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ALGORITHMIC CLASSIFICATION ON THE EXPANSION OF FRACTIONS IN NEGATIVE RATIONAL BASE

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Abstract. This paper rigorously explores the expansion of fractions in the unorthodox number system with a rational negative base $\frac{-N_b}{D_b}$, building on the work of Lucia Rossi and Jörg M. Thuswaldner on multiple number representations in such a base. Our objective is to establish a finite number of recurring expansions, using our novel theories and algorithms. We introduce definitions and conditions for four types of expansions, and present two distinct proofs for the Complete Residue System Theorem, our first main theorem. Our Second Main Theorem outlines the bounds of terminating and recurring expansions in any number system, providing a method to compute all expansions for any fraction $\frac{m}{n}$. These findings provide a thorough examination of fraction representations in the negative rational base system, enhancing understanding of its intricate characteristics.

Contents

1. Introduction

1.1. **Motivation.** Lucia Rossi and Jörg M. Thuswaldner's *A Number System with Base -3/2* [3] is the main inspiration for this work. The paper has introduced an algorithm for generating expansions in the system, and has shown several characteristics of it, such as $\mathcal{D}[-\frac{3}{2}] = \mathbb{Z}[\frac{1}{2}]$, the non-unique expansions of $\frac{m}{n}$ in base $-\frac{3}{2}$. To address the issue of overlapping expansions, the paper introduced a 2-adic extension for the production of nice tiling.
We illustrate the system's aforementioned overlapping using a typical example. The fraction $\frac{-2}{5}$ has distinct representations $(21.\overline{02})$, $(2.\overline{20})$ and $(0.\overline{1})$ in base $-\frac{3}{2}$. We are intrigued by the the multiple expansions, in the form of $(d_k \dots d_0.d_{-1}d_{-2}\dots)$, as before the introduction of a 2-adic expansion. Thus, we would like to fix a value for the number of representations, which we deduce to be finite. Our group believes that the preservation of unique expansions is not a necessity when we are not considering tiling. Our main goal in this paper is to produce a finite number of representations under certain circumstances.

1.2. **Literature Review.** The number system with a non-integer negative base takes references from previous works considering systems with negative bases and rational bases only.
William J. Gilbert and R. James Green's *Negative Based Number Systems* [2] The paper listed a few properties, of the representation of $\frac{m}{n}$, including the possibility of its having non-unique representations, and its absolute yielding of recurring or terminating expansions in any negative base. The paper has also presented an algorithm in which a rational number p q can be expanded into number systems with integer negative bases. We have also drawn and utilised the aforementioned algorithm from this paper, and apply it to a non-integer negative base, $-\frac{3}{2}$, which has not been explicitly explored in the cited work.
The paper did not explore the representation of $\frac{m}{n}$ in negative bases further, and we aim to discover more properties, in base $-\frac{N_b}{D_b}$, via considering the remainders generated, which m to n can yield recurring expansions and their properties, et cetera.

Shigeki Akiyama, Christiane Frougny, and Jacques Sakarovitch's *On the Representation of Numbers in a Rational Base* [1] uses the same method shown in [3] to represent positive integers and real numbers in a rational base. It considers arbitrary base $\frac{N_b}{D_b}$ and produces a tree containing the mappings for each a in $\mathcal{D} = \{0, \ldots, N_b - 1\}$ from $\mathbb{N}$ into itself. The paper takes a different direction to our work, and concerns itself with the Josephus problem and Mahler's $\frac{3}{2}$-problem. Also, we are concerned with negative rational base $-\frac{N_b}{D_b}$, and map elements as produced by our Reverse Algorithm.

2. $\frac{-N_b}{D_b}$ Number Base System

A number system is a structured approach to represent and manipulate numbers, utilizing symbols and rules to express quantities and perform mathematical operations. We acknowledge that different number systems exist due to their distinct purposes and properties.

Motivated by papers mentioned in Section 1, our curiosity is piqued by the exploration of number systems with different bases. Specifically, we aim to investigate the properties of these "unconventional bases." Upon conducting research, we discovered a lack of elementary treatment on the "basic properties" of such bases. Existing literature primarily focuses on advanced properties like "Tiling Theory" and "Cantor Set". This section aims to provide an elementary treatment of the "basic properties" of numbers in the $\frac{-N_b}{D_b}$ number system. By "basic properties", we refer to the analogous properties mentioned in Section 1 within the context of the base-10 system.

In Section 2.1, we will first define the number system $(\frac{-N_b}{D_b}, \mathcal{D})$ and define these four expansions of a fraction: **integer expansion**, **terminating expansion**, **primary recurring expansion** and **recurring expansion**.

In Section 2.2, we provide the necessary and sufficient conditions for a fraction to have an **integer expansion** and **terminating expansion** respectively.

In Sections 2.3 and 2.4, we will provide two proofs of the **Complete Residue System Theorem**, which is closely related to the analysis of fractions having primary recurring expansions; for instance, there is a technique that helps us find all fractions with primary recurring expansion for a fixed q.

In Section 2.3, we will establish the some necessary condition for fraction having primary recurring expansion. Then, we will state the Complete Residue System Theorem. In order to give a first proof of this theorem, we will establish the definition of β-expansion and present its relation with numerators of primary recurring expansion. Then, following some lemmas and the important Theorem 2.17, we will combine to show the First Proof of the Complete Residue System Theorem.

In Section 2.4, we will define the **Forward Algorithm** and **Reverse algorithm**, which is conducive for understanding fraction having primary recurring expansion. Then, we will explore the properties of the **Reverse Algorithm**, in particular, its naturally way of deciding whether a fraction have primary recurring expansion or not. After that, following some properties of residues while applying the **Reverse Algorithm**, we will following provide the Second Proof of the Complete Residue System Theorem.

A graphical representation of the **Reverse Algorithm** will be provided. This gives a natural and meaningful way of visualizing the "cycle" of primary recurring expansion.
Also, the **Reverse Algorithm** naturally provides a way to distinguish all primary recurring expansion for a fixed denominator q.
Section 2.5 will cover the second key of this paper – the number of representations. We will deduce the minimum and maximum numbers of representations a certain fraction $\frac{m}{n}$ has in an arbitrary $\frac{-N_b}{D_b}$ number system. Furthermore, we will provide a procedure for finding all terminating or recurring expansions of a particular fraction $\frac{m}{n}$.

2.1. **Number System and Expansion Definitions.** In this subsection, we aim to introduce the terminologies and definitions that we will be using while analyzing the base $\frac{-N_b}{D_b}$ number system. (The subscript b means base)
We are interested in understanding the "basic properties" and we aim to delve into the fundamental properties of number systems, encompassing necessary and sufficient conditions for various expansions. Specifically, we seek to explore conditions regarding the expansion of numbers with only an integer part, the expansion of numbers with a finite decimal part after the decimal point, the expansion of numbers with a recurring part after the decimal point, and other related properties. Upon doing so, we also found out that the literature lacks formal terms describing those expansions aforementioned. This leads us to the following definitions.
In the number systems with base $-\frac{N_b}{D_b}$, we will first define such a number system, then define the expansions that this paper prominently focuses on.
In Definition 2.1, we will first define the number system and the notation that we will be using when referring to the number system. Then, in Definition 2.2, we will define the four types of expansions that we will be analyzing throughout the paper.

Definition 2.1. Consider $N_b, D_b \in \mathbb{N}$ such that $N_b > D_b \geq 1$ and $\gcd(N_b, D_b) = 1$. Denote:

$$(d_k \dots d_0.a_1a_2\dots)_{\frac{-N_b}{D_b}} = \sum_{i=1}^{\infty} a_i \left(\frac{-N_b}{D_b}\right)^{-i} + \sum_{i=0}^{k} d_i \left(\frac{-N_b}{D_b}\right)^{i}$$

(i) Define the number system $(\frac{-N_b}{D_b}, \mathcal{D})$ with digits $d_i \in \mathcal{D} = \{0, 1, \dots, N_b - 1\}$ (i.e. N_b digits), where

$$(\frac{-N_b}{D_b}, \mathcal{D}) := \left\{ d \in \mathbb{Q} \mid d = (d_k \dots d_0.a_1a_2\dots)_{\frac{-N_b}{D_b}} \quad (k \in \mathbb{N}_{\geq 0}, d_i, a_i \in \mathcal{D}) \right\}$$

(ii) Define the set $\mathcal{D}\left[\frac{-N_b}{D_b}\right]$ within the number system $(\frac{-N_b}{D_b}, \mathcal{D})$ as:

$$\mathcal{D}\left[\frac{-N_b}{D_b}\right] := \left\{ d \in \mathbb{Q} \mid d = (d_k \dots d_0)_{\frac{-N_b}{D_b}} \quad (k \in \mathbb{N}_{\geq 0}, d_i, a_i \in \mathcal{D}) \right\}$$

In this paper, we focus on exploring the number system $(\frac{-N_b}{D_b}, \mathcal{D})$ defined in Definition 2.1(i). The set $\mathcal{D}\left[\frac{-N_b}{D_b}\right]$ defined in Definition 2.1(ii) is used for convenience in later Theorems.

Definition 2.2. (i) A non-zero $d \in \mathbb{Q}$ has an **integer expansion** if:

$$d := (d_t d_{t-1} \dots d_0)_{\frac{-N_b}{D_b}} = \sum_{i=0}^{t} d_i \left(\frac{-N_b}{D_b}\right)^i$$

for some $t \in \mathbb{Z}_{\geq 0}$, $d_i \in \mathcal{D}$ for all $0 \leq i \leq t$ and $d_t > 0$.

Note that the set $\mathcal{D}\left[\frac{-N_b}{D_b}\right]$ contains all integer expansions in the number system $(\frac{-N_b}{D_b}, \mathcal{D})$.

(ii) A non-zero $d \in \mathbb{Q}$ has a **terminating expansion** if there exists $y \in \mathbb{Z}_{\geq 0}$ such that $(-\frac{N_b}{D_b})^y d$ has an integer expansion.

(iii) A non-zero $d \in \mathbb{Q}$ has a **primary recurring expansion** if there exists $n \geq 1$ such that

$$d := (0.\overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}} = \sum_{i=1}^{\infty} \left[(\frac{-D_b}{N_b})^n\right]^i \sum_{j=1}^{n} s_j (\frac{-N_b}{D_b})^j$$

$$= \frac{\sum_{i=1}^{n} s_i (-D_b)^i (N_b)^{n-i}}{(N_b)^n - (-D_b)^n}$$

The period of d is defined to be the minimum possible length of the repetend $(\overline{s_1 \dots s_n})$. The definition is useful for our analysis in Section 2.2 and our proof in Sections 2.3 and 2.4.

Note that for the special case that $d = 0$, we will regard that 0 has both terminating expansion $0 = (0)_{\frac{-N_b}{D_b}}$ and primary recurring expansion $0 = (0.\bar{0})_{\frac{-N_b}{D_b}}$

(iv) A non-zero $d \in \mathbb{Q}$ has a **recurring expansion** if there exists $y \in \mathbb{Z}_{\geq 0}, n \in \mathbb{N}$ such that

$$d := (d_t \dots d_0.a_1 \dots a_y \overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}}$$
$$= \sum_{i=0}^{t} d_i \left(\frac{-N_b}{D_b}\right)^i + \sum_{i=1}^{y} a_i \left(\frac{-D_b}{N_b}\right)^i + \left(\frac{-D_b}{N_b}\right)^y \sum_{i=1}^{\infty} \left[(-\frac{D_b}{N_b})^n\right]^i \sum_{j=1}^{n} s_j (-\frac{N_b}{D_b})^j$$
$$= \sum_{i=0}^{t} d_i \left(\frac{-N_b}{D_b}\right)^i + \sum_{i=1}^{y} a_i \left(\frac{-D_b}{N_b}\right)^i + \left(\frac{-D_b}{N_b}\right)^y \left[\frac{\sum_{i=1}^{n} s_i (-D_b)^i (N_b)^{n-i}}{(N_b)^n - (-D_b)^n}\right]$$

Note that if $y = 0$, the term $\sum_{i=1}^{y} \left(\frac{-D_b}{N_b}\right)^i$ does not exists.

2.2. **Integer and Terminating expansion.** In this subsection, we aim to find the necessary and sufficient conditions for a rational number d to have an integer or terminating expansion.

With this goal in mind, we will first define a recurrence relation in Definition 2.3. This recurrence relation provides a method that allows us to find the digits $d_i \in \mathcal{D}$ of a fraction N_0, provided that it has an integer expansion.
Having a way to find the digits of a fraction N_0, the question of whether N_0 can be represented through our recurrence relation naturally arises, thus motivating us to find all possible values of N_0 where N_0 has integer expansion.
The answer to the question lies in Theorem 2.6. The following set equality, $\mathcal{D}\left[\frac{-N_b}{D_b}\right] = \mathbb{Z}\left[\frac{1}{D_b}\right]$, which we are going to prove in Theorem 2.6, gives a close-formed expression on the forms of fractions having integer expansion. That is, we have found out that the set of fractions having integer expansion is equal to the set of fractions containing only powers of D_b in the denominator.
Note that a large portion of the proof for Theorem 2.6 lies on proving the existence of an expansion for $d \in \mathbb{Z}[\frac{1}{D_b}]$.
Doing so, we have found out that the most difficult part of the proof lies in the existence of integer expansion for $d \in \mathbb{Z}$.
Therefore, we introduce Lemma 2.4 and Lemma 2.5, which specifically deal with such scenarios.
After proving the set equality, we would like to investigate whether the expansion for $d \in \mathcal{D}[\frac{-N_b}{D_b}]$ is unique or not.

This motivates us to have Theorem 2.7, which proves every expansion

$$d = (d_t \dots d_0)_{\frac{-N_b}{D_b}} \in \mathcal{D}[\frac{-N_b}{D_b}]$$

is unique.

After that, an example is provided to exemplify our theory and showcase how the Theorem 2.6 can be used to find an integer expansion of a number $N_0 \in \mathbb{Z}[\frac{1}{D_b}]$.
Finally, Corollary 2.9 provides a necessary and sufficient condition for an integer expansion.

Having completed our analysis on fractions with integer expansion, we would shift our focus to fractions with terminating expansions.

With some similar manipulation, Corollary 2.9 asserts the uniqueness of terminating expansion and provides a necessary and sufficient condition for a terminating expansion.

Definition 2.3. Denote $\mathbb{Z}[\frac{1}{D_b}] := \left\{ \frac{a}{(D_b)^l} \;\middle|\; a \in \mathbb{Z}, l \in \mathbb{Z}_{\geq 0} \right\}$. This notation is already well-established and frequently employed in mathematical literature.

Consider $N_0 = \frac{a}{(D_b)^l} \in \mathbb{Z}[\frac{1}{D_b}]$, we can find representation in $\mathcal{D}\left[\frac{-N_b}{D_b}\right]$ through the following recurrence relation which uses two equation.

$$N_0 = -N_b N_1 + d_0 \tag{GE1}$$

$$D_b N_i = -N_b N_{i+1} + d_i \qquad \text{for all } i \geq 1 \tag{GE2}$$

The idea of this recurrence relation is that we want to keep $N_i \in \mathbb{Z}[\frac{1}{D_b}]$ for all $i \geq 0$. The recurrence relation starts by inputting N_0 in (GE1).
The choice of $d_0 \in \mathcal{D}$ is chosen such that $a - (D_b)^l d_0 \equiv 0 \pmod{N_b}$. Since $\gcd\big((D_b)^l, N_b\big) = 1$, we know that such $d_0 \in \mathcal{D}$ exist uniquely such that the congruence above is well-defined.
The reason that we define $d_0 \in \mathcal{D}$ satisfying $a - (D_b)^l d_0 \equiv 0 \pmod{N_b}$ is because we want $N_1 \in \mathbb{Z}[\frac{1}{D_b}]$. Notice that $N_1 = \left(\frac{-1}{N_b}\right)\left(\frac{a}{(D_b)^l} - d_0\right) = \frac{-a+(D_b)^l d_0}{N_b(D_b)^l}$. The only way we can make $N_1 \in \mathbb{Z}[\frac{1}{D_b}]$ is for d_0 to satisfy the congruence equation above.
Afterwards, N_i will be inputted in (GE2) for all $i \geq 1$. Naturally we obtain $N_i = \frac{c}{(D_b)^{l-k}} \in \mathbb{Z}[\frac{1}{D_b}]$ where $0 \leq k \leq l$ and $c \in \mathbb{Z}$. The choice of $d_i \in \mathcal{D}$ is chosen such that $-c + (D_b)^{l-k-1} d_i \equiv 0 \pmod{N_b}$. Again, since $\gcd\big((D_b)^l, N_b\big) = 1$, we know that such $d_i \in \mathcal{D}$ exist uniquely such that the congruence above is well-defined.
Similarly, the reason that we define $d_i \in \mathcal{D}$ satisfying $-c + (D_b)^{l-k-1} d_i \equiv 0 \pmod{N_b}$ is because we want $N_{i+1} \in \mathbb{Z}[\frac{1}{D_b}]$. Notice that

$$N_{i+1} = \left(\frac{-1}{N_b}\right)\left((D_b)\frac{c}{(D_b)^{l-k}} - d_i\right) = \frac{-c + (D_b)^{l-k-1} d_i}{N_b (D_b)^{l-k-1}}.$$

The only way we can make $N_{i+1} \in \mathbb{Z}[\frac{1}{D_b}]$ is for d_i to satisfy the congruence equation above.
We call such recurrence relation as **Integer Expansion Recurrence Relation**.

Now, we want to show that if $N_i \in \mathbb{Z}$, after applying **Integer Expansion Recurrence Relation** on N_i, then the expansion itself will terminate. With doing so, we observed that following interesting pattern when apply $N_i \in \mathbb{Z}$.

Lemma 2.4. *Given that $N_i \in \mathbb{Z}$ for some $i \geq 1$. After applying the **Integer Expansion Recurrence Relation** on N_i:*

(i) If $|N_i| > \frac{N_b - 1}{N_b - D_b}$, then $|N_{i+1}| < |N_i|$
(ii) If $0 < N_i \leq \frac{N_b - 1}{N_b - D_b}$, then $N_{i+1} = -N_i + 1$
(iii) If $-\frac{N_b - 1}{N_b - D_b} \leq N_i < 0$, then $N_{i+1} = -N_i$

Proof. *Proof of Lemma 2.4(i)*

We consider (GE2) as $i \geq 1$.
By assumption, $|N_i| > \frac{N_b - 1}{N_b - D_b}$.

By taking the same considerations as above, we have,

$$\begin{aligned}|N_{i+1}| &\leq |-\frac{D_b}{N_b}N_i| + |\frac{d_i}{N_b}| \\ &\leq |N_i|\left(\frac{D_b}{N_b} + \frac{N_b-1}{|N_i|N_b}\right) && (d_i \leq N_b - 1) \\ &< |N_i|\left(\frac{D_b}{N_b} + \frac{N_b-1}{N_b}\frac{N_b-D_b}{N_b-1}\right) && (\frac{1}{|N_i|} < \frac{N_b-D_b}{N_b-1}) \\ &= |N_i|\end{aligned}$$

We conclude that if $|N_i| > \frac{N_b-1}{N_b-D_b}$ then $|N_{i+1}| < |N_i|$ for all $i \geq 1$

Proof of Lemma 2.4(ii)

By definition of picking d_i in Definition 2.3, we know we must have $d_i \in \mathcal{D}$ satisfying $-N_i + (D_b)^{-1}d_i \equiv 0 \pmod{N_b}$, we know $d_i = (D_b - N_b)N_i + N_b$ is the only solution satisfying that.

By assumption, $0 < N_i \leq \frac{N_b-1}{N_b-D_b}$. We know that such choice d_i is well-defined because

$$N_b = (D_b - N_b)(0) + N_b > (D_b - N_b)N_i + N_b \geq (D_b - N_b)\frac{N_b - 1}{N_b - D_b} + N_b$$
$$N_b > (D_b - N_b)N_i + N_b = d_i \geq 1$$

which shows that $d_i \in \mathcal{D}$
By direct substitution to (GE2),

$$\begin{aligned}N_{i+1} &= -\frac{1}{N_b}\left(D_bN_i - ((D_b - N_b)N_i + N_b)\right) \\ &= -\frac{1}{N_b}\left(D_bN_i - (D_b - N_b)N_i - N_b\right) \\ &= -\frac{1}{N_b}\left(N_bN_i - N_b\right) \\ &= -N_i + 1 \in \mathbb{Z}\end{aligned}$$

Notice that $d_i \in \mathcal{D}$ is unique. And this concludes Lemma 2.4(ii).

Proof of Lemma 2.4(iii)
By definition of picking d_i in Definition 2.3, we know we must have $d_i \in \mathcal{D}$ satisfying $-N_i + (D_b)^{-1}d_i \equiv 0 \pmod{N_b}$, we know $d_i = (D_b - N_b)N_i$ is the only solution satisfying that.

We know that such choice d_i is well-defined because

$$-\frac{N_b - 1}{N_b - D_b} \leq N_i < 0$$
$$N_b - 1 \geq (D_b - N_b)N_i = d_i > 0$$

which shows that $d_i \in \mathcal{D}$
By direct substitution to (GE2),

$$\begin{aligned} N_{i+1} &= -\frac{1}{N_b}\left(D_bN_i - ((D_b - N_b)N_i)\right) \\ &= -\frac{1}{N_b}\left(D_bN_i - (D_b - N_b)N_i\right) \\ &= -\frac{1}{N_b}\left(N_bN_i\right) \\ &= -N_i \end{aligned}$$

Notice the choice for d_i is unique. And this concludes Lemma 2.4(iii). □

Lemma 2.5. *It is given that $N_0 \in \mathbb{Z} \setminus \{0\}$. After applying **Integer Expansion Recurrence Relation** on N_0, there exist $j \in \mathbb{N}$ such that $N_j = 0$.*

Proof. Suppose $N_0 \in \mathbb{Z} \setminus \{0\}$. Apply $N_0 \in \mathbb{Z} \setminus \{0\}$ to Equation (GE1). By the choice of d_0 picked in Definition 2.3, we know that $N_1 \in \mathbb{Z}$.
Applying Lemma 2.4(i) if needed, we know that there exist $m \in \mathbb{N}$ such that $|N_m| \leq \frac{N_b-1}{N_b-D_b}$.
Without loss of generality, we can assume $N_m > 0$.
(If $N_m > 0$, then apply Lemma 2.4(iii) so that $N_{m+1} > 0$).

Applying Lemma 2.4(ii) and (iii) respectively,

$$N_{m+1} = -N_m + 1$$
$$N_{m+2} = -(-N_m + 1) = N_m - 1$$

Hence, $N_{m+2} = N_m - 1 < N_m$.
This gives $N_m > N_{m+2} = N_m - 1 > N_{m+4} = N_m - 2 > \dots$

Now, apply the step of **Integer Expansion Recurrence Relation** $2N_m - 2$ more times, we obtain $N_{m+(2N_m-2)} = 1$
Now we simply apply Lemma 2.4(ii) once which gives $N_{m+(2N_m-1)} = 0$
We have shown that the recurrence relation will "terminate" at some point. □

Theorem 2.6. *The set $\mathcal{D}\left[\frac{-N_b}{D_b}\right]$ in the number system $(\frac{-N_b}{D_b}, \mathcal{D})$ have the following equality:*

$$\mathcal{D}\left[\frac{-N_b}{D_b}\right] = \mathbb{Z}\left[\frac{1}{D_b}\right]$$

Proof.
The inclusion $\mathcal{D}\left[\frac{-N_b}{D_b}\right] \subseteq \mathbb{Z}[\frac{1}{D_b}]$ can be observed by taking common denominator on $d = (d_t \dots d_0)_{\frac{-N_b}{D_b}} = \sum_{i=0}^{k} d_i \left(\frac{-N_b}{D_b}\right)^i$ for some $t \in \mathbb{Z}_{\geq 0}$, and thus is trivial.

Now, considering the reverse inclusion, we first consider the case $N_0 \in \mathbb{Z}$ For $N_0 \in \mathbb{Z}$, by Lemma 2.4 and Lemma 2.5 to find a well-defined integer expansion of $N_0 = (d_{j-1} \dots d_0)_{\frac{-N_b}{D_b}}$ which is in the set $\mathcal{D}[\frac{-N_b}{D_b}]$ and that the digits $d_i \in \mathcal{D}$ is uniquely decided.

