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Silver Award

Cutting Twisted Solid Tori (TSTs)

Team members:	Kai Lai Chan, Tsz Nam Chan, Ho Yin Lau, Kai Shing Mok, Yiu Shing Wong
Teacher:	Mr. Sai Hung Chan
School:	Sha Tin Government Secondary School

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Abstract. In the paper, we generalize the process of cutting a Möbius strip and similar strips to a larger extent than Möbius, Listing, Ball-Coxeter and Fatehi's papers. We generalize the object from a strip to a "twisted solid torus" (which we abbreviate to tst) and consider the result after cutting it. The Argand diagram, together with the usage of complex numbers, has been used to describe the lines in the cross section of tst. In our derivation, we have used a technique of checking the concurrence of lines defined by parametric equations by applying the concept of pole-polar duality from inversive geometry. Euler's celebrated formula on graphs has also been employed. Then we study the resultant objects formed from the cutting process and call them "knotted tst". We then deduce a general formula for the number of different knotted tsts. After that, we consider the links that are formed from the cutting of tsts, which we call "tst links". General forms of their braid words, Seifert matrices and Alexander polynomials are then deduced. Then we generalize the results further and consider cutting a tst in the form of a non-trivial knot and study the resultant links. Finally, we study the cutting of combinations of more than one tsts in the form of virtual knots, which we call "tst products", and derive a general formula for the result.

Index of Notations

$d(\tau)$	a chosen positive denominator for the twist turn of the tst before cut (τ)
$D(\tau)$	denominator of the simplest expression of $\tau (D(\tau) = \dfrac{d(\tau)}{g(\tau)})$
$\varDelta$	one-half of the chosen denominator for the twist turn of tst before cut $(\varDelta = \dfrac{d(\tau)}{2})$
$g(\tau)$	the greatest common divisor of the chosen $d(\tau)$ and corresponding $n(\tau)$
G	partial tst of a tst $\varGamma$

γ	integral curves satisfying the slope field equation for a tst
$\mathit{\Gamma}$	tst; i.e. set of integral curves satisfying conditions on the initial value
I	number of type I knotted tst(s) in the knotted tst sum $\Phi(n(\tau), d(\tau), M)$
$l(s)$	a line in $\mathit{\Lambda}(n(\tau), d(\tau), M)$, the cross section of the tst at $t = 0$ after cutting
L	set of sets of lines resultant from the cutting of a tst (l)
$\mathit{\Lambda}$	cross section of tst at $t = 0$ after cutting
m	one-half of the number of portion(s) a given tst is cut into $(m = \frac{M}{2})$
M	number of portion(s) a given tst is cut into
μ	order of multiple cutting
$n(\tau)$	numerator of the twist turn of tst before cut under the chosen denominator $(n(\tau) = \tau d(\tau))$
$N(\tau)$	numerator of the simplest expression of the twist turn of tst before cut $(N(\tau) = \frac{n(\tau)}{g(\tau)})$
P	number of type II knotted tst(s) in the knotted tst sum $\Phi(n(\tau), d(\tau), M)$
Φ	resultant knotted tst sum of after Φ-cutting
σ	surface resultant from cutting
t	the real variable specifying the position along tst
τ	twist turn of tst before cut
$<\tau>$	tst with twist turn τ
θ	twisting function used to define a tst for a given twist turn
z	the complex variable specifying position in the cross section of tst
i, k, r, C, s, u	integral- or real-valued parameters

Convention

The sign conventions for knot or link crossings in our paper are as follows:

Positive

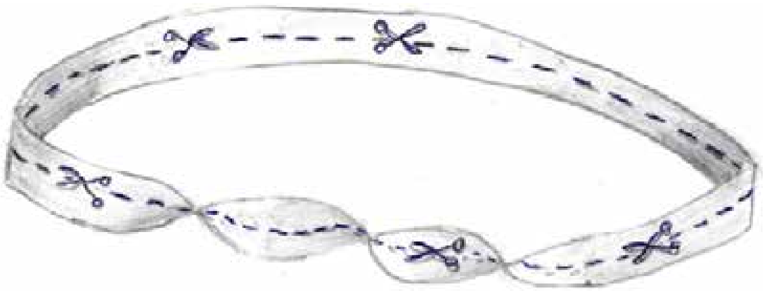
Negative

1. Introduction

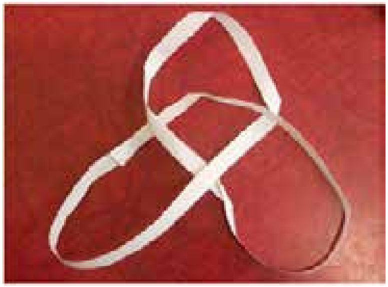
The Möbius strip was discovered by Möbius which was announced in 1858. It is formed by joining two edges of a rectangle with a half-twist, in either the clockwise or anticlockwise directions. It now becomes a well-known fact that when it is cut into half, it gives only 1 loop but not 2.

[Image taken from haggisthesheep.wordpress.com]

Listing and Tait generalized this idea in their book [5] in 1847 and the same idea appears in Ball-Coxeter's paper [1] in 1987. Instead of one half-twist, two or more half-twists are given to the strip. When these strips are cut along one-half, one-third, and one-quarter of the width, the results contain a pattern that depends on whether the number of half-twists given is odd or even. An example is given below: giving the strip three half twist, and then cut along half the width.
Here is the appearance of the strip before cut:



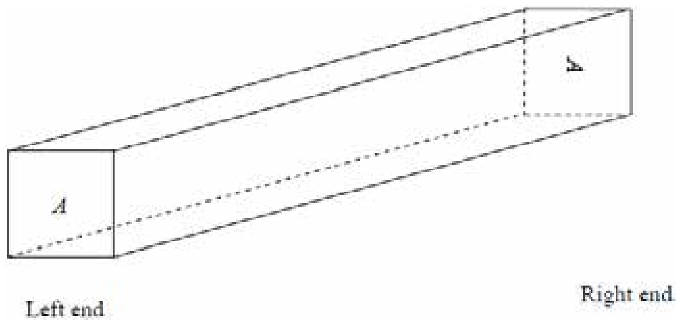
And its appearance after cut:



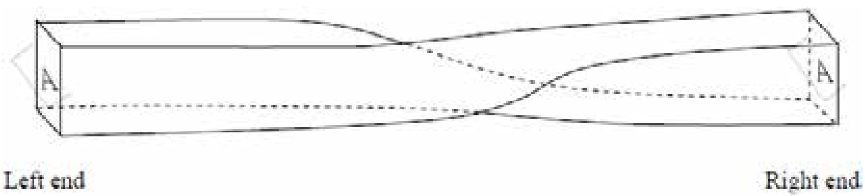
[Image taken from mathcraft.wonderhowto.com]

The knot formed is exactly the trefoil knot.

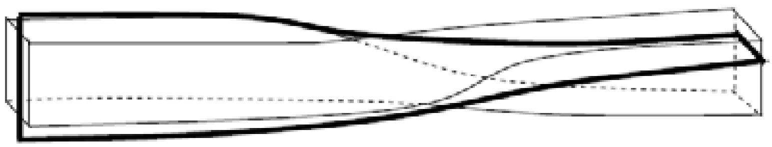
Considerations beyond the Möbius strip were made by Fatehi [3] in 2010. He considered a generalized version of strips with polygonal cross sections. His results, however, are incomplete. He only presents the cross section after cutting, but the "angles of twist", in addition to the number of different strips of the resultant object, are not evaluated. An example is given below: first fix one end of a cuboid and then twist the other end by $\frac{\pi}{2}$ radians in the direction as defined below, and then cut along one-third of the width:



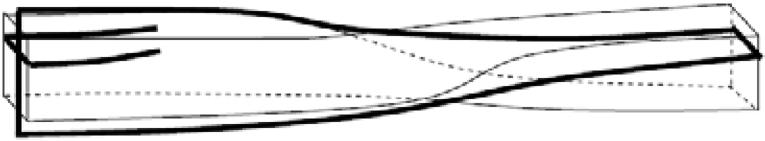
The two A's originally do not have the same orientation, but after we fix the left end and then rotate the right end, we can make their orientation agree:



Then we can connect the right end to the left end. For easy visualization, we do not draw it in this paper, but the reader is reminded that the two ends are actually joined.
We can then insert our blade onto the left end, and then cut the "twisted cuboid" while rotating the blade:

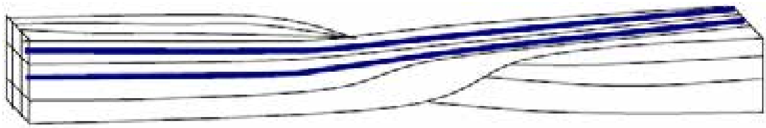


Because the right end is connected to the left, we are brought back to where we have started:

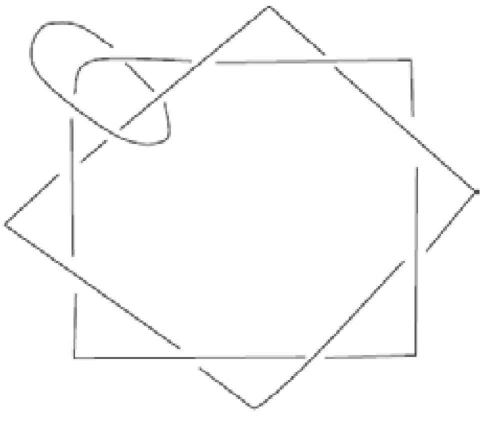


Then we continue this process and eventually we get an object with its cross section same as what is described in Fatehi's paper as a "3×3 matrix":

Although the cutting process separates the twisted cuboid into nine partitions, in reality there are only three connected portions linked to each other.
For the sake of simplicity, we first shrink the nine partitions to nine curves. As an example, we have shrunk two of the partitions into two blue curves:

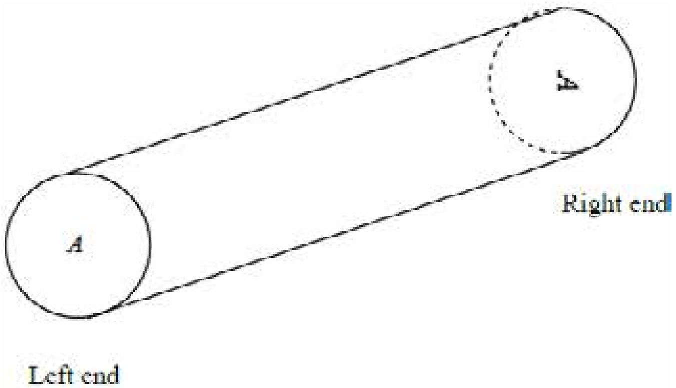


Then recalling the fact that the left and right ends of the twisted cuboid are connected to each other, we join the nine curves suitably according to how they are connected. After a series of continuous deformation, we would obtain:

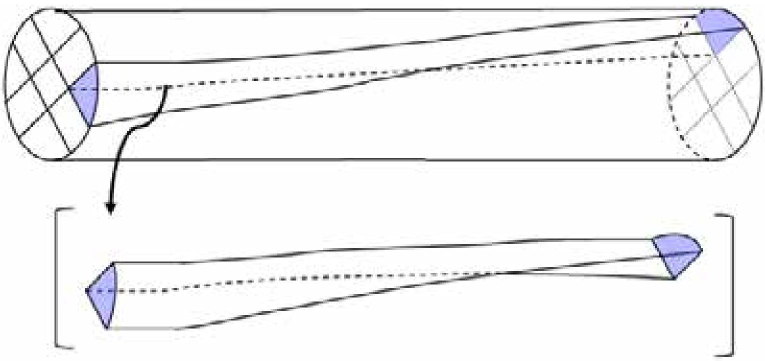


We can see that there are exactly three closed curves in the above diagram. These three curves represent precisely objects that can be individually deformed to twisted cuboids.

In our project, we make an attempt at generalizing the above results to cutting other similar objects. As the boundary of any d-sided polygon is homeomorphic to S^1, we generalize the idea of cutting a twisted prism to cutting a "twisted solid torus". A twisted solid torus is basically the same as a cylinder with one end twisted, then having its two ends joined together. In this way, we can allow the "angle of twist" to be $\alpha = 2\pi\tau$ where τ is any arbitrary real number.
For example, the process of cutting the "twisted cuboid" illustrated above can be transformed corresponding into cutting a twisted solid torus as illustrated below:



Note that the left end is fixed, the right end is rotated by $\frac{\pi}{2}$ radians (i.e. $\tau = \frac{1}{4}$) in the direction such that the two A's are in the same orientation after rotation, and then the two ends are connected. After cutting, one of the "nine partitions" are isolated and given below:



We go on considering the results of such cutting, and deduce general formulae to represent the results that are laid out in this paper.

After that, we study the link formed by cutting these twisted solid tori, and deduce a general form of their braid words, Seifert matrices and Alexander polynomial.

Next we study the result of cutting a further generalized version of these objects, namely combining more than one twisted solid tori to form a virtual knot.

2. Twisted solid torus

Notation 1. *$f(1-)$ is used to denote $\lim_{h \to 1^-} f(h)$ provided that it exists.*

We first choose the interval $[0,1)$ as the representative class of elements in the group R/Z so as to define a twisted solid torus in $\mathbf{C} \times [0,1)$:

Definition 2. *$<\tau>= \{(\gamma(t), t) : \gamma(t) = \gamma(0)\exp[2\pi i\theta(t)]$ and $|\gamma(0)| \leq 1\}$, where $\theta(t) : [0,1) \to \boldsymbol{R}$ is some function which has a continuous derivative with $\theta(0) = 0$ and $\theta(1-) = \tau$, is called a twisted solid torus (tst).*

Definition 3. *The function $\theta(t)$ in the definition of the tst $<\tau>$ is called its twisting function.*

Definition 4. *The number τ is called the twist turn of $<\tau>$.*

First we consider a coordinate system (z,t) in $\mathbf{C} \times [0,1)$: (A cylinder drawn notwithstanding, the reader is reminded that the left and right ends are connected to form a torus)

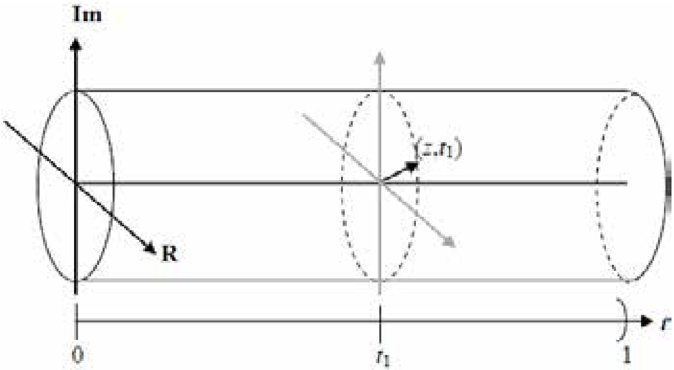


ILLUSTRATION 1. A coordinate system for a tst.

Note that in fact for the function $\gamma(t) = \gamma(0)\exp[2\pi i\theta(t)]$,

$$\gamma'(t) = 2\pi i\theta'(t)\gamma(0)\exp[2\pi i\theta(t)] = 2\pi i\theta'(t)\gamma(t)$$

So it is a solution to the following differential equation:

$$\frac{dz}{dt} = 2\pi i\theta'(t)z$$

In differential form, this is the same as:

$$dz = 2\pi i\theta'(t)zdt$$

If we plot this as a vector field, we obtain what is shown in the following illustration:

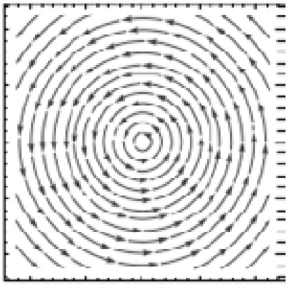
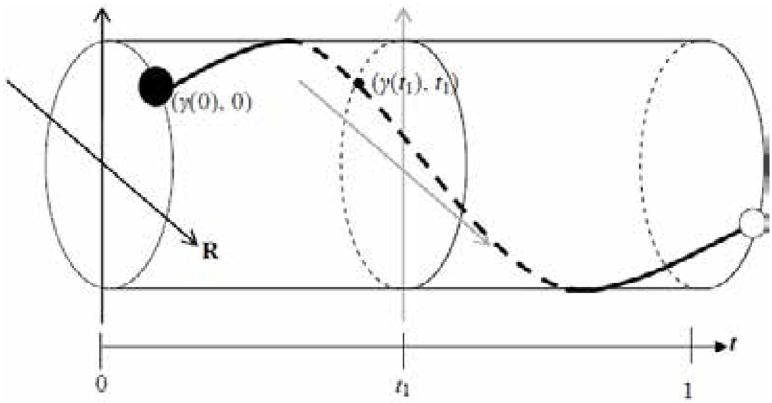


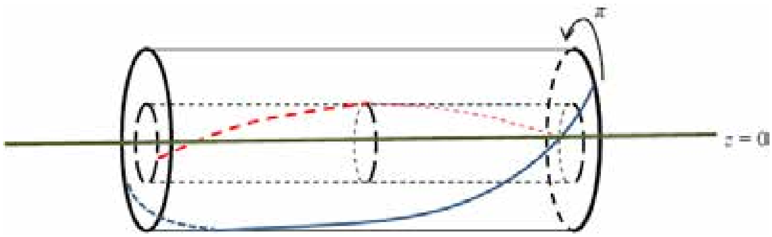
ILLUSTRATION 2. Visualizing the vector field.

