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ON THE SUMMATION OF FRACTIONAL PARTS AND ITS APPLICATION

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Abstract. The summation of fractional parts is an old topic in number theory since the time of G.H.Hardy and J.E.Littlewood (see [3]). Throughout the years, many mathematicians have contributed to the estimation of the sum $\sum_{n\leq N}\{\alpha n\}$, where α is an irrational number. In Section 2, we estimate the fractional part sum of certain non-linear functions, which can be applied to refine an existing bound of the discrepancy. In Section 3, we continue to make use of the sum in order to study the distribution of quadratic residues and 'relatively prime numbers' modulo integers.

1. Introduction

Definition 1. [5, p.1] *For a real number x, let $\lfloor x \rfloor$ denote the greatest integer $\leq x$. Then, the fractional part $\{x\}$ is defined as $x - \lfloor x \rfloor$.*

Definition 2. [5, p.1] *Let $x_1, \ldots, x_N$ be a sequence contained in $[0, 1)$. Then, for any subinterval $[\alpha, \beta)$ of $[0, 1)$, the counting function $A([\alpha, \beta); N)$ is defined as*

$$|\{x_1, \ldots, x_N\} \cap [\alpha, \beta)|.$$

Definition 3. [5, Chapter 2, Definition 1.1] *Let $x_1, \ldots, x_N$ be a sequence contained in $[0, 1)$. Then, the discrepancy D_N of the sequence is defined as*

$$\sup_{0\leq\alpha<\beta\leq 1} \left|\frac{A([\alpha, \beta); N)}{N} - (\beta - \alpha)\right|.$$

Definition 4. [5, Chapter 2, Definition 1.2] *Let $x_1, \ldots, x_N$ be a sequence contained in $[0, 1)$. Then, the discrepancy D_N^* of the sequence is defined as*

$$\sup_{0<\alpha\leq 1} \left|\frac{A([0, \alpha); N)}{N} - \alpha\right|.$$

Theorem 5 (**Koksma's Inequality**). [5, Chapter 2, Theorem 5.1, Example 5.1] *Let $f(x)$ be a continuously differentiable function on $[0,1]$ and suppose we are given N points $x_1,\dots,x_N$ in $[0,1)$ with discrepancy D_N^*. Then,*

$$|\frac{1}{N}\sum_{n\le N} f(x_n)-\int_0^1 f(t)dt|\le D_N^*\int_0^1 |f'(t)|dt.$$

Remark 6. *It shows that the summation of fractional parts is in fact closely related to the discrepancy.*

Theorem 7 (**Niederreiter**). [7, Theorem 4.1] *Let $f(x)$ be a strictly increasing function defined for $x\ge 1$ which has a continuous derivative for $x\ge x_0$. Furthermore, $\lim_{x\to\infty} f(x)=\infty, \lim_{x\to\infty} xf'(x)=\infty$, and $f'(x)$ tends monotonically to zero as $x\to\infty$. If D_N^* denotes the discrepancy of the partial sequence of fractional parts $\{f(1)\},\dots,\{f(N)\}$, then $D_N^*=O(f(N)/N)+O(1/Nf'(N))$.*

Remark 8. *From author's point of view, the part applying L'Hospital's Rule of the proof is not rigorous enough since the limit may not exist. The following theorem is the corrected form.*

Theorem 9. *Let $f(x)$ be a strictly increasing function defined for $x\ge 1$ which has a continuous derivative for $x\ge x_0$. Furthermore, $\lim_{x\to\infty} f(x)=\infty$, $f'(x)$ and $xf'(x)$ tend monotonically to zero and infinity respectively as $x\to\infty$. Then, $D_N^*=O(f(N)/N)+O(1/Nf'(N))$.*

Remark 10. *The proof is omitted since we will prove a stronger form (see Theorem 26).*

Theorem 11. *Let $f(x)$ be a function satisfying the conditions of Theorem 9. Then,*

$$\sum_{n\le N}\{f(n)\}=\frac{1}{2}N+O(f(N))+O(\frac{1}{f'(N)}).$$

Proof. It follows immediately from Theorem 5 and Theorem 9. □

Remark 12. *We will prove a stronger form (see Theorem 18).*

2. Distribution of Sequences Modulo One

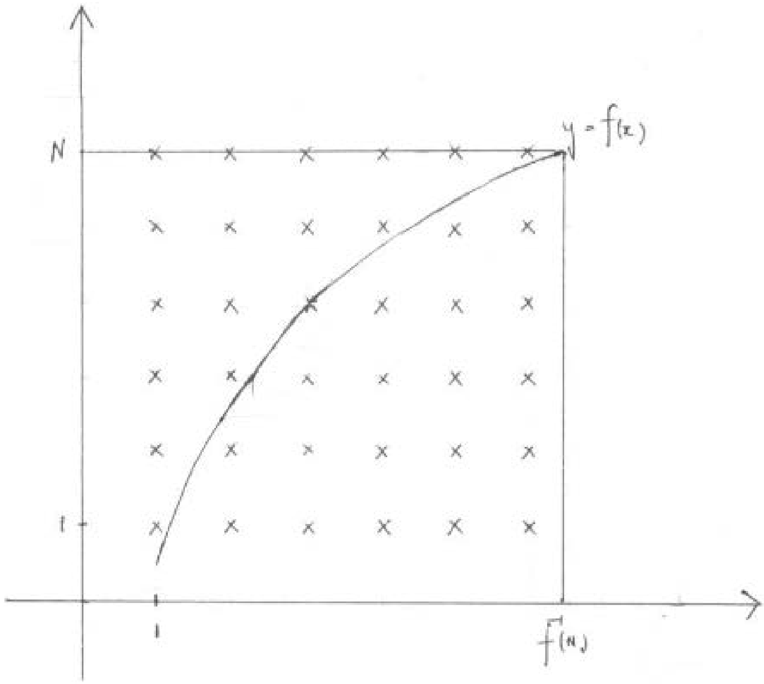
Lemma 13. *Let $f(x)$ be a strictly increasing, continuous function defined for $x\ge 1$ with $0\le f(1)<1$. Then, for all positive integers N,*

$$\sum_{n\le f^{-1}(N)}\lfloor f(n)\rfloor+\sum_{m\le N}\lfloor f^{-1}(m)\rfloor=Nf^{-1}(N)+O(N),$$

where f^{-1} denotes the inverse function of f.

[See reviewer's comment (2)]

Proof. We consider the figure below.



