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FERMAT POINT EXTENSION – LOCUS, LOCATION, AND LOCAL USE

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Abstract. Published in 1659, the solutions of Fermat Point problem help people find out the point at which the sum of distances to 3 fixed points in the plane is minimized. In this paper, we are going to further discuss the case when the number of fixed point is more than 3. Also, we would like to find out if there exists a way such that the location of point minimizing the sum of distances to more than three given points can be determined just by compasses and ruler, or approximated by mathematical methods.

1. Background

Suppose there is a natural disaster, the government has to build a temporary aid-offering centre. However, the victims are distributed in different locations. Where can we build the centre so that the needs of all the victims can be satisfied? Apart from building multiple centers (This may not be possible if the resources are limited in some countries), the only way out is to build the centre in a location such that the sum of the distances between the centre and the victims is the minimum, which in turn minimizes the time required for transport.

In fact, this can be applied to many different fields. Similar topic has been discussed in the past. **Fermat point** refers to the solution to the problem of finding a point F inside a triangle ΔABC such that the total distance from the three vertices to point F is minimum. Now, we are going to study

[1]This work is done under the supervision of the authors' teacher, Mr. Yiu-Kwong Lau

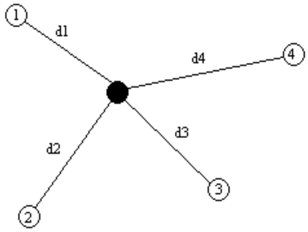


FIGURE 1. The black point is selected in a location such that $d1 + d2 + d3 + d4$ is the minimum

the optimum point in figures with more than 3 vertices.

2. Extension

As the location of Fermat Point in a triangle is well-known, we would like to study the location of Fermat Point in quadrilateral, pentagon, or even n-sided polygon. To generalize the name, we define the Fermat Point in an n-sided polygon as n-F Point.

2.1. Fermat Point in Quadrilateral (4-F Point)

Let P_1, P_2, P_3 and P_4 be the four points of a quadrilateral. We would like to discuss the location and locus of the 4-F Point of quadrilateral $P_1P_2P_3P_4$. We are able to find the 4-F Point with geometric proof in different cases.

Case 1: $P_1P_2P_3P_4$ is a convex quadrilateral

Consider a variable point P_x on the same plane, by triangle inequality:

- $P_1P_x + P_xP_3 \geqslant P_1P_3$ (equality holds iff P_1, P_x, P_3 are collinear, and P_x is located in between P_1 and P_3)
- $P_2P_x + P_xP_4 \geqslant P_2P_4$ (equality holds iff P_2, P_x, P_4 are collinear, and P_x is located in between P_2 and P_4).

Therefore, it is clearly that $\sum_{i=1}^{4} P_iP_x$ reaches minimum when P_x is the intersection point of lines P_1P_3 and P_2P_4. (cf. **Reviewer's Comment** 1)

Case 2: Three of the points are collinear (i.e. P_1, P_2, P_3 are collinear while P_4 is an arbitrary stationary point)

Consider a variable point P_x on the same plane, by triangle inequality:

- $P_1P_x + P_xP_3 \geqslant P_1P_3$ (equality holds iff P_1, P_x, P_3 are collinear, and P_x is located in between P_1 and P_3)
- $P_2P_x + P_xP_4 \geqslant P_2P_4$ (equality holds iff P_2, P_x, P_4 are collinear, and P_x is located in between P_2 and P_4).

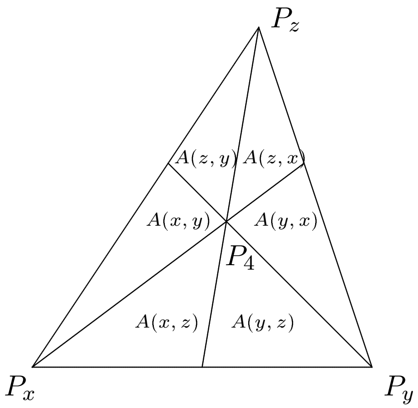
Therefore, it is clear that $\sum_{i=1}^{4} P_iP_x$ reaches minimum when P_x is the intersection point P_2 of lines P_1P_3 and P_2P_4. In fact, it is a boundary case of the situation mentioned in Case 1. (cf. **Reviewer's Comment** 2)

Case 3: $P_1P_2P_3P_4$ is a concave quadrilateral. Assume that P_4 is a stationary point located inside the triangle $\Delta P_1P_2P_3$.

It is found that P_4 is the 4-F Point which can be proven by the fact that for any variable point P_x inside triangle, $\sum_{i=1}^{4} P_iP_x \geqslant \sum_{i=1}^{3} P_iP_4$.

The diagram is constructed by the following method: Let P_1P_4 and P_2P_3 meet at N_1, P_2P_4 and P_1P_3 meet at N_2, P_3P_4 and P_1P_2 meet at N_3. A diagram is constructed by joining the lines on the left, which consists of a triangle divided into 6 parts.

To denote the parts of area, the symbol $A(x, y)$ is used to represent them, where x represents the near point and y represent the point in which side near N. Take the following diagram as an example: (cf. **Reviewer's Comment** 3)



The value $A(x, y)$ is assigned in the following way: If the region touch the vertex P_i, then i is assigned in the first coordinate (where $i = x, y$ or z). In addition, if the region does not touch the line P_iP_j, then the second coordinate is j.

To generalize the situation, without loss of generality, assume that the three exterior fixed points are P_x, P_y and P_z, while the fixed interior point is P_4 (here, $\{x, y, z\} = \{1, 2, 3\}$). It is easy to show that for any point P_n on the same plane,

$$P_xP_n + P_yP_n + P_zP_n + P_4P_n \geqslant P_xP_4 + P_yP_4 + P_zP_4,$$

and the equality holds iff $P_n = P_4$.

Proof.
Case i: P_n is located outside the triangle $\Delta P_xP_yP_z$.

Draw a line which separates P_n and triangle $\Delta P_xP_yP_z$. Let P_h be the orthogonal projection of P_n on that line. It is obvious that

$$P_xP_n + P_yP_n + P_zP_n + P_4P_n \geqslant P_xP_h + P_yP_h + P_zP_h + P_4P_h.$$

As a result, P_n can be "moved" until it lies on the perimeter or triangle $\Delta P_xP_yP_z$, which is case ii.

Case ii: P_n is located inside the triangle $\Delta P_x P_y P_z$.

Without the loss of generality, assume P_n is located inside $A(x, y)$. By triangle inequality,

$$P_x P_n + P_n P_4 \geqslant P_x P_4, \tag{1}$$

equality holds iff P_x, P_n and P_4 are collinear. Then, let $P_n P_y$ intersect $P_z P_4$ at P_b. By triangle inequality,

$$P_n P_y + P_n P_z = P_n P_b + P_b P_y + P_n P_z \geqslant P_b P_y + P_b P_z,$$

and equality holds iff P_n, P_b and P_z are collinear. Similarly,

$$P_b P_y + P_b P_z = P_b P_y + P_b P_4 + P_4 P_z \geqslant P_4 P_y + P_4 P_z,$$

equality holds iff P_b, P_4 and P_y are collinear. Hence

$$P_n P_y + P_n P_z \geqslant P_4 P_y + P_4 P_z \tag{2}$$

Summing (1) and (2),

$$P_x P_n + P_y P_n + P_z P_n + P_4 P_n \geqslant P_x P_4 + P_y P_4 + P_z P_4.$$

The equality holds iff $P_n = P_4$, which implies the deserved point is exactly P_4. □

2.2. Fermat Point in n-sided polygon, where $n \geqslant 5$ (n-F Point)

At this moment, no geometric or mathematical method is found to work out the n-Points Fermat Point. Therefore, computer programs were developed to find out the coordinates of the n-F Point in order to study its location regarding to those fixed points. For simplicity, the case "$n = 5$ and the polygon is convex (i.e. convex pentagon)" will be the main focus.