Then, consider the case $N_0 \in \mathbb{Z}[\frac{1}{D_b}] \setminus \mathbb{Z}$. Consider $N_0 \in \mathbb{Z}[\frac{1}{D_b}] \setminus \mathbb{Z}$, which we write as $\frac{a}{(D_b)^l}$ where $l > 0$. We want to show that by applying **integer expansion recurrence relation** on $N_0, N_1, \dots$, we can have $N_i \in \mathbb{Z}[\frac{1}{D_b}]$ for all i, and there exists j such that $N_j = 0$.
Consider Equations (GE1) and (GE2)
Substituting N_0 into Equation (GE1), we have

$$\begin{aligned} N_0 &= -N_b N_1 + d_0 \\ N_1 &= -\frac{1}{N_b}(N_0 - d_0) \\ &= -\frac{1}{N_b} N_0 + \frac{d_0}{N_b} \\ &= \frac{-a + (D_b)^l d_0}{N_b (D_b)^l} \end{aligned} \tag{GE1}$$

By Definition 2.3, there exists a unique solution $d_0 \in \mathcal{D}$ such that $-a + (D_b)^l d_0 \equiv 0 \pmod{N_b}$.

We now let $c_1 = \frac{-a+(D_b)^l d_0}{N_b} \in \mathbb{Z}$. Substituting N_1 in Equation (GE2), we have

$$\begin{aligned} N_2 &= -\frac{D_b}{N_b} N_1 + \frac{d_1}{N_b} \\ &= \frac{-D_b c_1}{N_b (D_b)^l} + \frac{d_1}{N_b} \\ &= \frac{-c_1 + (D_b)^{l-1} d_1}{N_b (D_b)^{l-1}} \end{aligned}$$

Again, by Definition 2.3, there exists a unique solution $d_0 \in \mathcal{D}$ such that $-c_1 + (D_b)^{l-1} d_1 \equiv 0 \pmod{N_b}$. Note that the power of (D_b) is reduced by one every time we conduct (GE2).

We now let $c_2 = \frac{-c_1+(D_b)^{l-1} d_1}{N_b} \in \mathbb{Z}$ and we substitute N_2 in Equation (GE2).

Iterating the above argument l times, through the process of finite descent, we find that $N_{l+1} \in \mathbb{Z}$ and find unique values $d_i \in \mathcal{D}$ for all $0 \leq i \leq l$.
Now, we have found that, by finite descent, $N_0, \ldots, N_l \in \mathbb{Z}[\frac{-N_b}{D_b}] \setminus \mathbb{Z}$ gives a unique choice of $d_i \in \mathcal{D}$.
Also, now that we have proven that once N_{l+1} has become an integer,
by Lemma 2.5, after applying N_{l+1} **Integer Expansion Recurrence Relation**, we also know there exists $j > l$ such that $N_j = 0$.
Now, by backwards substitution,

$$\begin{aligned} N_0 &= d_0 + \frac{-N_b}{D_b}\left[-N_b N_2 + d_1\right] \\ &= d_0 + \frac{-N_b}{D_b} d_1 + \left(\frac{-N_b}{D_b}\right)^2 \left[-N_b N_3 + d_2\right] \\ &\cdots \\ &= d_0 + \frac{-N_b}{D_b} d_1 + \left(\frac{-N_b}{D_b}\right)^2 d_2 + \cdots + \left(\frac{-N_b}{D_b}\right)^{j-1} d_{j-1} \end{aligned}$$

which shows that $N_0 = (d_{j-1} \ldots d_0)_{\frac{-N_b}{D_b}}$ has an well-defined expansion in $\mathcal{D}[\frac{-N_b}{D_b}]$. Notice that the choice of digit $d_i \in \mathcal{D}$ is chosen uniquely throughout N_0 to N_{j-1}.

Hence, we prove the inclusion $\mathcal{D}\left[\frac{-N_b}{D_b}\right] \supseteq \mathbb{Z}[\frac{1}{D_b}]$ and equivalently, every number $N_0 \in \mathbb{Z}\left[\frac{1}{D_b}\right]$ has an **integer expansion** representation in the number system $(\frac{-N_b}{D_b}, \mathcal{D})$.

□

Theorem 2.7. *Every number $N_0 \in \mathbb{Z}\left[\frac{1}{D_b}\right]$ has a unique* ***integer expansion*** *in the number system $(\frac{-N_b}{D_b}, \mathcal{D})$.*

Theorem 2.6 guarantees the existence of integer expansion for $N_0 \in \mathbb{Z}\left[\frac{1}{D_b}\right]$.

The following concerns the uniqueness part.

Let $d \in \mathbb{Q}$.
Suppose that $d = (a_k \ldots a_0)_{\frac{-N_b}{D_b}} = (b_m \ldots b_0)_{\frac{-N_b}{D_b}}$ for some $a_0, \ldots, a_k, b_0, \ldots, b_m \in \mathcal{D}$
That is,

$$\sum_{i=0}^{k} a_i \left(\frac{-N_b}{D_b}\right)^i = \sum_{i=0}^{m} b_i \left(\frac{-N_b}{D_B}\right)^i \tag{1}$$

Without loss of generality, we assume $k \leq m$
Then,

$$\sum_{i=0}^{m}(b_i - a_i)(\frac{-N_b}{D_b})^i = 0$$

where $a_{k+1}, \ldots, a_m = 0$.

By taking $c_i = a_i - b_i \in \{-(N_b - 1), -(N_b - 2), \ldots, 0, \ldots, N_b - 2, N_b - 1\}$, we have

$$-c_0 = \sum_{i=1}^{m} c_i(\frac{-N_b}{D_b})^i$$
$$-c_0(D_b)^m = \sum_{i=1}^{m} c_i(-N_b)^i(D_b)^{m-i}$$

Since N_b divides right-hand-side, but $|c_0| \leq |N_b - 1|$ and $\gcd(N_b, D_b) = 1$, we obtain $c_0 = 0$ is the only possible value for c_0.

We continue the argument by dividing both sides by $-N_b$

$$-c_1(D_b)^{m-1} = \sum_{i=2}^{m} c_i(-N_b)^i(D_b)^{m-i}$$

Again, we must have $c_1 = 0$.
By repeating the argument above $m-1$ times, it yields $c_i = 0$ for all $0 \leq i \leq m-1$
That is, we have $a_i = b_i$ for all $0 \leq i \leq m-1$
Then, for the equality to occur, we must have $a_m = b_m$. Hence, we have $a_i = b_i$ for all $0 \leq i \leq m$, which concludes our proof.

Example. Consider base $\frac{-9}{4}$. We can determine the integer expansion for the fraction $\frac{3}{64}$ by utilizing the recurrence relation equations (GE1) and (GE2).
By substituting $N_0 = \frac{3}{64}$ we have:

(GE1) $$\left(\frac{3}{64}\right) = -9\left(\frac{21}{64}\right) + 3$$

(GE2) $$4\left(\frac{21}{64}\right) = -9\left(\frac{3}{16}\right) + 3$$

(GE2) $$4\left(\frac{3}{16}\right) = -9\left(\frac{1}{4}\right) + 3$$

(GE2) $$4\left(\frac{1}{4}\right) = -9(0) + 1$$

By which we can conclude that the fraction $\frac{3}{64}$ has an integer expansion $\frac{3}{64} = (1333)_{\frac{-9}{4}}$ in base $\frac{-9}{4}$. Notice that the denominator of N_i "Descends" by D_b in every step in the sense that the power of (D_b) in the denominator decreases by 1 every time (except for (GE1)) until all powers of (D_b) is cancelled out, at which N_k is an integer for some $k \in \mathbb{Z}_{\geq 0}$.

Example. Consider base $\frac{-6}{5}$. Because we want to show that β–expansion can be very long, we will demonstrate by finding the integer expansion for the integer 35. Then by substituting $N_0 = 35$ in the recurrence relation, we have:

(GE1)	$(35) = -6\,(-5) + 5$	
(GE2)	$5\,(-5) = -6\,(5) + 5$	by Lemma 2.4 (iii)
(GE2)	$5\,(5) = -6\,(-4) + 1$	by Lemma 2.4 (ii)
(GE2)	$5\,(-4) = -6\,(4) + 4$	by Lemma 2.4 (iii)
(GE2)	$5\,(4) = -6\,(-3) + 2$	by Lemma 2.4 (ii)
(GE2)	$5\,(-3) = -6\,(3) + 3$	by Lemma 2.4 (iii)
(GE2)	$5\,(3) = -6\,(-2) + 3$	by Lemma 2.4 (ii)
(GE2)	$5\,(-2) = -6\,(2) + 2$	by Lemma 2.4 (iii)
(GE2)	$5\,(2) = -6\,(-1) + 4$	by Lemma 2.4 (ii)
(GE2)	$5\,(-1) = -6\,(1) + 1$	by Lemma 2.4 (iii)
(GE2)	$5\,(1) = -6\,(0) + 5$	by Lemma 2.4 (ii)

After applying **Integer Expansion Recurrence Relation**, we can conclude that 35 has an integer expansion of $35 = (51423324155)_{\frac{-6}{5}}$ in base $\frac{-6}{5}$. Notice that when $|N_1| \leq \frac{N_b-1}{N_b-D_b} = 5$, we apply Lemma 2.5, which gives that $N_m = N_2 = 5$. Now, we could apply the procedure given by Lemma 2.5: To apply Lemma 2.4(iii) and Lemma 2.4(ii) respectively in accordance to N_i such that the value of $|N_{i+2}|$ descends by 1 in every two steps.

Corollary 2.8. *Let $x \in \mathbb{Z}$ and $y \in \mathbb{N}$ such that* $\gcd(x, y) = 1$. *The fraction $\frac{x}{y}$ has an integer expansion if and only if*

$$y = \prod_{\substack{\hat{p}|D_b \\ \hat{p}\ prime}} \hat{p_i}^{\alpha_i}$$

for some $\alpha_i \in \mathbb{Z}_{\geq 0}$

Proof. Assume $\frac{x}{y}$ has an integer expansion.

Write

$$\frac{x}{y} = (d_t d_{t-1} \ldots d_0)_{\frac{-N_b}{D_b}} = \sum_{i=0}^{t} d_i \left(\frac{-N_b}{D_b}\right)^i = \frac{\sum_{i=0}^{t} d_i(-N_b)^i (D_b)^{t-i}}{(D_b)^t}$$

for some $t \in \mathbb{Z}_{\geq 0}$ and $d_i \in \mathcal{D}$ for all $0 \leq i \leq t$.

Now, as

$$y\left(\sum_{i=0}^{t} d_i(-N_b)^i(D_b)^{t-i}\right) = x(D_b)^t$$

Thus, we have $y \mid x(D_b)^t$. However, $\gcd(x,y) = 1$. By applying Euclid's lemma, we have $y \mid (D_b)^t$, which concludes one side of the proof.

It is well-known that for a prime p and nonzero integer n, we let $\nu_p(n)$ denote the largest integer e with p^e dividing n. The notation $\nu_p(n)$ will be used in the following.
Let

$$\text{ind}_{D_b}(\hat{p_i}) := \begin{cases} \dfrac{\alpha_i}{\nu_{\hat{p_i}}(D_b)}, & \nu_{\hat{p_i}}(D_b) \mid \alpha_i \\ \left\lfloor \dfrac{\alpha_i}{\nu_{\hat{p_i}}(D_b)} \right\rfloor + 1, & \nu_{\hat{p_i}}(D_b) \nmid \alpha_i \text{ and } \nu_{\hat{p_i}}(D_b) \neq 0 \\ 0, & \nu_{\hat{p_i}}(D_b) = 0 \end{cases}$$

Take $k = \max\bigl(\text{ind}_{D_b}(\hat{p_1}), \text{ind}_{D_b}(\hat{p_2}), \ldots, \text{ind}_{D_b}(\hat{p_l})\bigr)$ where y has l prime factors dividing D_b
That is, k is the smallest non-negative integer such that $y \mid (D_b)^k$.

$$\frac{x}{y} = \frac{x}{y}\left(\frac{\frac{(D_b)^k}{y}}{\frac{(D_b)^k}{y}}\right) = \frac{x\left(\frac{(D_b)^k}{y}\right)}{(D_b)^k}$$

where $x\dfrac{(D_b)^k}{y} \in \mathbb{Z}$ Since $\frac{x}{y} = \dfrac{x\left(\frac{(D_b)^k}{y}\right)}{(D_b)^k} \in \mathbb{Z}\left[\dfrac{1}{D_b}\right]$, we apply Theorem 2.7 and we know $\frac{x}{y}$ has an unique integer expansion. □

Corollary 2.9. *(i) Terminating expansion is unique.*

(ii) Let $x \in \mathbb{Z}$ and $y \in \mathbb{N}$ where $\gcd(x,y) = 1$. The number $\frac{x}{y}$ has a terminating expansion if and only if

$$y = \prod_{\substack{\hat{p} \mid D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} \prod_{\substack{\hat{q} \mid N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j}$$

for some $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$

Proof.
Proof of part (i) Consider the number

$$\frac{x}{y} = (d_s \ldots d_0.a_1 \ldots a_k)_{\frac{-N_b}{D_b}} = (e_t \ldots e_0.b_1 \ldots b_l)_{\frac{-N_b}{D_b}}$$

where $l \geq k$ and $a_k, b_l, d_s, e_t \neq 0$.

When we multiply $\frac{x}{y}$ by $(\frac{-N_b}{D_b})^l$, by Theorem 2.6, we know that the number $\frac{x}{y}(\frac{-N_b}{D_b})^l$ has an unique integer expansion. If $l > k$, then we have $l-k$ zeros directly on the rightmost digit for the number $(d_s \ldots d_0 a_1 \ldots a_k)(\frac{-N_b}{D_b})^{l-k}$. Since $\frac{x}{y}(\frac{-N_b}{D_b})^l$ has an unique integer expansion, we know that b_l would be zero.
This contradicts with our assumption.
Hence $l = k$, and it is not possible to have $l-k$ zeros directly on the rightmost digit for the number $\frac{x}{y}(\frac{-N_b}{D_b})^l$.
By a similar argument, we can also prove that $s = t$. Otherwise, $d_s = 0$ or $e_t = 0$, and contradiction occurs in both cases.

Now that we have proven $l = k$ and $s = t$. We know that $\frac{x}{y}(\frac{-N_b}{D_b})^l$ has an unique integer expansion. Hence, we have

$$d_s = e_s, d_{s-1} = e_{s-1}, \ldots, a_k = b_k$$

which concludes our proof.

Proof of part (ii)
Assume $\frac{x}{y}$ has a terminating expansion

$$\begin{aligned}\frac{x}{y} &= (d_t \ldots d_0.a_1 \ldots a_m)_{\frac{-N_b}{D_b}} \\ &= \sum_{i=0}^{t} d_i(\frac{-N_b}{D_b})^i + \sum_{i=1}^{m} a_i(\frac{-D_b}{N_b})^i \\ &= \frac{\left(\sum_{i=0}^{t} d_i(-N_b)^i(D_b)^{t-i}\right)(N_b)^m + \left(\sum_{i=1}^{m} a_i(-D_b)^i(N_b)^{m-i}\right)(D_b)^t}{(D_b)^t(N_b)^m}\end{aligned}$$

Combining the fact

$$x(D_b)^t(N_b)^m = y\left[\left(\sum_{i=0}^{t} d_i(-N_b)^i(D_b)^{t-i}\right)(N_b)^m + \left(\sum_{i=1}^{m} a_i(-D_b)^i(N_b)^{m-i}\right)(D_b)^t\right]$$

Thus, we have $y \mid x\left[(D_b)^t(N_b)^m\right]$. However, $\gcd(x, y) = 1$. By applying Euclid's lemma, we have $y \mid (D_b)^t(N_b)^m$.

As a result, we can express

$$y = \prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p}_i^{\gamma_i} \prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q}_j^{\delta_j}$$

where $\gamma_i, \delta_j \in \mathbb{Z}_{\geq 0}$.
Now, for the reverse direction, we assume that $\frac{x}{y}$ is a fraction and

$$y = \prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} \prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j}$$

for some $\alpha_i, \beta_j \in \mathbb{Z}_{\geq 0}$.

For simplicity, write $\prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} = P_{D_b}$ and $\prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j} = P_{N_b}$

Let

$$\operatorname{ind}_{N_b}(\hat{q_j}) := \begin{cases} \dfrac{\beta_j}{\nu_{\hat{q_j}}(N_b)}, & \nu_{\hat{q_j}}(N_b) \mid \beta_j \\ \left\lfloor \dfrac{\beta_j}{\nu_{\hat{q_j}}(N_b)} \right\rfloor + 1, & \nu_{\hat{q_j}}(N_b) \nmid \beta_j \text{ and } \nu_{\hat{q_j}}(N_b) \neq 0 \\ 0, & \nu_{\hat{q_j}}(N_b) = 0 \end{cases}$$

That is, we take $m = \max\big(\operatorname{ind}_{N_b}(\hat{q_1}), \operatorname{ind}_{N_b}(\hat{q_2}), \ldots, \operatorname{ind}_{N_b}(\hat{q_{l'}})\big)$ where y has l' prime factors dividing N_b.
Again, this meant that we take m to be the smallest non-negative integer such that $P_{N_b} \mid (N_b)^m$.

$$\left(\frac{-N_b}{D_b}\right)^m \left(\frac{x}{y}\right) = \frac{x\left[\dfrac{(-N_b)^m}{P_{N_b}}\right]}{(D_b)^m P_{D_b}} \tag{2}$$

where $\frac{(-N_b)^m}{P_{N_b}} \in \mathbb{Z}$
Then, by Corollary 2.8, we know that

$$\left(\frac{-N_b}{D_b}\right)^m \left(\frac{x}{y}\right) = \frac{x\left[\dfrac{(-N_b)^m}{P_{N_b}}\right]}{(D_b)^m P_{D_b}}$$

has an integer expansion.
After that, we can simply multiply both sides by $(\frac{-D_b}{N_b})^m$ which yields

$$\frac{x}{y} = (\frac{-D_b}{N_b})^m \frac{x\left[\dfrac{(-N_b)^m}{P_{N_b}}\right]}{(D_b)^m P_{D_b}}$$

which has a terminating expansion. □

2.3. **Primary Recurring Expansion and Complete Residue System Theorem.** In this subsection, we'll first reflect on the properties of primary recurring expansion, and then we will provide the first proof of the Complete Residue System Theorem, which is the main theorem of this paper.

Lemma 2.10. *Let* $\frac{p}{q} = (0.\overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}}$ *be a fraction having* ***primary recurring expansion*** *where* $\gcd(p,q) = 1$ *and* n *is a positive integer. Then*

(i) $\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2} \le \frac{p}{q} \le \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}$
(ii) $q \mid (N_b)^n - (-D_b)^n$
(iii) $D_b \mid p$
(iv) $\gcd(q, N_b D_b) = 1$
(v) $[-N_b(D_b)^{-1}]^n \equiv 1 \pmod q$

Proof.
Proof of (i) The upper bound of the value of a **primary recurring expansion** is $\big(0.\ \overline{[0]\ [N_b-1]}\ \big)_{\frac{-N_b}{D_b}} = \frac{(N_b-1)(\frac{-N_b}{D_b})^{-2}}{1-(\frac{-N_b}{D_b})^{-2}} = \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}$ by sum of geometric series with common ratio $(\frac{-D_b}{N_b})^2$. We obtain $\big(0.\ \overline{[0]\ [N_b-1]}\ \big)_{\frac{-N_b}{D_b}}$ by maximizing the positive component (digits with even power) and minimising the negative component (digits with odd power).
The lower bound of the value of a **primary recurring expansion** is

$$(0.\ \overline{[N_b-1]\ [0]}\ )_{\frac{-N_b}{D_b}} = \frac{(N_b-1)(\frac{-N_b}{D_b})^{-1}}{1-(\frac{-N_b}{D_b})^{-2}} = \frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}$$

by sum of geometric series with common ratio $(\frac{-D_b}{N_b})^2$. We obtain

$$(0.\ \overline{[N_b-1]\ [0]}\ )_{\frac{-N_b}{D_b}}$$

by minimizing the positive component (digits with even power) and maximizing the negative component (digits with odd power).
Thus we have

$$\text{(B)} \qquad \frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}q \le p \le \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q$$

Proof of (ii) and (iii) By Definition 2.2(iii),

$$\frac{p}{q} = \frac{\sum_{i=1}^{n} s_i(-D_b)^i(N_b)^{n-i}}{(N_b)^n-(-D_b)^n}$$

By rearranging terms, we have

$$p\big((N_b)^n-(-D_b)^n\big) = q(-D_b)\Big(\sum_{i=1}^{n} s_i(-D_b)^{i-1}(N_b)^{n-i}\Big)$$

Thus we have $q \mid p\big[(N_b)^n-(-D_b)^n\big]$. However, by assumption, $\gcd(p,q)=1$. We apply Euclid's Lemma to yield $q \mid \big[(N_b)^n-(-D_b)^n\big]$, thus concluding (ii).
Also, $D_b \mid p\big[(N_b)^n-(-D_b)^n\big]$. Obviously, $\gcd\Big(D_b\ ,\ (N_b)^n-(-D_b)^n\Big) = 1$. We apply Euclid's Lemma to yield $D_b \mid p$, which concludes (iii).
Proof of (iv) By Definition 2.1, we know N_b and D_b is coprime.
Let $\gcd(N_b, q) = g$, we know $\gcd(D_b, g) = 1$.

However, by Proposition 2.10(ii)

$$\begin{aligned} (N_b)^n - (-D_b)^n &\equiv 0 \pmod{g} \\ -(D_b)^n &\equiv 0 \pmod{g} \\ D_b &\equiv 0 \pmod{g} \end{aligned}$$

Thus, combining the fact $\gcd(D_b, g) = 1$ and $g \mid D_b$, we conclude $g = 1$.
Similarly, let $\gcd(D_b, q) = g'$, we can deduce that $N_b \equiv 0 \pmod{g'}$
Thus, showing $g = g' = 1$
Hence, $\gcd(q, N_b D_b) = 1$, which concludes our proof.
Proof of (v) By considering Proposition 2.10(ii),

$$\begin{aligned} (N_b)^n - (-D_b)^n &\equiv 0 \pmod{q} \\ (N_b)^n &\equiv (-D_b)^n \pmod{q} \\ (N_b)^n \left((-D_b)^{-1}\right)^n &\equiv 1 \pmod{q} \\ \left[-N_b(D_b)^{-1}\right]^n &\equiv 1 \pmod{q} \qquad (3) \end{aligned}$$

□

Now that we have established our lemmas above, we will dive into the core proof of the paper: Complete Residue System Theorem.
For context, we will talk about the motivation of the Complete Residue System and the process of its formation. Initially, our objective was to search for a necessary and sufficient condition for fractions to possess primary recurring expansions. Specifically, we aimed to determine all numerators of fractions that have primary recurring expansions for a fixed q. While we were unable to find such a condition, our investigation into fractions exhibiting primary recurring expansions yielded an intriguing observation regarding the numerators. We discovered that, for a fixed q, the numerators of fractions with primary recurring expansions, having a denominator of q, form a complete residue system mod$q$. Upon further examination, this notable property held true for all q where $\gcd(q, N_b D_b) = 1$. This seems natural, as it aligns back with our theory in Section 2.2 — The number $\frac{x}{y}$ has a terminating expansion if and only if y only contains factors of N_b and D_b.
That is, we know that among numerators that have the same residue mod$q$, there exists at least one such numerator, say p, such that $\frac{p}{q}$ has a primary recurring expansion. We believe this property is crucial for understanding fractions having primary recurring expansions.

Definition 2.11. Denote

$$\mathcal{P}_q := \{p \mid p \in \mathbb{Z}, q \in \mathbb{N}, \frac{p}{q} \text{ has primary recurring expansion } \}$$

Denote

$$\mathbb{P}_q := \{p \bmod q \mid p \in \mathcal{P}_q\}$$

It is obvious that, by definition, $\mathcal{P}_q$ forms complete residue system mod$q$ if and only if $\mathbb{P}_q = \mathbb{Z}_q$

Theorem 2.12 (Complete Residue System Theorem)**.** *If* $\gcd(q, N_b D_b) = 1$*, then* $\mathcal{P}_q$ *forms a complete residue system* mod$q$

Theorem 2.12 tends to be "naturally" correct on a class of q, where $q = (N_b)^{2n} - (-D_b)^{2n}$ and $n \in \mathbb{N}$. Therefore, it is our prioritized study target.
Hence, unless otherwise stated, from Definition 2.13 to Corollary 2.19 we will fix $q = (N_b)^{2n} - (-D_b)^{2n}$ where $n \in \mathbb{N}$.