It shows that

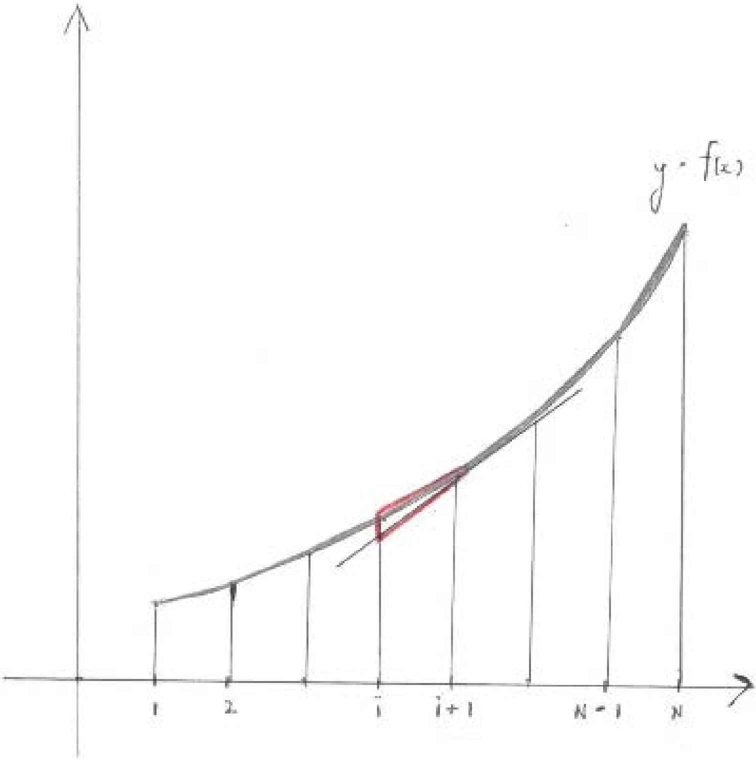
$$\sum_{n\le f^{-1}(N)} \lfloor f(n)\rfloor + \sum_{m\le N} \lfloor f^{-1}(m)\rfloor = N\lfloor f^{-1}(N)\rfloor + g(N),$$

where $g(N)$ denotes the number of positive integers $n \le N$ such that $f^{-1}(n)$ is an integer.
Obviously, $g(N) \le N$. Hence, the lemma follows. □

Lemma 14. *Let f be a function which is differentiable, increasing and convex on $[1, N]$ with $f(1) \ge 0$. Then,*

$$\sum_{n\le N} f(n) - \frac{1}{2}(f(1) + f(N)) - \int_1^N f(t)dt \le \frac{1}{2} f'(N).$$

Proof. We consider the figure below.



Notice that the shaded area representing the difference of the summation of trapezium and the integration is exactly

$$\sum_{n\le N} f(n) - \frac{1}{2}(f(1)+f(N)) - \int_1^N f(t)dt.$$

It remains to bound the shaded area between i and $i+1$ ($i = 1, 2, \ldots, N-1$). Obviously, the shaded area between i and $i+1$ is smaller than the red triangle. In fact,

$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2}(f(i) - (f(i+1) - f'(i+1))) \\ &= \frac{1}{2}(f'(i+1) - (f(i+1) - f(i))) \\ &\le \frac{1}{2}(f'(i+1) - f'(i)).\end{aligned}$$

The lemma then follows immediately from the telescoping sum from $i = 1$ to $N-1$. $\square$

Lemma 15. *Let f be a function satisfying the conditions of Theorem 9. Then,*

$$\lim_{x\to\infty}\frac{(f^{-1})'(x+1)}{(f^{-1})'(x)}=1.$$

Proof. Given that $xf'(x)$ tends monotonically to infinity as $x\to\infty$ and therefore for all $x\geq f(x_0), f^{-1}(x)f'(f^{-1}(x))=\frac{f^{-1}(x)}{(f^{-1})'(x)}$ tends monotonically to infinity as $x\to\infty$. It follows that $\frac{f^{-1}(x)}{(f^{-1})'(x)}\leq\frac{f^{-1}(x+1)}{(f^{-1})'(x+1)}$. or. equivalently,

$$\frac{(f^{-1})'(x+1)}{(f^{-1})'(x)}\leq\frac{f^{-1}(x+1)}{f^{-1}(x)}.$$

Thus, it remains to prove that $\lim_{x\to\infty}\frac{f^{-1}(x+1)}{f^{-1}(x)}=1$ since $\frac{(f^{-1})'(x+1)}{(f^{-1})'(x)}>1$.
By Lagrange's Mean Value Theorem,

$$\begin{aligned}\lim_{x\to\infty}\frac{f^{-1}(x+1)}{f^{-1}(x)}&=\lim_{x\to\infty}\exp(\log f^{-1}(x+1)-\log f^{-1}(x))\\&=\lim_{\xi\to\infty}\exp(\frac{(f^{-1})'(\xi)}{f^{-1}(\xi)})\\&=1,\end{aligned}$$

where $\xi\in(x,x+1)$. This completes the proof. □

Lemma 16. *Let $f(x)$ be a strictly increasing function defined for $x\geq 1$ which has a continuous derivative. Furthermore, $\lim_{x\to\infty}f(x)=\infty, f'(x)$ and $xf'(x)$ tend monotonically to zero and infinity respectively as $x\to\infty$. Then,*

$$\sum_{n\leq f^{-1}(N)}\{f(n)\}=\frac{1}{2}f^{-1}(N)+E(N)+O(N),$$

where $E(N):=\sum_{m\leq N}f^{-1}(m)-\frac{1}{2}(f^{-1}(1)+f^{-1}(N))-\int_1^N f^{-1}(t)dt$.

Proof. Without loss of generality, we can assume that $0\leq f(1)<1$ since $\{x\}$ is a function with period 1. We have

$$\sum_{n\leq f^{-1}(N)}\{f(n)\}=\sum_{n\leq f^{-1}(N)}f(n)-\sum_{n\leq f^{-1}(N)}\lfloor f(n)\rfloor.$$

By Lemma 13,

$$\sum_{n\leq f^{-1}(N)}\lfloor f(n)\rfloor+\sum_{m\leq N}\lfloor f^{-1}(m)\rfloor=Nf^{-1}(N)+O(N).$$

