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A MARKOV MODEL OF THE BUSY FOOTBRIDGE PROBLEM

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Abstract. The central problem we are investigating is based on a problem from the 2018 Singapore International Mathematics Challenge. It is about a mathematical model of the probabilities that the people on a footbridge from two sides meet. In our paper, we generalize the contest problem in various cases. We develop a Markov model then formulate a transition matrix to solve the generalized problem. Also, we define an expansion rule of the transition matrices to reduce the time complexity to compute. Furthermore, we propose a new topic on the expected number of collisions. We tackle the problem by performing Jordan decomposition. Lastly, we optimize the method of finding eigenvalues by observing the recursive relationships in transition matrices.

1. The Contest Problem

The central problem we are investigating is based on a problem from the 2018 Singapore International Mathematics Challenge [1] organized by the National University of Singapore. We construct a mathematical model about the probabilities that people on a two-lane system collide. We find that the change in probabilities due to different constraints is inspiring. The difficulty of calculation increases when there are more people walking on the footbridge.

As a result, we decide to use the knowledge on matrices and apply Markov Chain on the Busy Footbridge Problem. In this section, we introduce the Busy Footbridge problem and some patterns we observe.

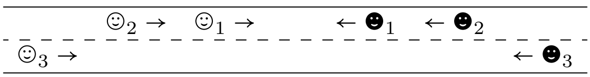
1.1. The Busy Footbridge Problem

Here comes the contest problem from Section C, the 2018 Singapore International Mathematics Challenge.

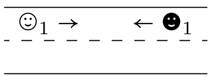
Consider a long East-West footbridge, where two groups of people from each side are heading to the opposite directions. Provided that the footbridge is just wide enough for two people to pass each other, and hence can be considered as having two lanes. People arrive in a steady stream from both ends and join either lane randomly with equal probability.

When two people walking in opposite directions meet, there is a ***collision*** if they are on the same lane and one of the two people must change to the other lane, with equal probability, before they can pass each other and continue walking. In the absence of collisions, people remain in their lane, and we assume that people walking in the same direction are sufficiently spread out that they do not overtake or interact with each other.

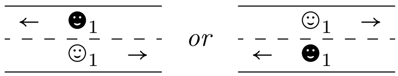
Definition 1. *We define $\smiley_m$ be the m^{th} person from the West and $\smiley$ be any person from the West. We define $\blacksmiley_n$ be the n^{th} person from the East and $\blacksmiley$ be any person from the East.*



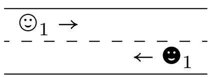
Example 2. *This example illustrates the collisions and changes in lanes when two people ($\smiley_1$ and $\blacksmiley_1$) meet.*
***Case* 1:** *When the two people are on the same lane, a collision occurs.*



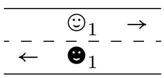
Case 1: outcomes *Either $\smiley_1$ changes lane and $\blacksmiley_1$ remains on the upper lane, or vice versa.*



***Case* 2:** *When the two people are on different lanes, no collision occurs.*



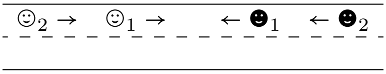
***Case* 2 outcome:** *They pass each other and remain on their original lanes.*



1.2. Illustrative examples

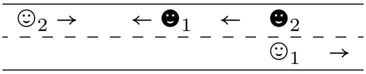
When we consider the footbridge with more people, the problem becomes complicated.

Example 3. *Suppose there are two people from each side and they are arranged in the following state, what is the probability that* $☺_2$ *collides with* $☻_2$*?*



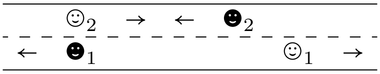
Note that $☺_1$ *must collide with* $☻_1$*. The positions of* $☻_1$ *and* $☻_2$ *after* $☺_1$ *passes through will be listed in the following cases.*

***Case* 1:** $☺_1$ *collides with* $☻_1$ *and changes to the lower lane.*

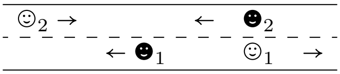


In order to collide with $☻_2$, $☺_2$ *must remain on the upper lane after colliding with* $☻_1$.

$$Pr(☺_2 \text{ collides with } ☻_2 \mid \text{Case 1}) = \frac{1}{2}$$

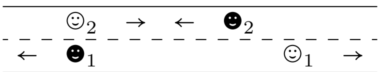


***Case* 2:** $☺_1$ *collides with* $☻_1$ *and remains on the upper lane. Then* $☺_1$ *collides with* $☻_2$ *and changes to the bottom lane.*

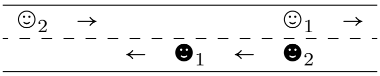


$☺_2$ *must collide with* $☻_2$.

$$Pr(☺_2 \text{ collides with } ☻_2 \mid \text{Case 2}) = 1$$

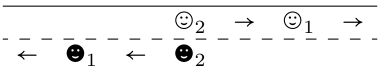


***Case* 3:** $☺_1$ *collides with* $☻_1$ *and remains on the upper lane. Then* $☺_1$ *collides with* $☻_2$ *and remains on the upper lane.*



$☺_2$ *cannot collide with* $☻_2$.

$$Pr(☺_2 \text{ collides with } ☻_2 \mid \text{Case 3}) = 0$$



Hence, in this example,

$$Pr(\smiley_2 \text{ collides with } \blacksmiley_2) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times 1 + \frac{1}{4} \times 0 = \frac{1}{2}$$

Example 4. *Suppose there are three people from each side and they are arranged in the following state. What is the probability that $\smiley_3$ collides with $\blacksmiley_3$?*

		$\smiley_1 \rightarrow$	$\leftarrow \blacksmiley_1$	$\leftarrow \blacksmiley_2$	
$\smiley_3 \rightarrow$	$\smiley_2 \rightarrow$				$\leftarrow \blacksmiley_3$

More cases and possible outcomes have to be considered. Hence, we have to adopt a better approach.

1.3. Mathematical modeling on the Busy Footbridge Problem

Definition 5. *We denote $p_{m,n}$ be the probability that the m^{th} person from the West collides with the n^{th} person from the East ($\smiley_m$ collides with $\blacksmiley_n$). We denote q_n be the limiting value of $p_{m,n}$ (the existence will be proved in Chapter 5) when m tends to infinity for all n, which is*

$$q_n = \lim_{m \to \infty} p_{m,n}$$

The main idea of the Busy Footbridge Problem is to model the configuration of the people and the collisions between them. For a footbridge with sufficiently many people walking on it, what is the probability that the n^{th} person from the East collides with the person very far away from the West (q_n)? In the Singapore International Mathematics Challenge Section C - Busy Footbridge Problem **Question 3**, we are going to calculate $p_{1,n}, p_{2,n}, q_1, q_2$ and q_3 for all people with equal probability to change lane and equal probability to enter the footbridge on one of the two lanes. For further investigation, we are going to generalize the problem to calculate $p_{m,n}$ and q_n for all m, n.

1.4. Monte Carlo Algorithm

Before solving the problem mathematically, we try to tackle it with Monte Carlo Algorithm, a randomized algorithm, to get a clearer concept. [See reviewer's comment (1a)] With the help of computer programs, we randomize the lane that $\smiley$ enters, the movement of $\smiley$ whether it remains on the original lane or changes lane in a collision and the initial positions of $\blacksmiley_n$. There are m $\smiley$ and we put m to be a sufficiently large number ($m = 100$ in our program) and we repeat the process 10^7 times. We count the number of collisions of $\smiley_m$ with each $\blacksmiley_n$.

$$q_n \approx \frac{\text{Number of Collisions of } \blacksmiley_n \text{ with } \smiley_m}{\text{Number of Times}}$$

We obtain the following results.

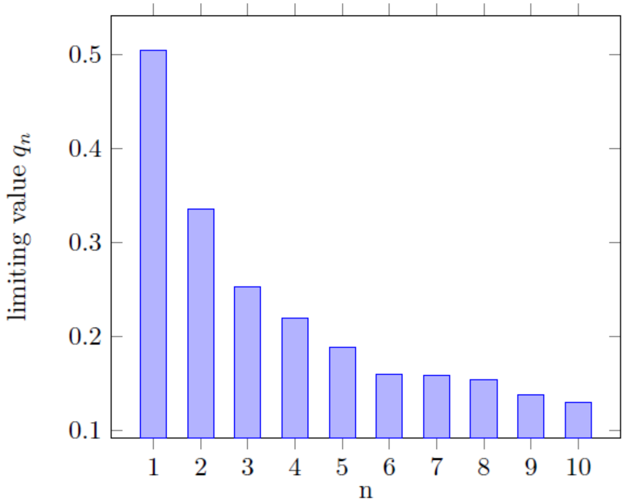


FIGURE 1. The results of q_1 and q_{10}

Observation 6. *The value of q_n seems to be $\dfrac{1}{n+1}$ for small n. From the above results, q_n decreases as n increases. In addition, the rate of decrease of q_n decreases when n increases. However, q_n starts to fluctuate without any easily observable patterns when n becomes larger. [See reviewer's comment (2b)]*

Observation 7. *In the change of the positions of the people from the East during the simulation, we conclude that the decrease in limiting value q_n is due to the increase of the length of consecutive people on the same lane, as it increases the difficulty of clearing out people in front of ☻$_n$.*

1.5. Motivation

We can approximate qn and find their patterns by Monte Carlo Algorithm. However, we would like to find the exact values of $p_{m,n}$ and q_n. In addition, when n and m become larger, the computer does not have sufficient capacity to compute. Hence, the Monte Carlo Algorithm is not efficient enough. We adopt the Markov Model to solve the problem.

2. Essential Background of Markov Chain

In this chapter, we want to introduce Markov Chain into our problem. The movement of people on the footbridge can be interpreted as states changing. People from one side can be in different configurations and we want to find the probability that one configuration changes to another.

2.1. Markov chain

A Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.

Definition 8. *A discrete-time Markov chain is a sequence of random variables $X_1, X_2, \ldots,$ with the probability of moving to the next state depends only on the present state and not on the previous states, such that*

$$Pr(X_{n+1} = x \mid X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = Pr(X_{n+1} = x \mid X_n = x_n)$$

while $Pr(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) > 0$.

2.2. Transition matrix

Definition 9. *Denote $s_{i,j}$ be the probability of moving from state j to state i in one step. The transition matrix (Markov matrix) is a square matrix with dimension $u \times u$ that contains all $s_{i,j}$.*

$$S = \begin{pmatrix} s_{0,0} & s_{0,1} & \cdots & s_{0,u-1} \\ s_{1,0} & s_{1,1} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ s_{u-1,0} & \cdots & \cdots & s_{u-1,u-1} \end{pmatrix}$$

Since the total transition probability from state j to all others is 1, the column sum is $\sum_{i=0}^{u-1} s_{i,j} = 1$.

3. Applying Markov chain to the Busy Footbridge Problem

As $p_{m,n}$ depends on not only the n^{th} person, but also other people on the lanes, we decide to investigate the positions of the whole set of people in the East, instead of one particular person.

