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Honorable Mention

Spherical Fagnano's Problem and its Extensions

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SPHERICAL FAGNANO'S PROBLEM AND ITS EXTENSIONS

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Abstract. In a given acute triangle, the inscribed triangle with minimum perimeter is the orthic triangle. This problem was proposed and solved using calculus by Fagnano in 1775.

Now we wonder, will the result remain unchange when the problem is discussed on a sphere? [See reviewer's comment (2)] In this paper, we will first try to find the answer of the "spherical Fagnano's problem". Based on our results in spherical triangle cases, we will go further to generalize the problem to spherical quadrilateral and n-sided spherical polygon in spherical geometry.

1. Introduction

In Euclidean geometry, Fagnano's problem is an optimization problem that was stated by Giovanni Fagnano in 1775: for a given acute triangle, determine the inscribed triangle with minimal perimeter.

Fagnano solved this question by using calculus and the answer is the orthic triangle. [See reviewer's comment (3)] Once the answer became known, several purely geometric solutions were also discovered. In addition, the solution of an obtuse triangle does not exist and it will degenerate to twice of the altitude from the obtuse angle. [See reviewer's comment (4)] Using the light reflection principle, the general problem of finding the shortest perimeter of an inscribed polygon within a polygon has also been discussed and solved comprehensively.

However, we are living on a sphere where the solution above in Euclidean geometry is not yet applicable and precise, especially in daily applications within a large scale. For the sake of enhancing the precision, we are going to discuss this problem in a new extend, from Euclidean to spherical, from triangle to N-gon.

Once we started to work on this question, we discovered that spherical geometry is quite different from our conception of Euclidean geometry. Apart from using the light reflection principle, we use stereographic projection method, which is able to project the polygon on a sphere to a 2-dimensional plane with angle preserving property, to help us solve this problem.

There are four main parts in this paper.

The first part is to introduce some fundamental spherical geometry, which is necessary to solve our problem.

The second part is the solution of spherical triangle problem. We attacked this problem by dividing the spherical triangles according to the number of obtuse angles in a triangle. In the case of acute spherical triangle (no obtuse angle), we attained the same result as in the Euclidean case. The orthic triangle is the triangle with minimum perimeter inscribed in acute spherical triangle. The other three cases with obtuse angles have also been proved to admit no solutions, but a degenerated one.

In the third part, we further discussed the problem in the case of spherical quadrilaterals. We proved that there is no solution in all kinds of spherical quadrilaterals, but we found an algorithm to obtain the degenerate solution.

In the last part, we tried to generalize the problem to n-sided spherical polygon. We found out that the answer is different due to the parity of sides. We proved that there is no solution in $2n$-sided spherical polygon. A solution can be obtained in a $2n + 1$ -sided spherical polygon iff it is a closed path of light.

2. What about on a sphere?

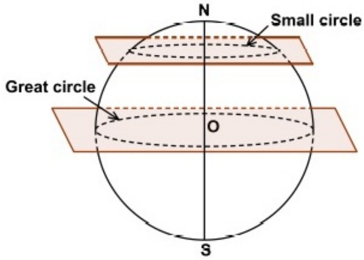

FIGURE 2.1

In this project, the sphere we are going to discuss is a unit sphere, and all the spherical polygons or spherical triangles discussed are having sides and angles in between 0 and π.

Definition 1. *A great circle is the intersection of a sphere with a plane through the centre of the sphere, otherwise, it is a small circle.*

[See reviewer's comment (5)]

Remark 2. *The geodesic, or the 'straight line' joining two points on the sphere is the great circle joining the points.*

Definition 3. *As shown in Fig 2.2, the spherical angle N, formed by two great circless SAN and SBN is defined to be the angle between the planes AON and BON, i.e. $\angle$ AOB.*

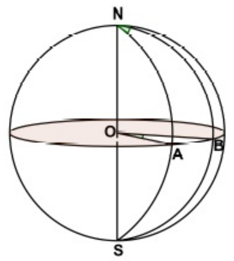

FIGURE 2.2

Definition 4. *A spherical polygon is the portion of a spherical surface bounded by the arcs of great cirlces. In particular, a spherical triangle is bounded by three arcs of great circles.*

Proposition 5. *(Trihedral angle Inequality) The sum of any two face angles of a trihedral angle is greater than the third face angle.*

[See reviewer's comment (6)]

Proof. Let $\angle ZOY = \alpha, \angle XOZ = \beta, \angle XOY = \gamma$. WLOG, assume $\alpha \geq \beta \geq \gamma > 0$, obviously, $\alpha + \gamma > \beta$ and $\alpha + \beta > \gamma$. Hence remains ro prove $\beta + \gamma > \alpha$.

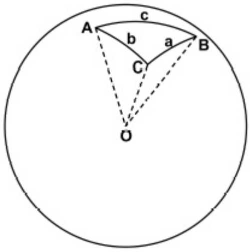

FIGURE 2.3

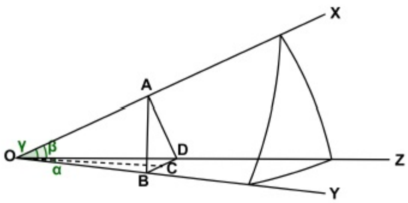

FIGURE 2.4

Case 1. $\alpha = \beta \geq \gamma > 0$. Trivial.

Case 2. $\alpha > \beta \geq \gamma > 0$.

First, randomly choose A, B on OX, OY respectively. Then construct OC on plane YOZ such that $\angle COZ = \beta$, $OC = OA$. Extend BC to meet OZ at D. Then by Euclidean triangle inequality,

$$AB + AD > BC + CD$$

Since AD and CD are corresponding sides of congruent triangles $\triangle OAD$ and $\triangle OCD$, then

$$AB > BC$$

Also, in $\triangle OAB$ and $\triangle OCB$, $OA = OC$, [See reviewer's comment (7)] then

$$\angle AOB > \angle COB$$
$$\gamma > \alpha - \beta$$
$$\therefore \beta + \gamma > \alpha$$

□

In Corollary 5 to Proposition 10, we consider the spherical triangle with sides a, b, c, opposite to $\angle A, \angle B, \angle C$ respectively.

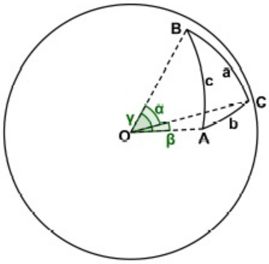

FIGURE 2.5

Corollary 6. Spherical Triangle Inequality *In a spherical triangle with sides a, b and c,*

$$\begin{aligned} a+b &> c \\ a+c &> b \\ b+c &> a \end{aligned}$$

Remark 7. *In the sphere, the shortest path between two points is the minor arc of the great circle passing through that two points.*

Proposition 8. *In the spherical triangle,*

$$\begin{aligned} \sin A &= \frac{\sin a}{\sin c} \\ \cos A &= \frac{\tan b}{\tan c} \\ \tan A &= \frac{\tan a}{\sin b} \end{aligned}$$

[See reviewer's comment (8)]

Proof. As shown in Fig 2.6

$$\sin A = \frac{B'C'}{A'B'} = \frac{\frac{B'C'}{OB'}}{\frac{A'B'}{OB'}} = \frac{\sin a}{\sin c}$$

$$\cos A = \frac{A'C'}{A'B'} = \frac{\frac{A'C'}{OA'}}{\frac{A'B'}{OA'}} = \frac{\tan b}{\tan c}$$

$$\tan A = \frac{B'C'}{A'C'} = \frac{\frac{B'C'}{OC'}}{\frac{A'C'}{OC'}} = \frac{\tan a}{\sin b}$$

□

Theorem 9. Spherical Pythagoras Theorem *As shown in Fig 2.6,*

$$\cos c = \cos a \cos b$$

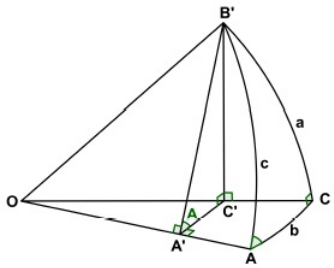

FIGURE 2.6

Proof. By Prop 8,

$$\sin A = \frac{\sin a}{\sin c}$$

$$\cos A = \frac{\tan b}{\tan c}$$

$$\tan A = \frac{\tan a}{\sin b}$$

$$\frac{\sin a}{\sin c} \times \frac{\tan c}{\tan b} = \frac{\tan a}{\sin b}$$

$$\therefore \cos c = \cos a \cos b$$

□

Proposition 10. Spherical Sine Law

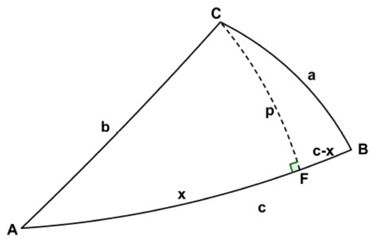

FIGURE 2.7

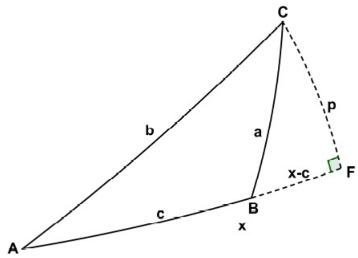

FIGURE 2.8

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

Proof. In either figure,

$$\sin p = \sin b \sin A = \sin a \sin B$$

$$\therefore \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}$$

Similarly,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

□

Proposition 11. Spherical Cosine Law *As shown in Fig 2.7 and Fig 2.8,*

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{1}$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos B \tag{2}$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C \tag{3}$$

Proof. In either figures,

$$\cos b = \cos p \cos x \tag{1}$$
$$\cos a = \cos p \cos(c - x) \tag{2}$$

2) ÷ 1),

$$\begin{aligned}\frac{\cos a}{\cos b} &= \frac{\cos(c-x)}{\cos x}\\ &= \frac{\cos c\cos x + \sin c\sin x}{\cos x}\\ &= \cos c + \sin c\tan x\\ &= \cos c + \sin c\tan b\cos A\\ \therefore \cos a &= \cos b\cos c + \sin b\sin c\cos A\end{aligned}$$

