Hang Lung Mathematics Awards
Vol. 10 (2023), pp. 153–186

STRUCTURE OF CRITICAL GROUPS OF CIRCULANT GRAPHS

A RESEARCH REPORT SUBMITTED TO THE SCIENTIFIC COMMITTEE OF THE HANG LUNG MATHEMATICS AWARDS

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2345266

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AUGUST 2023

Abstract. The critical group of a graph is defined as the torsion subgroup of the co-kernel of the Laplacian matrix of the graph. In this paper, we investigated the critical groups of two classes of unitary circulant graphs, which are Cayley graphs on the group of integers modulo n, with connecting set being the set of units modulo n. The explicit group structure of such graphs when n is product of two distinct primes and when n is a prime power, are computed using Ramanujan Sums. Furthermore, we investigated the critical groups of circulant graphs with fixed connecting sets, and expressed one of the components of the group as the greatest common divisor of real and imaginary parts of Chebyshev Polynomials.

Keywords. Cayley Graphs, Critical Group, Laplacian Matrix, Smith Normal Form, Circulant Graph, Ramanujan Sum, Chebyshev Polynomial

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1. Introduction and Main Results

The notion of the *critical group* of a graph, also known as the Jacobian group, the sandpile group or the Picard group, has been independently discussed by many authors [2][3][5]. It is an important algebraic invariant of a finite graph.

In this paper, we only consider finite simple graphs G with no loops and at most one edge between any two vertices. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G. We can view this group from a combinatorial perspective via the *chip-firing game*, first discussed by Biggs [3] (see also the survey by Glass and Kaplan [5]). It is defined informally as follows: for a graph G, assign an integer number of chips to every vertex. We are allowed to perform chip-firing moves (also known as the *toppling rule*) on any vertex except one. When we perform a chip-firing move on a vertex $v \in V(G)$, the number assigned to v is subtracted by the degree of v, and the number assigned to each neighbour of v is increased by 1. Writing this in mathematical terms, define a divisor by a function $f : V(G) \to \mathbb{Z}$. Any divisor on $V(G)$ is an element of $\mathbb{Z}^{|V(G)|}$. The degree of each divisor is f defined by $\deg(f) = \sum_{v \in G} f(v)$, and we can further define addition in this sense by $(f+g)(v) = f(v) + g(v)$. Notice that every chip-firing move preserves the degree. In fact, the set of all divisors of G with the same degree forms a free abelian group with $|V(G)| - 1$ generators. In this context, we focus on the group $\mathrm{Div}^0(G)$ of all divisors on G with degree 0. For each function $f \in \mathrm{Div}^0(G)$, we treat it as a column vector in $\mathbb{Z}^{|V(G)|}$. Whenever we perform a chip-firing move, it is equivalent to subtracting a column of $\mathbf{L}(G)$ from this vector. The chip-firing game is closely related to the abelian sandpile model, introduced by Bak et al. in [1]. It has also been discussed by Baker and Norine [2]. This allows us to visualise critical groups of graphs, but we are going to define it in another way for easier computation.

The second way to define the critical group $K(G)$ of graph G is to make use of its relation with the Laplacian matrix of G, $\mathbf{L}(G)$. For two given vertices $u, v \in V(G)$, denote a_{uv} the number of edges between u and v. The matrix $\mathbf{A} = \mathbf{A}(G) = \{a_{uv}\}_{u,v \in V(G)}$ is called the *adjacency matrix* of the graph G. The degree $d(v)$ of a vertex $v \in V(G)$ is defined by $d(v) = \sum_{u \in V(G)} a_{uv}$. Consider $\mathbf{D} = \mathbf{D}(G)$ diagonal matrix indexed by the elements of $V(G)$ with $d_{uv} = d(v)$. Then $\mathbf{L}(G) = \mathbf{D} - \mathbf{A}$ is called the *Laplacian matrix*, or simply the *Laplacian* of the graph G.

We also define the *cokernel* of the Laplacian by $\operatorname{coker}(\mathbf{L}(G)) = \mathbb{Z}^{|V(G)|}/\operatorname{im}(\mathbf{L}(G))$. This is a finitely generated abelian group. Since the Laplacian has the kernel generated by the column vector $(1, 1, \ldots, 1)^T$, its rank is $|V(G)| - 1$. This implies

$$\operatorname{coker}(\mathbf{L}(G)) \cong \left(\bigoplus_{i=1}^{|V(G)|-1} \mathbb{Z}/d_i\mathbb{Z} \right) \oplus \mathbb{Z}, \text{ with } d_1 \mid d_2 \mid \cdots \mid d_{|V(G)|-1}.$$

Such d_i are *invariant factors* of $\mathbf{L}(G)$, and the prime power factors of invariant factors are known as *elementary divisors*. We define $K(G)$ by the torsion subgroup of $\mathbb{Z}^{|V(G)|}/\operatorname{im}(\mathbf{L}(G))$, so that it is a finite abelian group. Thus,

$$K(G) \cong \bigoplus_{i=1}^{|V(G)|-1} \mathbb{Z}/d_i\mathbb{Z}, \text{ with } d_1 \mid d_2 \mid \cdots \mid d_{|V(G)|-1}.$$

We can compute the critical group $K(G)$ by finding the Smith Normal Form of the Laplacian $\mathbf{L}(G)$. Indeed, by considering $n \times n$ integer matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\det(\mathbf{P}) = \det(\mathbf{Q}) = \pm 1$ and $\mathbf{P}\mathbf{L}(G)\mathbf{Q} = \mathbf{D}$, where $\mathbf{D} = \operatorname{diag}(d_1, \ldots, d_{|V(G)|-1}, 0)$ is a diagonal matrix, we have $\operatorname{coker}(\mathbf{L}(G)) \cong \operatorname{coker}(\mathbf{D}) \cong \bigoplus_{i=1}^{|V(G)|-1} \mathbb{Z}/d_i\mathbb{Z}$. This means we can determine $K(G)$ directly from the SNF of $\mathbf{L}(G)$.

In this paper, we focus on the critical groups of *Cayley graphs* of abelian groups. In general, for a Cayley graph G built from an abelian group H with connection set $S \subset H$ such that the identity element of H is not in S, each $v \in V(G)$ is associated to a unique element $h_v \in H$. Then for all $x, y \in G$, there is an edge between x, y if and only if $h_x h_y^{-1} \in S$. Even though the set-up looks simple, critical groups of Cayley graphs are not completely solved. For instance, the explicit group structures of the Cayley graphs of $\mathbb{F}_2^n$ (even hypercubes) are still unknown. In our case, we are going to focus on Cayley graphs built from $\mathbb{Z}/n\mathbb{Z}$. By identifying each vertex of $V(G)$ with a unique integer from the set $\{1, 2, \ldots, n\}$, consider some non-empty connection set $S \subset \mathbb{Z}/n\mathbb{Z}$. Then, for all x, y satisfying $1 \leq x < y \leq n$, $xy \in E(G)$ if and only if $y - x \in S$. This is what is known as *circulant graphs.*

Our calculations are focused on two types of graphs. The first type is Unitary Circulant Graphs (UCG), which are Cayley graphs constructed from $\mathbb{Z}/n\mathbb{Z}$ with connection set $\{1 \leq a < n : (a, n) = 1\}$, i.e. set of units modulo n. We completely solve the critical group of UCG when $n = pq$, where $p < q$ are primes, as well as when $n = p^k$, where p is prime and $k \in \mathbb{N}$. To achieve this, we make use of the fact that the p-adic valuation of eigenvalues of the Laplacian can be transferred to that of invariant factors for Cayley graphs of abelian groups and p not dividing n. Here the eigenvalues can be written in terms of Ramanujan Sums. Furthermore, in particular for UCG, we handle the cases when such p divides the size of the graph as well. The second type is circulant graphs with fixed jumps. This was inspired by a suggested research project in Glass and Kaplan [5]. To be precise, let $s_1, s_2, \ldots, s_k$ be integers such that $1 \leq s_1 < s_2 < \cdots < s_k < \dfrac{n}{2}$. The graph

$C_n(s_1, s_2, \ldots, s_k)$ with n vertices $0, 1, 2, \ldots, n-1$ is defined by connecting each vertex i, where $0 \le i \le n-1$ to the vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k \pmod{n}$. Given that the critical group is isomorphic to a direct sum of finite abelian groups, we found an expression for the size of the smallest non-trivial component for graphs $C_n(1,3), C_n(2,3)$ and $C_n(1,2,3)$ for all $n \ge 7$ as the greatest common divisor (GCD) of real and imaginary parts of the two kinds of Chebyshev Polynomials T_n, U_n and U_{n-1}. This is made possible by using the fact that entries of the Smith Normal Form are related to the GCD of all $m \times m$ minors of the matrix, where $1 \le m < n$. To be precise, we have the following results:

Theorem 1. For primes $p < q$, if d is the largest common factor of $p-1, q-1$ satisfying $\left(d, \frac{p-1}{d}\right) = \left(d, \frac{q-1}{d}\right) = 1$,

$$\begin{aligned}
&K(\mathrm{UCG}(pq)) \\
&\cong (\mathbb{Z}/(pq-p-q)\mathbb{Z})^{pq-2p-2q+3} \oplus (\mathbb{Z}/(d(pq-p-q)\mathbb{Z}))^{p-1} \\
&\oplus \mathbb{Z}/((p-1)(pq-p-q))\mathbb{Z} \oplus (\mathbb{Z}/(q(p-1)(pq-p-q))\mathbb{Z})^{q-p-1} \\
&\oplus \mathbb{Z}\Big/\frac{q(p-1)(q-1)(pq-p-q)}{d}\mathbb{Z} \oplus \left(\mathbb{Z}\Big/\frac{pq(p-1)(q-1)(pq-p-q)}{d}\mathbb{Z}\right)^{p-2}.
\end{aligned}$$

Theorem 2. When $k \ge 2$,

$$\begin{aligned}
K(\mathrm{UCG}(p^k)) \cong\ & \mathbb{Z}/\left(p-1\right)\mathbb{Z} \oplus (\mathbb{Z}/(p-1)p^{k-1}\mathbb{Z})^{p^k-2p} \oplus \mathbb{Z}/(p-1)p^{2k-2}\mathbb{Z} \\
& \oplus (\mathbb{Z}/(p-1)p^{2k-1}\mathbb{Z})^{p-2}.
\end{aligned}$$

Theorem 3. For $n \ge 7$, the smallest non-trivial component of $K(C_n(2,3))$ has size

$$\frac{1}{52}\gcd(52d, -2s_n + 12t_n - 11u_n + v_n - 4n, 2u_n + 14v_n - 4n),$$

where $s_n, t_n, u_n, v_n \in \mathbb{Z}$, $d = \gcd(n, s_n, t_n, u_n, v_n)$, $s_n + i\sqrt{3}t_n = 4T_n\left(\frac{-3+i\sqrt{3}}{4}\right) - 4$ and $u_n + i\sqrt{3}v_n = 2U_{n-1}\left(\frac{-3+i\sqrt{3}}{4}\right)$.

Theorem 4. For $n \ge 7$, the smallest non-trivial component of $K(C_n(1,3))$ has size

$$\frac{1}{20}\gcd(20d, 2u_n + 6v_n + 18n, s_n + 3t_n + 10v_n, 10t_n + 10n),$$

where $s_n, t_n, u_n, v_n \in \mathbb{Z}$, $d = \gcd(n, s_n, t_n, u_n, v_n)$, $s_n + it_n = 2T_n\left(\frac{1+i}{2}\right) - 2$ and $u_n + iv_n = 2U_{n-1}\left(\frac{1+i}{2}\right)$.

Theorem 5. For $n \geq 7$, the smallest non-trivial component of $K(C_n(1,2,3))$ has size

$$\frac{1}{56}\gcd(56d, 2u_n + 14v_n + 52n, s_n + 7t_n + 28v_n, 28t_n + 28v_n + 28n)$$

where $s_n, t_n, u_n, v_n \in \mathbb{Z}$, $d = \gcd(n, s_n, t_n, u_n, v_n)$, $s_n + i\sqrt{7}t_n = 4T_n\left(\frac{-3+i\sqrt{7}}{4}\right) - 4$ and $u_n + i\sqrt{7}v_n = 2U_n\left(\frac{-3+i\sqrt{7}}{4}\right)$.

2. Definitions and Prerequisites

Here are definitions of some key terms and standard results that we need for our work that has not yet been mentioned in the previous section.

Definition (Euler Totient Function)**.** For $n \in \mathbb{N}$, the *Euler totient function* $\varphi(n)$ is defined to be the number of positive integers between 1 and n (inclusive), that is coprime with n. Formally,

$$\varphi(n) := \prod_{p|n}\left(1 - \frac{1}{p}\right).$$

Definition (Möbius Function)**.** For $n \in \mathbb{N}$, the *Möbius function* $\mu(n)$ is defined to be the sum of primitive n-th roots of unity. In fact,

$$\mu(n) = \begin{cases} +1 & n \text{ is square-free with even number of prime factors,} \\ -1 & n \text{ is square-free with odd number of prime factors,} \\ 0 & n \text{ is not square-free.} \end{cases}$$

Definition (p-adic valuation)**.** The p-adic valuation of $n \in \mathbb{N}$ is the maximum $r \in \mathbb{Z}$ such that $p^r \mid n$. It is denoted $\nu_p(n)$.

