Hang Lung Mathematics Awards
Vol. 4 (2010)

Hang Lung Mathematics Awards 2010

Silver Award

Curve Optimization Problem

Team Member: Ping Ngai Chung
Teacher: Ms. Mee Lin Luk
School: La Salle College

Hang Lung Mathematics Awards
Vol. 4 (2010), pp. 27–48

CURVE OPTIMIZATION PROBLEM

TEAM MEMBER
Ping Ngai Chung

TEACHER
Ms. Mee Lin Luk

SCHOOL
La Salle College

Abstract. In this project, we shall introduce a new quantity associated with any given shape on the plane: "optimal curve", which is defined as the shortest curve such that its convex hull fully covers a given shape S. Here curve can involve straight lines or union of straight lines. [See reviewer's comment (2)] We shall investigate on some properties of this kind of curve and also prove a theorem that among shapes with a given fixed length of perimeter, the circle has the maximal optimal curve. [See reviewer's comment (3)] Moreover, we will introduce an algorithm to find the shortest curve with convex hull equals a given shape in polynomial time.

1. Introduction

Isoperimetric problems refer to all kinds of mathematics problems that maximize or minimize a certain quantity given a fixed perimeter of a plane figure. The well-known classical isoperimetric problem is to determine a plane figure with the largest possible area given the length of its perimeter. The solution is indeed a circle. In this paper, we are interested in another type of isoperimetric problem. Before explaining the actual problem, we shall first introduce a kind of curve for each plane figure.

"Optimal curve" is defined as the shortest curve such that its convex hull fully covers a given shape S. Here curve can involve straight lines or union of straight lines.

The project starts with a known result about the optimal curve of a circle. Then we will solve the alternative isoperimetric problem: what is the plane figure with the longest possible optimal curve given the length of its perimeter? We shall show that the solution is indeed also a circle. After that, we will research on some properties of the optimal curve for a given shape. There is no general solution to the optimal curve of arbitrary plane figure, yet we can use the properties to limit the possible curves and resort the problem to a computer program. Then we will eventually try to solve the problem in polynomial time which is rather efficient for application.

Chapter 1 is the introduction to the project, with some definitions used throughout the project and two known theorems that are useful in the project.

Chapter 2 is about proving that the shape with greatest optimal curve given a fixed perimeter is actually the circle. [See reviewer's comment (4)] This is the main result of the project.

Chapter 3 is about finding the shortest curve such that its convex hull is a given convex polygon S. The shape of the curve turns out varying a lot for different polygons so we will introduce an algorithm to find the solution in polynomial time.

Before we proceed to the main content, a number of symbols and terms which will be widely used in the project are defined as follows:

Definitions: [See reviewer's comment (6)]

P:	arbitrary point set
N:	a curve
S:	a convex set, it may be called a "shape" in the paper
$C(S)$	convex hull of convex set S
$C(A_1A_2A_3\dots A_n)$:	convex hull of points $A_1, A_2, A_3, \dots A_n$.
$L(N)$:	length of a curve N
$G(S)$:	a curve with convex hull containing S, i.e. $S \subseteq C(G(S))$
$\gamma(S)$:	a collection of all curves with convex hull congruent to S
$\gamma_0(S)$:	shortest curve in $\gamma(S)$, i.e. $\gamma_0(S) \in \gamma(S)$ and $L(\gamma_0(S)) \leq L(N)\ \forall N \in \gamma(S)$
$\Gamma(S)$:	a collection of all curves with convex hull containing S, i.e. $\Gamma(S) = \{N \text{ is a curve} \mid S \subseteq C(N)\}$
$\Gamma_0(S)$:	shortest curve in $\Gamma(S)$, i.e. $\Gamma_0(S) \in \Gamma(S)$ and $L(\Gamma_0(S)) \leq L(N)\ \forall N \in \Gamma(S)$. We call this curve the "optimal curve" of shape S. This will be the major concern of the whole article. Note that the "optimal curve" may not be unique.
$\text{int}(S)$:	the interior part of shape S, i.e. $\text{int}(S)$ $= \{T \in \mathbb{R}^2 \backslash S \mid \text{for all line } l \text{ passing through } T,\ l \cap S \neq \emptyset\}$
$\text{ext}(S)$:	the exterior part of shape S, i.e. $\text{ext}(S) = \mathbb{R}^2 \backslash (S \cup \text{int}(S))$.
$l(AB)$:	the line passing through points A and B
$r(AB)$:	the ray starting from A and passing through B

$\overline{AB}$:	the line segment from A to B
$\widehat{AB}$:	A and B are points on a given curve, then $\widehat{AB}$ is the part of the curve with endpoints A and B.
$\angle ABC$:	the undirected angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ that is not greater than π
supp $\angle ABC$:	the undirected angle between $\overrightarrow{BA}$ and $\overrightarrow{CB}$ that is not greater than π
ref $\angle ABC$:	the undirected angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ that is greater than π and smaller than 2π.
$\mathrm{P}(\angle ABC)$:	$r(BA)$ and $r(BC)$ divide the $\mathbb{R}^2$ plane into two parts. $\mathrm{P}(\angle ABC)$ is the part where $\angle ABC$ is at (excluding $r(BA)$ and $r(BC)$). Similarly define $\mathrm{P}(\text{supp } \angle ABC)$ and $\mathrm{P}(\text{ref}\angle ABC)$.
"Supporting line" of shape S:	a line on the plane that intersect S on at least one point so that shape S lies on only one half-plane divided by this line.

Besides, there are two key theorems that will be used in the project:

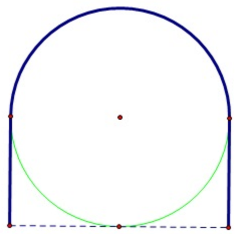
Theorem 1. *Given a point set, among the continuous shapes that completely covers the whole set, the convex hull of the set has the smallest perimeter.*

[See reviewer's comment (7)]

Proof. Firstly among any convex sets covering the point set, the convex hull has the smallest perimeter. This can be proved by contradiction. Suppose another convex set that covers the point set has smaller area. Since this convex set also covers the convex hull, we can find two points on the boundary of this set that are not connected by straight line on the boundary of the set. Then as long as we take the two points close enough, we can draw a straight line between them that does not intersect the convex hull. Then by replacing the original curve between the two points with the straight line we can get a convex set with smaller perimeter, contradiction.