Therefore, the solution to the differential equation, i.e. the integral curves, can be thought as curves that rotate about the line $z = 0$. From $t = 0$ to $t \to 1^-$, these curves rotate by $2\pi(\theta(1-) - \theta(0)) = 2\pi\tau$.
Hence, the twist turn τ of a tst represents the amount of twisting in it.
Hence, $<\tau>$ is a set of the following curves in $\mathbf{C} \times [0, 1)$: choose a point $\gamma(0)$ on or inside the unit circle at $t = 0$. Then the curve $(\gamma(t), t)$ is a spiral that has its starting point at $(\gamma(0), 0)$ and for $t \in [0, 1)$, it winds around the line $z = 0$. From $t = 0$ to $t \to 1^-$, it winds by $2\pi\tau$ radians anticlockwise in total.
The illustration below shows a curve in the tst $<\tau>$.

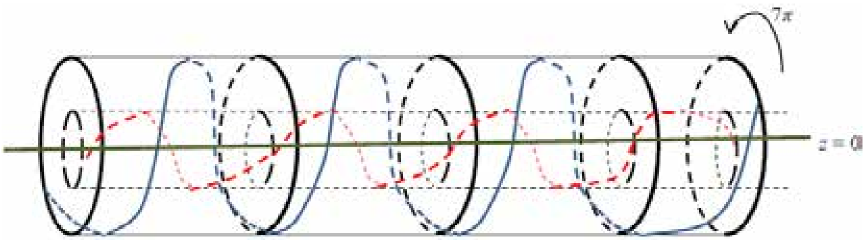


The following are two concrete examples that give an intuitive idea of a tst:

Example 5. *Two curves (in red and blue) in the tst* $\left\langle \dfrac{1}{2} \right\rangle$ *are shown below. They both rotate about the center line* $z = 0$ *(in green) by* $2\pi \cdot \dfrac{1}{2} = \pi$ *radians:*



Example 6. *Two curves (one in red and the other in blue) in the tst* $\left\langle \dfrac{7}{2} \right\rangle$ *are shown below. They both rotate about the center line* $z = 0$ *(in green) by* $2\pi \cdot \dfrac{7}{2} = 7\pi$ *radians:*



Besides, we define the equality of tsts as below:

Definition 7. *Two tsts* $<\tau_1>$ *and* $<\tau_2>$ *are* *equal* $\Leftrightarrow \tau_1 = \tau_2$. *We denote equality by the usual "=" sign.*

3. Mathematical Formulation of Cuts

Here is how we define "cutting a tst into M parts".
We first consider $\tau \in \mathbf{Q}$, so that for any given τ, we can choose any denominator $d(\tau) > 0$. We let the corresponding numerator $n(\tau)$, such that $n(\tau) = d(\tau)\tau$.
Insert the following set of $(M-1)$ parallel lines to the cross section at $t = 0$.

$$L_0 = \left\{ \begin{array}{cc|c} \exp \dfrac{\pi i(2q-M)}{2M} + s & \text{if } d(\tau) = 1 & \\ & & q = 1, 2, 3, \cdots, M-1, s \in \mathbf{R} \\ \exp \dfrac{2\pi i q}{M d(\tau)} + is\Big(\exp \dfrac{2\pi i}{d(\tau)} - 1\Big) & \text{if } d(\tau) > 1 & \end{array} \right\}$$

Then these $(M-1)$ lines divide the arc from 1 to $\exp \dfrac{2\pi i}{d(\tau)}$ into M parts with equal arclengths.

We define $l_{0,q}(s) = \begin{cases} \exp\dfrac{\pi i(2q-M)}{M} + s & \text{if } d(\tau)=1 \\ \exp\dfrac{2\pi i q}{Md(\tau)} + is\Big(\exp\dfrac{2\pi i}{d(\tau)} - 1\Big) & \text{if } d(\tau)>1 \end{cases}$ for a particular q, so that in fact

$$L_0 = \{l_{0,1}(s), l_{0,2}(s), l_{0,3}(s), \ldots, l_{0,M-1}(s)\}$$

This is shown in the following illustration:

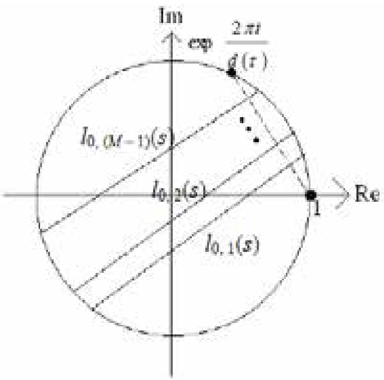
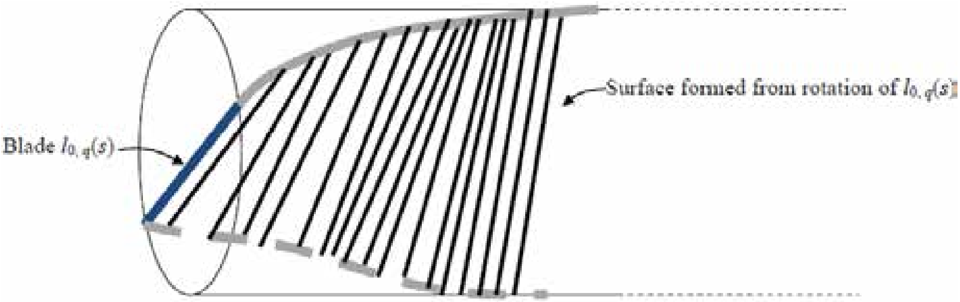


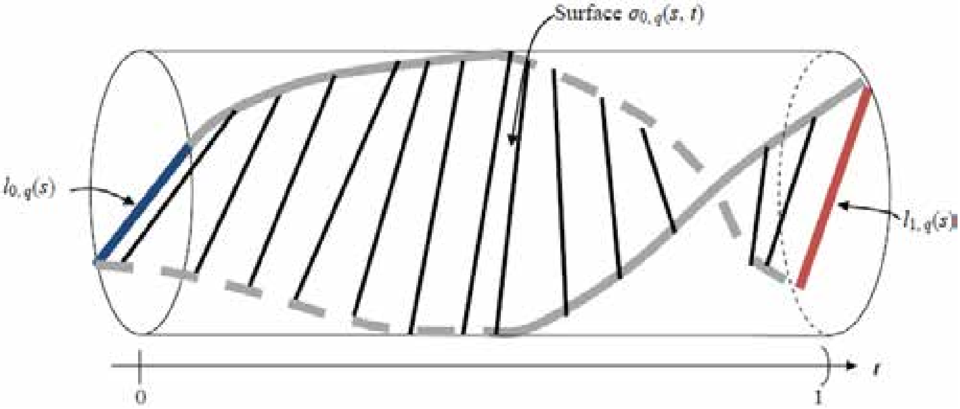
Illustration 3. Inserting the lines in the set L_0.

Then each of the lines $l_{0,q}(s)$ are perpendicular to the vector from 1 to $\exp\dfrac{2\pi i}{d(\tau)}$ and the line $l_{0,q}(s)$ passes through the point $\exp\dfrac{2\pi i q}{Md(\tau)}$.