Definition (Coset)**.** For a subgroup H of a group G and an element x of G, define xH to be the set $\{xh : h \in H\}$ and Hx to be the set $\{hx : h \in H\}$. A subset of G of the form xH for some x in G is said to be a *left coset* of H and a subset of the form Hx is said to be a *right coset* of H.

Definition (Quotient Group)**.** For a group G and a normal subgroup N of G, the *quotient group* of N in G, written as G/N, is the set of cosets of N in G. To define a binary operation on the set of cosets G/N, for each aN and bN in G/N, the product of aN and bN, $(aN)(bN)$ is $(ab)N$.

Definition (Cokernel)**.** The *cokernel* of a group homomorphism $f : A \to B$ between abelian groups is the quotient group $B/\operatorname{Im}(f)$.

Definition (p-Sylow Subgroup)**.** For a finite abelian group Γ and a prime p that divides $|\Gamma|$, the *p-Sylow subgroup* of Γ is the maximal subgroup of Γ where its size is a power of p.

Remark. In our cases, the p-Sylow subgroups of $K(G)$ are the direct sum of $\mathbb{Z}/m\mathbb{Z}$ where m is an elementary divisor of the group that is a power of p.

Definition (Characters of Finite Abelian Groups)**.** A character χ of a finite abelian group G is a homomorphism $\chi : G \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Theorem (Kirchoff's Matrix Tree Theorem)**.** [5, Theorem 5 and Corollary 3] For a given connected graph G with n labeled vertices, let λ_i, where $1 \leq i \leq n-1$, be the non-zero eigenvalues of its Laplacian matrix. Then the order of $K(G)$ is $\dfrac{1}{n}\displaystyle\prod_{i=1}^{n-1} \lambda_i$.

Definition (Characteristic Equation)**.** Suppose $(a_n)_{n\in\mathbb{N}}$ is an integer sequence with the linear recurrence relation $c_k a_{n+k} + c_{k-1}a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0$. Then, the recurrence relation has a *characteristic equation* defined to be the equation $c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0 = 0$.

Definition (Chebyshev Polynomials)**.** *Chebyshev polynomials of the first kind* T_n are defined by $T_n(\cos\theta) = \cos(n\theta)$, while *Chebyshev polynomials of the second kind* U_n are defined by $U_n(\cos\theta)\sin\theta = \sin\big((n+1)\theta\big)$.

3. Critical Groups of Unitary Circulant Graphs

Here we determine the critical groups of some integral circulant graphs. They are defined as follows:

Definition (Integral Circulant Graphs)**.** A circulant graph G is *integral* if all eigenvalues of the adjacency matrix of G are integers.

A criterion on whether a circulant graph is integral has been determined by So in [9]. Before we cite the criterion, define $G_n(d) = \{1 \leq k \leq n-1 : (k,n) = d\}$, where d is a divisor of n. Then, we have the following:

Proposition 1. Let G be a circulant graph on n vertices with symbol $S(G)$. Then G is integral if and only if $S(G)$ is a union of the $G_n(d)$.

Proof. See [9]. ■

We wish to focus on connected circulant graphs, and we can guarantee so if $S(G)$ consists of $G_n(1)$, as such graph would then include a cycle. Therefore, in this paper, we study a class of integral circulant graphs, which are graphs G with $S(G) = G_n(1)$. We call them Unitary Circulant Graphs (UCG), as the distance between connected vertices are all units modulo n. Such graphs are defined as follows:

Definition (Unitary Circulant Graphs)**.** The *unitary circulant graph* with n vertices
$G = \mathrm{UCG}(n)$ is a circulant graph such that if $V(G) = \{1, 2, 3, \ldots, n\}$, then $\forall x, y \in V(G)$, $xy \in E(G)$ if and only if $(|x-y|, n) = 1$.

In this paper, we determine the critical group of $\mathrm{UCG}(n)$ when $n = pq$ and $n = p^k$, where $p < q$ are primes and $k \in \mathbb{N}$. Firstly, on the case $n = pq$. As mentioned in the Introduction, the critical group of $\mathrm{UCG}(n)$ can be uniquely determined with the SNF of the Laplacian. To determine so, we need the eigenvalues of the

Laplacian. Ducey and Jalil showed that the eigenvalues of the adjacency matrix $\mathbf{A}_{pq}$ are actually sums of irreducible characters in [4]. In our case, as the graphs are built on $\mathbb{Z}/n\mathbb{Z}$, the characters are functions from $\mathbb{Z}/n\mathbb{Z}$ to the unit circle, i.e. $\{z \in \mathbb{C} : |z| = 1\}$.

Lemma 1. *All irreducible characters of $\mathbb{Z}/n\mathbb{Z}$ are given by $\chi_r(s) = e^{\frac{2\pi i r s}{n}}$, where $1 \leq r, s \leq n$.*

Proof. By definition, for any character $\chi : \mathbb{Z}/n\mathbb{Z} \to S^1$, $\chi(a+b) = \chi(a) \cdot \chi(b)$. In particular, $\chi(0) = 1$. Notice that

$$\begin{aligned} \chi(n \cdot 1) &= \chi(n) \\ \chi(1)^n &= \chi(0) \\ \chi(1)^n &= 1. \end{aligned}$$

This implies that for any character of $\mathbb{Z}/n\mathbb{Z}$, $\chi(1)$ is always a n-th root of unity. We then consider some ω that $\omega^n = 1$ and $\omega^m \neq 1$ for all $1 \leq m < n$. Then all the characters are $\chi_1, \chi_2, \ldots, \chi_n$, where for any $1 \leq r \leq n$, $\chi_r(1) = \omega^r$, and $\chi_r(s) = \chi_r(1)^s = \omega^{rs}$. ∎

From Lemma 1, we know that sums of irreducible characters are actually sums of n-th roots of unity, which is also called Ramanujan sums. With this fact, So explicitly computed the spectra of some integral circulant graphs, in particular UCG(n).

Lemma 2. *Let G be a circulant graph on n vertices with symbol $S(G) = G_n(1)$ where d is a proper divisor of n. With the convention $(0, n) = n, \varphi(1) = \mu(1) = 1$, the eigenvalues of the adjacency matrix $\mathbf{A}_{pq}$ are $\lambda_t(G) = \dfrac{\varphi(n)\mu\left(\frac{n}{(t,n)}\right)}{\varphi\left(\frac{n}{(t,n)}\right)}$, where $0 \leq t \leq n-1$.*

Lemma 3. *The Laplacian matrix $\mathbf{L}_{pq}$ of UCG(pq) has eigenvalues $pq - p - q$ with multiplicity $(p-1)(q-1)$, $(p-1)q$ with multiplicity $q-1$, $(q-1)p$ with multiplicity $p-1$ and 0 with multiplicity 1.*

Proof. For $0 \leq t \leq pq - 1$,

$$(t, pq) = \begin{cases} pq & t = 0 \\ p & t \equiv 0 \pmod{p} \\ q & t \equiv 0 \pmod{q} \\ 1 & \text{otherwise} \end{cases}$$

When $(t, pq) = pq$, there is only 1 possible value of t (which is 0), and

$$\lambda_t(\mathrm{UCG}(pq)) = \frac{\varphi(pq)\mu(1)}{\varphi(1)} = \varphi(pq) \text{ with multiplicity 1;}$$

when $(t, pq) = p$, there are $q-1$ possible values of t (which are kp where $1 \leq k \leq q-1$), and

$$\lambda_t(\mathrm{UCG}(pq)) = \frac{\varphi(pq)\mu(q)}{\varphi(q)} = 1 - p \text{ with multiplicity } q-1;$$

when $(t, pq) = q$, there are $p-1$ possible values of t (which are kq where $1 \leq k \leq p-1$), and

$$\lambda_t(\mathrm{UCG}(pq)) = \frac{\varphi(pq)\mu(p)}{\varphi(p)} = 1 - q \text{ with multiplicity } p-1;$$

when $(t, pq) = 1$, there are $\varphi(pq) = (p-1)(q-1)$ possible values of t, and

$$\lambda_t(\mathrm{UCG}(pq)) = \frac{\varphi(pq)\mu(pq)}{\varphi(pq)} = 1 \text{ with multiplicity } (p-1)(q-1).$$

Since the degree of $\mathrm{UCG}(pq)$ is $\varphi(pq) = (p-1)(q-1)$, the Laplacian $\mathbf{L}_{pq} = \varphi(pq)\mathbf{I} - \mathbf{A}_{pq}$, the eigenvalues of $\mathbf{L}_{pq}$ are exactly $(p-1)(q-1)$ minus that of $\mathbf{A}_{pq}$, with multiplicities remain unchanged. Hence we are done. ■

It follows immediately from the Kirchoff's Matrix Tree Theorem that the order of the critical group of $\mathrm{UCG}(pq)$ equals product of all non-zero eigenvalues (with multiplicities) divided by pq, which is $(pq-p-q)^{(p-1)(q-1)}(p-1)^{q-1}(q-1)^{p-1}p^{p-2}q^{q-2}$. Ducey and Jabel related the multiplicity of prime powers as an elementary divisor of $\mathbf{A}_{pq}$ and the number of eigenvalues of $\mathbf{A}_{pq}$ exactly divisible by the prime powers in [4]. In particular, this remains true for $\mathbf{L}_{pq}$, since it is a linear combination of $\mathbf{A}_{pq}$. To apply that into our case, we have the following:

Lemma 4. For a prime r that does not divide n, $k \in \mathbb{N}$, the multiplicity of r^k as an invariant factor of $\mathbf{L}_{pq}$ equals the number of eigenvalues of $\mathbf{L}_{pq}$ with r-adic valuation equal to k. Furthermore, the r-Sylow subgroup of $K(\mathrm{UCG}(pq))$ is isomorphic to the r-Sylow subgroup of $\bigoplus_{i=1}^{pq-1}(\mathbb{Z}/\lambda_i\mathbb{Z})$.

An example of the above result is that for any prime factor r of $pq-p-q$, the r-Sylow subgroup of $K(\mathrm{UCG}(pq))$ is isomorphic to the r-Sylow subgroup of $(\mathbb{Z}/(pq-p-q)\mathbb{Z})^{(p-1)(q-1)}$. Furthermore, Rushanan deduced that all components of the SNF of any diagonalisable matrix divdes the product of all non-zero distinct eigenvalues (ignoring multiplicities) in [8]. In particular, this holds for our Laplacian matrix $\mathbf{L}_{pq}$. Precisely,

Lemma 5. If $K(\mathrm{UCG}(pq)) \cong (\mathbb{Z}/s_1\mathbb{Z}) \oplus (\mathbb{Z}/s_2\mathbb{Z}) \oplus \ldots (\mathbb{Z}/s_r\mathbb{Z})$ with $s_1 \mid s_2 \mid \cdots \mid s_r$, then

$$s_r \mid (pq-p-q)(p-1)(q-1)pq.$$

With the help of Lemmas 4 and 5, we can first decompose $K(\mathrm{UCG}(pq))$ into direct sum of coprime components for certain values of p and q.

Proposition 2. If $q \not\equiv 1 \pmod{p}$, then for d the largest common factor of $p-1, q-1$ such that $\left(d, \frac{p-1}{d}\right) = \left(d, \frac{q-1}{d}\right) = 1$,

$$K(\mathrm{UCG}(pq)) \cong (\mathbb{Z}/pq-p-q\mathbb{Z})^{(p-1)(q-1)} \oplus (\mathbb{Z}/d\mathbb{Z})^{p+q-2} \oplus \left(\mathbb{Z}/\left(\frac{p-1}{d}\right)\mathbb{Z}\right)^{q-1} \oplus \left(\mathbb{Z}/\left(\frac{q-1}{d}\right)\mathbb{Z}\right)^{p-1} \oplus (\mathbb{Z}/p\mathbb{Z})^{p-2} \oplus (\mathbb{Z}/q\mathbb{Z})^{q-2}.$$

Proof. Let $p-1 = dP$ and $q-1 = dQ$. Note that for d to satisfy the above condition, for all prime factors t of d, $\nu_t(p) = \nu_t(q)$. Now let $K(\mathrm{UCG}(pq)) \cong (\mathbb{Z}/d_1\mathbb{Z})^{u_1} \oplus (\mathbb{Z}/d_2\mathbb{Z})^{u_2} \oplus \cdots \oplus (\mathbb{Z}/d_r\mathbb{Z})^{u_r}$, where $d_1, d_2, \ldots, d_r$ are powers of distinct primes. Then by Lemma 5, d_i divides s_r for each $1 \le i \le r$, so d_i divdes $(pq-p-q)(p-1)(q-1)pq$. It is clear that $pq-p-q, p-1, q-1$ are each coprime with p and q, and since $pq-p-q = (p-1)(q-1)-1$, it is also coprime with $p-1$ and $q-1$. Hence, by Lemma 5, this implies d_i either equals p or q, or is a prime power factor of $pq-p-q, d, P$ or Q. By Lemma 4, we can deduce that for each $1 \le i \le r$, if d_i divides $pq-p-q$, $u_i = (p-1)(q-1)$. Similarly, if $d_i = p, u_i = p-2$; if $d_i = q, u_i = q-2$; if $d_i = \frac{p-1}{d}, u_i = q-1$; if $d_i = \frac{q-1}{d}, u_i = p-1$. Finally, if d_i divides d, $u_i = p-1+q-1 = p+q-2$, taking the multiplicities from $(\mathbb{Z}/p\mathbb{Z})$ and $(\mathbb{Z}/q\mathbb{Z})$ together. Combining everything together, we have the desired result. ■

The proof above is unable to help with the case when $q \equiv 1 \pmod{p}$ because $q-1$ would also contain factors of p, and if $\nu_p(q-1) > 1$, we might have components of the form $(\mathbb{Z}/p^2\mathbb{Z})$ or even higher powers of p. So we need one more result before coming back to the case when $q \equiv 1 \pmod{p}$.