3.1. Introduction of the Busy Footbridge Problem solution

We denote V_n^T be the vector with dimension 1×2^n which the i^{th} entry represents the probability that ☺ collides with ☻$_n$ in state i. We denote $L_n^{(m-1)}$ be the vector with dimension $2^n \times 1$ which the i^{th} entry represents the probability that the state i occurs after the first $m-1$ ☺ passes all n ☻. A Markov matrix S_n is introduced to encode the probabilities of transitions between each state. We find $p_{m,n}$ by the following equation.

$$p_{m,n} = V_n^T L_n^{(m-1)} = V_n^T (S_n)^{m-1} L_n^{(0)}$$

[See reviewer's comment (1b)]

3.2. Interpretation of the Busy Footbridge Problem

Since the movement of people on each side of the footbridge is relative, we can interpret the people from the East side (☻) are standing instead of walking, while the people from the West side (☺) are passing the people from the East side (☻) discretely one by one.

Notice that the probability that ☺$_m$ collides with ☻$_n$ depends on the people in front of ☻$_n$, but independent on those behind. So, when calculating $p_{m,n}$, we can interpret that there are only n ☻'s.

3.3. Binary states representation

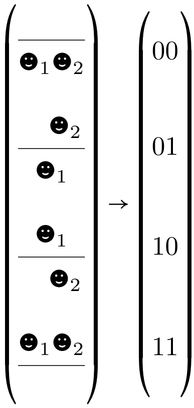
We introduce the binary representation to arrange the states orderly. Without the diagrams of smilies on the footbridge, the binary representation is very important to understand the theorems and lemmas.

Definition 10. *We denote $\Omega^{(n)}$ be the sample space of states and $R^{(n)} \in \Omega^{(n)}$ be a state of n people from the East, where*

$$R^{(n)} = (r_1, r_2, \dots, r_n), \quad r_i = \begin{cases} 1, & \text{if the } i^{th} \text{ person is on the upper lane,} \\ 0 & \text{otherwise.} \end{cases}$$

We denote $|R^{(n)}|$ be a binary number which represents a state. We denote input *states* be the configurations of people from the East before the m^{th} person from the West passes, while *output states* be the configurations of people from the East after the m^{th} person from the West passes.

Example 11. *This example illustrates the correspondence of the states for $n = 2$ into binary state representation.*



3.4. Formulation of transition matrix in Busy Footbridge Problem

A transition matrix is generated such that the rows represent the output states, while the columns represent the input states. Each cell in the matrix represents the probability of each input state changing to each output state.

Definition 12. *Denote S_n be the transition matrix with n people from the East. We denote $s_{n_{i,j}}$ be the probability that the j^{th} input state changes to the i^{th} output state. Therefore,*

$$S_n = \begin{pmatrix} s_{n_{0,0}} & s_{n_{0,1}} & \cdots & s_{n_{0,2^{n-1}-1}} \\ s_{n_{1,0}} & s_{n_{1,1}} & \cdots & s_{n_{1,2^{n-1}-1}} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n_{2^{n-1}-1,0}} & s_{n_{2^{n-1}-1,1}} & \cdots & s_{n_{2^{n-1}-1,2^{n-1}-1}} \end{pmatrix}$$

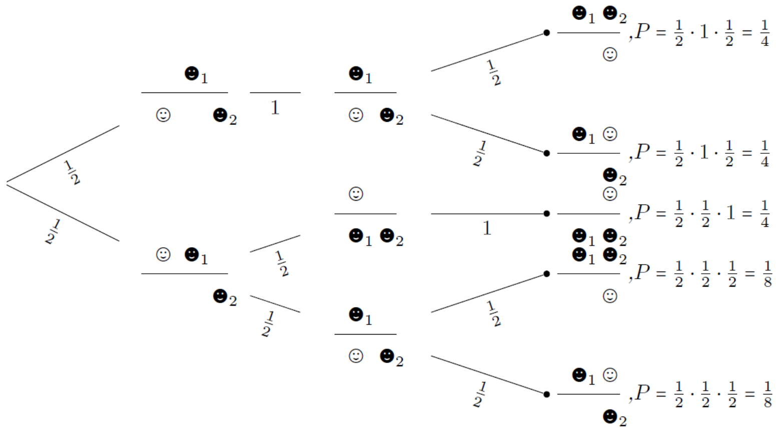
Example 13. *This example illustrates the probabilities that each input state changes to each output state in S_2. Each column index represents a possible input state while each row index represents a possible output state.*

	00	01	10	11
00	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
01	0	$\frac{3}{8}$	0	$\frac{1}{8}$
10	$\frac{1}{8}$	0	$\frac{3}{8}$	0
11	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{4}$

3.5. Forming transition matrix by exhaustion method

We perform exhaustion to find S_n. For each input state, we simulate the movement of a ☺. We enumerate all possible outcomes of its movements, move on to the next position and simulate it recursively and obtain the probabilities of state changes.

Example 14. *The following example illustrates the probabilities of the input state:* (10)*, changing to each output state, which are the probabilities of the* 2^{nd} *column in* S_2*.*



$$s_{2_{0,2}} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$
$$s_{2_{1,2}} = 0$$
$$s_{2_{2,2}} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$
$$s_{2_{3,2}} = \frac{1}{4}$$

We obtain the following results by performing exhaustion.

$$S_1 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$S_2 = \begin{pmatrix} \frac{3}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{3}{8} & 0 & \frac{1}{8} \\ \frac{1}{8} & 0 & \frac{3}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{3}{4} \end{pmatrix}$$

$$S_3 = \begin{pmatrix} \frac{3}{4} & \frac{3}{8} & \frac{3}{8} & \frac{3}{16} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{3}{8} & 0 & \frac{3}{16} & 0 & \frac{1}{8} & 0 & \frac{1}{16} \\ 0 & 0 & \frac{3}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{3}{16} & \frac{3}{8} & 0 & 0 & \frac{1}{16} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{16} & 0 & 0 & \frac{3}{8} & \frac{3}{16} & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 & 0 & \frac{3}{16} & 0 & 0 \\ \frac{1}{16} & 0 & \frac{1}{8} & 0 & \frac{3}{16} & 0 & \frac{3}{8} & 0 \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{3}{16} & \frac{3}{8} & \frac{3}{8} & \frac{3}{4} \end{pmatrix}$$

Note that the exhaustion method takes $O(2^n \times 2^n)$ time to perform.

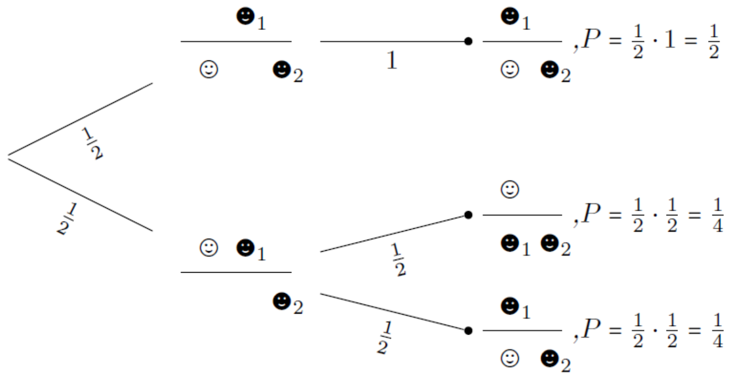
3.6. Forming collision vector by exhaustion method

Definition 15. *Denote V_n^T be the vector with dimension 1×2^n which the i^{th} entry represents the probability that a person from the West collides with the n^{th} person from the East in state i.*

$$V_n = \begin{pmatrix} v_{n_0} \\ v_{n_1} \\ \vdots \\ v_{n_{2^n-1}} \end{pmatrix}$$

We perform exhaustion to find V_n. For each input state, we simulate the movement of ☺. We enumerate all possible outcomes of its movements, move on to the next position and simulate it recursively and obtain the probability that ☺ collides with ☻$_n$.

Example 16. *The following example illustrates the probability of a person from the West collides with the n^{th} person in the input state:* (10)*, which is the probability of the 2^{nd} cell in V_2.*



$$v_{2_2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

We obtain the following results by performing exhaustion.

$$V_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad V_2 = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}, \quad V_3 = \begin{pmatrix} \frac{1}{8} \\ \frac{7}{8} \\ \frac{5}{8} \\ \frac{3}{8} \\ \frac{3}{8} \\ \frac{5}{8} \\ \frac{7}{8} \\ \frac{1}{8} \end{pmatrix}$$

Note that exhaustion method also takes $O(2^n \times 2^{2^n})$ time to perform.

3.7. Forming initial vector

Definition 17. *We denote $L_n^{(m-1)}$ be the vector with dimension $2^n \times 1$ which the i^{th} entry represents the probability that the state i occurs after the $m-1^{th}$ person* *from the West passes the n^{th} person from the East.*

$$L_n^{(0)} = \begin{pmatrix} \frac{1}{2^n} \\ \vdots \\ \frac{1}{2^n} \end{pmatrix}$$

The product of the transition matrix and the initial vector represents the probabilities that each state occurs after the first person from the West passes $R^{(n)}$. Inductively, the product of the transition matrix raised to an m^{th} power and the initial vector represents the probabilities that each state occurs after m people from the West passes $R^{(n)}$.

$$L_n^{(m)} = (S_n)^m L_n^{(0)}$$

Example 18. *We will calculate $p_{10,2}$.*

$$\begin{aligned} p_{10,2} &= \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}^T \begin{pmatrix} \frac{3}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{3}{8} & 0 & \frac{1}{8} \\ \frac{1}{8} & 0 & \frac{3}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{3}{4} \end{pmatrix}^9 \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}^T \begin{pmatrix} \frac{440235}{1048576} & \frac{439209}{1048576} & \frac{434601}{1048576} & \frac{433579}{1048576} \\ \frac{85077}{1048576} & \frac{86103}{1048576} & \frac{88663}{1048576} & \frac{89685}{1048576} \\ \frac{89685}{1048576} & \frac{88663}{1048576} & \frac{86103}{1048576} & \frac{85077}{1048576} \\ \frac{433579}{1048576} & \frac{434601}{1048576} & \frac{439209}{1048576} & \frac{440235}{1048576} \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \\ &= \frac{174763}{524288} \end{aligned}$$

The above calculation takes $O((2^n)^3 \times lg(m))$. When n becomes large, the computing time increases exponentially. However, when m becomes large, the computing time increases logarithmically. Note that $p_{m,n} = p_{n,m}$. So, if $n > m$, we can calculate $p_{n,m}$ instead of $p_{m,n}$ in order to reduce the time complexity.