The description of programs and underlying mathematical principle are included in the appendix of this report.

3. Major Findings

3.1. Clues from Computer Program Results

At the beginning, 4 points were fixed to make a square. When the remaining variable point is located at different location, the 5-F point varies within a bounded area. Apart from that, some other interesting results have been found. The finding can be said in this way:

1.1 If the variable point is located inside the bounded area or on the boundary, the location of the 5-F point is exactly the variable point locates; If the variable point is located outside the bounded area, the location of the 5-F point is on the boundary of that area.

1.2 Such bounded area is determined by the 4 fixed points already, and is independent from the variable point. (cf. **Reviewer's Comment** 4)

1.3 Also, for the 4 fixed points if any 3 of them are chosen and the Fermat point of those 3 points are found, the four 3-F points also lie on the boundary of the bounded area (locus of 5-F points).

After that, the cases of different $n > 5$ were also studied and the following observation and hypothesis were drawn:

1.4 When $(n-1)$ of n points are fixed and the remaining one varies, locus of n-F point forms a bounded area.

1.5 Such bounded area is determined by the $(n-1)$-fixed points already, and is independent from the variable point. (cf. **Reviewer's Comment** 4)

1.6 (Generalization of 1.1) When the variable point is located inside the bounded area, n-F point is located at the same place as the variable point; when the variable point is located outside the bounded area, the n-F point is located on the boundary of the bounded area.

1.7 (Generalization of 1.3) For those $(n-1)$ points, $(n-1)$ different groups of $(n-2)$ points can be chosen, and their $(n-2)$-F points are all located on the boundary of the bounded area.

3.2. Theorems by Mathematical Methods

To determine whether the hypotheses are true or not, mathematical methods are used to prove the above findings.

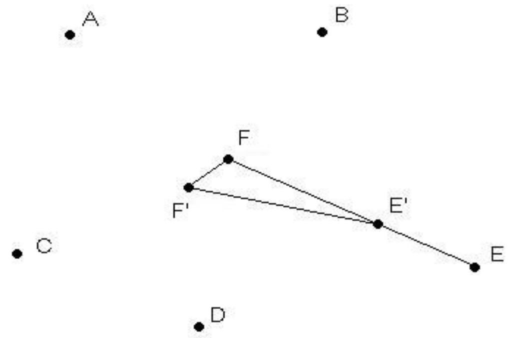
Theorem 1. *For any n points on a plane, where $(n-1)$ points are fixed and the remaining one varies, a total number of $C_{n-2}^{n-1} = n-1$ groups of $(n-2)$ points can be chosen. Using these groups of points, we can find out a total number of $(n-1)$ $(n-2)$-F points, and those $(n-2)$-F points lie on the boundary of bounded area formed by the locus of n-F points when the n-th point varies. (Hypothesis 1.5)*

Proof. The proof mainly depends on the triangle inequality.

Lemma 2. *(cf.* ***Reviewer's Comment*** *5) For any n points on a plane, consisting of fixed points P_1, P_2, ..., P_{n-1} and variable point P_n, if the variable point moves outside the bounded area of n-F point locus and moves along the line projected from n-F points towards the variable point P_n, the location of n-F point remains the same.*

Proof of Lemma 2. To make it easier, the case $n = 5$ is selected as an example first.

As shown in the figure, A, B, C and D are the 4 fixed points and E is the variable point. Assume F is the 5-F point of A, B, C, D and E. E' is an arbitrary point on line EF. If Lemma 2 is true, then F is also the 5-F point of A, B, C, D and E'.



We argue by contradiction. Assume the 5-F point of A, B, C, D and E' is F' different from F. By definition,

$$AF + BF + CF + DF + E'F > AF' + BF' + CF' + DF' + E'F'$$

$$\begin{aligned} AF + BF + CF + DF + (E'F + E'E) &> AF' + BF' + CF' + DF' + (E'F' + E'E) \\ AF + BF + CF + DF + EF &> AF' + BF' + CF' + DF' + (E'F' + E'E) \\ &> AF' + BF' + CF' + DF' + EF' \end{aligned}$$

By triangle inequality. A contradiction is resulted, as F is the 5-F point of A, B, C, D and E so $AF + BF + CF + DF + EF$ should reach minimum.

For the general case, assume the n points are $P_1, \ldots, P_n$ respectively. Let $P_1, \ldots, P_{n-1}$ be fixed and P_n is the variable point. Let F be the n-F point of the points $P_1, \ldots, P_n$. Assume an arbitrary point P'_n moves along the line P_nF. Similarly, let F' be the n-F point of points $P_1, \ldots, P_{n-1}$ and P'_n.

$$\sum_{i=1}^{n-1} P_iF + P'_nF > \sum_{i=1}^{n-1} P_iF' + P'_nF'$$

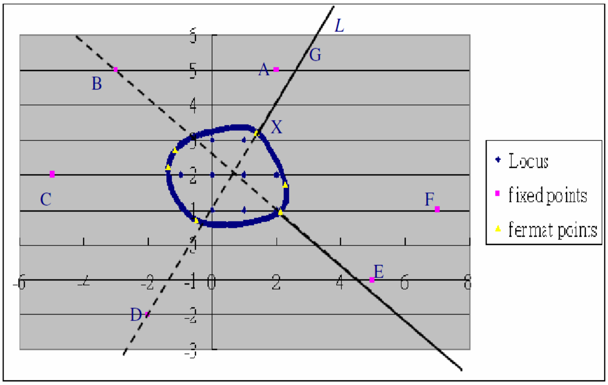
$$\sum_{i=1}^{n-1} P_iF + P'_nF + P_nP'_n > \sum_{i=1}^{n-1} P_iF' + P'_nF' + P_nP'_n$$

$$\sum_{i=1}^{n-1} P_iF + P_nF > \sum_{i=1}^{n} P_iF' + P_nP_n'$$
$$\sum_{i=1}^{n} P_iF > \sum_{i=1}^{n} P_iF'$$

By triangle inequality. Contradiction results again and this finishes the proof of Lemma 2. □

Lemma 3. *For any n points on a plane, including $(n-1)$ fixed points and 1 variable point, we choose $(n-2)$ points from the set of fixed points. When the variable point moves to a certain position such that it lies on the straight line formed by the $(n-2)$-F point and the remaining fixed point, the n-F point is the same as that $(n-2)$-F point when the variable point moves outward along the line.*

Proof of Lemma 3. To make it easier, the following case is first considered.



Here $n = 7$. A, B, C, D, E and F are the fixed points and G is the variable point. $n-2 = 5$ points, namely A, B, C, E and F, are chosen. The remaining fixed point is D. Let X be the 5-F point of A, B, C, E and F. When the 7^{th} point G moves along Line L, the 2-F point of D and G lies on the line segment DG.