Lemma 6. If $q \equiv 1 \pmod{p}$, there are exactly $p-1$ invariant factors in $K(\mathrm{UCG}(pq))$ that are divisible by p.

Proof. We know that the number of invariant factors that are not congruent to 0 $\pmod{p}$ is equal to the rank of $\mathbf{L}_{pq}$ over $\mathbb{F}_p$, the finite field of order p. So the number of invariant factors that are congruent to $0 \pmod{p}$ is equal to pq minus the rank of $\mathbf{L}_{pq}$ over $\mathbb{F}_p$. Next, consider the matrix $\mathbf{L}_{pq} - p\mathbf{I}$. Since we are working over $\mathbb{F}_p$, the rank remains unchanged after the transformation, and all eigenvalues are subtracted by p. Now we look at the eigenvalues of $\mathbf{L}_{pq} - p\mathbf{I}$ that are multiples of p. They are $p(q-1)-p = p(q-2)$ with multiplicity $p-1$, and $-p$ with multiplicity 1. This means $\nu_p(\det(\mathbf{L}_{pq} - p\mathbf{I})) = p$. Hence, there are at most p elementary factors in $\mathbf{L}_{pq} - p\mathbf{I}$ that are divisible by p, and so so there are at most p invariant factors in $\mathbf{L}_{pq} - p\mathbf{I}$ that are divisible by p.

On the other hand, we claim that there are at least p invariant factors in $\mathbf{L}_{pq} - p\mathbf{I}$ that are divisible by p. Indeed, recall that $\det(\mathbf{L}_{pq}) = (pq-p-q)^{(p-1)(q-1)}(p-1)^{q-1}(q-1)^{p-1}p^{p-2}q^{q-2}$, which means

$$\nu_p(\det(\mathbf{L}_{pq})) = (p-1)\nu_p(q-1) + p - 2.$$

By Lemma 5, for each invariant factor s_i, $1 \le i \le r$, $\nu_p(s_i) \le \nu_p(q-1)+1$. We cannot have fewer than $p-1$ invariant factors that are divisible by p, then their total p-adic valuation would be at most $(p-2)\nu_p(q-1)+p-2$, which is strictly smaller than what we have. Therefore, we must have at least $p-1$ non-zero invariant factors that are divisible by p. Going back to $\mathbf{L}_{pq}-p\mathbf{I}$, there are at least p invariant factors that are divisible by p. This implies there are exactly p invariant factors in $\mathbf{L}_{pq}-p\mathbf{I}$ that are divisible by p. Since the ranks of $\mathbf{L}_{pq}-p\mathbf{I}$ and $\mathbf{L}_{pq}$ are equal, we know there are exactly $p-1$ (not including the 0) invariant factors in the critical group that are divisible by p. ■

With this lemma, we are ready to determine the critical group of UCG(pq) completely.

Theorem 1. For primes $p<q$, if d the largest common factor of $p-1, q-1$ such that $\left(d, \dfrac{p-1}{d}\right)=\left(d, \dfrac{q-1}{d}\right)=1$,

$$\begin{aligned}&K(\mathrm{UCG}(pq))\\&\cong(\mathbb{Z}/pq-p-q\mathbb{Z})^{pq-2p-2q+3}\oplus(\mathbb{Z}/d(pq-p-q)\mathbb{Z})^{p-1}\\&\oplus\mathbb{Z}/(p-1)(pq-p-q)\mathbb{Z}\oplus(\mathbb{Z}/q(p-1)(pq-p-q)\mathbb{Z})^{q-p-1}\\&\oplus\mathbb{Z}/\frac{q(p-1)(q-1)(pq-p-q)}{d}\mathbb{Z}\oplus(\mathbb{Z}/\frac{pq(p-1)(q-1)(pq-p-q)}{d}\mathbb{Z})^{p-2}.\end{aligned}$$

Proof. From Proposition and Lemma, we know that for all primes $p<q$,

$$\begin{aligned}K(\mathrm{UCG}(pq))\cong(\mathbb{Z}/(pq-p-q)\mathbb{Z})^{(p-1)(q-1)}\oplus(\mathbb{Z}/d\mathbb{Z})^{p+q-2}\oplus(\mathbb{Z}/\frac{p-1}{d}\mathbb{Z})^{q-1}\\\oplus(\mathbb{Z}/\frac{q-1}{d}\mathbb{Z})^{p-1}\oplus(\mathbb{Z}/p\mathbb{Z})^{p-2}\oplus(\mathbb{Z}/q\mathbb{Z})^{q-2}.\end{aligned}$$

For $K(\mathrm{UCG}(pq))$ to be isomorphic to the form $(\mathbb{Z}/s_1\mathbb{Z})^{u_1}\oplus(\mathbb{Z}/s_2\mathbb{Z})^{u_2}\oplus(\mathbb{Z}/s_3\mathbb{Z})^{u_3}\oplus(\mathbb{Z}/s_4\mathbb{Z})^{u_4}\oplus(\mathbb{Z}/s_5\mathbb{Z})^{u_5}\oplus(\mathbb{Z}/s_6\mathbb{Z})^{u_6}$, with $s_1\mid s_2\mid\cdots\mid s_6$, consider the multiplicities of invariant factors in descending order. We have $(p-1)(q-1)>p+q-2>q-1>q-2>p-1>p-2$. Then following this order, we know that for $1\le i\le 6, u_i$ satisfy

$$\begin{aligned}u_1+u_2+u_3+u_4+u_5+u_6&=(p-1)(q-1)\\u_2+u_3+u_4+u_5+u_6&=p+q-2\\u_3+u_4+u_5+u_6&=q-1\\u_4+u_5+u_6&=q-2\\u_5+u_6&=p-1\\u_6&=p-2\end{aligned}$$

Solving, we have $u_1=pq-2p-2q+3, s_2=p-1, s_3=1, s_4=q-p-1, s_5=1, s_6=p-2$. Also, we have

$$\begin{aligned}s_1&=pq-p-q\\s_2&=d(pq-p-q)\\s_3&=(p-1)(pq-p-q)\\s_4&=q(p-1)(pq-p-q)\end{aligned}$$

$$s_5 = \frac{q(p-1)(q-1)(pq-p-q)}{d}$$
$$s_6 = \frac{pq(p-1)(q-1)(pq-p-q)}{d}$$

Putting them together we get the desired result. ■

Example. When $n = 2p$, where $p \geq 3$ is a prime,

$$K(\mathrm{UCG}(n)) \cong \mathbb{Z}/\left(p-2\right)\mathbb{Z} \oplus (\mathbb{Z}/p(p-2)\mathbb{Z})^{p-3} \oplus \mathbb{Z}/\left(p(p-1)(p-2)\right)\mathbb{Z}.$$

Example. When $n = 3p$, where $p \geq 5$ is a prime,

$$\begin{aligned} K(\mathrm{UCG}(n)) \cong\ & (\mathbb{Z}/2p-3\mathbb{Z})^{p-3} \oplus (\mathbb{Z}/2(2p-3)\mathbb{Z})^3 \oplus (\mathbb{Z}/2p(2p-3)\mathbb{Z})^{p-4} \\ & \oplus \mathbb{Z}/\left(p(p-1)(2p-3)\right)\mathbb{Z} \oplus \mathbb{Z}/\left(3p(p-1)(2p-3)\right)\mathbb{Z}. \end{aligned}$$

Example. When $n = 5p$, where $p \geq 7$ is a prime,

$$\begin{aligned} \text{if } p \equiv 1 \pmod 4, K(\mathrm{UCG}(n)) \cong & (\mathbb{Z}/4p-5\mathbb{Z})^{3p-7} \oplus (\mathbb{Z}/4(4p-5))\mathbb{Z})^5 \\ & \oplus (\mathbb{Z}/4p(4p-5)\mathbb{Z})^{p-6} \oplus \mathbb{Z}/\left(p(p-1)(4p-5)\right)\mathbb{Z} \\ & \oplus (\mathbb{Z}/5p(p-1)(4p-5)\mathbb{Z})^3; \\ \text{if } p \equiv 3 \pmod 4, K(\mathrm{UCG}(n)) \cong & (\mathbb{Z}/4p-5\mathbb{Z})^{3p-7} \oplus (\mathbb{Z}/2(4p-5))\mathbb{Z})^4 \\ & \oplus \mathbb{Z}/\left(4(4p-5)\right)\mathbb{Z} \oplus (\mathbb{Z}/4p(4p-5)\mathbb{Z})^{p-6} \\ & \oplus \mathbb{Z}/\left(2p(p-1)(4p-5)\right)\mathbb{Z} \\ & \oplus (\mathbb{Z}/10p(p-1)(4p-5)\mathbb{Z})^3. \end{aligned}$$

Next, we determine the decomposition of the critical group of $\mathrm{UCG}(n)$ when $n = p^k$, where p is a prime, and $k \in \mathbb{N}$. Proposition 3 settles the case when $k = 1$.

Proposition 3.

$$K(\mathrm{UCG}(p)) \cong (\mathbb{Z}/p\mathbb{Z})^{p-2}.$$

Proof. By definition, the Laplacian matrix L_p of the complete graph is

$$\begin{pmatrix} p-1 & -1 & -1 & \dots & -1 \\ -1 & p-1 & -1 & \dots & -1 \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & -1 & -1 & \dots & p-1 \end{pmatrix}$$

Let the SNF of the matrix be $\mathrm{diag}(s_1, s_2, \dots, s_{p-1}, 0)$, with $s_1 \mid s_2 \mid \cdots \mid s_{p-1}$. Then for $u_r = \prod_{i=1}^{r} s_i$, where $1 \le r \le p-1$, u_r is the greatest common divisor of all $r \times r$ minors of the Laplacian matrix. First, it is clear that $u_1 = 1$, so $s_1 = 1$. Next, note that the 2×2 submatrices must be one of the following forms:

$$\begin{pmatrix} p-1 & -1 \\ -1 & p-1 \end{pmatrix}, \begin{pmatrix} p-1 & -1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & p-1 \\ -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & -1 \\ p-1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & p-1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Then the 2×2 minors must be either $p^2 - 2p, p, -p$ or 0, and their GCD is p. So $s_1 s_2 = p$, i.e. $s_2 = p$. Since $s_2 \mid s_j$, for all $3 \le j \le p-1$, $s_j \ge p$. This implies the GCD of all $(p-1) \times (p-1)$ minors is at least p^{p-2}. On the other hand, for the $(p-1) \times (p-1)$ submatrix M of L_p, notice that by subtracting each of rows 1 to $p-2$ by row $p-1$, and then adding each of columns 1 to $p-2$ to row $p-1$, we have

$$\begin{aligned} \det(M) &= \begin{vmatrix} p & 0 & 0 & \dots & 0 & -p \\ 0 & p & 0 & \dots & 0 & -p \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p & -p \\ -1 & -1 & -1 & \dots & -1 & p-1 \end{vmatrix} \\ &= \begin{vmatrix} p & 0 & 0 & \dots & 0 & 0 \\ 0 & p & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p & 0 \\ -1 & -1 & -1 & \dots & -1 & 1 \end{vmatrix} \\ &= p^{p-2} \end{aligned}$$

Therefore the GCD of all $(p-1) \times (p-1)$ minors is exactly p^{p-2}, which means $s_j = p$ for every $2 \le j \le p-1$. That means the SNF is $\mathrm{diag}(1, p, p, \dots, p, 0)$, and so the critical group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{p-2}$. ∎

Next we derive the decomposition form of $K(\mathrm{UCG}(p^k))$ when $k \ge 2$.

Theorem 2. When $k \geq 2$,

$$K(\mathrm{UCG}(p^k)) \cong \mathbb{Z}/\,(p-1)\,\mathbb{Z} \oplus (\mathbb{Z}/(p-1)p^{k-1}\mathbb{Z})^{p^k-2p} \oplus \mathbb{Z}/(p-1)p^{2k-2}\mathbb{Z} \oplus (\mathbb{Z}/(p-1)p^{2k-1}\mathbb{Z})^{p-2}.$$

Proof. Denote $\mathbf{L}_{p^k}$ the Laplacian of the required graph. Consider $\mathbf{L}_{p^k}$ as a $\mathbb{Z}$-linear operator $L : \mathbb{Z}^{p^k} \to \mathbb{Z}^{p^k}$. $\operatorname{coker}(\mathbf{L}_{p^k})$ is an abelian group generated by elements $x_1, x_2, \ldots, x_{p^k}$, such that by defining

$$a_{qi} = x_{pi-p+l}, 1 \leq q \leq p, 1 \leq i \leq p^{k-1},$$

and $a_{0i} = a_{pi}$, they satisfy equations $R(q,i)$, where $1 \leq q \leq p, 1 \leq i \leq p^{k-1}$, defined by

$$R(q,i) : (p-1)p^{k-1}a_{qi} = \sum_{j=1}^{p^{k-1}} \sum_{1 \leq m \leq p, m \neq q} a_{mj}.$$

This holds because for each vertex v, when a chip-firing move is performed on it, its degree decreases by $\deg(v) = \varphi(p^k) = (p-1)p^{k-1}$, and all neighbours of v increase by 1. Since the critical group is the torsion part of the cokernel of the Laplacian, the expressions arose above are equal to 0 here. To study its structure, we reduce equations $R(q,i)$. For each $2 \leq r \leq p^{k-1}$, by putting $R(q,1)$ into $R(q,r)$,

$$(p-1)p^{k-1}a_{q1} = (p-1)p^{k-1}a_{qr} = \sum_{j=1}^{p^{k-1}} \sum_{1 \leq m \leq p, m \neq q} a_{mj},$$

so we have equations $S(q,r) : (p-1)p^{k-1}(a_{q1} - a_{qr}) = 0$ for any $1 \leq q \leq p, 2 \leq r \leq p^{k-1}$.