These lines can be thought as "blades". When one of the "blades", $l_{0,q}(s)$ (shown in blue), say, cuts along the tst, the blade rotates about the line $z = 0$ while going through the tst. As it rotates continuously, a surface is formed:



The blade eventually arrives at $t \to 1^{-}$:



The surface $\sigma_{0,q}(s,t)$ can then be represented by the following equation:

$$\sigma_{0,q}(s,t) = l_{0,q}(s) \exp\left[2\pi i \theta(t)\right]$$

This means that it is the line $l_{p,q}(s)$ rotating along the $z = 0$ line that forms the surface.
If we consider the $(M-1)$ lines in L_0, i.e. the lines $l_{0,1}(s)$, $l_{0,2}(s)$, $l_{0,3}(s)$, …, $l_{0,M-1}(s)$ rotating at the same time, we obtain one surface for each line, so we have $(M-1)$ surfaces in total.
The surface $\sigma_{0,q}(s,t)$ intersects the cross section at $t \to 1^{-}$ of the tst in a line (shown in red)
Similarly, each surface $\sigma_{0,q}(s,t)$, where $q = 1, 2, 3, \ldots, M-1$, intersects the cross section at $t \to 1^{-}$ of the tst at exactly one line. We can find the line by evaluating $t \to 1^{-}$ in the function $\sigma_{0,q}(s,t)$:

$$\sigma_{0,q}(s, 1-) = l_{0,q}(s) \exp\left[2\pi i \theta(1-)\right] = l_{0,q}(s) \exp\left(2\pi i \tau\right)$$

We denote this line by $l_{1,q}(s)$.

$$l_{1,q}(s) = l_{0,q}(s) \exp\left(2\pi i \tau\right)$$

The line is exactly what is obtained by rotating all points lying on $l_{0,q}(s)$ about 0 by $2\pi\tau$ radians in the anti-clockwise direction.
If we let the blades keep going in the positive t direction (with their orientation unchanged), they then arrive at the position where $t = 1$.
But since $1 + \mathbf{Z} = 0 + \mathbf{Z}$, the plane $t = 1$ is the same as that at $t = 0$. In other words, the blades are now back to the position where $t = 0$.
We then define $L_1 = \{l_{1,1}(s), l_{1,2}(s), l_{1,3}(s), \ldots, l_{1,M-1}(s)\} = L_0 \exp\left(2\pi i \tau\right)$.
Keeping the process going on, we can obtain $L_2, L_3, \ldots$,
where $l_{p+1,q} = l_{p,q} \exp\left(2\pi i \tau\right)$ and $L_p = \{l_{p,q} \mid q = 1, 2, 3, \ldots, M-1\}$.
We can then prove the following proposition by induction:

Proposition 8. $L_p = L_0 \exp\left(2\pi i \tau p\right)$ *for nonnegative integers* $p = 0, 1, 2, 3, \ldots$

Proof. We prove it by induction. When $p = 0$, this is obviously true, since $\exp[2\pi i\tau(0)] = 1$. Assume $L_p = L_0 \exp(2\pi i\tau p)$. Consider $L_{p+1} = L_p \exp(2\pi i\tau) = L_0 \exp[2\pi i\tau(p+1)]$. The result therefore holds by the principle of induction. □

We observe that after several times of rotation, the lines in the set L_0 returns to original position. When this occurs, we stop rotating the lines and end this cutting process.
In other words, we stop the cutting when we find some positive integer p such that $z \mapsto z \exp 2\pi i\tau(p+1)$ is the identity map, as this gives $L_{p+1} = L_0$.

Lemma 9 (Least Rotations to Original)**.** *For the equation $z = z \exp 2\pi i\tau r$, the smallest positive integral solution is $r = \dfrac{d(\tau)}{g(\tau)}$ where $g(\tau) \equiv gcd(|n(\tau)|, d(\tau))$.*

Proof. To solve $\exp 2\pi i\tau r = 1$, we have:

$$
\begin{aligned}
& \tau r \in \mathbf{Z} \\
\Leftrightarrow\ & \frac{n(\tau)}{d(\tau)} r \in \mathbf{Z}, \text{ since } p > 0 \\
\Leftrightarrow\ & d(\tau) \mid (n(\tau) r) \\
\Leftrightarrow\ & n(\tau) r \equiv 0 \ (\mathrm{mod}\ d(\tau)) \\
\Leftrightarrow\ & r\frac{n(\tau)}{g(\tau)} \equiv 0 \left(\mathrm{mod}\ \frac{d(\tau)}{g(\tau)}\right), \text{ where } g(\tau) \equiv \gcd\left(|n(\tau)|, d(\tau)\right) \text{ as defined.} \\
\Leftrightarrow\ & r \equiv 0 \left(\mathrm{mod}\ \frac{d(\tau)}{g(\tau)}\right) \text{ since } \frac{n(\tau)}{g(\tau)} \text{and } \frac{d(\tau)}{g(\tau)} \text{ are coprime to each other}
\end{aligned}
$$

Hence, the smallest positive integral solution is
$r = \min\limits_{k \in \mathbf{Z}^+} \left(\dfrac{d(\tau)}{g(\tau)} k\right) = \dfrac{d(\tau)}{g(\tau)}$. □

With this Least Rotations to Original Lemma, in order to consider the result of cutting process, we only need the sets $L_0, L_1, L_2, \ldots, L_{D(\tau)-1}$, where $D(\tau) = \dfrac{d(\tau)}{g(\tau)}$.
From now on, we focus on the cross section of the tst at t = 0 and put the cross section in a plane.
Then we introduce a set defined as follows:

$$\{L_0, L_1, L_2, \ldots, L_{D(\tau)-1}\}$$

We use the notation $\Lambda(n(\tau), d(\tau), M)$ to denote a graph with the lines in these sets together with the unit circle $|z| = 1$. Note that this is exactly the same as the cross section at $t = 0$ of the tst after cut.
This set contains the lines in the set L_0 rotated about the point 0 by $2\pi\tau p$ radians anticlockwise, where p is an integer between 0 and $D(\tau) - 1$ inclusive.

Writing explicitly, the lines contained in the sets $L_0, L_1, L_2, \ldots, L_{p-1}$ have the following form:

$$l_{p,q}(s) = \exp \frac{2\pi i(pn(\tau)M + q)}{Md(\tau)} + is[\exp\,(2\pi i\tau p)]\left(\exp \frac{2\pi i}{d(\tau)} - 1\right)$$

where $0 \leq p \leq d(\tau) - 1$ and $1 \leq q \leq M - 1$ and p, q are integers.

Example 10. *This is how $\Lambda(1, 6, 3)$ is constructed step by step:*
We first draw the circle $|z| = 1$:

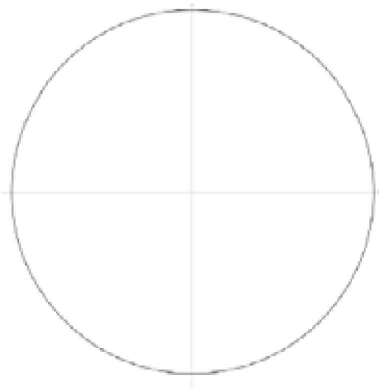


Illustration 4. The unit circle $|z| = 1$.

Then we divide the arc from the point 1 to $\exp \frac{2\pi i}{d(\tau)} = \exp \frac{2\pi i}{6}$ *into 3 parts with equal arclengths:*

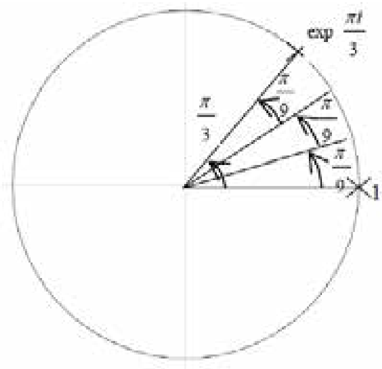


Illustration 5. Dividing the arc into 3 parts.

Then we calculate the number of blades, which is $M - 1 = 3 - 1 = 2$, and insert the lines in the set L_0:

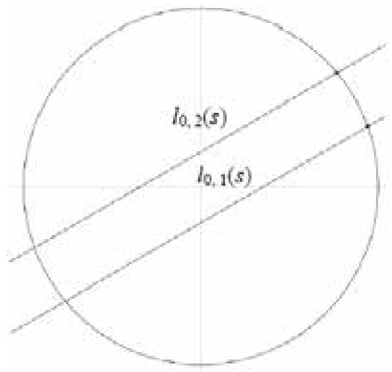


ILLUSTRATION 6. Inserting the lines from the set L_0.

Afterwards, we rotate the lines in the set L_0 about 0 by $2\pi\tau$ radians, where $\tau = \dfrac{n(\tau)}{d(\tau)} = \dfrac{1}{6}$. So we rotate the lines by $\dfrac{\pi}{3}$ radians to obtain the set of lines L_1:

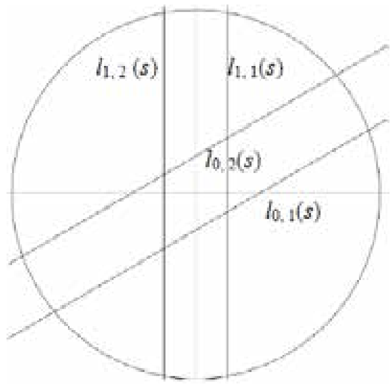


ILLUSTRATION 7. Inserting the lines from the set L_1.