The total distance from X to the seven points A, B, C, D, E, F and G
$= AX + BX + CX + DX + EX + FX + GX$
$= (AX + BX + CX + EX + FX) + (DX + GX)$.

By definition, $AX + BX + CX + EX + FX$ is minimized and $DX + GX$ is minimized. Therefore, $AX + BX + CX + DX + EX + FX + GX$ is minimized and X is the 7-F point of A, B, C, D, E, F and G so X lies on the boundary of the bounded area.

Now the general case is considered. Let X be the $(n-2)$-F point of P_1, $P_2 \ldots P_{n-2}$.

When the n-th point P_n moves along the line formed by the X and P_{n-1} such that X is located between P_{n-1} and P_n.

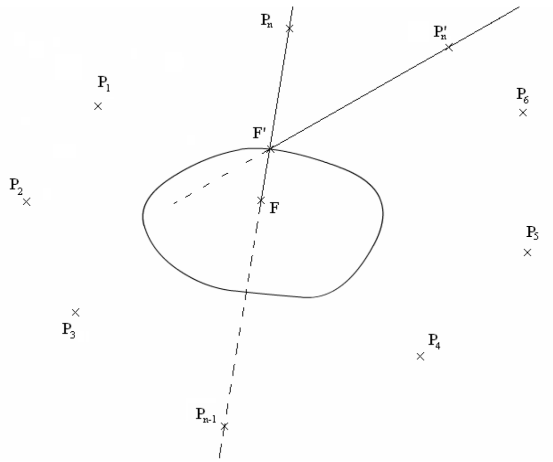
The total distance from X to $P_1, P_2 \ldots P_n$

$$\sum_{i=1}^{n-2} P_iX + P_{n-1}X + P_nX.$$

By definition $\sum_{i=1}^{n-2} P_iX$ is minimized and $P_{n-1}X + P_nX$ is minimized as X, P_{n-1} and P_n are collinear. Therefore, $\sum_{i=1}^{n} P_iX$ is minimized and X is the n-F point of $P_1, P_2, \ldots, P_n$. Thus X lies on the boundary of the bounded area. □

By these two lemmata, theorem 1 can be proved in this way.

From Lemma 3, it is ensured that the $(n-2)$-F point of P_1 to P_{n-2} must lies inside the bounded area or on the boundary. By contradiction, assume that such $(n-2)$-F point, namely F, lies inside the bounded area. Construct a line L_1 passing through F and P_{n-1} and let such line intersect with the boundary at point F'. By Lemma 3, if P_n moves along L_1, the n-F point remains unchanged and is exactly the point F.



Assume that when the n-F point is F', P_n is located at the position P'_n. Construct the line L_2 by joining P'_n and F'. When P_n moves along line L_2, the n-F point is also F' by lemma 2. Therefore when P_n moves along line L_2 and reaches point F', the n-F point is also F'. A contradiction is resulted since F' lies on L_1 and the n-F point should be located at F. This finishes the proof of theorem 1. □

Theorem 4. *Given points $\{P_1, \ldots, P_n\}$ and their n-F point F, $\sum_{i=1}^{n} \frac{FP_i}{|FP_i|} = 0$, i.e. the sum of unit vectors FP_i is zero.*

Proof. The proof makes use of trigonometric functions.

Lemma 5. *For any n points on a plane, including $(n-1)$ fixed points and 1 variable point, a line joining the variable point and n-F point is unique(by Lemma 1.1). Let the set of fixed points be $P_1, \ldots P_{n-1}$ and P_n is variable. Assume a point F moves along that line and for all angles made by the line P_iF and the line joining the variable point and n-F point be θ_i. Point F is exactly the n-F point if and only if $\sum_{i=1}^{n} \cos\theta_i = 0$.*

Proof of Lemma 5. A coordinate plane can be induced into the system by putting x-axis be the line formed by the n-F point and P_n. Let the coordinates of point P_i be (a_i, b_i) for all $i = 1, 2, ..n$ respectively. The angle $\angle P_iFP_n$ is θ_i for all $i = 1, 2, ..n$.

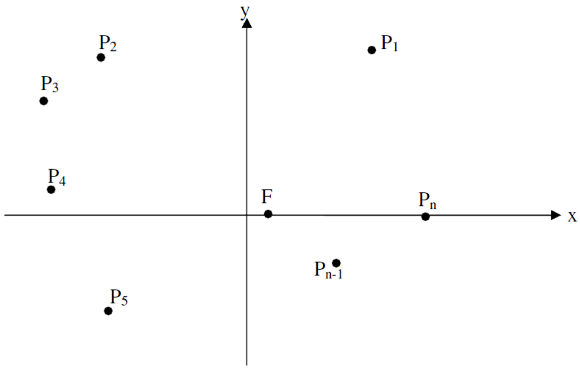


FIGURE 2. (In the figure, only $P_1, \ldots, P_5$, P_{n-1} and P_n are drawn).

Let point $F(x, 0)$ be a movable point on the x-axis. By distance formula, the sum of distance S from points $P_1 \ldots P_n$ to F is

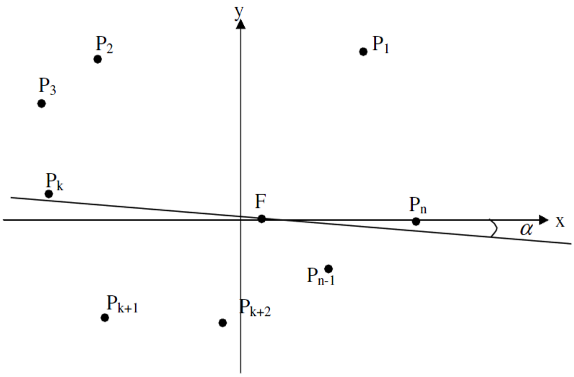
$$S = \sum_{i=1}^{n} \sqrt{(x-a_i)^2 + b_i^2}.$$

$$\begin{aligned} \Rightarrow \frac{dS}{dx} &= \sum_{i=1}^{n} \frac{2(x-a_i)}{2\sqrt{(x-a_i)^2 + b_i^2}} \\ &= \sum_{i=1}^{n} \frac{(x-a_i)}{\sqrt{(x-a_i)^2 + b_i^2}} \\ &= \sum_{i=1}^{n} \cos\theta_i \end{aligned}$$

Therefore, when S reaches minimum, $\frac{dS}{dx} = 0$ implying $\sum_{i=1}^{n} \cos\theta_i = 0$. $\square$

Lemma 6. *(Continued from Lemma 5) Among the set of fixed points be P_1 to P_{n-1} and P_n is variable, draw a line passes through P_n. Let points be P_1 to P_k lies on one side from the line and the others on the other side. Assume a point F moves along that line and for all angles made by the line P_iF and the line joining the variable point and n-F point be θ_i. We have*

$$\sum_{i=1}^{k} \sin\theta_i = \sum_{i=k+1}^{n} \sin\theta_i.$$



Proof of Lemma 6. Assume that point P_1 to P_k lies on one side from the x-axis and points P_{k+1} to P_{n-1} lie on another side. By lemma 5,

$$\sum_{i=1}^{n} \cos\theta_i = 0 \Rightarrow \sum_{i=1}^{k} \cos\theta_i + \sum_{i=k+1}^{n} \cos\theta_i = 0.$$