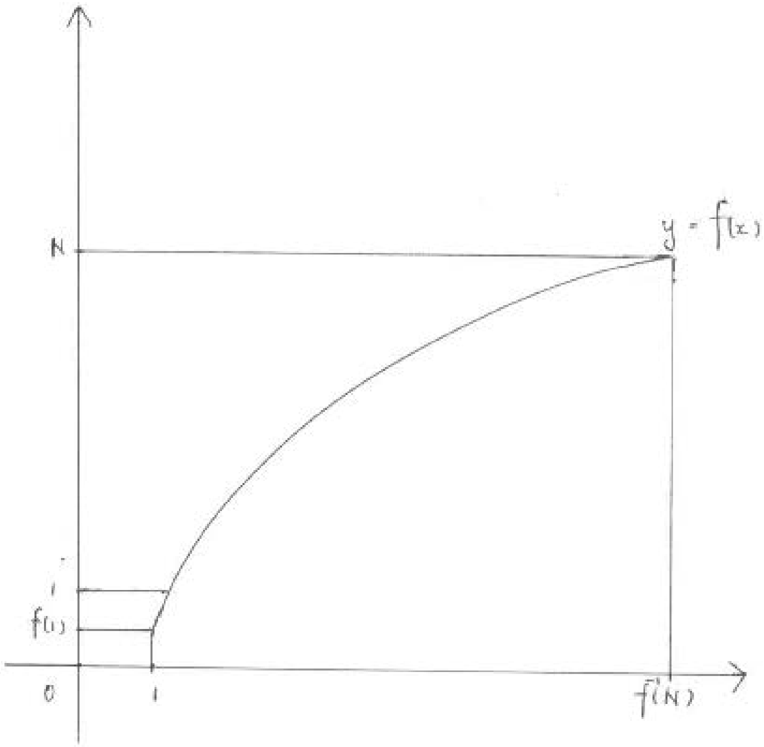
Therefore,

$$\begin{aligned}\sum_{n\le f^{-1}(N)} \{f(n)\} &= \sum_{n\le f^{-1}(N)} f(n) + \sum_{m\le N} f^{-1}(m) - Nf^{-1}(N) + O(N)\\ &= \sum_{n\le f^{-1}(N)} f(n) + \int_1^N f^{-1}(t)dt + \frac{1}{2}(f^{-1}(1)+f^{-1}(N))\\ &\quad + E(N) - Nf^{-1}(N) + O(N).\end{aligned}$$

By Euler's Summation Formula [1, Theorem 3.1],

$$\sum_{1<n\le f^{-1}(N)} f(n) = \int_1^{f^{-1}(N)} f(t)dt + \int_1^{f^{-1}(N)} \{t\}f'(t)dt + N\{f^{-1}(N)\} - f(1)\{1\}.$$

We consider the figure below.



It shows that

$$\int_1^{f^{-1}(N)} f(t)dt + \int_1^N f^{-1}(t)dt + \int_{f(1)}^1 f^{-1}(t)dt + f(1) = Nf^{-1}(N).$$

Hence,

$$\sum_{n\leq f^{-1}(N)}\{f(n)\} = Nf^{-1}(N) - f(1) - \int_{f(1)}^{1} f^{-1}(t)dt + O(\int_{1}^{f^{-1}(N)} f'(t)dt)$$
$$- Nf^{-1}(N) + \frac{1}{2}f^{-1}(N) + E(N) + O(N)$$
$$= \frac{1}{2}f^{-1}(N) + E(N) + O(f(f^{-1}(N)) - f(1)) + O(N)$$
$$= \frac{1}{2}f^{-1}(N) + E(N) + O(N).$$

This completes the proof. □

Lemma 17. *Let f be a function satisfying the conditions of Lemma 16. Then, there exists a constant $c > 0$ such that for all $\epsilon > 0$ and sufficiently large N (depend on ϵ), we have*

$$|\sum_{n\leq N}\{f(n)\} - \frac{1}{2}N| \leq \frac{1+\epsilon}{f'(N)} + cf(N).$$

Proof. Suppose that $f^{-1}(M) \leq N < f^{-1}(M+1)$, where M is a positive integer. Then,

$$\sum_{n\leq N}\{f(n)\} = \sum_{n\leq f^{-1}(M)}\{f(n)\} + \sum_{f^{-1}(M)<n\leq N}\{f(n)\}$$
$$= \frac{1}{2}f^{-1}(M) + E(M) + O(M) + \sum_{f^{-1}(M)<n\leq N}\{f(n)\}$$
$$= \frac{1}{2}N + E(M) + O(M) + \sum_{f^{-1}(M)<n\leq N}(\{f(n)\} - \frac{1}{2}).$$

Now, we bound the sum on the right hand side and we conclude that

$$|\sum_{f^{-1}(M)<n\leq N}(\{f(n)\} - \frac{1}{2})| \leq \frac{1}{2}(f^{-1}(M+1) - f^{-1}(M))$$
$$\leq \frac{1}{2}(f^{-1})'(M+1).$$

By Lemma 15, for all $\epsilon > 0$ and sufficiently large M (depend on ϵ), we have

$$(f^{-1})'(M+1) \leq (1+\epsilon)(f^{-1})'(M).$$

Therefore,

$$\sum_{f^{-1}(M)<n\leq N}(\{f(n)\} - \frac{1}{2}) \leq (\frac{1}{2} + \frac{\epsilon}{2})(f^{-1})'(M).$$

By applying Lemma 14 with f^{-1}, we obtain $0 \leq E(M) \leq \frac{1}{2}(f^{-1})'(M)$.
Notice that $M \leq f(N)$ and $(f^{-1})'(M) \leq (f^{-1})'(f(N)) = 1/f'(N)$ and hence the lemma follows (ϵ are not necessarily the same). □

Theorem 18. *Let f be a function satisfying the conditions of Theorem 9. Then, there exists a constant $c > 0$ such that for all $\epsilon > 0$ and sufficiently large N (depend on ϵ), we have*

$$|\sum_{n\leq N}\{f(n)\} - \frac{1}{2}N| \leq \frac{1+\epsilon}{f'(N)} + cf(N).$$

Proof. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \geq x_0, \\ f'(x_0)(x-x_0)+f(x_0) & \text{if } 1 \leq x \leq x_0. \end{cases}$$

We have

$$\begin{aligned}\sum_{n\leq N}\{f(n)\} &= \sum_{n\leq x_0}\{f(n)\} + \sum_{x_0<n\leq N}\{F(n)\} \\ &= \sum_{n\leq x_0}(\{f(n)\}-\{F(n)\}) + \sum_{n\leq N}\{F(n)\}.\end{aligned}$$

By triangle inequality,

$$|\sum_{n\leq N}\{f(n)\} - \frac{1}{2}N| \leq |\sum_{n\leq N}\{F(n)\} - \frac{1}{2}N| + |\sum_{n\leq x_0}(\{f(n)\}-\{F(n)\})|.$$

On the other hand, since $F(x)$ satisfies the conditions of Lemma 16, by Lemma 17,

$$|\sum_{n\leq N}\{F(n)\} - \frac{1}{2}N| \leq \frac{1+\epsilon}{F'(N)} + cF(N).$$

The theorem then follows since $\sum_{n\leq x_0}(\{f(n)\}-\{F(n)\})$ is bounded. □