For every $2 \leq q \leq p$, subtracting each of $R(q,1)$ by $R(1,1)$ gives equations $U(q)$, defined by

$$U(q) : (p-1)p^{k-1}a_{q1} - (p-1)p^{k-1}a_{11} + \sum_{j=1}^{p^{k-1}} (a_{qj} - a_{1j}) = 0$$

Rearranging $U(q)$ to make $a_{qp^{k-1}}$ the subject,

$$a_{qp^{k-1}} = (p-1)p^{k-1}(a_{11} - a_{q1}) + \sum_{j=1}^{p^{k-1}} a_{1j} - \sum_{j=1}^{p^{k-1}-1} a_{qj}$$

Then we put it into $S(q, p^{k-1})$, and since $(p-1)p^{k-1}(a_{11} - a_{1j}) = (p-1)p^{k-1}(a_{q1} - a_{qj}) = 0$,
we get

$$(p-1)p^{k-1}\left(a_{q1} - (p-1)p^{k-1}(a_{11} - a_{q1}) - \sum_{j=1}^{p^{k-1}} a_{1j} + \sum_{j=1}^{p^{k-1}-1} a_{qj}\right) = 0$$

$$(p-1)p^{k-1}\left(a_{q1} - (p-1)p^{k-1}(a_{11} - a_{q1}) - p^{k-1}a_{11} + (p^{k-1} - 1)a_{q1}\right) = 0$$

$$(p-1)p^{k-1}(a_{q1} - p(p^{k-1})a_{11} + (p(p^{k-1}) - 1)a_{q1}) = 0$$

$$(p-1)p^{2k-1}(a_{q1} - a_{11}) = 0$$

From what we have obtained above, we can rewrite $K(\mathrm{UCG}(p^k))$ as

$$\begin{aligned}\operatorname{coker}(\mathbf{L}_{p^k}) \cong \langle & a_{1i}, a_{qr} : 1 \le i \le p^{k-1}, 2 \le q \le p, 1 \le r \le p^{k-1}-1 \mid \\ & (p-1)p^{k-1}(a_{q1} - a_{qr}) = 0 : 1 \le q \le p, 2 \le r \le p^{k-1}-1; \\ & (p-1)p^{2k-1}(a_{q1} - a_{11}) = 0 : 2 \le q \le p; R(1,1)\rangle\end{aligned}$$

So it remains to work on $R(1,1)$. First, rewrite the equation as

$$(p-1)p^{k-1}a_{11} - \sum_{q=2}^{p}\left(a_{qp^k-1} + \sum_{j=1}^{p^{k-1}-1} a_{qj}\right) = 0$$

Put $U(q)$ into $R(1,1)$,

$$\begin{aligned}&(p-1)p^{k-1}a_{11} \\ &-\sum_{q=2}^{p}\left((p-1)p^{k-1}(a_{11}-a_{q1}) + \sum_{j=1}^{p^{k-1}} a_{1j} - \sum_{j=1}^{p^{k-1}-1} a_{qj} + \sum_{j=1}^{p^{k-1}-1} a_{qj}\right) &= 0 \\ &(p-1)p^{k-1}a_{11} - (p-1)p^{k-1}\sum_{q=2}^{p}(a_{11}-a_{q1}) - (p-1)\sum_{j=1}^{p^{k-1}} a_{1j} &= 0\end{aligned}$$

At this point, we are unable to make $a_{1p^{k-1}}$ as the subject of the equation because there is a coefficient $p-1$ that cannot be cancelled out. Here we let $\mathbf{M}$ be another Z-linear operator $\mathbf{M} : \mathbb{Z}^{p^k-p+1} \to \mathbb{Z}^{p^k-p+1}$ representing the reduced relations. Thus $\operatorname{coker}(\mathbf{L}_{p^k}) = \operatorname{coker}((p-1)\mathbf{M})$. It suffices to determine the Smith Normal Form of $(p-1)\mathbf{M}$. To proceed, note that $\operatorname{coker}(\mathbf{M})$ is isomorphic to

$$\begin{aligned}&\langle a_{1i}, a_{qr} : 1 \le i \le p^{k-1}, 2 \le q \le p, 1 \le r \le p^{k-1}-1 \mid \\ &p^{k-1}(a_{q1} - a_{qr}) = 0 : 1 \le q \le p, 2 \le r \le p^{k-1}-1; \\ &p^{2k-1}(a_{q1} - a_{11}) = 0 : 2 \le q \le p; T(1,1)\rangle\end{aligned}$$

where $T(1,1)$ is the equation

$$p^{k-1}a_{11} - p^{k-1}\sum_{q=2}^{p}(a_{11}-a_{q1}) - \sum_{j=1}^{p^{k-1}} a_{1j} = 0$$

$$a_{1p^{k-1}} = p^{k-1}a_{11} - p^{k-1}\sum_{q=2}^{p}(a_{11}-a_{q1}) - \sum_{r=1}^{p^{k-1}-1} a_{1r}.$$

Put $T(1,1)$ into $S(1,p^{k-1})$,

$$p^{k-1}\left(a_{11} - p^{k-1}a_{11} + \sum_{q=2}^{p} p^{k-1}(a_{11}-a_{q1}) + \sum_{r=1}^{p^{k-1}-1} a_{1r}\right) = 0$$

$$p^{k-1}\left((1 - p^{k-1} + (p-1)p^{k-1})a_{11} - p^{k-1}\sum_{q=2}^{p} a_{q1} + (p^{k-1}-1)a_{11}\right) = 0$$

$$(p-1)p^{2k-2}a_{11} - p^{2k-2}\sum_{q=2}^{p} a_{q1} = 0$$

$$p^{2k-2}\sum_{q=2}^{p}(a_{11} - a_{q1}) = 0.$$

Now coker($\mathbf{M}$) becomes

$$\begin{aligned}
&\langle a_{qr} : 1 \le q \le p, 1 \le r \le p^{k-1} - 1 \mid \\
&p^{k-1}(a_{q1} - a_{qr}) = 0 : 1 \le q \le p, 2 \le r \le p^{k-1} - 1; \\
&p^{2k-1}\,(a_{q1} - a_{11}) = 0 : 2 \le q \le p; \\
&p^{2k-2}\sum_{q=2}^{p}(a_{11} - a_{q1}) = 0\rangle.
\end{aligned}$$

For each $2 \le q \le p$, let $b_q = a_{q1} - a_{11}$. Then the torsion subgroup of coker($\mathbf{M}$) is isomorphic to

$$\begin{aligned}
&\langle a_{11}, a_{qr}, b_i : 1 \le q \le p, 2 \le r \le p^{k-1} - 1, 2 \le i \le p \mid \\
&p^{k-1}(b_q + a_{11} - a_{qr}) = 0 : 1 \le q \le p, 2 \le r \le p^{k-1} - 1; \\
&p^{2k-1}b_q = 0 : 2 \le q \le p; \\
&p^{2k-2}\sum_{q=2}^{p} b_q = 0\rangle.
\end{aligned}$$

For the equations $p^{2k-1}b_q = 0$, $2 \le q \le p$ and $p^{2k-2}\sum_{q=2}^{p} b_q = 0$, note that the sum of the first $q-1$ equations implies the last equation, which means one of the equations $p^{2k-1}b_q = 0$ is redundant. So, from all the equations, the torsion subgroup of coker($\mathbf{M}$) has a subgroup that is isomorphic to $(\mathbb{Z}/p^{k-1}\mathbb{Z})^{p(p^{k-1}-2)} \oplus \mathbb{Z}/p^{2k-2}\mathbb{Z} \oplus (\mathbb{Z}/p^{2k-1}\mathbb{Z})^{p-2}$. But since there are $p^k - p$ generators and the rank of the subgroup we obtained is $p^k - p + 1$, coker($\mathbf{M}$) is isomorphic to

$$\mathbb{Z} \oplus (\mathbb{Z}/p^{k-1}\mathbb{Z})^{p^k-2p} \oplus \mathbb{Z}/p^{2k-2}\mathbb{Z} \oplus (\mathbb{Z}/p^{2k-1}\mathbb{Z})^{p-2}.$$

Thus the Smith Normal Form of $\mathbf{M}$ is diag$(1, p^{k-1}, ..., p^{k-1}, p^{2k-2}, p^{2k-1}, ..., p^{2k-1}, 0)$ where there are $p^k - 2p$ occurrences of p^{k-1} and $p-2$ occurrences of p^{2k-1}. Now since the Smith Normal Form of $(p-1)\mathbf{M}$ is $(p-1)$ times that of $\mathbf{M}$, i.e. diag$(p-1, (p-1)p^{k-1}, ..., (p-1)p^{k-1}, (p-1)p^{2k-2}, (p-1)p^{2k-1}, ..., (p-1)p^{2k-1}, 0)$, where there are $p^k - 2p$ occurrences of $(p-1)p^{k-1}$ and $p-2$ occurrences of $(p-1)p^{2k-1}$. Thus, we have

$$\begin{aligned}
K(\mathrm{UCG}(p^k)) \cong\ &\mathbb{Z}/\,(p-1)\,\mathbb{Z} \oplus (\mathbb{Z}/(p-1)p^{k-1}\mathbb{Z})^{p^k-2p} \oplus \mathbb{Z}/(p-1)p^{2k-2}\mathbb{Z} \\
&\oplus (\mathbb{Z}/(p-1)p^{2k-1}\mathbb{Z})^{p-2},
\end{aligned}$$

as desired. ∎

Example. When $n = 2^k, k \geq 2$,

$$K(\mathrm{UCG}(2^k)) \cong (\mathbb{Z}/2^{k-1}\mathbb{Z})^{2^k-4} \oplus \mathbb{Z}/2^{2k-2}\mathbb{Z}.$$

Example. When $n = 3^k, k \geq 2$,

$$K(\mathrm{UCG}(3^k)) \cong \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/2(3^{k-1})\mathbb{Z})^{3^k-6} \oplus \mathbb{Z}/2(3^{2k-2})\mathbb{Z} \oplus \mathbb{Z}/2(3^{2k-1})\mathbb{Z}.$$

4. Critical Groups of Circulant Graphs with Fixed Jumps

In this section, we turn to compute the critical group for circulant graphs with fixed jumps, i.e. the connecting set of the Cayley graph built on $\mathbb{Z}/n\mathbb{Z}$ is independent of n. Hou, Woo and Chen computed explicitly the critical group for $C_n(1,2)$ in [6], which is isomorphic to

$$\mathbb{Z}/(n, F_n)\mathbb{Z} \oplus \mathbb{Z}/F_n\mathbb{Z} \oplus \mathbb{Z} \Big/ \frac{nF_n}{(n, F_n)}\, \mathbb{Z},$$

where F_n is the n^{th} Fibonacci number, and $F_1 = F_2 = 1$. In this paper, we focus on the other circulant graphs that connect vertices with distance at most 3, which are $C_n(1,3), C_n(2,3)$ and $C_n(1,2,3)$.

Consider the circulant graph $C_n(2,3)$, where every vertex labelled i is adjacent to the vertices $i \pm 2$ and $i \pm 3$, addition taken modulo n. Here $K(C_n(2,3))$ is the quotient group of $\mathbb{Z}^n$ by the subgroup spanned by $n-1$ elements expressing the toppling rules

$$x_{i-3} + x_{i-2} - 4x_i + x_{i+2} + x_{i+3},$$

where $x_i \in \mathbb{Z}^n$ is a vector with entries all equal to 0 except the i-th entry. To simplify the subgroup, choose vertex 6 as the root. Let $\bar{x}_i$ be the image of x_i in $C_n(2,3)$, then $\bar{x}_6 = 0$ in $C_n(2,3)$. Applying the toppling rule to other vertices, we have

$$(*) \qquad \begin{cases} \bar{x}_1 = 4\bar{x}_{n-2} - \bar{x}_{n-4} - \bar{x}_{n-5} - \bar{x}_n, \\ \bar{x}_2 = 4\bar{x}_{n-1} - \bar{x}_{n-3} - \bar{x}_{n-4} - \bar{x}_1, \\ \bar{x}_3 = 4\bar{x}_n - \bar{x}_{n-2} - \bar{x}_{n-3} - \bar{x}_2, \\ \bar{x}_4 = 4\bar{x}_1 - \bar{x}_{n-1} - \bar{x}_{n-2} - \bar{x}_3, \\ \bar{x}_5 = 4\bar{x}_2 - \bar{x}_n - \bar{x}_{n-1} - \bar{x}_4, \\ \bar{x}_i = 4\bar{x}_{i-3} - \bar{x}_{i-5} - \bar{x}_{i-6} - \bar{x}_{i-1} \quad \text{for each } 7 \leq i \leq n. \end{cases}$$

From this system, we can tell there are at most 5 generators for $C_n(2,3)$. Indeed, each $\bar{x}_i$ can be expressed as a linear combination of $\bar{x}_j$, where $1 \leq j \leq 5$. For $7 \leq i \leq n$, we define A_i, B_i, C_i, D_i, E_i such that

$$\bar{x}_i = A_i\bar{x}_1 + B_i\bar{x}_2 + C_i\bar{x}_3 + D_i\bar{x}_4 + E_i\bar{x}_5,$$

with initial conditions

$$(A_1, A_2, A_3, A_4, A_5, A_6) = (1,0,0,0,0,0)$$
$$(B_1, B_2, B_3, B_4, B_5, B_6) = (0,1,0,0,0,0)$$
$$(C_1, C_2, C_3, C_4, C_5, C_6) = (0,0,1,0,0,0)$$

$$(D_1, D_2, D_3, D_4, D_5, D_6) = (0, 0, 0, 1, 0, 0)$$
$$(E_1, E_2, E_3, E_4, E_5, E_6) = (0, 0, 0, 0, 1, 0)$$

and they each satisfy the recurrence relation

$$S_{i+6} + S_{i+5} - 4S_{i+3} + S_{i+1} + S_i = 0$$

where S is a placeholder for A, B, C, D and E.