It is similar for the other two formulae. □

2.1. Stereographic Projection

For any point $P(a, b, c)$ on the unit sphere $a^2+b^2+c^2 = 1$, the stereographic projection of P is $P'(x, y, 0)$ such that $\overrightarrow{NP'} = k\overrightarrow{NP}$ for some $k \in \mathbb{R}$. [See reviewer's comment (9)]

$$(x, y, -1) = k(a, b, c-1)$$

$$\therefore \quad k = \frac{1}{1-c}, x = \frac{a}{1-c}, y = \frac{b}{1-c}$$

$$\therefore \quad P'\left(\frac{a}{1-c}, \frac{b}{1-c}, 0\right)$$

Theorem 12. *Any small circles or great circles on the sphere will become a circle or straight line on the plane via stereographic projection.*

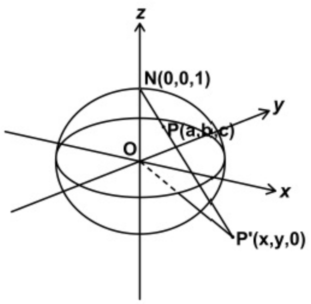

FIGURE 2.9

Proof. Any circle on a sphere can be represented by

$$x^2 + y^2 + z^2 = 1 \tag{1}$$

$$Ax + By + Cz = D \tag{2}$$

Let $P(x, y, z)$ be a point on the circle and $P'(x', y', 0)$ be its stereographic projection, then

$$x' = \frac{x}{1-z}, \quad y' = \frac{y}{1-z}$$

$$\therefore x'^2 + y'^2 = \frac{x^2+y^2}{(1-z)^2} = \frac{1-z^2}{(1-z)^2} = \frac{1+z}{1-z}$$

$$\begin{aligned} \therefore z &= \frac{x'^2+y'^2-1}{x'^2+y'^2+1} \\ x &= \frac{2x'}{x'^2+y'^2+1} \\ y &= \frac{2y'}{x'^2+y'^2+1} \end{aligned}$$

Subst. it in (2) [See reviewer's comment (10)]

$$\begin{aligned} 2Ax' + 2By' + C(x'^2+y'^2-1) &= D(x'^2+y'^2+1) \\ (C-D)x'^2 + (C-D)y'^2 + 2Ax' + 2By' - (C+D) &= 0 \end{aligned}$$

which is an equation of a circle or straight line. □

Theorem 13. *Stereographic projection is conformal, i.e. angle preserving.*

Proof. According to Fig 2.10, due to symmetry, α is equal to β, the angle N on the plane π_1, and since π_1 and π_2 are parallel, as in Fig 2.11,

$$NN'//P'P'' \quad \text{and} \quad NN''//P'P'''$$

in 3-dimensional sense,

$$\begin{aligned} \therefore \beta &= \gamma \\ \therefore \alpha &= \gamma \end{aligned}$$

[See reviewer's comment (11)] □

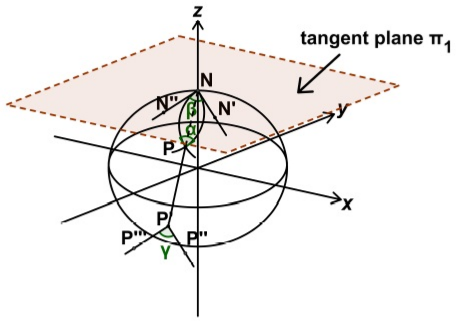

FIGURE 2.10

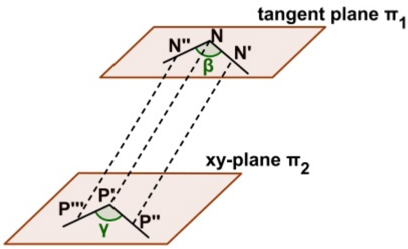

FIGURE 2.11

Theorem 14. *Suppose P,Q are diametrically opposite points on the unit sphere, and P', Q' are their stereographic projections respectively, then P', Q' are in the opposite derection about O and*

$$OP' \times OQ' = 1$$

[See reviewer's comment (12)]

Proof. As shown in Fig 2.12, since $\angle QNP = \frac{\pi}{2}$ and $\angle NOP' = \frac{\pi}{2}$, then

$$\triangle NOP' \sim \triangle Q'ON.$$

$$\therefore \frac{OP'}{NO} = \frac{NO}{OQ'}$$

$$\therefore OP' \times OQ' = 1 \times 1 = 1$$

∎

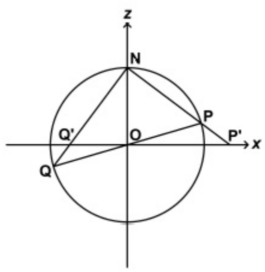

FIGURE 2.12

Remark 15. *Theorem 9 to Theorem 14 are important because if we want to study the phenomenon on the sphere via stereographic projection, we should know the connection between them.*

3. TRIANGLES

In this section, all $A', B', \ldots$ represent the stereographic projection of the corresponding points $A, B, \ldots$ on the sphere.

Theorem 16. *On a unit sphere, the shorteset path from a point A to a great circle C is the minor arc of the great circle passing through A and perpendicular to C.*

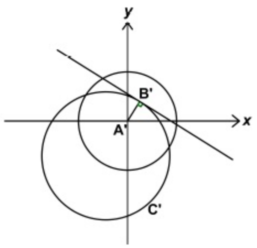

FIGURE 3.1

Proof. WLOG let A be the south pole. By Theorem 12, the stereographic projection of C is a circle C' on the xy-plane. Then as shown in Fig 3.1, by Euclidean geometry, the distance between A' and a point B' on C' is minimum when $A'B'$ is the minor line segment perpendicular to C'. So by Theorem 14 the result follows. [See reviewer's comment (13)] $\square$

Lemma 17. *On the unit sphere, any side of an acute spherical $\triangle ABC$ (i.e. every angle is smaller than $\frac{\pi}{2}$) is less than $\frac{\pi}{2}$.*

Proof. In view of stereographic projection, WLOG, let A' be the origin, B' lies on the positive X-axis and C' lies in quadrant I. Then B', C' should lie inside the unit circle, for otherwise, there are just two cases. [See reviewer's comment (14)]

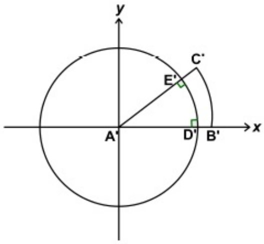

FIGURE 3.2

Case 1. As shown in Fig 3.2 both B' and C' lie on or outside the circle, then by calculating the area of spherical $\triangle ABC$ and $\triangle ADE$, we have

$$\begin{aligned}
\text{Area of spherical } \triangle ABC &> \text{Area of spherical } \triangle ADE \\
\angle A + \angle B + \angle C - \pi &> \angle A + \frac{\pi}{2} + \frac{\pi}{2} - \pi \\
\angle B + \angle C &> \pi
\end{aligned}$$

which contradicts the assumption that both $\angle B$ and $\angle C$ are acute.

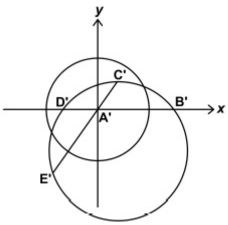

FIGURE 3.3

[See reviewer's comment (15)]

Case 2. Either B' or C' lies on or outside the circle, WLOG, assume B' lies outside and C' lies inside, then as shown in Fig 3.3, suppose D', E' are the stereographic projections of the diametrically opposite points of B, C respectively. By Theorem 14 and converse of power chord theorem, B', C', D', E' are concyclic, and by Theorem 12, the circle $BC'B''C''$ is the stereographic projection of the great circle passing through B and C. Similarly, by Theorem 12 again, the straight line $A'B'$ and $A'C'$ are the arcs of great circles passing through A, B and A, C respectively. Therefore the 'triangle' $A'B'C'$ is the stereographic projection of the spherical $\triangle ABC$.

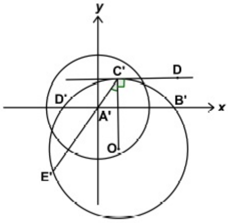

FIGURE 3.4

As shown in Fig 3.4, since $A'B' \geq 1$ and $A'D' \leq 1$, the mid-point of $B'D'$ lies in the right hand side of A', and since $A'C' < 1$ and $A'E' > 1$, the mid-point of $C'E'$ lies in quadrant III, then the center O of the circle $B'C'D'E'$ must lie in quadrant IV.