Lemma 19. *Let f be a function satisfying the conditions of Lemma 20. Then,*

$$\lim_{x\to\infty}\frac{(f^{-1})''(x)}{(f^{-1})'(x)} = 0.$$

Proof. Given that $xf'(x)$ tends monotonically to infinity as $x \to \infty$ and therefore for all $x \geq f(x_0)$, $\frac{1}{f^{-1}(x)f'(f^{-1}(x))} = \frac{(f^{-1})'(x)}{f^{-1}(x)}$ tends monotonically to zero as $x \to \infty$. Consequently, we have

$$\frac{d}{dx}\frac{(f^{-1})'(x)}{f^{-1}(x)} \leq 0.$$

Thus, by quotient rule,

$$\frac{f^{-1}(x)(f^{-1})''(x) - (f^{-1})'(x)(f^{-1})'(x)}{(f^{-1}(x))^2} \leq 0.$$

Notice that $f^{-1}(x)$ is a convex function and hence $(f^{-1})''(x) \geq 0$. Rearranging the terms above gives

$$\frac{(f^{-1})''(x)}{(f^{-1})'(x)} \leq \frac{(f^{-1})'(x)}{f^{-1}(x)}.$$

Since the left hand side is nonnegative, the lemma follows. □

Lemma 20. *Let $f(x)$ be a strictly increasing function defined for $x \geq 1$ which has a continuous nonnegative third derivative. Furthermore, $\lim_{x\to\infty} f(x) = \infty$, $f'(x)$ and $xf'(x)$ tend monotonically to zero and infinity respectively as $x \to \infty$. Then,*

$$\sum_{n\leq f^{-1}(N)} \{f(n)\} = \frac{1}{2}f^{-1}(N) + \frac{1}{12}(f^{-1})'(N) + O((f^{-1})''(N)) + O(N).$$

Proof. By the proof of Lemma 16, we have

$$\sum_{n\leq f^{-1}(N)} \{f(n)\} = \sum_{n\leq f^{-1}(N)} f(n) + \sum_{m\leq N} f^{-1}(m) - Nf^{-1}(N) + O(N)$$

and

$$\sum_{1<n\leq f^{-1}(N)} f(n) = \int_1^{f^{-1}(N)} f(t)dt + \int_1^{f^{-1}(N)} \{t\}f'(t)dt + N\{f^{-1}(N)\} - f(1)\{1\}.$$

By Euler-Maclaurin Formula [6, Theorem B.5],

$$\sum_{1<m\leq N} f^{-1}(m) = \int_1^N f^{-1}(t)dt + \frac{1}{2}(f^{-1}(N) - f^{-1}(1)) + \frac{1}{12}((f^{-1})'(N) - (f^{-1})'(1)) + \frac{1}{6}\int_1^N B_3(\{t\})(f^{-1})'''(t)dt,$$

where $B_3(t)$ is the third Bernoulli polynomial.
In [6, Corollary B.4], $B_3(\{t\})$ is proved to be bounded. Therefore,

$$\begin{aligned}\sum_{n\leq f^{-1}(N)} \{f(n)\} &= \int_1^{f^{-1}(N)} f(t)dt + \int_1^N f^{-1}(t)dt - Nf^{-1}(N) + \frac{1}{2}f^{-1}(N)\\ &\quad + \frac{1}{12}(f^{-1})'(N) + O(\int_1^{f^{-1}(N)} f'(t)dt) + O(\int_1^N (f^{-1})'''(t)dt) + O(N)\\ &= \frac{1}{2}f^{-1}(N) + \frac{1}{12}(f^{-1})'(N) + \int_1^{f^{-1}(N)} f(t)dt + \int_1^N f^{-1}(t)dt\\ &\quad - Nf^{-1}(N) + O((f^{-1})''(N)) + O(N).\end{aligned}$$

The remaining part is similar to the proof of Lemma 16. □

Lemma 21. *Let f be a function satisfying the conditions of Lemma 20. Then, there exists a constant $c > 0$ such that for all $\epsilon > 0$ and sufficiently large N (depend on ϵ), we have*

$$|\sum_{n\leq N}\{f(n)\} - \frac{1}{2}N| \leq \frac{\frac{7}{12}+\epsilon}{f'(N)} + cf(N).$$

Proof. It is similar to the proof of Lemma 17. □

Theorem 22. *Let $f(x)$ be a strictly increasing function defined for $x \geq 1$ which has a continuous nonnegative third derivative for $x \geq x_0$. Furthermore, $\lim_{x\to\infty} f(x) = \infty$, $f'(x)$ and $xf'(x)$ tend monotonically to zero and infinity respectively as $x \to \infty$.*

Then, there exists a constant $c > 0$ such that for all $\epsilon > 0$ and sufficiently large N (depend on ϵ), we have

$$|\sum_{n\leq N}\{f(n)\} - \frac{1}{2}N| \leq \frac{\frac{7}{12}+\epsilon}{f'(N)} + cf(N).$$

Proof. It is similar to the proof of Theorem 18 □

Lemma 23. *Let D_N denote the discrepancy of the partial sequence of fractional parts $\{f(1)\}, \ldots, \{f(N)\}$. For all $t \in [0,1]$, if*

$$R(t;N) := \sum_{n\leq N}\{f(n)+t\} - \frac{1}{2}N,$$

then

$$D_N = \frac{1}{N}\sup_{t_1,t_2\in[0,1]}|R(t_1;N) - R(t_2;N)|.$$

Proof. Suppose that $t_1 < t_2$, then

$$\begin{aligned}\sum_{n\leq N}\{f(n)+t_1\} - \sum_{n\leq N}\{f(n)+t_2\} &= (\sum_{n\leq N}(f(n)+t_1) - \sum_{n\leq N}\lfloor f(n)+t_1\rfloor) \\ &\quad - (\sum_{n\leq N}(f(n)+t_2) - \sum_{n\leq N}\lfloor f(n)+t_2\rfloor) \\ &= (t_1-t_2)N - (\sum_{n\leq N}\lfloor f(n)+t_1\rfloor - \sum_{n\leq N}\lfloor f(n)+t_2\rfloor) \\ &= (t_1-t_2)N - A([1-t_1, 1-t_2);N).\end{aligned}$$

Now, let α, β be $1-t_1, 1-t_2$ respectively. Therefore, $0 \leq \alpha < \beta \leq 1$ and

$$\sum_{n\leq N}\{f(n)+t_1\} - \sum_{n\leq N}\{f(n)+t_2\} = (\beta-\alpha)N - A([\alpha,\beta);N).$$

On the other hand,

$$\sum_{n\leq N}\{f(n)+t_1\} - \sum_{n\leq N}\{f(n)+t_2\} = R(t_1;N) - R(t_2;N).$$

The lemma then follows immediately from the definition of discrepancy. □

Criterion 24. *The sequence $(f(n))$ is equidistributed modulo 1 if and only if for all $t \in [0,1]$, we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N}\{f(n)+t\} = \frac{1}{2}.$$

Proof. In [5, Chapter 1, Lemma 1.1], the sequence $(f(n)+t)$ is proved to be equidistributed modulo 1 if $(f(n))$ is equidistributed modulo 1. Then, in [5, Chapter 1, Theorem 1.1], it shows that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\le N}\{f(n)+t\}=\frac{1}{2}.$$