For non-convex sets, we can take their convex hull to form a convex set with perimeter shorter than theirs, which in term has perimeter longer than or equal to that of the convex hull of the original set. So the result follows. □

Theorem 2. *The optimal curve of circle is a shape like this:*

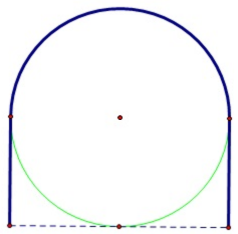


For this theorem, it is a known result due to H. Joris in the paper [2] Theorem 2. I will not reproduce the proof here again due to its length and the proof itself does not contribute to the following context. The result, though, plays an essential role in the project.

2. The shape with greatest optimal curve given a fixed length of perimeter

[See reviewer's comment (5)]

In Chapter 1, we have introduced that the optimal curve, i.e. the shortest curve with convex hull covering a given circle has a shape like this (the blue line):

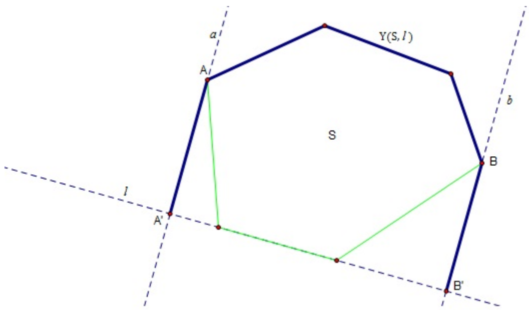


It is a "yurk"-like shape. It is determined only by the given shape (a circle in this case), and a supporting line. In general, given an arbitrary connected shape, we can define a *yurk* $Y(S, l)$ for each supporting line l.

Definition 3. *For a given shape S and its supporting line l,*

1. *Choose two supporting lines a, b that are perpendicular to l. Then each of a, b touches S on at least one point from the definition of supporting lines. Choose one of the touching points on a and label it as A. Similarly label the point B on b. (it is obvious that the choice of the touching point will not affect $Y(S, l)$)*
2. *Label $A' = a \cap l$ and $B' = b \cap l$. We define the **yurk** $\mathbf{Y}(\mathbf{S}, \mathbf{l}) = \overline{A'A} \cup \widehat{AB} \cup \overline{BB'}$.*

Example of $Y(S,l)$ for a given hexagon S:



Definition 4. *Define* **width** *as the distance between two parallel supporting lines of shape S, and* **d(S,l)** *is the width of S in the direction perpendicular to l.*

[See reviewer's comment (8)]

We shall first consider a special case for shape S:

Theorem 5. $L(Y(S_0,l)) = \frac{p}{2} + d(S_0,l)$, *where S_0 is a centrosymmetric convex set, i.e. a convex set with rotational symmetry of order 2 with respect to a point O on the plane.*

[See reviewer's comment (9)]

Proof. Consider an supporting line p of S_0 that is perpendicular to line l, then if we perform a rotation of 180 degrees with respect to O, then we will get another supporting line p' of S_0.

Label A as one of the points at which p intersects S_0, then the rotational image of A, labelled as A', will also be one of the intersection points of p' and S_0. Since A' is the image of A, so A, O, A' collinear. Furthermore, $AO = OA'$.

Label the projection of A, O, A' on l as H, O', H' respectively.

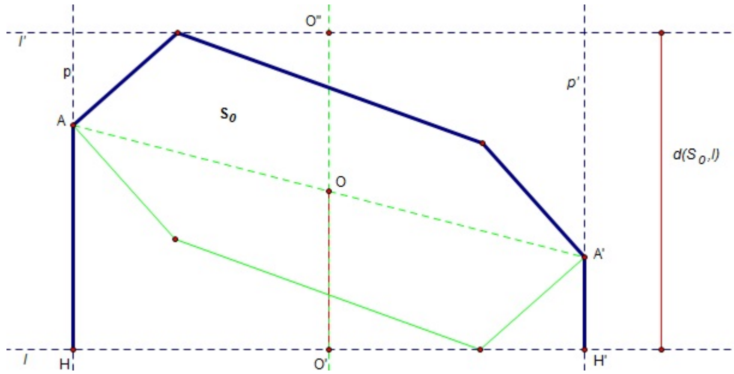
Then since S_0 is centrosymmetric, so every line m passing through O will cut the perimeter of S into two congruent parts, i.e. AA' cuts the perimeter into half. Let the length of perimeter be P, then $L(\widehat{AA'}) = \frac{p}{2}$.

Moreover, as O is midpoint of AA', O' is midpoint of HH', so $AH + A'H' = 2OO'$. Notice that if we rotate OO' and l by 180 degrees with respect to O to get OO'' and l' respectively, then as $OO' \perp l$ and $OO'' \perp l'$, $O'O''$ is indeed the width of shape S_0 in the direction perpendicular to l. So $2OO' = O'O'' = d(S_0,l)$.

Therefore:

$$L(Y(S_0,l)) = L(\overline{AH} \cup \widehat{AA'} \cup \overline{A'H'}) = AH + L(\widehat{AA'}) + A'H'$$
$$= \frac{p}{2} + 2OO' = \frac{p}{2} + d(S_0,l).$$

□



Theorem 6. $\min_l L(Y(S_0,l)) \leq \left(\frac{1}{2}+\frac{1}{\pi}\right) P$ *for all centrosymmetric set* S_0.

Proof. By Theorem 5, $\min_l L(Y(S_0,l)) = \frac{p}{2} + \min_l(d(S_0,l)) = \frac{p}{2} + 2\min_l(OO')$.

(O' is defined as in the proof of Theorem 5)

In fact OO' is the distance from O to line l. To make OO' the shortest, we should first notice that O' must lie on S_0. It is because if otherwise, suppose OO' intersects S_0 at K, we can choose the supporting line l_0 which touches S_0 at K instead. This would be a shorter choice than l. From the convexity of S_0 this line exists.

So all we have to do is to find a point K on S_0 such that OS_0 is minimal, and then take the supporting line which touches S_0 at K. So the length of OK is $d' = \frac{1}{2}\min_l(d(S_0,l))$.

If we draw a circle with radius d' and centre O, then this circle should lies completely inside shape S. So the length of perimeter of this circle is less than that of S_0, i.e. $2\pi(d') \leq P$, $d' \leq \frac{p}{2\pi}$ by Theorem 1.

