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The Generalized Tower of Hanoi Problem

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THE GENERALIZED TOWER OF HANOI PROBLEM

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Abstract. In this paper, we look into a generalized version of the well-known Tower of Hanoi problem. We will investigate the shortest methods of traversing between any two valid configurations of discs in the standard problem, as well as in some variants.

1. Introduction

1.1. Background

The Tower of Hanoi problem, also known as the Tower of Brahma or Lucas' Tower problem, is a well-known solitaire game. The problem was introduced by mathematician Édouard Lucas in 1883. Since then, there have been many variants invented upon the standard game.

The standard Tower of Hanoi problem can be summarized as thus:

There are three upright pegs, and n cylindrical discs of distinct sizes. Each disc has a hole in it such that it can be placed on any peg. At the start of the game, all the discs are placed on the same peg in descending order of size, such that the uppermost disc is the smallest. [See reviewer's comment (2)] The goal is to move all the discs to another peg, moving one disc at a time from one peg to another, with the conditions that

(a) Only the top disc on each peg can be moved;
(b) A disc cannot be placed on top of a smaller disc.

The algorithm for solving this standard game, as well as the fewest number of moves required is well-known. However, little is known about the fewest number of moves required to move the discs from any one valid configuration of discs to another. In this project, we aim to solve this problem for the standard game as well as for some of the more popular variants.

1.2. Notation and reformulation

All variables within this report are implicitly assumed to be positive integers unless otherwise stated.

To reduce the complexity of this report, we shall reformulate the base game thusly:

Label the three pegs 0, 1 and 2, and label the n discs $1, 2, 3, \ldots, n$ in ascending order of size, such that the disc labelled 1 is the smallest disc. In a game with n discs, each configuration will be represented by an n-digit ternary code with possible leading zeros, where the m-th digit from the right represents the peg that disc m is placed upon. The term 'code' refers to an n-digit ternary code.

For example, in a 4-disc game, a configuration could be represented by 1210. Here disc 1 (the smallest disc) is placed on peg 0, discs 4 and 2 are placed on peg 1 (in that order), and the disc 3 is placed on peg 2.

Disc m can be moved from peg P to peg Q, where m, P, Q are variables, if and only if the rightmost $(m-1)$ digits of the current code do not contain P or Q. (1)

A path from configuration A to configuration B is a sequence of configurations and the moves required such that the first configuration is configuration A and the last configuration is configuration B, and each configuration can be achieved in one valid move from the previous configuration. A non-repeating path is such a path with all distinct configurations.

The shortest path is the path between two given configurations with the fewest number of configurations involved.

The path length is the number of moves used in a given path. The shortest path length is the path length of the shortest path between two given configurations.

If the shortest path length from configuration A to configuration B is 1, then we consider configuration B to be adjacent to configuration A.

To traverse from configuration A to configuration B is to arrange a sequence of moves such that configuration A can be changed to configuration B.

The base case refers to the standard rules as set out above. A variant refers to modification of the rules in any way such that a move that would not be valid in the base case is now valid, or a move that would be valid in the base case is now not valid.

The specific case for a variant of the Tower of Hanoi problem refers to the problem of the shortest path from all discs on one peg to all discs on another peg. The general case for a variant refers to the problem of the shortest path from any valid configuration to another.

The notation $00\ldots0$ (m) means m '0's in a row. $11\ldots1$ (m) and $22\ldots2$ (m) are defined similarly. An expression like $12022\ldots2$ (m) means 120 followed by m '2's in a row.

1.3. Project Outline

This paper consists of six chapters.

Chapter 1 (the current chapter) is a brief introduction of our project.

In Chapter 2, we state and prove a few properties of the base game, and introduce the specific variants we are going to discuss.

2. Preliminary Work

Proposition 1. *At any time, the discs on each peg are arranged in ascending order of size such that the top disc is the smallest.*

Proof. This is evident from the condition that no disc can be put on a smaller disc. □

Proposition 2. *The reformulation of the game rules in line (1) above is valid.*

Proof. Line (1) is equivalent to stating that a disc can be moved if and only if

(a') no discs with smaller labelled values are on the origin peg of the disc to be moved; and
(b') no discs with smaller labelled values are on the destination peg of the disc to be moved.

Condition (a) is equivalent to stating that the disc to be moved is the smallest one on its origin peg, according to Proposition 1. Therefore condition (a) is equivalent to condition (a').

Condition (b) is equivalent to stating that the disc to be moved must be smaller than the original top disc on the destination peg. This is then equivalent to stating that the moved disc will be the smallest disc on the destination peg once it is moved, according to Proposition 1, which is then equivalent to condition (b'). □

Proposition 3. *There is a one-one correspondence between the ternary n-digit codes and the possible configurations of the n discs.*

Proof. Since the discs must be arranged in ascending order of size with the smallest on top, there is no ambiguity as to the configuration of discs given an n-digit code. Thus a function exists mapping the codes to the possible configurations of discs.