Example 19. *The following is an example that $n > m$:*

$$
\begin{aligned}
&p_{3,5} = p_{5,3}\\
&= \begin{pmatrix} \frac{1}{8} \\ \frac{7}{8} \\ \frac{5}{8} \\ \frac{3}{8} \\ \frac{3}{8} \\ \frac{5}{8} \\ \frac{7}{8} \\ \frac{1}{8} \end{pmatrix}^T
\begin{pmatrix}
\frac{3}{4} & \frac{3}{8} & \frac{3}{8} & \frac{3}{16} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{16} \\
0 & \frac{3}{8} & 0 & \frac{3}{16} & 0 & \frac{1}{8} & 0 & \frac{1}{16} \\
0 & 0 & \frac{3}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 \\
0 & 0 & \frac{3}{16} & \frac{3}{8} & 0 & 0 & \frac{1}{16} & \frac{1}{8} \\
\frac{1}{8} & \frac{1}{16} & 0 & 0 & \frac{3}{8} & \frac{3}{16} & 0 & 0 \\
0 & \frac{1}{16} & 0 & 0 & 0 & \frac{3}{16} & 0 & 0 \\
\frac{1}{16} & 0 & \frac{1}{8} & 0 & \frac{3}{16} & 0 & \frac{3}{8} & 0 \\
\frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{3}{16} & \frac{3}{8} & \frac{3}{8} & \frac{3}{4}
\end{pmatrix}^4
\begin{pmatrix} \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{8} \\ \frac{7}{8} \\ \frac{5}{8} \\ \frac{3}{8} \\ \frac{3}{8} \\ \frac{5}{8} \\ \frac{7}{8} \\ \frac{1}{8} \end{pmatrix}^T
\begin{pmatrix}
\frac{14955}{32768} & \frac{13973}{32768} & \frac{12785}{32768} & \frac{11871}{32768} & \frac{11121}{32768} & \frac{10335}{32768} & \frac{9243}{32768} & \frac{8517}{32768} \\
\frac{757}{32768} & \frac{1739}{32768} & \frac{1839}{32768} & \frac{2753}{32768} & \frac{1455}{32768} & \frac{2241}{32768} & \frac{2373}{32768} & \frac{3099}{32768} \\
\frac{245}{32768} & \frac{139}{32768} & \frac{239}{32768} & \frac{65}{32768} & \frac{303}{32768} & \frac{129}{32768} & \frac{261}{32768} & \frac{27}{32768} \\
\frac{1451}{32768} & \frac{1557}{32768} & \frac{2545}{32768} & \frac{2719}{32768} & \frac{2481}{32768} & \frac{2655}{32768} & \frac{3483}{32768} & \frac{3717}{32768} \\
\frac{3717}{32768} & \frac{3483}{32768} & \frac{2655}{32768} & \frac{2481}{32768} & \frac{2719}{32768} & \frac{2545}{32768} & \frac{1557}{32768} & \frac{1451}{32768} \\
\frac{27}{32768} & \frac{261}{32768} & \frac{129}{32768} & \frac{303}{32768} & \frac{65}{32768} & \frac{239}{32768} & \frac{139}{32768} & \frac{245}{32768} \\
\frac{3009}{32768} & \frac{2373}{32768} & \frac{2241}{32768} & \frac{1455}{32768} & \frac{2753}{32768} & \frac{1839}{32768} & \frac{1739}{32768} & \frac{757}{32768} \\
\frac{8517}{32768} & \frac{9243}{32768} & \frac{10335}{32768} & \frac{11121}{32768} & \frac{11871}{32768} & \frac{12785}{32768} & \frac{13937}{32768} & \frac{14955}{32768}
\end{pmatrix}
\begin{pmatrix} \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \end{pmatrix}\\
&= \frac{269}{1024}
\end{aligned}
$$

With the formula, we are able to compute $p_{m,n}$ for all m and n. However, it takes a long time to compute the transition matrix, as well as the collision vector. So, we are going to use another method to compute the transition matrix in the next chapter. In addition, the probability that a person changes lanes in a collision and the probability that a person enters one of the two lanes are fixed and both are $\frac{1}{2}$ in this chapter. We would like to further investigate the problem with variable probabilities.

4. Generating Transition Matrix, Initial Vector and Collision Vector

In this chapter, we will generate the transition matrix S_n and the collision vector V_n with another method. We try to observe the patterns between S_1, S_2, S_3 and so on. We propose a method to generate S_n and V_n recursively. In addition, we generalize the problem and method for the probabilities that a person changes lanes ρ in a collision and enters lanes σ.

4.1. Forming Markov matrix recursively

We are going to demonstrate and prove the transition matrix expansion rule which generates S_n recursively.

Notation 20. *We write the transition matrix S_n as the following:*

$$S_n = \begin{pmatrix} s_{n_{0,0}} & s_{n_{0,1}} & \cdots & s_{n_{0,2^n-1}} \\ s_{n_{1,0}} & s_{n_{1,1}} & \cdots & s_{n_{1,2^n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n_{2^n-1,0}} & s_{n_{2^n-1,1}} & \cdots & s_{n_{2^n-1,2^n-1}} \end{pmatrix}$$

Define

$$M_{n_{i,j}} = \begin{pmatrix} s_{n_{2i,2j}} & s_{n_{2i,2j+1}} \\ s_{n_{2i+1,2j}} & s_{n_{2i+1,2j+1}} \end{pmatrix}$$

Then

$$S_n = \begin{pmatrix} s_{n_{0,0}} & s_{n_{0,1}} & \cdots & s_{n_{0,2^n-1}} \\ s_{n_{1,0}} & s_{n_{1,1}} & \cdots & s_{n_{1,2^n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n_{2^n-1,0}} & s_{n_{2^n-1,1}} & \cdots & s_{n_{2^n-1,2^n-1}} \end{pmatrix}$$
$$= \begin{pmatrix} M_{n_{0,0}} & M_{n_{0,1}} & \cdots & M_{n_{0,2^n-1}} \\ M_{n_{1,0}} & M_{n_{1,1}} & \cdots & M_{n_{1,2^n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n_{2^n-1,0}} & M_{n_{2^n-1,1}} & \cdots & M_{n_{2^n-1,2^n-1}} \end{pmatrix}$$

Definition 21. *Let $R^{(n)} \in \Omega^{(n)}$. Define $(R^{(n)})_*, (R^{(n)})^* \in \Omega^{(n+1)}$ by the followingL*

- *$(R^{(n)})_*$ refers to appending one more person on the* **lower lane** *on the right of the n^{th} person in $R^{(n)}$, which is appending $r_{n+1} = 0$ to $R^{(n)}$.*
- *$(R^{(n)})^*$ refers to appending one more person on the* **upper lane** *on the right of the n^{th} person in $R^{(n)}$, which is appending $r_{n+1} = 0$ to $R^{(n)}$.*

Observation 22. *When ☺ passes $R^{(n-1)}$, and $R^{(n-1)}$ changes to a new state $R'^{(n-1)} \in \Omega^{(n-1)}$, the person from the West will be on the different lane of the $(n-1)^{th}$ person from the East just after they meet each other. If a collision occurs, either one will move to the other lane. If there is no collision, they will remain on their original lane, which are on different lanes.*

Example 23. *This example illustrates* **Observation 22**. *According to the tree diagram in Example 14, suppose* (10) *be the input state, the possible output states are* (11) *or* (10) *or* (00). *We can see that the person from the West is on the different lane of the* 2^{nd} *person from the East. That is,*

$$\begin{array}{c} ☻_1 \quad ☻_2 \\ \hline \qquad ☺ \end{array} \quad or \quad \begin{array}{c} ☻_1 \quad ☺ \\ \hline \qquad ☻_2 \end{array} \quad or \quad \begin{array}{c} \qquad ☺ \\ \hline ☻_1 \quad ☻_2 \end{array}$$

Lemma 24. *The relations between the elements in* S_{n-1} *and the elements in* S_n *can be represented by the following:*

$$s_{n-1_{i,j}} = s_{n_{2i,2j}} + s_{n_{2i+1,2j}} = s_{n_{2i,2j+1}} + s_{n_{2i+1,2j+1}}$$

Proof. Let $A, B \in \Omega^{(n-1)}$ and $|A| = j$ and $|B| = i$. Then $|A_*| = 2j$ and $|A^*| = 2j + 1$. After a ☺ passes $R^{(n)}$, $s_{n-1_{i,j}}$ represents the probability of A changing to B. Just after the person passes or collides with the $(n-1)^{th}$ person from the East, A_* changes into either B_* or B^* , so the sum of the probabilities that A_* changes to B_* or B^* equal to the probability that A changes to B. Similarly, the sum of the probabilities that A^* changes to B_* or B^* also equal to the probability that A changes to B. □

Example 25. *Suppose* $A = (10)$ *and* $B = (00)$ *and* $Pr(B \mid A) = s_{2_{0,2}} = \frac{3}{8}$. *By* **Observation 22**, ☺ *is on the upper lane when it just passes* ☻$_2$. *We can divide the problem into the following cases.*

Case 1: A_* *changes to* B_*. $Pr(B_* \mid A_*) = 1 \times \dfrac{3}{8} = \dfrac{3}{8}$ *as* ☻ *is on the lower lane,* ☺ *does not collide with* ☻$_3$. *So, the state remains unchanged.*

Case 2: A_* *changes to* B^*. $Pr(B^* \mid A_*) = 0 \times \dfrac{3}{8} = 0$. ☺ *has to collide with* ☻$_2$ *and remain on the bottom lane in order to collide with* ☻$_3$ *and to change* ☻$_3$ *to the upper lane. So, it is unable to change to* B^*.'

Case 3: A^* *changes to* B^* *or* B_*, $Pr(B^* \mid A^*) = Pr(B_* \mid A^*) = \dfrac{1}{2} \times \dfrac{3}{8} = \dfrac{3}{16}$, ☺ *must collide with* ☻$_3$ *and* ☻$_3$ *has* $\dfrac{1}{2}$ *to be on the upper lane or lower lane.*

Then,

$$Pr(B_* \text{ or } B^* \mid A^*) = s_{3_{0,5}} + s_{3_{1,5}} = \frac{3}{16} + \frac{3}{16} = \frac{3}{8} = s_{2_{0,2}}$$

and

$$Pr(B_* \text{ or } B^* \mid A_*) = s_{3_{0,4}} + s_{3_{1,4}} = \frac{3}{8} + 0 = \frac{3}{8} = s_{2_{0,2}}$$

Theorem 26. *(Expansion Rule)*

$$\begin{pmatrix} M_{n_{i,j}} \\ M_{n_{i+1,j}} \end{pmatrix} = \begin{pmatrix} s_{n_{2i,2j}} & s_{n_{2i,2j+1}} \\ s_{n_{2i+1,2j}} & s_{n_{2i+1,2j+1}} \\ s_{n_{2i+2,2j}} & s_{n_{2i+2,2j+1}} \\ s_{n_{2i+3,2j}} & s_{n_{2i+3,2j+1}} \end{pmatrix} = \begin{pmatrix} s_{n-1_{i,j}} & \frac{1}{2} \times s_{n-1_{i,j}} \\ 0 & \frac{1}{2} \times s_{n-1_{i,j}} \\ \frac{1}{2} \times s_{n-1_{i+1,j}} & 0 \\ \frac{1}{2} \times s_{n-1_{i+1,j}} & s_{n-1_{i+1,j}} \end{pmatrix},$$

where $i \equiv 0 \pmod 2$.