Definition 2.13. Remaining the assumption that $q = (N_b)^{2n} - (-D_b)^{2n}$. Take $\beta \in \mathbb{Z}_q$ such that $\beta \equiv -N_b(D_b)^{-1} \pmod q$.
Define

(i) $(t_{2n-1}, t_{2n-2}, \ldots, t_1, t_0)_\beta$ to be a β**-expansion** mod$q$ of an integer k if

$$k \equiv (t_{2n-1}, t_{2n-2}, \ldots, t_1, t_0)_\beta := \sum_{i=0}^{2n-1} t_i\beta^i \pmod q$$

(ii) Denote

$$\mathcal{T} := \{\sum_{i=0}^{2n-1} t_i\beta^i \mod q \quad | \quad t_i \in \mathbb{Z} \text{ and } 0 \le t_i \le N_b - 1 \text{ for } 0 \le i \le 2n-1\}$$

to be the set collection of residue mod$q$ of β-expansion mod$q$
Denote

$$\mathcal{S} := \{\sum_{i=1}^{2n} s_i\beta^{2n-i} \mod q \quad | \quad s_i \in \mathbb{Z} \text{ and } 0 \le s_i \le N_b - 1 \text{ for } 1 \le i \le 2n\}$$

Note that $\mathcal{T}$ is defined for convenient notation in later theorems.
Obviously, $\mathcal{T} = \mathcal{S}$. For the sake of clarity, we include below the correspondence of expansions $\mathcal{T}$ and $\mathcal{S}$:

place value β^i	β^{2n-1}	β^{2n-2}	β^{2n-3}	$\ldots$	β^4	β^3	β^2	$\beta 1$	β^0
t_i	t_{2n-1}	t_{2n-2}	t_{2n-3}	$\ldots$	t_4	t_3	t_2	t_1	t_0
s_i	s_1	s_2	s_3	$\ldots$	s_{2n-4}	s_{2n-3}	s_{2n-2}	s_{2n-1}	s_{2n}

In short, $(t_{2n-1}, t_{2n-2}, \ldots, t_1, t_0)_\beta = (s_1, s_2, \ldots, s_{2n-1}, s_{2n})_\beta \in \mathcal{T}$ if and only if that β-expansion $(t_{2n-1}, t_{2n-2}, \ldots, t_1, t_0)_\beta = (s_1, s_2, \ldots, s_{2n-1}, s_{2n})_\beta \in \mathcal{S}$

The β-expansion has a close relationship with fractions having primary recurring expansion. Given the same digits s_i, by definition of β-expansion, the residue formed by the β-expansion mod$q$:

$$(t_{2n-1}, t_{2n-2}, \ldots, t_1, t_0)_\beta = (s_1, s_2, \ldots, s_{2n-1}, s_{2n})_\beta = \sum_{i=0}^{2n-1} t_i\beta^i \pmod q$$

is the same as the residue of the numerator of a primary recurring expansion $\alpha \sum_{i=1}^{2n} s_i \left(\frac{-N_b}{D_b}\right)^{2n-i}$, where $\alpha = (D_b)^{2n}$.
Hence, the notion of β-expansion is beneficial to understand the residues of numerators of fraction having primary recurring expansion.
Also, the length of the β-expansion $(t_{2n-1}, \ldots, t_0)_{\frac{-N_b}{D_b}}$ is fixed to be $2n$. The fact that the β-expansion has $2n$ is vital in the proving Theorem 2.17, which shows that

an β-expansion mod$q$ for the integer $k+1$ given that the previous integer k has an β-expansion mod$q$.
That is, the method we propose in Theorem 2.17 will not work if we simply take β-expansion to have length n.

Lemma 2.14. *Remaining the assumption that $q=(N_b)^{2n}-(-D_b)^{2n}$. Given that $k \equiv \alpha \sum_{i=1}^{2n} s_i \beta^{2n-i} \pmod{q}$ where $\alpha = {D_b}^{2n}$. Then, retaining the digits s_i, we know $k \equiv \alpha \sum_{i=1}^{2n} s_i \left(\frac{-N_b}{D_b}\right)^{2n-i} \pmod{q}$ and $\dfrac{\alpha \sum_{i=1}^{2n} s_i (\frac{-N_b}{D_b})^{2n-i}}{q}$ has a primary recurring expansion.*

Proof. It is, trivial, that by definition of β, we know that $k \equiv \alpha \sum_{i=1}^{2n} s_i \left(\frac{-N_b}{D_b}\right)^{2n-i}$ $\pmod{q}$.
Then, we know that $\dfrac{\alpha \sum_{i=1}^{2n} s_i \left(\frac{-N_b}{D_b}\right)^{2n-i}}{q} = \dfrac{\sum_{i=1}^{2n} s_i(-D_b)^i (N_b)^{2n-i}}{(N_b)^{2n}-(-D_b)^{2n}}$ has primary recurring expansion by Definition 2.2(iii). □

Theorem 2.15. *Remains the assumption on $q=(N_b)^{2n}-(-D_b)^{2n}$. If $\mathcal{T}=\mathbb{Z}_q$, then $\mathcal{T}=\mathbb{P}_q=\mathbb{Z}_q$*

Proof. The inclusion $\mathbb{P}_q \subseteq \mathcal{T}=\mathbb{Z}_q$ is trivial. It is because $\mathbb{P}_q$ is the set collection of residues of the numerator of primary recurring expansion mod$q$.
Now, we want to show the inclusion $\mathcal{T} \subseteq \mathbb{P}_q$.

Assume p' has a β-expansion mod$q$

Write $p' \equiv (t_{2n-1}, \dots, t_0)_\beta \pmod{q} \in \mathcal{T}$ where $t_i \in \mathcal{D}$ for all $0 \le i \le 2n-1$
Then we know $p' \equiv (s_1, \dots, s_{2n})_\beta \pmod{q} \in \mathcal{S}$ where $s_i \in \mathcal{D}$ for all $1 \le i \le 2n$.

Since $\gcd(\alpha, q)=1$ and $\mathcal{T}=\mathbb{Z}_{(N_b)^{2n}-(-D_b)^{2n}}$.
Then the set

$$\left\{\alpha \sum_{i=1}^{2n} s_i \beta^{2n-i} \mod q \quad | \quad s_i \in \mathbb{Z} \text{ and } 0 \le s_i \le N_b - 1 \text{ for } 1 \le i \le 2n\right\}$$
$$= \mathbb{Z}_{(N_b)^{2n}-(-D_b)^{2n}} = \mathcal{T} = \mathcal{S}$$

Hence, write $p' \equiv \alpha \sum_{i=1}^{2n} {s_i}' \beta^{2n-i} \pmod{q}$ for some ${s_i}' \in \mathcal{D}$

Then Apply Lemma 2.14, we know that $\dfrac{\alpha \sum_{i=1}^{2n} s_i{}'(\frac{-N_b}{D_b})^{2n-i}}{q}$ has a primary recurring expansion.

But, by the same Lemma, we know $p' \equiv \alpha \sum_{i=1}^{2n} s_i{}' \beta^{2n-i} \equiv \alpha \sum_{i=1}^{2n} s_i \left(\frac{-N_b}{D_b}\right)^{2n-i}$ $\pmod q \in \mathbb{P}_q$
Hence $\mathcal{T} \subseteq \mathbb{P}_q$, which concludes our proof. □

Now that we have established a strong connection between the set of of β-expansion and $\mathbb{P}_q$ given that $\mathcal{T}$ forms complete residue system $\bmod q$. We would now spend a considerable amount of effort on showing that the set of β expansion mod$q$ forms a complete residue system $\bmod q$. That is, we are proving the set equality $\mathcal{T} = \mathbb{Z}_q$. To achieve this, we will combine Lemma 2.16 and Theorem 2.17, which gives the final form of the proof in Corollary 2.18.

We discover that the hardest part of the proof lies on Theorem 2.17. This says that if k has a β-expansion mod$q$, then $k+1$ also has a β-expansion mod$q$.

However, before we can do that, we have to first define a few well-defined "actions" that we can do to manipulate the β-expansion.

Therefore, we would introduce Lemma 2.16, which introduces three key congruence identities. The three congruence identities are crucial as they will be repeatedly used in Theorem 2.17. In short, Lemma 2.16 allows us to do well-defined "actions" in Theorem 2.17.

Lemma 2.16. *It is given that $q = (N_b)^{2n} - (D_b)^{2n}$, the following congruence identities hold:*

$$N_b \equiv -D_b\beta \pmod q \tag{I1}$$

$$N_b \equiv D_b\beta^2 + (N_b - D_b)\beta \pmod q \tag{I2}$$

$$\beta^{2n} \equiv 1 \pmod q \tag{I3}$$

Proof.
For (I1), By Definition 2.13, we know $\beta \equiv -N_b(D_b)^{-1} \pmod q$. Also, $(D_b)^{-1} \bmod q$ is well-defined because $(q, D_b) = 1$ by Lemma 2.10(iv). Then simple rearrangement concludes (i).

For (I2),

$$\begin{aligned} D_b\beta^2 + (N_b - D_b)\beta &\equiv (N_b)^2(D_b)^{-1} + N_b - (N_b)^2(D_b)^{-1} \\ &\equiv N_b \pmod q \end{aligned}$$

Note that (I3) comes from the following:

$$\begin{aligned}\beta^{2n} - 1 &\equiv \left(-N_b(D_b)^{-1}\right)^{2n} - 1 \\ &\equiv (N_b)^{2n}(D_b)^{-2n} - (D_b)^{2n}(D_b)^{-2n} \\ &\equiv (D_b)^{-2n}\left((N_b)^{2n} - (D_b)^{2n}\right) \\ &\equiv 0 \pmod q\end{aligned}$$

□

Now that we have proved these three congruence identities, we would use them for the hardest part of the proof for the set equality $\mathcal{T} = \mathbb{Z}_q$. As mentioned above, the hardest part lies in how to find a well-defined β-expansion mod$q$ of $k+1$ given a well-defined β-expansion of k.

The following Theorem 2.17 investigates how a well-defined β-expansion of k, could lead to another well-defined β-expansion for $k+1$.

When adding 1 each time to the β-expansion, we found out that the most difficult part lies on some β-expansion of k such that the β-expansion of $k+1$ has carrying issue (See Case 2 of Theorem 2.17). To tackle this problem, we will be repeatedly using the three congruence identities found in Lemma 2.16.

Theorem 2.17. *If an integer k has a β-expansion* mod$q$*, then $k+1$ also has a β-expansion* mod$q$

Proof.
By definition, $k \equiv (t_{2n-1}, t_{2n-2}, \dots, t_1, t_0)_\beta \pmod q$

- **Case 1** If $t_0 \le N_b - 2$, then

$$k+1 \equiv (\ t_{2n-1}, t_{2n-2}, \dots t_1,\ (t_0+1)\ )_\beta \pmod q$$

  where $t_0 + 1 \le N_b - 1$ and $k+1$ has a well-defined β-expansion mod$q$
- **Case 2** Assume $t_0 = N_b - 1$ Then

$$k+1 \equiv (\ t_{2n-1}, t_{2n-2}, \dots t_1,\ (0)\ )_\beta + N_b \pmod q$$

However, it is not yet a well-defined β-expansion mod$q$ as we have an extra N_b at the end.
Then, we observe the following pattern when determining the value of other values of s_i

(i) If $t_1 \ge D_b$, then using congruence equation (I1)

$$k+1 \equiv (\ t_{2n-1}, t_{2n-2}, \dots,\ (t_1 - D_b)\ , 0)_\beta \pmod q$$

where $t_1 - D_b \ge 0$ and $k+1$ has a well-defined β-expansion $\bmod q$
This is because $-D_b\beta \equiv N_b \pmod q$ from (I1).

(ii) If $t_1 < D_b$, then using congruence equation (I2)

$$k+1 \equiv (\ t_{2n-1}, t_{2n-2}, \dots,\ \ (t_2 + D_b)\ \ ,\ \ (t_1 + N_b - D_b)\ \ , 0)_\beta \pmod q$$

where $0 \le t_1 + N_b - D_b \le N_b - 1$.

However, a well-defined β-expansion $\bmod q$ of $k+1$ depends on the value of $t_2 + D_b$. Notably, if $t_2 + D_b < N_b$, then $k+1$ has a well-defined β-expansion mod$q$. But if $t_2 + D_b \ge N_b$, we need to further manipulate the expression $t_2 + D_b$ and β-expansion mod$q$ in order for $k+1$ to have a well-defined β-expansion mod$q$

With the consideration of inequality assumptions on different t, we divide all possible $(t_{2n-1}, t_{2n-2}, \dots t_1, t_0)_\beta$ into the following cases.

- **Case 2A**
  There exists an integer H such that $0 \le H < n-1$ such that
  $$\begin{cases} t_{2i-1} < D_b \text{ for all } 0 < i \le H \\ t_{2i} \ge N_b - D_b \text{ for all } 0 < i \le H \\ t_{2H+1} \ge D_b \text{ or } t_{2H+2} < N_b - D_b \end{cases}$$
- **Case 2B**
  For all $i = 1, 2, \dots, n-1$ $\begin{cases} t_{2i-1} < D_b, \\ t_{2i} \ge N_b - D_b \end{cases}$

□

Example on Case 2A
We visualize the proving logic of Case 2A by setting an example when $H = 1$. We focus on t_1, t_2, t_3, t_4. Consider

$$k+1 \equiv (\ t_{2n-1}, t_{2n-2}, \dots, t_1,\ \ 0\ \ )_\beta + N_b \pmod q$$

By definition, $\begin{cases} t_1 < D_b \\ t_2 \ge N_b - D_b \\ t_3 \ge D_b \text{ or } t_4 < N_b - D_b \end{cases}$

Step 1:
Consider $i = 1$. Since $i \le H = 1$, we are concerning the first and second inequality. Observe t_1. we can't apply (I1), and cannot form $(\ t_{2n-1}, t_{2n-2}, \dots, t_2, (t_1 - D_b),\ \ 0\ \ )_\beta$ since $t_1 - D_b < 0$. Therefore only(I2) can be applied.
Step 2:
Perform (I2). Then

$$\begin{aligned} k+1 &\equiv (t_{2n-1}, t_{2n-2}, \dots, t_3, (t_2 + D_b), (t_1 + N_b - D_b),\ \ 0)_\beta \pmod q \\ &\equiv (t_{2n-1}, t_{2n-2}, \dots, t_3, (t_2 + D_b - N_b), (t_1 + N_b - D_b),\ \ 0)_\beta + N_b\beta^2 \pmod q \end{aligned}$$

Note we subtract N_b from the place value for β^2 and then we add back $N_b\beta^2$ at the end.
Notice $N_b - 1 \ge t_2 + D_b - N_b, t_1 + N_b - D_b \ge 0$. Then we are done with the digits t_2 and t_1 as $t_2, t_1 \in \mathcal{D}$. We are also done with the first and second inequality

$$\begin{cases} t_{2i-1} < D_b \text{ for all } 0 < i \leq H \\ t_{2i} \geq N_b - D_b \text{ for all } 0 < i \leq H \end{cases}$$

within the simultaneous inequality

$$\begin{cases} t_{2i-1} < D_b \text{ for all } 0 < i \leq H \\ t_{2i} \geq N_b - D_b \text{ for all } 0 < i \leq H \\ t_{2H+1} \geq D_b \text{ or } t_{2H+2} < N_b - D_b \end{cases}$$

Write $t_2 + D_b - N_b = t_2{}'$ and $t_1 + N_b - D_b = t_1{}'$

Then we consider how can t_3 or t_4 absorb the $N_b\beta^2$ to form a well-defined β-expansion mod$q$.
That is we are considering the third inequality $t_{2H+1} \geq D_b$ or $t_{2H+2} < N_b$ within the simultaneous inequality.

(a) If $t_3 \geq D_b$. Also, if we have the case where both $t_3 \geq D_b$ and $t_4 < N_b - D_b$, then we consider the case for the $t_3 \geq D_b$. Then apply (I1), which yields:

$$k + 1 \equiv (\ t_{2n-1}, t_{2n-2}, \ldots, (t_3 - D_b), t_2{}', t_1{}',\ \ 0\ )_\beta \pmod q$$

where $0 \leq t_3 - D_b, t_2{}', t_1{}' \leq N_b - 1$.
This is because by (I1), $N_b\beta^2 \equiv -D_b\beta^3 \pmod q$
in which t_3 absorbs $-D_b\beta^3$ to form a well-defined β-expansion mod$q$.

(b) If $t_4 < N_b - D_b$ only. Then we can only apply (I2), where

$$k + 1 \equiv (\ t_{2n-1}, t_{2n-2}, \ldots, (t_4 + D_b), (t_3 + N_b - D_b), t_2{}', t_1{}',\ \ 0\ )_\beta \pmod q$$

where $0 \leq t_4 + D_b, t_3 + N_b - D_b, t_2{}', t_1{}' \leq N_b - 1$.
This is because by (I2), $N_b\beta^2 \equiv D_b\beta^4 + (N_b - D_B)\beta^3 \pmod q$.
Hence, we conclude that; in any subcases, where k has a well-defined β-expansion mod$q$ satisfying the simultaneous inequality in Case 2A, we know that $k + 1$ also has a well-defined β-expansion mod$q$.

Therefore, under the inequality assumption of Case 2A, we conclude that if k lies in Case 2A, we have the following general formulas:

Case 2A(i) If $t_{2H+1} \geq D_b$, then we have the following general formula

$$\begin{aligned} k + 1 \equiv \\ (t_{2n-1}, t_{2n-2}, \ldots, t_{2H+2}, (t_{2H+1} - D_b), (t_{2H} + D_b - N_b), (t_{2H-1} + N_b - D_b), \\ (t_{2H-2} + D_b - N_b), (t_{2H-3} + N_b - D_b), \ldots \\ \ldots, (t_2 + D_b - N_b), (t_1 + N_b - D_b),\ \ 0\ )_\beta \pmod q \end{aligned}$$

where $0 \leq (t_{2H+1} - D_b) \leq N_b - 1$ and $0 \leq (t_{2i} + D_b - N_b), (t_{2i-1} + N_b - D_b) \leq N_b - 1$ for all $0 < i \leq H$.
Hence, the above is a well-defined β-expansion mod$q$

Case 2A(ii) If $t_{2H+2} < N_b - D_b$, then we have the following general formula.

$$k+1 \equiv$$
$$(t_{2n-1}, t_{2n-2}, \ldots, (t_{2h+2}+D_b), (t_{2H+1}+N_b-D_b), (t_{2H}+D_b-N_b), (t_{2H-1}+N_b-D_b), (t_{2H-2}+D_b-N_b), (t_{2H-3}+N_b-D_b), \ldots \ldots, (t_2+D_b-N_b), (t_1+N_b-D_b), \ 0 \ )_\beta \pmod q$$

where $0 \le (t_{2H+2}+D_b), (t_{2H+1}+N_b-D_b) \le N_b - 1$ and
$0 \le (t_{2i}+D_b-N_b), (t_{2i-1}+N_b-D_b) \le N_b - 1$ for all $0 < i \le H$.
Hence, the above is a well-defined β-expansion mod$q$.

On Case 2B

The difference between Case 2A and Case 2B is that the former has $H < n-1$, whilst the latter includes $H = n-1$.
Hence we can apply the same proving procedure from Case 2A until the index $H = n-1$, hence continuing the proving procedure until t_{2n-2} and t_{2n-3}.
Applying the argument in Case 2A yields:

$$k+1 \equiv$$
$$(t_{2n-1}, (t_{2n-2}+D_b-N_b), (t_{2n-3}+N_b-D_b), (t_{2n-4}+D_b-N_b), \ldots, (t_2+D_b-N_b), (t_1+N_b-D_b), \ 0 \ )_\beta + N_b\beta^{2n-2} \pmod q$$

where $0 \le (t_{2i}+D_b-N_b), (t_{2i-1}+N_b-D_b) \le N_b - 1$ for all $0 < i \le H$.

Case 2B If $t_{2n-1} < D_b$, again, (I1) should not be used as taking

$$N_b\beta^{2n-2} \equiv -D_b\beta^{2n-1}\beta$$

will yield $(t_{2n-1} - D_b) < 0$.
Hence, we perform (I2) on $N_b\beta^{2n-2}$, which yields

$$k+1 \equiv$$
$$D_b\beta^{2n} + ((t_{2n-1}+N_b-D_b), (t_{2n-2}+D_b-N_b), (t_{2n-3}+N_b-D_b), (t_{2n-4}+D_b-N_b), \ldots \ldots, (t_2+D_b-N_b), (t_1+N_b-D_b), \ 0 \ )_\beta \pmod q$$

where $0 \le (t_{2i}+D_b-N_b), (t_{2i-1}+N_b-D_b) \le N_b - 1$ for all $0 < i \le H$.
This is because by (I2), $N_b\beta^{2n-2} \equiv D_b\beta^{2n} + (N_b - D_B)\beta^{2n-1} \pmod q$.

But notice from (I3), we have $\beta^{2n} \equiv 1 \pmod q$. Hence we write

$$k+1 \equiv$$
$$(\ (t_{2n-1}+N_b-D_b), (t_{2n-2}+D_b-N_b), (t_{2n-3}+N_b-D_b), (t_{2n-4}+D_b-N_b), \ldots \ldots, (t_2+D_b-N_b), (t_1+N_b-D_b), \ D_b \ )_\beta \pmod q$$

where $0 \le (t_{2i}+D_b-N_b), (t_{2i-1}+N_b-D_b) \le N_b - 1$ for all $0 < i \le H$.
Hence, the above is a well-defined β-expansion mod$q$ and also a general formula for Case 2B.

Obviously if $t_{2n-1} \geq D_b$, then we can apply (I1) and we have $(t_{2n-1} - D_b) \geq 0$ and the $N_b\beta^{2n-2}$ will be cancelled out to form a well-defined β-expansion $\bmod q$.

Corollary 2.18. *Given* $q = (N_b)^{2n} - (-D_b)^{2n}$. *Then* $\mathbb{Z}_q = \mathcal{T}$

Proof.
The inclusion $\mathcal{T} \subseteq \mathbb{Z}_q$ is trivial.
This is because $\mathcal{T}$ takes residues in modulo q of β-expansion $\bmod q$, naturally $\mathcal{T} \subseteq \mathbb{Z}_q$

For the inclusion $\mathcal{T} \supseteq \mathbb{Z}_q$ We first claim that every natural number has a β-expansion $\bmod q$.
We will proceed with induction.

Base case: Notice that $1 \equiv (0, 0, \ldots, 1)_\beta \pmod q$ has a well-defined β-expansion $\bmod q$.
Inductive step: Now, we proceed with induction on k.
Applying Theorem 2.17 in the inductive step finishes the proof. Hence, every natural number has a well-defined β-expansion $\bmod q$.
Therefore, every natural number $k \in \mathbb{Z}_q$ has a well-defined β-expansion $\mod q$. But we know $\mathcal{T}$ contains all residues β-expansion. Hence these well-defined β-expansion $\bmod q$ forms a subset of $\mathcal{T}$, that is $\mathbb{Z}_q \subseteq \mathcal{T}$ □

Corollary 2.19. *Given* $q = (N_b)^{2n} - (-D_b)^{2n}$. *The set* $\mathcal{T} = \mathbb{P}_q = \mathbb{Z}_q$, *i.e. (*$\mathcal{P}_q$ *forms a complete residue system* $\bmod q$*)*

Proof. Apply Corollary 2.18, we have $\mathcal{T} = \mathbb{Z}_q$.
Then apply Theorem 2.15, we have $\mathcal{T} = \mathbb{P}_q = \mathbb{Z}_q$. □
Now, we want to show that the numerators of primary recurring expansions form the complete residue system $\mathcal{P}_{(N_b)^{2n}-(-D_b)^{2n}} = \mathbb{Z}_{(N_b)^{2n}-(-D_b)^{2n}}$ doesn't only occur when $q = (N_b)^{2n} - (-D_b)^{2n}$.
In the following theorem, we will no longer fix $q' = (N_b)^{2n} - (-D_b)^{2n}$.
We will prove that as long as $(q', N_bD_b) = 1$, then $\mathcal{P}_{q'} = \mathbb{Z}_{q'}$

Theorem 2.20. *(i) If* $\gcd(q', N_bD_b) = 1$. *Then there exists* $m \in \mathbb{N}$ *such that* $q' \mid (N_b)^{2m} - (-D_b)^{2m}$

(ii) If $\mathcal{P}_y$ *forms complete residue system* $\bmod y$ *and* $x \mid y$. *Then* $\mathcal{P}_x$ *forms complete residue system* $\bmod x$.

Proof.

Proof of (i) By the assumption $(q', N_bD_b) = 1$, we have $(q', D_b) = 1$. Then, since $\left[-(D_b)^{-1}\right]^{-1} \equiv D_b \pmod{q'}$ is well-defined, $-(D_b)^{-1} \pmod{q'}$ is also well-defined. Combining this with the fact that $(q', N_b) = 1$, we know that $-N_b(D_b)^{-1} \pmod{q'}$ is also well-defined and $\gcd(-N_b(D_b)^{-1}, q') = 1$.