Since each disc in a possible configuration is placed on exactly one peg labelled from 0 to 2, there exists a unique code for every configuration. Thus a function exists as well in the backwards direction, and we are done. □

We shall now note some of the properties of the base game. We consider these properties characteristic of the Tower of Hanoi problem, and thus we shall investigate variants that share these characteristics below.

Property 4. *It is possible to determine the possible configurations that can be reached in one move given the current configuration (a code).*

For example, this property is not present in a variant that adds the condition 'At the m-th move, only odd-labelled discs can be moved if m is odd and only even-labelled discs can be moved if m is even', due to the added variable of the move number.

Property 5. *Each configuration is determined uniquely by a code.*

For example, this property is not present in a variant that changes condition (b) to 'A disc can be placed on another disc only if the disc to be placed is at most one size larger than the other disc', due to violation of Proposition 1 (which then leads to violation of Proposition 3)

We also wish to preserve the conditions of the original Tower of Hanoi game (i.e. each variant we consider will have additional constraints added to the base rules instead of replacing them).

We locate the following types of modifications, which seemed the most obvious:

(a) Restricted moves. We can disallow certain types of moves.
(b) Restricted configurations. We can disallow certain configurations of discs altogether.

Of each modification, we designed the most intuitive variants that satisfy Property 4 and Property 5 for discussion below.

(a) Directed moves. In this variant, we define the direction of pegs $0 \to 1 \to 2 \to 0$ as clockwise (and $0 \to 2 \to 1 \to 0$ as anticlockwise). We then label each disc such that each disc can only move in one direction, either clockwise or anticlockwise.

(b) Restricted configurations. In this variant, we colour each disc one of black or white, and add the constraint that a disc cannot lie on top of a disc of the same colour.

3. The Base Case

3.1. The specific case

The specific case for the base case is well known. A unique algorithm (aside from symmetry) for the solution of this case using the shortest path is

(1) Move disc 1 clockwise.
(2) Find the number of pegs with discs on them. If it is 2 or more, go to step 3. Otherwise, go to step 4.
(3) Consider the second smallest top disc on the three pegs, which exists due to step 2. Move this disc to the unique other peg that does not have disc 1 on it. Go to step 1.
(4) The goal is reached.

Below, the 'base case algorithm' means the above algorithm.

It is similarly well-known that the number of moves in this algorithm is $2^n - 1$.

Proposition 6. *The final configuration is all discs on the peg that is one step clockwise from the original peg, if n is odd; and all discs on the peg that is one step anticlockwise from the original peg, if n is even.*

Proof. Without loss of generality assume that the starting configuration is $00\dots0$ (n). We proceed by induction. For $n = 1$, the result is clear. For $n > 1$, note that disc n will never be moved in the above algorithm unless all of discs $1, 2, \dots, n-1$ are on the same peg that is distinct from the peg disc n lies on (otherwise the second smallest top disc cannot be disc n). Consider discs $1, 2, \dots, n-1$. By the induction assumption, the first time this occurs (after $2^{n-1} - 1$ moves) the configuration will be $011\dots1$ $(n-1)$ if n is even and $022\dots2$ $(n-1)$ if n is odd. Then the next move results in $211\dots1$ $(n-1)$ if n is even and $122\dots2$ $(n-1)$ if n is odd. Again, by the induction assumption, after $2^{n-1}-1$ moves, the configuration is $22\dots2$ (n) if n is even and $11\dots1$ (n) if n is odd, and we are done. □

It should follow that if step (1) in the base case algorithm is modified to move disc 1 anticlockwise (define this as the 'inverse base case algorithm') then the exact opposite result will be obtained. The final configuration will be all discs on the peg that is one step clockwise from the original peg, if n is even; and all discs on the peg that is one step anticlockwise from the original peg, if n is odd.

This proposition will be used in the analysis of the general case below.

3.2. The general case

For ease of discussion, we let the starting configuration be $a_n a_{n-1} \dots a_1$ and the final configuration be $b_n b_{n-1} \dots b_1$, where $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are all either 0, 1 or 2.

Proposition 7. *The largest moving disc in a non-repeating path cannot move between the same two pegs more than once.*

Proof. Let there be n discs and let the largest moving disc be disc m. Without loss of generality let it move between pegs 0 and 1. The configurations before and after the move must be $a_n a_{n-1} \dots a_{m+1} 022 \dots 2\ (m-1)$ to $a_n a_{n-1} \dots a_{m+1} 122 \dots 2\ (m-1)$ in some order, given that the discs $m+1, m+2, \dots, n$ started in the configuration $a_n a_{n-1} \dots a_{m+1}$. It then follows that if a move between these two pegs occurs again, at least one configuration must repeat. □

Proposition 8. *The shortest path length from* $00 \dots 0\ (n)$, $11 \dots 1\ (n)$ *or* $22 \dots 2\ (n)$ *to any configuration is less than or equal to* $2^n - 1$.