To prove the converse, by the definition of $R(t;N)$ in Lemma 23,

$$\sum_{n\le N}\{f(n)+t_1\}-\sum_{n\le N}\{f(n)+t_2\}=R(t_1;N)-R(t_2;N).$$

On the other hand, it is given that

$$\lim_{N\to\infty}\frac{1}{N}(\sum_{n\le N}\{f(n)+t_1\}-\sum_{n\le N}\{f(n)+t_2\})=0.$$

Since t_1,t_2 above are chosen arbitrarily, by Lemma 23,

$$\lim_{N\to\infty}D_N=\lim_{N\to\infty}\frac{1}{N}\sup_{t_1,t_2\in[0,1]}|R(t_1;N)-R(t_2;N)|=0.$$

Therefore, the sequence $(f(n))$ is equidistributed modulo 1 (see [5, Chapter 2, Theorem 1.1]) ☐

Remark 25. *We can apply the criterion to prove a weaker form of Fejer's Theorem* [5, Chapter 1, Corollary 2.1]*: If the function $f(x)$ satisfies the conditions of Theorem 9, then the sequence $(f(n))$ is equidistributed modulo 1.*

Theorem 26. *Let f be a function satisfying the conditions of Theorem 9. Then, there exists a constant $c>0$ such that for all $\epsilon>0$ and sufficiently large N (depend on ϵ), we have*

$$D_N\le\frac{\frac{3}{2}+\epsilon}{Nf'(N)}+\frac{cf(N)}{N}.$$

Proof. By the proof of Lemma 17, there exists a constant $c>0$ such that for all $\epsilon>0$ and sufficiently large N (depend on ϵ), we have

$$-(\frac{\frac{1}{2}+\epsilon}{f'(N)}+cf(N))\le R(t;N)\le\frac{1+\epsilon}{f'(N)}+cf(N).$$

Therefore, by Lemma 23,

$$D_N=\frac{1}{N}\sup_{t_1,t_2\in[0,1)}|R(t_1;N)-R(t_2;N)|\le\frac{\frac{3}{2}+2\epsilon}{Nf'(N)}+\frac{2cf(N)}{N}.$$

The theorem then follows immediately (ϵ,c are not necessarily the same). ☐

Remark 27. *This theorem is a stronger form of Theorem 9 since $D_N^*\le D_N$ (see* [5, Chapter 2, Theorem 1.3]*) and one of the implied constants is explicit.*

Theorem 28. *Let f be a function satisfying the conditions of Theorem 22. Then, there exists a constant $c>0$ such that for all $\epsilon>0$ and sufficiently large N (depend on ϵ), we have*

$$D_N \leq \frac{1+\epsilon}{Nf'(N)} + \frac{cf(N)}{N}.$$

Proof. It is similar to the proof of Theorem 26. □

Remark 29. *There are common functions satisfying the conditions of Theorem 22. In fact, we have the following corollaries.*

Corollary 30. *Let D_N be the discrepancy of the sequence of fractional parts $\{\alpha n^\sigma\}$ where $n=1,2,\ldots,\alpha>0, 0<\sigma<\frac{1}{2}$. Then, for all $\epsilon>0$ and sufficiently large N (depend on ϵ), we have*

$$|\sum_{n\leq N}\{\alpha n^\sigma\} - \frac{1}{2}N| \leq \frac{\frac{7}{12}+\epsilon}{\alpha\sigma}N^{1-\sigma}$$

and

$$D_N \leq \frac{1+\epsilon}{\alpha\sigma}N^{-\sigma}.$$

Corollary 31. *Let D_N be the discrepancy of the sequence of fractional parts $\{\alpha(\log n)^\sigma\}$ where $n=1,2,\ldots,\alpha>0,\sigma>1$. Then, for all $\epsilon>0$ and sufficiently large N (depend on ϵ), we have*

$$|\sum_{n\leq N}\{\alpha(\log n)^\sigma\} - \frac{1}{2}N| \leq \frac{\frac{7}{12}+\epsilon}{\alpha\sigma}N(\log N)^{1-\sigma}$$

and

$$D_N \leq \frac{1+\epsilon}{\alpha\sigma}(\log N)^{1-\sigma}.$$

3. Distribution of Sequences Modulo Integers

Lemma 32. *Let p be an odd prime, then*

$$\sum_{n=0}^{p-1}\{\frac{n^2}{p}+\frac{1}{q}\} = (\frac{1}{2}+\frac{1}{q})p - \frac{1}{2} + \frac{1}{p}\sum_{n=0}^{p-1} n\left(\frac{n}{p}\right) - \sum_{\substack{n=0 \\ \{\frac{n^2}{p}\}+\frac{1}{q}\geq 1}}^{p-1} 1,$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol.

Proof. We have

$$\sum_{n=0}^{p-1}\{\frac{n^2}{p}+\frac{1}{q}\} = (\sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{q}<1}}^{p-1} + \sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{q}\geq 1}}^{p-1})\{\frac{n^2}{p}+\frac{1}{q}\}$$
$$= \sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{q}<1}}^{p-1}(\{\frac{n^2}{p}\}+\frac{1}{q}) + \sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{q}\geq 1}}^{p-1}(\{\frac{n^2}{p}\}+\frac{1}{q}-1)$$
$$= \sum_{n=0}^{p-1}\{\frac{n^2}{p}\} + \sum_{n=0}^{p-1}\frac{1}{q} - \sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{q}\geq 1}}^{p-1} 1.$$

Notice that $\sum_{n=0}^{p-1}\{\frac{n^2}{p}\}$ is twice the sum of quadratic residues (mod p) divided by p. Therefore,

$$\sum_{n=0}^{p-1}\{\frac{n^2}{p}+\frac{1}{q}\} = 2\sum_{n=0}^{p-1}\frac{n}{p}(\frac{1}{2}(1+\left(\frac{n}{p}\right))) + \frac{p}{q} - \sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{q}\geq 1}}^{p-1} 1$$
$$= \frac{1}{p}\sum_{n=0}^{p-1} n\left(\frac{n}{p}\right) + \frac{1}{2}(p-1) + \frac{p}{q} - \sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{q}\geq 1}}^{p-1} 1$$
$$= (\frac{1}{2}+\frac{1}{q})p - \frac{1}{2} + \frac{1}{p}\sum_{n=0}^{p-1} n\left(\frac{n}{p}\right) - \sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{q}\geq 1}}^{p-1} 1.$$

This completes the proof. □

Theorem 33. *Let* $L(1)$ *denote the sum*

$$\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{k}{p}\right).$$

[See reviewer's comment (3)]
Suppose that prime $p \equiv 3 \pmod 4$ *and* $p \neq 3$*. Then, we have the following results:*