When the line is rotated by α, where $\alpha \neq n\pi$ for any integer n, the equation becomes

$$\sum_{i=1}^{k} \cos(\theta_i + \alpha) + \sum_{i=k+1}^{n} \cos(\theta_i - \alpha) = 0$$

$$\sum_{i=1}^{k} (\cos\theta_i \cos\alpha - \sin\theta_i \sin\alpha)$$
$$+ \sum_{i=k+1}^{n} (\cos\theta_i \cos\alpha + \sin\theta_i \sin\alpha) = 0$$

$$\sum_{i=1}^{k} \cos\theta_i \cos\alpha - \sum_{i=1}^{k} \sin\theta_i \sin\alpha$$
$$+ \sum_{i=k+1}^{n} \cos\theta_i \cos\alpha + \sum_{i=k+1}^{n} \sin\theta_i \sin\alpha = 0$$

$$\sum_{i=1}^{k}\cos\theta_i\cos\alpha+\sum_{i=k+1}^{n}\cos\theta_i\cos\alpha$$
$$-\sum_{i=1}^{k}\sin\theta_i\sin\alpha+\sum_{i=k+1}^{n}\sin\theta_i\sin\alpha=0$$
$$\cos\alpha\sum_{i=1}^{n}\cos\theta_i+\sin\alpha(-\sum_{i=1}^{k}\sin\theta_i+\sum_{i=k+1}^{n}\sin\theta_i)=0$$

By Lemma 5, $\sum_{i=1}^{n}\cos\theta_i=0$. Therefore

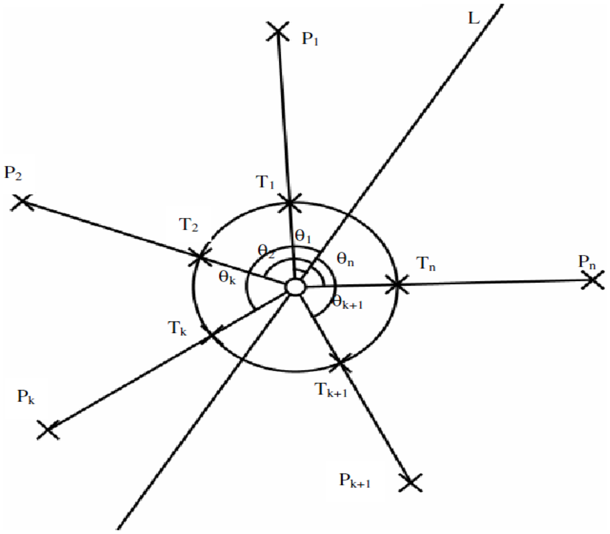
$$\sin\alpha(-\sum_{i=1}^{k}\sin\theta_i+\sum_{i=k+1}^{n}\sin\theta_i)=0$$
$$-\sum_{i=1}^{k}\sin\theta_i+\sum_{i=k+1}^{n}\sin\theta_i=0$$
$$\sum_{i=1}^{k}\sin\theta_i=\sum_{i=k+1}^{n}\sin\theta_i$$

As a result, a desired α can be taken to get the desired line and it finishes the proof. □

(Continue to the proof of Theorem 4) Draw a circle Γ with radius r centered at F such that Γ intersect with FP_i at T_i for all $i=1,2,\dots,n$. Construct a line L passing through F such that $\{T_1\dots T_k\}$ lie on one side of the arc while $\{T_{k+1}\dots T_n\}$ lie on the other arc. Let the angle between L and FT_i be ϕ_i for all $i=1,2,\dots,n$. Let $\hat{\imath}$ be the unit vector on L and $\hat{\jmath}$ be the unit vector perpendicular to L.

For all $i=1,2,\dots,n$, we have:

$$FT_i=|FT_i|\cos\phi_i\hat{\imath}+|FT_i|\sin\phi_i\hat{\jmath}=r\cos\phi_i\hat{\imath}+r\sin\phi_i\hat{\jmath}.$$



$$\sum_{i=1}^{n} FT_i = \sum_{i=1}^{k} FT_i + \sum_{i=k+1}^{n} FT_i$$

$$\sum_{i=1}^{n} FT_i = \sum_{i=1}^{k}(r\cos\phi_i\hat{i} + r\sin\phi_i\hat{j}) + \sum_{i=k+1}^{n}(r\cos\phi_i\hat{i} + r\sin\phi_i\hat{j})$$

$$= r\sum_{i=1}^{k}\cos\phi_i\hat{i} + r\sum_{i=1}^{k}\sin\phi_i\hat{j} + r\sum_{i=k+1}^{n}\cos\phi_i\hat{i} + r\sum_{i=k+1}^{n}\sin\phi_i\hat{j}$$

$$= r\sum_{i=1}^{n}\cos\phi_i\hat{i} + r\sum_{i=1}^{k}\sin\phi_i\hat{j} + r\sum_{i=k+1}^{n}\sin\phi_i\hat{j}$$

$$= 0 + r\sum_{i=1}^{k}\sin\phi_i\hat{j} + r\sum_{i=k+1}^{n}\sin\phi_i\hat{j}\ (\textit{By lemma } 5)$$

$$\sum_{i=1}^{n} FT_i = r\sum_{i=1}^{k} \sin\phi_i \hat{j} - r\sum_{i=1}^{k} \sin\phi_i \hat{j} \text{ } (By\ lemma\ 6)$$

$$\sum_{i=1}^{n} FT_i = 0$$

$$\sum_{i=1}^{n} \frac{FT_i}{r} = 0$$

$$\sum_{i=1}^{n} \frac{FT_i}{|FT_i|} = 0$$

And it finishes the proof of theorem 4. □

4. Application

4.1. Approximate the location of the shortest distance point by compass, ruler and mathematical methods

The method of finding the 3-F point and 4-F point is well-known and simple. Here is a simple idea of constructing them:

4.2. 3-F point

Assume the 3 fixed points are A, B and C respectively. Construct the points D and E outside triangle ΔABC such that triangles ΔDAB and ΔEAC are equilateral. The intersection of lines DC and EB is exactly the 3-F point.

4.3. 4-F point

For a convex quadrilateral, the intersection of the diagonals is the 4-F Point; for a concave quadrilateral, the point located inside the triangle formed by the other 3 points is the 4-F point. The proof has been printed in the 'Extension' part.

As a result, the following paragraphs mainly focus on the method of constructing the n-F point by compass and ruler, where $n \geqslant 5$, and a special case for $n = 5$.

4.4. n-F Point

4.4.1. When $n \geqslant 5$

Consider $\{P_1, P_2, \ldots, P_{n-1}\}$ are fixed and P_n varies. By Theorem 1, $(n-1)$ $(n-2)$-F points can be found out and they can be linked by a straight line so that the locus of n-F point is approximated. Name the $(n-2)$-F points which does not consider the distance to the points P_n and P_i be R_i(where $i = 1, \ldots, n-1$). Name the line joining R_i and P_i be L_i. where L_i only refers to the line segment projected away from R_i in the opposite side from the bounded area. By Lemma 2, when P_n moves along L_i, the n-F point is always R_i. Therefore, the area outside bounded area can be divided into $(n-1)$ parts by $L_1, \ldots, L_{n-1}$.