Lemma 7. For all integers $n \geq 7$,

$$B_n = A_n + A_{n-1} \tag{1}$$
$$C_n = B_{n-1} = A_{n-1} + A_{n-2} \tag{2}$$
$$D_n = -B_{n+3} = -A_{n+3} - A_{n+2} \tag{3}$$
$$E_n = -B_{n+2} = -A_{n+2} - A_{n+1} \tag{4}$$
$$6 - n = A_n + 3A_{n-1} + 5A_{n-2} + 3A_{n-3} + A_{n-4} \tag{5}$$

Proof. We will proceed with induction on n.

For (1), since
$B_7 = 4B_4 - B_6 - B_2 - B_1 = -1$ and $A_7 = 4A_4 - A_6 - A_2 - A_1 = -1$ and $A_6 = 0$,
The base case is done.

For $n > 6$, suppose (1) holds for some positive integer k. Then, for $n = k + 1$,

$$\begin{aligned}
B_{k+1} &= 4B_{k-2} - B_k - B_{k-4} - B_{k-5} \\
&= 4(A_{k-2} + A_{k-3}) - (A_k + A_{k-1}) - (A_{k-4} + A_{k-5}) - (A_{k-5} + A_{k-6}) \\
&= -A_k - A_{k-1} + 4A_{k-2} + 4A_{k-3} - A_{k-4} - 2A_{k-5} - A_{k-6} \\
&= -(A_k + A_{k-1} - 4A_{k-3} + A_{k-5} + A_{k-6}) + 4A_{k-2} - 4A_{k-4} - A_{k-5} \\
&= 0 - (A_{k+1} + A_k - 4A_{k-2} + A_{k-4} + A_{k-5}) + A_{k+1} + A_k \\
&= A_{k+1} + A_k
\end{aligned}$$

Therefore by induction, (1) holds for all natural numbers $n \geq 7$.

For (2), since by calculation

$$C_7 = 4C_4 - C_6 - C_2 - C_1 = 0, C_8 = 4C_5 - C_7 - C_3 - C_2 = -1,$$

and

$$A_6 = 4A_3 - A_5 - A_1 - A_0 = 0, A_7 = 4A_4 - A_6 - A_2 - A_1 = -1,$$
$$A_8 = 4A_5 - A_7 - A_3 - A_2 = 1$$

The base case is done.

For $n > 6$, suppose (2) holds for some positive integer k. Then, for $n = k + 1$,

$$\begin{aligned}
C_{k+1} &= 4C_{k-2} - C_k - C_{k-4} - C_{k-5} \\
&= 4B_{k-3} - B_{k-1} - B_{k-5} - B_{k-6} \\
&= B_k = B_{(k+1)-1}
\end{aligned}$$

Therefore by induction, (2) holds for all $n \in \mathbb{N}$.
For (3), since

$$B_8 = 4B_5 - B_7 - B_3 - B_2 = 0, B_9 = 4B_6 - B_8 - B_4 - B_3 = 0,$$

The base case is done.

For $n > 6$, suppose (3) holds for some positive integer k. Then, for $n = k + 1$,

$$\begin{aligned} D_{k+1} &= 4D_{k-2} - D_k - D_{k-4} - D_{k-5} \\ &= 4(-B_{k+1}) - (-B_{k+3}) - (-B_{k-1}) - (-B_{k-2}) \\ &= -(4B_{k+1} - B_{k+3} - B_{k-1} - B_{k-2}) \\ &= B_{k+4} = B_{(k+1)+3} \end{aligned}$$

Therefore by induction, (3) holds for all $n \in \mathbb{N}$.

For (4), since

$$B_7 = 4B_4 - B_6 - B_2 - B_1 = -1 \text{ and } A_7 = 4A_4 - A_6 - A_2 - A_1 = -1$$

$$\text{and } A_6 = 0,$$

The base case is done.

For $n > 6$, suppose (4) holds for some positive integer k. Then, for $n = k + 1$,

$$\begin{aligned} E_{k+1} &= 4E_{k-2} - E_k - E_{k-4} - E_{k-5} \\ &= 4(-B_k) - (-B_{k+2}) - (-B_{k-2}) - (-B_{k-3}) \\ &= -(4B_k - B_{k+2} - B_{k-2} - B_{k-3}) \\ &= B_{k+3} = B_{(k+1)+2} \end{aligned}$$

Therefore by induction, (4) holds for all $n \in \mathbb{N}$.

For (5), it is easy to see that it holds for $n = 7$. The base case is done.

For $n > 7$, Suppose (5) holds for some positive integer k. Then, for $n = k + 1$,

$$\begin{aligned} &A_{n+1} + 3A_n + 5A_{n-1} + 3A_{n-2} + A_{n-3} \\ &= (4A_{k-2} - A_k - A_{k-4} - A_{k-5}) + 3A_n + 5A_{n-1} + 3A_{n-2} + A_{n-3} \\ &= 2A_k + 5A_{k-1} + 7A_{k-2} + A_{k-3} - A_{k-4} - A_{k-5} \\ &= 2(A_k + 3A_{k-1} + 5A_{k-2} + 3A_{k-3} + A_{k-4}) \\ &\quad - (A_{k-1} + 3A_{k-2} + 5A_{k-3} + 3A_{k-4} + A_{k-5}) \\ &= 2(6 - n) - (6 - (n - 1)) = 6 - (n + 1) \end{aligned}$$

Therefore by induction, (5) holds for all $n \in \mathbb{N}$. Hence we are all done. ∎

From $(*)$, iterating the relations to $\bar{x}_{n+i} = \bar{x}_i$, where $1 \le i \le 5$, we can build a new system between generators $\bar{x}_i$

$$\begin{cases} \bar{x}_1 = A_{n+1}\bar{x}_1 + B_{n+1}\bar{x}_2 + C_{n+1}\bar{x}_3 + D_{n+1}\bar{x}_4 + E_{n+1}\bar{x}_5 \\ \bar{x}_2 = A_{n+2}\bar{x}_1 + B_{n+2}\bar{x}_2 + C_{n+2}\bar{x}_3 + D_{n+2}\bar{x}_4 + E_{n+2}\bar{x}_5 \\ \bar{x}_3 = A_{n+3}\bar{x}_1 + B_{n+3}\bar{x}_2 + C_{n+3}\bar{x}_3 + D_{n+3}\bar{x}_4 + E_{n+3}\bar{x}_5 \\ \bar{x}_4 = A_{n+4}\bar{x}_1 + B_{n+4}\bar{x}_2 + C_{n+4}\bar{x}_3 + D_{n+4}\bar{x}_4 + E_{n+4}\bar{x}_5 \\ \bar{x}_5 = A_{n+5}\bar{x}_1 + B_{n+5}\bar{x}_2 + C_{n+5}\bar{x}_3 + D_{n+5}\bar{x}_4 + E_{n+5}\bar{x}_5 \end{cases}$$

Hence we have a relation matrix $M(2,3)$ between generators $\bar{x}_i$, $1 \le i \le 5$. By row and column reductions,

$$\begin{aligned}
&M(2,3) \\
&= \begin{bmatrix} A_{n+1}-1 & B_{n+1} & C_{n+1} & D_{n+1} & E_{n+1} \\ A_{n+2} & B_{n+2}-1 & C_{n+2} & D_{n+2} & E_{n+2} \\ A_{n+3} & B_{n+3} & C_{n+3}-1 & D_{n+3} & E_{n+3} \\ A_{n+4} & B_{n+4} & C_{n+4} & D_{n+4}-1 & E_{n+4} \\ A_{n+5} & B_{n+5} & C_{n+5} & D_{n+5} & E_{n+5}-1 \end{bmatrix} \\
&\sim \begin{bmatrix} A_{n+1}-1 & B_{n+1} & B_n & -B_{n+4} & -B_{n+3} \\ A_{n+2} & B_{n+2}-1 & B_{n+1} & -B_{n+5} & -B_{n+4} \\ A_{n+3} & B_{n+3} & B_{n+2}-1 & -B_{n+6} & -B_{n+5} \\ A_{n+4} & B_{n+4} & B_{n+3} & -B_{n+7}-1 & -B_{n+6} \\ A_{n+5} & B_{n+5} & B_{n+4} & -B_{n+8} & -B_{n+7}-1 \end{bmatrix} \\
&\sim \begin{bmatrix} A_{n+1}-1 & A_n+1 & B_n & -B_{n+4} & -B_{n+3} \\ A_{n+2} & A_{n+1}-1 & B_{n+1} & -B_{n+5} & -B_{n+4} \\ A_{n+3} & A_{n+2} & B_{n+2}-1 & -B_{n+6} & -B_{n+5} \\ A_{n+4} & A_{n+3} & B_{n+3} & -B_{n+7}-1 & -B_{n+6} \\ A_{n+5} & A_{n+4} & B_{n+4} & -B_{n+8} & -B_{n+7}-1 \end{bmatrix} \scriptstyle C2 \to C2-C1 \\
&\sim \begin{bmatrix} A_{n+1}-1 & A_n+1 & A_{n-1}-1 & B_{n+4} & B_{n+3} \\ A_{n+2} & A_{n+1}-1 & A_n+1 & B_{n+5} & B_{n+4} \\ A_{n+3} & A_{n+2} & A_{n+1}-1 & B_{n+6} & B_{n+5} \\ A_{n+4} & A_{n+3} & A_{n+2} & B_{n+7}+1 & B_{n+6} \\ A_{n+5} & A_{n+4} & A_{n+3} & B_{n+8} & B_{n+7}+1 \end{bmatrix} \begin{matrix} \scriptstyle C3 \to C3-C2 \\ \scriptstyle C4 \to -C4 \\ \scriptstyle C5 \to -C5 \end{matrix} \\
&\sim \begin{bmatrix} A_{n+1}-1 & A_n+1 & A_{n-1}-1 & B_{n+3} & B_{n+4} \\ A_{n+2} & A_{n+1}-1 & A_n+1 & B_{n+4} & B_{n+5} \\ A_{n+3} & A_{n+2} & A_{n+1}-1 & B_{n+5} & B_{n+6} \\ A_{n+4} & A_{n+3} & A_{n+2} & B_{n+6} & B_{n+7}+1 \\ A_{n+5} & A_{n+4} & A_{n+3} & B_{n+7}+1 & B_{n+8} \end{bmatrix} \scriptstyle C4 \leftrightarrow C5 \\
&\sim \begin{bmatrix} A_{n+1}-1 & A_n+1 & A_{n-1}-1 & B_{n+3} & 3A_{n+2}+n \\ A_{n+2} & A_{n+1}-1 & A_n+1 & B_{n+4} & 3A_{n+3}+n \\ A_{n+3} & A_{n+2} & A_{n+1}-1 & B_{n+5} & 3A_{n+4}+n \\ A_{n+4} & A_{n+3} & A_{n+2} & B_{n+6} & 3A_{n+5}+n \\ A_{n+5} & A_{n+4} & A_{n+3} & B_{n+7}+1 & 3A_{n+6}+n \end{bmatrix} \begin{matrix} \scriptstyle C5 \to C5+2C4+3C1+C2 \\ \scriptstyle C5 \to -C5 \end{matrix}
\end{aligned}$$

$$\sim \begin{bmatrix} A_{n+1}-1 & A_n+1 & A_{n-1}-1 & A_{n+2} & 3A_{n+2}+n \\ A_{n+2} & A_{n+1}-1 & A_n+1 & A_{n+3} & 3A_{n+3}+n \\ A_{n+3} & A_{n+2} & A_{n+1}-1 & A_{n+4} & 3A_{n+4}+n \\ A_{n+4} & A_{n+3} & A_{n+2} & A_{n+5} & 3A_{n+5}+n \\ A_{n+5} & A_{n+4} & A_{n+3} & A_{n+6} & 3A_{n+6}+n \end{bmatrix} C4 \to C4+C3+3C2+5C1+C5$$

$$\sim \begin{bmatrix} A_{n+1}-1 & A_n+1 & A_{n-1}-1 & A_{n+2} & n \\ A_{n+2} & A_{n+1}-1 & A_n+1 & A_{n+3} & n \\ A_{n+3} & A_{n+2} & A_{n+1}-1 & A_{n+4} & n \\ A_{n+4} & A_{n+3} & A_{n+2} & A_{n+5} & n \\ A_{n+5} & A_{n+4} & A_{n+3} & A_{n+6} & n \end{bmatrix} C5 \to C5-3C4$$

$$\sim \begin{bmatrix} A_{n+2} & A_{n+1}-1 & A_n+1 & A_{n-1}-1 & n \\ A_{n+3} & A_{n+2} & A_{n+1}-1 & A_n+1 & n \\ A_{n+4} & A_{n+3} & A_{n+2} & A_{n+1}-1 & n \\ A_{n+5} & A_{n+4} & A_{n+3} & A_{n+2} & n \\ A_{n+6} & A_{n+5} & A_{n+4} & A_{n+3} & n \end{bmatrix} \begin{matrix} C4 \leftrightarrow C3 \\ C3 \leftrightarrow C2 \\ C2 \leftrightarrow C1 \end{matrix}$$

$$\sim \begin{bmatrix} A_{n+2} & A_{n+1}-1 & A_n+1 & A_{n-1}-1 & n \\ A_{n+3} & A_{n+2} & A_{n+1}-1 & A_n+1 & n \\ A_{n+4} & A_{n+3} & A_{n+2} & A_{n+1}-1 & n \\ A_{n+5} & A_{n+4} & A_{n+3} & A_{n+2} & n \\ n & n & n & n & -13n \end{bmatrix} \begin{matrix} R5 \to R5+3R4+5R3+3R2+R1 \\ R5 \to -R5 \end{matrix}$$