(a) The number of quadratic residues (mod p) between 0 and $\frac{1}{3}p$ *is*

$$\begin{cases} \frac{1}{6}(p-1) + \frac{\sqrt{p}}{\pi}L(1) & \text{if } p \equiv 1 \pmod 3, \\ \frac{1}{6}(p-2) + \frac{\sqrt{p}}{2\pi}L(1) & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

(b) The number of quadratic residues (mod p) between $\frac{2}{3}p$ *and p is*

$$\begin{cases} \frac{1}{6}(p-1) - \frac{\sqrt{p}}{\pi}L(1) & \text{if } p \equiv 1 \pmod 3, \\ \frac{1}{6}(p-2) - \frac{\sqrt{p}}{2\pi}L(1) & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Proof. We shall prove part (b) first. Only the case that $p \equiv 1 \pmod 3$ is proved since the proofs are similar. Recall that the Fourier series of $\{x\}$ $(x \notin \mathbb{Z})$ can be expressed as

$$\frac{1}{2} - \frac{1}{2\pi i}\sum_{k\neq 0}\frac{e^{2\pi i k x}}{k}.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{p-1}\{\frac{n^2}{p}+\frac{1}{3}\} &= \sum_{n=0}^{p-1}(\frac{1}{2} - \frac{1}{2\pi i}\sum_{k\neq 0}\frac{e^{2\pi i k(\frac{n^2}{p}+\frac{1}{3})}}{k})\\ &= \frac{1}{2}p - \frac{1}{2\pi i}\sum_{k\neq 0}\frac{1}{k}\sum_{n=0}^{p-1}e^{2\pi i k(\frac{n^2}{p}+\frac{1}{3})}\\ &= \frac{1}{2}p - \frac{1}{2\pi i}\sum_{k\neq 0}\frac{\zeta_3^k}{k}G(k;p),\end{aligned}$$

where $G(k;p)$ is the Gauss sum.
We shall calculate the sum on the right hand side first. Recall that (see [1, p.195]) $G(k;p) = \left(\frac{k}{p}\right)i\sqrt{p}$ if $p \equiv 3 \pmod 4$ and $p \nmid k$. It follows that

$$\begin{aligned}\sum_{k\neq 0}\frac{\zeta_3^k}{k}G(k;p) &= \sum_{\substack{k\neq 0\\ p\nmid k}}\frac{\zeta_3^k}{k}\left(\frac{k}{p}\right)i\sqrt{p} + \sum_{\substack{k\neq 0\\ p\mid k}}\frac{\zeta_3^k}{k}p\\ &= \sum_{k\neq 0}\frac{\zeta_3^k}{k}\left(\frac{k}{p}\right)i\sqrt{p} + \sum_{\substack{k\neq 0\\ p\mid k}}\frac{\zeta_3^k}{k}p.\end{aligned}$$

Since $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = -1$, we have $\left(\frac{-k}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{k}{p}\right) = -\left(\frac{k}{p}\right)$. Thus, the sum can be written as

$$i\sqrt{p}\sum_{k=1}^{\infty}\frac{\zeta_3^k+\zeta_3^{-k}}{k}\left(\frac{k}{p}\right) + p\sum_{k=1}^{\infty}\frac{\zeta_3^{pk}+\zeta_3^{-pk}}{pk}.$$

Now, we shall calculate the sums above separately. We have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\zeta_3^k+\zeta_3^{-k}}{k}\left(\frac{k}{p}\right) &= -\sum_{\substack{k=1\\ 3\nmid k}}^{\infty}\frac{1}{k}\left(\frac{k}{p}\right) + 2\sum_{k=1}^{\infty}\frac{1}{3k}\left(\frac{3k}{p}\right)\\ &= -\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{k}{p}\right) + 3\sum_{k=1}^{\infty}\frac{1}{3k}\left(\frac{3k}{p}\right).\end{aligned}$$

By the Law of Quadratic Reciprocity, if $p \equiv 1 \pmod 3$, then $\left(\frac{3}{p}\right) = -1$.
Thus,

$$\sum_{k=1}^{\infty} \frac{\zeta_3^k + \zeta_3^{-k}}{k}\left(\frac{k}{p}\right) = -2L(1).$$

On the other hand, since $p \equiv 1 \pmod 3$, it follows that $pk \equiv k \pmod 3$, and so

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta_3^{pk} + \zeta_3^{-pk}}{k} &= \sum_{k=1}^{\infty} \frac{\zeta_3^k + \zeta_3^{-k}}{k}\\ &= \sqrt{3}iL(1,\chi)\\ &= \sqrt{3}i(\frac{1}{9}\pi\sqrt{3})\\ &= \frac{\pi i}{3},\end{aligned}$$

where χ is the Dirichlet character (mod 3) such that $\chi(2) = -1$ and $L(1,\chi)$ is the Dirichlet L-function.
Finally, by combining the sums, we have

$$\begin{aligned}\sum_{n=0}^{p-1}\{\frac{n^2}{p} + \frac{1}{3}\} &= \frac{1}{2}p - \frac{1}{2\pi i}(-2i\sqrt{p}L(1) + \frac{\pi i}{3})\\ &= \frac{1}{2}p + \frac{\sqrt{p}}{\pi}L(1) - \frac{1}{6}.\end{aligned}$$

By Lemma 32,

$$(\frac{1}{2} + \frac{1}{3})p - \frac{1}{2} + \frac{1}{p}\sum_{n=0}^{p-1} n\left(\frac{n}{p}\right) - \sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{3}\geq 1}}^{p-1} 1 = \frac{1}{2}p + \frac{\sqrt{p}}{\pi}L(1) - \frac{1}{6}.$$

In fact, we have the following identity [2, p.8]: If $p \equiv 3 \pmod 4$, then

$$L(1) = -\frac{\pi}{p^{\frac{3}{2}}}\sum_{n=0}^{p-1} n\left(\frac{n}{p}\right).$$

Hence, after rearrangement, we obtain

$$\sum_{\substack{n=0\\ \{\frac{n^2}{p}\}+\frac{1}{3}\geq 1}}^{p-1} 1 = \frac{1}{3}(p-1) - \frac{2\sqrt{p}}{\pi}L(1).$$

Notice that the sum on the left hand side is twice the number of quadratic residues (mod p) between $\frac{2p}{3}$ and p. Then, part (b) follows immediately.
Next, we shall prove part (a). Suppose that q is a quadratic residue (mod p), then $-q$ must not be. Therefore, the number of quadratic residues (mod p) between 0 and $\frac{1}{3}p$ is

$$\frac{1}{3}(p-1) - (\frac{1}{6}(p-1) - \frac{\sqrt{p}}{\pi}L(1)) = \frac{1}{6}(p-1) + \frac{\sqrt{p}}{\pi}L(1).$$

This completes the proof. □

[See reviewer's comment (4)]