Consider P_n is located outside the bounded area and between L_k and L_{k+1}. To approximate the location of n-F point, a reference line can be constructed in a parallel direction to line R_kR_{k+1}. Let the reference line intersect L_k and L_{k+1} at G_k and G_{k+1} respectively. We can construct a point F between R_k and R_{k+1} such that $R_kF : FR_{k+1} = G_kP_n : P_nG_{k+1}$. F is called the approximated n-F point.

4.4.2. When $n = 5$.

First, we can find out the 3-F Point(R_4) of $P_1P_2P_3$ by compass and ruler.

Similarly, construct the other 3 3-F Points (R_1, R_2 and R_3).

Then, we can join the diagonals. We would like to find out a point X on the diagonal such that P_1XP_3 or P_2XP_4 equals $120°$, i.e. X is the 3-F point of $P_1P_3P_5$ or $P_2P_4P_5$.

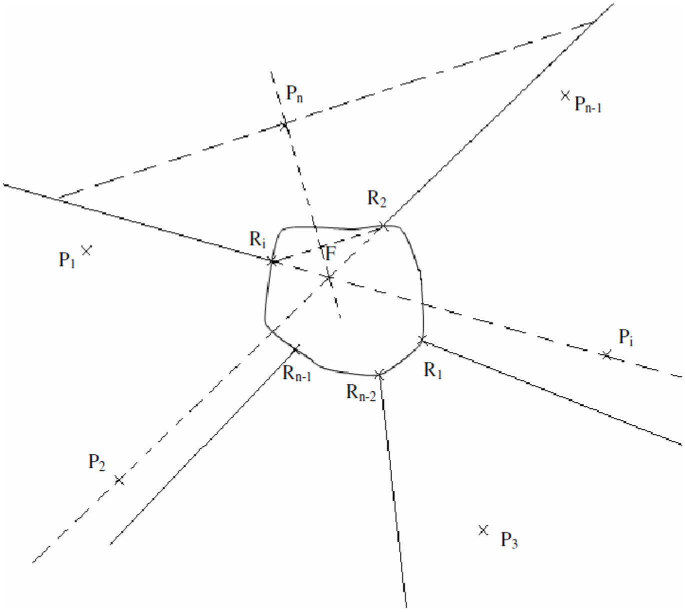
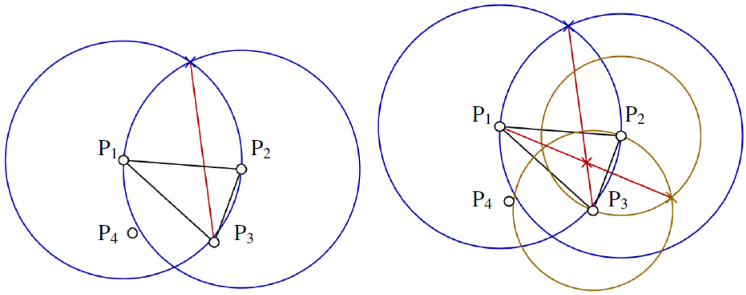


FIGURE 3. A diagram showing Case 1



1. Join the diagonal (P_2P_4) first.

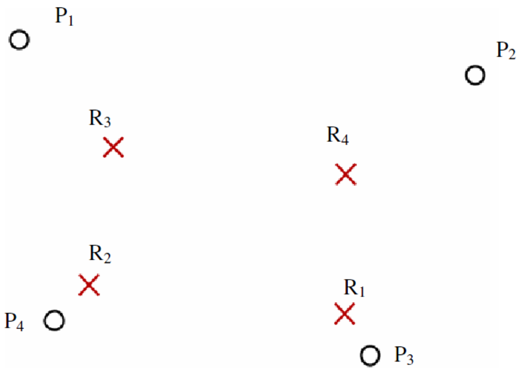
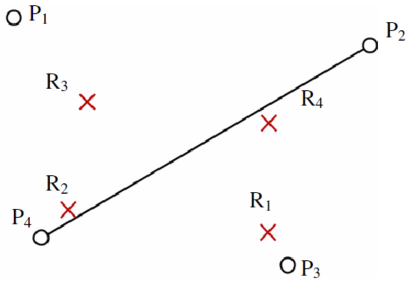
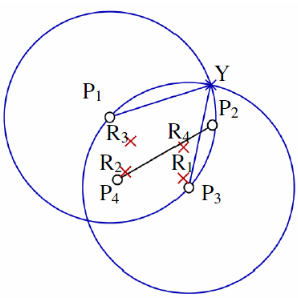


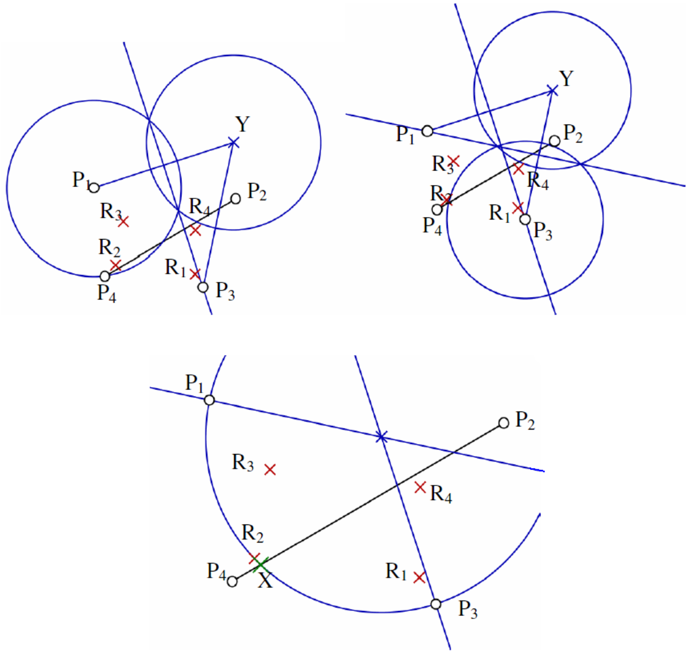
FIGURE 4. (The red crosses are the 3-F points.)



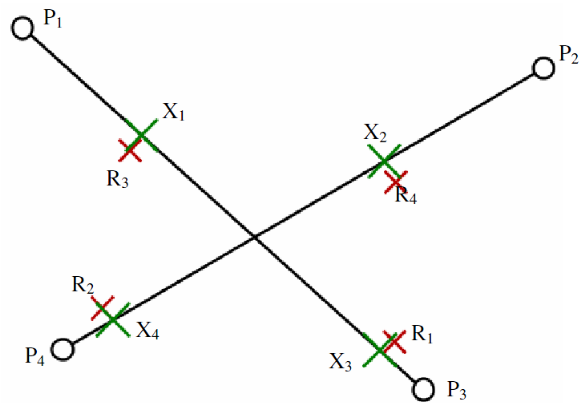
2. Using P_1P_3, construct an equilateral triangle ΔP_1P_3Y.



3. Construct the circumcircle of triangle P_1P_3Y. So for any point Z on the minor arc P_1P_3, $\angle P_1ZP_3 = 180° - 60° = 120°$ (opp. $\angle$, cyclic quad.)
4. Therefore, the intersection point of the circle and P_2P_4 is the desired point X.