Proof. Suppose $A, B \in \Omega^{(n-2)}$. In S_{n-1}, since $i \equiv 0 \pmod 2$, there exists a $|B_*| = i$ and $|(B_*)^*| = 2i+1$. For $j \equiv 0 \pmod 2$, there exists a $|A_*| = j$ so $|(A_*)_*| = 2j$. Since $(A_*)_*$ is unable to change to $(B_*)^*$, $s_{n_{2i+1,2j}} = 0$.

For $j \equiv 1 \pmod 2$, there exists a $|A^*| = j$ so $|(A^*)_*| = 2j$. Since $(A^*)_*$ is unable to change to $(B_*)^*$, $s_{n_{2i+1,2j}} = 0$. By Lemma 24,

$$s_{n_{2i,2j+1}} + s_{n_{2i+1,2j+1}} = s_{n-1_{i,j}} \quad \text{and} \quad s_{n_{2i,2j}} = s_{n-1_{i,j}}$$

Similarly, $s_{n_{2i+2,2j+1}} = 0$ and $s_{n_{2i+3,2j+1}} = s_{n-1_{i+1,j}}$.

Suppose $C \in \Omega^{(n-1)}, D \in \Omega^{(n-2)}$. In S_{n-1}, since $i \equiv 0 \pmod 2$, there exist a $|D_*| = i$ so $|(D_*)_*| = 2i$ and $|(D_*)^*| = 2i + 1$. There also exists a $|C| = j$ and $|C^*| = 2j + 1$.

Given that a person from the West changes the people from the East from state C to D_*, by **Observation 22**, that person must be on the upper lane after passing the $(n-1)^{th}$ person, and must collide with the n^{th} person from the East. The n^{th} person from the East has equal probability to change or remain after the collision which leads to two outcomes, $|(D_*)_*| = 2i$ and $|(D_*)^* = 2i + 1$. So, there is equal probability of $(D_*)_*$ and $(D_*)^*$ to occur. By **Lemma 24**,

$$s_{n_{2i,2j+1}} = s_{n_{2i+1,2j+1}} = \frac{1}{2} s_{n-1_{i,j}}$$

Similarly,

$$s_{n_{2i+2,2j}} = s_{n_{2i+3,2j}} = \frac{1}{2} s_{n-1_{i+1,j}}$$

□

In the above situation, that is with a $\frac{1}{2}$ probability which a person changes lane in a collision and $\frac{1}{2}$ probability that a person chooses the lane, we are able to use the above methods to generate the transition matrix S_n for all n.

Using Theorem 26, we can find S_n in terms of $S_{(n-1)}$. Therefore, we could generate the transition matrix S_n for all n recursively.

4.2. General extension of transition matrix

We generalize the above theorems, with various probabilities that a person enters the footbridge on one of the two lanes and changes lane in a collision.

Definition 27. *Denote σ be the probability that people from both sides enter the footbridge on the upper lane. Denote ρ_n be the probability that the n^{th} person from the East remains on the original lane in a collision occurs.*

We first represent S_1 with ρ_1 and σ

$$S_1 = \begin{pmatrix} 1-(1-\sigma)(1-\rho_1) & \sigma(1-\rho_1) \\ (1-\sigma)(1-\rho_1) & 1-\sigma(1-\rho_1) \end{pmatrix}$$

Theorem 28.

$$\begin{pmatrix} M_{n_{i,j}} \\ M_{n_{i+1,j}} \end{pmatrix} = \begin{pmatrix} s_{n_{2i,2j}} & s_{n_{2i,2j+1}} \\ s_{n_{2i+1,2j}} & s_{n_{2i+1,2j+1}} \\ s_{n_{2i+2,2j}} & s_{n_{2i+2,2j+1}} \\ s_{n_{2i+3,2j}} & s_{n_{2i+3,2j+1}} \end{pmatrix} = \begin{pmatrix} s_{n-1_{i,j}} & (1-\rho_n)s_{n-1_{i,j}} \\ 0 & \rho_n s_{n-1_{i,j}} \\ \rho_n s_{n-1_{i+1,j}} & 0 \\ (1-\rho_n)s_{n-1_{i+1,2}} & s_{n-1_{i+1,j}} \end{pmatrix}$$

where $i \equiv 0 \pmod 2$.

Proof. The proof is similar to **Theorem 26**. Suppose $A, B \in \Omega^{(n-2)}$. In S_{n-1}, since $i \equiv 0 \pmod 2$, there exists a $|B_*| = i$ and $|(B_*)^*| = 2i+1$.

For $j \equiv 0 \pmod 2$, there exists a $|A_*| = j$ so $|(A_*)_*| = 2j$. Since $(A_*)_*$ is unable to change to $(B_*)^*$, $s_{n_{2i+1,2j}} = 0$.

For $j \equiv 1 \pmod 2$, there exists a $|A^*| = j$ so $|(A^*)_*| = 2j$. Since $(A^*)_*$ is unable to change to $(B_*)^*$, $s_{n_{2i+1,2j}} = 0$. By **Lemma 24**,

$$s_{n_{2i,2j+1}} + s_{n_{2i+1,2j+1}} = s_{n-1_{i,j}} \quad \text{and} \quad s_{n_{2i,2j}} = s_{n-1_{i,j}}$$

Similarly, $s_{n_{2i+2,2j+1}} = 0$ and $s_{n_{2i+3,2j+1}} = s_{n-1_{i+1,j}}$.

Suppose $C \in \Omega^{(n-1)}, D \in \Omega^{(n-2)}$. In S_{n-1}, since $i \equiv 0 \pmod 2$, there exists a $|D_*| = i$ so $|(D_*)_*| = 2i$ and $|(D_*)^*| = 2i+1$. There also exists a $|C| = j$ and $|C^*| = 2j+1$.

Given that a person from the West changes the people from the East from state C to D_*, by **Observation 22**, the person from the West must be on the upper lane after passing the $(n-1)^{th}$ person, and must collide with the n^{th} person from the East. The n^{th} person from the East has probability ρ_n to remain after the collision which leads to two outcomes, $|(D_*)_*| = 2i$ and $|(D_*)^*| = 2i+1$. By **Lemma 24**,

$$s_{n_{2i,2j+1}} = (1-\rho_n)s_{n-1_{i,j}} \quad \text{and} \quad s_{n_{2i+1,2j+1}} = \rho_n s_{n-1_{i,j}}$$

Similarly,

$$s_{n_{2i+2,2j}} = \rho_n s_{n-1_{i,j}} \quad \text{and} \quad s_{n_{2i+3,2j}} = (1-\rho_n)s_{n-1_{i,j}}$$

□

4.3. Forming collision vector recursively

Appending $r_{n+1} = 0$ or 1 to $R^n \in \Omega^n$, which is appending $(n+1)^{th}$ people to V_n, we can obtain V_{n+1}. We expand each cell in V_n to two cells at the corresponding position to obtain V_{n+1}.

Notation 29. *We write the collision vector V_n as the following:*

$$V_n = \begin{pmatrix} v_{n_0} \\ v_{n_1} \\ \vdots \\ v_{n_{2^n-1}} \end{pmatrix}$$

Define

$$U_{n_i} = \begin{pmatrix} v_{n_{2i}} \\ v_{n_{2i+1}} \end{pmatrix}$$

Then

$$V_n = \begin{pmatrix} v_{n_0} \\ v_{n_1} \\ \vdots \\ v_{n_{2^n-1}} \end{pmatrix} = \begin{pmatrix} U_{n_0} \\ U_{n_1} \\ \vdots \\ U_{n_{2^n-1}} \end{pmatrix}$$

Lemma 30.

$$v_{n_{2i}} + v_{n_{2i+1}} = 1$$

Proof. Suppose $A \in \Omega^{(n-1)}$ and $|A| = i$. Then $|A_*| = 2i$ and $|A^*| = 2i+1$. Just after a person from the West meets the $(n-1)^{th}$ person from the East, that person from the West can be on either the upper lane or the lower lane. It collides with the n^{th} person in A_* if he is on the lower lane or with the n^{th} person in A^* if he is on the upper lane, so the sum of the probabilities of colliding with the n^{th} person in states A_* and A^* is 1. □

Theorem 31. *(Expansion Rule)*

$$\begin{pmatrix} U_{n_i} \\ U_{n_{i+1}} \end{pmatrix} = \begin{pmatrix} v_{n_{2i}} \\ v_{n_{2i+1}} \\ v_{n_{2i+2}} \\ v_{n_{2i+3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \times v_{n-1_i} \\ 1 - \frac{1}{2} \times v_{n-1_i} \\ 1 - \frac{1}{2} \times v_{n-1_{i+1}} \\ \frac{1}{2} \times v_{n-1_{i+1}} \end{pmatrix}$$

where $i \equiv 0 \pmod 2$.

Proof. In V_{n-1}, for $i \equiv 0 \pmod 2$, there exists a $|A_*| = i$. When we append the n^{th} person from the East on the lower lane, the state can be represented by $(A_*)_*$. The collision with the n^{th} person from the East only occurs if the person from the West collides with the $(n-1)^{th}$ person and remains on the lower lane. The probability that the person from the West remains on its original lane after the collision is $1 - \frac{1}{2} = \frac{1}{2}$.

$$v_{n_{2i}} = \frac{1}{2} v_{n-1_i}$$

From **Lemma 30**,

$$v_{n_2 i} + v_{n_{2i+1}} = 1 \quad \text{and} \quad \frac{1}{2} v_{n-1_i} + v_{n_{2i+1}} = 1 \quad \text{and} \quad v_{n_{2i+1}} = 1 - \frac{1}{2} v_{n-1_i}$$

Similarly, for $i \equiv 1 \pmod 2$,

$$v_{n_{2i+3}} = \frac{1}{2} v_{n-1_{i+1}} \quad \text{and} \quad v_{n_{2i+2}} = 1 - \frac{1}{2} v_{n-1_{i+1}}$$

□

4.4. General extension of collision vector

We first represent V_1 in terms of σ.

$$V_i = \begin{pmatrix} 1 - \sigma \\ \sigma \end{pmatrix}$$

Theorem 32.

$$\begin{pmatrix} U_{n_i} \\ U_{n_{i+1}} \end{pmatrix} = \begin{pmatrix} v_{n_{2i}} \\ v_{n_{2i+1}} \\ v_{n_{2i+2}} \\ v_{n_{2i+3}} \end{pmatrix} = \begin{pmatrix} (1 - \rho_{n-1}) v_{n-1} \\ 1 - (1 - \rho_{n-1}) v_{n-1} \\ 1 - (1 - \rho_{n-1}) v_{n-1_{i+1}} \\ (1 - \rho_{n-1}) v_{n-1_{i+1}} \end{pmatrix}$$

where $i \equiv 0 \pmod 2$.