The coprime condition then allows us to apply Euler's Theorem, which yields

$$\begin{aligned}\left[-N_b(D_b)^{-1}\right]^{2\phi(q')} &\equiv 1 \pmod{q'}\\ (N_b)^{2\phi(q')}\left((-D_b)^{-1}\right)^{2\phi(q')} &\equiv 1 \pmod{q'}\\ (N_b)^{2\phi(q')} &\equiv (-D_b)^{2\phi(q')} \pmod{q'}\\ (N_b)^{2\phi(q')}-(-D_b)^{2\phi(q')} &\equiv 0 \pmod{q'}\end{aligned}$$

Therefore, $q' \mid (N_b)^{2\phi(q')} - (-D_b)^{2\phi(q')}$ where $\phi(q')$ is the Euler-Toitent function.

Proof of (ii) Since $\mathcal{P}_y$ forms complete residue system mod$y$ by assumption, we know that there exists $p \in \mathcal{P}_y$ such that $p = k\frac{y}{x} + g_k y$ for all $k \in \mathbb{Z}_x$ where g_k is an integer.

Collect those p in the set $\mathbb{O}$

$$\mathbb{O} := \{0, \ , \frac{y}{x} + g_1 y \ , \ 2\frac{y}{x} + g_2 y \ , \ \dots \ , \ (x-1)\frac{y}{x} + g_{x-1} y\}$$

We know that

$$0 \ , \ \frac{\frac{y}{x} + g_1 y}{y} \ , \ \frac{2\frac{y}{x} + g_2 y}{y} \ , \dots \ , \ \frac{(x-1)\frac{y}{x} + g_{x-1} y}{y}$$

are fractions having primary recurring expansion.

Now, we know that

$$0 = 0 \ , \ \frac{\frac{y}{x} + g_1 y}{y} = \frac{1 + g_1 x}{x} \ , \ \frac{2\frac{y}{x} + g_2 y}{y} = \frac{2 + g_2 x}{x}, \ \dots \ , \ \frac{(x-1)\frac{y}{x} + g_{x-1} y}{y} = \frac{(x-1) + g_{x-1} x}{x}$$

are fraction having primary recurring expansion.

Then $0 \ , \ 1 + g_1 x \ , \ 2 + g_2 x \ , \ \dots \ , \ (x-1) + g_{x-1} x \in \mathcal{P}_x$.
Then, it is obvious that $\mathcal{P}_x$ forms complete residue system mod$x$ as, after taking modulo x, we obtain every residue mod$x$.

□

Having established the necessary theorems and corollaries, we return to prove the complete residue system theorem.

Theorem 2.12. If $\gcd(q', N_b D_b) = 1$, then

$$\mathcal{P}_{q'} := \{p \mid \frac{p}{q'} \text{ has primary recurring expansion } \}$$

forms a complete residue system modq'

Proof. Assume that $\gcd(q', N_b D_b) = 1$. Apply Theorem 2.20(i), there exists $n \in \mathbb{N}$ such that $q' \mid (N_b)^{2n} - (-D_b)^{2n}$. Since $q' \mid (N_b)^{2n} - (-D_b)^{2n}$ for some $n \in \mathbb{N}$

and by Corollary 2.19, we know that $\mathcal{P}_{(N_b)^{2n}-(-D_b)^{2n}}$ forms complete residue system $\bmod (N_b)^{2n} - (-D_b)^{2n}$ and Then, applying Theorem 2.20(ii), $\mathcal{P}_{q'}$ also forms complete residue system $\bmod q'$, which concludes our proof.

□

Example. The existence of an integer expansion for every integer has been well established. However, the existence of a β-expansion for all residues remains uncertain, unless utilizing Theorem 2.17.

The concern is indeed valid. As demonstrated in the previous example, even a relatively small integer like 35 has an integer expansion with 11 digits in base $\frac{-6}{5}$. To obtain a β-expansion with a digit length of $2n$, it is necessary to apply the congruence identities outlined in Lemma 2.16.

In the subsequent example, we will illustrate the β-expansion of residues modulo 65 in base $\frac{-3}{2}$.
We observe that the β-expansion mod65 of residues from 0 to 5 corresponds to their integer expansions and requires fewer than $2n = 4$ digits. The first difficulty naturally arises when dealing with larger integers such as 6, as the length of their integer expansions may exceed the desired number of digits.
To attain a β-expansion with 4 digits, we must follow the procedures outlined in Theorem 2.17.

Given that the number 5 has a β-expansion mod65 in the form $5 \equiv (0, 2, 1, 2)_{\frac{-3}{2}} \pmod{65}$, we would like to find the β-expansion mod65 of the residue 6.

Since $t_0 = N_b - 1 = 2$, we are in Case 2.
But, we know $\begin{cases} t_1 = 1 < D_b \\ t_2 = 2 \geq N_b - D_b \\ t_3 = 0 < D_b \end{cases}$
Since for all $i = 1, 2$, we have $\begin{cases} t_{2i-1} < D_b, \\ t_{2i} \geq N_b - D_b \end{cases}$
By definition, we are in Case 2B.
However, we know Case 2A and Case 2B differs only on the case for $t_{2(n-1)+1} = t_{2(2-1)+1} = t_3$.
Hence, we apply Case 2A for t_1 and t_2.
Hence we know

$$\begin{aligned} 5 + 1 &\equiv \big(t_3, (t_2 + D_b - N_b), (t_1 + N_b - D_b), 0\big)_\beta + N_b\beta^{2n-2} \pmod{65} \\ &\equiv \big(0, 1, 2, 0\big)_\beta + 3\beta^2 \pmod{65} \end{aligned}$$

Know that $t_{2n-1} = t_3 = 0 < D_b$, hence we can only perform (I2) on $3\beta^{2n-2}$, which yields

$$5 + 1 \equiv 2\beta^4 + ((0 + 3 - 2), 1, 2, \ \ 0\ \ )_\beta \pmod{65}$$

Refer to (I3), we have $\beta^4 \equiv 1 \pmod{65}$. Hence,

$$5 + 1 \equiv (1, 1, 2, \ 2 \ )_\beta \pmod{65}$$

which is a well-defined β-expansion mod$q$.
Now that we have determined 6 has a β-expansion mod65 where $6 \equiv (1, 1, 2, 2)_\beta \pmod{65}$, we would like to find the β-expansion mod65 of the residues 7.
Since $t_0 = N_b - 1 = 2$, we are in Case 2.
But, we know $\begin{cases} t_1 = 2 = D_b \\ t_2 = 1 = N_b - D_b \\ t_3 = 1 < D_b \end{cases}$
Since, for $H = 0$, we know $t_{2H+1} = t_1 = D_b$, we are in Case 2A(i).
Then, apply the general formula

$$\begin{aligned} 6 + 1 &\equiv (1, 1, (t_1 - D_b), 0)_\beta \pmod{65} \\ &\equiv (1, 1, 0, 0)_\beta \pmod{65} \end{aligned}$$

To reduce redundancy, we only list the β-expansion of residues mod65 up to 10 below. Nevertheless, we know that other residues from 10 to 65 have at least 1 corresponding β-expansion mod$q$.

Residue	β-expansion $\pmod{65}$
0	$(0,0,0,0)_\beta$
1	$(0,0,0,1)_\beta$
2	$(0,0,0,2)_\beta$
3	$(0,2,1,0)_\beta$
4	$(0,2,1,1)_\beta$
5	$(0,2,1,2)_\beta$
6	$(1,1,2,2)_\beta$
7	$(1,1,0,0)_\beta$
8	$(1,1,0,1)_\beta$
9	$(1,1,0,2)_\beta$
10	$(2,0,1,2)_\beta$

2.4. **Reverse Algorithm - A second proof for the Complete Residue System theorem.** In the following subsection, we will provide a second proof to Theorem 2.12 by utilizing the Reverse Algorithm, which can also efficiently decide whether a rational number has a primary recurring expansion. It should be noted that throughout this subsection, q is strictly fixed such that $\gcd(q, N_b D_b) = 1$ in order to attain a recurring expansion for the sake of proof.
We will first define the algorithms that will be used in our proof.

We will first define the **Forward Algorithm** in Definition 2.21, on account of its close relationship with the expansion of a fraction $\frac{p}{q}$ to its right-hand-side. Second of all, the **Forward Algorithm** is highly relevant to fractions with primary recurring expansions. We can find a necessary and sufficient condition between a primary recurring expansion, its digits a_i and "remainders" r_i in Theorem 2.22.

That is, if $p = r_k$ for some k, then naturally through the **Forward algorithm** $\frac{p}{q}$ has primary recurring expansion. In addition, given that $\frac{p}{q} = (0.\overline{a_1 \dots a_k})_{\frac{-N_b}{D_b}}$ has primary recurring expansion; subsequently, after applying p, q and $a_1 \dots a_k$ to the **Forward algorithm**, naturally $\frac{r_1}{q}, \dots \frac{r_{k-1}}{q}$ can attain primary recurring expansions as well. The details will be provided in Corollary 2.23.

After defining the **Forward Algorithm**, we will define the **Reverse Algorithm** on Definition 2.24. The **Reverse Algorithm** is observed to be interdependent with the **Forward Algorithm**. The relationship will be mentioned in the paragraph before Definition 2.24 where we discuss the motivation; and also Definition 2.24 itself.

Definition 2.21 (Forward Algorithm)**.** Let $b = \frac{-N_b}{D_b}$ and we define the following procedure as the **Forward Algorithm**: [2]

Input: Alg($p, q, \mathbb{A} = \{a_1, a_2, \dots, a_k\}$*)*, where $p \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathcal{D}$ such that $a_i q + r_i \equiv 0 \pmod{D_b}$ for all $1 \leq i \leq k$
Output: $r_1, r_2, \dots, r_k$

$$\begin{aligned} bp &= a_1 q + r_1 \\ br_1 &= a_2 q + r_2 \\ &\dots \\ br_{k-2} &= a_{k-1} q + r_{k-1} \\ br_{k-1} &= a_k q + r_k \\ &\dots \end{aligned}$$

The **Forward Algorithm** is closely related to the expansions of p and r_i for $1 \leq i \leq k$.

More specifically, we have

$$\begin{aligned} \frac{p}{q} &= (0.a_1)_{\frac{-N_b}{D_b}} + \left(\frac{-D_b}{N_b}\right) \frac{r_1}{q} \\ \frac{r_i}{q} &= (0.a_{i+1})_{\frac{-N_b}{D_b}} + \left(\frac{-D_b}{N_b}\right) \frac{r_{i+1}}{q} \end{aligned}$$

And that

$$\frac{p}{q} = (0.a_1 \dots a_k)_{\frac{-N_b}{D_b}} + \left(\frac{-D_b}{N_b}\right)^k \frac{r_k}{q}$$

According to our definition, we know r_i is an integer for all i.
We are aware that the stipulation $a_i q + r_i \equiv 0 \pmod{D_b}$ for all $1 \leq i \leq k$ is not technically a strict requirement for finding the expansion for $\frac{p}{q}$. However, we will

consider such a stipulation while defining our algorithm for convenience.

In Section 3, we will forgo the stipulation $a_iq + r_i \equiv 0 \pmod{D_b}$ and consider the case where r_i may not be an integer.

Theorem 2.22. *Let $b = \frac{-N_b}{D_b}$. There exists $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}$ where $a_i \in \mathcal{D}$ and at least one a_i is non-zero such that $p = r_k$ and $r_i \neq r_j$ for all $1 \leq i, j < k$ and $i \neq j$, if and only if $\frac{p}{q} = (0.\overline{a_1 \ldots a_k})$ is a fraction having a primary recurring expansion with period k.*

Proof. Assume there exists $\mathbb{A} = \{a_1, a_2, \ldots, a_k\}$ where $a_i \in \mathcal{D}$ and at least one a_i is non-zero such that $p = r_k$ and $r_i \neq r_j$ for all $1 \leq i, j < k$ and $i \neq j$ for all $j \in \mathbb{N}$ We perform the **Forward Algorithm** on $\mathrm{Alg}(p, q, \mathbb{A} = \{a_1, a_2, \ldots, a_k\})$.

$$
\begin{aligned}
bp &= a_1q + r_1 && p = b^{-1}(a_1q + r_1) = q(b^{-1}a_1) + b^{-1}r_1 \\
&\ldots \\
br_{k-1} &= a_kq + r_k && p = q(b^{-1}a_1 + b^{-2}a_2 + b^{-3}a_3 + \cdots + b^{-(k-1)}a_{k-1} + b^{-k}a_k) \\
& && \quad + b^{-k}r_k \\
&\ldots \\
br_{2k-1} &= a_{2k}q + r_{2k} && p = q(b^{-1}a_1 + b^{-2}a_2 + b^{-3}a_3 + \cdots + b^{-(k-1)}a_{k-1} + b^{-k}a_k)+ \\
& && \quad + qb^{-k}(b^{-1}a_{k+1} + b^{-2}a_{k+2} + b^{-3}a_{k+3} + \ldots \\
& && \quad + b^{-(k-1)}a_{2k-1} + b^{-k}a_{2k}) + b^{-2k}r_{2k} \\
&\ldots && \ldots
\end{aligned}
$$

Thus, by taking $a_{hk+1} = a_1$, $a_{hk+2} = a_2$, $\ldots$, $a_{(h+1)k-1} = a_{k-1}$, $a_{(h+1)k} = a_k$ for all $h \in \mathbb{N}$, we obtain

$$
\begin{aligned}
p &= q\sum_{i=0}^{\infty} b^{-ik}\left(\sum_{j=1}^{k} b^{-j}a_j\right) \\
&= \frac{\left(\sum_{j=1}^{k}\left(\frac{-D_b}{N_b}\right)^j a_j\right)q}{1-\left(\frac{-D_b}{N_b}\right)^k} \\
&= \frac{\sum_{j=1}^{k} a_j(-D_b)^i(N_b)^{k-i}}{(N_b)^k - (-D_b)^k}(q)
\end{aligned}
$$

$\dfrac{p}{q} = \dfrac{\sum_{j=1}^{k} a_j(-D_b)^i(N_b)^{k-i}}{(N_b)^k - (-D_b)^k}$ is a fraction with a primary recurring expansion, as it satisfies Definition 2.2(iii).

Assume that $\frac{p}{q} = (0.\overline{a_1 \dots a_k})$ is a fraction having a primary recurring expansion with period k.
Then, we can simply substitute it back into the **Forward Algorithm**, and that the result $p = r_k$ and $r_i \neq r_j$ for all $1 \leq i, j < k$ where $i \neq j$ comes naturally. □

Corollary 2.23. *If $\frac{p}{q}$ is a fraction having a primary recurring expansion with period k, then after performing* ***Forward Algorithm*** *on $\frac{p}{q}$, the fractions $\frac{p}{q}, \frac{r_1}{q}, \frac{r_2}{q}, \dots, \frac{r_{k-1}}{q}$ must be distinct fractions such that they have primary recurring expansions with same period k.*

Proof. Consider the forward algorithm defined in Definition 2.21 and Theorem 2.22. Note that if $\frac{p}{q} = (0.\overline{a_1 \dots a_k})_{\frac{-N_b}{D_b}}$ has a primary recurring expansion, there exists a set $\mathbb{A} = \{a_1, a_2, \dots, a_k\}$ such that $p = r_k$ and $r_i \neq r_j$ for all $1 \leq i, j < k$ and $i \neq j$.
We perform the **Forward Algorithm** on $\frac{r_1}{q}$ with the input

$$Alg(r_1, q, \mathbb{A} = \{a_2, \dots, a_k, a_1\}).$$

Taking $a_{hk+1} = a_2$, $a_{hk+2} = a_3$, … , $a_{(h+1)k-1} = a_k$, $a_{(h+1)k} = a_1$ for all $h \in \mathbb{N}$, we obtain

$$\begin{aligned} r_1 = {} & q(b^{-1}a_2 + b^{-2}a_3 + b^{-3}a_4 + \dots + b^{-(k-1)}a_k + b^{-k}a_1) \\ & + qb^{-k}(b^{-1}a_2 + b^{-2}a_3 + b^{-3}a_4 + \dots + b^{-(k-1)}a_k + b^{-k}a_1) \\ & + qb^{-2k}(b^{-1}a_2 + b^{-2}a_3 + b^{-3}a_4 + \dots + b^{-(k-1)}a_k + b^{-k}a_1) \\ & + qb^{-3k}(b^{-1}a_2 + b^{-2}a_3 + b^{-3}a_4 + \dots + b^{-(k-1)}a_k + b^{-k}a_1) \\ & + \dots \end{aligned}$$

Repeat the steps above in Theorem 2.22. We obtain, $\frac{r_1}{q} = (0.\overline{a_2a_3 \dots a_ka_1})_{\frac{-N_b}{D_b}}$, which is a well-defined primary recurring expansion with period k. Similarly, we can utilize the procedure to find the primary recurring expansions for $\frac{r_2}{q}, \dots, \frac{r_{k-1}}{q}$
By assumption $r_i \neq r_j$ for all $1 \leq i, j < k$ and $i \neq j$, such fractions are all distinct. □

However, the **Forward Algorithm** has limitations. There might be more than one a_i that satisfies $a_iq + r_i \equiv 0 \pmod{D_b}$. For example, if $(N_b - 1)q + r_i \equiv 0 \pmod{D_b}$, then $(N_b - D_b - 1)q + r_i \equiv 0 \pmod{D_b}$ while $(N_b - 1) \in \mathcal{D}$ and $(N_b - D_b - 1) \in \mathcal{D}$. This implies that even if $\frac{p}{q}$ has a primary recurring expansion, one "wrong" choice of a_i will lead to failure of finding the target expansion. To avoid wasting time doing trial and error to find out the correct set of $\mathbb{A} = \{a_1, a_2, \dots, a_k\}$, we develop the "**Reverse Algorithm**".
With the **Forward Algorithm**, we play with the set of $\mathbb{A} = \{a_1, \dots, a_k\}$ to try find such $\mathbb{A}$ such that $\frac{p}{q}$ has primary recurring expansion. Then, we try "reverse engineering" the process, that is, given a fraction $\frac{N_0}{q}$, we try to substitute backward to the "last step" of the **Forward Algorithm** $br_{k-1} = a_kq + r_k$. But now we

substitute N_0 for r_k, $a_0{}'$ for a_k and get N_1 for r_{k-1}. While doing so, we obtain $N_1 = \frac{-D_b}{N_b}(a_0{}'q + N_0)$.
With the digits $a_0{}' \in \mathcal{D}$, to keep $N_1 \in \mathbb{Z}$, we observed that the equation $a_0{}'q + N_0 \equiv 0 \pmod{N_b}$. However, we notice that the congruence equation $a_0{}'q + N_0 \equiv 0 \pmod{N_b}$ has only one solution for $a_0{}' \in \mathcal{D}$.
Below is a motivating example, which exemplifies the importance of the **Reverse Algorithm**. Consider base $\frac{-3}{2}$, we apply the our new "**Reverse Algorithm**" on $N_0 = 7$. (The actual Definition will be provided below)

$$N_1 = -8 = \left(\frac{-2}{3}\right)[(1)(5) + 7] \tag{4}$$

$$N_2 = 2 = \left(\frac{-2}{3}\right)[(1)(5) - 8] \tag{5}$$

$$N_3 = -8 = \left(\frac{-2}{3}\right)[(2)(5) + 2] \tag{6}$$

But by reversing the order, we have the format for the **Forward Algorithm**

$$\left(\frac{-3}{2}\right)(-8) = [(2)(5) + 2]$$
$$\left(\frac{-3}{2}\right)(2) = [(1)(5) - 8]$$

Notice $N_1 = N_3$ and $N_1 \neq N_2$ and $N_2 \neq N_3$. Thus, we apply Theorem 2.22, and know that $\frac{-8}{5} = (0.\overline{21})_{\frac{-3}{2}}$.
We see while testing for N_0 in our "**Reverse Algorithm**", we magically get that $\frac{-8}{5}$ has primary recurring expansion. We also find out that Equation (5) and Equation (6) "suggest" digits for the primary recurring expansion of $\frac{N_1}{q} = \frac{N_3}{q}$.
In conclusion, we know that such "**Reverse Algorithm**" allows us to find a primary recurring expansion, while the digits of the "**Reverse Algorithm**" are the digits for that particular primary recurring expansion.
This motivates us to define a **Reverse Algorithm** and find properties of such algorithm.

Definition 2.24 (Reverse Algorithm)**.** We define the following procedure as the **Reverse Algorithm**.
Input: $N_0 \in \mathbb{Z}$, *where* $N_0 \in \left[\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}q \ , \ \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q\right]$
Output: $\{a_0, a_1, \dots\}, \{N_1, N_2, \dots\}$

$$\mathbb{Z} \ni N_{i+1} = \frac{-D_b}{N_b}(a_i q + N_i) \tag{R.A.}$$

where $a \in \mathcal{D}$ such that $a_i q + N_i \equiv 0 \pmod{N_b}$.

Note that the existence and uniqueness of $a_i \in \mathcal{D}$ in the above congruence equation (R.A.) is guaranteed, as $(q, N_b) = 1$. The restriction $N_0 \in \mathbb{Z}$ gives a unique solution of $N_i \in \mathbb{Z}$ for all $i \geq 0$.

The **Forward Algorithm** gives birth to the **Reverse Algorithm**. Knowing that the **Forward Algorithm** gives an expansion on $\frac{p}{q}$ and $\frac{r_i}{q}$ for all $1 \leq i \leq k-1$, we define the **Reverse Algorithm**. As mentioned above, we know the **Reverse Algorithm** gives the digits a_i for some fraction $\frac{N_i}{q}$. With the above definition, a_i, which is uniquely defined, suggests the digit choice $\mathbb{A} = \{a_1, a_2, \dots a_k\}$ for $\frac{N_i}{q}$. Then, we can apply the **Forward Algorithm** to deduce a primary recurring expansion for $\frac{N_i}{q}$.

The restriction $N_0 \in \mathbb{Z}$, where $N_0 \in \left[\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}q \ , \ \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q\right]$, allows us to deduce the properties of N_i in the following propositions.
Recall Lemma 2.10(i). To have a primary recurring expansion for $\frac{p}{q}$,
we have,

$$\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}q \leq p \leq \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q \tag{B}$$

Now that we have restricted N_0 in bound (B), after applying (R.A.) on N_0, we deduce the following property on N_i for $i > 0$.

Lemma 2.25. *It is given that N_i satisfies* (B), *i.e.*

$$N_i \in \left[\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}q \ , \ \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q\right],$$

and N_{i+1} is the resultant after performing (R.A.) *on N_i. Then N_{i+1} also satisfies* (B).

Proof.
We first consider the lower bound of N_i.
From bound (B), we have

$$\begin{aligned}
N_i &\leq \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q \\
aq + N_i &\leq (\frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2} + a_i)q \\
N_{i+1} = \frac{-D_b}{N_b}(aq+N_i) &\geq \frac{-D_b}{N_b}\left(\frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2} + N_b - 1\right)q \qquad (\text{take } a_i = N_b - 1) \\
&= \frac{-D_b}{N_b}\left(\frac{(N_b-1)((D_b)^2+(N_b)^2-(D_b)^2)}{(N_b)^2-(D_b)^2}\right)q \\
&= -D_b\left(\frac{(N_b-1)(N_b)}{(N_b)^2-(D_b)^2}\right)q
\end{aligned}$$

Note that we take $a_i = N_b - 1$ to obtain the lower bound of N_i

Now we consider the upper bound of N_i
From (B), we have

$$N_i \geq \frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}q$$

$$aq+N_i \geq \frac{(-N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}+a_iq$$

$$N_{i+1} = \frac{-D_b}{N_b}(aq+N_i) \leq \frac{-D_b}{N_b}\left(\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}+0\right)q \qquad (\text{take } a_i=0)$$

$$= \frac{-D_b}{N_b}\left(\frac{(N_b-1)((D_b)^2+(N_b)^2-(D_b)^2)}{(N_b)^2-(D_b)^2}\right)q$$

$$= \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q$$

□

We have found the amazing property that once N_0 is inside bound (B), then every iteration N_i afterward also satisfies bound (B). This property shows that all iterations N_i lie within the interval $\left[\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}q\ ,\ \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q\right]$. That means that after a certain iteration, we know there must exist $N_i = N_j$ for some $i,j \in \mathbb{N}$. Referring back to our motivating example above, we know that such $\frac{N_i}{q} = \frac{N_j}{q}$ has primary recurring expansion. In the lemma below, we will give a formal proof of the existence of $N_i = N_j$ for some $i,j \in \mathbb{N}$. We also would like to prove that the occurrence of the equality $N_x = N_y$ implies that $\frac{N_x}{q} = \frac{N_y}{q}$ has a primary recurring expansion.