Proof. We proceed by induction. For $n = 1$, the claim is obvious.

For $n > 1$, without loss of generality let the starting configuration be $00 \dots 0\ (n)$. Either $b_n = 0$ or $b_n \neq 0$. If $b_n = 0$, we can simply choose not to move the disc n and use the shortest path from $00 \dots 0\ (n-1)$ to $b_{n-1} \dots b_1$, which has $2^{n-1} - 1$ or fewer moves according to the induction assumption. Thus this case is trivially solved.
If $b_n \neq 0$, without loss of generality let $b_n = 1$. Use the base case algorithm to reach the configuration $022 \dots 2\ (n-1)$, using $2^{n-1} - 1$ moves. Then we can reach the adjacent configuration $122 \dots 2\ (n-1)$ using a total of 2^{n-1} so far. Using the induction assumption, we can then reach $1b_{n-1} \dots b_1$ using no more than $2^{n-1} + 2^{n-1} - 1 = 2^n - 1$ moves. □

Corollary 9. *The shortest path length between any two n-digit configurations* $a_n a_{n-1} \dots a_1$ *and* $b_n b_{n-1} \dots b_1$ *is no more than* $2^n - 1$.

Proof. Choose a number from $\{0, 1, 2\}$ such that it is not identical to a_n or b_n. Without loss of generality let it be 0. Then the distance from $a_n a_{n-1} \dots a_1$ to $a_n 00 \dots 0\ (n-1)$ is at most $2^{n-1} - 1$, the distance from $a_n 00 \dots 0\ (n-1)$ to $b_n 00 \dots 0\ (n-1)$ is at most 1, and the distance from $b_n 00 \dots 0\ (n-1)$ to $b_n b_{n-1} \dots b_1$ is at most $2^{n-1} - 1$, for a total distance of at most $2^{n-1} - 1 + 1 + 2^{n-1} - 1 = 2^n - 1$. □

Corollary 10. *The shortest path between* $a_n a_{n-1} \dots a_1$ *and* $b_n b_{n-1} \dots b_1$*, where some nonnegative number integer* m *satisfies "for all* i *such that* $n - m < i \leq n$*, we have* $a_i = b_i$*", does not include any moves of discs* i *where* $n - m < i \leq n$.

Proof. The condition means that the starting and final configuration of the largest m discs are identical. It follows from corollary 9 that the shortest path length is no more than $2^{n-m} - 1$.

Assume to the contrary that disc i was moved during some shortest path, for some i satisfying $n - m < i \leq n$; moreover, assume this i is the largest such i satisfying the above condition. Without loss of generality we may assume $a_i = b_i = 0$, and that the first move of disc i was moving from peg 0 to peg 1. The configuration representing this must then be $a_n a_{n-1} \ldots a_{i+1} 022 \ldots 2$ $(i-1)$ to $a_n a_{n-1} \ldots a_{i+1} 122 \ldots 2$ $(i-1)$, since none of the discs $i+1, i+2, \ldots, n$ will have moved by the assumption that the chosen i is the largest possible.

If disc i ever moves from peg 1 to peg 0, it violates Proposition 7. Thus it cannot be a shortest path.

Otherwise, disc i never moves from peg 1 to peg 0, so it must have moved to peg 2 from peg 1 (since it needs to return to peg 0). The configuration representing this must be $a_n a_{n-1} \ldots a_{i+1} 100 \ldots 0$ $(i-1)$ to $a_n a_{n-1} \ldots a_{i+1} 200 \ldots 0$ $(i-1)$. However, the shortest distance between $a_n a_{n-1} \ldots a_{i+1} 122 \ldots 2$ $(i-1)$ and $a_n a_{n-1} \ldots a_{i+1} 100 \ldots 0$ $(i-1)$ is $2^{i-1} - 1$. Thus the number of total moves must be at least 2^{i-1}, adding in the move of disc i from peg 1 to peg 2. Since $n - m < i$, it follows that $2^{n-m} - 1 < 2^{i-1}$, so this path cannot be the shortest path, reaching a final contradiction. □

Corollary 11. *The shortest path between $a_n a_{n-1} \ldots a_1$ and $b_n b_{n-1} \ldots b_1$, where $a_1 = a_2 = \cdots = a_n \neq b_n$, moves disc n exactly once.*

Proof. Without loss of generality assume $a_1 = a_2 = \cdots = a_n = 0$, $b_n = 1$. Assume disc n was moved more than once. By Proposition 7, disc n must have moved from peg 0 to peg 2, then to peg 1.

