary condition for the product not to be a square. Unluckily, we then discovered the limitations of the third approach when the number of consecutive integers is very large. It may be due to the roughness of our estimation. Although we couldn't complete the proof of our conjecture, we all enjoyed the process of formulating conjectures and thinking new ideas of solving problems through the cooperation among our team members in the past few months.

[1]This work is done under the supervision of the authors' teacher, Mr. Chi-Keung Lai.

1. Introduction

In this report, we would like to test our conjecture that the product of ($n \geqslant 2$) consecutive positive integers is never a square. In the past few months, we had developed three approaches to investigate our conjecture.

The first approach: Through investigating the lengths of consecutive products, we wished to find some patterns for generalization. However, we had only proved the products with lengths up to 12 and found hard to go further.

The second approach: We had proved that the product of consecutive positive integers is never a square if there is at least a prime among them. Later, we would explore these consecutive products without primes among them to see whether our conjecture is still true.

The third approach: After reading an academic journal [1] written by P. Erdos and J.L. Selfridge, we tried to find another method to prove our conjecture is true for $n \geqslant 2$.

2. Investigation Background

Our schoolmates had proved that the consecutive products are never a square for length $n = 2, 3, 4$ few years ago. After proving more cases, we believe that the product of any consecutive positive integers is never a square and started our research.

3. Research Results

3.1. The first approach

Theorem 1. *The product of five consecutive positive integers is never a square.*

Proof. Considering the product of five consecutive positive integers, $a \times b \times c \times d \times e$, we assume the product is a square. Now we investigate our conjecture case by case as follows:

Type 1: b and d having a common factor 2.

Case 1: a and d having common factor 3.
Since only 2 and 3 can be the common factors of the five consecutive positive integers, both c and e are relatively prime with the other three numbers. If the product is a square, c and e must be squares too. However, it is impossible that the difference between two consecutive squares is two. Thus our assumption is false.

Case 2: b and e having a common factor 3.
The proof for case 2 is the same as above. It can be done when we consider that a and c are relatively prime with the other three numbers.

Case 3: Only c is a multiple of 3.
a and c are now relatively prime. The proof for case 3 is the same as above.

Type 2: a, c and e having a common factor 2.

Case 1: 3 is not a common factor among the five numbers.
b and d are now relatively prime. The proof for case 4 is the same as above.

Case 2: a and d having a common factor 3.
Let $a = 3k$ and $b = 3k + 1$. As b is relatively prime with other four numbers, it must be a square if the product is a square. Since b is odd, so its remainder is 1 when it is divided by 8 (as the remainder is 1 when any odd square is divided by 8). Thus $b = 24h + 1$. On the other hand, $e = 24h + 4 = 4(6h + 1)$ and $6h + 1$ is odd and relatively prime with the other four numbers. Therefore, if the consecutive product is a square, $5h + 1$ must be a square too. Since $6h + 1$ is a square, e is also a square. b and e must be 1 and 4 respectively since their difference is 3. However, the product is not a square after checking.

Case 3: b and e having a common factor 3.
Let $b = 3k$ and $d = 3k + 2$. Since d and the other four numbers are relatively prime, it must not be a square (as the reminder is never 2 when a square is divided by 3). Hence the assumption is false. The six cases have already exhausted all the situations of the occurrence of possible common factors. We found that the assumption is always wrong, so their product is never a square. □

Theorem 2. *The product of six consecutive positive integers is never a square.*

Proof. Now we consider all the remainders when squares are divided by 30. For any $k > 30$, $(30m+k)^2 = 30(30m^2+2km)+k^2$, then $(30m+k)^2 \equiv k \pmod{30}$. Therefore it is sufficient to consider the remainders when square numbers $1^2, 2^2, \ldots, 30^2$ are divided by 30. By checking, we found that the remainders of square numbers must not be 7, 13, 17, 23 and 29 when they are divided by 30. We can say that all integers in the forms $(30n+7)$, $(30n+13)$, $(30n+17)$, $(30n+23)$ or $(30n+29)$ must not be squares. They also do not have factors 2, 3, and 5. For any six consecutive positive integers, if there exists a number in the form of $(30n+7)$, $(30n+13)$, $(30n+17)$, $(30n+23)$ or $(30n+29)$, this number must be relatively prime with others and non-square. The product of those six consecutive positive integers is never a square. Therefore their product can only be expressed as either