To further solve for A_n, from the recurrence relation, we obtain the characteristic equation

$$\lambda^6 + \lambda^5 - 4\lambda^3 + \lambda + 1 = 0 \tag{6}$$
$$(\lambda-1)^2(\lambda^4 + 3\lambda^3 + 5\lambda^2 + 3\lambda + 1) = 0$$
$$\lambda^2 + 2 + \frac{1}{\lambda^2} + 3(\lambda + \frac{1}{\lambda}) + 5 - 2 = 0$$

Let $k = \lambda + \dfrac{1}{\lambda}$, we have

$$k^2 + 3k + 3 = 0$$
$$k = \frac{-3 \pm i\sqrt{3}}{2}.$$

Consider the equation

$$(*) \qquad \lambda^4 + 3\lambda^3 + 5\lambda^2 + 3\lambda + 1 = 0.$$

Note that if the z is a root of $(*)$, then the roots of $(*)$ are $z, \dfrac{1}{z}, \overline{z}, \dfrac{1}{\overline{z}}$.

Then, for $k_1, k_2, k_3, k_4, k_5, k_6 \in \mathbb{C}$, all solutions to the recurrence equation are in the form

$$k_1 z^n + k_2 z^{-n} + k_3 \overline{z}^n + k_4 \overline{z}^{-n} + k_5 n + k_6.$$

Now take z such that $z + z^{-1} = \dfrac{-3+i\sqrt{3}}{2}$. Let $z = e^{i\theta}$, where $\theta \in \mathbb{C}$.

Then $z+z^{-1}=2\cos\theta$, so $\cos\theta=\dfrac{-3+i\sqrt{3}}{4}$. Denote $\dfrac{-3+i\sqrt{3}}{4}=\alpha$.

$$\begin{aligned} z^n+z^{-n} &= 2\cos n\theta \\ &= 2T_n(\cos\theta) \\ &= 2T_n\left(\frac{-3+i\sqrt{3}}{4}\right) \\ &= 2T_n(\alpha) \end{aligned}$$

$z-z^{-1}=2i\sin\theta$. Then,

$$\begin{aligned} z^n-z^{-n} &= 2i\sin n\theta \\ &= 2i\sin\theta U_{n-1}(\cos\theta) \\ &= (z-z^{-1})U_{n-1}(\alpha) \end{aligned}$$

Similarly, $\overline{z}+\overline{z}^{-1} = \dfrac{-3-i\sqrt{3}}{4}=\overline{\alpha}$. Then,

$$\begin{aligned} \overline{z}^n+\overline{z}^{-n} &= 2T_n(\overline{\alpha}) \\ \overline{z}^n-\overline{z}^{-n} &= (\overline{z}-\overline{z}^{-1})U_{n-1}(\overline{\alpha}) \end{aligned}$$

Now note that

$$T_n(\overline{\alpha})=\overline{T_n(\alpha)} \text{ and } U_{n-1}(\overline{\alpha})=\overline{U_{n-1}(\alpha)},$$

and for any $w\in\mathbb{C}$,

$$\mathrm{Re}(w)=\frac{w+\overline{w}}{2} \text{ and } \mathrm{Im}(w)=\frac{w-\overline{w}}{2i}.$$

We define s_n,t_n,u_n,v_n the numbers satisfying $s_n+i\sqrt{3}t_n=4T_n(\alpha)-4$ and $u_n+i\sqrt{3}v_n=2U_{n-1}(\alpha)$. Then,

$$\begin{aligned} s_n &= 4\,\mathrm{Re}(T_n(\alpha))-4 \\ &= 4(\frac{T_n(\alpha)+\overline{T_n(\alpha)}}{2})-4 \\ &= 4(\frac{z^n+z^{-n}}{4}+\frac{\overline{z}^n+\overline{z}^{-n}}{4})-4 \\ &= z^n+z^{-n}+\overline{z}^n+\overline{z}^{-n}-4 \end{aligned}$$

$$\begin{aligned} t_n &= \frac{4}{\sqrt{3}}\,\mathrm{Im}(T_n(\alpha) \\ &= \frac{4}{\sqrt{3}}(\frac{T_n(\alpha)-\overline{T_n(\alpha)}}{2i}) \\ &= \frac{2}{\sqrt{3}}(\frac{z^n+z^{-n}-\overline{z}^n-\overline{z}^{-n}}{2}) \\ &= \frac{z^n+z^{-n}-\overline{z}^n-\overline{z}^{-n}}{i\sqrt{3}} \end{aligned}$$

$$\begin{aligned}
u_n &= 2\operatorname{Re}(U_{n-1}(\alpha) \\
&= 2(\frac{U_{n-1}(\alpha)+U_{n-1}(\overline{\alpha})}{2}) \\
&= \frac{z^n - z^{-n}}{z - z^{-1}} + \frac{\overline{z}^n - \overline{z}^{-n}}{\overline{z} - \overline{z}^{-1}}
\end{aligned}$$

$$\begin{aligned}
v_n &= \frac{2}{\sqrt{3}}(\operatorname{Im}(U_{n-1}(\alpha))) \\
&= \frac{2}{\sqrt{3}}(\frac{U_{n-1}(\alpha)-U_{n-1}(\overline{\alpha})}{2i}) \\
&= \frac{1}{i\sqrt{3}}(\frac{z^n - z^{-n}}{z - z^{-1}} + \frac{\overline{z}^n - \overline{z}^{-n}}{\overline{z} - \overline{z}^{-1}})
\end{aligned}$$

Therefore s_n, t_n, u_n, v_n can be expressed in the form $k_1 z^n + k_2 z^{-n} + k_3 \overline{z}^n + k_4 \overline{z}^{-n} + k_5 n + k_6$, i.e. they satisfy the recurrence relation

$$S_{i+6} + S_{i+5} - 4S_{i+3} + S_{i+1} + S_i = 0.$$

Lemma 8. For all $n \in \mathbb{N}$, s_n, t_n, u_n, v_n are all integers.

Proof. We prove this by induction on n. For $0 \leq n \leq 5$, we observe the following.

$$\begin{aligned}
(s_0, s_1, s_2, s_3, s_4, s_5) &= (0, -7, -5, 5, -21, 8) \\
(t_0, t_1, t_2, t_3, t_4, t_5) &= (0, 1, -3, 3, 3, -16) \\
(u_0, u_1, u_2, u_3, u_4, u_5) &= (0, 2, -3, 1, 6, -16) \\
(v_0, v_1, v_2, v_3, v_4, v_5) &= (0, 0, 1, -3, 4, 0).
\end{aligned}$$

For $n \geq 6$, we use the recurrence relation, $x_n + x_{n+1} - 4x_{n+3} + x_{n+5} + x_{n+6} = 0$, i.e.

$$x_n = -x_{n-1} + 4x_{n-3} - x_{n-5} - x_{n-6}.$$

By induction, we proved that s_n, t_n, u_n, v_n are integers for all n. ■

To compute the sizes of non-trivial components of $K(C_n(2,3))$, we look at the greatest common divisor (GCD) of minors of the relation matrix $M(2,3)$. In particular, the size of the smallest non-trivial component is simply the GCD of all entries of the relation matrix. With the help of Sage, we obtained that for all $n \in \mathbb{N}$,

$$\begin{aligned}
52A_{n+2} &= -2s_n + 12t_n - 11u_n + v_n - 4n \\
52A_{n+3} &= -s_n - 7t_n + 2u_n - 12v_n - 4n \\
52A_{n+4} &= 2u_n + 14v_n - 4n \\
52A_{n+5} &= s_n + 7t_n + 2u_n - 12v_n - 4n \\
52(A_{n+1} - 1) &= 10s_n - 8t_n + 15u_n + 27v_n - 4n \\
52(A_n + 1) &= -17s_n - 15t_n + 2u_n - 64v_n - 4n \\
52(A_{n-1} - 1) &= 8s_n + 56t_n - 50u_n + 66v_n - 4n
\end{aligned}$$

Let $d_n(2,3)$ be the required GCD. Then we have

$$\begin{aligned}
&d_n(2,3)\\
&= \gcd(A_{n+2}, A_{n+3}, A_{n+4}, A_{n+5}, A_{n+1}-1, A_n+1, A_{n-1}-1, n, -13n)\\
&= \frac{1}{52}\gcd(52A_{n+2}, 52A_{n+3}, 52A_{n+4}, 52A_{n+5},\\
&\qquad\qquad 52(A_{n+1}-1), 52(A_n+1), 52(A_{n-1}-1), 52n)
\end{aligned}$$

Proposition 4. For $d_n = \gcd(s_n, t_n, u_n, v_n, n)$,

$$d_n(2,3) = \frac{1}{52}\gcd(52d_n, -2s_n + 12t_n - 11u_n + v_n - 4n, 2u_n + 14v_n - 4n).$$

Proof. Note that

$$\begin{aligned}
52(A_{n-1}-1) + 2(52n) + 12(52A_{n+3}) + 9(52A_{n+4}) + 4(52A_{n+5}) &= 0\\
52(A_n+1) - 2(52n) - 4(52A_{n+2}) - 12(52A_{n+3}) - 8(52A_{n+4}) - 3(52A_{n+5}) &= 0\\
52(A_{n+1}-1) + 3(52A_{n+2}) + 5(52A_{n+3}) + 3(52A_{n+4}) + (52A_{n+5}) + 52n &= 0
\end{aligned}$$

Therefore $52(A_{n-1}-1)$, $52(A_n+1)$ and $52(A_{n+1}-1)$ can be ignored. Also,

$$\begin{aligned}
3(52A_{n+5}) + 10(52A_{n+3}) - 3(52(A_n+1)) - 5(52A_{n+4})&\\
-(52(A_{n+1}-1)) - 5(52A_{n+2}) + 52(A_{n-1}-1) &= 52s_n\\
52(A_{n+1}-1) + 5(52A_{n+2}) - 6(52A_{n+3})&\\
-52A_{n+4} - 52(A_{n-1}-1) + 2(52A_{n+5}) &= 52t_n\\
3(52A_{n+5}) - 5(52A_{n+3}) + 3(52A_{n+4}) - 52(A_{n-1}-1) &= 52u_n\\
2(52A_{n+4}) - 52A_{n+3} - 52A_{n+5} &= 52v_n
\end{aligned}$$

Then we are done. ∎

Next, we consider the circulant graph $C_n(1,3)$, where every vertex labelled i is adjacent to the vertices $i\pm1$ and $i\pm3$, addition taken modulo n. Here $K(C_n(2,3))$ is the quotient group of $\mathbb{Z}^n$ by the subgroup spanned by $n-1$ elements expressing the toppling rules

$$x_{i-3} + x_{i-1} - 4x_i + x_{i+1} + x_{i+3},$$

where $x_i \in \mathbb{Z}^n$ is a vector with entries all equal to 0 except the i-th entry. Using a similar argument to what we did on $C_n(2,3)$, we have at most 5 generators $\bar{x}_i$, $1 \le i \le 5$.