Then the path must include the moves $011 \ldots 1$ $(n-1) \to 211 \ldots 1$ $(n-1)$ and $200 \ldots 0$ $(n-1)$ to $100 \ldots 0$ $(n-1)$. However, the shortest path length from $00 \ldots 0$ (n) to $011 \ldots 1$ $(n-1)$ is $2^{n-1} - 1$, one move is needed to reach $211 \ldots 1$ $(n-1)$, the shortest path length from $211 \ldots 1$ $(n-1)$ to $200 \ldots 0$ $(n-1)$ is $2^{n-1} - 1$, and one more move is needed to move to $100 \ldots 0$ $(n-1)$, for a total of at least $2^{n-1} - 1 + 1 + 2^{n-1} - 1 + 1 = 2^n$ moves, reaching contradiction. □

The algorithm for the shortest path between $a_n a_{n-1} \ldots a_1$ and $b_n b_{n-1} \ldots b_1$, where $a_1 = a_2 = \cdots = a_n$, is then within reach.

Case I: If $a_n \neq b_n$, let $c_n = 3 - a_n - b_n$. c_n will be the number from $\{0, 1, 2\}$ not equal to either a_n or b_n. Then disc n moves exactly once by Corollary 11, from a_n to b_n. Then the move must be from $a_n c_n c_n \ldots c_n$ $(n-1)$ to $b_n c_n c_n \ldots c_n$ $(n-1)$, for a total of $2^{n-1} - 1 + 1 = 2^{n-1}$ moves to reach $b_n c_n c_n \ldots c_n$ $(n-1)$. Afterwards, disc

n does not move, so we simply consider the shortest path from $c_n c_n \dots c_n$ $(n-1)$ to $b_{n-1} \dots b_1$.

Case II: If $a_n = b_n$, then by Corollary 10 disc n never moves, so simply consider the shortest path from $a_{n-1} \dots a_1$ to $b_{n-1} \dots b_1$.

We have a recursive algorithm. We used c_i to store the current location of the i-th disc right after the $(i+1)$-th disc is put into place. Note that case II is invoked when $c_m = b_{m-1}$ for some m, as can be seen from the last line of the algorithm. Thus, we simply define $c_{n+1} = a_1 = a_2 = \cdots = a_n$, and add "let $c_n = c_{n+1}$" to the instructions in Case II. Then Case I is invoked only when $c_{m+1} = b_m$, where m is the number of digits in the currently considered code (i.e. the label of the largest currently considered disc).

Also note that if we consider all numbers to be modulo 3, then "set $c_n \equiv -a_n - b_n$" suffices for Case I, and if $c_{m+1} \equiv b_m$, then setting $c_n \equiv -c_{n+1} - b_n$ also works to set $c_n = c_{n+1}$.

Here is the algorithm written explicitly, given an n-digit ternary code for the final configuration and a number a from $\{0, 1, 2\}$ for the starting configuration:

Step 1: Set i equal to n. Set c_{n+1} equal to a. Set A equal to 0. Go to Step 2.
Step 2: Is $b_i \equiv c_{i+1}$? If yes, go to Step 4. If no, go to Step 3.
Step 3: Add 2^{i-1} to A. Go to Step 4.
Step 4: Set $c_i \equiv -c_{i+1} - b_i$. Decrease i by 1. If $i = 0$, go to Step 5. If not, go to Step 2.
Step 5: Output A as the number of moves needed.

A quick method to calculate the shortest path length is to recursively calculate c_m for each m from n to 1. Below is an example calculating the shortest path length from 0000000 to 2022121.

i	8	7	6	5	4	3	2	1
b_i	N/A	2	0	2	2	1	2	1
$b_i = c_{i+1}$?	N/A	No	No	Yes	Yes	No	No	Yes
c_i	0	1	2	2	2	0	1	1
A	0	64	96	96	96	100	102	102

The answer is 102 moves.

Now we can generalise this for any two configurations $a_n a_{n-1} \dots a_1$ and $b_n b_{n-1} \dots b_1$.

We may assume $a_n \neq b_n$; otherwise, by Corollary 10, disc n never moves, so it is equivalent to considering the configurations $a_{n-1} \dots a_1$ and $b_{n-1} \dots b_1$.

Define $c_n = 3 - a_n - b_n$; since the largest disc cannot move between the same two pegs twice, then either disc n moves from a_n to b_n or from a_n to c_n to b_n.