$$(30n) \times (30n+1) \times (30n+2) \times (30m+3) \times (30n+4) \times (30n+5)$$

or

$$(30n+1) \times (30n+2) \times (30m+3) \times (30n+4) \times (30n+5) \times (30n+6).$$

Since $(30n+1)$ does not have factors 2, 3 and 5, $(30n+1)$ is relatively prime with the others. Hence $(30n+1)$ must be a square in order to make the product a square.

For all odd numbers $(2n+1)$,

$$(2n+1)^2 = 4n^2+4n+1 = 4n(n+1) = 8k+1 \quad \text{since } n(n+1) \text{ is even.}$$

Since $30n+1$ is a square odd number, therefore

$$\begin{aligned} 30n+1 &= 8k+1 \\ \Rightarrow \quad 15n &= 4k \\ \Rightarrow \quad n &= 4M \\ \Rightarrow \quad k &= 15M \\ \Rightarrow 30n+1 &= 120M+1. \end{aligned}$$

Now the product of these 6 consecutive positive integers could be written as

$$(120M) \times (120M+1) \times (120M+2) \times (120M+3) \times (120M+4) \times (120M+5)$$

or

$$(120M+1) \times (120M+2) \times (120M+3) \times (120M+4) \times (120M+5) \times (120M+6).$$

In this new form, $(120M+4) = 4(30M+1)$. $(30M+1)$ do not have factors 2, 3, and 5. $(30M+1)$ must be relatively prime with other five integers. Moreover, $(30M+1)$ must be a square in order to make the product a square. This means $(120M + 4)$ must be a square too. However, as we have mentioned that $(120M + 1)$ is a square number, it is impossible for two squares $(120M + 1)$ and $(120M + 4)$ with a difference of 3 except $1\times2\times3\times4\times5\times6$. However, $1\times2\times3\times4\times5\times6$ is not a square. Conclusively, the product of six consecutive positive integers is never a square. □

Theorem 3. *The product of seven consecutive positive integers is never a square.*

Proof. This proof of Theorem 3 is similar to that of Theorem 2 by considering the remainders of square numbers divided by 30. Only the product $(30n) \times (30n+1) \times (30n+2) \times (30n+3) \times (30n+4) \times (30n+5) \times (30n+6)$ is possibly a square.

By considering and in the same way as the proof of Theorem 2, we can conclude that the product of seven consecutive positive integers is never a square. □

Theorem 4. *The product of eight consecutive positive integers is never a square.*

Proof. Let the consecutive positive integers be $(n-3)$, $(n-2)$, $(n-1)$, n, $(n+1)$, $(n+2)$, $(n+3)$ and $(n+4)$,

$$M = (n-3) \times n \times (n+2) \times (n+3) = n^4 + 2n^3 - 9n^2 - 18n,$$

and

$$N = (n-2) \times (n-1) \times (n+1) \times (n+4) = n^4 + 2n^3 - 9n^2 - 2n + 8.$$

$$\begin{aligned} M \times N &= n^8 + 4n^7 - 14n^6 - 56n^5 + 49n^4 + 196n^3 - 36n^2 - 144n \\ &\leqslant \left(\frac{M+N}{2}\right)^2 = (n^4 - 2n^3 - 9n^2 - 10n + 4)^2 \end{aligned} \tag{1}$$

Besides,

$$\left(\frac{M+N}{2} - 1\right)^2 = n^8 + 4n^7 - 14n^6 - 56n^5 + 47n^4 + 192n^3 + 46n^2 - 60n + 9.$$