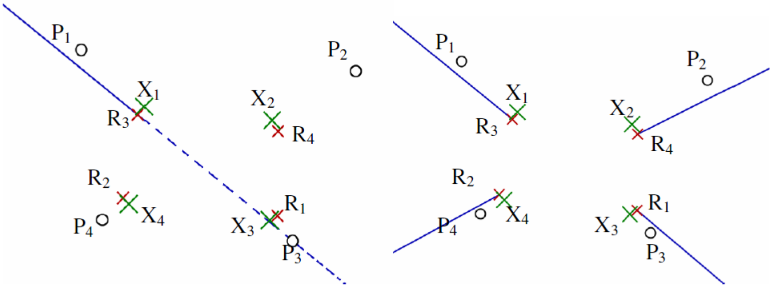


Similarly, a total number of 4 points X can be constructed, name X_1 to X_4.

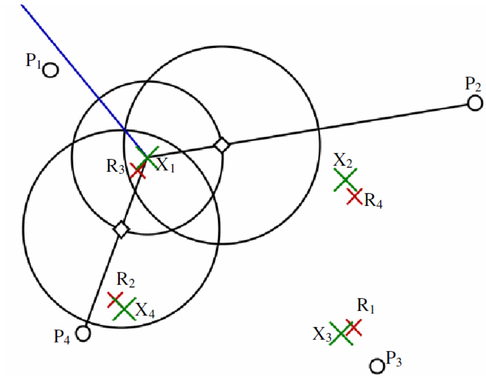


Using similar method in Case 1, we can construct those L lines (i.e. joining R_i to P_i, the blue lines in the diagram).

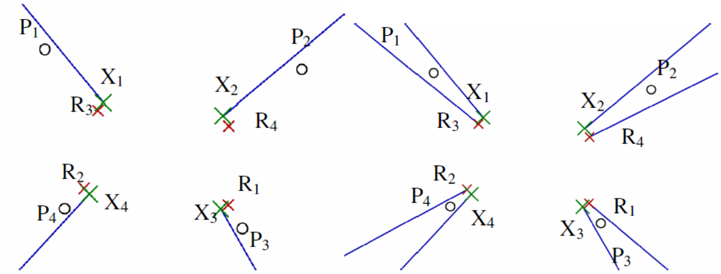
Then, using compass and ruler, we can construct the angle bisector of ext. $\angle P_2X_1P_4$. Since $\angle P_2X_1P_4 = 120°$, for all P_5 lies on the angle

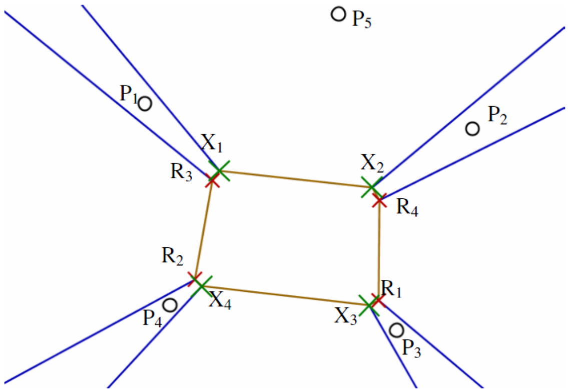


bisector, the n-F point is X_1 (by Lemma 3).

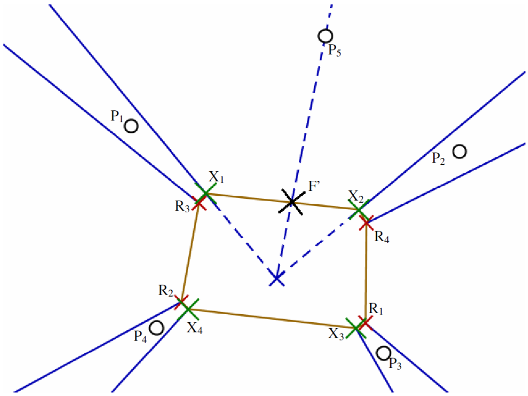


Therefore we know that X_1 lies on the boundary of the bounded area. Similar case for X_2, X_3 and X_4.





As a result, the bounded area can be approximated by 8 points. Using similar method mentioned in Case 1, we can approximate the location of 5-F points on a more precisely approximated boundary.



Percentage Error due to Approximation

To check the accuracy of the above-mentioned approximation, a group of test data is designed for the case $n = 5$.

The coordinates of five fixed points:
$P_1 = (3, 8)$, $P_2 = (8, 3)$, $P_3 = (5, -5)$, $P_4 = (-4, -4)$, $P_5 = (-5, 3)$

Using the program developed by us, the coordinates of the 5-F Point F is $(1.426, 1.070)$.

By distance formula, $\sum_{i=1}^{5} FP_i = 35.13761$.

Using the approximation method mentioned, the coordinates of the 5-F Point F' is $(1.31483, 0.64845)$, calculated by the program C.a.R.

By distance formula, $\sum_{i=1}^{5} FP_i = 35.17251$.

Therefore, the percentage error of the approximation in terms of sum of total distances

$$= \frac{35.17251 - 35.13761}{35.13761} \times 100 = 0.099.$$

Result shows that such method is very accurate.

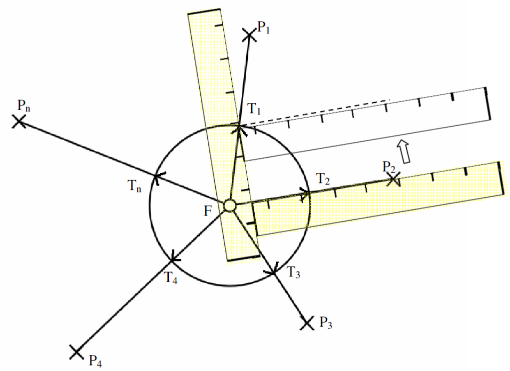
Determine the shortest distance point by compass and ruler

Given fixed points $\{P_1, P_2, \ldots, P_n\}$ and a point F, we can determine if the point F is exactly the n-F point of $\{P_1, P_2, \ldots, P_n\}$ or not.

Firstly, draw a circle centered at F with a radius R smaller than any FP_i for all $i = 1, 2, \ldots, n$. Assume the circle intersect FP_i at T_i .By theorem 4, we can deduce that $\sum_{i=1}^{n} FT_i = 0$.

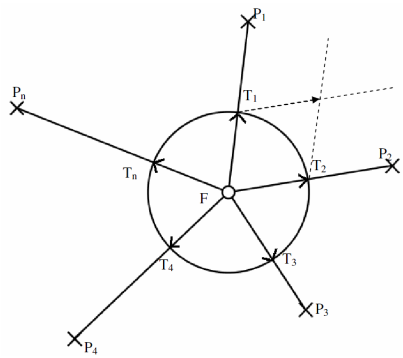
Secondly, translate FT_2 to the position T_1. By this tip-to-tail method, if the vector FT_n 'returns' to the original position F, it shows the sum of vectors is zero and F is the n-F point. Otherwise, F is not the n-F point.

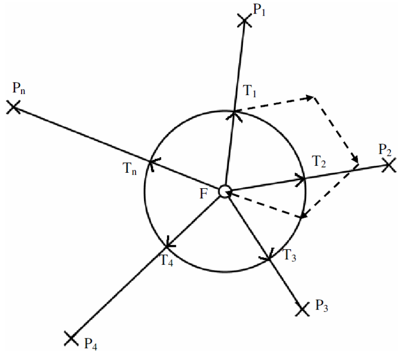
Steps 1: Draw the circle and use ruler to translate the vectors by drawing the parallel lines.