If disc n moves from a_n to b_n, it must be from $a_n c_n c_n \dots c_n$ $(n-1)$ to $b_n c_n c_n \dots c_n$ $(n-1)$. Using the above algorithm twice can find the number of moves from $a_n a_{n-1} \dots a_1$ to $a_n c_n c_n \dots c_n$ $(n-1)$, and from $b_n c_n c_n \dots c_n$ $(n-1)$ to $b_n b_{n-1} \dots b_1$. Adding one to their sum is a possible shortest path length;

If disc n moves from a_n to c_n to b_n, then the two moves involved must be from $a_n b_n b_n \dots b_n$ $(n-1)$ to $c_n b_n b_n \dots b_n$ $(n-1)$, and $c_n a_n a_n \dots a_n$ $(n-1)$ to $b_n a_n a_n \dots a_n$ $(n-1)$. Using the above algorithm twice can find the number of moves from $a_n a_{n-1} \dots a_1$ to $a_n b_n b_n \dots b_n$ $(n-1)$, and from $b_n a_n a_n \dots a_n$ $(n-1)$ to $b_n b_{n-1} \dots b_1$. The shortest path distance from $c_n b_n b_n \dots b_n$ $(n-1)$ to $c_n a_n a_n \dots a_n$ $(n-1)$ is $2^{n-1} - 1$.

Thus the possible shortest path length is $2^{n-1} - 1 + 1 + 1 = 2^{n-1} + 1$ plus the sum of the two algorithm results.

Now, the shortest path length must be one of the two possible shortest path lengths, so simply take the smaller result. We have a complete algorithmic method to find the shortest path length between any two configurations.

4. The Clockwise/Anticlockwise Restriction Case

4.1. Preliminary Work

This case restricts all discs to either move in the direction $0 \to 1 \to 2 \to 0$ (clockwise) or $0 \to 2 \to 1 \to 0$ (anticlockwise). We colour each disc either black or white, and add the restriction that white discs only travel clockwise while black discs only travel anticlockwise. In this variant, if configuration A is adjacent to configuration B, then configuration B cannot be adjacent to configuration A.

Proposition 12. *There is exactly one non-repeating path from any configuration to any other configuration, as long as the largest disc is on different pegs in the starting and final configurations.*

Proof. We proceed by induction. When $n = 1$, the claim is obvious.

If $n > 1$, let the starting configuration be $a_n a_{n-1} \dots a_1$ and the final configuration be $b_n b_{n-1} \dots b_1$. Without loss of generality assume $a_n = 0$ and disc n is white.

Case I: $b_n = 1$. Then disc n must have moved from peg 0 to peg 1. Note that since this variant is a more stringent version of the base case, many of its properties still hold - in particular, Proposition 7 clearly holds - thus the only move for disc n is from peg 0 to peg 1. Then the move must be from $022 \dots 2$ $(n-1)$ to $122 \dots 2$ $(n-1)$. There are no other moves for disc n, so the case is reduced to $n-1$ discs, from $a_{n-1} \dots a_1$ to $22 \dots 2$ $(n-1)$ and from $22 \dots 2$ $(n-1)$ to $b_{n-1} \dots b_1$. By the induction assumption, there is exactly one path.

Case II: $b_n = 2$. By Proposition 7, disc n moved from peg 0 to peg 1, then from peg 1 to peg 2. The moves must be from $022\ldots2$ $(n-1)$ to $122\ldots2$ $(n-1)$ and from $100\ldots0$ $(n-1)$ to $200\ldots0$ $(n-1)$. There are no other moves for disc n, so the case is reduced to $n-1$ discs, from $a_{n-1}\ldots a_1$ to $22\ldots2$ $(n-1)$, from $22\ldots2$ $(n-1)$ to $00\ldots0$ $(n-1)$ and from $00\ldots0$ $(n-1)$ to $b_{n-1}\ldots b_1$. By the induction assumption, there is exactly one path. $\square$

Given this proposition, we have that any non-repeating path must be the shortest path in this variant, as long as the largest disc is on different pegs in the starting and final configurations.

4.2. The specific case

Note that since this variant lacks reversibility, there are two different values of the shortest path length from all discs on one given peg to all discs on another given peg, depending on whether the discs moved one peg clockwise or one peg anticlockwise. We define two sequences $\{s_n\}$ and $\{l_n\}$, where s_i is the clockwise shortest path length for i discs, and l_i is the anticlockwise shortest path length for i discs.

We start this section with a specific example, where all discs are white.

Proposition 13. *If all discs are white, then $s_n = 2l_{n-1}+1$ and $l_n = 2l_{n-1}+s_{n-1}+1$ for all $n > 1$. [See reviewer's comment (3)]*

Proof. Proceed by induction. Without loss of generality assume the discs start all on peg 0. When moving disc n to peg 1, the other discs must be on peg 2, i.e. moved one peg anticlockwise. Then the other discs are moved one peg anticlockwise from peg 2 to peg 1 collectively to end up on the same peg as the largest disc. Thus $s_n = 2l_{n-1} + 1$.