Since

$$\begin{aligned} & M \times N - \left(\frac{M+N}{2} - 1\right)^2 \\ &= 2n^4 + 4n^3 - 82n^2 - 9 \\ &= (n-6)(2n^3 + 16n^2 + 14n) - 9 > 0 \quad \text{for } n \geqslant 7, \end{aligned}$$

we have

$$M \times N > \left(\frac{M+N}{2} - 1\right)^2. \tag{2}$$

Combining the results of (1) and (2), when $n \geqslant 7$, we get $\left(\frac{M+N}{2}\right)^2 > M \times N > \left(\frac{M+N}{2} - 1\right)^2$. Since $M \times N$ lies between two consecutive squares, $M \times N$ must not be a square. Moreover,

$$1 \cdot 2 \cdot 3 \cdots 8, 2 \cdot 3 \cdot 4 \cdots 9 \text{ and } 3 \cdot 4 \cdot 5 \cdots 10$$

are not square. Therefore, the product of eight consecutive positive integers is never a square. □

Theorem 5. *The product of nine consecutive positive integers is never a square.*

Proof. Now we consider the product of any nine consecutive positive integers a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 and a_9. We now investigate their product $\prod_{i=1}^{9} a_i$ in the following cases:

Case 1: Both a_1 and a_8 are multiples of 7.
Among a_2, a_3, a_4, a_5, a_6, a_7, at least one a_j cannot be in the form $(30n+7)$, $(30n+11)$, $(30n+17)$, $(30n+23)$ or $(30n+29)$ for otherwise a_j must be relatively prime with other numbers because any number in these forms are not multiples of 2, 3, 5 and 7. As mentioned above, the number is never a square and thus $\prod_{i=1}^{9} a_i$ is never a square too.

By the above reasons, $\{a2, a3, a4, a5, a6, a7\}$ must be either in the form of $\{30n, 30n+1, 30n+2, 30n+3, 30n+4, 30n+5\}$ or $\{30n+1, 30n+2, 30n+3, 30n+4, 30n+5, 30n+6\}$. In both cases, $30n+1$ must be relatively prime with other numbers and hence it must be a square. As mentioned above, this number written in the form of $120n+1$, and the third number behind it is $120n+4$ which can be written as $4(30n+1)$. As the factor $30n+1$ is relative prime with other numbers, it must be a square again. However, it is impossible for two squares $(120M+1)$ and $(120M+4)$ to have a difference of 3 except 9!, which is luckily not a square.

Case 2: a_1 and a_9 are multiples of 7.
It can be proved similarly by investigate a_3, a_4, a_5, a_6, a_7, a_8 instead of a_2,

a_3, a_4, a_5, a_6, a_7 as in case 1.

Case 3: There exists only one multiple of 7 in nine numbers.
It can be proved similarly by investigate a_1, a_2, a_3, a_4, a_5, a_6 instead of a_2, a_3, a_4, a_5, a_6, a_7 as in case 1. □

Theorem 6. *The product of ten consecutive positive integers is never a square.*

Proof. Now we consider ten consecutive positive integers a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 and a_{10}.

Case 1: If a_i and a_{i+7} are multiples of 7, then the method used in Theorem 5 can be used to investigate $ai + 1$, a_{i+2}, a_{i+3}, a_{i+4}, a_{i+5}, a_{i+6} instead and the conclusion can be similarly drawn.

Case 2: If only one of them is a multiple of 7, then any other six numbers can be investigated and the conclusion can be similarly drawn. □

Theorem 7. *The product of eleven consecutive positive integers is never a square.*

Proof. Since the prime factors of eleven positive integers are 2, 3, 5 and 7, the proof is similar to that of Theorem 6. □

Theorem 8. *The product of twelve consecutive positive integers is never a square.*

Proof. We now investigate the product of twelve positive integers a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 and a_{10}, a_{11}, a_{12}.

Case 1: a_1 and a_2 having factor 11.
If a_i and a_{i+7} are multiples of 7, we only consider the six numbers a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}, a_{i+5}, a_{i+6} and conclusion can be similarly drawn.