For $7 \le i \le n$, we define A_i, B_i, C_i, D_i, E_i such that

$$\bar{x}_i = A_i\bar{x}_1 + B_i\bar{x}_2 + C_i\bar{x}_3 + D_i\bar{x}_4 + E_i\bar{x}_5.$$

Using a similar induction argument, for all $n \in \mathbb{N}$, we know that

$$B_n = A_{n-1} \tag{7}$$
$$C_n = A_n + A_{n-2} \tag{8}$$
$$D_n = -C_{n+3} = -A_{n+3} - A_{n+1} \tag{9}$$
$$E_n = -B_{n+3} = -A_{n+2} \tag{10}$$

Then we can construct another relation matrix $M(1,3)$, similar to $M(2,3)$ for $C_n(2,3)$, on generators $\bar{x}_i$, $1 \le i \le 5$ here. By row and column reductions,

$$
\begin{aligned}
&M(1,3)\\
&= \begin{bmatrix} A_{n+1}-1 & B_{n+1} & C_{n+1} & D_{n+1} & E_{n+1} \\ A_{n+2} & B_{n+2}-1 & C_{n+2} & D_{n+2} & E_{n+2} \\ A_{n+3} & B_{n+3} & C_{n+3}-1 & D_{n+3} & E_{n+3} \\ A_{n+4} & B_{n+4} & C_{n+4} & D_{n+4}-1 & E_{n+4} \\ A_{n+5} & B_{n+5} & C_{n+5} & D_{n+5} & E_{n+5}-1 \end{bmatrix}\\
&\sim \begin{bmatrix} A_{n+1}-1 & A_n & A_{n+1}+A_{n-1} & -A_{n+4}-A_{n+2} & -A_{n+3} \\ A_{n+2} & A_{n+1}-1 & A_{n+2}+A_n & -A_{n+5}-A_{n+3} & -A_{n+4} \\ A_{n+3} & A_{n+2} & A_{n+3}+A_{n+1}-1 & -A_{n+6}-A_{n+4} & -A_{n+5} \\ A_{n+4} & A_{n+3} & A_{n+4}+A_{n+2} & -A_{n+7}-A_{n+5}-1 & -A_{n+6} \\ A_{n+5} & A_{n+4} & A_{n+5}+A_{n+3} & -A_{n+8}-A_{n+6} & -A_{n+7}-1 \end{bmatrix}\\
&\sim \begin{bmatrix} A_{n+2} & A_{n+1}-1 & A_n & A_{n-1}+1 & n \\ A_{n+3} & A_{n+2} & A_{n+1}-1 & A_n & n \\ A_{n+4} & A_{n+3} & A_{n+2} & A_{n+1}-1 & n \\ A_{n+5} & A_{n+4} & A_{n+3} & A_{n+2} & n \\ n & n & n & n & -10n \end{bmatrix}
\end{aligned}
$$

From the recurrence relation, we obtain the characteristic equation

$$\lambda^6+\lambda^4-4\lambda^3+\lambda^2+1=0. \tag{11}$$

Let $k=\lambda+\frac{1}{\lambda}$, we have

$$
\begin{aligned}
\lambda^6+\lambda^4-4\lambda^3+\lambda^2+1&=0\\
(\lambda-1)^2(\lambda^4+2\lambda^3+4\lambda^2+2\lambda+1)&=0\\
\lambda^2+2+\frac{1}{\lambda^2}+2(\lambda+\frac{1}{\lambda})+4-2&=0\\
k^2+2k+2&=0\\
k&=-1\pm i
\end{aligned}
$$

By a similar argument to the graph $C_n(2,3)$, we define $s_n+it_n=2T_n\left(\frac{-1+i}{2}\right)-2$ and $u_n+iv_n=U_{n-1}\left(\frac{-1+i}{2}\right)$, where $s_n,t_n,u_n,v_n\in\mathbb{Z}$. Then, using a similar computation method, we know

$$
\begin{aligned}
20A_{n+5}&=s_n+3t_n+2u_n-4v_n-2n\\
20A_{n+4}&=2u_n+6v_n-2n\\
20A_{n+3}&=-s_n-3t_n+2u_n-4v_n-2n\\
20A_{n+2}&=-2s_n+4t_n-8u_n-4v_n-2n\\
20(A_{n+1}-1)&=7s_n+t_n+2u_n+16v_n-2n\\
20(A_n+1)&=-4s_n-12t_n+22u_n-14v_n-2n\\
20(A_{n-1}+1)&=-15s_n+15t_n-38u_n-24v_n-2n
\end{aligned}
$$

Let $d_n(1,3)$ be the required GCD in this case. Again, using a similar argument, we have

$$\begin{aligned} d_n(1,3) &= \gcd(A_{n+2}, A_{n+3}, A_{n+4}, A_{n+5}, A_{n+1}-1, A_n+1, A_{n-1}+1, n, -10n) \\ &= \frac{1}{20}\gcd(20A_{n+2}, 20A_{n+3}, 20A_{n+4}, 20A_{n+5}, \\ &\qquad 20(A_{n+1}-1), 20(A_n+1), 20(A_{n-1}-1), 20n) \end{aligned}$$

Proposition 5. For $d_n = \gcd(s_n, t_n, u_n, v_n, n)$,

$$d_n(1,3) = \frac{1}{20}\gcd(20d_n, 2u_n + 6v_n + 18n, s_n + 3t_n + 10v_n, 10t_n + 10n)$$

Proof. Note that

$$\begin{aligned} 20(A_n+1) - 20n - 2(20A_{n+5}) - 3(20A_{n+4}) - 6(20A_{n+3}) &= 0 \\ 20(A_{n-1}+1) - 20n - 2(20A_{n+4}) - 3(20A_{n+3}) - 6(20A_{n+2}) &= 0 \\ 20(A_{n+1}-1) + 20n + (20A_{n+5}) + 2(20A_{n+4}) + 4(20A_{n+3}) + 2(20A_{n+2}) &= 0 \end{aligned}$$

Hence, $20(A_n+1)$, $20(A_{n-1}+1)$ and $20(A_{n+1}-1)$ can be ignored. Also,

$$\begin{aligned} -3(20n) - 2(20A_{n+5}) - 12(20A_{n+4}) - 10(20A_{n+3}) - 6(20A_{n+2}) &= 20s_n \\ 20n + 4(20A_{n+5}) + 4(20A_{n+4}) + 2(20A_{n+2}) &= 20t_n \\ 20n + 3(20A_{n+5}) + 4(20A_{n+4}) + 3(20A_{n+3}) &= 20u_n \\ -20A_{n+5} + 2(20A_{n+4}) - 20A_{n+3} &= 20v_n \\ 20n + 20A_{n+4} &= 2u_n + 6v_n + 18n \\ 20A_{n+4} - 20A_{n+3} &= s_n + 3t_n + 10v_n \\ 20n + 2(20A_{n+5}) + 2(20A_{n+4}) + 20A_{n+2} &= 10t_n + 10n \end{aligned}$$

Then we are done. ■

Finally, we consider the circulant graph $C_n(1,2,3)$, where every vertex labelled i is adjacent to the vertices $i \pm 1$, $i \pm 2$ and $i \pm 3$, addition taken modulo n. Here $K(C_n(1,2,3))$ is the quotient group of $\mathbb{Z}^n$ by the subgroup spanned by $n-1$ elements expressing the toppling rules

$$x_{i-3} + x_{i-2} + x_{i-1} - 6x_i + x_{i+1} + x_{i+2} + x_{i+3},$$

where $x_i \in \mathbb{Z}^n$ is a vector with entries all equal to 0 except the i-th entry. Using a similar argument to what we did on $C_n(2,3)$, we have at most 5 generators $\bar{x}_i$, $1 \le i \le 5$. For $7 \le i \le n$, we define A_i, B_i, C_i, D_i, E_i such that

$$\bar{x}_i = A_i\bar{x}_1 + B_i\bar{x}_2 + C_i\bar{x}_3 + D_i\bar{x}_4 + E_i\bar{x}_5.$$

Using a similar induction argument, for all $n \in \mathbb{N}$, We have

$$B_n = A_n + A_{n-1} \tag{12}$$
$$C_n = A_n + A_{n-1} + A_{n-2} \tag{13}$$
$$D_n = -C_{n+3} = -A_{n+3} - A_{n+2} - A_{n+1} \tag{14}$$
$$E_n = -B_{n+2} = -A_{n+2} - A_{n+1} \tag{15}$$

Then we can construct another relation matrix $M(1,2,3)$, similar to $M(2,3)$ for $C_n(2,3)$, on generators $\bar{x}_i$, $1 \le i \le 5$ here. By row and column reductions,

$$M(1,2,3) = \begin{bmatrix} A_{n+1}-1 & B_{n+1} & C_{n+1} & D_{n+1} & E_{n+1} \\ A_{n+2} & B_{n+2}-1 & C_{n+2} & D_{n+2} & E_{n+2} \\ A_{n+3} & B_{n+3} & C_{n+3}-1 & D_{n+3} & E_{n+3} \\ A_{n+4} & B_{n+4} & C_{n+4} & D_{n+4}-1 & E_{n+4} \\ A_{n+5} & B_{n+5} & C_{n+5} & D_{n+5} & E_{n+5}-1 \end{bmatrix}$$

$$\sim \begin{bmatrix} A_{n+1}-1 & A_n+1 & C_{n+1} & D_{n+1} & E_{n+1} \\ A_{n+2} & A_{n+1}-1 & C_{n+2} & D_{n+2} & E_{n+2} \\ A_{n+3} & A_{n+2} & C_{n+3}-1 & D_{n+3} & E_{n+3} \\ A_{n+4} & A_{n+3} & C_{n+4} & D_{n+4}-1 & E_{n+4} \\ A_{n+5} & A_{n+4} & C_{n+5} & D_{n+5} & E_{n+5}-1 \end{bmatrix}$$

$$\sim \begin{bmatrix} A_{n+2} & A_{n+1}-1 & A_n+1 & A_{n-1} & n \\ A_{n+3} & A_{n+2} & A_{n+1}-1 & A_n+1 & n \\ A_{n+4} & A_{n+3} & A_{n+2} & A_{n+1}-1 & n \\ A_{n+5} & A_{n+4} & A_{n+3} & A_{n+2} & n \\ n & n & n & n & -14n \end{bmatrix}$$

From the recurrence relation, we obtain the characteristic equation

$$\lambda^6 + \lambda^5 + \lambda^4 - 6\lambda^3 + \lambda^2 + \lambda + 1 = 0. \tag{16}$$

Let $k = \lambda + \frac{1}{\lambda}$, we have

$$\begin{aligned} \lambda^6 + \lambda^5 + \lambda^4 - 6\lambda^3 + \lambda^2 + \lambda + 1 &= 0 \\ (\lambda-1)^2(\lambda^4 + 3\lambda^3 + 6\lambda^2 + 3\lambda + 1) &= 0 \\ \lambda^2 + \frac{1}{\lambda^2} + 3(\lambda + \frac{1}{\lambda}) + 6 &= 0 \\ k^2 + 3k + 4 &= 0 \\ k &= \frac{-3 \pm i\sqrt{7}}{2} \end{aligned}$$

By a similar argument to the graph $C_n(2,3)$, we define

$$s_n + i\sqrt{7}t_n = 4T_n\left(\frac{-3+i\sqrt{7}}{4}\right) - 4$$

and

$$u_n + i\sqrt{7}v_n = 2U_n\left(\frac{-3+i\sqrt{7}}{4}\right)$$

, where s_n, t_n, u_n, v_n are integers. Then, using a similar computation method, we know that

$$\begin{aligned} 56A_{n+5} &= s_n + 7t_n + 2u_n - 14v_n - 4n \\ 56A_{n+4} &= 2u_n + 14v_n - 4n \\ 56A_{n+3} &= -s_n - 7t_n + 2u_n - 14v_n - 4n \end{aligned}$$

$$\begin{aligned}
56A_{n+2} &= -2s_n + 14t_n - 12u_n - 4n \\
56(A_{n+1} - 1) &= 11s_n - 7t_n + 16u_n - 56v_n - 4n \\
56(A_n + 1) &= -18s_n - 42t_n + 16u_n - 140v_n - 4n \\
56(A_{n-1}) &= -5s_n + 133t_n - 110u_n + 98v_n - 4n
\end{aligned}$$

Let $d_n(1,2,3)$ be the required GCD in this case. Again, using a similar argument, we have

$$\begin{aligned}
& d_n(1,2,3) \\
=& \gcd(A_{n+2}, A_{n+3}, A_{n+4}, A_{n+5}, A_{n+1} - 1, A_n + 1, A_{n-1}, n, -14n) \\
=& \frac{1}{56}\gcd(56A_{n+2}, 56A_{n+3}, 56A_{n+4}, 56A_{n+5}, 56(A_{n+1} - 1), 56(A_n + 1), 56A_{n-1}, 56n) \\
=& \frac{1}{56}\gcd(-2s_n + 14t_n - 12u_n - 4n, -s_n - 7t_n + 2u_n - 14v_n - 4n, 2u_n + 14v_n - 4n, \\
& s_n + 7t_n + 2u_n - 14v_n - 4n, 11s_n - 7t_n + 16u_n - 56v_n - 4n, \\
& -18s_n - 42t_n + 16u_n - 140v_n - 4n, -5s_n + 133t_n - 110u_n + 98v_n - 4n, 56n).
\end{aligned}$$

Proposition 6. For $d_n = \gcd(s_n, t_n, u_n, v_n, n)$,

$$d_n(1,2,3) = \frac{1}{56}\gcd(56d_n, 2u_n + 14v_n + 52n, s_n + 7t_n + 28v_n, 28t_n + 28v_n + 28n)$$

Proof. Note that

$$\begin{aligned}
56(A_{n-1}) - 56(A_n + 1) + 56(A_{n+1} - 1) + 4(56n)& \\
+7(56A_{n+5}) + 17(56A_{n+4}) + 31(56A_{n+3}) &= 0 \\
56(A_n + 1) - 2(56n) - 3(56A_{n+5}) - 8(56A_{n+4}) - 15(56A_{n+3}) - 3(56A_{n+2}) &= 0 \\
56(A_{n+1} - 1) + 56n + 56A_{n+5} + 3(56A_{n+4}) + 6(56A_{n+3}) + 3(56A_{n+2}) &= 0
\end{aligned}$$