Proof. The proof is similar to **Theorem 31**. In V_{n-1}, for $i \equiv 0 \pmod 2$, there exists a $|A_*| = i$. When we append the n^{th} person from the East on the lower lane, the state can be represented by $(A_*)_*$. The collision with the n^{th} person in the East only occurs if the person from the West collides with the $(n-1)^{th}$ person and remains on the lower lane. The probability that the person from the West remains on the original lane after the collision is $1 - \rho_{n-1}$.

When we go through the whole process for other cases, we have a conclusion of **Lemma 30**. Each person from the East can have different ρ, and we can use the same method to solve the problem, as the last person is independent to the previous $n - 1$ people. □

4.5. General extension of initial vector

Appending $r_{n+1} = 0$ or 1 to $R^n \in \Omega^n$, which is appending $(n+1)^{th}$ people to $L_n^{(0)}$, we can obtain $L_{n+1}^{(0)}$. We expand each cell in $L_n^{(0)}$ to two cells at the corresponding position to obtain $L_{n+1}^{(0)}$.

Notation 33. *We write the collision vector $L_n^{(0)}$ as the following:*

$$L_n = \begin{pmatrix} l_{n_0} \\ l_{n_1} \\ \vdots \\ l_{n_{2^n-1}} \end{pmatrix}$$

Define

$$H_{n_i} = \begin{pmatrix} l_{n_{2i}} \\ l_{n_{2i+1}} \end{pmatrix}$$

Then

$$L_n = \begin{pmatrix} l_{n_0} \\ l_{n_1} \\ \vdots \\ l_{n_{2^n-1}} \end{pmatrix} = \begin{pmatrix} H_{n_0} \\ H_{n_1} \\ \vdots \\ H_{n_{2^n-1}} \end{pmatrix}$$

We first represent L_1 in terms of σ

$$L_1 = \begin{pmatrix} 1-\sigma \\ \sigma \end{pmatrix}$$

Theorem 34. *(Expansion Rule)*

$$H_{n_i} = \begin{pmatrix} l_{n_{2i}} \\ l_{n_{2i+1}} \end{pmatrix} = \begin{pmatrix} (1-\sigma) l_{n-1_i} \\ \sigma l_{n-1_i} \end{pmatrix}$$

Proof. Suppose $A \in \Omega^{(n-1)}$ and $|A| = i$. The $|A_*| = 2i$ and $|A^*| = 2i+1$. When we append the n^{th} person from the East on the lower lane, A changes to either A_* or A^*. The n^{th} person from the East has probability $(1-\sigma)$ to enter the footbridge on the lower lane, changing the state to A_*. Similarly, it has probability σ to enter the footbridge on the upper lane, changing the state to A^*. Therefore,

$$l_{n_{2i}} = (1-\sigma) l_{n-1} \quad \text{and} \quad l_{n_{2i+1}} = \sigma l_{n-1_i}$$

$\square$

Using the above lemmas, we are now able to generate the transition matrices and collision vector with the Expansion rules recursively instead of performing exhaustion. By looking at S_1 and V_1, we can avoid the large amount of calculations and generate S_n and V_n for larger n. In addition, we are able to compute the $p_{m,n}$ with variable probabilities of changing lanes and entering lanes.

5. Finding the Limiting Value by Ergodic Theory

Recall q_n be a limiting value of $p_{m,n}$ when m tends to infinity for all n. Ergodic Theory is the study of a long-term average behavior of systems evolving in time. In order to obtain the limiting value q_n, we apply the Ergodic Theory to the Markov Model of the Busy Footbridge Problem.

5.1. Ergodic Theory

Definition 35. *A square matrix P is called regular if there exists an integer n such that all entries of P^n are positive.*

Lemma 36. *The Markov chain in the Busy Footbridge Problem with $0 < \rho_i < 1$, $\forall i \in \{1, 2, ..., n\}$ is a regular Markov chain.*

Proof. To prove the Markov chain is regular, we prove that S_n is accessible, which any state is possible to go to every state with a finite number of steps. We introduce an n-step algorithm to change one state to another with n people on the East. In the first step, a person from the West enters the footbridge on the same lane to the n^{th} person from the East. The person from the West remains on the original lane before he meets the n^{th} person. Then, he eventually collides with the n^{th} person from the East, the position of the n^{th} person after the collision depends on the output state.

Similarly, for the i^{th} step, a person from the West enters the footbridge on the same lane to the $(n-i+1)^{th}$ person from the East. The person from the West remains on the original lane before he meets the $(n-i+1)^{th}$ person. Then, the person from the West eventually collides with the $(n-i+1)^{th}$ person from the East, the position of the $(n-i+1)^{th}$ person after the collision depends on the output states. If the person from the West collides with the $(n-i+2)^{th}$ to the n^{th} person from the East, the person from the West changes lane and does not affect the positions of the $(n-i+2)^{th}$ to the n^{th} person from the East.

Therefore, after n steps, we can change any state to another arbitrary state. So, all states are accessible and all entries in $(S_n)^n$ are positive. Hence, the Markov chain is regular. [See reviewer's comment (1c)] □

Theorem 37. *Let P be a regular Markov matrix and π be the stationary probability vector of P. π is defined as a probability distribution (i.e. π is an eigenvector of*

P, associated with eigenvalue 1), which is

$$P\pi = \pi$$

For an ergodic Markov chain, there exists a unique probability vector and it is strictly positive.

Theorem 38. *Let P be the transition matrix for a regular chain. Then $\lim_{m\to\infty} P^m$ approaches a limiting matrix $\prod$ with all column be the same vector π, which is strictly positive (i.e., the components are all positive and they sum to one).*

[See reviewer's comment (2a)]

Note that our problem is a regular Markov matrix by **Lemma 36**. Using **Theorem 37**, we have the following corollary.

Corollary 39. *$q_n = \lim_{m\to\infty} p_{m,n}$ exists for all n and $\lim_{m\to\infty} (S_n)^m$ exists for all n.*

5.2. Computation of q_n

Definition 40. *We denote π_n be the stationary probability vector of S_n, which is*

$$\pi_n = \begin{pmatrix} \pi_{n_1} \\ \pi_{n_2} \\ \vdots \\ \pi_{n_{2n}} \end{pmatrix}$$

$\lim_{m\to\infty} (S_n)^m$ for all n reaches a stationary state, in which the row vectors are constant vectors. So, the initial probability is independent. So, with V_n and π_n, we can calculate q_n by the following equation:

$$q_n = V_n^T \cdot \pi_n$$

Example 41. *The example illustrates the computation of q_1, q_2 and q_3.*

Computing q_1: *Recall* $V_1^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$\begin{aligned}
\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} &= \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \quad \textit{and} \quad \pi_1 + \pi_2 = 1 \\
\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\
q_1 &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\
&= \frac{1}{2}
\end{aligned}$$

Computing q_2: *Recall* $V_2^T = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$

$$\begin{aligned}
\begin{pmatrix} \frac{3}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{3}{8} & 0 & \frac{1}{8} \\ \frac{1}{8} & 0 & \frac{3}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} &= \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} \quad \textit{and} \quad pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \\
\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} &= \begin{pmatrix} \frac{5}{12} \\ \frac{1}{12} \\ \frac{1}{12} \\ \frac{5}{12} \end{pmatrix} \\
q_2 &= \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{5}{12} \\ \frac{1}{12} \\ \frac{1}{12} \\ \frac{5}{12} \end{pmatrix} \\
&= \frac{1}{3}
\end{aligned}$$

Computing q_3: *Recall* $V_2^T = \begin{pmatrix} \frac{1}{8} & \frac{7}{8} & \frac{5}{8} & \frac{3}{8} & \frac{3}{8} & \frac{5}{8} & \frac{7}{8} & \frac{1}{8} \end{pmatrix}$

$$\begin{pmatrix} \frac{3}{4} & \frac{3}{8} & \frac{3}{8} & \frac{3}{16} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{3}{8} & 0 & \frac{3}{16} & 0 & \frac{1}{8} & 0 & \frac{1}{16} \\ 0 & 0 & \frac{3}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{3}{16} & \frac{3}{8} & 0 & 0 & \frac{1}{16} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{16} & 0 & 0 & \frac{3}{8} & \frac{3}{16} & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 & 0 & \frac{3}{16} & 0 & 0 \\ \frac{1}{16} & 0 & \frac{1}{8} & 0 & \frac{3}{16} & 0 & \frac{3}{8} & 0 \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{3}{16} & \frac{3}{8} & \frac{3}{8} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \\ \pi_7 \\ \pi_8 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \\ \pi_7 \\ \pi_8 \end{pmatrix}$$

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \\ \pi_7 \\ \pi_8 \end{pmatrix} = \begin{pmatrix} \frac{77}{216} \\ \frac{13}{216} \\ \frac{1}{216} \\ \frac{17}{216} \\ \frac{17}{216} \\ \frac{1}{216} \\ \frac{13}{216} \\ \frac{77}{216} \end{pmatrix}$$

$$q_3 = \begin{pmatrix} \frac{1}{8} & \frac{7}{8} & \frac{5}{8} & \frac{3}{8} & \frac{3}{8} & \frac{5}{8} & \frac{7}{8} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} \frac{77}{216} \\ \frac{13}{216} \\ \frac{1}{216} \\ \frac{17}{216} \\ \frac{17}{216} \\ \frac{1}{216} \\ \frac{13}{216} \\ \frac{77}{216} \end{pmatrix}$$

$$= \frac{7}{27}$$

The sequence of q_n continues with $q_4 = \dfrac{53}{243} \approx 0.2181, q_5 = \dfrac{419}{2187} \approx 0.1916.$

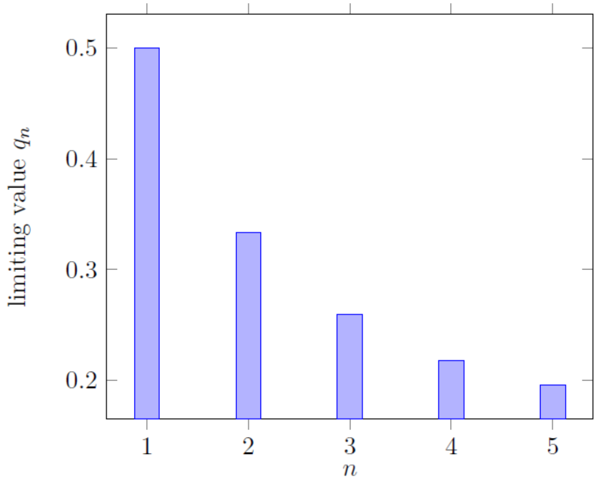


FIGURE 2. The results of q_n by Ergodic Theory

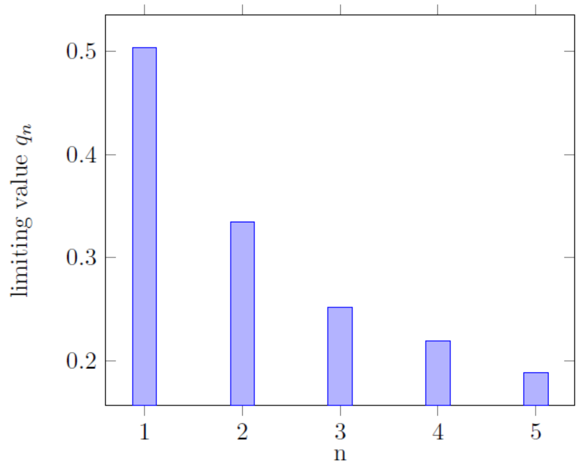


FIGURE 3. The results of q_n by Monte Carlo Algorithm

[See reviewer's comment (2c)]

6. Expected Number of Collisions

In this chapter, we would like to investigate the expected number of collisions between $\smiley_n$ with $\blacksmiley_s$. As we are considering a finite number of $\smiley_s$, Ergodic theory is not suitable in this chapter. We introduce Jordan normal form of S_n to solve the problem.