So $\angle A'C'D' = \frac{\pi}{2} + \angle A'C'O' > \frac{\pi}{2}$ which contradicts to the assumption. [See reviewer's comment (16)]

By Case I and Case II, we get $AB, AC < \dfrac{\pi}{2}$. Then by letting B be the south pole (origin), using the same argument, we can prove that $BC < \dfrac{\pi}{2}$. □

Theorem 18. Acute spherical triangle *For an acute spherical $\triangle ABC$, the inscribed $\triangle PQR$ with the minimum perimeter is the 'orthic' spherical $\triangle ABC$.*

[See reviewer's comment (17)]

Proof. As shown in Fig 3.5, on the sphere, we reflect the spherical triangle ABC about great circles AB and AC. Let P be a point on BC, P' and P'' be its images, then $\angle P'AP'' = 2\angle A < \pi$ and

$$AP = AP' = AP''$$

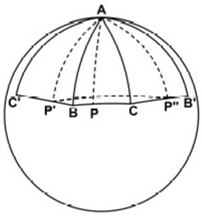

FIGURE 3.5

By Lemma 17, $AB, BP < \dfrac{\pi}{2}$, then by spherical cosine law, [See reviewer's comment (18)]

$$\cos AP = \cos AB \cos BP + \sin AB \sin BP \cos B > 0$$

$$\therefore AP < \frac{\pi}{2}$$

By spherical cosine law again,

$$\begin{aligned}\cos P'P'' &= \cos^2 AP + \sin^2 AP \cos 2A \\ &= \cos^2 AP + \sin^2 AP(1 - 2\sin^2 A) \\ &= 1 - 2\sin^2 AP \sin^2 A\end{aligned}$$

$$\begin{aligned}&\text{Then } P'P'' \text{ is minimum} \\ \iff &1 - \cos P'P'' \text{ is minimum} \\ \iff &\sin^2 AP \text{ is minimum} \\ \iff &\sin AP \text{ is minimum (by previous observation } AP < \frac{\pi}{2} \\ \iff &AP \text{ is minimum} \\ \iff &AP \perp BC \text{ (by Theorem 16)}\end{aligned}$$

Then as shown in Fig 3.6, since $P'RQP''$ is a great circle,

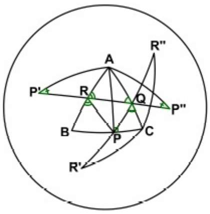

FIGURE 3.6

$$\angle ARQ = \angle BRP \quad , \quad \angle AQR = \angle CQP$$

Since the spherical triangle $AP'P''$ is isosceles,

$$\angle AP'P'' = \angle AP'P''$$
$$\angle APR = \angle APQ$$

$$\therefore \angle BPR = \frac{\pi}{2} - \angle APR = \frac{\pi}{2} - \angle APQ = \angle CPQ$$

Then reflect R about BC and AC to R' and R'' respectively. Since $\angle BPR = \angle CPQ$ and $\angle AQR = \angle CQP$, $R'PQR''$ is a great circle, then the spherical triangle $CR'R''$ is isosceles, imply

$$\angle CR'R'' = \angle CR''R'$$

So we get

$$\angle CRP = \angle CRQ$$

$$\therefore CR \perp AB$$

Similarly, $BQ \perp AC$. □

Lemma 19. Angle-side Relationship in spherical Triangle *If triangle ABC is a spherical triangle with $\angle A \leq \angle B \leq \angle C$, and let $BC = a$, $CA = b$, $AB = c$ then $a \leq b \leq c$.*

Proof. There are only four cases.

Case 1. $0 < \angle A \leq \angle B \leq \angle C \leq \dfrac{\pi}{2}$ (No obtuse angle)

Since sine function is strictly increasing in $[0, \dfrac{\pi}{2}]$,

$$0 < \sin A \leq \sin B \leq \sin C$$

Then by spherical sine law, [See reviewer's comment (19)]

$$0 < \sin a \leq \sin b \leq \sin c$$

$$\therefore a \leq b \leq c \text{ (By Lemma 17)}$$

Case 2. $0 < \angle A \leq \angle B \leq \dfrac{\pi}{2} \leq \angle C < \pi$ (One obtuse angle)

In view of stereograhic projection, let A' be the origin, suppose on the contrary that $c \leq b$, then there are 3 cases.

Case 2.1. $\dfrac{\pi}{2} \leq c \leq b$

As shown in Fig 3.7, since $c \leq b$, we have $A'B' \leq A'C'$.

Let D, E be the diametrically opposite points of B, C respectively, and let PQ, RS be the perpendicular bisectors of $B'D', C'E'$ respectively. Then as shown in Fig 3.8,

$$\begin{aligned} A'P &= \frac{1}{2}\left(A'B' - \frac{1}{A'B'}\right) \\ &\leq \frac{1}{2}\left(A'C' - \frac{1}{A'C'}\right) \\ &= A'R \end{aligned}$$

Therefore by Euclidean geometry of right-angled triangle,

$$A'P \leq A'R < A'S$$

[See reviewer's comment (20)]

Therefore O is below $A'B'$.

Then $\angle B$ is obtuse which contradicts the assumption.

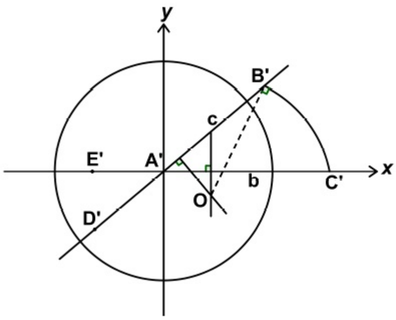

FIGURE 3.7

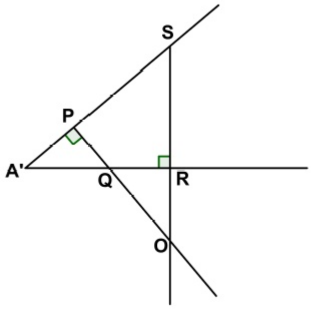

FIGURE 3.8

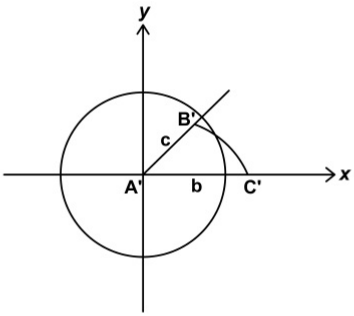

FIGURE 3.9

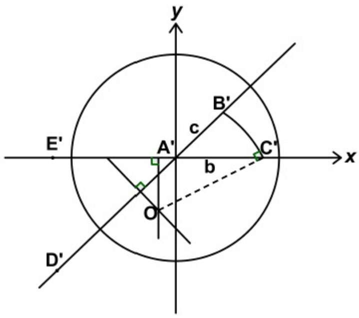

FIGURE 3.10

Case 2.2. $c \leq \dfrac{\pi}{2} \leq b$

As shown in Fig 3.9, this is just the case in the proof of Lemma 17 (case 2), which results in contradiction to the assumption.

Case 2.3. $b \leq c \leq \dfrac{\pi}{2}$

As shown in Fig 3.10, Fig 3.11, by similar argument as Case 2.1 [See reviewer's comment (21)], we get

$$A'R \leq A'P \leq A'Q$$

then O is below $A'C'$, and $\angle C$ is acute which contradict to the assumption.

By combining the Cases 2.1 to 2.3, we get $b < c$.

Lastly, we need to prove $a \leq b$. Suppose in the contrary that $b < a$, we have 3 more cases:

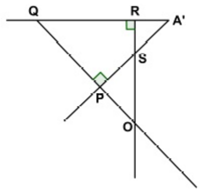

FIGURE 3.11

Case 2.4. $\frac{\pi}{2} \leq b \leq a$

By calculating the area of spherical triangle ABC, we get $\angle A + \angle B > \pi$ which contradicts to the assumption.

Case 2.5. $b < \frac{\pi}{2} < a$

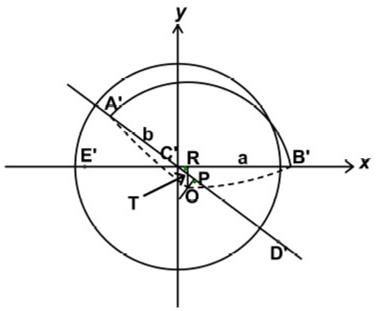

FIGURE 3.12

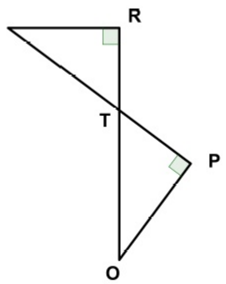

FIGURE 3.13

As shown in Fig 3.12, since $0 < \angle A \leq \angle B \leq \dfrac{\pi}{2}$, O must lie in below $C'B'$ and $A'D'$, then by Euclidean geometry of right-angled triangle,

$$OP < OT < OR$$

Then

$$\sin \angle OA'P = \frac{OP}{OA'} < \frac{OR}{OB'} = \sin \angle OB'R$$
$$\therefore \angle OA'P < \angle OB'R$$
$$\therefore \angle A = \frac{\pi}{2} - \angle OA'P > \frac{\pi}{2} - \angle OB'R = \angle B$$

which contradicts to the assumtion.

Case 2.6. $b < a < \dfrac{\pi}{2}$

By spherical sine law, $\sin b < \sin a$ implies $\sin \angle B < \sin \angle A$, then $\angle B < \angle A$ which contradicts to the assumption.

By combining the cases 2.4 to 2.6, we get $a \leq b$ and so $a \leq b < c$.

Case 3. 0 $0 < \angle A \leq \frac{\pi}{2} < \angle B \leq \angle C < \pi$ (Two obtuse angles)

As shown in Fig 3.14, the spherical triangle DBC is the supplementary spherical $\triangle ABC$ with respect to A.

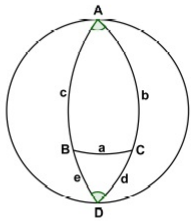

FIGURE 3.14

Then by Lemma 17 and Case 1, $\angle DCB \leq \angle DBC < \dfrac{\pi}{2}$ and $\angle BDC = \angle BAC \leq \frac{\pi}{2}$ imply

$$a \leq \frac{\pi}{2}, \quad e \leq d < \frac{\pi}{2}$$
$$\pi - e \geq \pi - d > \frac{\pi}{2}$$
$$\therefore a \leq \frac{\pi}{2} < b \leq c$$

Case 4. $\dfrac{\pi}{2} < \angle A \leq \angle B \leq \angle C < \pi$ (Three obtuse angles)

As shown in Fig 3.15, the spherical triangle DBC is the supplementary spherical triangle ABC with respect to A.

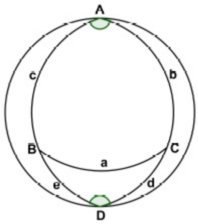

FIGURE 3.15

Since the spherical triangle DBC has only one obtuse angle and

$$\angle DCB \leq \angle DBC < \frac{\pi}{2},$$

by Case 2, $e \leq d$ and so $c \geq b$.

Using the similar argument, if we consider the supplementary spherical triangle ABC with respect to C, we get $a \leq b$. □

Theorem 20. Spherical triangle with 1 obtuse angle *For a spherical triangle ABC with acute $\angle A, \angle B$ and obtuse $\angle C$, the inscribed spherical triangle PQR with minimum perimeter does not exist and it will degenerate to the altitude from C to AB.*

Proof. Let P be a point on BC and P', P'' be the reflection points of P about AB and AC respectively.