Theorem 34. *Suppose that prime $p \equiv 3 \pmod 4$. Then, we have the following results:*

(a) The number of quadratic residues (mod p) between 0 and $\frac{1}{4}p$ is

$$\begin{cases} \frac{1}{8}(p-3) & \text{if } p \equiv 3 \pmod 8, \\ \frac{1}{8}(p-3) + \frac{\sqrt{p}}{2\pi}L(1) & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

(b) The number of quadratic residues (mod p) between $\frac{3}{4}p$ and p is

$$\begin{cases} \frac{1}{8}(p-3) & \text{if } p \equiv 3 \pmod 8, \\ \frac{1}{8}(p-3) - \frac{\sqrt{p}}{2\pi}L(1) & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

Proof. It is similar to the proof of Theorem 33. □

Definition 35. *If $1 \le \alpha \le n$, then $\varphi_\alpha(n)$ is defined as the number of integers between α and n (including α) that are relatively prime to n. In particular, when $\alpha = 1, \varphi_1(n)$ is exactly the Euler's totient function $\varphi(n)$.*

Lemma 36. *If $n, q > 1$, we have*

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{n} \{\frac{m}{n} + \frac{1}{q}\} = (\frac{1}{2} + \frac{1}{q})\varphi(n) - \varphi_{(1-1/q)n}(n).$$

Proof. We have

$$\begin{aligned} \sum_{\substack{m=1 \\ (m,n)=1}}^{n} \{\frac{m}{n} + \frac{1}{q}\} &= (\sum_{\substack{m=1 \\ (m,n)=1 \\ \frac{m}{n}+\frac{1}{q}<1}}^{n} + \sum_{\substack{m=1 \\ (m,n)=1 \\ \frac{m}{n}+\frac{1}{q}\ge 1}}^{n})\{\frac{m}{n} + \frac{1}{q}\} \\ &= \sum_{\substack{m=1 \\ (m,n)=1 \\ m<(1-1/q)n}}^{n} (\frac{m}{n} + \frac{1}{q}) + \sum_{\substack{m=1 \\ (m,n)=1 \\ m\ge(1-1/q)n}}^{n} (\frac{m}{n} + \frac{1}{q} - 1) \\ &= \frac{1}{n} \sum_{\substack{m=1 \\ (m,n)=1}}^{n} m + \frac{1}{q}\varphi(n) - \varphi_{(1-1/q)n}(n) \\ &= \frac{1}{n}(\frac{1}{2}n\varphi(n)) + \frac{1}{q}\varphi(n) - \varphi_{(1-1/q)n}(n) \\ &= (\frac{1}{2} + \frac{1}{q})\varphi(n) - \varphi_{(1-1/q)n}(n). \end{aligned}$$

This completes the proof. □

Lemma 37. *Let χ be a nonprincipal character (mod n). Then, we have*

$$\sum_{k=1}^{\infty} \frac{\chi(k)}{k} c_n(k) = L(1,\chi)(\mu * \chi)(n),$$

where $c_n(k)$ is the Ramanujan sum.

Proof. We begin with the following identity [4, Theorem 271]:

$$c_n(k) = \sum_{\substack{d|n \\ d|k}} \mu(\frac{n}{d})d.$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\chi(k)}{k} c_n(k) &= \sum_{k=1}^{\infty} \frac{\chi(k)}{k} \sum_{\substack{d|n \\ d|k}} \mu(\frac{n}{d})d \\
&= \sum_{d|n} \sum_{m=1}^{\infty} \frac{\chi(md)}{md} \mu(\frac{n}{d})d \\
&= \sum_{m=1}^{\infty} \frac{\chi(m)}{m} \sum_{d|n} \mu(\frac{n}{d})\chi(d) \\
&= L(1,\chi)(\mu * \chi)(n).
\end{aligned}$$

This completes the proof. □

Lemma 38.

(a) *Let χ be the Dirichlet character (mod 3) such that $\chi(2) = -1$. Then,*

$$(\mu * \chi)(n) = \begin{cases} 0 & \text{if } 9 \mid n \text{ or } n \text{ has a prime factor } p \equiv 1 \pmod 3, \\ \lambda(n) 2^{\omega_3(n)} & \text{otherwise,} \end{cases}$$

where $\lambda(n)$ is the Liouville function and $\omega_3(n)$ denotes the number of distinct prime factors $p \neq 3$ of n.

(b) *Let χ be the Dirichlet character (mod 4) such that $\chi(3) = -1$. Then,*

$$(\mu * \chi)(n) = \begin{cases} 0 & \text{if } 4 \mid n \text{ or } n \text{ has a prime factor } p \equiv 1 \pmod 4, \\ \lambda(n) 2^{\omega_2(n)} & \text{otherwise,} \end{cases}$$

where $\omega_2(n)$ denotes the number of distinct odd prime factors of n.

Proof. We shall prove part (a) only since the proofs are similar. The convolution of two multiplicative functions is still a multiplicative function, so the value of $(\mu * \chi)(n)$ for all positive integers n are determined by the value at powers of primes. We have

$$(\mu * \chi)(p^k) = \sum_{d|p^k} \mu(\frac{p^k}{d})\chi(d).$$

If $p \neq 3$, then since $\mu(p^k) = 0$ for all $k \geq 2$, it follows that

$$(\mu * \chi)(p^k) = \chi(p^k) - \chi(p^{k-1}) = \chi(p)^{k-1}(\chi(p) - 1).$$

Therefore,

$$(\mu * \chi)(p^k) = \begin{cases} 0 & \text{if } p \equiv 1 \pmod 3, \\ (-1)^k 2 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

If $p = 3$, then it is not difficult to conclude that

$$(\mu * \chi)(3^k) = \begin{cases} -1 & \text{if } k = 1, \\ 0 & \text{if } k \geq 1. \end{cases}$$

Then, part (a) follows immediately. □

Theorem 39. *If $n \geq 3$, then*

$$\varphi_{2n/3}(n) = \begin{cases} \frac{1}{3}\varphi(n) & \text{if } 9 \mid n \text{ or } n \text{ has a prime factor } p \equiv 1 \text{ (mod 3)}, \\ \frac{1}{3}\varphi(n) + \frac{1}{6}\lambda(n)2^{\omega_3(n)} & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned} \sum_{\substack{m=1 \\ (m,n)=1}}^{n} \{\frac{m}{n} + \frac{1}{3}\} &= \sum_{\substack{m=1 \\ (m,n)=1}}^{n} (\frac{1}{2} - \frac{1}{2\pi i} \sum_{k \neq 0} \frac{e^{2\pi i k (\frac{m}{n} + \frac{1}{3})}}{k}) \\ &= \frac{1}{2}\varphi(n) - \frac{1}{2\pi i} \sum_{k \neq 0} \frac{e^{\frac{2\pi i k}{3}}}{k} \sum_{\substack{m=1 \\ (m,n)=1}}^{n} e^{2\pi i \frac{m}{n} k} \\ &= \frac{1}{2}\varphi(n) - \frac{1}{2\pi i} \sum_{k \neq 0} \frac{\zeta_3^k}{k} c_n(k). \end{aligned}$$