If we continue the above process, we can obtain the set L_3, L_4 *and* L_5*. As* $D(\tau)-1 = 6-1 = 5$*, we can stop the cutting process and obtain the following diagram, which is exactly* $\Lambda(1,6,3)$*:*

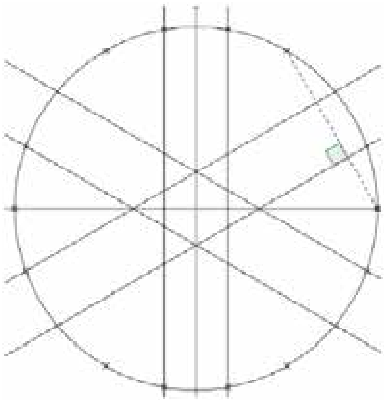


Illustration 8. $\Lambda(1,6,3)$.

Example 11. $\Lambda(3,5,4)$ *can be obtained in the same manner and it appears in the following form:*

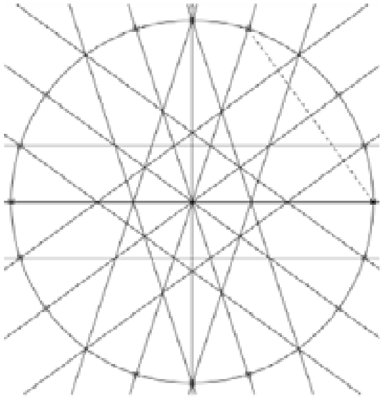


Illustration 9. $\Lambda(3,5,4)$.

4. Studying the Result of Cutting

In this section we will present our approach to study the result of cutting. Firstly we make more definitions on the cross section $\Lambda(n(\tau), d(\tau), M)$:

Definition 12 (Graph of $\Lambda(n(\tau), d(\tau), M)$)**.** *The graph of* $\Lambda(n(\tau), d(\tau), M)$ *is* $\Lambda(n(\tau),\ d(\tau),\ M)$ *with all points and line segments outside the unit circle discarded.*

Definition 13 (Vertex, edge, face)**.** *On the graph of $\Lambda(n(\tau), d(\tau), M)$:*
A vertex is an intersection point of a line $l_{p1,q1}$ with the unit circle $|z| = 1$ or with another line $l_{p2,q2}$, where $l_{p1,q1}$ and $l_{p2,q2}$ are distinct lines;
an edge is a line segment or arc that connects only two vertices; and
a face is a simply connected region bounded by edges in which there are no other edges or vertices.

For example, the graph of $\Lambda(1, 6, 3)$ is:

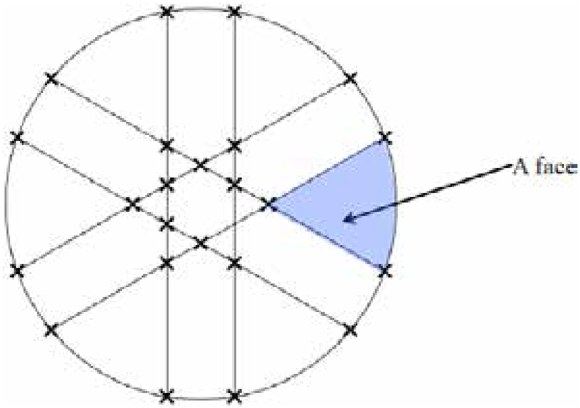


Illustration 10. Graph of $\Lambda(1, 6, 3)$.

In the graph above, each vertex is marked with a cross (x), and each edge is a line segment or arc connecting two vertices with no other vertices on it, and an example of the 19 faces is indicated above. We then consider the resultant mathematical object from cutting a tst. Shown below is the example of cutting the tst $\left\langle \dfrac{1}{4} \right\rangle$ with the chosen denominator $d(\tau)$ to be 4:

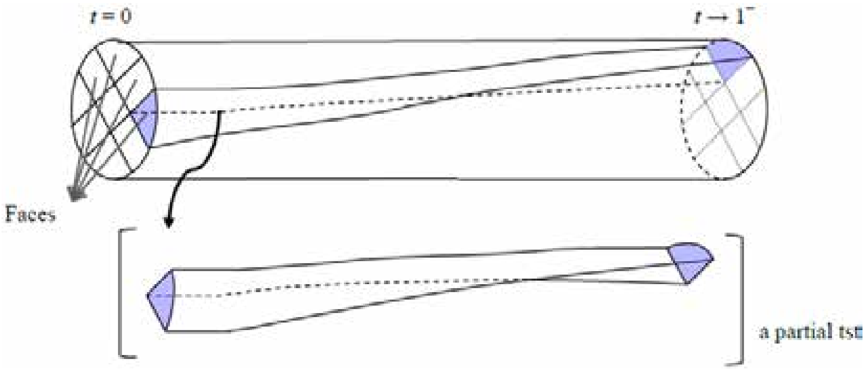
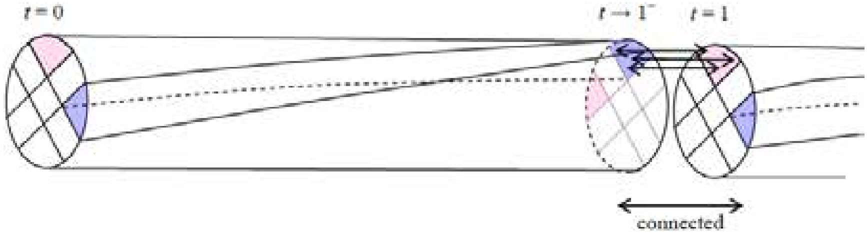
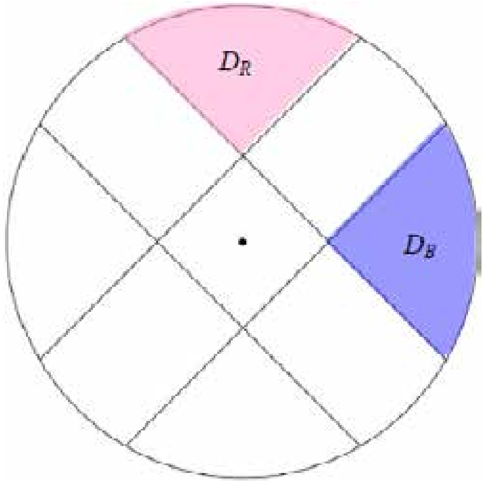


Illustration 11. Partial tsts are formed after cutting.

Definition 14. *For a face D in the graph of $\Lambda(n(\tau), d(\tau), M)$, the partial tst $G(D)$ is the set of all curves $(\gamma(t), t)$ in the tst $<\tau>$ where $\gamma(t)$ is a curve in $<\tau>$ and $\gamma(0) \in D$.*

After that, we observe that some partial tsts are connected to each other (since the plane at $t \to 1^-$ is connected to the plane at $t = 0$), as seen in the following illustration:

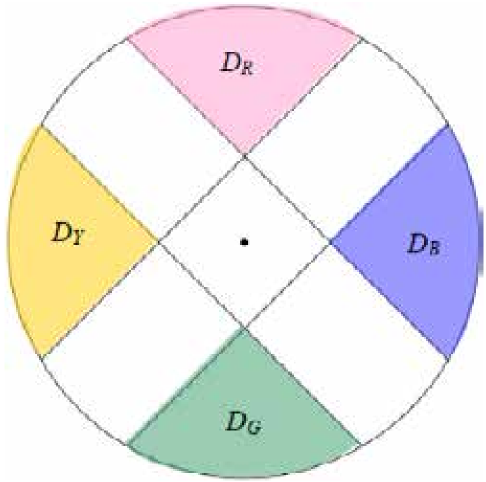




Denote the red and blue faces by D_R and D_B respectively. Then we observe that the partial tsts $G(D_R)$ and $G(D_B)$ are connected to each other. Moreover, D_R is obtained by rotating the face D_B by $2\pi\tau$ radians about the origin.This thus motivates us to define "connected" tsts:

Definition 15. *The partial tst $G(D_1)$ is said to be connected to the partial tst $G(D_2)$ if D_1 and D_2 are on the same $\Lambda(n(\tau), d(\tau), M)$ and the face D_2 is obtained by rotating the face D_1 by $2\pi\tau$ radians about the point 0.*

We also note that some collection of partial tsts are connected to one another and no any others. If they are isolated, they can be taken as a whole to give a closed object as explained in the following:



If we denote the blue, red, yellow and green faces on the left by D_B, D_R, D_Y and D_G respectively, then $G(D_B)$ is connected to $G(D_R), G(D_R)$ to $G(D_Y), G(D_Y)$ to $G(D_G)$ and, finally, $G(D_G)$ to $G(D_R)$, because the faces D_R, D_Y and D_G and D_B is obtained after successively rotating the face D_B by $2\pi\tau$ radians about 0. Hence these mutually connected partial tsts form a closed object.