When moving the largest disc to peg 2, it in fact moves clockwise twice. Before the first move from peg 0 to peg 1, the other discs must be on peg 2, i.e. moved one peg anticlockwise. Then the other discs must be moved one peg clockwise from peg 2 to peg 0 to allow disc n to move from peg 1 to peg 2. Finally, the other discs move from peg 0 to peg 2, which is one peg anticlockwise. Thus $l_n = 2l_{n-1} + s_{n-1} + 1$. [See reviewer's comment (4)]

Now we attempt to solve $s_n = 2l_{n-1} + 1$ and $l_n = 2l_{n-1} + s_{n-1} + 1$ for a general formula. $l_n = 2l_{n-1} + s_{n-1} + 1 = 2l_{n-1} + 2l_{n-2} + 3$; then considering the sequence $\{l_n+1\}$, we have that $l_n+1 = 2(l_{n-1}+1)+2(l_{n-2}+1)$. The characteristic equation $x^2 - 2x - 2 = 0$ solves for the roots $1+\sqrt{3}$ and $1-\sqrt{3}$; solving for coefficients with the first two term $l_1 = 2$ and $l_2 = 7$ yields the general formula as [See reviewer's comment (4)]

$$l_n + 1 = \frac{2\sqrt{3}+3}{6}(1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n, \text{ so}$$
$$l_n = \frac{2\sqrt{3}+3}{6}(1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n - 1.$$

Solving for s_n is then simple and yields $s_n = \frac{\sqrt{3}+3}{6}(1+\sqrt{3})^n + \frac{3-\sqrt{3}}{6}(1-\sqrt{3})^n - 1$.

Other colouring methods of the discs yield a similar series of recurrence relations that can then be solved. Of particular note is the alternate colouring scheme, where disc m is white if m is odd and disc m is black otherwise. This solves for the recurrence relations:

$l_n = 2s_{n-1} + 1$ if n is even and $l_n = 2l_{n-1} + s_{n-1} + 2$ otherwise;

and $s_n = 2l_{n-1} + 1$ if n is odd and $s_n = 2s_{n-1} + l_{n-1} + 2$ otherwise. Considering the sequence $\{d_n\}$ where $d_n = s_n$ if n is odd and $d_n = l_n$ if n is even yields the recurrence relation $d_n = 2d_{n-1} + 1$, which together with $d_1 = 1$ solves for $d_n = 2^n - 1$. Hence, either the base case algorithm or the inverse algorithm must in fact satisfy the restrictions of this alternating colour scheme. It is easily checked that it is the base case algorithm that satisfies these restrictions (since the smallest disc is white).

The remaining values are solved easily: $s_n = 2^{n+1} - 2$ when n is even and $l_n = 2^{n+1} - 2$ when n is odd. This can be used to prove the property that the shortest paths from $00\ldots0$ (n) to $11\ldots1$ (n) and $11\ldots1$ (n) to $22\ldots2$ (n) in fact do not repeat any configuration, as follows: Join the two paths to form a path from $00\ldots0$ (n) to $22\ldots2$ (n). This path must satisfy the alternate colour scheme's restrictions. Assume to the contrary that there is a repetition. Then by removing the cycle from the joined path, we have a non-repeating path with less than $2 \times (2^n - 1) = 2^{n+1} - 2$ moves. This contradicts Proposition 12's proof that there is only one non-repeating path. □

4.3. The general case

Proposition 14. *$s_n \leq 2l_n \leq 4s_n$ for any n and any colouring scheme.*

Proof. Assume this is false, i.e. $s_n > 2l_n$ ir $l_n > 2s_n$ for some colouring scheme and some n. Without loss of generality assume $s_n > 2l_n$. By first using the unique non-repeating path to move all discs one peg anticlockwise, then doing so again, we have a path that moves all discs one peg clockwise. By removing all cycles from this path, we get a non-repeating path with at most $2l_n$ moves. Thus $s_n > 2l_n$ cannot be true. □

Corollary 15. *For any colouring scheme, the shortest path between* $a_n a_{n-1} \dots a_1$ *and* $b_n b_{n-1} \dots b_1$*, where some nonnegative integer* m *satisfies "for all* i *such that* $n - m < i \leq n$*, we have* $a_i = b_i$*", does not include any moves of discs* i *where* $n - m < i \leq n$*.*

Proof. The condition means that the starting and final configuration of the largest m discs are identical.

Assume to the contrary that a disc i moves during a shortest path, where $n - m < i \leq n$.
Assume also that this i is the largest such i. Note that there may be more than one non-repeating path, since the largest disc is on the same peg in the starting and final configurations.