Case 2: When 11 is not a common factor of any two of the twelve numbers, the theorem can be proved similarly. □

3.2. The second approach

Bertrand's Postulate [2]: When x is a real number equal to or greater than 1, there exists at least one prime p such that $x < p \leqslant 2x$.

Theorem 9. *The product of consecutive positive integers is never a square if there is a prime among them.*

Proof. Let $a_1 < a_2 < \cdots < a_n$ be n consecutive positive integers for $n \geqslant 2$.

If there is at least one prime among them, we let the greatest prime be a_i. Assume the product of the consecutive numbers is a square, there must exist integers k and m such that $a_k = ma_i$ for otherwise the product cannot be a square. i.e. $a_i < a_k = ma_i \leqslant a_n$. Obviously, $2a_i$ exists in the sequence. However, by the Bertrand's Postulate, there exists at least one prime p such that $a_i < p < 2a_i \leqslant a_n$, which contradict to our assumption that a_i is the greatest prime.

Hence for any n consecutive positive integers containing at least one prime, their product is never a square. □

Corollary 10. *For any consecutive positive integers that contains prime number(s), their product is never a power of any integer.*

Proof. Similar to that of Theorem 9. □

Corollary 11. *For natural number n greater than 1, $n!$ is never a power of any integer.*

Proof. Since $n!$ must contain a prime number 2, $n!$ is never a power of any integer by Corollary 1. □

Corollary 12. *For natural number n greater than 1, $n!!$ is never as power of any integer.*

Proof. Case 1: When n is an even number, let n be $2m$ for some natural number m. So $n!! = (2m)!! = 2m \times m!$. If $m \geqslant 3$, then there exists a greatest prime number p in $\{1, 2, \ldots, m\}$ and hence $n!!$ cannot be written as a power of any integer.

Case 2: Since 1×3 is not a square. For any odd number greater than 3, there exists the greatest prime number p greater than 3 in the set $\{1, 3, 5, \ldots, n\}$. $\{1, 3, 5, \ldots, n\}$ must contain $3p$ in order to make the product a square. However, by Bertrand's Postulate, there exists a prime number p_0 such that

$p < p_0 < 3p$, which leads to a contradiction. Thus $n!!$ cannot be written as a power of any integer. □

Corollary 13. *For any m consecutive positive integers, their product is never a power of any integer if the first number is smaller than m.*

Proof. Let the product be $(m-k)(m-k+1)\cdots(2m-k-1)$ where $0 < k < m$ and k, $m \in \mathbb{N}$. By Bertrand's Postulate, there exists a prime number such that $m-k < p < 2m-2k < 2m-k-1$. Applying the Corollary 1, the result follows. □

3.3. The third approach

In July, we knew that a famous mathematician named Erdos had already proved our conjecture in 1939. The conjecture lasted for 150 years. We were happy that we have little sense of mathematics to get in touch with the problem that mathematicians were interested in. At the same time, we were all worried about what we should do next. Finally we decided to continue the project and try to find an alternative way. Then, we had the third approach.

In the third approach we investigate the conjecture by filling blanks. Erdos in [1] stated that if the product of the n consecutive numbers is a square, then their square-free parts are distinct. (square-free part of a number is the greatest factor of the number containing no square factors like 4, 9, 16 etc). We now illustrate our third approach using 13 consecutive numbers as an example. By considering the prime factors of their square-free integers, we can prove our conjecture on 13 consecutive numbers by exhaustion: Let the 13 consecutive positive integers be a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, each of which can be written as $a_i = b_i k_i^2$, where k_i^2 is the greatest square factor of a_i and b_i is the square-free part of a_i.

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}

Let's consider b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13} and count the maximum total number of prime factors they have. In the 13 boxes above, 2 appears at most 7 times, 3 at most 5 times, 5 at most 3 times, 7 at most 2 times and 11 at most 2 times. The total number of prime factors is 19. If there exists a prime factor greater than or equal to 13 among b_i's, the product of ai will never be a square. Therefore all prime factors greater than or equal to 13 can be neglected.