From this example, we can see that from $\Lambda(n(\tau), d(\tau), M)$, we must be able to obtain a finite sequence of k partial tsts $G(D_1), G(D_2), G(D_3), \ldots, G(D_k)$ from a tst such that, for example, $G(D_1)$ is connected to $G(D_2)$, and the last partial tst $G(D_k)$ is connected to the first one $G(D_1)$. We are thus motivated to define the following:

Definition 16. *A* <u>*knotted tst*</u> *is a sequence of distinct partial tsts* $G(D_1)$, $G(D_2)$, $G(D_3)$, ..., $G(D_k)$ *where for each* r *satisfying* $1 \leq r \leq k-1$, $G(D_r)$ *is connected to* $G(D_{r+1})$ *and* $G(D_k)$ *is connected to* $G(D_1)$.

We can now define the twist turn for a knotted tst. To deduce its formula, we first consider how we can deduce the twist turn of a given tst Γ with its twisting function $\theta(t)$ found.
Recall that all the curves $(\gamma(t), t) \in \Gamma$ share the same twisting function $\theta(t)$. Therefore we can consider only one instead of all the curves $(\gamma(t), t)$. Take a particular

$\gamma(t)$ which has $\gamma(0) \neq 0$, and then the twist turn τ is given by:

$$\begin{aligned}
\tau &= \tau - 0 \\
&= \theta(1-) - \theta(0) \\
&= \lim_{h \to 1^-} \int_0^h \theta'(t)dt \\
&= \int_{\theta(0)}^{\theta(1-)} \frac{\gamma \cdot 2\pi i d\theta}{2\pi i \gamma} \\
&= \int_{\gamma(0)}^{\gamma(1-)} \frac{d\gamma}{2\pi i \gamma} \\
&= \frac{1}{2\pi i} \int_{\gamma[0,1)} \frac{d\gamma}{\gamma}
\end{aligned}$$

We now analogously define the twist turn of a partial tst by:

Definition 17. *The twist turn of a partial tst $G(D)$ is twt $G(D) = \frac{1}{2\pi i} \int_{\gamma[0,1)} \frac{d\gamma}{\gamma}$ where $(\gamma(t), t)$ is a curve in $G(D)$ with $\gamma(0) \neq 0$.*

Note that "twt" is short for "twist turn".
We also define the twist turn of a knotted tst as the sum of twist turns of all its constituent partial tsts:

Definition 18. *The twist turn of a knotted tst $\Gamma_K = G(D_1)$, $G(D_2)$, $G(D_3)$, …, $G(D_K)$ is twt $\Gamma_K =$twt $G(D_1)+$twt $G(D_2)+$twt $G(D_3) + \ldots +$twt $G(D_k)$*

We will also use the following notation to symbolize a knotted tst with twist turn τ:

Notation 19. *A knotted tst with twist turn τ is denoted by $<\tau>_K$.*

The subscript K is used to distinguish between knotted tsts and tsts.

We are then using the number of different knotted tsts, in addition to each of their twist turns, to present the result of cutting a general tst.
In general, cutting a tst may give more than one knotted tsts. We could have studied the finite set that contains them, but this is not a satisfactory construction, since in some cases more than one of them have the same twist turn. Counting the number of knotted tsts (instead of *different* knotted tsts) formed is not the same as the cardinality of the set.
For example, $\{<2>_K, <2>_K, <1>_K\} = \{<2>_K, <1>_K\}$ is a set with cardinality 2. However, if we cut some tst and obtain 2 knotted tsts being $<2>_K$ and 1 tst being $<1>_K$, we have obtained 3 knotted tsts instead of 2. To resolve this problem, we consider a multiset that contains all the knotted tsts formed and use notations from the paper [7] to make the following definitions:

Definition 20. *τ_1>$_K$ + <τ_2>$_K$≡ [<τ_1>$_K$ _ <τ_2>$_K$] is the multiset that contains <τ_1>$_K$ and <τ_2>$_K$, called the sum of the two knotted tsts.*

Sums can be added to form new multisets, i.e. for the multisets A and B, $A+B$ is a multiset C satisfying

$$m_A(x) + m_B(x) = m_C(x)$$

for all elements x in A and B, where $m_D(x)$ is the multiplicity of the element x in some multiset D, i.e. the number of times x appears in D. For example, the multiplicity of 3 in the multiset $D = [3\_3\_3\_5\_2\_1\_7\_3]$ is $m_D(3) = 4$.

Proposition 21. *If S_1 and S_2 are two knotted tst sums, then*

$$S_1 + S_2 = S_2 + S_1 \text{ and } (S_1 + S_2) + S_3 = S_1 + (S_2 + S_3).$$

Proof. This follows from the commutativity and associativity of addition of positive integers. □

Notation 22. *We write a knotted tst sum $S = \left[\underbrace{\langle\tau\rangle_K\text{-}\langle\tau\rangle_K\text{-}\langle\tau\rangle_K\text{-}\cdots\text{-}\langle\tau\rangle_K}_{k\ times}\right]$ as k <τ>$_K$.*

This notion of knotted tst sums allows us to express the result of tst-cutting more clearly and simply, as it is obvious that cutting a tst gives us a knotted tst sum. Hence, we define:

Notation 23. *$\Phi(n(\tau), d(\tau), M)$ denotes the knotted tst sum resultant from cutting a tst <τ> represented as $\left\langle \dfrac{n(\tau)}{d(\tau)} \right\rangle$ in M parts.*

The careful reader may observe that taking the result of cutting a tst as a knotted tst sum is not enough to determine how the knotted tsts are linked to each other. This is discussed in the section on tst links. We also notice that there is a close relationship between the tst sums $\Phi(n(\tau), d(\tau), M)$ and $\Phi(-n(\tau), d(\tau), M)$, so we make the following definitions.

Definition 24. *The conjugate of a knotted tst <τ>$_K$ is <$-\tau$>$_K$ and is denoted by (<τ>$_K$)*,.*

The conjugate of a knotted tst sum is then defined (inductively) as:

Definition 25. *If $S = \sum_{i=1}^{k} \langle\tau_i\rangle_K$, then its conjugate is $S^* = \sum_{i=1}^{k} (\langle\tau_i\rangle_K)^*$.*

Definition 26. *The conjugate of the sum of knotted tst sums S_1 and S_2 is defined by $(S_1 + S_2)^* = {S_1}^* + {S_2}^*$.*

Example 27. *The conjugate of* $(2 <0.25>_K +15 <0.5>_K)^* = (2 <0.25>_K)^* + (15 <0.5>_K)^* = 2 <-0.25>_K +15 <-0.5>_K$.

This conjugate notion will be useful as soon as we develop the "Basis Formula" in the next section.

5. The twist turns of knotted tsts in $\Phi(n(\tau), d(\tau), M)$

First we notice the following fact:

Theorem 28 (Twist turn of knotted tst)**.** *If the number of partial tsts in a knotted tst Γ_K in the sum $\Phi(n(\tau), d(\tau), M)$ is r, then twt $\Gamma_K = r\tau$.*

Proof. Recall from definition that the twist turn of a knotted tst is the sum of the twist turns of its partial tsts. But as these partial tsts are from the same tst $<\tau>$, they share the same twisting function $\theta(t)$ for which $\theta(1-) - \theta(0) = \tau$.
Hence each of the partial tsts in the knotted tst has a twist turn of τ.
In other words, twt $\Gamma_K = \underbrace{\tau + \tau + \tau + \cdots + \tau}_{r\ \tau'\text{s}}$, so the stated result follows. □

Therefore, we can now suppose the number of partial tsts in the knotted tst is k. Then we recall the fact that if $G(D_1)$ is connected $G(D_2)$, D_2 can be obtained by rotating D_1 by $2\pi\tau$ radians about the point 0. Hence, if we name the partial tsts in the knotted tst by $G(D_1), G(D_2), \ldots, G(D_K)$, then $D_2, D_3, \ldots, D_k$ are in fact the faces obtained after successive rotations of the face D_1 by $2\pi\tau$ radians about the point 0. Since all of them are distinct, k is the smallest number of rotations required for the face to return to its original position, and our mission is to find out the value of k.
Before we go deep into the derivation, we recall that for a point z in the graph $\Lambda(n(\tau), d(\tau), M)$, there is a curve $(\gamma(t), t)$ in the tst $<\tau>$ where $\gamma(0) = z$. If we trace along the curve and consider the point where it intersects the plane $t \to 1^-$, we would be at the point $(\gamma(1-), 1^-) = (\gamma(0) \exp 2\pi i\theta(1-), 1^-) = (z \exp 2\pi i\tau, 1^-)$. Note that $z \exp 2\pi i\tau$ is exactly obtained by rotating the point z $2\pi i\tau$ radians about the point 0. If we trace instead a face D, then the curve $(\gamma(t), t)$ would be replaced by the partial tst $G(D)$. Then the partial tst $G(D)$ intersects the plane at $t \to 1^-$ in the face D' where D' is obtained by rotating the point z $2\pi i\tau$ radians about the point 0. Since the plane at $t \to 1^-$ is connected to the one at $t = 0$, the face D' would appear in the graph of $\Lambda(n(\tau), d(\tau), M)$, and by definition, $G(D)$ is connected to $G(D')$.
With this idea in mind, we separately consider two cases to find the value of k:

Case I The face D contains the point 0.
Observing the following illustration gives us the proposition that follows.