Hence $56(A_{n+1} - 1)$, $56(A_n + 1)$ and $56(A_{n-1})$ can be ignored. Also,

$$\begin{aligned}
-7(56n) - 7(56A_{n+5}) - 42(56A_{n+4}& \\
-35(56A_{n+3}) - 14(56A_{n+2}) &= 56s_n \\
56n + 5(56A_{n+5}) + 6(56A_{n+4}) + 56A_{n+3} + 2(56A_{n+2}) &= 56t_n \\
2(56n) + 7(56A_{n+5}) + 14(56A_{n+4}) + 7(56A_{n+3}) &= 56u_n \\
2(56A_{n+4}) - 56A_{n+5} - 56A_{n+3} &= 56v_n \\
56n + 56A_{n+4} &= 2u_n + 14v_n + 52n \\
56A_{n+4} - 56A_{n+3} &= s_n + 7t_n + 28v_n \\
56n + 2(56A_{n+5}) + 4(56A_{n+4}) + 56A_{n+2} &= 28t_n + 28v_n + 28n
\end{aligned}$$

Hence we are done. ■

5. Discussion

The p-adic approach used to compute $K(\mathrm{UCG}(pq))$ is actually supposed to work for abelian Cayley graphs in general, even it is not integral. However, the computation of p-adic valuation on non-integral eigenvalues (mostly not real numbers) is highly technical, and so we tried other methods to determine the critical group for circulant graphs with fixed jumps. In fact, Mednykh suggested much simpler expressions for the first few non-trivial components of $K(C_n(1,3)), K(C_n(2,3))$ and $K(C_n(1,2,3))$ in [7]. However, we could not find a complete proof in it, and obtained another expression instead. Nevertheless, we have tried to compute the actual values of the GCDs with a computer, and the numerical values obtained from our expressions matched those computed using the expressions Mednykh stated. (see Appendix). Hence, we believe one final step to completely obtain Mednykh's results is to prove that the two GCDs are actually equal with rigorous mathematical arguments. For $n \in \mathbb{N}$, $s_n, t_n, u_n, v_n \in \mathbb{Z}$ defined as above, let $d = \gcd(s_n, t_n, u_n, v_n, n)$. Then we conjecture the following:

Conjecture 1. For $d_n = \gcd(s_n, t_n, u_n, v_n, n)$,

$$\gcd(52d_n, -2s_n + 12t_n - 11u_n + v_n - 4n, 2u_n + 14v_n - 4n) = 52d_n.$$

Conjecture 2.

- If $4 \nmid n$, $\gcd(20d_n, 2u_n + 6v_n + 18n, s_n + 3t_n + 10v_n, 10t_n + 10n) = 20d_n$.
- If $4 \mid n$ and $8 \nmid n$, $\gcd(20d_n, 2u_n + 6v_n + 18n, s_n + 3t_n + 10v_n, 10t_n + 10n) = 5d_n$.
- If $8 \mid n$, $\gcd(20d_n, 2u_n + 6v_n + 18n, s_n + 3t_n + 10v_n, 10t_n + 10n) = 10d_n$.

Conjecture 3.

- If $6 \nmid n$, $\gcd(56d_n, 2u_n + 14v_n + 52n, s_n + 7t_n + 28v_n, 28t_n + 28v_n + 28n) = 56d_n$
- If $6 \mid n$, $\gcd(56d_n, 2u_n + 14v_n + 52n, s_n + 7t_n + 28v_n, 28t_n + 28v_n + 28n) = 28d_n$

From the results of our computation, we are actually quite close to a much simpler expression. In fact, we are also able to prove that they are equal for some values of n. Indeed, for Conjecture 1, it suffices to prove that $52d_n \mid -2s_n + 12t_n - 11u_n + v_n - 4n = 52A_{n+2}$ and $52d_n \mid 2u_n + 14v_n - 4n = 52A_{n+4}$. If $(d_n, 52) = 1$, we are done because by definition, 52 divides $52A_{n+2}$ and $52A_{n+4}$, and d_n divides s_n, t_n, u_n, v_n, n because it is their GCD. With this we can already show that for all $n \in \mathbb{N}$ with $(n, 52) = 1$, i.e. $n \equiv \pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11 \pmod{26}$, Conjecture 1 is true. This is because if n does not share any prime factor with 52, none of its divisors will either. Using a similar argument, we can also show that Conjecture 2 is true for all n that is coprime with 20, i.e. $n \equiv \pm 1, \pm 3 \pmod{10}$, and Conjecture 3 is true for all n that is coprime with 56, i.e. $n \equiv \pm 1, \pm 3, \pm 5 \pmod{14}$.

6. Acknowledgement

We would like to express our sincere gratitude to Dr. Chi Ho Yuen and Mr. Yuen Ho Wong for their continuous guidance and advice. We would not have completed this project without their help.

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Appendix

$C_n(2,3)$

n	d	$R1$	$R2$	$gcd(d, R1, R2)$	$d(2,3)$
7	1	10	14	1	1
8	1	-17	-39	1	1
9	1	8	68	1	1
10	2	32	-12	2	2
11	1	-96	-147	1	1
12	3	120	396	3	3
13	1	15	-419	1	1
14	1	-390	-198	1	1
15	1	830	1726	1	1
16	1	-706	-3243	1	1
17	1	-878	2202	1	1
18	1	4063	4725	1	1
19	1	-6512	-17080	1	1
20	4	2560	24360	4	4
21	1	13568	-3943	1	1
22	1	-38032	-63336	1	1
23	1	45087	153849	1	1
24	3	11634	-157266	3	3
25	1	-159810	-103358	1	1
26	1	324030	698337	1	1
27	1	-253030	-1260122	1	1
28	1	-393265	756177	1	1
29	1	1632664	2040588	1	1
30	2	-2496608	-6820452	2	2

Where

$$R1 = \frac{1}{52}(s_n + 7t_n + 26v_n)$$

$$R2 = \frac{1}{52}(26t_n + u_n + 33v_n + 24n)$$

$C_n(1,3)$

n	d	$R3$	$R4$	$R5$	$gcd(d, R3, R4, R5)$	$d(1,3)$
7	1	7	4	24	1	1
8	8	16	8	-12	4	4
9	1	-7	-24	-31	1	1
10	1	6	12	104	1	1
11	1	58	51	-39	1	1
12	4	-52	-111	-214	1	1
13	1	-34	17	462	1	1
14	1	262	295	-34	1	1
15	1	-225	-488	-1248	1	1
16	16	-352	-128	1992	8	8
17	1	1249	1600	689	1	1
18	1	-758	-2008	-6736	1	1
19	1	-2398	-1641	8065	1	1
20	4	5844	8241	7534	1	1
21	1	-1658	-7503	-34450	1	1
22	1	-14326	-12669	29470	1	1
23	1	26183	40508	55832	1	1
24	8	2608	-23576	-168068	4	4
25	1	-79431	-82040	88433	1	1
26	1	110606	190036	354392	1	1
27	1	65338	-45269	-782087	1	1
28	4	-416612	-481951	137938	1	1
29	1	430334	846945	2055390	1	1
30	15	564750	134415	-3452610	15	15

Where

$$R3 = \frac{1}{20}(2u_n + 6v_n + 18n)$$

$$R4 = \frac{1}{20}(s_n + 3t_n + 10v_n)$$

$$R5 = \frac{1}{20}(10t_n + 10n)$$

$C_n(1, 2, 3)$

n	d	$R6$	$R7$	$R8$	$gcd(d, R6, R7, R8)$	$d(1, 2, 3)$
7	7	21	21	49	7	7
8	1	1	-21	-125	1	1
9	1	-40	-42	119	1	1
10	1	156	195	295	1	1
11	1	-135	-292	-1210	1	1
12	4	-282	-148	1698	2	2
13	1	1371	1652	1239	1	1
14	7	-2016	-3388	-10486	7	7
15	1	-1028	987	20231	1	1
16	1	11531	12558	-3094	1	1
17	1	-23553	-35085	-81780	1	1
18	2	6781	30333	213809	1	1
19	1	87631	80849	-161577	1	1
20	1	-244217	-331849	-549563	1	1
21	7	210322	454538	2058637	7	7
22	1	564922	354599	-2607471	1	1
23	1	-2311405	-2876328	-2718996	1	1
24	24	3158220	5469624	18175620	12	12
25	1	2488981	-669240	-32448947	1	1
26	1	-20046658	-22535640	-942252	1	1
27	1	38043158	58089815	145712749	1	1
28	7	-4474351	-42517510	-352313332	7	7
29	1	-157184551	-152710201	217939394	1	1
30	2	404317307	561501857	1023866011	1	1

Where

$$R6 = \frac{1}{56}(2u_n + 14v_n + 52n)$$

$$R7 = \frac{1}{56}(s_n + 7t_n + 28v_n)$$

$$R8 = \frac{1}{56}(28t_n + 28v_n + 28n)$$

Code Appendix. All codes are compiled on https://sagecell.sagemath.org/

Code for calculating s_n, t_n, u_n, v_n of the graph $C_n(2,3)$

```
s=[0,-7,-5,5,-21,8]
t=[0,1,-3,3,3,-16]
u=[0,2,-3,1,6,-16]
v=[0,0,1,-3,4,0]
for i in range(6,1000):
    s.append(4*s[i-3]-s[i-5]-s[i-6]-s[i-1])
    t.append(4*t[i-3]-t[i-5]-t[i-6]-t[i-1])
    u.append(4*u[i-3]-u[i-5]-u[i-6]-u[i-1])
    v.append(4*v[i-3]-v[i-5]-v[i-6]-v[i-1])

for i in range(0,1000):
    print(i,s[i],t[i],u[i],v[i])
```

Code for calculating s_n, t_n, u_n, v_n of the graph $C_n(1,3)$

```
s=[0,-3,-4,3,-4,-13]
t=[0,1,-2,-1,8,-9]
u=[0,1,-1,-1,4,-3]
v=[0,0,1,-2,0,6]
for i in range(6,1000):
    s.append(4*s[i-3]-s[i-4]-s[i-6]-s[i-2])
    t.append(4*t[i-3]-t[i-4]-t[i-6]-t[i-2])
    u.append(4*u[i-3]-u[i-4]-u[i-6]-u[i-2])
    v.append(4*v[i-3]-v[i-4]-v[i-6]-v[i-2])

for i in range(0,1000):
    print(i,s[i],t[i],u[i],v[i])
```

Code for calculating s_n, t_n, u_n, v_n of the graph $C_n(1,2,3)$

```
s=[0,-7,-7,14,-35,-7]
t=[0,1,-3,2,9,-31]
u=[0,2,-3,-1,15,-32]
v=[0,0,1,-3,3,6]
for i in range(6,1000):
    s.append(6*s[i-3]-s[i-4]-s[i-6]-s[i-2]-s[i-5]-s[i-1])
    t.append(6*t[i-3]-t[i-4]-t[i-6]-t[i-2]-t[i-5]-t[i-1])
    u.append(6*u[i-3]-u[i-4]-u[i-6]-u[i-2]-u[i-5]-u[i-1])
    v.append(6*v[i-3]-v[i-4]-v[i-6]-v[i-2]-v[i-5]-v[i-1])

for i in range(0,1000):
    print(i,s[i],t[i],u[i],v[i])
```

Code for calculating SNF of UCG(n)

```
def comp(n):
    a = [];
    for i in range(1,ceil(n/2)):
        if(gcd(i,n)==1):
            a.append(i)
```

```
    k = len(a)
    sk = 2*a[k-1]
    A = matrix(sk)
    A[sk-1,sk/2] = 2*k
    for i in range(0,sk-1):
        A[i,i+1] = 1
    for i in range(0,k):
        A[sk-1,sk/2-a[i]] = -1
        if(sk/2+a[i]<sk):
            A[sk-1,sk/2+a[i]] = -1
    S = A^n - matrix.identity(sk)
    return(S.smith_form()[0])
for i in range(3,139):
    print(i,comp(i),sep='\n')
```

Code for calculating reduced SNF of UCG(n), i.e. the n-th integer in the SNF is the product of the first n integers in the reduced SNF

```
def comp(n):
    a = [];
    for i in range(1,ceil(n/2)):
        if(gcd(i,n)==1):
            a.append(i)
    k = len(a)
    sk = 2*a[k-1]
    A = matrix(sk)
    A[sk-1,sk/2] = 2*k
    for i in range(0,sk-1):
        A[i,i+1] = 1
    for i in range(0,k):
        A[sk-1,sk/2-a[i]] = -1
        if(sk/2+a[i]<sk):
            A[sk-1,sk/2+a[i]] = -1
    S = (A^n - matrix.identity(sk)).smith_form()[0]
    b = [S[0][0]]
    for i in range(1,sk):
         b.append(S[i][i]/S[i-1][i-1])
    return(b)
for i in range(3,139):
    print(i,comp(i),'\n',sep='\n')
```

REVIEWERS' COMMENTS

This paper studies the intriguing subject of the critical group of a graph, a concept rooted in algebraic graph theory which establishes a link between the analysis of graphs and the exploration of abelian groups. It aims to explicitly determine the structure of such a group in two types: unitary circulant graphs, and circulant graphs of fixed jumps. The first type is broken into two cases: $n = pq$ and $n = p^k$ for primes p, q with $p < q$. The proof techniques here rely on computations using the Laplacian and applying Kirchoff's Matrix Tree Theorem. For the second case the authors consider three families of circulant graphs, each with the property that vertices with distance at most 3 are connected. The proof techniques here rely on direct computation, considering each of the three cases separately.

The paper generally receives good review from referees. Some suggested editorial improvement to explain what computations were being done and why they were done in such a way.