6.1. Attempt on finding the expected number of collisions

The answer will be the sum of the probabilities that ☻$_n$ collides with the first m ☺$_s$, which can be calculated by the following: [See reviewer's comment (1d)]

$$\begin{aligned}\text{Expected number of collisions} &= \sum_{i=1}^{m} p_{i,n} \\ &= \sum_{i=1}^{m} V^T (S_n)^{i-1} L_n^{(0)} \\ &= V^T \left(\sum_{i=1}^{m} (S_n)^{i-1} \right) L_n^{(0)}\end{aligned}$$

Generally, we can solve the summation by the formula of sum of geometric series, that is

$$\sum_{i=1}^{m} (S_n)^{i-1} = ((S_n)^m - I) \times ((S_n) - I)^{-1}$$

[See reviewer's comment (1e)]

However, by **Theorem 37**, 1 is an eigenvalue of S_n for all n, and $\det(S_n - I) = 0$. Hence, $(S_n - I)^{-1}$ does not exist. So, the geometric sum formula is not applicable. In order to find the sum faster, we decide to use Jordan decomposition as an alternative to compute the summation.

6.2. Jordan decomposition

Definition 42. *The matrix $J_{n,i}$ is a Jordan block. In a Jordan block, it has λ_i's on the diagonal, 1's on the superdiagonal and 0's elsewhere. A Jordan matrix is a block matrix that has Jordan blocks down its block diagonal and is zero elsewhere.*

For example, let $J_{n-1}, J_{n-2}, \ldots, J_{n,p}$ be Jordan blocks, which each $J_{n,i}$ is in the form

$$J_{n,i} = \begin{pmatrix} \lambda_i & 1 & & & 0 \\ & \lambda_i & 1 & & \\ & & \lambda_i & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda_i \end{pmatrix}$$

The Jordan matrix J_n is in the form

$$J_n = \begin{pmatrix} J_{n,1} & & & 0 \\ & J_{n,2} & & \\ & & \ddots & \\ 0 & & & J_{n,p} \end{pmatrix}$$

We want to use Jordan decomposition to transform the S_n to $P_n J_n P_n^{-1}$, where J_n is a Jordan matrix.

To calculate S_n^i, we have

$$(S_n)^i = (P_n J_n P_n^{-1})^i = P_n (J_n)^i P_n^{-1}$$

Then

$$\begin{aligned} \text{Expected number of collisions} &= V^T \left(\sum_{i=1}^{m} (S_n)^{i-1} \right) L_n^{(0)} \\ &= V^T P_n \left(\sum_{i=1}^{m} (J_n)^{i-1} \right) P_n^{-1} L_n^{(0)}, \end{aligned}$$

where $(J_n)^i$ is comparatively easier to compute than $(S_n)^i$ for large i.

However, in order to perform Jordan decomposition, we have to solve for eigenvalues and eigenvectors of S_n. Normally, to find the eigenvalues of S_n, the determinant of $(S_n - \lambda I)$ have to be calculated, which has a time complexity of $O((2^n)!)$. Also, the characteristic polynomial of degree 2^n has to be solved. Therefore, we develop a fast method to find eigenvalues, by observing recursive relationships of S_n and its corresponding eigenvalues.

Example 43. *We demonstrate the Jordan decomposition for S_2.*

$$\begin{aligned} S_2 &= \begin{pmatrix} \frac{3}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{3}{8} & 0 & \frac{1}{8} \\ \frac{1}{8} & 0 & \frac{3}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{3}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{12} & \frac{1}{16} & \frac{1}{2} & \frac{5}{12} \\ -\frac{1}{12} & -\frac{1}{16} & 0 & \frac{1}{12} \\ -\frac{1}{12} & \frac{1}{16} & 0 & \frac{1}{12} \\ \frac{1}{12} & -\frac{1}{16} & -\frac{1}{2} & \frac{5}{12} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -5 & -5 & 1 \\ 0 & -8 & 8 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

In addition, we could obtain a general form of $p_{m,n}$ by the following, in order to answer the original contest problem,

$$p_{m,n} = V^T (S_n)^{m-1} L_n^{(0)} = V^T P_n (J_n)^{m-1} P_n^{-1} L_n^{(0)}$$

Example 44. *This example illustrates the computation of* $p_{m,1}$ *and* $p_{m,2}$.

Computing $p_{m,1}$

$$\begin{aligned} p_{m,1} &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}^T \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}^{m-1} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}^{m-1} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2^{m-1}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2} \end{aligned}$$

Computing $p_{m,2}$

$$\begin{aligned} p_{m,2} &= \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}^T \begin{pmatrix} \frac{3}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{3}{8} & 0 & \frac{1}{8} \\ \frac{1}{8} & 0 & \frac{3}{8} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{3}{4} \end{pmatrix}^{m-1} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & -8 & \frac{1}{5} \\ -1 & -1 & 8 & \frac{1}{5} \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{m-1} \begin{pmatrix} \frac{1}{12} & -\frac{5}{12} & -\frac{5}{12} & \frac{1}{12} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{5}{12} & \frac{5}{12} & \frac{5}{12} & \frac{5}{12} \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 & \frac{4}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{4^{m-1}} & 0 & 0 & 0 \\ 0 & \frac{1}{2^{m-1}} & 0 & 0 \\ 0 & 0 & \frac{1}{2^{m-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{16} \\ 0 \\ 0 \\ \frac{5}{16} \end{pmatrix} \\ &= \frac{1}{3} - \frac{1}{4^{m+1}} \end{aligned}$$

6.3. Observation of patterns of eigenvalues

Recall the generalized S_1 from Chapter 4.2:

$$S_1 = \begin{pmatrix} 1-(1-\sigma)(1-\rho_1) & \sigma(1-\rho_1) \\ (1-\sigma)(1-\rho_1) & 1-\sigma(1-\rho_1) \end{pmatrix}$$

To find the Jordan form of S_1, we have to find its eigenvalues λ. We can find λ by solving the following equation:

$$\det(S_1 - \lambda I) = 0$$

$$\begin{vmatrix} 1-(1-\sigma)(1-\rho_1)-\lambda & \rho(1-\rho_1) \\ (1-\sigma)(1-\rho_1) & 1-\sigma(1-\rho_1)-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\rho_1-\lambda) = 0$$

$$\lambda = \rho_1 \quad \text{or} \quad 1$$

We find σ is independent to all eigenvalues. By the Expansion rules in **Theorem 28** with the second person from the East changes lane with probability ρ_2 in a collision, the eigenvalues of S_2 are $1, \rho_1, \rho_2$ and $\rho_1\rho_2$ by solving $\det(S_2 - \lambda I) = 0$. So, we speculate that there is a relationship between eigenvalues of S_1 and S_2. If λ is the eigenvalue of S_1, λ and $\rho_2\lambda$ are also the eigenvalues of S_2. We rewrite S_1 to be in the following form to avoid clumsy expressions in the following calculations.

$$S_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then, we try to prove that for any eigenvalue λ in S_1, both λ and $\rho_2\lambda$ are the eigenvalues for S_2. With the expansion rule in **Theorem 28**, we can generate S_2 in terms of a, b, c, d.

	00	01	10	11
00	a	$(1-\rho_2)a)$	b	$(1-\rho_2)b$
01	0	$\rho_2 a$	0	$\rho_2 b$
10	$\rho_2 c$	0	$\rho_2 d$	0
11	$(1-\rho_2)c$	c	$(1-\rho_2)d$	d

We can find the eigenvalues by solving $\det(S_2 - \lambda I) = 0$. We first evaluate $S_2 - \lambda I$

$$
\begin{aligned}
S_2 - \lambda I &= \begin{pmatrix} a & (1-\rho_2)a & b & (1-\rho_2)b \\ 0 & \rho_2 a & 0 & \rho_2 b \\ \rho_2 c & 0 & \rho_2 d & 0 \\ (1-\rho_2)c & c & (1-\rho_2)d & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \\
&= \begin{pmatrix} a-\lambda & (1-\rho_2)a & b & (1-\rho_2)b \\ 0 & \rho_2 a-\lambda & 0 & \rho_2 b \\ \rho_2 c & 0 & \rho_2 d-\lambda & 0 \\ (1-\rho_2)c & c & (1-\rho_2)d & d-\lambda \end{pmatrix}
\end{aligned}
$$

We transform $S_2 - \lambda I$ with the following row and column operations, as they only affect the sign of $\det(S_2 - \lambda I)$.

$$
\begin{aligned}
&\begin{pmatrix} a-\lambda & (1-\rho_2)a & b & (1-\rho_2)b \\ 0 & \rho_2 a-\lambda & 0 & \rho_2 b \\ \rho_2 c & 0 & \rho_2 d-\lambda & 0 \\ (1-\rho_2)c & c & (1-\rho_2)d & d-\lambda \end{pmatrix} \\
\sim &\begin{pmatrix} a-\lambda & a-\lambda & b & b \\ 0 & \rho_2 a-\lambda & 0 & \rho_2 b \\ \rho_2 c & 0 & \rho_2 d-\lambda & 0 \\ c & c & d-\lambda & d-\lambda \end{pmatrix} \begin{matrix} r_1 + r_2 \to r_1 \\ \\ \\ r_3 + r_4 \to r_4 \end{matrix} \\
\sim &\begin{pmatrix} a-\lambda & a-\lambda & b & b \\ c & c & d-\lambda & d-\lambda \\ 0 & \rho_2 a-\lambda & 0 & \rho_2 b \\ \rho_2 c & 0 & \rho_2 d-\lambda & 0 \end{pmatrix} \begin{matrix} \\ r_3 \to r_2 \\ r_4 \to r_3 \\ r_2 \to r_4 \end{matrix} \\
\sim &\begin{pmatrix} a-\lambda & b & a-\lambda & b \\ c & d-\lambda & c & d-\lambda \\ 0 & \rho_2 b & \rho_2 a-\lambda & 0 \\ \rho_2 c & 0 & 0 & \rho_2 d-\lambda \end{pmatrix} \begin{matrix} \\ c_3 \to c_2 \\ c_4 \to c_3 \\ c_2 \to c_4 \end{matrix}
\end{aligned}
$$