Steps 2: Find out the intersection and we can successfully translate the vector.

Steps 3: Check whether the vectors 'return' to F to determine F is n-F point or not.





5. Amendment (2011)

In 2011, we have rewritten some of the statements and proofs in order to enhance the readability of the report, as well as introduced two fundamental theorems to provide a more rigorous background of the n-F point problem at the suggestion of the reviewers.

Lemma 7. *(Re-statement of Lemma 2): Let F be the n-F point of $P_1, \ldots,$ P_{n-1} and P_n. Joining FP_n and let E' be on the line segment FP_n. Then F is the n-F point of $P_1, \ldots, P_{n-1}$ and E'.*

Proof. Please refer to the proof of Lemma 2. □

Theorem 8. *(Existence of the n-F Point): Let $P_1, \ldots, P_{n-1}$ and P_n be n points on a plane. Then there exist a point F such that the sum of distance from F to $P_1, \ldots, P_{n-1}$ and P_n is minimum over the whole plane.*

Proof. We introduce the Cartesian coordinates to the statement. Let (x_i, y_i) be the coordinates of P_i. Let D be the distance function, i.e.

$$D(x,y) = \sum_{i=1}^{n} \sqrt{(x-x_i)^2 + (y-y_i)^2}.$$

The theorem is thus saying that D has a global minimum, attained at some minimizing points. Note that D is a continuous function.

Take $r = \max_{1 \leqslant i \leqslant n} \sqrt{x_i^2 + y_i^2}$, C_r be the circle centered at the origin with radius r, and $C = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \leqslant r\}$. Note that $P_i \in C$ for all i satisfying $1 \leqslant i \leqslant n$ and C is closed and bounded in $\mathbb{R}^2$. Since D is continuous, the restriction map $D|_C$ attains a global minimum, namely m, in some points inside C.

Let $X = (a,b)$ be a point lying outside C. Define $X' = (\frac{ra}{\sqrt{a^2+b^2}}, \frac{rb}{\sqrt{a^2+b^2}})$, the intersection point between C_r and line OX. This yield $P_iX' < P_iX$ for all i satisfying $1 \leqslant i \leqslant n$, hence $m \leqslant D(X') \leqslant D(X)$. Combining the cases, m is the global minimum of D. Since D can attain m in some points inside C, the minimizing points exist. □

Theorem 9. *(Uniqueness of the n-F Point): Let $P_1, \ldots, P_{n-1}$ and P_n be n fixed points on a plane. The n-F is unique if the points are not collinear.*

Proof. Assume the n-F point is not unique and let A and B be two of the n-F point.

Claim 10. *The mid-point between A and B should have a shorter sum of distance.*

Proof of Claim. Consider triangle ΔABP_i. Let C be the mid-point of AB. Construct Q_i by extending P_iC such that $P_iC = CQ_i$. □

By triangle inequality, we have $AP_i + AQ_i \geqslant P_iQ_i$ and $BP_i + BQ_i \geqslant P_iQ_i$. Note that as $P_1, \ldots, P_{n-1}$ and P_n are not collinear, equality sign does not hold for some i. Using the fact that $P_iC = CQ_i$ and combining the equations, we have $AP_i + BP_i \geqslant 2CP_i$, hence

$$2\sum_{i=1}^{n} AP_i = \sum_{i=1}^{n} AP_i + \sum_{i=1}^{n} BP_i = \sum_{i=1}^{n}(AP_i + BP_i) > 2\sum_{i=1}^{n} CP_i$$

and thus $\sum_{i=1}^{n} AP_i > \sum_{i=1}^{n} CP_i$, contradicting the fact that A is the n-F point. $\square$

Note that if all of the fixed points are collinear, the n-F point may not necessarily be unique. We lay the focus on the case that they are not collinear for this project.

6. Conclusion

To conclude, the topic of this paper has been discussed in Mathematical field for a long time. Here, the paper shows that with elementary calculation, such as triangle inequality, simple differentiation and 'tip-to-tail' method in vector, can develop some lemmas and theorems which help us probe into the problem in different perspectives.

Overall, the necessary and sufficient condition for a point to be the n-F point of n given points is found in Theorem 4. A mathematical approximation method (with accuracy over 99) which can be implemented with compass and ruler is also discovered.

Although the geometric approach to find the n-F point of n given points is still unknown, it is hoped that the findings throughout this paper are the stepping-stones to solve the problem.

7. Appendix

7.1. Description of the program

A program using Free Pascal which is useful in finding out the coordinates of the optimum point (The total distance between the point and the given points is the minimum.) has been written. The outputs are corrected to 3 decimal places. **Sample input and output:**

Sample Input:

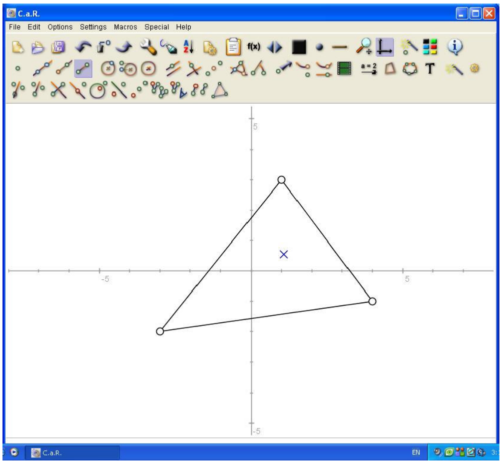
```
Enter the total number of given points: 3
```

Sample Output:

```
Enter the total number of given points: 3
Enter the coordinates of the given points:
For example:
2   3
4   5

-3   -2
1   3
4   -1
The coordinates of the optimum point are:
1.085   0.551
```

Then, a dynamic geometry program called 'Compass and Ruler' (C.a.R.) is used to plot the points so as to study the result obtained.



Algorithm of the program

First, the mean of the x-coordinates and that of the y-coordinates are found. It is called the “first point”.

Then, the points around the “first point” are tested whether the total distance is smaller than that of the “first point”.

If so, similar things are done for the “better point” until there is no other “better point”.

Proof of the validation of the program. Firstly, “if $f(x) + g(x) = h(x)$ then $f'(x) + g'(x) = h'(x)$” has to be shown.

$$\begin{aligned}
h'(x) &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + G(x+\Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

By this principle, if $h(x) = \sum_{i=1}^{n} f_i(x)$, then $h'(x) = \sum_{i=1}^{n} f_i'(x)$. By induction, $h^{(j)}(x) = \sum_{i=1}^{n} f_i^{(j)}(x)$, where $h^{(j)}(x)$ stands for the the j-th derivative of $h(x)$.

Now, let $h(x, y)$ represents the function of the summation of distance from the point (x, y) to a number of fixed points. Now "$h(x, y)$ only has an absolute minimum but no any relative minimums" is going to be proved.

$$h(x, y) = \sum_{i=1}^{n} \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

In the program, y_i is treated as a constant first.