Then since $\angle P'BC = 2\angle B < \pi$ and $\angle BCP'' = 2\angle C > \pi$, the shortest path from P' to P'' is $(P' \to C \to P'')$, which is equivalent to the spherical triangle PCR, as shown in Fig 16. [See reviewer's comment (22)]

Then by spherical triangle inequality, $2CR$ must shorter than the perimeter of spherical $\triangle PCR$, and by Theorem 16, CR is minimum when $CR \perp AB$, which is the degenerated case.

So it will be minimum when AP is minimum, that is $AP \perp BC$.

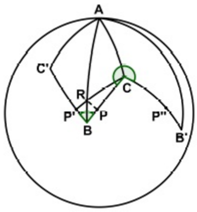

FIGURE 3.16

$\square$

Theorem 21. Spherical triangle with 2 obtuse angles *For a spherical triangle ABC with acute $\angle A$ and obtuse $\angle B$, $\angle C$, then the inscribed spherical triangle PQR with minimum perimeter does not exist and it will degenerate to the side BC, the side included by the two obtuse angles.*

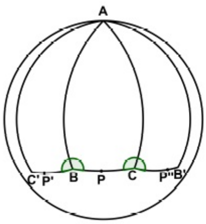

FIGURE 3.17

Proof. Let P be a point on BC, and P', P'' be the reflection point of P about AB, AC respectively. Then as shown in Fig 3.17, since $\angle P'BC = 2\angle B > \pi$ and $\angle BCP'' = 2\angle C > \pi$, the shortest path from P' to P'' is $(P' \to B \to C \to P'')$ which is equivalent to the length $2BC$. So the inscribed triangle PQR with minimum perimeter is degenerated to side BC. $\square$

Theorem 22. Spherical triangle with 3 obtuse angles *For a spherical triangle ABC with three obtuse angles, the inscribed spherical triangle PQR with minimum perimeter does not exist and it will degenerate to the side opposite to the smallest angle.*

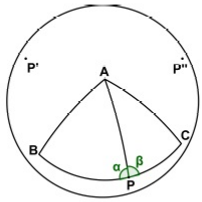

FIGURE 3.18

Proof. Let P be a variable point on BC, and $P'P''$ be the reflecting points, then since $\angle P'AP'' = 2\angle A > \pi$, then $P'P''$ lies outside the spherical triangle ABC. So by spherical triangle inequality, the shortest path from P' to P'' must be $(P' \to A \to P'')$ or $(P' \to B \to C \to P'')$, which corresponds to the degenerated side AP or BC.

And as shown in Fig 3.18, either α or β will less than or equal to $\frac{\pi}{2}$, then by Lemma 19, $AB < AP$ or $AC < AP$.

Therefore AP is minimum when P coincides with B or C. Then compare AB, AC and BC, by Lemma 19 again, the degenerated inscribed spherical triangle with minimum perimeter is the side opposite to the smallest angle. □

4. Spherical Quadrilateral

In Euclidean geometry, it's proved that the inscribed quadrilateral with minimum perimeter exists iff the quadrilateral is cyclic and the centre of the circumcircle is inside the quadrilateral, as stated in the following theorem.

Theorem 23. *Quoted as Reference [2]*

If $ABCD$ is a Euclidean quadrilateral with

(i) $\angle A + \angle C = \pi$ *and the centre of the circumcircle is inside the quadrilateral. Then the inscribed quadrilateral with minimum perimeter exists and has infinite solutions. [See reviewer's comment (23)]*
(ii) $\angle A + \angle C = \pi$ *and the centre of the circumcircle is outside the quadrilateral. Then the inscribed quadrilateral with minimum perimeter does not exist and it will degenereate as a triangle passing through the two adjacent obtuse angles.*
(iii) $\angle A + \angle C > \pi$ *and* $\angle A \geq \angle C$
Then the inscribed quadrilateral with minimum perimeter does not exist and it will degenerate as a triangle passing through A or the line AC.

Proof. Omitted. □

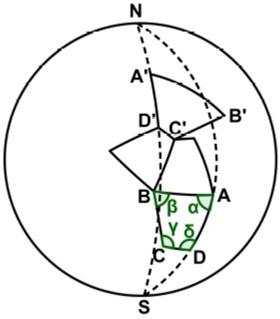

FIGURE 4.1

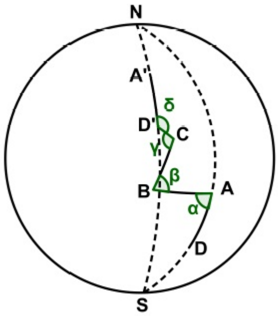

FIGURE 4.2

However, in spherical geometry, when a quadrilateral is reflected three times, the corresponding sides will never be parallel, as shown in Fig 4.1. And in this reflection, we are just interested in sides of reflection, as shown in Fig 4.2, and we call this zig-zag path the '**expanded form**' of the spherical quadrilateral. Moreover, we call the 2-gon NS the '**expanded 2-gon**' of the spherical quadrilateral. [See reviewer's comment (24)]

Theorem 24. *In a 2-gon NS, AB and $A'B'$ are two equal segments on the 2 sides. [See reviewer's comment (25)]*

(a) If P, P' are variable points on $AB, A'B'$ respectively such that $AP = A'P'$, as shown in Fig 4.3, then PP' will be minimum when it coincides with AA' or BB'.
(b) If P, P' are variable points on $AB, A'B'$ respectively such that $AP = B'P'$, as shown in Fig 4.4, then PP' will be minimum when $NP = NP'$, or the degenerated solution AB' or BA'.

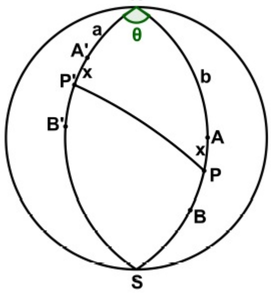

FIGURE 4.3

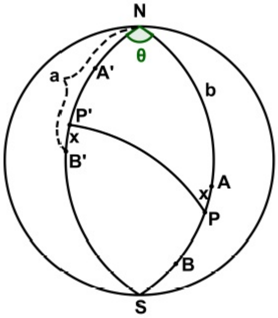

FIGURE 4.4

Proof.

(a) By spherical cosine law,

$$\begin{aligned}\cos PP' &= \cos(a+x)\cos(b+x) + \sin(a+x)\sin(b+x)\cos\theta \\ \frac{d\cos PP'}{dx} &= -\cos(a+x)\sin(b+x) - \sin(a+x)\cos(b+x) \\ &\quad + \sin(a+x)\cos(b+x)\cos\theta + \cos(a+x)\sin(b+x)\cos\theta \\ &= (\cos\theta - 1)\sin(a+b+2x)\end{aligned}$$

And since $0 < \theta < \pi$ and $a+x, b+x < \pi$, then $\cos\theta - 1 < 0$ and $0 < a+b+2x < 2\pi$.

$a+b+2x$	$a+b+2x<\pi$	$a+b+2x=\pi$	$a+b+2x>\pi$
$\frac{d\cos PP'}{dx}$	-	0	+

$$\begin{aligned}&\text{So } PP' \text{ is maximum} \\ \iff& \cos PP' \text{ is minimum} \\ \iff& a+b+2x=\pi \\ \iff& NP+NP'=\pi \\ \iff& PP' \text{ is passing through the centre of the 2-gon.}\end{aligned}$$

Then PP' is minimum when it conincides with AA' or BB'. [See reviewer's comment (26)]

(b) By spherical cosine law,

$$\begin{aligned}\cos PP' &= \cos(a-x)\cos(b+x) + \sin(a-x)\sin(b+x)\cos\theta \\ \frac{d\cos PP'}{dx} &= \sin(a-x)\cos(b+x) - \cos(a-x)\sin(b+x) \\ &\quad + \sin(a-x)\cos(b+x)\cos\theta + \cos(a-x)\sin(b+x)\cos\theta \\ &= (\cos\theta + 1)\sin(a-b-2x)\end{aligned}$$

Similarly, since $0 < \theta < \pi$ and $0 < a-x, b+x < \pi$, then $\cos\theta + 1 > 0$ and $-\pi < a-b-2x < \pi$.

$a-b-2x$	$a-b-2x<\pi$	$a-b-2x=\pi$	$a-b-2x>\pi$
$\frac{d\cos PP'}{dx}$	+	0	-

$$\begin{aligned}&\text{So } PP' \text{ is maximum} \\ \iff& \cos PP' \text{ is minimum} \\ \iff& a-b-2x=0 \\ \iff& NP=NP'\end{aligned}$$

If $x = \dfrac{(a-b)}{2}$ exceeds the range $0 < x < AB$, then PP' will be minimum when it coincides with AB' or BA'.

□

Theorem 25. Spherical Quadrilateral with 3 acute angles *For a spherical quadrilateral $ABCD$ with 3 acute angles, the inscribed spherical quadrilateral $PQRS$ with minimum perimeter does not exist and it will degenerate to the spherical triangle passing through the obtuse angle.*

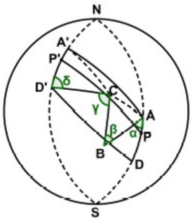

FIGURE 4.5

[See reviewer's comment (27)]

Proof. WLOG, let γ be the obtuse angle, and P be a variable point on AP, then its expanded form is shown in Fig 4.5.

Since $\angle DBD' = 2\beta < \pi$, we know B is below DD'. Then

$$\begin{aligned}
&\angle SDB + \angle SD'B \\
&= \pi - \angle BDA + \pi - \angle BD'A' \\
&= 2\pi - (\angle BDA + \delta + \angle BD'C) \\
&= 2\pi - 2\delta \\
&> \pi \\
\therefore\ &\angle SDD' + \angle SD'D \\
&= 2\pi - 2\delta + 2\angle BDD' \\
&> \pi
\end{aligned}$$

$\therefore DD'$ lies above the centre of the expanded 2-gon.