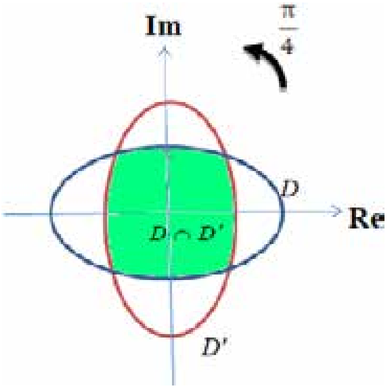


Illustration 12. An example of a face in the containing 0 in the graph of $\Lambda(n(\tau), d(\tau), M)$ being not rotational symmetric of order being a multiple of $d(\tau)/g(\tau)$.

Proposition 29. *If there is a face that contains 0 in the graph of $\Lambda(n(\tau), d(\tau), M)$, then it is rotational symmetric of order being a multiple of $\frac{d(\tau)}{g(\tau)}$, where $g(\tau) = gcd(\ |n(\tau)|, d(\tau)\ )$.*

Proof. Assume that D (in blue) is a face that contains the point 0 and that D is not rotational symmetric of order a multiple of $\frac{d(\tau)}{g(\tau)}$. Then D' (in red), the face obtained by rotating D by $2\pi\tau$ radians about the point 0, also appears in the graph of $\Lambda(n(\tau), d(\tau), M)$, according to the reasoning above. But since D is not rotational symmetric of order a multiple of $\frac{d(\tau)}{g(\tau)}$, the faces D and D' are not the same and $D \cap D'$ (in green) is a region bounded by edges. Therefore there are edges in D and D', making D not a face. This contradicts to our initial assumption that D is a face. Hence, if D is a face that contains the point 0, D must be rotational symmetric of order a multiple of $\frac{d(\tau)}{g(\tau)}$. □

Since D is rotational symmetric of order $\frac{d(\tau)}{g(\tau)}$, rotating D by $2\pi\tau$ radians about the point 0 once gives the same face D. Hence, for the face D containing 0, the smallest number k of rotations by $2\pi\tau$ radians about the point 0 for it to return to its original position is 1.
Therefore, for faces in Case I, the partial tst $G(D)$ is connected to itself and forms a knotted tst individually, whose twist turn is τ. Therefore, each of these faces corresponds to one knotted tst in the knotted tst sum, namely $<\tau>_K$.

Case II The face D does not contain the point 0.
We first claim the following propositions:

Proposition 30. *In $\Lambda(n(\tau), d(\tau), M)$, let the smallest number of rotations by $2\pi\tau$ rad about the point 0 for a* ***point*** *to return to its original position be $k > 0$. Then for every non-zero point in the graph of $\Lambda(n(\tau), d(\tau), M)$, the least positive integral value of such k is the same.*

Proof. k is the smallest integer that satisfies the equation $z \exp(2\pi i\tau k) = z$. This reduces to $\exp(2\pi i\tau k) = 1$ if $z \neq 0$. Hence k is the smallest positive integer satisfying $\exp(2\pi i\tau k) = 1$. This argument is independent of z, so it is true for all $z \neq 0$. □

Proposition 31. *In $\Lambda(n(\tau), d(\tau), M)$, let the smallest number of rotations by $2\pi\tau$ rad about the point 0 for a* ***face*** *to return to its original position be $k > 0$. Then for every region not containing 0, the least positive integral value of such k is the same.*

Proof. By the last proposition, consider all points in the face not containing 0. They have the same k. □

By the Least Rotations to Original lemma in Section 3, the smallest positive integral solution to the equation $\exp(2\pi i\tau k) = 1$ is $k = \dfrac{d(\tau)}{g(\tau)}$, where $g(\tau) \equiv \gcd(|n(\tau)|, d(\tau))$. Therefore, for a face D in Case II, the partial tst $G(D)$ does not individually become a knotted tst; however, every $D(\tau)$ such partial tsts belong to one knotted tst $\mathit{\Gamma}_K$, whose twist turn is twt $\mathit{\Gamma}_K = \tau\dfrac{d(\tau)}{g(\tau)} = \dfrac{n(\tau)}{d(\tau)}\dfrac{d(\tau)}{g(\tau)} = \dfrac{n(\tau)}{g(\tau)}$. Therefore, for each face D in this case, $G(D)$ is an element of some knotted tst $\left\langle \dfrac{n(\tau)}{g(\tau)} \right\rangle_K$.
From the above reasoning, we can now conclude that there are *only two* types of knotted tst in the knotted tst sum $\Phi(n(\tau), d(\tau), M)$, namely:

Type I A knotted tst with only one partial tst $G(D)$, where D is a face in $\Lambda(n(\tau), d(\tau), M)$ containing the point 0.

Type II Any other knotted tsts in the sum.

We denote the number of type I and II knotted tsts in $\Phi(n(\tau), d(\tau), M)$ by I and P respectively.
Under this classification, the twist turns of type I and II knotted tsts are respectively:

$$\text{I.}\frac{n(\tau)}{d(\tau)} = \tau \qquad \text{II.}\frac{n(\tau)}{g(\tau)}$$

In this way, we can take $\left\langle \dfrac{n(\tau)}{d(\tau)} \right\rangle_K$ and $\left\langle \dfrac{n(\tau)}{g(\tau)} \right\rangle_K$ as "basic elements" for the knotted tst sum and we have the following theorem:

Theorem 32 (Basis Formula)**.** *For all $n(\tau), d(\tau), M \in \boldsymbol{Z}, d(\tau) > 0$, $\Phi(n(\tau), d(\tau), M) = I\left\langle \dfrac{n(\tau)}{d(\tau)} \right\rangle_K + P\left\langle \dfrac{n(\tau)}{g(\tau)} \right\rangle_K$, where I and P are the respective multiplicity of the two elements in the knotted tst sum.*

Proof. Use the fact that $\left\langle \dfrac{n(\tau)}{d(\tau)} \right\rangle_K$ and $\left\langle \dfrac{n(\tau)}{g(\tau)} \right\rangle_K$ are the only types of elements in the knotted tst sum. □

Theorem 33 (Conjugate Cutting)**.** $\Phi(-n(\tau), d(\tau), M) = [\Phi(n(\tau), d(\tau), M]^*$.

Proof. The tst $<-\tau>$ can be obtained by taking the conjugate of the tst $<\tau>$. After cutting $<-\tau>$, by the Basis Formula, each resultant tst is either $<-\tau>_K$ or $\left\langle -\dfrac{n(\tau)}{g(\tau)} \right\rangle_K$, each of which is the conjugate of the strips formed from Φ-cutting $<\tau>$, with the multiplicity of each knotted tst unchanged. □

Example 34. *If $\Phi(1, 2, 3) = P <1>_K + I <0.5>_K$, then $\Phi(-1, 2, 3) = [\Phi(1, 2, 3)]^* = (P <1>_K + I <0.5>_K)^* = P <-1>_K + I <-0.5>_K$.*

This allows us to consider only the case with $\tau > 0$ and then use the Conjugate Cutting Theorem to deduce the result for $\tau < 0$.

6. Deducing a Formula for Multiplicity I of $<\tau>$ in Sum $\Phi(n(\tau), d(\tau), M)$

For simplicity, from now on, we are going to use the following notation:

Notation 35. $N(\tau) = \dfrac{n(\tau)}{g(\tau)}$ *and* $D(\tau) = \dfrac{d(\tau)}{g(\tau)}$

Observe the graphs $\Lambda(1, 6, 3)$ and $\Lambda(3, 5, 4)$. Notice that the region containing 0 is found in the graph when M is odd. Besides, some lines $l_{p,q}$ intersect at the point 0 when M is even.