So $\min_l L(Y(S_0,l)) = \frac{p}{2} + \min_l(d(S_0,l)) = \frac{p}{2} + 2d' \leq \frac{p}{2} + 2 \times \frac{p}{2\pi} = \left(\frac{1}{2}+\frac{1}{\pi}\right) P$. □

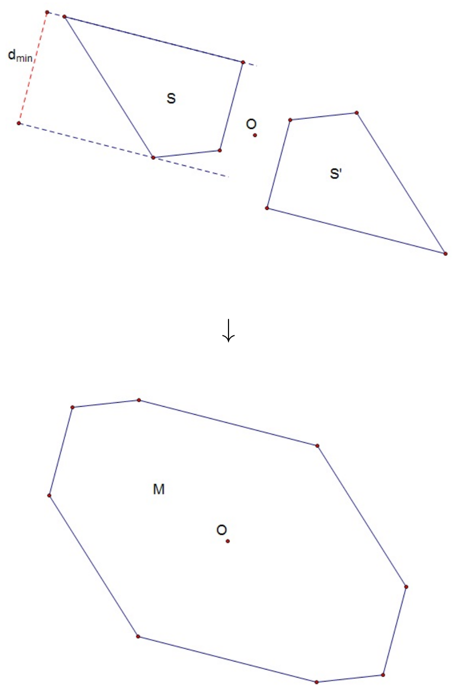
Theorem 7. $\min_l L(Y(S,l)) \leq \left(\frac{1}{2} + \frac{1}{\pi}\right) P$ *for all convex set* S.

Proof. Let d_{min} be the minimum width of convex set S, i.e. there exist parallel supporting lines l_0, l_0' such that distance between them is d_{min}, while the distance between any other pairs of parallel supporting lines of convex set S is not less than d_{min}.

Pick an arbitrary point O on the plane.

Let convex set S' be the rotational image of convex set S by 180 degrees with respect to point O. Now consider the *Minkowski Sum* of $S + S' = M$ (See Appendix 1 for definition and properties):

From the properties of Minkowski Sum we know that the width of M is at least $2d_{min}$ (Theorem 15) and its perimeter is $2P$ (Theorem 16). Moreover M is centrosymmetric (Theorem 17), so there exists a point O such that M has rotational symmetry of order 2 with respect to O. [See reviewer's comment (10)]

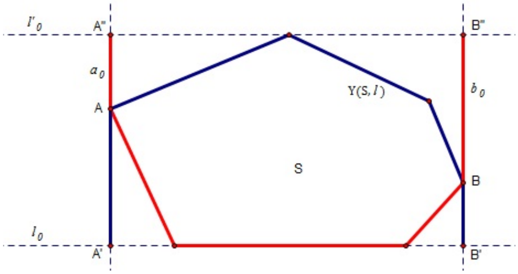


Then by Theorem 5, $L(Y(M,L)) = \frac{2P}{2} + d(M,l) = P + d(M,l)$.

By Theorem 6,

$$\min_l L((Y(M,L)) \le \left(\frac{1}{2}+\frac{1}{\pi}\right)(2P)$$
$$\min_l (P + d(M,l)) \le \left(\frac{1}{2}+\frac{1}{\pi}\right)(2P)$$
$$P + \min_l (d(M,l)) \le \left(\frac{1}{2}+\frac{1}{\pi}\right)(2P)$$
$$P + 2d_{min} \le P + \frac{2P}{\pi}$$
$$d_{min} \le \frac{P}{\pi}$$

Back to the convex set S.
Label the two supporting lines of S that are perpendicular to l_0 as a_0, b_0. a_0 intersects l_0, l_0' at A', A'' respectively, b_0 intersects l_0, l_0' at B', B'' respectively. The touching points of shape S on a_0, b_0 are A and B respectively.



Notice that:

$$\begin{aligned} L(Y(S,l_0)) + L(Y(S,l_0')) &= A'A + L(\widehat{AB}) + BB' + A''A + P - L(\widehat{AB}) + BB'' \\ &= P + 2d(S,l) = P + 2d_{min} \end{aligned}$$

$$\begin{aligned} \therefore \min_l L(Y(S,l)) &\le \min\{L(Y(S,l_0)), L(Y(S,l_0'))\} \le \frac{1}{2}\left(L(Y(S,l_0)) + L(Y(S,l_0'))\right) \\ &= \frac{1}{2}(P + 2d_{min}) \le \frac{1}{2}\left(P + 2\frac{P}{\pi}\right) = \left(\frac{1}{2}+\frac{1}{\pi}\right)P \end{aligned}$$

□

Now we shall prove the main theorem of the project:

Given a fixed length of perimeter, the shape with maximum optimal curve is a circle.

Definition 8. *Define a function* $\mathbf{r}(\mathbf{S}) = \frac{\text{L(optimal curve of S)}}{\text{L(perimeter of S)}}$.

Theorem 9. *$r(S) \leq r(C)$ for all convex set S, where C is a circle on the plane.*

Proof. By Theorem 2, for a circle C with radius r, $r(C) = \dfrac{\pi+2}{2\pi} = \dfrac{1}{2} + \dfrac{1}{\pi}$. So it is left to prove that for all shape S on the Euclidean plane, $r(S) \leq \dfrac{1}{2} + \dfrac{1}{\pi}$. Since for arbitrary convex set S, the convex hull of $Y(S, l)$ can cover the whole shape S, so we have:

$$r(S) = \frac{\text{L(optimal curve of S)}}{\text{L(perimeter of S)}} \leq \frac{\min_l L(Y(S,l))}{\text{L(perimeter of S)}} \leq \frac{\left(\frac{1}{2}+\frac{1}{\pi}\right)P}{P} = \frac{1}{2} + \frac{1}{\pi} = r(C)$$

□

It is now left to prove that the equality case of the inequality in Theorem 9 holds if and only if S is a circle.

For this, we have to re-consider the equality cases in the various inequalities. For the equality in Theorem 9 to hold, all the equality cases in the intermediate inequalities have to hold.

In Theorem 6, the equality case of the inequality $d' \leq \dfrac{P}{2\pi}$ holds if and only if the shape S is exactly a circle (by Theorem 1). So the shape M in Theorem 7 is a circle, i.e. with constant width. So the shape S in Theorem 7 also has constant width, which is the radius of the circle M (by Theorem 15).