By Theorem 24 (a), the variable point P should be as close to A as possible.

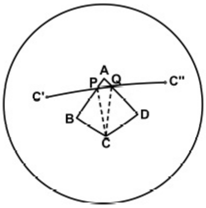

FIGURE 4.6

However, since $\angle ACA' = 2\gamma > \pi$, C is below AA' and then the minimum path must pass through C. Therefore if we reflect C about AB, BD to get C', C'' respectively, as shown in Fig 4.6, then

$$\angle C'BD = 2\beta - \angle CBD < 2\beta < \pi$$

and

$$\angle BDC'' = 2\delta - \angle BDC < 2\delta < \pi$$

So P, Q are above B, D respectively. Then the degenerated inscribed quadrilateral is the spherical triangle CPQ. □

Theorem 26. Spherical quadrilateral with 2 opposite acute angles and 2 opposite obtuse angles *For a spherical quadrilateral $ABCD$ with 2 opposite acute angles, the inscribed spherical quadrilateral $PQRS$ with minimum perimeter does not exist and it will degenerate to the diagonal passing through the 2 obtuse angles.*

Proof. WLOG, let α, γ be the opposite obtuse angles, then since $\beta, \delta < \pi$, similar to the proof of Theorem 25, the variable point P should be as close to A as possible.

However, since $\angle C'BD = 2\gamma > \pi$, C is below AA', so the minimum path must pass through C. By symmetry, if we change the role of α, β and γ, then we get the minimum path must pass through A. So the degenerated inscribed spherical quadrilateral is the diagonal passing through A and C. □

Note that the two cases just stated before have solutions exactly the same as when it is in Euclidean geometry, as stated in Theorem 23. For the spherical quadrilateral, we have 3 more cases:

Case 1. 2 adjacent acute angles and 2 obtuse angles

Case 2. 1 angle and 3 obtuse angles

Case 3. 4 obtuse angles

However, for these types of spherical quadrilaterals, the solutions inscribed spherical triangle with minimum perimeter are different from the cases in Euclidean geometry, as stated in Theorem 23, even though their are cyclic spherical quadrilateral.

Theorem 27. *A spherical quadrilateral $ABCD$ is cyclic iff the sums of the opposite angles are equal. (i.e. $\angle A + \angle C = \angle B + \angle D$, but they must larger than π)*

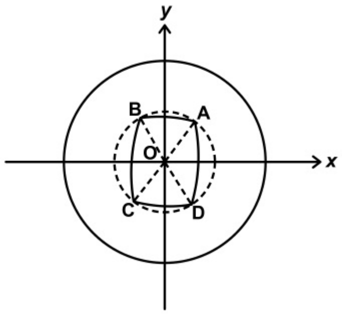

FIGURE 4.7

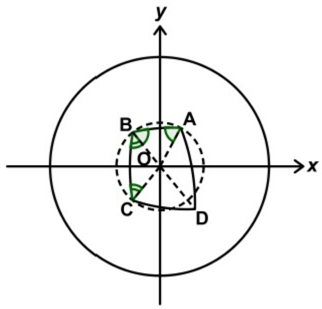

FIGURE 4.8

Proof.
$\Rightarrow$) WLOG, as shown in Fig 4.7, let the centre of the circumcirlce of $ABCD$ be the origin in the stereographic projection. Then $OA = OB = OC = OD$.

$$\therefore \angle OAB = \angle OBA, \angle OBC = \angle OCB, \angle OCD = \angle ODC, \angle OAD = \angle ODA$$

$$\begin{aligned}\therefore \angle A + \angle C &= (\angle OAB + \angle OAD) + (\angle OCB + \angle OCD)\\ &= \angle OBA + \angle ODA + \angle OBC + \angle ODC\\ &= \angle B + \angle D\end{aligned}$$

$\Leftarrow$) Since any 3 points on a sphere are concyclic, WLOG, let the centre of the circumcircle of A, B, C be origin in the stereographic projection. Then

$$OA = OB = OC.$$

$$\therefore \angle OAB = \angle OBA, \quad \angle OBC = \angle OCB$$

and since $\angle A + \angle C = \angle B + \angle D$,

$$\therefore \angle OAD + \angle OCD = \angle D = \angle ODA + \angle ODC$$

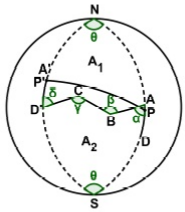

FIGURE 4.9

If $\angle OAD > \angle ODA$, then $\angle OCD < \angle ODC$, by Lemma 19, $OA < OD < OC$, which is a contradiction. Similarly $\angle OAD < \angle ODA$ will result in contradiction. Therefore, $\angle OAD = \angle ODA$, then $OD = OA$, A, B, C, D is concyclic. $\square$

As we have stated before, the three more cases are different from the Euclidean cases, even thoungh they are cyclic, solution does not exist. Consider the expanded form of a cyclic spherical quadrilateral, as shown in Fig 4.9, then $\alpha + \gamma = \beta + \delta$. And we let the area of $NABCD'$, $SABCD'$ be A_1, A_2 respectively, then

$$\begin{aligned} A_1 &= \theta + (\pi - \alpha) + \beta + (2\pi - \gamma) + \delta - 3\pi \\ &= \theta - \alpha + \beta - \gamma + \delta = \theta \\ A_2 &= \theta + \alpha + (2\pi - \beta) + \beta + \gamma + (\pi - \delta) - 3\pi \\ &= \theta + \alpha - \beta + \gamma - \delta = \theta \end{aligned}$$

Then $A_1 = A_2$ and so the expanded form of $ABCD$ is almost at the centre of the expanded 2-gon.

By Theorem 24 (a), since AA' and DD' are near to the centre.

We cannot determine whether we need the variable point P to be close to A or close to D. ————————————————————(*)

Then if we consider another spherical quadrilateral $ABCD$ which $\alpha + \gamma \sim \beta + \delta$, then it is very similar to a cyclic spherical quadrilateral. So it should have the same property (*). This is very different from the Euclidean case, as in that case, we need to make P close to A if $\alpha + \gamma > \beta + \delta$ and vise versa. But in spherical geometry, P may need to close to D even if $\alpha + \gamma > \beta + \delta$.

In the following passage, we will state a counter example in each of the three cases. However, before we do so, we must prove two more theorems about stereographic projection.

Theorem 28. *Let Q be the reflection point of P about the great circle Γ. When presented in the stereographic projection, Q' is the inversion point of P' with respect to the circle Γ'.*

Proof. Let $P = (x_0, y_0, z_0)$ and $Q = (x, y, z)$, then as shown in Fig 4.10

$$\begin{aligned}\overrightarrow{PQ} &= (x - x_0, y - y_0, z - z_0)\\ &= k(A, B, C)\end{aligned}$$

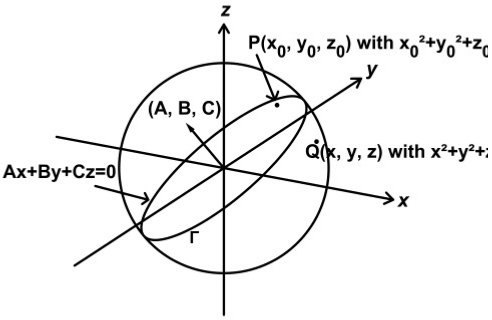

FIGURE 4.10

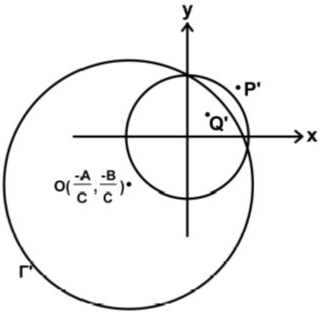

FIGURE 4.11

$$\therefore \begin{cases} x = x_0 + kA \\ y = y_0 + kB \\ z = z_0 + kC \end{cases}$$

So we have

$$\begin{aligned}(x_0 + kA)^2 + (y_0 + kB)^2 + (z_0 + kC)^2 &= 1\\ \therefore k(2Ax_0 + kA^2 + 2By_0 + kB^2 + 2Cz_0 + kC^2) &= 0\\ k = 0 \text{ (rejected) or } k &= -\frac{2(Ax_0 + By_0 + Cz_0)}{A^2 + B^2 + C^2}\\ Q &= (x_0 + kA, y_0 + kB, z_0 + kC)\end{aligned}$$

Then let Γ be the intersetion of $Ax + By + Cz = 0$ with the sphere. As shown in Fig 4.11, by the proof of Theorem 12, we know the equation of Γ' is

$$\begin{aligned}x'^2 + y'^2 + \frac{2A}{C}x' + \frac{2B}{C}y' - 1 &= 0\\ \left(x' + \frac{A}{C}\right)^2 + \left(y' + \frac{B}{C}\right)^2 &= \frac{A^2 + B^2 + C^2}{C^2}\\ \therefore \text{Center of } \Gamma' &= \left(-\frac{A}{C}, -\frac{B}{C}\right)\\ \therefore \text{Radius of } \Gamma' &= \sqrt{\frac{A^2 + B^2 + C^2}{C^2}}\end{aligned}$$

and we have

$$P' = \left(\frac{x_0}{1-z_0}, \frac{y_0}{1-z_0}\right)$$
$$Q' = \left(\frac{x_0+kA}{1-z_0-kC}, \frac{y_0+kB}{1-z_0-kC}\right)$$

Then

$$\overrightarrow{OP'} = \left(\frac{Cx_0+A-Az_0}{C(1-z_0)}, \frac{Cy_0+B-Bz_0}{C(1-z_0)}\right)$$
$$\overrightarrow{OQ'} = \left(\frac{x_0+kA}{1-z_0-kC}+\frac{A}{C}, \frac{y_0+kB}{1-z_0-kC}+\frac{B}{C}\right)$$
$$= \left(\frac{Cx_0+A-Az_0}{C(1-z_0-kC)}, \frac{Cy_0+B-Bz_0}{C(1-z_0-kC)}\right)$$