Without loss of generality assume disc i started on peg 0, and without loss of generality assume disc i moves from peg 0 to peg 1, to peg 2, and back to peg 0. Then the moves must be from $a_n a_{n-1} \dots a_{i+1} 022 \dots 2$ $(i-1)$ to $a_n a_{n-1} \dots a_{i+1} 122 \dots 2$ $(i-1)$, from $a_n a_{n-1} \dots a_{i+1} 100 \dots 0$ $(i-1)$ to $a_n a_{n-1} \dots a_{i+1} 200 \dots 0$ $(i-1)$, and finally from $a_n a_{n-1} \dots a_{i+1} 211 \dots 1$ $(i-1)$ to $a_n a_{n-1} \dots a_{i+1} 011 \dots 1$ $(i-1)$. There is a minimum of $min\{s_{i-1}, l_{i-1}\}$ moves between either of $a_n a_{n-1} \dots a_{i+1} 122 \dots 2$ $(i-1)$ to $a_n a_{n-1} \dots a_{i+1} 100 \dots 0$ $(i-1)$ and between $a_n a_{n-1} \dots a_{i+1} 200 \dots 0$ $(i-1)$ and $a_n a_{n-1} \dots a_{i+1} 211 \dots 1$ $(i-1)$. Let a moves be required to traverse from $a_n a_{n-1} \dots a_1$ to $a_n a_{n-1} \dots a_{i+1} 022 \dots 2$ $(i-1)$, and b moves be required to traverse from $a_n a_{n-1} \dots a_{i+1} 011 \dots 1$ to $b_n b_{n-1} \dots b_1$. Thus, there are at least $2 \times min\{s_{i-1}, l_{i-1}\} + 3 + a + b$ moves in this shortest path.

Now consider an alternate path, using the shortest path from $a_n a_{n-1} \dots a_1$ to $a_n a_{n-1} \dots a_{i+1} 022 \dots 2$ $(i-1)$, then to $a_n a_{n-1} \dots a_{i+1} 011 \dots 1$ $(i-1)$, then finally to $b_n b_{n-1} \dots b_1$. The path length is at most $a + b + max\{s_{i-1}, l_{i-1}\}$. Using Proposition 14, we have that $max\{s_{i-1}, l_{i-1}\} < 2 \times min\{s_{i-1}, l_{i-1}\} + 3$, so we have a shorter, possible repeating path compared to the shortest path, which is a clear contradiction. □

The algorithm for the shortest path between $a_n a_{n-1} \dots a_1$ and $b_n b_{n-1} \dots b_1$, where $a_1 = a_2 = \cdots = a_n$, is yet again within reach.

Case I: If $a_n \neq b_n$, without loss of generality assume $a_n = 0$ and disc n is white. Then if $b_n = 1$, disc n moves exactly once, from $022 \dots 2$ $(n-1)$ to $122 \dots 2$ $(n-1)$, for a total of $l_{n-1} + 1$ moves so far (with s_0, l_0 defined as 0). Afterwards, disc n does not move, so we simply consider the shortest path from $22 \dots 2$ $(n-1)$ to $b_{n-1} \dots b_1$. Otherwise, $b_n = 2$, so disc n moves exactly twice, from $022 \dots 2$ $(n-1)$ to $122 \dots 2$ $(n-1)$, and then from $100 \dots 0$ $(n-1)$ to $200 \dots 0$ $(n-1)$, for a total of $s_{n-1} + l_{n-1} + 2$ moves so far. Afterwards, again, disc n does not move, so we simply consider the shortest path from $00 \dots 0$ $(n-1)$ to $b_{n-1} \dots b_1$.

Case II: If $a_n = b_n$, then by Corollary 15 disc n never moves, so simply consider the shortest path from $a_{n-1} \ldots a_1$ to $b_{n-1} \ldots b_1$.

We have a recursive algorithm. By calculating the values of s_m, l_m for each m as described in section 4.2, we can calculate the distance easily. Here we demonstrate for the case with all discs white, using ternary system in the following algorithm, given an n-digit ternary code for the final configuration and a number a from $\{0, 1, 2\}$ for the starting configuration. We shall use c_i to store the current location of the i-th disc right after the $(i + 1)$-th disc is put into place, as before.

Step 1: Set i equal to n. Set c_{n+1} equal to a. Set A equal to 0. Go to Step 2.
Step 2: What is $b_i - c_{i+1}$ congruent to? If 0, go to Step 3. If 1, go to Step 4. If 2 go to Step 5.
Step 3: Set $c_i \equiv c_{i+1}$. Go to Step 6.
Step 4: Set $c_i \equiv b_i + 1$. Add $l_{n-1} + 1$ to A. Go to Step 6.
Step 5: Set $c_i \equiv b_i + 1$. Add $s_{n-1} + l_{n-1} + 2$ to A. Go to Step 6.
Step 6: Decrease i by 1. If $i = 0$, go to Step 7. If not, go to Step 2.
Step 7: Output A as the number of moves needed.