Let $(y - y_i)^2 = c^2$ and $x_i = b_i \forall i$ (for easy understanding only, i.e. b_i is constant). The equation becomes

$$h(x) = \sum_{i=1}^{n} \sqrt{(x - b_i)^2 + c^2} = \sum_{i=1}^{n} \sqrt{x^2 - 2b_i x + b_i^2 + c^2}.$$

Let $f_i(x) = \sum_{i=1}^{n} \sqrt{x^2 - 2b_i x + b_i^2 + c^2}$. Consider the geometric meaning of $f_i(x)$, it is the distance from the point (x, y) to the point (b_i, y). From this perspective, it is obvious that $f_i(x)$ is an decreasing function when $x < b_i$, a stationary point at $x = b_i$, and an increasing function when $x > b_i$, i.e. $f_i'(x)$ is an increasing function.

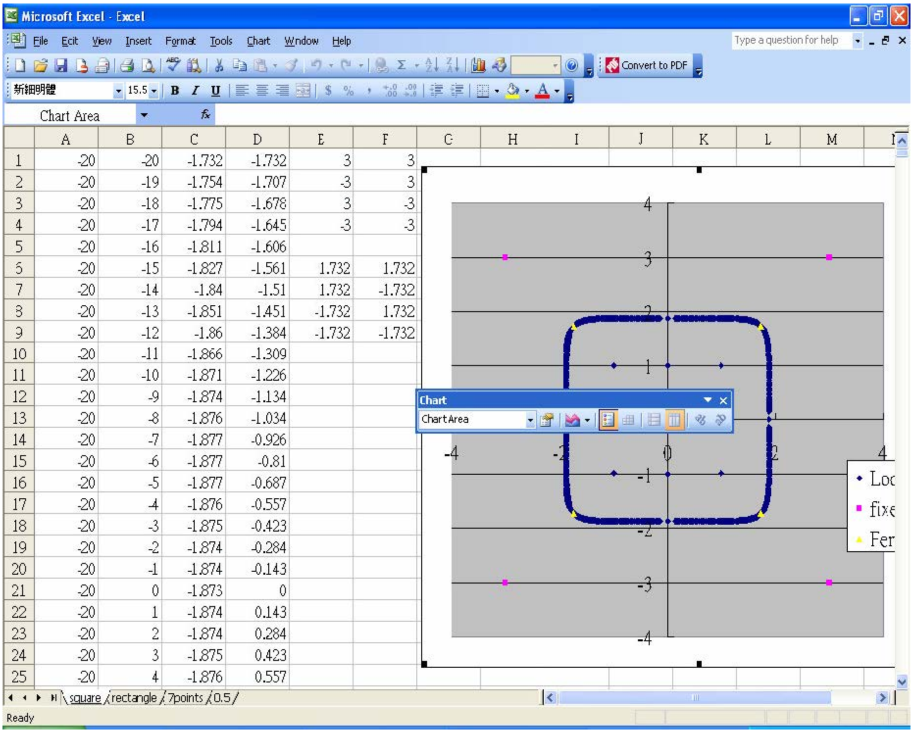
$$h(x) = \sum_{i=1}^{n} f_i(x).$$

Since $f_i'(x)$ are increasing $\forall i$, $h'(x)$ is also increasing. Hence $h'(x)$ only passes through x-axis at one point, implying $h(x)$ has only one turning point, so its minimum point is unique. Therefore, our hypothesis is true and the reclusive program can be used to find out the point.

Different Versions of the Program

Using Microsoft Excel

Version	Input	Output
Distance3.exe// Distance3.5.exe (*debugged*)	The coordinates of n points	The coordinates of the n-F point
Distance6.exe// Distance6.5.exe (*debugged*)	The coordinates of the $n-1$ points	The coordinates of the n-F point with the given $n-1$ points and $(-10,-10)$, $(-9.9,-10)$, $\ldots$, $(10, 9.9)$ **OR** $(10, 10)$ in txt files



	A	B	C	D	E	F
1	-20	-20	-1.732	-1.732	3	3
2	-20	-19	-1.754	-1.707	-3	3
3	-20	-18	-1.775	-1.678	3	-3
4	-20	-17	-1.794	-1.645	-3	-3
5	-20	-16	-1.811	-1.606		
6	-20	-15	-1.827	-1.561	1.732	1.732
7	-20	-14	-1.84	-1.51	1.732	-1.732
8	-20	-13	-1.851	-1.451	-1.732	1.732
9	-20	-12	-1.86	-1.384	-1.732	-1.732
10	-20	-11	-1.866	-1.309		
11	-20	-10	-1.871	-1.226		
12	-20	-9	-1.874	-1.134		
13	-20	-8	-1.876	-1.034		
14	-20	-7	-1.877	-0.926		
15	-20	-6	-1.877	-0.81		
16	-20	-5	-1.877	-0.687		
17	-20	-4	-1.876	-0.557		
18	-20	-3	-1.875	-0.423		
19	-20	-2	-1.874	-0.284		
20	-20	-1	-1.874	-0.143		
21	-20	0	-1.873	0		
22	-20	1	-1.874	0.143		
23	-20	2	-1.874	0.284		
24	-20	3	-1.875	0.423		
25	-20	4	-1.876	0.557		

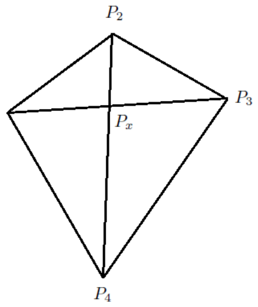
The data generated by the program is copied to Microsoft Excel, and the data is used to produce a XY(Scatter) chart. Those charts help us to make reasonable hypothesis. □

REFERENCES

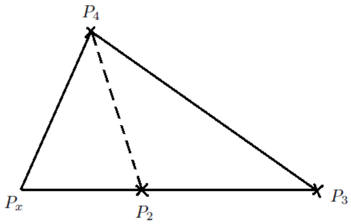
[1] Plastria, Frank. *4-point Fermat location problems revisited. New proofs and extensions of old results*, IMA Journal of Management Mathematics **17** (2006), 387–396
[2] Weiszfeld, E. *Sur le point pour lequel la somme des distances de n points donnes est minimum*, Tohoku Math. Journal **43** (1937), 355–386.

Reviewer's Comments

1. The following figure should be inserted under case 1 on page 2:



2. The following figure should be inserted under case 2 on page 2:



3. Originally the diagram shown is coloured to distinguish the smaller triangles; for convenience of printing, we distinguished the smaller triangles by suitable labels as shown.

4. On page 5, items 1.2 and 1.5 can be deleted.

5. The reviewer suggests the following restatement of Lemma 2:
   Lemma 2 *Let F be the n-F point of $A_1, \ldots, A_n$ and E' is a point on the line segment FA_n. Then F is also the n-F point of $A_1, \ldots, A_{n-1}, E'$.*

6. The reviewer suggests the following restatement of Lemma 3:
   Lemma 3 *For any n points on a plane, including $(n-1)$ fixed points and 1 variable point, we choose $(n-2)$ points from the set of fixed points. When the variable point moves to a certain position such that it lies on the straight line formed by the $(n-2)$-F point and the remaining fixed point, the n-F point is the same as this $(n-2)$-F point when the variable point moves outward along the line. For example,*

take $n = 7$. A, B, C, D, E, F *are fixed points.* G *is a variable point. Let* X *be the* 5-F *point of* A, B, C, E *and* F. *Then* X *lies on the boundary of locus of the* 7-F *points.*