Then clearly O, P' and Q' are collinear, also we have

$$\begin{aligned}
OP' \times OQ' &= \overrightarrow{OP'} \cdot \overrightarrow{OQ'} \\
&= \frac{(Cx_0+A-Az_0)^2+(Cy_0+B-Bz_0)^2}{C^2(1-z_0)(1-z_0-kC)} \\
&= \frac{C^2(1+z_0)+2ACx_0+2BCy_0+(A^2+B^2)(1-z_0)}{C_2(1-z_0-kC)} \\
&= \frac{A^2+B^2+C^2-z_0(A^2+B^2+C^2)+2C(Ax_0+By_0)+2C^2z_0}{C^2(1-z_0-kC)} \\
&= \frac{A^2+B^2+C^2-z_0(A^2+B^2+C^2)+2C(Ax_0+By_0+Cz_0)}{C^2(1-z_0-kC)} \\
&= \frac{(A^2+B^2+C^2)(1-z_0+\frac{2C(Ax_0+By_0+Cz_0)}{A^2+B^2+C^2})}{C^2(1-z_0-kC)} \\
&= \frac{A^2+B^2+C^2}{C^2}
\end{aligned}$$

$\therefore Q'$ is the inversion point of P' with respect to Γ'. □

Theorem 29. Distance formula on stereographic projection *Let point $A'(x_1, y_1)$ and point $B'(x_2, y_2)$ be the stereographic projection of A and B on the sphere respectively. Then the minor arc AB on the sphere is*

$$2\arctan\sqrt{\frac{(x_1-x_2)^2+(y_1+y_2)^2}{(1+x_1x_2+y_1y_2)^2+(x_1y_2-x_2y_1)^2}}$$

Proof. By Theorem 12,

$$A = \left(\frac{2x_1}{x_1^2+y_1^2+1}, \frac{2y_1}{x_1^2+y_1^2+1}, \frac{x_1^2+y_1^2-1}{x_1^2+y_1^2+1}\right)$$
$$B = \left(\frac{2x_2}{x_2^2+y_2^2+1}, \frac{2y_2}{x_2^2+y_2^2+1}, \frac{x_2^2+y_2^2-1}{x_2^2+y_2^2+1}\right)$$

$$\begin{aligned}\cos AB &= |\overrightarrow{OA}||\overrightarrow{OB}|\cos AB \\ &= \overrightarrow{OA}\cdot\overrightarrow{OB} \\ &= \frac{4x_1x_2+4y_1y_2+(x_1^2+y_1^2-1)(x_2^2+y_2^2-1)}{(x_1^2+y_1^2+1)(x_2^2+y_2^2+1)}\end{aligned}$$

Since $\cos AB = \dfrac{1-\tan^2\frac{AB}{2}}{1+\tan^2\frac{AB}{2}}$,

$$\begin{aligned}\tan^2\frac{AB}{2} &= \frac{1-\cos AB}{1+\cos AB} \\ &= \frac{2(x_1^2+y_1^2)+2(x_2^2+y_2^2)-4x_1x_2-4y_1y_2}{2(x_1^2+y_1^2)(x_2^2+y_2^2)+2+4x_1x_2+4y_1y_2} \\ &= \frac{(x_1-x_2)^2+(y_1+y_2)^2}{(1+x_1x_2+y_1y_2)^2+(x_1y_2-x_2y_1)^2} \\ \therefore AB &= 2\arctan\sqrt{\frac{(x_1-x_2)^2+(y_1+y_2)^2}{(1+x_1x_2+y_1y_2)^2+(x_1y_2-x_2y_1)^2}}\end{aligned}$$

□

In the following, we will use GeoGebra to present the counter examples. All the spherical quarilaterals presented are almost cyclic, as stated before.

Counter Example 30. *2 adjacent acute angles and 2 obtuse angles (Attached file: counter example 1.ggb)*

By using Theorem. 28, we can construct A'' and A''' which are the reflection points of A' about $B'C'$ and $C'D'$ respectively. Similarly, we can construct B'' and B'''. By using distance formula, the spherical distance of $B''B'''$ is shorter than that of $A''A'''$ although $\alpha+\gamma>\beta+\delta$.

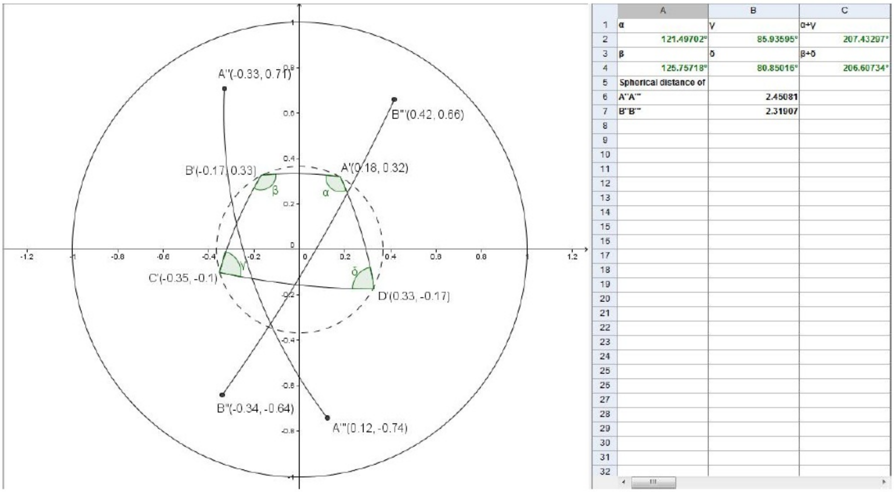

FIGURE 4.12

Counter Example 31. *1 acute angle and 3 obtuse angles (Attached file: counter example 2.ggb)*

By using Theorem 28, we can construct A'' and A''' which are the reflection points of A' about $B'C'$ and $C'D'$ respectively. By using distance formula, the spherical distancee of $2B'D'$ is shorter than that of $A''A'''$ although $\alpha + \gamma > \beta + \delta$.

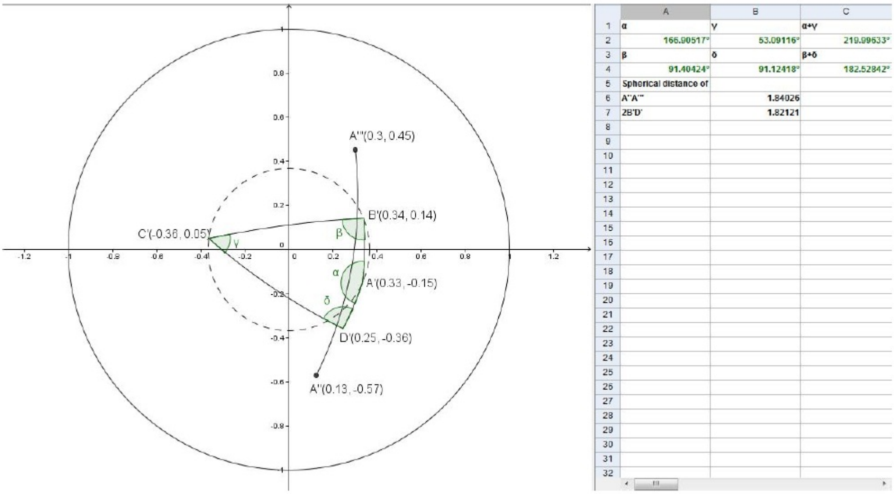

FIGURE 4.13

Counter Example 32. *4 obtuse angles (Attached file: counter example 3.ggb)*

By using distance formula, the spherical distancee of $B'D'$ is shorter than that of $A'C'$ although $\alpha+\gamma>\beta+\delta$.

Theorem 33. Spherical quadrilateral with 2 adjacent acute angles and 2 obtuse angles *For a spherical quadrilateral $ABCD$ with 2 adjacent acute angles and 2 obtuse angles, the inscribed spherical quadrilateral $PQRS$ with minimum perimeter does not exist and it will degenerate to the spherical triangle passing through one or both of the obtuse angles.*

Proof. WLOG, let β, γ be the obtuse angles, and its expanded form is shown in Fig 4.15. By property (*), we cannot determine whether we need the variable point close to A or D. And since $\angle DBD' = 2\beta, \angle ACA' = 2\gamma$ which are larger than π, we know B is above DD' and C is below AA', the shortest path must pass through B or C. Then we need to compare the length of the shortest path passing through B with that passing through C.

Let us use the point C as an example, we have an algorithm to find out the shortest path passing through C. As shown in Fig 4.16, if we reflect C about AB, AD to get C', C'' respectively, then the spherical triangle $AC'C''$ is isosceles with vertical angle 2α, then if we let the base angle be θ, as shown in Fig 4.17, we have

$$\tan\theta = \frac{\tan x}{\sin y}, \tan\alpha = \frac{\tan y}{\sin x}$$

$$\therefore \quad \tan\theta\tan\alpha = \frac{1}{\cos x\cos y} = \frac{1}{\cos AC}$$

$$\therefore \quad \theta = \arctan\frac{1}{\tan\alpha\cos AC}$$

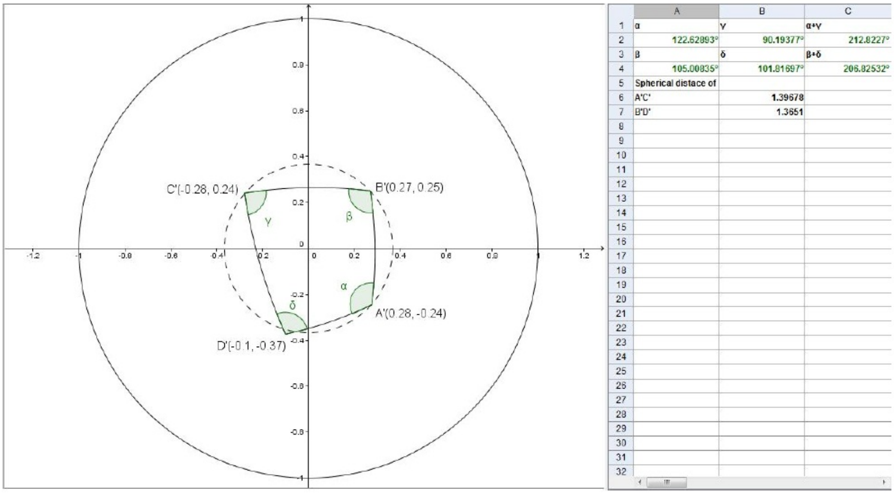

FIGURE 4.14

Case 1. $\angle ACB > \theta$

Since $\angle ACB > \theta = \angle ACP$, this is just the original case as shown in Fig 4.16, then the shortest path is just the triangle passing through C only with

$$\text{Perimeter} = C'C'' = \arccos(\cos^2 AC + \sin^2 AC \cos 2\alpha)$$

Case 2. $ACB \leq \theta$

This is the case shown in Fig 4.18, and the shortest path is the triangle passing through both B and C with

$$\begin{aligned}\text{Perimeter} &= C'B + BC'' \\ &= BC + \arccos(\cos AB \cos AC'' + \sin AB \sin AC'' \cos \angle BAC'') \\ &= BC + \arccos(\cos AB \cos AC + \sin AB \sin AC \cos(\alpha + \angle CAD))\end{aligned}$$