As Erdos said, all b_i's are distinct. To fill in these 13 boxes, we have to use the smallest number of factors with different combinations to finish the job. Firstly we fill in the boxes b_1, b_2, b_3, b_4, b_5 with 2, 3, 5, 7 and 11 respectively.

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
2	3	5	7	11								

There are 14 prime factors left. We then fill in the box b_6 with 1 (a_6 is a square number).

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
2	3	5	7	11	1							

There are 7 empty boxes. For minimizing the number of prime factors used, empty boxes must be filled with 2 or more prime numbers such that all b_i's are distinct. Although there are exactly 14 prime numbers, some of them must be equal no matter how you arrange due to the deficiency of prime factors. This contradicts that all b_i's are distinct. Thus the product of 13 consecutive positive integers is never a square.

A different method of filling boxes is provided as follows:

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
2	3	5	7	11	1	2	2	2	2	2	2	3
						7	5	5	3	3	3	
							11					

Now $b_{10} = b_{11} = b_{12}$ will contradict to that all b_i's must be distinct. Unluckily, this method is impractical when the number of consecutive numbers is very large. Therefore we try to generalize the above method of counting and obtain a necessary condition for the product to be a square.

Let n consecutive positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and $a_i = b_i k_i^2$, where k_i^2 is the greatest square factor of a_i and b_i is the square-free part of a_i.

Now we want to estimate the maximum prime total available.

Consider a particular prime $p < n$, if the product is a square, then the number $S_n(p)$ of prime factor p available in the product of the square-free parts is given by

$$S_n(p) = Q_n(p) + \frac{(-1)^{Q_n(p)} - 1}{2},$$

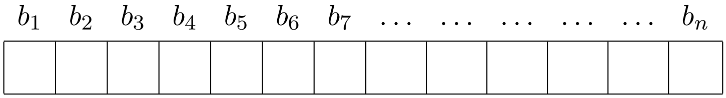
where

$$Q_n(p) = \sum_{\substack{k \in \mathbb{N} \\ p^{2k} \leqslant n}} \left(\left[\frac{n}{p^{2k-1}} \right] - \left[\frac{n}{p^{2k}} \right] + \operatorname{sgn} \left(\frac{n}{p^{2k-1}} - \left[\frac{n}{p^{2k-1}} \right] \right) \right).$$

Then the total number of primes available for n consecutive integers is simply called prime total $T(n)$ given by

$$T(n) = \sum_{p<n} S_n(p).$$

On the other hand, we shall find a lower bound for the prime total. We now fill up the n boxes with the primes or their products such that none of the boxes have the same number.



To use less prime factors, we first assume that there is a square among the consecutive integers (one of b_i's can be 1) and put each single prime factor into each of the $\pi(n)$ boxes. Then we put the products $p_i p_j$'s (where $i \neq j$) into the $(n - \pi(n))$ empty boxes. Thus we needs at least $\{\pi(n) + 2(n - \pi(n)) - 1\}$ prime factors. Hence the necessary condition for the product of consecutive positive integers to be square is

$$\begin{aligned} \pi(n) + 2(n - \pi(n)) - 1 &\leqslant T(n) \\ 2n - \pi(n) - 1 &\leqslant T(n) \end{aligned} \tag{3}$$

Let's investigate the case when $n = 13$, L.S. $= 19 > 18 =$ R.S.. (3) cannot be held when $n = 13$ and therefore the product of 13 consecutive positive integers is never a square.

As (3) is a **necessary** condition for the conjecture to be valid, then we can rephrase the theorem as follows:

Theorem 14. *If* $2n-\pi(n)-1 > T(n)$*, then the product of any* n *consecutive positive integers is never a square.*

In other words, we have found a **sufficient** condition for the consecutive product not to be a square. We have not completely solved the problem yet. We then investigated the inequality in the Theorem 10. When n is very large, we studied whether $\dfrac{T(n)+\pi(n)+1}{n}$ is convergent to a real number smaller than 2.