We denote transformed matrix as the following:

$$(S_2 - \lambda I)' = \left(\begin{array}{cc|cc} a-\lambda & b & a-\lambda & b \\ c & d-\lambda & c & d-\lambda \\ \hline 0 & \rho_2 b & \rho_2 a - \lambda & 0 \\ \rho_2 c & 0 & 0 & \rho_2 d - \lambda \end{array}\right) = \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right)$$

where A', B', C', D' are block matrices. We observe that $A' = B' = (S_1 - \lambda I)$. Hence $A'B' = B'A'$. So, $\det(S_2 - \lambda I) = \det(D'A' - C'B')$ (by [2]).

$$\begin{aligned} |\det(S_2 - \lambda I)| &= |\det(S_2 - \lambda I)'| \\ &= |\det(D'A' - C'B')| \\ &= |\det(D' - C') \det A'| \\ &= \left| \begin{vmatrix} \rho_2 a - \lambda & -\rho_2 b \\ -\rho_2 c & \rho_2 d - \lambda \end{vmatrix} \times \det A' \right| \\ &= \left| \begin{vmatrix} \rho_2 a - \lambda & -\rho_2 b \\ -\rho_2 c & \rho_2 d - \lambda \end{vmatrix} \times (-1)^2 \times \det A' \right| \\ &= \left| \begin{vmatrix} \rho_2 a - \lambda & \rho_2 b \\ \rho_2 c & \rho_2 d - \lambda \end{vmatrix} \times \det A' \right| \\ &= |\det(\rho_2 S_1 - \lambda I) \det(S_1 - \lambda I)| \end{aligned}$$

Since λ is an eigenvalue of S_1, $\det(S - \lambda I) = 0$. Then,

$$\det(S_2 - \lambda I) = 0$$

and the eigenvalues of S_2 are those of S_1.

Now, we try to prove $\rho_2 \lambda$ are also the eigenvalues of S_2 by proving $\det(S_2 - \rho_2 \lambda I) = 0$. Using the same row operation above, we can form $(S_2 - \rho_2 \lambda I)'$ and divide it into A'', B'', C'', D'' as the following:

$$(S_2 - \rho_2 \lambda I)' = \left(\begin{array}{cc|cc} a-\rho_2\lambda & b & a-\rho_2\lambda & b \\ c & d-\rho_2\lambda & c & d-\rho_2\lambda \\ \hline 0 & \rho_2 b & \rho_2 a - \rho_2\lambda & 0 \\ \rho_2 c & 0 & 0 & \rho_2 d - \rho_2\lambda \end{array}\right) = \left(\begin{array}{c|c} A'' & B'' \\ \hline C'' & D'' \end{array}\right)$$

Note that $A'' = B''$. We have

$$
\begin{aligned}
|\det(S_2 - \rho_2\lambda I)| &= |\det(S_2 - \rho_2\lambda I)'| \\
&= |\det(D''A'' - C''B'')| \\
&= |\det(D'' - C'')\det A''| \\
&= \left| \begin{vmatrix} \rho_2 a - \rho_2\lambda & -\rho_2 b \\ -\rho_2 c & \rho_2 d - \rho_2\lambda \end{vmatrix} \times \det A'' \right| \\
&= \left| \begin{vmatrix} \rho_2 a - \rho_2\lambda & -\rho_2 b \\ \rho_2 c & \rho_2 d - \rho_2\lambda \end{vmatrix} \times (-1)^2 \times \det A'' \right| \\
&= \left| (\rho_2)^2 \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \times \det A'' \right| \\
&= |(\rho_2)^2 \det(S_1 - \lambda I)\det(S_1 - \rho_2\lambda I)|
\end{aligned}
$$

Again, for the eigenvalue λ in S_1, $\det(S_1 - \lambda I) = 0$. So,

$$\det(S_2 - \rho_2\lambda I) = 0$$

The eigenvalues of S_2 are those of S_1 times ρ_2.

6.4. General eigenvalues for Markov matrix

In order to find all eigenvalues for S_n, we denote new notations for S_{n-1} and S_n.

Notation 45.

$$
S_{n-1} = \left(\begin{array}{cccc|cccc}
a_{0,0} & \cdots & \cdots & a_{0,2^{n-2}-1} & b_{0,0} & \cdots & \cdots & b_{0,2^{n-2}-1} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
a_{2^{n-2}-1,0} & \cdots & \cdots & a_{2^{n-2},2^{n-2}-1} & b_{2^{n-2}-1,0} & \cdots & \cdots & b_{2^{n-2}-1,2^{n-2}-1} \\
\hline
c_{0,0} & \cdots & \cdots & c_{0,2^{n-2}-1} & d_{0,0} & \cdots & \cdots & d_{0,2^{n-2}-1} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
c_{2^{n-2}-1,0} & \cdots & \cdots & c_{2^{n-2},2^{n-2}-1} & d_{2^{n-2}-1,0} & \cdots & \cdots & d_{2^{n-2}-1,2^{n-2}-1}
\end{array}\right)
$$

and by the Expansion rule, we have

$$S_n = \left(\begin{array}{cccc|cccc}
a_{0,0} & (1-\rho_n)a_{0,1} & \cdots & a_{0,2^{n-1}-1} & b_{0,0} & \cdots & \cdots & (1-\rho_{n+1})b_{0,2^{n-2}-1} \\
0 & \rho_n a_{0,1} & \cdots & \rho_{n+1}a_{0,2^{n-1}-1} & \vdots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots \\
(1-\rho_n)a_{2^{n-2}-1,0} & a_{2^{n-1}-1,0} & \cdots & a_{2^{n-1}-1,2^{n-2}-1} & (1-\rho_n)b_{2^{n-1}-1,0} & \cdots & \cdots & b_{2^{n-2}-1,2^{n-2}-1} \\
c_{0,0} & (1-\rho_n)c_{0,1} & \cdots & c_{0,2^{n-1}-1} & d_{0,0} & \cdots & \cdots & (1-\rho_{n+1})d_{0,2^{n-2}-1} \\
0 & \rho_n c_{0,1} & \cdots & \rho_{n+1}c_{0,2^{n-1}-1} & \vdots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots \\
(1-\rho_n)c_{2^{n-2}-1,0} & c_{2^{n-1}-1,0} & \cdots & c_{2^{n-1}-1,2^{n-2}-1} & (1-\rho_n)d_{2^{n-1}-1,0} & \cdots & \cdots & d_{2^{n-2}-1,2^{n-2}-1}
\end{array}\right)$$

Definition 46. *We define r_i and c_j be the i^{th} row and the j^{th} column. We define $(S_n - \lambda I)'$ from $(S_n - \lambda I)$ by the following operations (*).*

$$\textbf{Step 1:}\begin{cases} r_{4i} + r_{4i+1} \to r_{4i} \\ r_{4i+2} + r_{4i+3} \to r_{4i+3} \end{cases} \qquad \forall i \in \{0, 1, \dots, 2^{n-2} - 1\}$$

$$\textbf{Step 2:}\begin{cases} r_{4i} \to r_{2i} \\ r_{4i+1} \to r_{2^{n-1}+2i} \\ r_{4i+2} \to r_{2^{n-1}+2i+1} \\ r_{4i+3} \to r_{2i+1} \end{cases} \qquad \forall i \in \{0, 1, \dots, 2^{n-2} - 1\}$$

$$\textbf{Step 3:}\begin{cases} c_{4j} \to c_{2j} \\ c_{4j+1} \to c_{2^{n-1}+2j} \\ c_{4j+2} \to c_{2^{n-1}+2j+1} \\ c_{4j+3} \to c_{2j+1} \end{cases} \qquad \forall i \in \{0, 1, \dots, 2^{n-2} - 1\}$$

We define $(S_n - \rho_n\lambda I)'$ from $(S_n - \rho_n\lambda I)$ by the same operations.

Lemma 47.

$$(S_n - \lambda I)' = \left(\begin{array}{cccc|}
a_{0,0} - \lambda & a_{0,1} & \dots & b_{0,2^{n-2}-1} \\
a_{1,0} & a_{1,1} - \lambda & \dots & b_{1,2^{n-2}-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_{2^{n-2}-1,0} & c_{2^{n-1}-1,1} & \dots & d_{2^{n-2}-1,2^{n-2}-1} - \lambda \\
0 & \rho_n a_{0,1} & \dots & \rho_0 b_{0,2^{n-2}-1} \\
\rho_0 a_{1,0} & \ddots & & \vdots \\
\vdots & & \ddots & \vdots \\
\rho_n c_{2^{n-2}-1,0} & \dots & \dots & 0
\end{array}\right.$$

$$\left.\begin{array}{cccc}
a_{0,0} - \lambda & a_{0,1} & \dots & b_{0,2^{n-2}-1} \\
a_{1,0} & a_{1,1} - \lambda & \dots & b_{1,2^{n-2}-1} \\
\vdots & & \ddots & \vdots \\
c_{2^{n-2}-1,0} & c_{2^{n-2}-1,1} & \dots & d_{2^{n-2}-1,2^{n-2}-1} - \lambda \\
\rho_n a_{0,0} - \lambda & \dots & \dots & 0 \\
\vdots & \ddots & & \vdots \\
\vdots & & \ddots & \vdots \\
0 & \dots & \dots & d_{2^{n-2}-1,2^{n-2}-1} - \lambda
\end{array}\right)$$

Furthermore, we denote

$$\left(\begin{array}{c|c} A'_n & B'_n \\ \hline C'_n & D'_n \end{array}\right) = (S_n - \lambda I)'$$

$$(S_n - \rho_n \lambda I)' = \left(\begin{array}{cccc|}
a_{0,0} - \rho_n \lambda & a_{0,1} & \dots & b_{0,2^{n-2}-1} \\
a_{1,0} & a_{1,1} - \rho_n \lambda & \dots & b_{1,2^{n-2}-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_{2^{n-2}-1,0} & c_{2^{n-1}-1,1} & \dots & d_{2^{n-2}-1,2^{n-2}-1} - \rho_n \lambda \\
0 & \rho_n a_{0,1} & \dots & \rho_0 b_{0,2^{n-2}-1} \\
\rho_0 a_{1,0} & \ddots & & \vdots \\
\vdots & & \ddots & \vdots \\
\rho_n c_{2^{n-2}-1,0} & \dots & \dots & 0
\end{array}\right.$$

$$(\left.\begin{array}{cccc}
a_{0,0} - \rho_n \lambda & a_{0,1} & \dots & b_{0,2^{n-2}-1} \\
a_{1,0} & a_{1,1} - \rho_n \lambda & \dots & b_{1,2^{n-2}-1} \\
\vdots & & \ddots & \vdots \\
c_{2^{n-2}-1,0} & c_{2^{n-2}-1,1} & \dots & d_{2^{n-2}-1,2^{n-2}-1} - \rho_n \lambda \\
\rho_n a_{0,0} - \rho_n \lambda & \dots & \dots & 0 \\
\vdots & \ddots & & \vdots \\
vdots & & \ddots & \vdots \\
0 & \dots & \dots & d_{2^{n-2}-1,2^{n-2}-1} - \rho_n \lambda
\end{array}\right)$$

Furthermore, we denote

$$\left(\begin{array}{c|c} A''_n & B''_n \\ \hline C''_n & D''_n \end{array}\right) = (S_n - \rho_n \lambda I)'$$

Proof. The results can be obtained by the computation of the operations. □

Remark 48. *The example of the computation refers to Chapter 6.3.*

Theorem 49. *The eigenvalues of S_n are the eigenvalues of S_{n-1} and the eigenvalues of S_{n-1} times ρ_n.*

Proof. We have $A'_n = B'_n$ and $A'_n B'_n = B'_n A'_n$. So,

$$\begin{aligned}|\det(S_n - \lambda I)| &= |\det(D'_n A'_n - C'_n B'_n)|\\ &= |\det(D'_n - C'_n)\det A'_n|\\ &= |\det(D'_n - C'_n)\det(S_{n-1} - \lambda I)|\\ &= |\det(\rho_n S_{n-1} - \lambda I)\det(S_{n-1} - \lambda I)|\end{aligned}$$

Since λ is an eigenvalue of S_{n-1}, $\det(S_{n-1} - \lambda I) = 0$. Then $\det(S_n - \lambda I) = 0$. Hence, the eigenvalues of S_n are those of S_{n-1}.