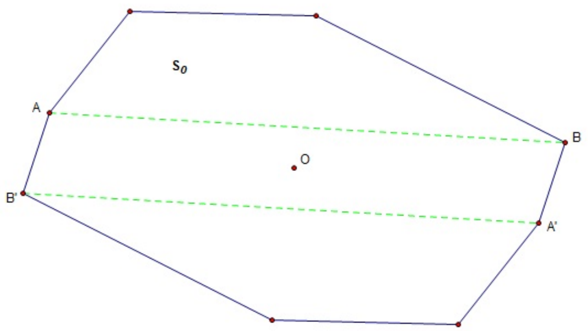
Now re-consider the inequality in the proof of Theorem 7:

$$\begin{aligned}\min_l L(Y(S,l)) &\leq \min\{L(Y(S,l_0)), L(Y(S,l_0'))\} \leq \frac{1}{2}\left(L(Y(S,l_0)) + L(Y(S,l_0'))\right) \\ &= \frac{1}{2}(P + 2d_{min}) \leq \frac{1}{2}\left(P + 2\frac{P}{\pi}\right) = \left(\frac{1}{2} + \frac{1}{\pi}\right)P\end{aligned}$$

The second inequality sign reveals that $L(Y(S,l_0)) = L(Y(S,l_0'))$ when equality holds. Since the shape S is of *constant width*, l_0 can be any supporting line of shape S. So this requires $L(Y(S,l_0)) = L(Y(S,l_0'))$ to hold for all the supporting lines of shape S. So S is a *centrosymmetric* shape.

It is left to show that the only shape of <u>constant width</u> and is <u>centrosymmetric</u> is circle.

For this, let a shape of constant width has a *diameter* (the line connecting two extreme points in a given direction) AB. AB must pass through the center of symmetry. Otherwise, its central image $A'B'$ is another diameter in the same direction. In the parallelogram $ABA'B'$ one of the angles A or B is not less than $90°$. Assuming it's A, from $\triangle BAB'$, $BB' > AB$. But this leads to a contradiction, because in a shape of constant width no two points may be at the distance exceeding its diameter (common width in any direction.)



Therefore, all diameters of a centrally symmetric shape of constant width pass through the center of symmetry. Because of the symmetry, each of the diameters is divided in half by that point. So it is a circle with diameter equals to the width.

Therefore, the shape with greatest optimal curve given a fixed length of perimeter is **circle**.

3. The shortest curve such that its convex hull is a given convex polygon S

In this chapter, we will first introduce three properties of the shortest curve and then suggest an algorithm to find the curve for a general polygon S given the coordinates of all its vertices.

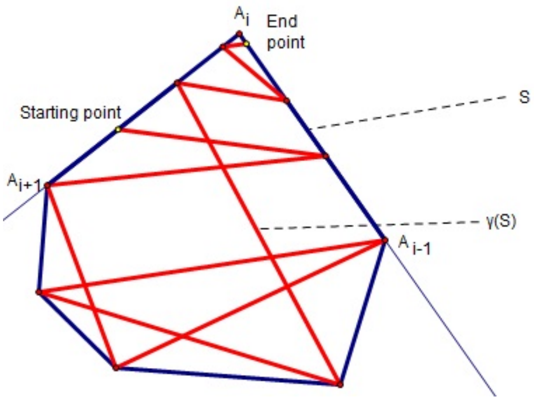
Suppose the polygon S has vertices $A_1, A_2, A_3, \dots A_n$, where A_i and A_{i+1} are consecutive $\forall i = 1, 2, 3, \dots, n$ and $A_{n+1} = A_1$.

Lemma 10. *The shortest curve passes through A_i $\forall A_1, A_2, A_3, \dots A_n$.*

Proof. We shall prove by contradiction, i.e. assume that the shortest curve does not pass through A_i for some $i \in \{1, 2, 3, , n\}$.

Denote the curve by $\gamma_0(S)$.

Since the convex hull of the curve is exactly shape S, no points on the curve should be in the exterior part of S. Since S is convex, the whole shape S is in the same half-plane divided by $l(A_i A_{i1})$ and also in the same half-plane divided by $l(A_i A_{i+1})$. Thus $\gamma_0(S)$ is in the region $P(\angle A_{i-1} A_i A_{i+1}) \cup r(A_i A_{i-1}) \cup r(A_i A_{i+1})$.



i.e. $C(\gamma_0(S))$ is in the region $P(\angle A_{i-1}A_iA_{i+1}) \cup r(A_iA_{i-1}) \cup r(A_iA_{i+1})$ too.

Notice that $A_i \in S \subseteq C(\gamma_0(S))$.

If A_i is inside $C(\gamma_0(S))$, then for all point $T \in C(\gamma_0(S))$, $l(A_iT)$ cuts the plane into two half-planes such that the intersection of each of the half plane and $C(\gamma_0(S))$ is not empty. However as $A_{i-1} \in C(\gamma_0(S))$ and the whole $C(\gamma_0(S))$ is on one half-plane of $l(A_iA_{i-1})$ only, this yields a contradiction.

So A_i lies on the boundaries of $C(\gamma_0(S))$, WLOG suppose the line which includes the side on which A_i be l_0.

Yet as $A_i \notin \gamma_0(S)$, there exist two points $M, N \in \gamma_0(S)$ which lies on l_0. Since $A_i \in MN$, M, N lies on different sides of A_i on l_0. But this contradict to the fact that $C(\gamma_0(S))$ is in the region $P(\angle A_{i-1}A_iA_{i+1}) \cup r(A_iA_{i-1}) \cup r(A_iA_{i+1})$.

[See reviewer's comment (11)] □

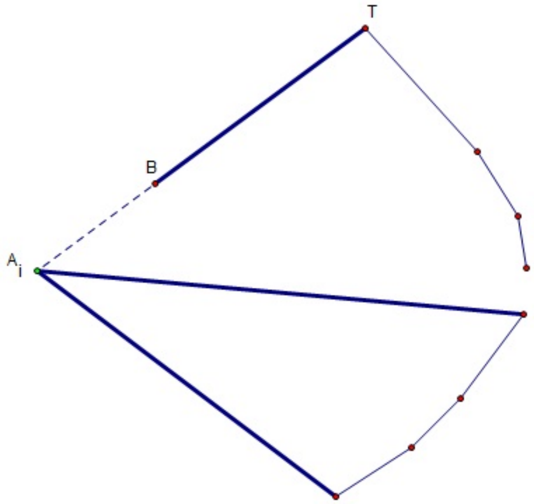
Lemma 11. *The shortest curve will pass each vertex at most one time.*

Proof. Proof by contradiction.