Lemma 2.26. *It is given that N_0 satisfies* (B)*.*

*(i) After performing the **Reverse Algorithm** on N_0 for finitely many times, there exists $j > i \geq 0$ such that $N_i = N_j$ for some $i,j \in \mathbb{Z}_{\geq 0}$, where $N_a \neq N_b$ for distinct a,b and $i \leq a,b < j$.*

*(ii) Perform **Reverse Algorithm** finitely many times on N_0. If $N_x = N_y$ for some $y > x \geq 0$ where $N_a \neq N_b$ for distinct a,b and $x \leq a,b < y$. Then, $\frac{N_x}{q}$ has primary recurring expansion with period $y-x$.*

Proof.
Proof of (i)
Assume such i,j does not exist. Then $N_i \neq N_j$ for all distinct $i,j \geq 0$.
Then, $N_0 \neq N_1 \neq N_2 \neq \cdots \neq N_{\lfloor \frac{(N_b-1)D_b}{N_b-D_b}q\rfloor+1}$. By Lemma 2.25, we know

$$N_0, N_1, \ldots, N_{\lfloor \frac{(N_b-1)D_b}{N_b-D_b}q\rfloor+1} \in \left[\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}q\ ,\ \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q\right],$$

so the interval $\left[\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}q\ ,\ \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q\right]$ contains at least $\lfloor \frac{(N_b-1)D_b}{N_b-D_b}q\rfloor + 2$ distinct integers.

However, the length of the interval is

$$\frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q - \frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}q = \frac{(N_b-1)D_b}{N_b-D_b}q$$

at which there are at most $\left\lfloor \frac{(N_b-1)D_b}{N_b-D_b}q \right\rfloor + 1$ distinct integers within the bound (B).

Contradiction occurs.

Hence, there exists $j > i \geq 0$ such that $N_i = N_j$ for some $i, j \in \mathbb{Z}_{\geq 0}$.
where $N_a \neq N_b$ for distinct a, b and $i \leq a, b < j$.
Then, by the well-ordering principle, there exists smallest j such that $N_i = N_j$, and this concludes the proof.

Proof of (ii)
We perform **Reverse Algorithm** y times on N_0.
Assume $N_x = N_y$ for some $y > x \geq 0$, where $N_a \neq N_b$ for distinct a, b and $x \leq a, b < y$.
Omitting the first x times of **Reverse Algorithm**, we have

$$\begin{aligned} N_{x+1} &= \frac{-D_b}{N_b}(a_x q + N_x) \\ N_{x+2} &= \frac{-D_b}{N_b}(a_{x+1} q + N_{x+1}) \\ &\cdots \\ N_y &= \frac{-D_b}{N_b}(a_{y-1} q + N_{y-1}) = N_x \end{aligned}$$

Now, we take $\mathbb{A} = \{a_{y-1}, a_{y-2}, \ldots, a_x\}$, where $a_l \in \mathcal{D}$ for all $x \leq l \leq y-1$
By substituting $\mathbb{A} = \{a_{y-1}, a_{y-2}, \ldots, a_x\}$ into the **Forward Algorithm** and reversing its order,

$$\begin{aligned} \frac{-N_b}{D_b}N_y &= a_{y-1}q + N_{y-1} \\ &\cdots \\ \frac{-N_b}{D_b}N_{x+2} &= a_{x+1}q + N_{x+1} \\ \frac{-N_b}{D_b}N_{x+1} &= a_x q + N_x \end{aligned}$$

We know $N_y = N_x$. But we also know $N_a \neq N_b$ for distinct a, b such that $x \leq a, b < y$ by assumption. Combining both, we apply Theorem 2.22, which shows that $\frac{N_y}{q} = \frac{N_x}{q} = (0.\overline{a_{y-1}\ldots a_x})$ has a primary recurring expansion with period $y - x$.
□

From Lemma 2.26(i), we know that, after applying **Reverse Algorithm** finitely many times, there must exist a fraction $\frac{N_j}{q}$ having primary recurring expansion for some $j \in N$.
Then, from Lemma 2.26(ii), we also know that, after applying the **Reverse Algorithm** such that the iterations N_x and N_y are equal, a primary recurring expansion occurs.
Therefore, we know that some $\frac{N_i}{q}$ do not have primary recurring expansions while others do; the idea is to classify which of those $\frac{N_i}{q}$ has primary recurring expansion and when exactly do we have one. Throughout our exploration, we have found out that after certain iterations of the **Reverse Algorithm**, say i times, $\frac{N_i}{q}, \frac{N_{i+1}}{q}, \dots$ also attain primary recurring expansions. Additionally, the ones "before", those $\frac{N_0}{q}, \frac{N_1}{q}, \dots, \frac{N_{i-1}}{q}$, do not.
Therefore, in Theorem 2.27 and Remark 2.28, we will formalize such statements.

Theorem 2.27. *It is given that N_0 satisfies* (B) *and performs the **Reverse Algorithm** j times. If there exists $j > i \geq 0$ such that $N_i = N_j$ for some $i, j \in \mathbb{Z}_{\geq 0}$, and $N_i \neq N_k$ for all $i < k < j$. Then, all the fractions $\frac{N_i}{q}, \frac{N_{i+1}}{q}, \dots, \frac{N_{j-1}}{q}$ have primary recurring expansions with period $j - i$.*

there exists $j > i \geq 0$ such that $N_i = N_j$ for some $i, j \in \mathbb{Z}_{\geq 0}$, where $N_a \neq N_b$ for distinct a, b and $i \leq a, b < j$
Proof. Substitute $x = i, y = j$ in Lemma 2.26(ii), we know $\frac{N_i}{q}$ has primary recurring expansion with period $j - i$.

Since $N_i = N_j$, it follows that $N_{i+m} = N_{j+m}$ for all $0 \leq m \leq j - i - 1$.
This is because the **Reverse Algorithm** gives unique values of N_k for all $k \geq 0$, hence the above can be easily obtained.

Note that $N_a \neq N_b$ for distinct a, b such that $i + m \leq a, b < j + m$ easily follows.

Then, after applying $x = i + m$, $y = j + m$ into Lemma 2.26(ii) for all $0 \leq m \leq j - i - 1$, we know that the fractions $\frac{N_i}{q}, \frac{N_{i+1}}{q}, \dots, \frac{N_{j-1}}{q}$ have primary recurring expansions with period $j - i$.
In particular,

$$\frac{N_i}{q} = (0.\overline{a_{j-1} \dots a_i})$$

$$\frac{N_{i+1}}{q} = (0.\overline{a_i a_{j-1} \dots a_{i+1}})$$

$$\dots$$

$$\frac{N_{j-1}}{q} = (0.\overline{a_{j-2} a_{j-3} \dots a_{j-1}})$$

∎

Remark 2.28. *We note that* $\frac{N_u}{q}$ *does not have primary recurring expansion for all* $0 \leq u < i$. *This is because there does not exist* $N_u = N_v$ *for all* $u < i$ *where* $u < v$, *we know* $\frac{N_u}{q}$ *doesn't have primary recurring expansion.*

After knowing which of those $\frac{N_i}{q}$ have primary recurring expansions, we will introduce some basic properties that are direct consequences of **Reverse Algorithm**. These properties act as the stepping stones for Lemma 2.30, which is crucial for understanding the residues' cycles of numerators in primary recurring expansions.

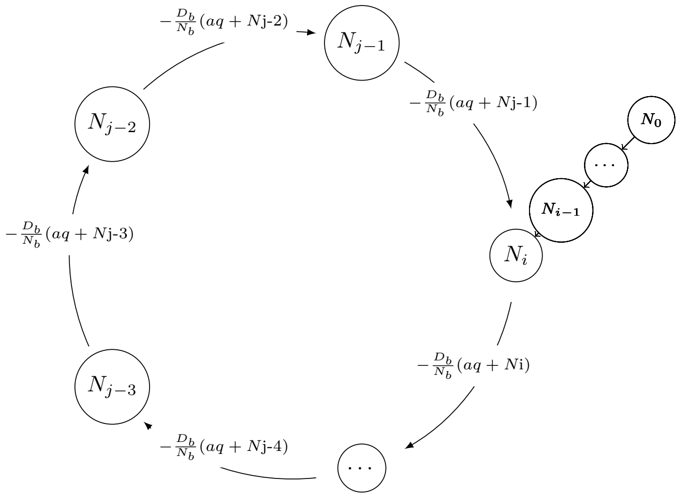


FIGURE 1. *A "spiral"-shaped generated by inputting* N_0 *into the reverse algorithm, in which* $N_i = N_j$.
It should be noted that the arrows are going down the algorithm, taking the steps labelled.

We provide a geometric interpretation to visualize the **Reverse Algorithm** and its process in Figure 1.
We notice that the **Reverse Algorithm** starts at N_0. For each iteration, we apply the congruence equation (R.A.) $N_{i+1} = \frac{-D_b}{N_b}(a_i q + N_i)$.
By the visualization, it is easy to see that $\frac{N_0}{q}, \ldots, \frac{N_{i-1}}{q}$ does not having primary recurring expansion as they do not belong to the "cycle" within the "spiral"-shaped figure. We also know that $\frac{N_i}{q}, \ldots, \frac{N_{j-1}}{q}$ also have primary recurring expansion as they stand within the "cycle".

Lemma 2.29. *It is given that* $\beta \equiv \left[-N_b(D_b)^{-1}\right] \pmod{q}$ *where* $\beta \in \mathbb{Z}_q$. *Applying the* ***Reverse Algorithm*** *on* N_i, *we deduce the following properties.*

(i) $N_{i+1} \equiv \beta^{-1} N_i \pmod{q}$
(ii) $N_{i+\text{ord}_q(\beta)} \equiv N_i \pmod{q}$
(iii) If $\frac{p}{q}$ *has a primary recurring expansion with period* n. *Then, after applying* $N_0 = p$ *in the* ***Reverse Algorithm*** n *times, we have* $(\beta^{-1})^n \equiv 1 \pmod{q}$

Proof.
Proof of (i):
Consider the Equation (R.A.). Taking modulo q, we have

$$N_{i+1} \equiv \beta^{-1} N_i \pmod q$$

Proof of (ii):
Since $\gcd(\beta, q) = 1$, we have $\mathrm{ord}_q(\beta)$ exists.
Hence, we can deduce the following congruence equation

$$\beta^{\mathrm{ord}_q(\beta)} \equiv \left[-N_b(D_b)^{-1}\right]^{\mathrm{ord}_q(\beta)} \equiv 1 \pmod q$$

After applying the **Reverse Algorithm** $\mathrm{ord}_q(\beta)$ times, we obtain

$$\begin{aligned} N_{i+\mathrm{ord}_q(\beta)} &\equiv (\beta^{-1})^{\mathrm{ord}_q(\beta)} N_i \pmod q \\ &\equiv N_i \pmod q \end{aligned}$$

which concludes our proof.

Proof of (iii):
By Theorem 2.27, we have $N_0 = N_n$. However, by continuously applying 2.29(i), we know $N_n \equiv (\beta^{-1})^n N_0 \pmod q$. Hence $(\beta^{-1})^n \equiv 1 \pmod q$ □
Having proven the basic properties about **Reverse Algorithm** through Lemma 2.29 and its relation with primary recurring expansion through Lemma 2.26 and Theorem 2.27; we will investigate on the relationship between residues of each iteration of **Reverse Algorithm** mod$q$.
We observe that a cycle of residues repeats itself through each iteration of the **Reverse Algorithm**.

For instance, we apply $N_0 = 11$ and $q = 7$ in base $\frac{-3}{2}$ to the **Reverse Algorithm** until $N_i = N_j$ for some $j > i \geq 0$.
In order to save time and space, which is required by calculation, we construct a table of N_i and its residues after applying **Reverse Algorithm**.

i	0	1	2	3	4	5	6	7	...
N_i	11	-12	8	-10	2	-6	4	-12	...
N_i mod q	4	2	1	4	2	1	4	2	...

Note that $N_1 = N_7$, hence we can simply apply **Reverse Algorithm** 7 times, and we find all the primary recurring expansion that there is to offer.

We observe that the residue cycle $2, 1, 4$ repeated twice for primary recurring expansion.
This interesting observation motivates us to the following Lemma 2.30(i), which shows this phenomenon occurs indefinitely in the **Reverse Algorithm**.

Besides, we observe that the residues formed by $N_i, \dots, N_{j-1}$, where i, j are smallest possible integers, encompass the residues formed by $N_0, \dots, N_{j-1}$. That is, we have $N_0 \bmod q = 4 \in \{N_1 \bmod q \ , \dots, \ N_6 \bmod q\} = \{2, 1, 4\}$ Therefore, knowing that $\frac{N_i}{q}, \dots, \frac{N_{j-1}}{q}$ form primary recurring expansions, this motivates us to show that the residues $N_i, \dots, N_{j-1} \bmod q$ are same as those as $N_0, \dots, N_{i-1} \bmod q$ in Lemma 2.30(ii).

Lemma 2.30. *It is given that N_0 satisfies* (B)*, and we perform the* ***Reverse Algorithm*** *finitely many times on N_0. Given that there exists $i \geq 0$ such that $N_i = N_j$ for some $j > i \geq 0$, where $N_i \neq N_k$ for all $i < k < j$.*

- *(i) If $\widetilde{a} \equiv \widetilde{b} \pmod{j-i}$ where $a \leq b$ for some $\widetilde{a}, \widetilde{b} \in \mathbb{Z}_{\geq 0}$. Then, after applying the* ***Reverse Algorithm*** *$\widetilde{b}$ times on N_0, we obtain $N_{\widetilde{a}} \equiv N_{\widetilde{b}} \pmod q$*
- *(ii) For all $a \in \mathbb{Z}_{\geq 0}$, there exists c where $i \leq c \leq j-1$ such that after applying the* ***Reverse Algorithm*** *finitely many times on N_0, we obtain $N_a \equiv N_c \pmod q$*

Proof.
Proof of (i):
Assume $\widetilde{a} \equiv \widetilde{b} \pmod{j-i}$.
Write $\widetilde{b} = k(j-i) + \widetilde{a}$ for some positive integer k.
Then, by Lemma 2.29(i) and (iii), we have $N_x \equiv \left(\beta^{-1}\right)^x N_0 \pmod q$ and $\left(\beta^{-1}\right)^{j-i} \equiv 1 \pmod q$.

Consider

$$\begin{aligned} N_{\widetilde{b}} &\equiv \left(\beta^{-1}\right)^{\widetilde{b}} N_0 \\ &\equiv \left(\beta^{-1}\right)^{k(j-i)+\widetilde{a}} N_0 \\ &\equiv \left(\beta^{-1}\right)^{\widetilde{a}} N_0 \\ &\equiv N_{\widetilde{a}} \pmod q \end{aligned}$$

which concludes our proof.

Proof of (ii):
Note that the set of indices $\{i, \dots, j-1\}$ forms a complete residue system $\bmod\ j-i$ That is, for all $a \in \mathbb{Z}_{\geq 0}$, there exists $c \in \mathbb{N}$ where $i \leq c \leq j-1$ and $a \equiv c \pmod{j-i}$.
Apply Lemma 2.30(i). For all $a \in \mathbb{Z}_{\geq 0}$, there exists $c \in \mathbb{N}$ such that $N_a \equiv N_c \pmod q$ where $i \leq c \leq j-1$. □

Having established the necessary definitions and lemmas, we will now proceed to demonstrate the Second proof for the complete residue system theorem.

Theorem 2.12.
We remain the assumption that $\gcd(q, N_b D_b) = 1$, then

$$\mathcal{P}_q := \{p \mid \frac{p}{q} \text{ has primary recurring expansion } \}$$

forms a complete residue system modq'.

Proof.
First, we want to discover subsets of $\mathbb{P}_q$ through the **Reverse Algorithm**.
Pick an arbitrary element $N_0 \in \mathbb{Z}_q$. Obviously, N_0 satisfies bound (B).
Perform the **Reverse Algorithm** $\lfloor \frac{(N_b-1)D_b}{N_b-D_b} q \rfloor + 2$ times on N_0, which is, theoretically, the longest possible length such that $N_i = N_j$ for some $j > i \geq 0$ in Lemma 2.26(i) as there are at most $\left\lfloor \frac{(N_b-1)D_b}{N_b-D_b} q \right\rfloor + 1$ distinct integers within the bound (B).

By Lemma 2.26(i) and the well-ordering principle, there exists the smallest non-negative integer i such that $N_i = N_j$ for some $j > i \geq 0$, where $N_i \neq N_k$ for all $i < k < j$.

Applying Lemma 2.27, then $\frac{N_i}{q}, \frac{N_{i+1}}{q}, \ldots \frac{N_{j-1}}{q}$ also have primary recurring expansion with period $j - i$.
We denote $n_z = N_z \bmod q$ where N_z are the resultant after performing **Reverse Algorithm** on N_0
Denote $\mathbb{C}_1 := \{n_k \ \mid \ i \leq k \leq j-1\}$.

That is, $\mathbb{C}_1$ is defined to be the set of residues of numerators $N_i, \ldots, N_{j-1} \bmod q$ for the fractions $\frac{N_i}{q}, \ldots, \frac{N_{j-1}}{q}$ having primary recurring expansion after applying **Reverse Algorithm** on N_0.

Recall Definition of $\mathbb{P}_q$ in Definition 2.11., we know $\mathbb{C}_1 \subseteq \mathbb{P}_q$ by definition of $\mathbb{C}_1$

Then, we pick an arbitrary element $N_0 \in \mathbb{Z}_q \setminus \mathbb{C}_1$. Obviously, N_0 satisfies bound (B).

Perform the **Reverse Algorithm** $\lfloor \frac{(N_b-1)D_b}{N_b-D_b} q \rfloor + 2$ times on ${N_0}'$.

We denote ${n_z}' = {N_z}' \bmod q$ where ${N_z}'$ are the resultant after performing **Reverse Algorithm** on ${N_0}'$
Similarly, denote $\mathbb{C}_2 := \{{n_k}' \ \mid \ i \leq k \leq j-1\}$.

Notice that $\mathbb{C}_2 \neq \emptyset$. This is because, by Lemma 2.30(ii), we know there exists $i \leq l \leq j-1$ such that $N_l \equiv N_0 \pmod{q}$.
Again, I know $\mathbb{C}_2 \subseteq \mathbb{P}_q$ by definition of $\mathbb{C}_2$

Repeat the above process of picking $N_0 \in \mathbb{Z}_q \setminus (\mathbb{C}_1 \cup \mathbb{C}_2 \cup \cdots \cup \mathbb{C}_k)$, applying **Reverse Algorithm** and constructing the set $\mathbb{C}_{k+1}$. Again, by a similar argument, we know that $\mathbb{C}_k$ is non-empty and that $\mathbb{C}_k \subseteq \mathbb{P}_q$. After applying the procedure for finitely many times, we obtain $\mathbb{Z}_q \setminus \bigcup_{i=1}^{l} \mathbb{C}_i = \emptyset$ for some $l \in \mathbb{N}$.

The idea is that we would like to fill up elements in $\mathbb{Z}_q$ via $\mathbb{C}_1, \mathbb{C}_2, \ldots$ until the union of $\mathbb{C}$ equals to $\mathbb{Z}_q$. We know we could do so because every time we pick an element in $\mathbb{Z}_q$, but different from those the previous $\mathbb{C}$.

Now we know that $\mathbb{Z}_q \setminus \bigcup_{i=1}^{l} \mathbb{C}_i = \emptyset$. That is, every residue $n \in \mathbb{Z}_q$, then $n \in \mathbb{C}_k$ for some $1 \leq k \leq l$.
But, we also know that $\bigcup_{i=1}^{l} \mathbb{C}_i \subseteq \mathbb{P}_q$ by the definition of those $\mathbb{C}_i$.
This meant that every $n \in \mathbb{C}_i$ for all $1 \leq i \leq l$, there exists a corresponding p such that $p \equiv n \pmod q$ and that $\frac{p}{q}$ has primary recurring expansion.
However, we also know that $\mathbb{P}_q \subseteq \mathbb{Z}_q = \bigcup_{i=1}^{l} \mathbb{C}_i$ because $\mathbb{P}_q$ contains element mod$q$.
Thus we have the relation $\bigcup_{i=1}^{l} \mathbb{C}_i = \mathbb{P}_q = \mathbb{Z}_q$.
Notice that $\mathbb{Z}_q = \mathbb{P}_q$ is logically equivalent to $\mathcal{P}_q$ forms complete residue system mod$q$ in Definition 2.11. Thus, this concludes our proof. □

Example. Suppose we $q = 7$ in base $\frac{-3}{2}$. We would like to exemplify that $\bigcup_{i=1}^{l} \mathbb{C}_i = \mathbb{P}_q = \mathbb{Z}_q$ for some $l \in \mathbb{N}$.
Consider take $N_0 = 5 \in \mathbb{Z}_7$.

The following table shows the results of **Reverse Algorithm** after performing it on N_0 until $N_i = N_j$ for some $j > i \geq 0$. Note that in Second proof of Complete Residue System Theorem, we perform the **Reverse Algorithm** $|\frac{N_b-1}{N_b-D_b}|+2$ times. This is for "safety", because we know that there must exist some $j > i \geq 0$ such that $N_i = N_j$ after performing that many time. Hence, in actual numerical calculation, we rarely perform the **Reverse Algorithm** strictly $|\frac{N_b-1}{N_b-D_b}| + 2$ times.

i	0	1	2	3	4
N_i	5	-8	-4	-2	-8
n_i	5	6	3	5	6

Notice that $N_1 = N_4$. Hence, this time, we just need to perform the **Reverse Algorithm** four times.
Now, from Theorem 2.27, we know that $\frac{5}{7}$ does not have primary recurring expansion while $\frac{-8}{7}, \frac{-4}{7}, \frac{-2}{7}$ also having primary recurring expansion.
Now, by definition, $\mathbb{C}_1 = \{6, 3, 5\}$.

We know that $\mathbb{C}_1 \subseteq \mathbb{P}_7$ by definition of $\mathbb{C}_1$.

Now, we take $N_0 \in \mathbb{Z}_7 \setminus \{6, 3, 5\}$, say $N_0 = 4$.

The following table shows the results of **Reverse Algorithm** after applying it on $N_0 = 4$ until $N_i = N_j$ for some $j > i \geq 0$.

i	0	1	2	3	4	5	6
N_i	4	-12	8	-10	2	6	4
n_i	4	2	1	4	2	1	4

Notice that $N_0 = N_6$. Hence, this time, we just need to perform the **Reverse Algorithm** six times.
Then, from Theorem 2.27, the fractions $\frac{4}{7}, \frac{-12}{7}, \frac{8}{7}, \frac{-10}{7}, \frac{2}{7}, \frac{6}{7}$ all having primary recurring expansion.
Now, by definition, $\mathbb{C}_2 = \{4, 2, 1\}$. Again, by definition of $\mathbb{C}_2$, we know that $\mathbb{C}_2 \subseteq \mathbb{P}_7$.
Now we know that $\mathbb{Z}_7 \setminus (\mathbb{C}_1 \cup \mathbb{C}_2) = \emptyset$. We could not take out any more element from $\mathbb{Z}_7 \setminus (\mathbb{C}_1 \cup \mathbb{C}_2)$
We know that $\mathbb{C}_1 \cup \mathbb{C}_2 \subseteq \mathbb{P}_7$ by definition of $\mathbb{C}_i$.
But we also know $\mathbb{P}_7 \subseteq \mathbb{Z}_7 = (\mathbb{C}_1 \cup \mathbb{C}_2)$ because $\mathbb{P}_7$ contains element mod7.

Thus, $(\mathbb{C}_1 \cup \mathbb{C}_2) = \mathbb{P}_7 = \mathbb{Z}_7$.

That is, we know the set of residues of primary recurring expansion after applying $N_0 \in \mathbb{Z}_7$ to the **Reverse Algorithm** and the set of residues of primary recurring expansion after applying $N_0 \in \mathbb{Z}_7 \setminus \mathbb{C}_1$ to the **Reverse Algorithm** combine to form a complete residue system mod7. That is, $(\mathbb{C}_1 \cup \mathbb{C}_2) = \mathbb{P}_7 = \mathbb{Z}_7$.