We have $A''_n = B''_n$, and $A''_n B''_n = B''_n A''_n$. So,

$$\begin{aligned}&|\det(S_n - \rho_n \lambda I)|\\ &= |\det(D''_n A''_n - C''_n B''_n)|\\ &= |\det(D''_n - C''_n)\det A''_n|\\ &= \left|(\rho_n)^{2^{n-1}}\begin{vmatrix} a_{0,0}-\lambda & -a_{0,1} & a_{0,2} & \cdots & \cdots & -b_{0,2^{n-2}-1}\\ -a_{1,0} & a_{1,1}-\lambda & -a_{1,2} & \cdots & \cdots & b_{0,2^{n-2}-1}\\ a_{2,0} & -a_{2,1} & \ddots & & & \vdots\\ \vdots & \vdots & & \ddots & & \vdots\\ \vdots & \vdots & & & \ddots & \vdots\\ -c_{2^{n-2}-1,0} & c_{2^{n-2}-1,1} & \cdots & \cdots & \cdots & d_{2^{n-2}-1,2^{n-2}-1}-\lambda \end{vmatrix}\right|\\ &\quad\times|\det(S_{n-1} - \rho_n\lambda I)|\\ &= \left|(\rho_n)^{2^{n-1}}\begin{vmatrix} a_{0,0}-\lambda & a_{0,1} & a_{0,2} & \cdots & \cdots & -b_{0,2^{n-2}-1}\\ a_{1,0} & a_{1,1}-\lambda & a_{1,2} & \cdots & \cdots & b_{0,2^{n-2}-1}\\ a_{2,0} & a_{2,1} & \ddots & & & \vdots\\ \vdots & \vdots & & \ddots & & \vdots\\ \vdots & \vdots & & & \ddots & \vdots\\ c_{2^{n-2}-1,0} & c_{2^{n-2}-1,1} & \cdots & \cdots & \cdots & d_{2^{n-2}-1,2^{n-2}-1}-\lambda \end{vmatrix}\right|\\ &\quad\times\left|(-1)^{2^n-1}\det(S_{n-1} - \rho_n\lambda I)\right|\\ &= |(\rho_n)^{2^n-1}\det(S_{n-1} - \lambda I)\det(S_{n-1} - \rho_n\lambda I)|\end{aligned}$$

Since λ is an eigenvalue of S_{n-1}, $\det(S_{n-1} - \lambda I) = 0$. Then $\det(S_n - \rho_n\lambda I) = 0$. The eigenvalues of S_n are those of S_{n-1} times ρ_n. □

By the method above, we could acquire eigenvalues of S_n from eigenvalues of S_1, which only has a time complexity of $O(2^n \times n)$. We could then solve for the eigenvectors of S_n by solving $(S_n - \lambda I)v = 0$, where v is the eigenvector.

By doing numerical experiments, we find that for any eigenvalues with algebraic multiplicity higher than 1, there is only 1 eigenvector. [See reviewer's comment (2d)] We propose the following conjecture:

Conjecture 50. *The dimension of the eigenspace corresponding to each eigenvalue of S_n is* 1*.*

Assuming the conjecture holds, we can determine J_n of the transition matrix S_n immediately. Even if the conjecture does not hold, we can still find J_n by the standard method.

7. Special Cases

In the above investigation, we focus on the person with a probability $0 < \rho < 1$ that a person changes lane in a collision. So, in this section, we are going to deal with the extreme cases where $\rho = 0$ or $\rho = 1$. We denoted a person with $\rho = 0$ as a 'polite person', and a person with $\rho = 1$ as a 'stubborn person'. These cases are considered in the 2018 Singapore International Mathematics Challenge Section C **Question 1**.

7.1. Polite person

Consider there is a polite person from the East. If a collision occurs between the person from the West and the polite person from the East, the polite person changes lane while the person from the West remains on the original lane. Let ☻$_{n_0}$ be a polite person. After ☺ meets ☻$_{n_0}$, the position of ☺ remains unchanged and ☺ meets ☻$_{n_0+1}$ on the lane after ☺ meets ☻$_{n_0-1}$. In other words, the configuration of the people from East behind ☻$_{n_0}$ is independent on ☻$_{n_0}$. So, we can interpret the situation by eliminating the polite person ☻$_{n_0}$ from the East.

If the n^{th} person is a polite to $R^{(n-1)}$, we can simply ignore and eliminate that person. Then, $S_{n-1} = S_n$.

7.2. Stubborn person

Consider there is a stubborn person from the East. If a collision occurs between the person from the West and the stubborn person, the stubborn person must remain on the original lane while the person from the West changes to the other lane. Fixed an integer n_0. Let ☻$_{n_0}$ be a stubborn person. After ☺ meets pass through ☻$_1$ to ☻$_{n_0}$, the position of the ☺ must on the different lane of ☻$_{n_0}$. So, we can interpret this situation as ☻$_{0+1}$ as the first person from the East.

If the n^{th} person is stubborn, we treat the $n+1^{th}$ person from the East as the first person from the East. In addition, ☺ meets ☻$_{n+1}$ on the lane different from ☻$_n$.

So σ is fixed. Then,

$$S_{n+1} = S_1 \quad \text{and}$$

$$\sigma = \begin{cases} 1, & \text{if the stubborn person is on bottom lane,} \\ 0, & \text{otherwise.} \end{cases}$$

7.3. Possible extensions

In our paper, we consider the footbridge with 2 lanes only. We may extend the problem to more lanes. We can also develop a Markov model to solve the problem. For example, if there are 3 lanes, we are able construct a transition matrix S_1 with dimension 3×3. We can also define similar expansion rules to find S_n, V_n and L_n recursively.

8. Summary on Busy Footbridge Problem

In Chapter 3, we can compute $p_{m,n}$ for all m and n. We generate transition matrices, collision vectors, and initial vectors to solve the Busy Footbridge Problem. In Chapter 4, we define expansion rules in order to generate S_n, V_n and L_n respectively. In addition, we are able to compute $p_{m,n}$ for various ρ and σ. In Chapter 5, we can apply the Ergodic Theory to obtain the limiting value q_n efficiently. We compare the exact value of q_n to the approximate results obtained by Monte Carlo Algorithm. The results are similar but not exactly equal to $\frac{1}{n+1}$. In Chapter 6, we further investigate the problem on the expected number of collisions. We apply Jordan Decomposition to ease our calculation. In Chapter 7, we discuss special cases of the problem where $\rho = 0$ and 1.

Finally, we answer the original contest problem using our Markov Model, namely

$$\begin{aligned} p_{m,1} &= \frac{1}{2}; \\ p_{m,2} &= \frac{1}{3} - \frac{1}{4^{m+1}}; \\ q_1 &= \frac{1}{2}; \\ q_2 &= \frac{1}{3}; \\ q_3 &= \frac{7}{27} \end{aligned}$$

REFERENCES

[1] The Singapore International Mathematics Challenge 2018 Committee, *Singapore International Mathematics Challenge 2018*, page 3.
https://www.nushigh.edu.sg/qql/slot/u90/file/simc/mathmodel/SIMC2018ChallengeQn.pdf

[2] John R. Silvester, *Determinant of Block matrices*, http://www.ee.iisc.ac.in/people/faculty/prasantg/downloads/blocks.pdf

[See reviewer's comment (2e)]

Reviewer's Comments

This article addressed a mathematical model of the probabilities that the people on a footbridge from two sides meet. Here are some of the reviewer's comments:

1. Novelty and methodology: The paper generalised the model to various cases and applied a Markov model to solve those problems. The authors compared the results generated by Monte Carlo Algorithm with those obtained from Markov chain method. The idea and method look fine, yet it missed some details and steps in explaining the claimed results. Here are some of the reviewer's suggestions:
   (a) The authors claimed to use Monte Carlo algorithm in approximating q_n. Yet there is no explanation on neither why Monte Carlo algorithm is used nor why it can approximate q_n.
   (b) It is not clear why $p_{m,n}$ can be given by such formula. The authors should give more explanations on it as this is a very important formula in the paper. Similarly, the authors should explain more in obtaining the formula for q_n.
   (c) In order to prove that the Markov chain is regular, one needs to show that for SOME integer n, the corresponding matrix P^n has only positive entries (according to Definition 35). Yet the authors only mentioned in the final line that "So, all states are accessible and all entries in $(S_n)^n$ are positive." Does it mean that it is true for ALL $n \in \mathbb{N}$? If the reviewer is correct, the authors tried to show "for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ (which may depend on n) such that $(S_n)^m$ contains only positive entries." Yet the authors did not explain it clearly in the paper.
   (d) The authors mentioned that "The answer will be the sum of the probabilities that (black smiling face)$_n$ collides with the first m (white smiling faces)." Why is it true? At least the reviewer is not sure why the sum always gives a positive integer.
   (e) "Generally, we can solve the summation by the formula of sum of geometric series", IN GENERAL such formula is NOT true unless $((S_n) - I)^{-1}$ exists (in particular not true for the case in the paper).
2. Organisation: The organisation is fine except that the authors missed the sources or references in supporting the arguments. Here are some examples:
   (a) It would be better if the authors can provide the reference or sources for Theorem 5.3 and Theorem 5.4.
   (b) The authors mentioned that "However, q_n starts to fluctuate without any easily observable patterns when n becomes larger." I am curious on how the fluctuation is as the reviewer cannot see such fluctuation from Figure 1.
   (c) The authors should pinpoint the difference between the values of q_n obtained by different methods.
   (d) The authors mentioned "By doing numerical experiments, we find that for any eigenvalues with algebraic multiplicity higher than 1, there is only

1 eigenvector." The authors should illustrate those claimed numerical examples in supporting the argument.

(e) The listed references are not sufficient. The authors should include more references used in the work.