Assume that the curve pass vertex A_i more than once.

Case 1) A_i is the endpoint of $\gamma_0(S)$

Then A_i is of at least degree 3. Let A_iT be the line segment which starts with A_i and is the first segment of $\gamma_0(S)$. If we choose a point B in A_iT, then as A_i is still of at least degree 2, so $C(\gamma_0(S)) = C(\gamma_0(S) \backslash BA_i)$, yet $L(\gamma_0(S)) > L(\gamma_0(S) \backslash BA_i)$, so we can replace $\gamma_0(S)$ by $\gamma_0(S) \backslash BA_i$. This contradicts the minimality of $\gamma_0(S)$.



Case 2) A_i is not the endpoint of $\gamma_0(S)$

Then A_i is of at least degree 4.

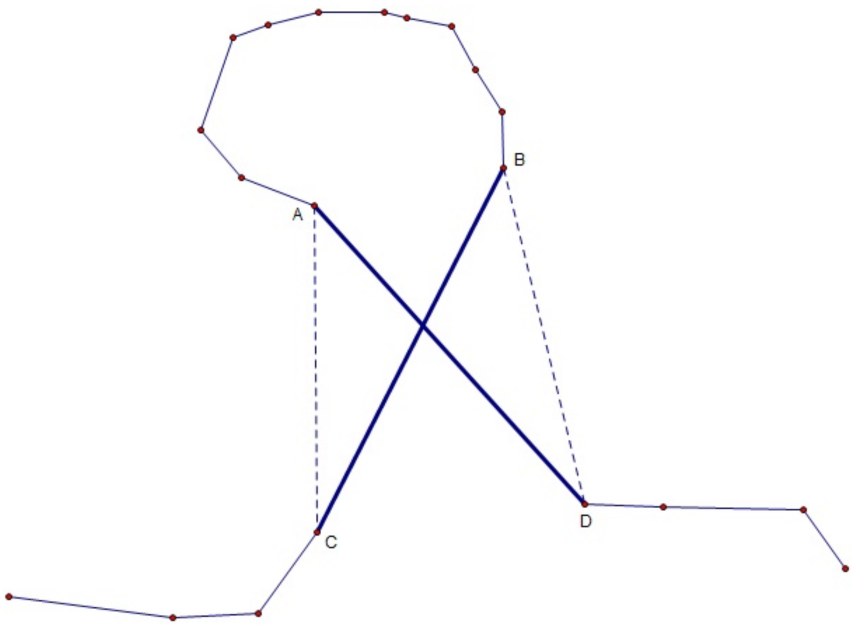
Let BA_i and A_iC be two consecutive line segments of $\gamma_0(S)$. If we replace BA_i and A_iC by BC, then as the degree of A_i is still at least 2, the convex hull will not change, but the total length of the curve would be shorter as $BA_i + A_iC > BC$, this again contradicts the minimality of $\gamma_0(S)$.

[See reviewer's comment (12)] $\square$

Lemma 12. *There is no self-intersection on the shortest curve.*

Proof. This is again proved by contradiction.

Suppose there exist two lines AD and BC in $\gamma_0(S)$ which intersect each other.



Let E be the intersection point of AD and BC, then by triangle inequality,

$$AE + EC > AC, BE + ED > BD,$$

so $AD + BC = AE + ED + BE + EC > AC + BD$. Moreover since the convex hull of AD and BC is $C(ABCD)$, while the convex hull of AC and BD is also $C(ABCD)$, so we can construct a new curve $\gamma_0'(S) = (\gamma_0(S) \cup AC \cup BD) \backslash AD \backslash BC$. It has the same convex hull as $\gamma_0(S)$, yet shorter, this contradicts to the minimality of $\gamma_0(S)$. □

From Lemma 10 we can conclude that the curve $\gamma_0(S)$ is the union of line segments whose endpoints are the vertices of S. From Lemma 11 we know that $\gamma_0(S)$ passes each vertex exactly once, so there are $n - 1$ line segments in $\gamma_0(S)$. From simple counting we know that the number of such curves is only $n!$. The shortest among these $n!$ curves would be $\gamma_0(S)$.

In general the shape of the shortest curves can be complicated, so we would try an algorithmic approach to find the shortest curve for a general polygon.

Since we have limited the answer to be the shortest among a finite number of curves, a simple algorithm would certainly be finding the length of all the $n!$ curves and take the minimum. However, the time complexity for this algorithm would be at least $O(n!)$, provided that we can find an $O(1)$ algorithm to find the length of curves.

However, the time complexity of this algorithm would be *non-polynomial* and thus not efficient and applicable for large n. So in the following, we would introduce an algorithm in *polynomial time* on n using the three lemmas.

First we shall develop a recursion relation using Lemma 12. We first choose one of the n vertices as a starting point of the curve. WLOG we shall choose A_1. Label it as γ_1 on the curve. Then we shall notice that there are only two choices of γ_2, which are A_2 or A_n. It is because if we choose neither of them, say we choose $A_i, i \neq 2, n$, then by Lemma 12 the curve will not intersect itself, so the curve will then completely be on either the left side or the right side of $l(A_1 A_i)$. But as A_2 and A_n are on different sides of $l(A_1 A_i)$, so at least one of them cannot be reached by the curve. This is a contradiction to Lemma 10.

Using similar logic we can show that when k vertices are chosen, there are only two choices of the $(k+1)$th vertex. That is the vertex so that all the chosen vertices are consecutive. A rigorous description is as follows:

Define a function $R : \{1, 2, 3, , n\}^3 \to \mathbb{R}$ as follows:
$R(S, T, L)$ is the length of shortest curve that passes through vertices $A_S, A_{S+1}, \ldots, A_T$ if $S \leq T$, or $A_S, A_{S+1}, \ldots, A_{n+T}$ if $S > T$ and ends at point A_L. From the observation in the above paragraph, $L = S$ or T.