As before, here is an example of traversing from 0000000 to 2012121:

i	8	7	6	5	4	3	2	1
b_i	N/A	2	0	1	2	1	2	1
$b_i - c_{i+1}$	N/A	2	0	1	0	2	0	2
c_i	0	0	0	2	2	2	2	2
A	0	1223	1223	1342	1342	1363	1363	1365

The answer is 1365 moves.
Note that it is necessary to reverse the colours considered for the reverse case, i.e. when $b_1 = b_2 = \cdots = b_n$, but $a_1 = a_2 = \cdots = a_n$ is not necessarily true.

Now we can generalise this for any two configurations $a_n a_{n-1} \ldots a_1$ and $b_n b_{n-1} \ldots b_1$.

We may assume $a_n \neq b_n$; otherwise, by Corollary 10, disc n never moves, so it is equivalent to considering the configurations $a_n a_{n-1} \ldots a_1$ and $b_n b_{n-1} \ldots b_1$.

Depending on the colour, either disc n on peg a_n can be moved to b_n in one move, in which case sum up the results from both sides and add one; or it requires two moves, in which case sum up the results, add two, and add s_{n-1} or l_{n-1}, whichever is appropriate. It is noted that there is only one method in this case and there is no need to take the minimum.

Thus we have a complete algorithmic method to find the shortest path length between any two configurations.

5. The Black/White Restriction Case

5.1. Preliminary Work

In this case, each disc is coloured one of black and white, and no two discs of the same colour can be stacked on top of each other.

Proposition 16. *There are exactly two possible moves from any configuration.*

Proof. This is easily tested by writing down all possible scenarios of topmost discs in terms of size and colour. □

Thus we have that the graph of possible moves and configurations is composed entirely of cycles.

Note that Proposition 7 holds as before. Note also that if disc n is to move, we have that all of discs $1, 2, \ldots, n-1$ must be of alternating colours. This is easily seen since, when disc n moves from a peg to another peg, all of the smaller discs must be on another peg, which cannot be done if they do not have alternating colours.

Thus we will only consider the case where all discs have alternating colours. Otherwise, if the alternating colour pattern fails at disc m (i.e. disc m and disc $m-1$ have the same colour) then disc $m+1$ can never move.

5.2. The specific and general cases

Proposition 17. *The n-th move in the base case algorithm moves the disc m such that m satisfies $2^m \mid n$ and 2^{m+1} does not divide n.*

Proof. We proceed via induction. The case for $n = 1$ is obvious.

For $n > 1$, the first $2^{n-1} - 1$ moves cannot encounter any contradiction by the induction assumption. On the 2^n-th move, the n-th disc is moved, satisfying the condition. At the k-th move after the 2^n-th move, the m-th disc is moved where $2^m \mid k$ and $2^{m=1}$ does not divide k. Then $2^m \mid k + 2^n$ and 2^{m+1} does not divide $k + 2^n$, since $n > m$. The proof is complete. □

Proposition 18. *All configurations encountered in the base case algorithm satisfy the alternating colour property.*

Proof. We proceed by induction.

For $n = 1$ and 2, the conclusion is obvious.

For $n > 2$, consider the bottommost disc. It is given that the base case algorithm satisfies that odd-numbered discs only move clockwise and even-numbered discs only move anticlockwise, as was proved above. Without loss of generality, the largest disc n can be assumed to be odd. Consider the first $2^{n-1} - 1$ moves. If an odd disc i is placed upon disc n, it has moved an number of moves that is divisible by 3. Consider the disc $i + 1$. It will have moved a number of moves that is also divisible by 3. $\square$

Reviewer's Comments

Grammatical mistakes and typos

1 The reviewer has comments on the wordings, which have been amended in this paper.
2 descending → ascending
3 The formula for l_n should be $l_n = 2l_{n-1} + s_{n-1} + 2$
4 Same, the formula for l_n should be $l_n = 2l_{n-1} + s_{n-1} + 2$

Comments

The paper generalized the famous Tower of Hanoi problem. The author investigated the numbers of moves needed to change between different arrangements of discs. An algorithm for the computation was given. Some variations of the problem with different sets of rules were also studied.

This research problem is interesting. The paper is well-written and the organization of is good. A suggestion is that the uniqueness of shortest path between different configurations in the standard game should be discussed. There are minor mistakes in some of the statements and proofs. For instance, proposition 16 is not correct for the configuration 12. There are also mistakes in the numbering of theorems.