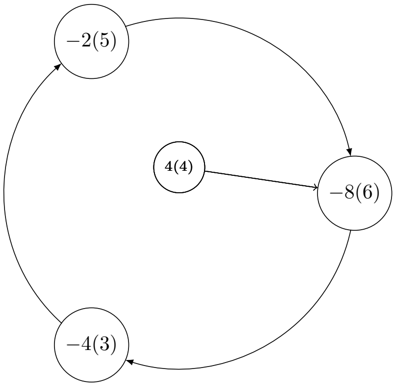


FIGURE 2. *A "spiral"-liked shape figure is generated by inputting 5 into the reverse algorithm, in which $N_1 = N_4$; We take q=7 in base $\frac{-3}{2}$*

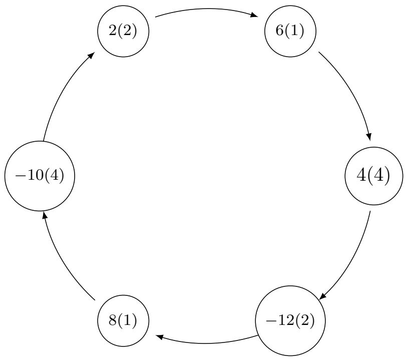


FIGURE 3. *A "spiral"-liked shape figure is generated by inputting 4 into the reverse algorithm, in which $N_0 = N_6$; We take q=7 in base $\frac{-3}{2}$*

Remark 2.31. *When performing* $N_0 = 5 \in \mathbb{Z}_q$, *even though* $\frac{5}{q}$ *does not having primary recurring expansion, the residue* $n_0 = 5$ *also belongs to the set* $\mathbb{C}_1 = \{6, 3, 5\}$. *That is,*

$$\{n_k \mid 0 \le k \le j-1\} = \mathbb{C}_1$$

In fact, by Lemma 2.30(ii), we know that after performing the ***Reverse Algorithm*** *with input* $N_0 \in \mathbb{Z}_q \setminus (\mathbb{C}_1 \cup \mathbb{C}_2 \cup \cdots \cup \mathbb{C}_k)$ *the set of residues*

$$\{n_k \mid 0 \le k \le j-1\} = \mathbb{C}_{k+1}$$

Remark 2.32. *If* $\frac{p}{q}$ *has an primary recurring expansion, then its expansion is unique.*
This can be easily shown as, by the ***Reverse Algorithm***, *we know that the digit choice is unique for every iteration*

2.5. **Number of representation in the form of terminating or recurring expansion.** In this subsection, we will determine the number of representations in the form of terminating or recurring expansion.
As mentioned in the introduction, the paper *A Number System with Base -3/2* [3] has determined there to be multiple expansions in the form of $(d_k \dots d_0.d_{-1}d_{-2}\dots)_{\frac{-3}{2}}$ – in fact, it claims that there are "infinitely many" [3]. However, there are interesting bounds that can be observed if only the terminating and recurring expansions are considered, and we can conclude a fixed value for the number of representations, which is finite.

We will first prove that every rational number has at least 1 terminating or recurring expansion in Theorem 2.34.

Then, in Theorem 2.35, we are going to prove that there are at most $\lfloor \frac{N_b-1}{N_b-D_b} \rfloor + 1$ number of terminating or recurring expansion.

We will address the non-terminating, non-recurring expansions of $\frac{p}{q}$ in a later section, which we believe to have infinitely many.

Lemma 2.33. *It is given that* N_0 *satisfies* (B), *and we perform the* ***Reverse Algorithm*** *finitely many times on* N_0. *There are at most* $\lfloor \frac{N_b-1}{N_b-D_b} \rfloor + 1$ *distinct integers such that they all have the same residue* $\bmod D_b q$.

Proof. Since the length of the interval (B)(ii) is $\frac{(N_b-1)}{N_b-D_b} D_b q$. We divide by $D_b q$ to find the number of distinct integers $\bmod D_b q$. Hence there are at most $\lfloor \frac{(N_b-1)D_b}{N_b-D_b} \rfloor + 1$ distinct integer such that they have the same residue $\bmod D_b q$ □

Theorem 2.34. *For all* $\frac{m}{n} \in \mathbb{Q}$, *where* $\gcd(m, n) = 1$. *The number* $\frac{m}{n}$ *has at least 1 representation with terminating expansion or recurring expansion.*

Proof.

Case (I). When

$$n = \prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} \prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j}$$

for some $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$. Then apply Corollary 2.9 (ii). We know that $\frac{m}{n}$ has a representation with terminating expansion.

Case (ii). When

$$n = q \prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} \prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j}$$

for some $\mathbb{N} \ni q \neq 1$, $\gcd(q, N_b D_b) = 1$ and $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$.

Refer to Corollary 2.9(ii) for the Definition of $\text{ind}_{N_b}(\hat{q_j})$
We take $y = \max\big(\text{ind}_{N_b}(\hat{q_1}), \text{ind}_{N_b}(\hat{q_2}), \ldots, \text{ind}_{N_b}(\hat{q_{l'}})\big)$ where y has l' prime factors dividing N_b. That is, we have y is the smallest non-negative integer such that $P_{N_b} \mid (N_b)^y$.

For simplicity, write,

$$\prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} = P_{D_b} \text{ and } \prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j} = P_{N_b}$$

Then we know,

$$\left(\frac{-N_b}{D_b}\right)^y \left(\frac{m}{n}\right) = \frac{\left[\frac{(-N_b)^y}{P_{N_b}}\right] m}{(D_b)^y P_{D_b}(q)} \tag{7}$$

where $\dfrac{(-N_b)^y}{P_{N_b}} \in \mathbb{Z}$ We know $\gcd\Bigg((D_b)^y P_{D_b} \ , \ q\Bigg) = 1$ by our definition in Case 2.

We apply Bezout's Lemma, and hence there exist $a, p \in \mathbb{Z}$ such that

$$qa + (D_b)^y \left[P_{D_b}\right] \ (p) = 1$$

$$q\left(am\left[\frac{(-N_b)^y}{P_{N_b}}\right]\right) + (D_b)^y P_{D_b} \ \left(pm\left[\frac{(-N_b)^y}{P_{N_b}}\right]\right) = \left[\frac{(-N_b)^y}{P_{N_b}}\right] m \tag{8}$$

Substituting Equation (8) in the numerator in Equation (7),

$$\left(\frac{-N_b}{D_b}\right)^y \left(\frac{m}{n}\right) = \frac{q\left(am\left[\frac{(-N_b)^y}{P_{N_b}}\right]\right) + (D_b)^y P_{D_b} \ \left(pm\left[\frac{(-N_b)^y}{P_{N_b}}\right]\right)}{(D_b)^y P_{D_b}(q)}$$

$$= am\frac{\left[\frac{(-N_b)^y}{P_{N_b}}\right]}{(D_b)^y P_{D_b}} + pm\frac{\left[\frac{(-N_b)^y}{P_{N_b}}\right]}{q}$$

Write $a' = am\left[\frac{(-N_b)^y}{P_{N_b}}\right] \in \mathbb{Z}$ and $p' = pm\left[\frac{(-N_b)^y}{P_{N_b}}\right] \in \mathbb{Z}$.
We have

$$\left(\frac{-N_b}{D_b}\right)^y\left(\frac{m}{n}\right) = \frac{a'}{(D_b)^y P_{D_b}} + \frac{p'}{q}$$

Now, apply Complete Residue System Theorem (Theorem 2.12)
There exists $c \in \mathbb{Z}$ such that $p' \equiv c \pmod q$, where $\frac{c}{q}$ has primary recurring expansion.
Obviously, $\frac{c-p'}{q} \in \mathbb{Z}$. Write,

$$\begin{aligned}\left(\frac{-N_b}{D_b}\right)^y\left(\frac{m}{n}\right) &= \frac{a'}{(D_b)^y P_{D_b}} + \frac{p'}{q}\\ &= \frac{a'}{(D_b)^y P_{D_b}} - \frac{c-p'}{q} + \frac{c}{q}\end{aligned}$$

This means that $\dfrac{a'}{(D_b)^y P_{D_b}} - \dfrac{c-p'}{q} \in \mathbb{Z}[\dfrac{1}{D_b}]$, while $\dfrac{c}{q}$ has a primary recurring expansion.

Apply Theorem 2.6 and Definition 2.2.
For some $t \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{N}$, we write,

$$\begin{aligned}\left(\frac{-N_b}{D_b}\right)^y\left(\frac{m}{n}\right) &= (d_t \dots d_0)_{\frac{-N_b}{D_b}} + (0.\overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}}\\ &= (d_t \dots d_0\ .\ \overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}}\\ \frac{m}{n} &= \left(\frac{-D_b}{N_b}\right)^y (d_t \dots d_0\ .\ \overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}}\end{aligned}$$

Note that if $t \geq y$, then,

$$\left(\frac{-D_b}{N_b}\right)^y (d_t \dots d_0\ .\ \overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}} = (d_{t-y} \dots d_y\ .\ d_{y-1} \dots d_0\overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}}$$

.

And if $t < y$, then

$$\left(\frac{-D_b}{N_b}\right)^y (d_t \dots d_0\ .\ \overline{s_1 \dots s_n}) = (0\ .\ \underbrace{0 \dots 0}_{y-t \text{ times}}\ d_t \dots d_0\overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}}$$

.

which is a representation with recurring expansion. □

Having known the minimum number of representations $\frac{m}{n}$ has, we naturally proceed to determine its maximum number of representations.

Theorem 2.35. *Let $\frac{m}{n} \in \mathbb{Q}$. If $\frac{m}{n}$ has recurring expansion, then $\frac{m}{n}$ has at most $\lfloor \frac{N_b-1}{N_b-D_b} \rfloor + 1$ recurring expansions.*

Proof. One can easily show that a fraction $\frac{m}{n} \in \mathbb{Q}$ cannot be both terminating expansion and recurring expansion.
Assume one has both terminating and recurring expansion. By Lemma 2.9(ii), we know

$$n = \prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} \prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j}$$

where $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$ and hence $\gcd\left((N_b)^n - (-D_b)^n, \prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} \prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j}\right) = 1$ as $\frac{m}{n}$ has terminating expansion.

However, we know that $\left(\frac{-N_b}{D_b}\right)^y \left(\frac{m}{n}\right)$ has a primary recurring expansion, and that by Theorem 2.6, we know that one can write that $\left(\frac{-N_b}{D_b}\right)^y \frac{m}{n} = \frac{v}{(D_b)^l}$ for some $v \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$.

By rearrangement, we have $m(-N_b)^y(D_b)^l = vn(D_b)^y$, and hence $n \mid m(-N_b)^y(D_b)^l$. But we know that $m \neq \prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} \prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j}$ for some $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$ because if it is, then $\frac{m}{n}$ would only have a terminating expansion, which contradicts with our assumption that $\frac{m}{n}$ has both terminating and recurring expansion.

Now, by our assumption of the theorem, we know that $\frac{m}{n}$ must not have terminating expansion when it already has a recurring one.

Thus, we write

$$n = q \prod_{\substack{\hat{p}|D_b \\ \hat{p} \text{ prime}}} \hat{p_i}^{\alpha_i} \prod_{\substack{\hat{q}|N_b \\ \hat{q} \text{ prime}}} \hat{q_j}^{\beta_j}$$

for some $\mathbb{N} \ni q \neq 1$, $\gcd(q, N_b D_b) = 1$ and $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$.

Then, by Case (ii) of Theorem 2.34, we have $y = \max\bigl(\text{ind}(1), \text{ind}(2), \ldots, \text{ind}(l')\bigr)$ where y has l' prime factors dividing N_b, and there exist $a', p' \in \mathbb{Z}$

$$\left(\frac{-N_b}{D_b}\right)^y \left(\frac{m}{n}\right) = \frac{\left[\frac{(-N_b)^y}{P_{Nb}}\right] m}{(D_b)^y P_{D_b}(q)} = \frac{a'}{(D_b)^y P_{D_b}} + \frac{p'}{q}$$

By Case (ii) of Theorem 2.34, we know that there exist $p' \equiv c \pmod{q}$ such that $\frac{a'}{(D_b)^y P_{D_b}} + \frac{p'}{q} = \frac{a'}{(D_b)^y P_{D_b}} - \frac{c-p'}{q} + \frac{c}{q}$, where $\frac{c}{q}$ has primary recurring expansion.

Write $\dfrac{a'}{(D_b)^y P_{D_b}} - \dfrac{c-p'}{q} = \dfrac{b}{(D_b)^y P_{D_b}} \in \mathbb{Z}[\dfrac{1}{D_b}]$

Rewriting

$$\left[\frac{(-N_b)^y}{P_{N_b}}\right] m = qb + (D_b)^y P_{D_b}(c) \tag{9}$$

Knowing that such c exists, we would like to let there be $\widetilde{b}$ and $\widetilde{c}$ respectively, in which $\frac{\widetilde{b}}{(D_b)^y P_{D_b}}$ forms a fraction having integer expansion, and $\frac{\widetilde{c}}{q}$ has a primary recurring expansion.
We would like to investigate the maximum possible values of $\widetilde{c}$ such that $\widetilde{c} \equiv c \pmod q$, and find all possible $\widetilde{c}$ such that the above is fulfilled. This is because, for any $\widetilde{b} \in \mathbb{Z}$, the fraction $\frac{\widetilde{b}}{(D_b)^y P_{D_b}}$ has integer expansion by Theorem 2.8.

In the following, by fixing $\frac{\widetilde{c}}{q}$ to have a primary recurring expansion, $\widetilde{c}$ has to fulfil the following conditions such that $\frac{\widetilde{c}}{q}$ has primary recurring expansion.

Consider Equation (9).

Taking modulo q,

$$\left[\frac{(-N_b)^y}{P_{N_b}}\right] m \equiv (D_b)^y P_{D_b}(\widetilde{c}) \pmod q$$

By rearranging,

$$\widetilde{c} \equiv m \left[\frac{(-N_b)^y}{P_{N_b}}\right] (D_b)^{-y} \left(P_{D_b}\right)^{-1} \pmod q \tag{10}$$

Knowing that $\frac{\widetilde{c}}{q}$ has a primary recurring expansion, by Lemma 2.10, we know

$$\widetilde{c} \equiv 0 \pmod{D_b} \tag{11}$$

Consider (10) and (11), after applying Chinese remainder theorem, we know

$$\widetilde{c} \equiv u \pmod{D_b q} \tag{12}$$

where u is an integer such that $0 \le u < D_b q$.

However, by bound (B), we know all potential values of $\widetilde{c}$ such that $\frac{\widetilde{c}}{q}$ has primary recurring expansion lies on the following bound.

$$\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(-D_b)^2} q \le \widetilde{c} \le \frac{(N_b-1)(D_b)^2}{(N_b)^2-(-D_b)^2} q$$

Now, we can show that the potential value of $\widetilde{c} \le \lfloor \frac{N_b-1}{N_b-D_b} \rfloor + 1$.

We can now apply Lemma 2.33, which shows the potential value of $\widetilde{c}$ such that there are at most $\lfloor \frac{N_b-1}{N_b-D_b} \rfloor + 1$ many $\frac{\widetilde{c}}{q}$.

That is, for $\widetilde{c}$ to be inbound, such $\widetilde{c}$ are in the form

$$\underbrace{\frac{-(N_b-1)(N_b)(D_b)-\alpha}{(N^b)^2-(-D_b)^2}q+u,\dots,u-D_bq,u,u+D_bq,\dots,\frac{(N_b-1)(D_b)^2-\beta}{(N^b)^2-(-D_b)^2}q+u}_{\text{at most }\lfloor\frac{N_b-1}{N_b-D_b}\rfloor+1\text{ terms}}$$

where α is the smallest positive integer such that

$$\begin{cases}-(N_b-1)(N_b)(D_b)-\alpha\equiv 0 \pmod{(N_b)^2-(-D_b)^2}\\ \alpha\equiv 0 \pmod{D_b}\end{cases}$$

and β is the smallest positive integer such that

$$\begin{cases}(N_b-1)(D_b)^2-\beta\equiv 0 \pmod{(N_b)^2-(-D_b)^2}\\ \beta\equiv 0 \pmod{D_b}\end{cases}$$

Note that both congruence simply ensure both $\frac{-(N_b-1)(N_b)(D_b)-\alpha}{(N^b)^2-(-D_b)^2}$ and $\frac{(N_b-1)(D_b)^2-\beta}{(N^b)^2-(-D_b)^2}$ are integer and that both $\frac{-(N_b-1)(N_b)(D_b)-\alpha}{(N^b)^2-(-D_b)^2}q$ and $\frac{(N_b-1)(D_b)^2-\beta}{(N^b)^2-(-D_b)^2}q$ are multiples of D_bq, and that both α and β are well-defined.

$\frac{\widetilde{c}}{q}$ has exactly $\lfloor\frac{N_b-1}{N_b-D_b}\rfloor+1$ many primary recurring expansions occurs when all the inbound residues $\widetilde{c}$, after having been divided by q, have primary recurring expansion.

That is, all

$$\frac{\frac{-(N_b-1)(N_b)(D_b)-\alpha}{(N^b)^2-(-D_b)^2}q+u}{q},\dots,\frac{u-D_bq}{q},\frac{u}{q},\frac{u+D_bq}{q},\dots,\frac{\frac{(N_b-1)(D_b)^2-\beta}{(N^b)^2-(-D_b)^2}q+u}{q}$$

have primary recurring expansions. □

Then, we have found the minimum and maximum numbers of terminating, recurring expansions $\frac{m}{n}$, it will be interesting to go a step further and define the procedure for finding all terminating or recurring expansions of $\frac{m}{n}$.

Procedure for finding all terminating or recurring expansion of $\frac{m}{n}$

Consider $m\in\mathbb{Z}, n\in\mathbb{N}$ where $\gcd(m,n)=1$.

Case 1 If

$$y=\prod_{\substack{\hat{p}|D_b\\ \hat{p}\text{ prime}}}\hat{p}_i^{\alpha_i}\prod_{\substack{\hat{q}|N_b\\ \hat{q}\text{ prime}}}\hat{q}_j^{\beta_j}$$

for some $\alpha_i,\beta_i\in\mathbb{Z}_{\geq 0}$.

Then, we take $y=\max\big(\operatorname{ind}_{N_b}(q_1),\operatorname{ind}_{N_b}(q_2),\dots,\operatorname{ind}_{N_b}(q_{l'})\big)$ where y has l' prime factors dividing N_b.

Since we know $(\frac{-N_b}{D_b})^y\left(\frac{m}{n}\right)$ has integer expansion, we can refer to Theorem 2.6 for the procedure to find such integer expansion.

Case 2

$$n = q \prod_{\substack{\hat{p}|D_b \\ \hat{p}\text{ prime}}} \hat{p_i}^{\alpha_i} \prod_{\substack{\hat{q}|N_b \\ \hat{q}\text{ prime}}} \hat{q_j}^{\beta_j}$$

where $\mathbb{N} \ni q \neq 1$, $\gcd(q, N_b D_b) = 1$ and $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$.

Again, we take $y = \max\big(\mathrm{ind}_{N_b}(q_1), \mathrm{ind}_{N_b}(q_2), \ldots, \mathrm{ind}_{N_b}(q_{l'})\big)$ for some y has l' prime factors dividing N_b.

Then, there exist $b, c \in \mathbb{Z}$ such that

$$\left(\frac{-N_b}{D_b}\right)^y \left(\frac{m}{n}\right) = \frac{\left[\dfrac{(-N_b)^y}{P_{N b}}\right] m}{(D_b)^y P_{D_b}(q)} = \frac{b}{(D_b)^y P_{D_b}} + \frac{c}{q}$$

where $\frac{b}{(D_b)^y P_{D_b}}$ has integer expansion and $\frac{c}{q}$ has primary recurring expansion.

By taking the common denominator,

$$\left[\frac{(-N_b)^y}{P_{N_b}}\right] m = qb + (D_b)^y P_{D_b}(c) \tag{13}$$

By Theorem 2.35, the number of $\widetilde{c}$ such that $\frac{\widetilde{c}}{q}$ has primary recurring expansion is at most $\lfloor \frac{N_b - 1}{N_b - D_b} \rfloor + 1$ times. We would like to find all $\widetilde{c}$.

Use the necessary condition from Equation (10) and (11) at Theorem 2.35, we know such $\widetilde{c}$ must satisfy the following.

$$\begin{cases} \widetilde{c} \equiv m \left[\dfrac{(-N_b)^y}{P_{N_b}}\right] (D_b)^{-y} \left(P_{D_b}\right)^{-1} \pmod q \\ \widetilde{c} \equiv 0 \pmod{D_b} \end{cases}. \tag{C1}$$

Also, it must satisfy our bound (B).

$$\frac{-(N_b - 1)(N_b)(D_b)}{(N_b)^2 - (-D_b)^2} q \leq \widetilde{c} \leq \frac{(N_b - 1)(D_b)^2}{(N_b)^2 - (-D_b)^2} q \tag{C2}$$

Denote $\mathcal{C}_q$ be the set of $\widetilde{c}$ that satisfy both (C1) and (C2).

The purpose of the new set $\mathcal{C}_q$ is to eliminate values of $\widetilde{c}$ where it is impossible for $\frac{\widetilde{c}}{q}$ to have primary recurring expansion. Hence, it allows us to reduce the time needed to test whether $\frac{\widetilde{c}}{q}$ has primary recurring expansion

We will now verify the validity of potential primary recurring expansion $\frac{\widetilde{c}}{q}$ by input $\widetilde{c} = N_0$ in the **Reverse Algorithm**.

Pick $\widetilde{c}_i \in \mathcal{C}_q$ for all N_i

After applying **Reverse Algorithm** finitely many times individually. If there exist $j \in \mathbb{N}$ such that $N_0 = N_j$, then by Theorem 2.26(ii), we know such $\frac{\widetilde{c}_i}{q}$ has primary recurring expansion.

We know $\mathcal{J} = \left\{ \widetilde{c}_i \in \mathcal{C}_q \quad | \quad \frac{\widetilde{c}_i}{q} \text{ has primary recurring expansion} \right\}$.

Then we substitute all $p \in \mathcal{J}$ in (16) to find the value of b respectively.

Now we obtain a set of solutions

$$\mathbb{S} = \left\{ (b,p) \in \mathbb{Z}^2 \quad | \quad \left(\frac{-N_b}{D_b}\right)^y \left(\frac{m}{n}\right) = \frac{b}{(D_b)^y P_{D_b}} + \frac{p}{q} \right\}$$

where $\frac{b}{(D_b)^y P_{D_b}}$ has integer expansion and $\frac{p}{q}$ has primary recurring expansion.

Then, refer to Theorem 2.6 for the procedure for integer expansion. Also, refer to Theorem 2.26(ii) to find the digits of the primary recurring expansion $\frac{p}{q}$.

Now, we can simply divide $\left(\frac{-N_b}{D_b}\right)^y$ both sides and obtain

$$\frac{m}{n} = \left(\frac{-D_b}{N_b}\right)^y (d_t \dots d_0 \ . \ \overline{s_1 \dots s_n})_{\frac{-N_b}{D_b}}$$

Then, we would refer to the procedure in Theorem 2.34 to find the recurring expansion of $\frac{m}{n}$, and we are done.

Example. Consider a fixed composite base, say $\dfrac{-21}{16}$. We want to find the number of terminating expansion or recurring expansion for the fractions $\frac{-13}{1332}, \frac{-25}{1332}$ respectively.

Consider $\frac{m}{n} = \frac{-13}{1332}$.
We know, by prime factorization, that $1332 = 2^2 3^2 (37)$,

Take $y = max(\text{ind}_{N_b}(3)) = 2$
Then, there exist $b, c \in \mathbb{Z}$ such that

$$\left(\frac{-21}{16}\right)^2 \left(\frac{-13}{1332}\right) = \frac{\left[\frac{(-21)^2}{3^2}\right](-13)}{(16)^2 2^2 (37)} = \frac{b}{(16)^2 2^2} + \frac{c}{37}$$

where $\frac{b}{(16)^2 (2^2)}$ has integer expansion and $\frac{c}{37}$ has primary recurring expansion.

By taking common denominator,

$$-637 = (37)b + (16)^2 2^2 \ \ c \tag{14}$$

Now, we would like to find all $\widetilde{c}$ such that $\frac{\widetilde{c}}{q}$ has primary recurring expansion.

By our procedure above, we know such $\widetilde{c}$ must satisfy the following.

$$\text{(C1)} \qquad \begin{cases} \widetilde{c} \equiv -637(16)^{-2}\left(2^2\right)^{-1} \pmod{37} \\ \widetilde{c} \equiv 0 \pmod{16} \end{cases}.$$

This simplifies to

$$\text{(C1)} \qquad \begin{cases} \widetilde{c} \equiv 13 \pmod{37} \\ \widetilde{c} \equiv 0 \pmod{16} \end{cases}.$$

Apply Chinese Remainder Theorem, we have

$$\widetilde{c} \equiv 272 \pmod{592}$$

Also, it must satisfy our bound (B).

$$\text{(C2)} \qquad \frac{-(21-1)(21)(16)}{21^2-(-16)^2}(37) = -1344 \leq \widetilde{c} \leq \frac{(21-1)(16)^2}{21^2-(-16)^2}(37) = 1024$$

Then $\mathcal{C}_q = \{-912, -320, 272, 864\}$
We will now verify the validity of potential primary recurring expansion $\frac{\widetilde{c}}{q}$ by input $N_0 = \widetilde{c}$ in the **Reverse Algorithm**.