Notice that $c_n(k) = c_n(-k)$ and therefore

$$\begin{aligned} \sum_{k \neq 0} \frac{\zeta_3^k}{k} c_n(k) &= \sum_{k=1}^{\infty} \frac{\zeta_3^k - \zeta_3^{-k}}{k} c_n(k) \\ &= \sqrt{3} i \sum_{k=1}^{\infty} \frac{\chi(k)}{k} c_n(k), \end{aligned}$$

where χ is the Dirichlet character (mod 3) such that $\chi(2) = -1$.
By Lemma 37,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\chi(k)}{k} c_n(k) &= L(1, \chi)(\mu * \chi)(n) \\ &= (\frac{1}{9}\pi\sqrt{3})(\mu * \chi)(n). \end{aligned}$$

Thus,

$$\sum_{\substack{m=1\\(m,n)=1}}^{n}\{\frac{m}{n}+\frac{1}{3}\}=\frac{1}{2}\varphi(n)-\frac{1}{2\pi i}(\sqrt{3}i)(\frac{1}{9}\pi\sqrt{3})(\mu*\chi)(n)$$
$$=\frac{1}{2}\varphi(n)-\frac{1}{6}(\mu*\chi)(n).$$

By Lemma 36, we have

$$(\frac{1}{2}+\frac{1}{3})\varphi(n)-\varphi_{2n/3}(n)=\frac{1}{2}\varphi(n)-\frac{1}{6}(\mu*\chi)(n)$$

and therefore

$$\varphi_{2n/3}(n)=\frac{1}{3}\varphi(n)+\frac{1}{6}(\mu*\chi)(n).$$

The theorem then follows immediately from Lemma 38. □

Theorem 40. *If $n>4$, then*

$$\varphi_{3n/4}(n)=\begin{cases}\frac{1}{4}\varphi(n) & \text{if } 4\mid n \text{ or } n \text{ has a prime factor } p\equiv 1 \pmod 4,\\ \frac{1}{4}\varphi(n)+\frac{1}{4}\lambda(n)2^{\omega_2(n)} & \text{otherwise.}\end{cases}$$

Proof. It is similar to the proof of Theorem 39. □

Theorem 41. *For $x>1$ we have*

$$\sum_{n\le x}\varphi_{2n/3}(n)=\frac{1}{\pi^2}x^2+O(x\log x)$$

and

$$\sum_{n\le x}\varphi_{3n/4}(n)=\frac{3}{4\pi^2}x^2+O(x\log x).$$

Therefore, the average order (see [1]*, p.1) of $\varphi_{2n/3}(n)$ and $\varphi_{3n/4}(n)$ are $\frac{1}{\pi^2}n$ and $\frac{3}{4\pi^2}n$ respectively.*

Proof. We only prove the first formula since the proofs are similar. By Theorem 39,

$$\sum_{n\le x}\varphi_{2n/3}(n)=\sum_{n\le 3}\varphi_{2n/3}(n)+\sum_{3<n\le x}\varphi_{2n/3}(n)$$
$$=\frac{1}{3}\sum_{n\le x}\varphi(n)+\frac{1}{6}\sum_{n\le x}(\mu*\chi)(n)+O(1).$$

Since $|\mu(k)|\le 1, |\chi(k)|\le 1$, it follows that

$$|(\mu*\chi)(n)|\le\sum_{d|n}|\mu(\frac{n}{d})\chi(d)|\le d(n).$$

Thus,

$$\sum_{n\le x}\varphi_{2n/3}(n)=\frac{1}{3}\sum_{n\le x}\varphi(n)+O(\sum_{n\le x}d(n))+O(1).$$

In fact, we have the following asymptotic formulas for the partial sums of $d(n)$ and $\varphi(n)$ (see [1, Theorem 3.3, Theorem 3.7]):

$$\sum_{n\le x} d(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x}),$$

$$\sum_{n\le x} \varphi(n) = \frac{3}{\pi^2}x^2 + O(x\log x). \quad (x>1)$$

The theorem then follows. □

Remark 42. *The theorem can be refined. We can show that (see* [1, Theorem 3.10]*)*

$$\begin{aligned}\sum_{n\le x}(\mu * \chi)(n) &= \sum_{n\le x}\chi(n)M(\frac{x}{n})\\ &\le \sum_{n\le x}|M(\frac{x}{n})|,\end{aligned}$$

where $M(x) := \sum_{n\le x}\mu(n)$*.*
By the Prime Number Theorem, we have $M(x) = o(x)$ *(see* [1, Theorem 4.14]*).*
Then, it is not difficult to conclude that

$$\sum_{n\le x}\varphi_{2n/3}(n) = \frac{1}{3}\sum_{n\le x}\varphi(n) + o(x\log x)$$

and

$$\sum_{n\le x}\varphi_{3n/4}(n) = \frac{1}{4}\sum_{n\le x}\varphi(n) + o(x\log x).$$

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Reviewer's Comments

General Comments

The use of English in this report was extremely good. So only a few mistakes were found. The flow of the report was good, but one needs some clarifications at certain parts.

In this report, definitions, lemmas, theorems and remarks without explaining the logical relationship or discussing between them were found. We don't know the aims of the sections and how this report was organised. In the introduction one should have an outline of the report and literature review. In each section, we don't know the purpose. Also we may not know the linkage between parts II and III if no elaboration involves. For the ordinary readers, they may get lose easily since they don't know what's going on.

The author introduced a lot of advanced mathematics terms: equidistributed modulo 1, Legendre's symbol, Gauss's sum, quadratic residue, the law of quadratic reciprocity, Dirichlet's L-function, Ramanujan's sum, Liouville's function, divisor function, Euler's phi function and Dirichlet's character. All of the terms listed above were appeared without mentioning their definitions. Background and some reviews are expected.

Theorem 33 was a big theorem. One may need some further elaborations. What is the usage of the condition $p \neq 3$? $p \equiv 3 \pmod 4$ was assumed, then $p \equiv 1$ or $2 \pmod 3$ were also introduced. One may use Chinese Remainder Theorem to combine those congruences and present in a remark.

Section names "Distribution of Sequences Modulo One" and "Distribution of Sequences Modulo Integers" were unclear.

The author may introduce big-O and little-o notations in Section 1. The dependance of implied constant should be discussed at each occurrence.

Mistakes

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. Page 2, Lemma 13: what if $f = 1 - \exp(-x)$?
3. Page 13, Theorem 33: Is $L(1)$ convergent?
4. Page 15, the proof of Theorem 33: justify the first three equalities.