Now let's estimate the value of $T(n)$.

Since $Q_n(p) \geqslant \left\{\left[\frac{n}{p}\right] - \left[\frac{n}{p^2}\right]\right\}$ and $S_n(p) \geqslant Q_n(p) - 1$, we have

$$T(n) \geqslant \sum_{p<n} \left\{\left[\frac{n}{p}\right] - \left[\frac{n}{p^2}\right]\right\} - \pi(n)$$

and

$$\begin{aligned}
&\frac{T(n)+\pi(n)+1}{n}\\
=\ &\frac{\sum_{p<n} \left\{\left[\frac{n}{p}\right] - \left[\frac{n}{p^2}\right]\right\} - \pi(n) + 1 + \pi(n)}{n}\\
\geqslant\ &\frac{\sum_{p<n} \left\{\frac{n}{p} - \frac{n}{p^2} - 1\right\} + 1}{n}\\
=\ &\frac{\sum_{p<n} \left\{\frac{n}{p} - \frac{n}{p^2}\right\} - \pi(n) + 1}{n}\\
=\ &\frac{\sum_{p<n} \frac{n}{p} - \sum_{p<n} \frac{n}{p^2} + 1 - \pi(n)}{n}
\end{aligned}$$

$$= \sum_{p<n} \frac{1}{p} - \sum_{p<n} \frac{1}{p^2} + \frac{1}{n} - \frac{\pi(n)}{n}.$$

Unfortunately, $\sum_{p\leqslant n} \frac{1}{p^2}$, $\frac{1}{n}$ and $\frac{\pi(n)}{n}$ are convergent [3] while $\sum_{p\leqslant n} \frac{1}{p}$ is divergent. Hence

$$\frac{\sum_{p<n} \left\{ \left[\frac{n}{p}\right] - \left[\frac{n}{p^2}\right] \right\} + 1 + \pi(n)}{n}$$

diverges to infinity instead of converging to a real number smaller than 2. Thus this method does not work when the length of the product is too large and need to be further modified in the future.

4. Research progress

In our first approach, we had tried many examples of consecutive products and wished to find some patterns for generalization. Although there were some patterns which helped us to prove few more cases (e.g. from the proof of lengths 6 and 7 , and the proof of lengths 9 up to 12), we could not generalize it. Besides, when the number of consecutive integers increased, it was more difficult for us to prove it. Because of this, we could only give proofs of lengths up to 12 consecutive integers. Thus we tried to find another approach.

In the second approach, we had used the Bertrand's Postulate to prove that if the consecutive numbers consist of prime number(s), then their product is not a square. Using this theorem, we had arrived at three beautiful results: "The consecutive product cannot be power of any integers if at least one of them is a prime number", "For any n, $n!$ and $n!!$ is never a power of any integers" and "For m consecutive integers, if the first number is smaller than m, then their product is never a power of any integers". However, for any consecutive numbers without prime, we couldn't prove it.

At the very beginning, we thought that the conjecture is open. However, in July , we found that our conjecture had been already proved by the famous mathematician P. Erdos in 1939. Then Erdos and another mathematician J.L. Selfridge took 9 years to prove a more general statement "Any product of consecutive integers cannot be a power of any integers" in 1975. Although our conjecture was already proved, we did not give up and tried our best to see if there were alternative proofs. So we kept on researching and developing our third approach.

In the third approach, we found a method of filling boxes and obtained a sufficient condition for the consecutive product not to be square. However, the method didn't work when n is large. The main reason may be that our estimation on the amount of the prime factors of the square-free integers was too rough and needed to be further modified in the future.

5. Conclusion

Although the three approaches only partially solved our conjecture, we had found some interesting results ourselves. In the past few months, we all enjoyed the process of formulating conjectures and thinking new ideas of solving problems through cooperation. We also had a taste on the confusion and happiness of doing research.

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