Then we can develop the following relations:

$$R(S,T,S) = \min(R(S+1,T,S+1) + A_S A_{S+1}, R(S+1,T,T) + A_S A_T)$$
$$R(S,T,T) = \min(R(S,T-1,S) + A_S A_T, R(S,T-1,T-1) + A_{T-1} A_T)$$

With the base case that $R(N, N, N) = 0 \ \forall N \in \{1, 2, 3, , n\}$, we can develop an dynamic programming algorithm with pseudo code like this:

(please note that in the following code, D, S, T and X are all taken modulo n, i.e. negative values are added by n, while values larger than n are subtracted by n)

$L(P, Q) =$ length of $A_P A_Q$

for D from 0 to $n-1$ by 1
    for T from 1 to n by 1
        IF $D = 0$ then $R(T-D, T, T) := 0$;
        ELSE
        $S := T - D$
        $R(S, T, T) := \min(R(S, T-1, S) + L(S, T), R(S, T-1, T-1)$
                $+L(T-1, T))$
    for S from 1 to n by 1
        $T := S + D$
        $R(S, T, S) := \min(R(S+1, T, S+1) + L(S, S+1), R(S+1, T, T)$
                $+L(S, T)$

MIN $= \infty$
for X from 1 to N by 1
    IF $R(X, X-1, X) <$ MIN then MIN $:= R(X, X-1, X)$
    IF $R(X, X-1, X-1) <$ MIN then MIN $:= R(X, X-1, X-1)$

output MIN

In this way we can find the minimum curve in time complexity $O(n^2)$.

The complete code in C++ and the program description is attached in the Appendix 2 of the project.

Conclusion

In this project, the main result is to prove that the convex set with greatest optimal curve given a fixed perimeter is a circle. We have also introduced an algorithmic approach to find the curve with convex hull exactly equal to a given shape. Most of the contents in these two parts are original.

The first result is a pure mathematical discussion and the result is believed to be new. The introduction of the Minkowski Sum is probably the most critical part of the proof. I have taken the proof of the Theorem of Barbier as reference and found it quite useful.

The second part, however, is quite different from the first part, in terms of both the result and the approach. The original goal of this part is indeed to find the actual optimal curve for a general shape. However, on the way that we do the research, we find that the shape of the optimal curve can actually vary a lot even for very similar shapes. The number of combinations is too voluminous that a pure logical discussion may not be able to find an elegant and useful result. We have thought of determining the conditions of the choice of different curves, yet no good properties can be found. So we eventually resort to an algorithmic approach, which seem to be more practical. Though this may reduce the elegancy of the project, we believe this can be of practical importance in other curve optimization problems.

A close variation of Theorem 2, the forest problem, is once open for a few decades ([1]). So it is not surprising that the more general result of considering the optimal curve for general shapes is complicated. I sincerely hope that the algorithm I introduced as well as the conclusion made in Chapter 2 would be useful in tackling this problem.

Last but not least, I would like to thank my teacher advisor, Ms. Luk Mee Lin, for her precious opinions on my project, and also Prof. Oliver Knill of Harvard University for introducing me the topic and giving some insights.

REFERENCES

[1] S. R. Finch and J. E. Wetzel. *Lost in a Forest*, Am. Math. Monthly, **111** (2004), no.8, p.645-654, MR2091541 (2006d:51016)
[2] H. Joris. *Le Chasseur Perdu dans la Forêt*, Elem. Math, **35** (1980), p.1-14, MR0559167 (81d:52001)
[3] O. Knill, *Curve Optimization Problems*, http://abel.math.harvard.edu/∼knill/various/wallstreet/index.html

Appendix 1: Minkowski Sum

For two given convex set A and B on the Argand plane, the Minkowski Sum $A+B$ is defined as follows:

$$A+B=\{a+b \mid a\in A, b\in B\}.$$

Here is a list of facts of Minkowski Sum which are used in the project.
(Unless otherwise specified, all the definitions used here are as mentioned in the main context.)

Theorem 13. *The shape of $A+B$ is independent to the choice of origin of the Argand plane.*

Proof. For convenience, denote the set X when O is chosen as origin as X_O. So for two different origins, say O and O', choose arbitrary point $x\in(A+B)_O$. Then there exist $a\in A_O, b\in B_O$ such that $a+b=x$ by definition.
Since $\overrightarrow{O'O}+a\in A_{O'}, \overrightarrow{O'O}+b\in B_{O'}$ so:

$$2\overrightarrow{O'O}+x=2\overrightarrow{O'O}+a+b=(\overrightarrow{O'O}+a)+(\overrightarrow{O'O}+b)\in(A+B)_{O'}$$

This means that every points in $(A+B)_{O'}$ is a simple translation of $(A+B)_O$ by $2\overrightarrow{O'O}$. Thus they have the same shape.

$\square$

Theorem 14. *If A and B are convex sets, then $A+B$ is a convex set.*

Proof. We can define convex set rigorously as follows:
A point set S is ***convex*** if $\forall x, y \in S, dx + (1-d)y \in S \ \forall d \in [0,1]$. This is just a rephrase of the usual definition of convexity: the line segment joining any pair of points of S lies entirely in S.

Then pick arbitrary two points, $y \in (A+B)$. [See reviewer's comment (13)] Suppose $a_x, a_y \in A$ and $b_x, b_y \in B$ such that $a_x + b_x = x$ and $a_y + b_y = y$. Since A and B are convex sets, so for $d \in [0,1]$, $da_x + (1-d)a_y \in A$ and $db_x + (1-d)b_y \in B$. Therefore

$$\begin{aligned} dx + (1-d)y &= d(a_x + b_x) + (1-d)(a_y + b_y) \\ &= (da_x + (1 - day + dbx + 1 - dby \\ &\in A + B \text{ by definition. [See reviewer's comment (14)]} \end{aligned}$$

So $(A+B)$ is also convex. $\square$

Theorem 15. $d(A,l) + d(B,l) = d(A+B,l)$. *In other words, the width of $(A+B)$ in the direction perpendicular to line l equals the sum of the widths of A and B in the same direction.*

Proof. In this proof, we shall put the whole problem onto a Cartesian plane instead of Argand plane using the well-known correspondence between the two of them for convenience.