$\widetilde{c}_i$	-912	-320
	$N_1 = 272 = \frac{-16}{21}[(15)(37) - 912]$	$N_1 = -320 = \frac{-16}{21}[(20)(37) - 320]$
	$N_2 = -320 = \frac{-16}{21}((4)(37) + 272)$	
	$N_3 = -320 = \frac{-16}{21}((20)(37) - 320)$	
	$N_2 = N_3$	$N_0 = N_1$

$\widetilde{c}_i$	272	864
	$N_1 = -320 = \frac{-16}{21}((4)(37) + 272$	$N_1 = -912 = \frac{-16}{21}((9)(37) + 864)$
	$N_2 = -320 = \frac{-16}{21}((20)(37) - 320)$	$N_2 = 272 = \frac{-16}{21}((15)(37) - 912)$
		$N_3 = -320 = \frac{-16}{21}((4)(37) + 272)$
		$N_4 = -320 = \frac{-16}{21}((20)(37) - 320)$
	$N_1 = N_2$	$N_3 = N_4$

By **Reverse Algorithm**, we found out that only $\frac{-320}{37}$ has primary recurring expansion while the others are not.
Hence, we know $\mathcal{J} = \{-329\}$
Then we substitute all $p = -272$ in Equation (16), and obtain

$$\begin{aligned} -637 &= (37)b + (16)^2 2^2 \ \ (-320) \\ b &= 8839 \end{aligned}$$

Now we obtain a set of solutions

$$\mathbb{S} = \{(8839, -320)\}$$

where $\frac{8839}{(16)^2 2^2}$ has integer expansion and $\frac{-320}{37}$ has primary recurring expansion.

Then, refer to Theorem 2.6 for the procedure for integer expansion and we have $\frac{8839}{1024} = ([4]\ \ [5]\ \ [3]\ \ [13])_{\frac{-21}{16}}$ where $[d]$ denotes digit and $d \in \mathcal{D}$.

Also, refer to Theorem 2.26(ii), we know $\frac{-320}{37} = (0.\overline{[20]})_{\frac{-21}{16}}$.

Now, we can simply divide $\left(\frac{-21}{16}\right)^2$ both sides and obtain

$$\begin{aligned} \frac{-13}{1332} &= \left(\frac{-16}{21}\right)^2 ([4]\ \ [5]\ \ [3]\ \ [13]\ \ .\ \ \overline{[20]})_{\frac{-21}{16}} \\ &= \left([4]\ \ [5]\ \ .\ \ \ [3]\ \ [13]\ \ \overline{[20]}\right)_{\frac{-21}{16}} \end{aligned}$$

This solution is the only solution on terminating or recurring expansion for $\frac{-25}{1332}$ in base$\frac{-21}{16}$.

Consider $\frac{m}{n} = \frac{-25}{1332}$ Take $y = max(\text{ind}_{N_b}(3)) = 2$. Then, there exist $b, c \in \mathbb{Z}$ such that

$$\left(\frac{-21}{16}\right)^2 \left(\frac{-25}{1332}\right) = \frac{\left[\frac{(-21)^2}{3^2}\right](-25)}{(16)^2 2^2 (37)} = \frac{b}{(16)^2 2^2} + \frac{c}{37}$$

where $\frac{b}{(16)^2(2^2)}$ has integer expansion and $\frac{c}{37}$ has primary recurring expansion.

By taking common denominator,

$$-1225 = (37)b + (16)^2 2^2 \ \ c \tag{15}$$

Following our procedure, we know such $\widetilde{c}$ must satisfy the following.

$$\text{(C1)} \qquad \begin{cases} \widetilde{c} \equiv -1225(16)^{-2}\left(2^2\right)^{-1} \pmod{37} \\ \widetilde{c} \equiv 0 \pmod{16} \end{cases}.$$

This simplifies to

$$\text{(C1)} \qquad \begin{cases} \widetilde{c} \equiv 25 \pmod{37} \\ \widetilde{c} \equiv 0 \pmod{16} \end{cases}.$$

Apply Chinese Remainder Theorem, we have

$$\widetilde{c} \equiv 432 \pmod{592}$$

Also, it must satisfy

$$\text{(C2)} \qquad -\frac{-(21-1)(21)(16)}{21^2-(-16)^2}(37) = -1344 \le \widetilde{c} \le \frac{(21-1)(16)^2}{21^2-(-16)^2}(37) = 1024$$

Then $\mathcal{C}_q = \{-1344, -752, -160, 432, 1024\}$
We will now verify the validity of potential primary recurring expansion $\frac{\widetilde{c}}{q}$ by input $N_0 = \widetilde{c}$ in the **Reverse Algorithm**.

$\widetilde{c}_i$	-1344	-752
	$N_1 = 1024 = \frac{-16}{21}[(0)(37) - 1344]$	$N_1 = 432 = \frac{-16}{21}[(5)(37) - 752]$
	$N_2 = -1344 = \frac{-16}{21}[(20)(37) + 1024]$	$-752 = \frac{-16}{21}[(15)(37) + 432]$
	$N_0 = N_2$	$N_0 = N_2$

$\widetilde{c}_i$	-160	432
	$N_1 = -160 = \frac{-16}{21}((10)(37) - 160)$	$N_1 = -752 = \frac{-16}{21}((15)(37) + 432)$
	$N_2 = -320 = \frac{-16}{21}((20)(37) - 320)$	$N_2 = 432 = \frac{-16}{21}((5)(37) - 752)$
	$N_0 = N_1$	$N_0 = N_2$

$\widetilde{c}_i$	1024
	$N_1 = -1344 = \frac{-16}{21}((20)(37) + 1024)$
	$N_2 = -1024 = \frac{-16}{21}((0)(37) - 1344)$
	$N_0 = N_2$

By our **Reverse Algorithm**, we found out all $\frac{-1344}{37}, \frac{-752}{37}, \frac{-160}{37}, \frac{432}{37}, \frac{1024}{37}$ has primary recurring expansion.

Hence, we know $\mathcal{J} = \{-1344, -752, -160, 432, 1024\}$
Then we substitute all $p \in \mathcal{J}$ in Equation (16), and obtain a set of solution

$$\mathbb{S} = \{(37163, -1344), (20779, -752), (4395, -160), (-11989, 432), (-28373, 1024)\}$$

where $\frac{b}{(16)^2 2^2}$ has integer expansion and $\frac{p}{37}$ has primary recurring expansion.

Now, we could easily deduce

First solution

$$\frac{-25}{1332} = \left(\frac{-21}{16}\right)^2 \left[\frac{37163}{(16)^2 2^2} + \frac{-1344}{37}\right]$$
$$= \left([16]\ [9]\ [1]\ .\ [5]\ [14]\ \overline{[20]\ [0]}\right)_{\frac{-21}{16}}$$

Second solution

$$\frac{-25}{1332} = \left(\frac{-21}{16}\right)^2 \left[\frac{20779}{(16)^2 2^2} + \frac{-752}{37}\right]$$
$$= \left([4]\ [6]\ .\ [0]\ [19]\ \overline{[15]\ [5]}\right)_{\frac{-21}{16}}$$

Third solution

$$\frac{-25}{1332} = \left(\frac{-21}{16}\right)^2 \left[\frac{4395}{(16)^2 2^2} + \frac{-160}{37}\right]$$
$$= \left([4]\ [6]\ .\ [0]\ [3]\ \overline{[10]}\right)_{\frac{-21}{16}}$$

Fourth solution

$$\frac{-25}{1332} = \left(\frac{-21}{16}\right)^2 \left[\frac{-11989}{(16)^2 2^2} + \frac{432}{37}\right]$$
$$= \left([4]\ [6]\ .\ [16]\ [8]\ \overline{[5]\ [15]}\right)_{\frac{-21}{16}}$$

Fifth solution

$$\frac{-25}{1332} = \left(\frac{-21}{16}\right)^2 \left[\frac{-28373}{(16)^2 2^2} + \frac{1024}{37}\right]$$
$$= \left([20]\ [11]\ .\ [11]\ [13]\ \overline{[0]\ [20]}\right)_{\frac{-21}{16}}$$

where $[d]$ means digit where $d \in \mathcal{D}$.

These are all the terminating or recurring expansion for $\frac{-25}{1332}$ in base $\frac{-21}{16}$.

In our paper, we will focus on the primary recurring expansions. To make our analysis easier, we aim to identify the potential values of p that result in a primary recurring expansion for $\frac{p}{q}$, rather than examining each expansion individually.

Procedure for finding all possible value of p such that $\frac{p}{q}$ has primary

recurring expansion given fixed q

By Lemma 2.10(iv), we know that if $\frac{p}{q}$ has primary recurring expansion , then $\gcd(q, N_b D_b) = 1$.

For a fraction $\frac{p}{q}$ to have a primary recurring expansion, we have two necessary conditions for such $\frac{p}{q}$.
From Lemma 2.10(iii),

$$p \equiv 0 \pmod{D_b} \tag{D1}$$

Also, recall form Lemma 2.10(i), $\frac{p}{q}$ must satisfy

$$\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(-D_b)^2}q \leq p \leq \frac{(N_b-1)(D_b)^2}{(N_b)^2-(-D_b)^2}q \tag{D2}$$

Denote $\mathbb{V} = \{p \mid p \text{ satisfy D1 and D2 and } q \nmid p\}$.

Pick an arbitrary element $p \in \mathbb{V}$. Take $N_0 = p$ and apply the **Reverse Algorithm** finitely many times, until there exists $j > i \geq 0$ such that $N_i = N_j$. Then by Theorem 2.27, we know the fractions $\frac{N_i}{q}, \ldots \frac{N_{j-1}}{q}$ have distinct primary recurring expansions. Also by Remark 2.28, we know $\frac{N_u}{q}$ does not have primary recurring expansion for all $0 < u < i$.

Denote $\mathbb{V}_1 = \{N_0, \ldots, N_{j-1}\}$
Then pick an element $p \in \mathbb{V} \setminus \mathbb{V}_1$
Then repeat the process taking for $N_0 = p$ and apply it to the **Reverse Algorithm** finitely many times. Then, we can again determine whether each $\frac{N_0}{q}, \ldots, \frac{N_{j-1}}{q}$ has primary recurring expansion.

Again, we denote $\mathbb{V}_2 = \{N_0, \ldots, N_{j-1}\}$

Repeat the above process of picking $p = N_0 \in \mathbb{V} \setminus (\mathbb{V}_1 \cup \mathbb{V}_2 \cup \cdots \cup \mathbb{V}_k)$, applying **Reverse Algorithm** and taking $\mathbb{V}_{k+1} = \{N_0, \ldots, N_{j-1}\}$ out of $\mathbb{V}$; until $\mathbb{V} \setminus \bigcup_{i=1}^{l} \mathbb{V}_i = \emptyset$ for some $l \in \mathbb{N}$.

We have found all primary recurring expansion for a fixed q. By Definition 2.11, the numerators N_i are in the set $\mathcal{P}_q$

Example. Consider base $\frac{-4}{3}$. We would like to find all fractions having primary recurring expansion for $q = 13$.

We consider the necessary condition for a primary recurring expansion $\frac{p}{13}$.

By the procedure above,

$$p \equiv 0 \pmod 3 \tag{D1}$$

and

$$\text{(D2)} \qquad \frac{-468}{7} \leq p \leq \frac{351}{7}$$

By definition, $\mathbb{V} = \{-66, -63, -60 \ldots, -42, -36, -3, 3, \ldots, 36, 42, 45, 48\}$.
Note that, by definition, $-39, 0, 39 \notin \mathbb{V}$ as they are divisible by 13.
Pick an arbitrary $p \in \mathbb{V}$, say -51. Take $N_0 = -51$ and apply the **Reverse Algorithm** until $N_i = N_j$ for some $j > i \geq 0$.

i	0	1	2	3	4	5	6	7
N_i	-51	9	-36	27	-30	3	-12	9

Note that $N_1 = N_7$, we perform the **Reverse Algorithm** 7 times and we have found all the primary recurring expansion for this input N_0. We know that $\frac{-51}{13}$ does not have primary recurring expansion, while the fractions $\frac{9}{13}.\frac{-36}{13}, \frac{27}{13}, \frac{-30}{13}, \frac{3}{13}, \frac{-12}{13}$ all have primary recurring expansion.

Now, pick $p \in \mathbb{V} \setminus \mathbb{V}_1 = \mathbb{V} \setminus \{-51, \ldots, -12\}$.

Say, we pick $p = N_0 = -60$ and apply the **Reverse Algorithm** until $N_i = N_j$ for some $j > i \geq 0$.

i	0	1	2	3	4	5	6	7	8	9
N_i	-60	45	-63	18	-33	-15	-21	6	-24	18

Note that $N_3 = N_9$, we perform the **Reverse Algorithm** 9 times and we have found all the primary recurring expansion for this input N_0. We know that $\frac{-60}{13}, \frac{45}{13}, \frac{-63}{13}$ doesn't have primary recurring expansion, while the fractions

$$\frac{18}{13}.\frac{-33}{13}, \frac{-15}{13}, \frac{-21}{13}.\frac{6}{13}, \frac{-24}{13}$$

all have primary recurring expansion.

Now, we would conduct a similar process as above, and pick $p \in \mathbb{V} \setminus (\mathbb{V}_1 \cup \mathbb{V}_2)$ and input to the **Reverse Algorithm**.
To save redundant calculation and space, in the following table, we collect all choices of N_0, all the element N_i that occur after N_0 is applied to the **Reverse Algorithm** and their respective set $\mathbb{V}_k$.
In the table, we denote PRE as primary recurring expansion. The parenthesized and sub-scripted parts of the number denote its residue mod13.

Trial	Picked element	N_i where $\frac{N_i}{q}$ has PRE	N_i where $\frac{N_i}{q}$ is not PRE
1	$-51_{(1)}$	$9_{(9)}, -36_{(3)}, 27_{(1)}, -30_{(9)}, 3_{(3)}, -12_{(1)}$	$-51_{(1)}$
2	$-60_{(5)}$	$18_{(5)}, -33_{(6)}, 15_{(2)}, -21_{(5)}, 6_{(6)}, -24_{(2)}$	$-60_{(5)}, 45_{(6)}, -63_{(2)}$
3	$-66_{(12)}$	$-9_{(4)}, -3_{(10)}, -27_{(12)}$	$-66_{(12)}, 30_{(4)}, -42_{(10)}, 12_{(12)}$
4	$-57_{(8)}$	$-18_{(8)}, -6_{(7)}, -15_{(11)}$	$-57_{(8)}, 33_{(7)}, -54_{(11)}, 21_{(8)}, -45_{(7)}, 24_{(11)]}$
5	$42_{(3)}$	N/A	$42_{(3)}$
6	$48_{(9)}$	N/A	$48_{(9)}$
7	$-48_{(4)}$	N/A	$-48_{(4)}, 36_{(10)}$

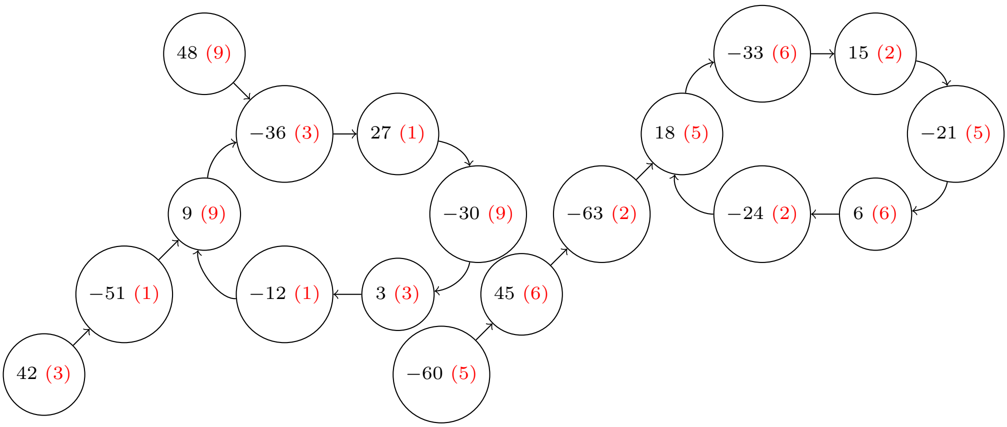
Where N/A implies that no new N_i such that $\frac{N_i}{q}$ has PRE is discovered with the corresponding input.
The following is a record of the set of $\mathbb{V}_k$ in chronological order of the picked element

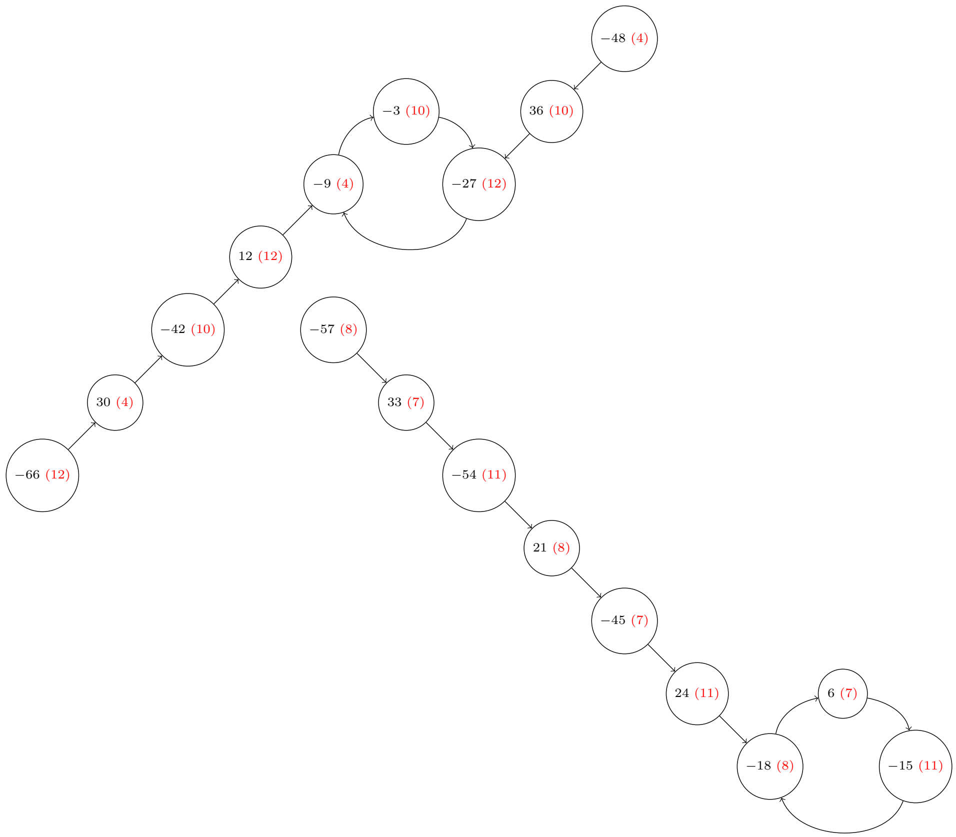
$$\begin{aligned}
\mathbb{V}_1 &= \{-51, 9, -36, 27, -30, 3, -12\} \\
\mathbb{V}_2 &= \{-60, 45, -63, 18, -33, 15, -21, 6, -24\} \\
\mathbb{V}_3 &= \{-66, 30, -42, 12, -9, -3, -27\} \\
\mathbb{V}_4 &= \{-57, 33, -54, 21, -45, 24, -18, -6, -15\} \\
\mathbb{V}_5 &= \{42, -51, 9, -36, 27, -30, 3, -12, 9\} \\
\mathbb{V}_6 &= \{48, -36, 27, -30, 3, -12\} \\
\mathbb{V}_7 &= \{-48, 36, -27, -9, -3\}
\end{aligned}$$

Now that we have every element $N_i \in \mathbb{V}$ has appeared and has been determined whether $\frac{N_i}{q}$ has primary recurring expansion, we have

$$\mathcal{P}_{13} = \{9, -36, 27, -30, 3, -12, 18, -33, 15, -21, 6, -24, -9, -3, -27, -18, -6, -15\}$$

, and we have classified all primary recurring expansion for a fixed $q = 13$.
Below, we provide a graphical representation of all the elements in $\mathbb{V}$, after being applied to the **Reverse Algorithm** for a fixed $q = 13$. The N_i with a "tail" is those $N_i \in \mathbb{V}$ such that $\frac{N_i}{q}$ does not have primary recurring expansion. Those N_i inside the "cycle" are those such that $\frac{N_i}{q}$ have primary recurring expansion.





Note that the above example provides a database of the primary recurring expansion $\mathcal{P}_{13}$. Now, we could determine the number of representations of the range of numerator given a fixed q in that base

Consider base $\frac{-4}{3}$. And we would like to find the number of representations for $m = 1, \ldots, 10$ given a fixed denominator $n = 454597 = (3)^2(4)^2(13)$.

We first apply the procedure similar to that of **Procedure for finding all terminating or recurring expansion of** $\frac{m}{n}$.

Notice I already have

$$\mathcal{P}_{13} = \{-36, -33, -30, -27, -24, -21, -18, -15, -12, -9, -6, -3, 3, 6, 9, 15, 18, 27\}$$

from above.

Consider $\frac{m}{n} = \frac{1}{(3)^2(4)^2(13)} = \frac{1}{1872}$.

Take $y = \max(\text{ind}_4(4)) = 2$

Then, there exist $b, c \in \mathbb{Z}$ such that

$$\left(\frac{-4}{3}\right)^2 \left(\frac{1}{(3)^2(4)^2(13)}\right) = \frac{\left[\frac{(-4)^2}{(4)^2}\right](1)}{(3)^2(3)^2(13)} = \frac{b}{(3)^2(3)^2} + \frac{c}{13}$$

where $\frac{b}{(3)^4}$ has integer expansion and $\frac{c}{13}$ has primary recurring expansion.

By taking common denominator,

$$1 = (13)b + (3)^4 \ c \tag{16}$$

By our procedure above, we know such $\widetilde{c}$ must satisfy the following.

$$\begin{cases} \widetilde{c} \equiv (1)(3)^{-4} \pmod{13} \\ \widetilde{c} \equiv 0 \pmod{3} \end{cases} . \tag{C1}$$

This simplifies to

$$\begin{cases} \widetilde{c} \equiv 9 \pmod{13} \\ \widetilde{c} \equiv 0 \pmod{3} \end{cases} . \tag{C1}$$

Apply Chinese Remainder Theorem, we have

$$\widetilde{c} \equiv 9 \pmod{39}$$

Also, it must satisfy

$$\begin{aligned} \frac{-(4-1)(4)(3)}{4^2-(-3)^2}(13) &= \frac{-468}{7} = -66.\overline{857142} \\ \le \widetilde{c} \le \frac{(4-1)(3)^2}{4^2-(-3)^2}(13) &= \frac{351}{7} = 50.\overline{142857} \end{aligned} \tag{C2}$$

Then $\mathcal{C}_{13} = \{-30, 9, 48\}$
We will now verify the validity of potential primary recurring expansion $\frac{\widetilde{c}}{q}$ directly by checking how many of those $\mathcal{C}_{13}$ are in $\mathcal{P}_{13}$.
Now, we can verify whether something is primary recurring expansion without inputting $N_0 = \widetilde{c}$ in the **Reverse Algorithm** for every single time, which saves time.

Note that the Complete Residue Theorem guarantees there must exist a $\widetilde{c} \in \mathcal{C}_{13}$ such that $\widetilde{c}$ is also in $\mathcal{P}_{13}$.

By comparing $\mathcal{C}_{13}$ and $\mathcal{P}_{13}$, we find out that $-30, 9$ is in both sets. This meant that we know $\mathcal{J} = \{-30, 9\}$. This means that there is only one pair of $(b, \widetilde{c})$ satisfying

$$\left(\frac{-4}{3}\right)^2 \left(\frac{1}{(3)^2(4)^2(13)}\right) = \frac{b}{(3)^2(3)^2} + \frac{c}{13}$$

where $\frac{b}{(3)^4}$ has integer expansion and $\frac{\widetilde{c}}{13}$ has primary recurring expansion.

Then we substitute all $p \in \mathcal{J}$ in Equation (16), and obtain a set of solution $\mathbb{S}$.
Now, substituting $\widetilde{c} = 9$ in (16), $b = -56$. Substituting $\widetilde{c} = -30$ in (16), $b = 187$.
Thus, we have

$$\mathbb{S} = \{(-56, 9), (187, -30)\}$$

where $\frac{b}{3^4}$ has integer expansion and $\frac{\tilde{c}}{13}$ has primary recurring expansion.