Then we can use similar argument to find out the shortest path passing through B and compare then to get the solution. $\square$

Theorem 34. Spherical quadrilateral with 1 acute angle and 3 obtuse angles *For a spherical quadrilateral with 1 acute angle and 3 obtuse angles, the inscribed spherical quadrilateral $PQRS$ with minimum perimeter does not exist and it will degenerate to either one of the following cases: (i) the spherical triangle passing through the obtuse angle opposite to the acute angle only, (ii)the spherical triangle passing through two adjacent obtuse angles, (iii) the diagonal passing the two obtuse angles.*

Proof. WLOG, let α be the acute angle. Similar to the proof in Theorem 4.8, the shortest path must pass through B, C of both.

Passing through C:

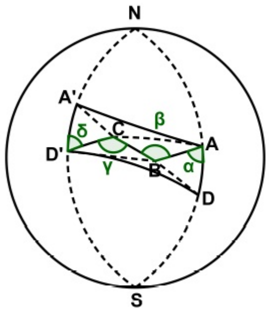

FIGURE 4.15

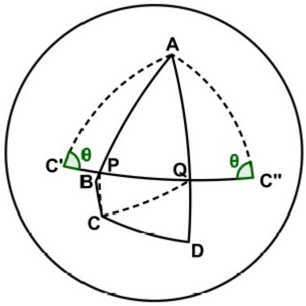

FIGURE 4.16

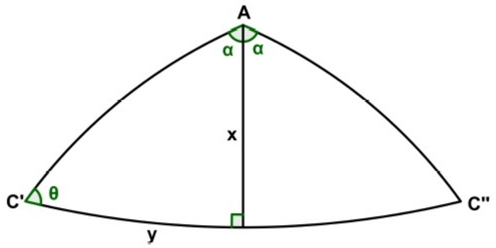

FIGURE 4.17

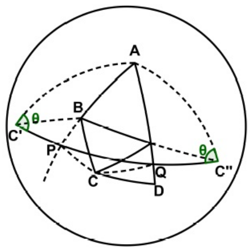

FIGURE 4.18

Similar to the proof in Theorem 4.8, if we let D to be also an obtuse angle.

Case 1. $\angle ACB > \theta$, $\angle ACD > \theta$

This is the case as shown in Fig 4.16, then the shortest path is the triangle passing through C only with

$$\text{Perimeter} = C'C'' = \arccos(\cos^2 AC + \sin^2 AC \cos 2\alpha)$$

Case 2. $\angle ACB \leq \theta$, $\angle ACD > \theta$

(If $\angle ACB > \theta, \angle ACD \leq \theta$, then we can change the role of B and D to get the same case.)

This is the case as shown in Fig 4.18, then the shortest path is the triangle passing through both B and C with

$$\begin{aligned}\text{Perimeter} &= C'B + BC'' \\ &= BC + \arccos(\cos AB \cos AC'' + \sin AB \sin AC'' \cos \angle BAC'') \\ &= BC + \arccos(\cos AB \cos AC + \sin AB \sin AC \cos(\alpha + \angle CAD))\end{aligned}$$

Case 3. $\angle ACB \leq \theta, \angle ACD \leq \theta$

This is the case as shown in Fig 4.19, whether it passes through B or D depends on the $\angle C'BD$ and $\angle BDC''$, and we have

$$\angle C'BD = 2\beta - \angle CBD$$
$$\angle BDC'' = 2\delta - \angle CDB$$

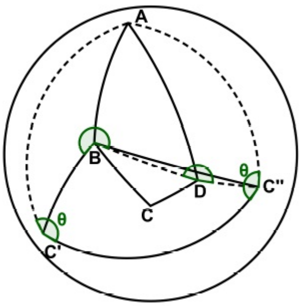

FIGURE 4.19

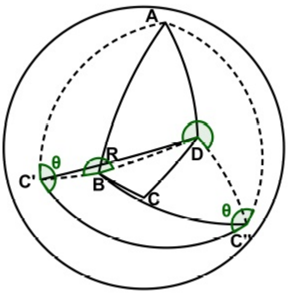

FIGURE 4.20

Case 3.1. $2\beta - \angle CBD \geq \pi$, $2\delta - \angle CDB \geq \pi$

The shortest path is the $\triangle CBD$, which must longer than $2BD$, where included in the case of the shortest path passing through B.

Case 3.2. $2\beta - \angle CBD \geq \pi$, $2\delta - \angle CDB < \pi$

(if $2\beta - \angle CBD < \pi, 2\delta - \angle CDB \geq \pi$, then we can change the role of B and D to get the same case.)

This is just the case in Fig 4.19, then the shortest path is the triangle passing throught both B and C with

$$\begin{aligned}\text{Perimeter} &= C'B + BC'' \\ &= BC + \arccos(\cos AB \cos AC'' + \sin AB \sin AC'' \cos \angle BAC'') \\ &= BC + \arccos(\cos AB \cos AC + \sin AB \sin AC \cos(\alpha + \angle CAD))\end{aligned}$$

Case 3.3. $2\beta - \angle CBD < \pi$, $2\delta - \angle CDB < \pi$

As shown in Fig 4.20, since $\angle ACB \leq \theta$, $\angle ACD \leq \theta$, we know that B, D must lie above $C'C''$, let R be the intersection of $C'D$ and AB. Since $\angle C'BD < \pi$, B must lie in below R. Then consider BC'', since $\angle BDC'' < \pi$, D must lie in below BC'', but by construction, D must lie above BC'', lead to a contradiction.

Passing through B:

Let B', B'' be the reflection of B about AD, CD respectively. Then since δ is obtuse, $\angle B'DB'' = 2\delta > \pi$. And since $\angle B'AC = 2\alpha - \angle BAC < \pi$, the shortest path will not pass through A. Therefore the shortest path is $(B \to D \to B)$ or $(B \to C \to B)$ where the path will touch a point on AD, and this case can be included in the case passing through C. So in this case, the perimeter of the shortest path is $2BD$.

After finding the perimeter of the shortest path passing through B or C, we can compare the result of them to get the solution. □

Theorem 35. Spherical quadrilateral with 4 obtuse angles *For a spherical quadrilateral with 4 obtuse angles, the inscribed spherical quadrilateral PQRS with minimum perimeter does not exist and it will degenerate to one of the diagonals.*

Proof. Since all the four angles are obtuse, and similarly to the proof of Theorem 33, we know that the shortest path must pass through B or C. Let us use B as an example, let B', B'' be the reflection of B about AD, CD respectively, then since δ is obtuse, $\angle B'DB'' = 2\delta > \pi$. Therefore the shortest path is $(B \to D \to B)$ or $(B \to A \to C \to B)$, but this path is obviously longer than $2AC$. So we can just compare the length of diagonal to get the solution. □

5. N-gon

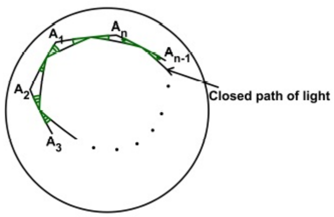

FIGURE 5.1

Definition 36. *An inscribed spherical polygon of a spherical polygon, where they have the same number of sides, is called a* <u>*closed path of light*</u> *iff all the angles of incidence are equal to the angles of reflection. (As shown in Fig 5.1)*

Theorem 37. *For a (2n + 1)-sided spherical polygon, an inscribed (2n + 1)-sided spherical polygon exists and having minimum perimeter iff it is a closed path of light.*

[See reviewer's comment (28)]

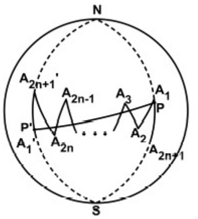

FIGURE 5.2

Proof.
$\Rightarrow$) Suppose the inscribed $(2n{+}1)$-sided polygon exists and has minimum perimeter, then we consider the expanded form of the original polygon together with the inscribed polygon, as shown in Fig 5.2. Since it has minimum perimeter, it must be a great circle passing through all the sides in the expanded form. Then for the sides $A_1A_2, A_2A_3, \ldots, A_{2n}A_{2n+1}$, by vertically opposite angles, the angles of incidence are equal to the angles of reflection. And by Theorem 24 (b), PP' is minimum when $NP = NP'$, so $\angle NPP' = \angle NP'P$, which are the angle of incidence and angle of reflection on the side A_1A_{2n+1}.