Suppose the angle between line l and the x-axis as θ. Suppose the normal form of the equations of the two supporting lines of A parallel to l as $x\cos\theta + y\sin\theta = p_A$ and $x\cos\theta + y\sin\theta = q_A$ (WLOG let $p_A < q_A$). Define p_B, q_B, p_{A+B} and q_{A+B} in a similar way. Then the width $d(A,l) = |p_A - q_A|$. Moreover, $\forall (x,y) \in A$,

$$p_A \le x\cos\theta + y\sin\theta \le q_A.$$

Similarly $d(B,l) = |p_B - q_B|$ and $\forall (x,y) \in B$,

$$p_B \le x\cos\theta + y\sin\theta \le q_B.$$

Please note that the equality in all the 4 inequalities hold for at least one pair of x and y.

Now for each point $(x,y) \in (A+B)$, there exist $(x_A, y_A) \in A$ and $(x_B, y_B) \in B$ such that $(x,y) = (x_A, y_A) + (x_B, y_B)$. As

$$p_A \le x_A\cos\theta + y_A\sin\theta \le q_A \text{ and } p_B \le x_B\cos\theta + y_B\sin\theta \le q_B,$$

so $p_A + p_B \le x\cos\theta + y\sin\theta \le q_A + q_B$. Pick $(x_A, y_A) \in A$ so that

$$p_A = x\cos\theta + y\sin\theta$$

and $(x_B, y_B) \in B$ so that

$$p_B = x\cos\theta + y\sin\theta.$$

Then $p_A + p_B = (x_A\cos\theta + y_A\sin\theta) + (x_B\cos\theta + y_B\sin\theta) = x\cos\theta + y\sin\theta$. Similarly we can verify that the equality in both inequalities hold for at least one pair of x and y.

Therefore $d(A+B, l) = (q_A + q_B) - (p_A + p_B) = d(A, l) + d(B, l)$. □

Theorem 16. *The perimeter of $(A+B)$ equals the sum of perimeters of A and B.*

Proof. We shall first consider the case when A and B are both convex polygons.

Arbitrarily pick one of the sides of A and label the line that it lies on as a. Suppose the angle between a and real axis is θ. For convenience multiply every points in A and B by $e^{-i\theta}$. Now by theorem 13 we can choose one ended point of the chosen side as the origin while the rest of the side lie on the positive real axis. Now translate B so that B lies completely above the real axis and touches the real axis on at least one point.

Let l_a be the length of the chosen side, so the part that A touches the real axis is $\{x \mid 0 \le \mathrm{Re}(x) \le l_a\}$.

Firstly since both A and B lies completely above the real axis, $(A+B)$ also lies completely above the real axis.

If none of the sides of B is parallel to a, suppose the point that B touches real axis as β. So the part that it touches the real axis would be $\{x \mid \beta \le \mathrm{Re}(x) \le \beta + l_a\}$. This means that $A+B$ would have a side with length l_a.

If one of the sides of B is parallel to a, suppose the part that B touches real axis as $\{x \mid \beta \le \mathrm{Re}(x) \le \beta + l_b\}$. Then the part that it touches the real axis would be $\{x \mid \beta \le \mathrm{Re}(x) \le \beta + l_a + l_b\}$. This means that $A+B$ would have a side with length $l_a + l_b$.

Since the side is arbitrarily chosen, we can do the same on every side of A and B. This means that for every side of A and B, there will be a side of $A+B$ with the same length and parallel to it, i.e. each side of A and B are translated to be a side of $A+B$. It is left to prove that each side of $A+B$ is parallel to one of the sides of A or B.

This is obvious using the proof by contradiction. If one of the sides of $A+B$ is not parallel to any of the sides of A and B, then the supporting lines of A and B parallel to this side would touch A and B at only one point. So by the definition of

Minkowski Sum the supporting line of $A+B$ parallel to this side would also touch the shape at only one point, contradict to the fact that it is a side of $A+B$.

So the sides of $A+B$ are composed of the sides of A and B only, so the perimeter of $A+B$ is equal to the sum of perimeter of A and B.

Now consider the general case of A and B. Let P_A and P_B be the perimeters of A and B respectively. We shall now construct polygon A' with n sides in the following way:

Choose a point on the boundary of A and label it as A_1. On the boundary of A, label a new point A_2 so that $L(\widehat{A_1A_2}) = \dfrac{P_A}{n}$. Now label A_{i+1} on the boundary of A so that A_{i+1} is on different side of A_i as $A_{i-1}, i = 2, \ldots, n$. A_{n+1} should then be equal to A_1.

Label the polygon $A_1A_2A_3 \ldots A_n$ as A'. B' is constructed in a similar way. Firstly A' is a convex polygon since A is convex. Also, due to the convexity of A, A' is completed covered by A.

So (perimeter of A')$\leq$(perimeter of A), (perimeter of B')$\leq$(perimeter of B).

On the other hand, for each A_iA_{i+1}, construct a line l_i that touches A and is parallel to A. Then $l_1, l_2, \ldots, l_n$ shall construct another polygon A'' with n sides. Moreover, A'' completely cover A. So (perimeter of A)$\leq$(perimeter of A''). Similarly construct B'' and we have (perimeter of B)$\leq$(perimeter of B'').

We shall then notice that as A covers A' and B covers B', then $A+B$ covers $A'+B'$. Similarly $A''+B''$ covers $A+B$. So

$$\text{(perimeter of } A'+B'\text{)} \leq \text{(perimeter of } A+B\text{)} \leq \text{(perimeter of } A''+B''\text{)}.$$

Notice that as n tends to infinity, A' tends to A'' and B' tends to B''. So $A'+B'$ tends to $A''+B''$, i.e. (perimeter of $A'+B'$) = (perimeter of $A''+B''$). So

$$\text{(perimeter of } A'+B'\text{)} = \text{(perimeter of } A''+B''\text{)} = \text{(perimeter of } A+B\text{)},$$
$$\text{(perimeter of } A'\text{)} = \text{(perimeter of } A''\text{)} = \text{(perimeter of } A\text{) and}$$
$$\text{(perimeter of } B'\text{)} = \text{(perimeter of } B''\text{)} = \text{(perimeter of } B\text{)}$$

Since A' and B' are polygons,

$$\text{perimeter of } A'+B' = \text{(perimeter of } A'\text{)+(perimeter of } B'\text{)},$$

so

$$\text{perimeter of } A+B = \text{(perimeter of } A\text{)+(perimeter of } B\text{)}.$$

Therefore the general case is solved. □

Theorem 17. *If convex set S' is the rotational image of convex set S by 180 degree with respect to a given point O, the $S+S'$ is centrosymmetric with centre O.*