Now, we could easily deduce the first and only solution.

First solution

$$\frac{1}{1872} = \left(\frac{-3}{4}\right)^2 \left[\frac{-56}{3^4} + \frac{9}{13}\right]$$
$$= \left([1]\ [2]\ [2]\ .\ [2]\ [0]\ \overline{[0]\ [1]\ [2]\ [1]\ [0]\ [3]}\right)_{\frac{-4}{3}}$$

Second solution

$$\frac{1}{1872} = \left(\frac{-3}{4}\right)^2 \left[\frac{187}{3^4} + \frac{-30}{13}\right]$$
$$= \left([1]\ [2]\ [2]\ .\ [2]\ [3]\ \overline{[1]\ [0]\ [3]\ [0]\ [1]\ [2]}\right)_{\frac{-4}{3}}$$

These are all the terminating or recurring expansion for $\frac{1}{1872}$ in base$\frac{-4}{3}$.
To save redundant calculation step, we will make a table listing the m, corresponding $\mathcal{C}_{13}$, its intersection with $\mathcal{P}_{13}$, which ultimately decides the number of terminating or recurring expansion for $\frac{m}{n}$.

m	$\mathcal{C}_{13}$	$\mathcal{C}_{13} \cap \mathcal{P}_{13}$	Number of recurring expansion
1	{-30,9,48}	{-30,9}	2
2	{-60,-21,18}	{-21,18}	2
3	{-51,-12,27}	{-12,27}	2
4	{-42,-3,36}	{-3}	1
5	{-33,6, 45}	{-33,6}	2
6	{-63,-24,15}	{-24,15}	2
7	{-54,-15,24}	{-15,}	1
8	{-45,-6,33}	{-6}	1
9	{-36,3,42}	{-36,3}	2
10	{-66,-27,12}	{-27}	1

3. Non-terminating, non-recurring expansion for $\frac{p}{q}$ in the Number System with Base $\frac{-N_b}{D_b}$

In this section, we will explore the non-terminating, non-recurring expansions for a fraction $\frac{p}{q}$, as they are relevant to its number of representations. The main theorem in this section aims to prove the existence of such expansions.
In order to obtain a non-terminating, non-recurring expansion for $\frac{p}{q}$ satisfying bound (B), we will provide, in this section, a way of picking a_i for $\frac{p}{q}$, which is achieved by inputting $\frac{p}{q}$ into a variation of the **Forward Algorithm** – the **Unconditional Forward Algorithm**.
The appearance of this variation has been indicated in Section 2.3.

Note that for a fraction $\frac{p}{q}$ satisfying bound (B), there are many different ways to achieve a non-terminating, non-recurring expansion. In the following, we provide a method of choosing a_i that we believe is the best in terms of the speed of convergence.

The method we use is to choose a_i such that $|r_i|$ is minimum. We could use a close form to describe that in the Lemma 3.5.

We will first define the **Unconditional Forward Algorithm**

Definition 3.1 (Unconditional Forward Algorithm)**.** Let $b = \frac{-N_b}{D_b}$ and we define the following procedure as the **Unconditional Forward Algorithm**:

Input: UnconAlg($p, q, \mathbb{A} = \{a_1, a_2, \ldots, a_k\}$*)*, where $p \in \mathbb{Z}, q \in \mathbb{N}$ and $a_i \in \mathcal{D}$
Output: $r_1, r_2, \ldots, r_k$

$$\begin{aligned} bp &= a_1 q + r_1 \\ br_1 &= a_2 q + r_2 \\ &\cdots \\ br_{k-2} &= a_{k-1} q + r_{k-1} \\ br_{k-1} &= a_k q + r_k \\ &\cdots \end{aligned}$$

We know that

$$\frac{p}{q} = (0.a_1 \ldots a_k)_{\frac{-N_b}{D_b}} + \left(\frac{-D_b}{N_b}\right)^k \frac{r_k}{q}$$

Is different from the original **Forward Algorithm**; as defined in Definition 3.1, we have forgone the stipulation $a_i q + r_i \equiv 0 \pmod{D_b}$, hence r_i does not necessarily have to be an integer. However, we restrict r_i such that $\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}(q) < r_i < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}(q)$ for all $i \geq 1$
That is, we no longer limit ourselves to finding terminating expansion or recurring expansion for $\frac{p}{q}$, but also non-terminating and non-recurring expansion.

We will now define the necessary lemma and corollary. We want to establish that r_k is bounded for all $k \in \mathbb{N}$ under certain circumstances.

Definition 3.2. We call $\frac{p}{q}$ has a integer-free expansion if it has an expansion in the form of $\frac{p}{q} = (0.a_1 a_2 \ldots)_{\frac{-N_b}{D_b}}$ or $(0.a_1 a_2 \ldots a_k)_{\frac{-N_b}{D_b}}$ for some $k \geq 1$

Lemma 3.3. *If $\frac{p}{q}$ has an integer-free expansion, then $\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2} < \frac{p}{q} < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}$, disregarding whether the expansion terminates or recurs.*

Proof. The following proof is very similar to Lemma 2.10 (i)
The upper bound of the value of an **integer-free expansion** is

$$\left(0.\overline{[0]\ [N_b-1]}\ \right)_{\frac{-N_b}{D_b}} = \frac{(N_b-1)(\frac{-N_b}{D_b})^{-2}}{1-(\frac{-N_b}{D_b})^{-2}} = \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}$$

by sum of geometric series with common ratio $(\frac{-D_b}{N_b})^2$. We obtain $\left(0.\overline{[0][N_b-1]}\right)_{\frac{-N_b}{D_b}}$ by maximizing the positive component (digits with even power) and minimising the negative component (digits with odd power).
The lower bound of the value of an **integer-free expansion** is

$$(0.\overline{[N_b-1][0]})_{\frac{-N_b}{D_b}} = \frac{(N_b-1)(\frac{-N_b}{D_b})^{-1}}{1-(\frac{-N_b}{D_b})^{-2}} = \frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}$$

by sum of geometric series with common ratio $(\frac{-D_b}{N_b})^2$. We obtain $(0.\overline{[N_b-1][0]})_{\frac{-N_b}{D_b}}$ by minimizing the positive component (digits with even power) and maximizing the negative component (digits with odd power). ☐

Lemma 3.4. *If $\frac{p}{q}$ has an integer-free expansion $(0.a_1a_2\ldots a_k\ldots)_{\frac{-N_b}{D_b}}$, after performing the **Unconditional Forward Algorithm** with the input $UnconAlg(p,q,\mathbb{A}=\{a_1,a_2,\ldots,a_k,\ldots\})$, then,*

- *(i) The fraction $\frac{r_i}{q}$ has an integer-free expansion for all $i\geq 1$.*
- *(ii) $\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2} < \frac{r_i}{q} < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}$ for all $i\geq 1$*

Proof.
Proof of (i) Assume $\frac{p}{q}=(0.a_1a_2\ldots a_k\ldots)_{\frac{-N_b}{D_b}}$. By Definition 3.1, $\frac{p}{q}=(0.a_1)_{\frac{-N_b}{D_b}}+\left(\frac{-D_b}{N_b}\right)\frac{r_1}{q}$.
We have

$$\begin{aligned}(0.a_1a_2\ldots a_k\ldots)_{\frac{-N_b}{D_b}} &= (0.a_1)_{\frac{-N_b}{D_b}}+\left(\frac{-D_b}{N_b}\right)\frac{r_1}{q}\\ (0.a_1)_{\frac{-N_b}{D_b}}+(0.0a_2\ldots a_k\ldots)_{\frac{-N_b}{D_b}} &= (0.a_1)_{\frac{-N_b}{D_b}}+\left(\frac{-D_b}{N_b}\right)\frac{r_1}{q}\\ (0.a_2\ldots a_k\ldots)_{\frac{-N_b}{D_b}}\left(\frac{-D_b}{N_b}\right) &= \left(\frac{-D_b}{N_b}\right)\frac{r_1}{q}\\ \frac{r_1}{q} &= (0.a_2\ldots a_k\ldots)_{\frac{-N_b}{D_b}}\end{aligned}$$

Therefore, $\frac{r_1}{q}$ has an integer-free expansion. Similarly, by the same token, if $\frac{r_1}{q}$ has an integer-free expansion, then $\frac{r_2}{q}$ has an integer-free expansion. By induction, we can conclude that $\frac{r_i}{q}$ has an integer-free expansion for all $i\geq 1$.

Proof of (ii)

By Lemma 3.4, $\frac{r_i}{q}$ has an integer-free expansion for all $i \geq 1$. Combining with the findings in Lemma 3.3, we can conclude $\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2} < \frac{r_i}{q} < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}$ for all $i \geq 1$

□

This is the reason we restrict $\frac{-(N_b-1)N_b(D_b)}{(N_b)^2-(D_b)^2}(q) < r_i < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}(q)$ for all $i \geq 1$ in the **Unconditional Forward Algorithm**.

Lemma 3.5. *Let $b = \frac{-N_b}{D_b}$. It is given $\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}q < r_i < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q$ for some $i > 0$.*

*If we perform the **Unconditional Forward Algorithm** for $\frac{r_i}{q}$ where*

$$a_{i+1} = \begin{cases} 0 \text{ if } r_i > 0 \\ \min\left[\lfloor \frac{br_i + \frac{q}{2}}{q} \rfloor, N_b - 1\right] \text{ if } r_i \leq 0 \end{cases}$$

then $\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}q < r_{i+1} < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q$.

Proof. Recall $br_i = a_{i+1}q + r_{i+1}$, or equivalently $r_{i+1} = br_i - a_{i+1}q$.
Now, we would split r_i into two different cases $r_i \geq 0$ and $r_i < 0$.
We want to show that $\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}q < r_{i+1} < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q$ for both cases.

Case 1 When

$$0 \leq r_i < \frac{(N_b - 1)(D_b)^2}{(N_b)^2 - (D_b)^2}q$$

By assumption, take $a_{i+1} = 0$.
We know

$$0 \geq br_i > \left(\frac{-N_b}{D_b}\right)\left(\frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}\right)q = \frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}q$$
$$\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}q < r_{i+1} = br_i \leq 0$$

Case 2 When

$$\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}q < r_i < 0$$

Multiplying both sides by b, we know

$$\frac{(N_b-1)(N_b)^2}{(N_b)^2-(D_b)^2}q = \left(\frac{-N_b}{D_b}\right)\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}q > br_i > 0 \tag{17}$$

By assumption, take $a = \min\left[\lfloor \frac{br_i + \frac{q}{2}}{q} \rfloor, N_b - 1\right]$.

Now, we may split into two more cases to analyze.

Case 2a If $br_i - (N_b - 1)q \geq \frac{-q}{2}$.
Then

$$\frac{br_i + \frac{q}{2}}{q} \geq \frac{(N_b - 1)q}{q} = N_b - 1$$

But we know $N_b - 1$ is integer, hence

$$\left\lfloor \frac{br_i + \frac{q}{2}}{q} \right\rfloor \geq \frac{(N_b - 1)q}{q} = N_b - 1$$

Then, we take $a_{i+1} = \min\left[\lfloor \frac{br_i + \frac{q}{2}}{q} \rfloor, N_b - 1\right] = N_b - 1$.

Then, apply Inequality (17)

$$\begin{aligned} r_{i+1} &= br_i - a_{i+1}q = br_i - (N_b - 1)q \\ &< \frac{-(N_b - 1)(N_b)^2}{(N_b)^2 - (D_b)^2} q - (N_b - 1)q \\ &= \frac{(N_b - 1)(D_b)^2}{(N_b)^2 - (D_b)^2} q \end{aligned}$$

Notice

$$\begin{aligned} r_i &= br_i - a_{i+1}q \\ &\geq \frac{-q}{2} \\ &= (-1)(1)(\frac{1}{2})q \\ &> (-1)\left[\frac{(N_b - 1)(D_b)}{N_b - D_b}\right]\left[\frac{N_b}{N_b + D_b}\right] q \\ &= \frac{-(N_b - 1)(N_b)(D_b)}{(N_b)^2 - (D_b)^2} q \end{aligned}$$

Case 2b If $br_i - (N_b - 1)q < \frac{-q}{2}$.
Then

$$\frac{br_i + \frac{q}{2}}{q} < \frac{(N_b - 1)q}{q} = N_b - 1$$

Obviously, $a_{i+1} = \min\left[\lfloor \frac{br_i + \frac{q}{2}}{q} \rfloor, N_b - 1\right] = \lfloor \frac{br_i + \frac{q}{2}}{q} \rfloor$.

Then,

$$\begin{aligned} r_{i+1} &= br_i - a_{i+1}q \\ &= br_i - \left(\left\lfloor \frac{br_i + \frac{q}{2}}{q} \right\rfloor\right) q \\ &\geq br_i - \left[br_i + \frac{q}{2}\right] \\ &= (-1)(1)(\frac{1}{2})q \\ &> (-1)\left[\frac{(N_b-1)(D_b)}{N_b - D_b}\right]\left[\frac{N_b}{N_b + D_b}\right] q \\ &= \frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2 - (D_b)^2} q \end{aligned}$$

It is obvious that $br_i - a_{i+1}q < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(-D_b)^2}$.

Combining Case 2a and Case 2b,

$$\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2 - (D_b)^2} q < r_{i+1} < \frac{(N_b-1)(D_b)^2}{(N_b)^2 - (D_b)^2} q,$$

which concludes our proof. □

Corollary 3.6. *Given that* $\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2} q < p < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2} q$.

If we perform the ***Unconditional Forward Algorithm*** *for* $\frac{r_i}{q}$ *where*

$$a_{i+1} = \begin{cases} 0 \text{ if } r_i > 0 \\ \min\left[\lfloor \frac{br_i + \frac{q}{2}}{q} \rfloor, N_b - 1\right] \text{ if } r_i \leq 0 \end{cases}$$

. Then $\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2} q < r_k < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2} q$ *for all* $k \in \mathbb{N}$.

Proof. If p is in (B), then by Lemma 3.5, r_1 is also in (B) by choosing

$$a_i = \begin{cases} 0 \text{ if } r_i > 0 \\ \min\left[\lfloor \frac{br_i + \frac{q}{2}}{q} \rfloor, N_b - 1\right] \text{ if } r_i \leq 0 \end{cases}.$$

Then we can easily show by induction that

$$\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2 - (D_b)^2} q < r_k < \frac{(N_b-1)(D_b)^2}{(N_b)^2 - (D_b)^2} q$$

for all $k \in \mathbb{N}$. □

Having established the necessary preliminaries, we will now move on to proving the main theorem in this section.

Theorem 3.7. *Suppose* $\frac{p}{q}$ *is a fraction and* $\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2} q < p < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2} q$. *There exists an integer-free, non-terminating, non-recurring expansion for* $\frac{p}{q}$.

Proof. By Lemma 3.5, we know that for every $\frac{p}{q}$ in bound (B), combined with a specific way to pick $\mathbb{A} = \{a_1, \ldots, a_j, \ldots\}$; after applying to **Unconditional Forward Algorithm**, $(0.a_1 \ldots a_j \ldots)_{\frac{-N_b}{D_b}}$ forms a well-defined expansion.

Note that if, after performing **Unconditional Forward Algorithm**, we obtain digits a_i such that there $\frac{p}{q}$ has a primary recurring expansion. Then, we can easily find a second way to pick a_i.
Then, by remark 2.32, for the same fraction $\frac{p}{q}$, we know the second method of picking a_i must not be a primary recurring expansion.

Now, notice if we substitute in the **Unconditional Forward Algorithm**, we have

$$\frac{p}{q} = \lim_{k\to\infty}\left[(0.a_1 \ldots a_k)_{\frac{-N_b}{D_b}} + \left(\frac{-D_b}{N_b}\right)^k \frac{r_k}{q}\right] = \lim_{k\to\infty}\left[\sum_{i=1}^{\infty} a_i \left(\frac{-D_b}{N_b}\right)^i + \left(\frac{-D_b}{N_b}\right)^k \frac{r_k}{q}\right]$$

Combining the fact $|\frac{-D_b}{N_b}| < 1$ and $\frac{r_k}{q}$ is bounded, we know $\lim_{k\to\infty}\left(\frac{-D_b}{N_b}\right)^k \frac{r_k}{q} = 0$, which yield

$$\frac{p}{q} = \lim_{k\to\infty}(0.a_1 \ldots a_k)_{\frac{-N_b}{D_b}} = \lim_{k\to\infty}\left[\sum_{i=1}^{\infty} a_i \left(\frac{-D_b}{N_b}\right)^i\right]$$

Hence, we deduce $\frac{p}{q} = \lim_{k\to\infty}(0.a_1 \ldots a_k)_{\frac{-N_b}{D_b}}$ to be an integer-free, non-terminating, non-recurring expansion. □

Example. In the following sample, we will show that by the method of choosing a_i such that $|r_i|$ is minimum in Lemma 3.5, the values of a_i is picked such that $\frac{p}{q}$ converges quickly.
Consider the base $\frac{-7}{3}$, we would like to find an integer-free, non-terminating, non-recurring expansion for $\frac{-30}{11}$. We know that $\frac{-(7-1)(7)(3)}{(7)^2-(3)^2}(11) < -30 < \frac{(7-1)(3)^2}{7^2-(3)^2}(11)$. Thus, we can apply the method of choosing

$$a_{i+1} = \begin{cases} 0 \text{ if } r_i > 0 \\ \min\left[\lfloor\frac{br_i+\frac{q}{2}}{q}\rfloor, N_b - 1\right] \text{ if } r_i \leq 0 \end{cases}$$

$$r_1 = \left(\frac{-7}{3}\right)(-30) - 6\,(11) = 4 \qquad a_1 = 6$$

$$r_2 = \left(\frac{-7}{3}\right)(4) - 0\,(11) = \frac{-28}{3} \qquad a_2 = 0$$

$$r_3 = \left(\frac{-7}{3}\right)\left(\frac{-28}{3}\right) - 2\,(11) = \frac{-2}{9} \qquad a_3 = 2$$

$$r_4 = \left(\frac{-7}{3}\right)\left(\frac{-2}{9}\right) - 0\,(11) = \frac{14}{27} \qquad a_4 = 0$$

$$r_5 = \left(\frac{-7}{3}\right)\left(\frac{14}{27}\right) - 0\,(11) = \frac{-98}{81} \qquad a_5 = 0$$

$$r_6 = \left(\frac{-7}{3}\right)\left(\frac{-98}{81}\right) - 0\,(11) = \frac{686}{243} \qquad a_6 = 0$$

$$r_7 = \left(\frac{-7}{3}\right)\left(\frac{686}{243}\right) - 0\,(11) = \frac{-4802}{729} \qquad a_7 = 0$$

$$r_8 = \left(\frac{-7}{3}\right)\left(\frac{-4802}{729}\right) - 1\,(11) = \frac{9557}{2187} \qquad a_8 = 1$$

$$r_9 = \left(\frac{-7}{3}\right)\left(\frac{9557}{2187}\right) - 0\,(11) = \frac{-66899}{6561} \qquad a_9 = 0$$

$$r_{10} = \left(\frac{-7}{3}\right)\left(\frac{-66899}{6561}\right) - 2\,(11) = \frac{35267}{19683} \qquad a_{10} = 2$$

In fact, this could go on forever without terminating. However, we will stop now and take these ten digits as an approximation for $\frac{-30}{11}$.

$(0.6020000102)_{\frac{-7}{3}} = (6)(\frac{-7}{3})^{-1} + (2)(\frac{-7}{3})^{-3} + (1)(\frac{-7}{3})^{-8} + (2)(\frac{-7}{3})^{-10}$
$\approx (-2.727515819)_{10} \approx (-2.\overline{72})_{10} = \frac{-30}{11}$
We can see that this approximation has a certain error, but the error will approach 0 as the number of digits taken approaches to infinity.

Remark 3.8. *Note that choosing a_{i+1} such that $|r_{i+1}|$ is minimum is not the only way to produce an r_{i+1} such that $\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}q < p < \frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q$. In fact, when there are multiple a_{i+1} that leads to inbound r_{i+1}, after performing **Unconditional Forward Algorithm** infinitely many times, there will be infinitely many different sets of $\mathbb{A} = \{a_1, \dots, a_j, \dots\}$, i.e. there are infinitely many non-terminating non-recurring expansion for $\frac{p}{q}$. However, these other ways of picking a_{i+1} do not converge to our target fraction as quickly as the method stated in 3.5.*

For any base $N_b > D_b \geq 1$. If

$$\frac{-(N_b-1)(D_b)^3}{(N_b)(N_b)^2-(D_b)^2}q \leq r_i < \left(\frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2} - \frac{D_b}{N_b}\right)q$$

Multiplying by b, we obtain

$$\frac{(N_b-1)(D_b)^2}{(N_b)^2-(D_b)^2}q \geq br_i > \left(\frac{-(N_b-1)(N_b)(D_b)}{(N_b)^2-(D_b)^2}+1\right)q$$

Now, we have at least two choices of $a_{i+1}=0$ or $a_{i+1}=1$ to pick from, both which makes r_{i+1} satisfies bound (B).
Now, that we know for some $\frac{r_i}{q}$, there are infinitely many representations for the fraction $\frac{r_i}{q}$
Note that in other bases and other scenarios, there might be even more choices of a_i that make r_{i+1} satisfy bound (B).

4. Conclusion and Area for Further Research

Conclusion.
In conclusion, through exploring the fundamental properties of a number system with base $\frac{-N_b}{D_b}$, we have found the interesting properties of the **Complete residue System Theorem**, and that by adopting terminating and recurring expansions of $\frac{m}{n}$, the case in which there are infinite representations will not occur. Furthermore, we deduce the minimum and maximum numbers of representations to be 1 and $\lfloor\frac{N_b-1}{N_b-D_b}\rfloor+1$ respectively.

Area for Further Research.
The interesting properties that lead to the Complete Residue System Theorem has been discovered and well-developed in this paper. However, we did not find a necessary and sufficient condition on finding the numerators of fraction having primary recurring expansion given a fixed q. We believe that the numerators of primary recurring expansion does not acquire an obvious pattern. The necessary and sufficient conditions are very useful as one can quickly determine the numerators of fraction having primary recurring expansion. This raises the efficiency for testing the recurring expansion of $\frac{m}{n}$ it eliminates the need for the two necessary conditions (10) and (11) as described in **Procedure for finding all terminating or recurring expansion of $\frac{m}{n}$.**
While dealing with primary recurring expansions, we have discovered that the period of a primary recurring expansion has an interesting feature. Recall that $\beta \in \mathbb{Z}_q$ is an integer such that $\beta \equiv -N_b(D_b)^{-1} \pmod q$.

Conjecture 1. *For all $N_b > D_b \geq 1$. If $\frac{p}{q}$ has primary recurring expansion, then the period of $\frac{p}{q}$ is either $\operatorname{ord}_q(\beta)$ or $2\operatorname{ord}_q(\beta)$*

Although this is not the main focus on finding number of representation, this conjecture is highly useful in related studies. For instance, if we know that $\frac{N_0}{q}$ has primary recurring expansion. Then, we know that we just need to perform the **Reverse Algorithm** for a maximum of $2\operatorname{ord}_q(\beta)$ times. We would also know that the cardinality of $\mathbb{C}_1,\ldots,\mathbb{C}_l$ would be $\mathrm{ord_q}(\beta)$ or $2\,\mathrm{ord_q}(\beta)$.

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5. Appendix

The following Python code gives all terminating and recurring expansion of a certain number $\frac{m}{n} \in \mathbb{Q}$ following **Procedure for finding all terminating or recurring expansion of** $\frac{m}{n}$ in Section 2.5. Users can input the desired base and fraction in the form of $\dfrac{m}{n}$. Users are also advised not to input numbers that are too large, for the sake of maximum recursion limits within looping functions.

The link to the code is provided below:
`https://drive.google.com/file/d/1gln-7T9yaz2glWXV7Y1tBV4NOss5Wyep/view?usp=drivesdk`

REVIEWERS' COMMENTS

The article focuses on the expansion of fractions on negative rational bases. The objective is to establish a finite number of recurring expansions using novel theories and algorithms. The authors introduce definitions and conditions for four types of expansions and present two distinct proofs for the Complete Residue System Theorem. The second main result outlines the bounds of terminating and recurring expansions in any number system and provides a method to compute all expansions for any fraction $\frac{m}{n}$.

One reviewer considers the report belongs to recreational mathematics, as unorthodox number systems were considered, yet there are several works along the direction of the report have appeared in the literature before, showing some interest from the community. Overall, the article apparently has sufficient research content, and is commendable as the work by high school students.