$\Leftarrow$) Suppose there exists a closed path of light, then when we consider the expanded form, use the same notation as in Fig 5.2, the equality of the angles of incidence and the angles of reflection imply PP' is a great circle and $\angle NPP'$ and $\angle NP'P$, this means $NP = NP'$, then by Theorem 24 (b) again, it is the inscribed polygon with minimum perimeter. $\square$

For a $2n$-sided spherical polygon, the solution is different from the Euclidean case, because the inscribed $2n$-sided spherical polygon will never exist, even

$$\sum_{k=1}^{n} A_{2k-1} = \sum_{k=1}^{n} A_{2k}$$

or even it is a regular spherical polygon! As stated in the following theorem,

Theorem 38. **$2n$-sided spherical polygon** *For a $2n$-sided spherical polygon, an inscribed $2n$-sided spherical polygon with minimum perimeter does not exist.*

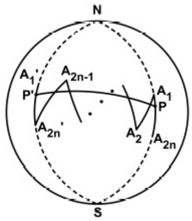

FIGURE 5.3

Proof. Consider the expanded form of the $2n$-sided polygon, as shown in Fig 5.3. Since there are no parallel great circle in spherical geometry, A_1A_{2n} and $A_1'A_{2n}'$ are not parallel. Then by Theorem 24 (a), the variable point P must close to A_1 or close to A_{2n}, so the shortest path must pass through at least one of the vertice, therefore the inscribed spherical polygon with minimum perimeter does not exist. $\square$

6. Conclusion

In spherical geometry, the solution of cases in acute triangles and one-obtuse-angle triangles will be similar to its solution in Euclidean geometry. However, on a sphere, a spherical triangle is able to consist more than one obtuse angle, the solution will be degenerated and approach to the side opposite to the smallest angle. [See reviewer's comment (29)]

For the spherical quadrilaterals, the solution will be very different from the Euclidean case, as it will never have an inscribed spherical quadrilateral with minimum perimeter. However, when we consider spherical quadrilateral in a small scale, such as a city compared with the Earth, then the case will be similar to the Euclidean geometry.

For example, if $ABCD$ is a small cyclic spherical quadrilaterals, then by the discussion after Theorem 27, the expanded form of the spherical quadrilateral will lie amost in the middle of the expanded 2-gon as shown in Fig 6.1 and Fig 6.2. And we can see that the two corresponding sides AD and $A'D'$ are almost parallel. Therefore, the variable point P and its image P' are having almost equal distance when P is sliding from A to D. This is similar to the case in Euclidean geometry. Then if A, B, C, D is not cyclic, by using the same notation as in the discussion after Theorem 27 again, we know $A_2 - A_1 = 2(\alpha + \beta - \gamma - \delta)$. But since it is small, the expanded form will lie near the pole of the expanded 2-gon as shown in Fig 6.3. So A_2 will tends to 2θ, and A_1 will tends to 0. So, $\alpha + \beta - \gamma - \delta \sim \theta$, this is similar to the Euclidean case.

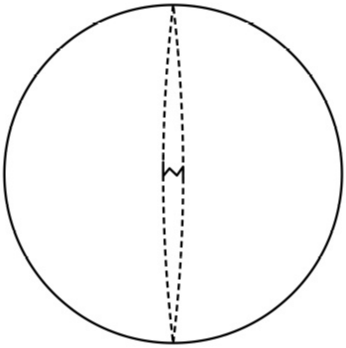

FIGURE 6.1

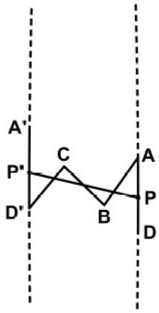

FIGURE 6.2

Lastly, we know that the Playfair's axiom in Euclidean geometry is different to that in spherical geometry, stating that there is no line can be drawn through any point not on a given line parallel to the given line on a sphere, and our conclusion on the $2n$-sided spherical polygon having the similar result. Also we know that this playfair's axiom in hyperbolic geometry is different from that in spherical and Euclidean geometry, so we can believe this problem can be extended to hyperbolic geometry with other different properties in the furture.

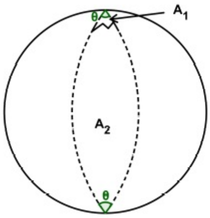

FIGURE 6.3

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Reviewer's Comments

Given an acute triangle, does there exist an inscribed triangle with minimum perimeter? This is Fagnano's Problem, proposed by Giovanni Fagnano in 1755. The answer is affirmative, and such a triangle is the orthic triangle, whose vertices are the feet of perpendiculars dropped from the three vertices of the given triangle to the respective corresponding sides. Inspired by this problem, the authors of this paper under review investigate a number of generalizations of Fagnano's Problem in the context of spherical geometry. They first review some basic spherical geometry/trigonometry, followed by their investigation of Fagnano's Problem for spherical triangles. They show, by means of the principle of light reflection, that if the spherical triangle has three acute angles then there exists an inscribed spherical triangle with minimum perimeter. However, if at least one angle of the given spherical triangle becomes obtuse then the desired inscribed triangle will degenerate into a segment of a great circle and thus Fagnano's Problem in this situation admits no solutions. The authors go on to analyse in detail Fagnano's Problem for spherical quadrilaterals in various cases (categorized by the acuteness of the angles of the quadrilaterials) by using the so-called expanded form, which is a path obtained by reflecting the given quadrilateral. They show that degeneration of the desired inscribed quadrilateral occurs and so there are no solutions. Finally the general case of spherical n-gons is studied and the authors carry over the expanded form technique to show that solutions exist when n is odd whereas degeneration occurs and thus no solutions exist when n is even. They also point out the possible research direction in the context of hyperbolic geometry.

In general the paper is well-organized. The authors give a good introduction by stating clearly the problem they would like to address and summarizing the structure of the paper. After proving a result of Fagnano's Problem for a spherical quadrilateral, the authors provide comparisons of this result with its analogue in the Euclidean case. In this way they give the readers an idea what makes the spherical Fagnano's Problem interesting. The conclusion is also well-written, and they relate their results in spherical geometry to those in Euclidean geometry by considering very small quadrilaterals. However, this paper is riddled with various issues in terms of clarity and grammar. There is a lack of connecting paragraphs between theorems/lemmata explaining what the authors want to do next. Very often some terms and notations are not defined or explained in the outset. The most notable example is that throughout the whole paper, they have never defined what orthic triangles mean! Besides many of their results concern the degeneration of the inscribed spherical triangles/quadrilaterals, but they never explain how this degeneration occurs when some vertices of the given triangle/quadrilateral become obtuse. It would be helpful if they could provide a step-by-step illustration of this degeneration process. Other issues of relatively minor nature include absence of suitable commas or full stops after lines of displayed equations and inconsistent way of highlighting terminology they would like to define (for example, they underline the terminology to be defined whereas in Definition 1, while they use bold type to highlight the terms 'expanded form' and 'expanded 2-gon'). The following are specific comments on the aforementioned issues.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. Change the first sentence of the second paragraph to 'Now we wonder whether the result will remain unchanged when the triangles are on a sphere.'
3. The authors should define orthic triangles.
4. The authors should either offer more explanations to the statement '...it will degenerate to twice of the altitude from the obtuse angle', or replace it with a more concise statement without going into details as they should leave the details in the latter sections rather than in the introduction.
5. The meaning of 'otherwise, it is a small circle' is not clear. By using the word 'otherwise' what conditions in the first part of Definition 1 do the author trying to negate? They could have said something along the line 'a small circle is the intersection of a sphere with a plane not containing the center'.
6. The authors should define 'trihedral angle', though I can guess its meaning from the picture.
7. The authors implicitly used sine law to get that $OA = OC$. They should have pointed that out.
8. The reviewer believes the authors are assuming that the spherical triangle is right-angled, but they fail to mention that.
9. The authors should define N, though one can easily guess that it means the north pole of the sphere.
10. Change 'Subst. it in (2)' to 'substitute the above equations to'. Never use unexplained abbreviations.
11. The authors should provide more explanation in the proof and more details of Fig. 2.10 which illustrates the proof. What are the two curves which intersect at N and P? What kind of symmetry is involved and why are α and β equal? What is the plane π_2? It seems to the reviewer that the proof is too short to be a correct one.
12. The reviewer suggests that the authors use '$\cdot$' in lieu of '$\times$', for one may easily mistake $\times$ for vector cross product in this formula.
13. It seems that the proof is predicated on the fact that stereographic projection preserves minimum distances, which the authors fail to mention.
14. It is a good idea to mention that the image of great circles through A' under stereographic projection is a straight line, as it is used often in the proof.
15. The authors should mention that they use Theorem 13 about the conformal property of stereographic projection. Besides, they should also give a reference about the formula for the area of a spherical triangle. Commas should be added to each line of displayed inequality.
16. The reviewer thinks by $\angle A'C'D'$ the authors actually mean $\angle A'C'D$.
17. Again, the authors should define 'orthic'.
18. Change '$BP < \frac{\pi}{2}$, then by spherical cosine law' to '$BP < \frac{\pi}{2}$. By spherical cosine law'.

19. The authors should cite Proposition 9, which is the spherical sine law, for the convenience of the readers.
20. The authors should define S.
21. Change 'similar argument as...' to 'an argument similar to that in Case 2.1'.
22. The reviewer believes by 'which is equivalent to the spherical triangle PCR' the authors mean 'which is the perimeter of the spherical triangle PCR.
23. The sentence does not read well (it does not make sense that 'the inscribed quadrilateral has infinite solutions'). It is better to change it to 'Then there are infinite many inscribed quadrilaterals with minimum perimeter'.
24. The reviewer suggests that the authors point out what makes the Fagnano's Problem in the planar and spherical case admit different solutions.
25. While the authors let AB and $A'B'$ be the two equal segments of two great circles on the 2 sides, in Fig. 4.1 these two segments are denoted AD and $A'D'$. They should fix this inconsistency.
26. It is not clear to the reviewer how the statement in the second last line implies the statement in the last line.
27. The labeling of the four vertices of the quadrilateral $ABCD$ is not in the right order.
28. Change '...spherical polygon exists and having minimum perimeter iff...' to 'spherical polygon with minimum perimeter exists iff...'.
29. Change 'a spherical triangle is able to consist more than one obtuse angle, the solution...' to 'there may be more than one obtuse angle in a spherical triangle. In this case, the solution...'.