Proof. Take O as the origin of the Argand plane. For every point $x \in S + S'$, there exist $a \in S$ and $b \in S'$ such that $a + b = x$. Moreover, as S and S' are rotational images with respect to O, so $-b \in S$ and $-a \in S'$, so $-x = -a - b \in S + S'$. Therefore $S + S'$ is centrosymmetric. $\square$

Appendix 2: The computer program:

The C++ code of the program is:

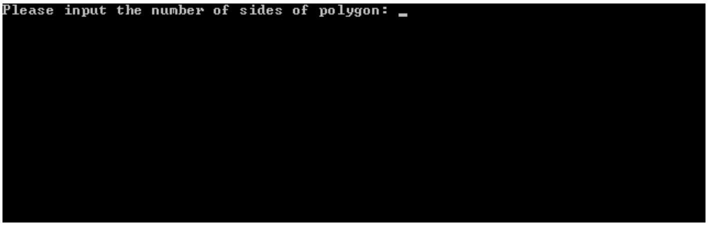
```cpp
#include <iostream>
#include <cmath>
#define MX 100

using namespace std;
int n;
double x_coor[MX], y_coor[MX];
double mini(double p, double q){return (p<q?p:q);}
double sq(double p){return (p*p);}
double L(int p, int q){
       return sqrt( sq(x_coor[p]-x_coor[q]) + sq(y_coor[p]-y_coor[q]) );}    //distance of A_p and A_q
double r[MX][MX][2];

int main(){
    cout << "Please input the number of sides of polygon: ";    cin >> n;
    cout << "Please input the x and y coordinates of each vertex clockwisely\n";
    for (int c = 0; c < n; c++){cout << c+1 << ": ";
    cin >> x_coor[c] >> y_coor[c];}                                          //input x and y coordinates
    for (int d = 0; d < n; d++){                  //loop for the number of vertices already on the curve
        for (int t = 0; t < n; t++){                                         //loop for the terminal vertex
            if (d == 0) {r[t][t][0] = 0; r[t][t][1] = 0;}                            //input initial case
            else {
                 int s = (n + t - d) % n;
                 r[s][t][1] = mini (r[s][(n+t-1)%n][0] + L(s,t), r[s][(n+t-1)%n][1] + L((n+t-1)%n,t));
            }
        }
        for (int ss = 0; ss < n; ss++){                                      //loop for the starting vertex
            int tt = (ss + d) % n;
            r[ss][tt][0] = mini (r[(ss+1)%n][tt][0] + L(ss,(ss+1)%n), r[(ss+1)%n][tt][1] + L(ss,tt));
        }
    }
    double ans = 10000000;
    for (int x = 0; x < n; x++){
        if (r[x][(x+n-1)%n][0] < ans) ans = r[x][(x+n-1)%n][0];                        //find minimum
        if (r[x][(x+n-1)%n][1] < ans) ans = r[x][(x+n-1)%n][1];                        //find minimum
    }
    cout << "The length of the shortest curve is: " << ans << endl;                    //output answer
    return 0;
}
```

The program runs in this way:

1. Input the number of sides of the polygon:



2 Input the coordinates of each vertex of the polygon **clockwisely**:

```
Please input the number of sides of polygon: 5
Please input the x and y coordinates of each vertex clockwisely
1: 0 0
2: 3 1
3: _
```

3 The program will output the length of the optimal curve:

```
Please input the number of sides of polygon: 5
Please input the x and y coordinates of each vertex clockwisely
1: 0 0
2: 3 1
3: 13 1
4: 16 0
5: 8 -1
The length of the shortest curve is: 17.0949
```

A slight adjustment on the data structure used in the program can actually produce the whole curve:

```
Please input the number of sides of polygon: 5
Please input the x and y coordinates of each vertex clockwisely
1: 0 0
2: 3 1
3: 13 1
4: 16 0
5: 8 -1
The length of the shortest curve is: 17.0949
The curve is along vertices: 1 2 5 3 4
```

Reviewer's Comments

The reviewer has some comments about the presentation of this paper, as well as the notations and typos.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. It is a little ambiguous to specify that "here curve can $\cdots$ union of straight lines". The reviewer suggests either deleting it or rewriting it as "here curves can involve unions of straight and curved lines".
3. It may be better to replace "maximal" by "longest".
4. It may be better to replace "greatest" by "longest".
5. It may be better to replace "greatest" by "longest".
6. Definitions for "int(S)" and "ext(S)" are confusing (with typos?) and not canonical;
   Definition for $P(\angle ABC)$ is not so mathematical although it can be understood. How about "$r(BA)$ and $r(BC)$ divide $\mathbb{R}^2$ plane into two sectors. $P(\angle ABC)$ is the minor sector with angle not greater than π "?
7. (a) it may be more precise to replace "continuous shapes" by "connected shapes";
   (b) in the 3rd line of the proof, "smaller area" should be "smallest perimeter";
   (c) the first part of the proof may be more readable together with a graph;
   (d) the second part, the conclusion of "convex hull has smaller perimeter than the original set" is not obvious, so a proof or a reference may be needed here.
8. "the direction perpendicular to l" is ambiguous and appears several times, and it should mean "the distance between two supporting lines parallel to l".
9. P should be defined first, i.e. the perimeter of S_0 (or perimeter of S for Theorem 7), and the notation P defined in section 1 should be deleted.
10. "there exists a point O" is not accurate, and it should be just the point O chosen before.
11. The proof is confusing.
    Firstly, the convex hull of the curve is assumed to be exactly S, i.e. $C(\gamma_0(S)) = S$, and it is obvious that A_i is on the boundary of S, which implies the first case "A_i inside $C(\gamma_0(S))$ does not make sense.
    The reviewer suggests rewriting the proof as follows.
    Since $A_i \in C(\gamma_0(S))$, there exist finite points $P_j \in \gamma_0(S), j = 1, \cdots, k$ such that A_i lies in the convex polygon Ω with vertexes P_j. Case 1: A_i lies in the interior of Ω; Case 2: A_i lies on one side (edge) of Ω, called l_0. Then derive the contradictions for both the two cases.
12. What is the definition of "degree of point A_i?
13. It should be "two points $x, y \in (A + B)$".
14. It should be "$(da_x + (1-d)a_y) + (db_x + (1-d)b_y) \in (A + B)$.
15. The references should be arranged in the alphabetical order.