Lemma 36 (Lines through center)**.** *In $\Lambda(n(\tau), d(\tau), M)$, there are some lines passing through 0 iff M is even.*

Proof. For $1 \le q \le M - 1$, the line $l_{p,q}$ is given by the equation

$$\begin{aligned} l_{p,q}(s) &= \exp\Big(\frac{2\pi i(pn(\tau)M + q)}{Md(\tau)}\Big) + is\Big[\exp\Big(\frac{2\pi ipn(\tau)}{d(\tau)}\Big)\Big]\Big[\exp\Big(\frac{2\pi i}{d(\tau)}\Big) - 1\Big] \\ &= \exp\Big(\frac{2\pi i(pn(\tau)M + q)}{Md(\tau)}\Big) - is\Big[\exp\Big(\frac{2\pi ipn(\tau)}{d(\tau)}\Big)\Big]\Big[1 - \exp\Big(\frac{2\pi i}{d(\tau)}\Big)\Big] \end{aligned}$$

We now have to solve the equation $l_{p,q}(s) = 0$.
Rearranging gives $\exp\Big(\dfrac{2\pi i(pn(\tau)M + q)}{Md(\tau)}\Big) = is\Big[\exp\Big(\dfrac{2\pi i pn(\tau)}{d(\tau)}\Big)\Big]\Big[1 - \exp\Big(\dfrac{2\pi i}{d(\tau)}\Big)\Big]$.
Multiplying both sides by $\exp\Big(\dfrac{2\pi i(-pn(\tau))}{d(\tau)}\Big)$, gives

$$\exp\Big[\frac{2\pi i(pn(\tau)M + q)}{Md(\tau)} - \frac{2\pi i pn(\tau)}{d(\tau)}\Big] = is\Big[1 - \exp\Big(\frac{2\pi i}{d(\tau)}\Big)\Big]$$

Simplifying gives:

$$\exp\Big(\frac{2\pi i q}{Md(\tau)}\Big) = is\Big[1 - \exp\Big(\frac{2\pi i}{d(\tau)}\Big)\Big] ---------- \quad (*)$$

Observe that the left hand side has a modulus 1 but $i\Big[1 - \exp\Big(\dfrac{2\pi i}{d(\tau)}\Big)\Big]$ does not. Given that s is any arbitrary real number, we can scale the number $i\Big[1 - \exp\Big(\dfrac{2\pi i}{d(\tau)}\Big)\Big]$ to have modulus 1, i.e. we can solve the equation $i\Big[1 - \exp\Big(\dfrac{2\pi i}{d(\tau)}\Big)\Big] = s' i \exp(2\pi i\zeta)$ for some ζ, and s' is the modulus of $i\Big[1 - \exp\Big(\dfrac{2\pi i}{d(\tau)}\Big)\Big]$. Then we note that after canceling the i's on both sides, the right hand side is exactly the polar form of a complex number. It follows that

$$\begin{aligned}
s'^2 &= \Big|1 - \exp\Big(\frac{2\pi i}{d(\tau)}\Big)\Big|^2 = \Big[1 - \exp\Big(\frac{2\pi i}{d(\tau)}\Big)\Big]\overline{\Big[1 - \exp\Big(\frac{2\pi i}{d(\tau)}\Big)\Big]} \\
&= \Big[1 - \exp\Big(\frac{2\pi i}{d(\tau)}\Big)\Big]\Big[1 - \exp\Big(-\frac{2\pi i}{d(\tau)}\Big)\Big] = 1 - \Big[\exp\Big(\frac{2\pi i}{d(\tau)}\Big) + \exp\Big(-\frac{2\pi i}{d(\tau)}\Big)\Big] + 1 \\
&= 2\Big(1 - \cos\frac{2\pi}{d(\tau)}\Big) = 4\sin^2\frac{2\pi}{2d(\tau)}
\end{aligned}$$

Since $s' \geq 0$, we take $s' = 2\sin\dfrac{2\pi}{2d(\tau)}$, since $d(\tau) > 0$. By division, we have

$$\begin{aligned}
\exp(2\pi i\zeta) &= \Big[1 - \exp\Big(\frac{2\pi i}{d(\tau)}\Big)\Big]\Big(\frac{1}{2}\csc\frac{2\pi}{2d(\tau)}\Big) \\
\exp(2\pi i\zeta) &= \Big[\Big(1 - \cos\frac{2\pi}{d(\tau)}\Big) - i\sin\frac{2\pi}{d(\tau)}\Big]\Big(\frac{1}{2}\csc\frac{2\pi}{2d(\tau)}\Big) \\
\exp(2\pi i\zeta) &= \Big(\frac{1 - \cos\frac{2\pi}{d(\tau)}}{2\sin\frac{2\pi}{2d(\tau)}}\Big) - i\frac{\sin\frac{2\pi}{d(\tau)}}{2\sin\frac{2\pi}{2d(\tau)}} = \frac{2\sin^2\frac{2\pi}{2d(\tau)}}{2\sin\frac{2\pi}{2d(\tau)}} - i\frac{2\sin\frac{2\pi}{2d(\tau)}\cos\frac{2\pi}{2d(\tau)}}{2\sin\frac{2\pi}{2d(\tau)}} \\
&= \sin\frac{2\pi}{2d(\tau)} - i\cos\frac{2\pi}{2d(\tau)} = \frac{1}{i}\Big(\cos\frac{2\pi}{2d(\tau)} + \sin\frac{2\pi}{2d(\tau)}\Big) \\
&= \frac{1}{i}\exp\Big(\frac{2\pi i}{2d(\tau)}\Big)
\end{aligned}$$

Hence, we have $i\exp(2\pi i\zeta) = \exp\Big(\dfrac{2\pi i}{2d(\tau)}\Big)$. Substitution into $(*)$
gives $\exp\Big(\dfrac{2\pi i q}{Md(\tau)}\Big) = ss'\exp\Big(\dfrac{2\pi i}{2d(\tau)}\Big)$

However, note that the left hand side of the equation has modulus one, while that on the right hand side is ss'. Hence we have $\exp\left(\frac{2\pi iq}{Md(\tau)}\right) = \exp\left(\frac{2\pi i}{2d(\tau)}\right)$. Therefore:

In $\Lambda(n(\tau), d(\tau), M)$, there are some lines passing through 0

$\Leftrightarrow \exp\left(\frac{2\pi iq}{Md(\tau)}\right) = \exp\left(\frac{2\pi i}{2d(\tau)}\right)$ for some integers q such that $1 \leq q \leq M-1$

$\Leftrightarrow \exp\left(2\pi i\frac{q}{Md(\tau)} - 2\pi i\frac{1}{2d(\tau)}\right) = 1$ for some integers q such that $1 \leq q \leq M-1$

$\Leftrightarrow \frac{2q-M}{2Md(\tau)} \in \mathbf{Z}$ for some integers q such that $1 \leq q \leq M-1$

$\Leftrightarrow 2q - M = k(2Md(\tau))$, where k is an integer for some integers q such that $1 \leq q \leq M-1$

Since $M < 2q - M < M$ and right hand side is a multiple of $2M$, we must have $k = 0$

$\therefore$ There are some lines passing through 0

$\Leftrightarrow 0 = 2q - M$ for some integers q such that $1 \leq q \leq M-1$

$\Leftrightarrow M$ is even.

$\square$

Now, we have gathered enough information to deduce a general formula for the multiplicity of $<\tau>_K$ in the sum $\Phi(n(\tau), d(\tau), M)$, which we recall to be the number of $<\tau>_K$ formed after cutting $<\tau>$ into M parts.

Theorem 37. *The multiplicity of* $<\tau>$ *in the sum* $\Phi(n(\tau), d(\tau), M)$ *is* $I = \frac{1-(-1)^M}{2}$.

Proof. The point 0 can only either be exactly a vertex, where $I = 0$, or inside one face, where $I = 1$. The former case is found when M is even, and the latter when M is odd.

Hence $I = \begin{cases} 1 & \text{if } M \text{ is odd} \\ 0 & \text{if } M \text{ is even} \end{cases}$, which can be combined to give $\frac{1-(-1)^M}{2}$. $\square$

7. More Properties of $\Lambda(n(\tau), d(\tau), M)$

Next we are going to calculate P, or the multiplicity of $<N(\tau)>_K$ in the sum $\Phi(n(\tau), d(\tau), M)$. In order to do so, we have to investigate more properties of $\Lambda(n(\tau), d(\tau), M)$.

Recall that we can discard every point, line segment and region outside the unit circle of $\Lambda(n(\tau), d(\tau), M)$ to obtain the graph of $\Lambda(n(\tau), d(\tau), M)$. Besides, we have defined a vertex, an edge and a face.

So we can deduce, from the graph of $\Lambda(1,6,3)$, the numbers of vertices, edges and faces.

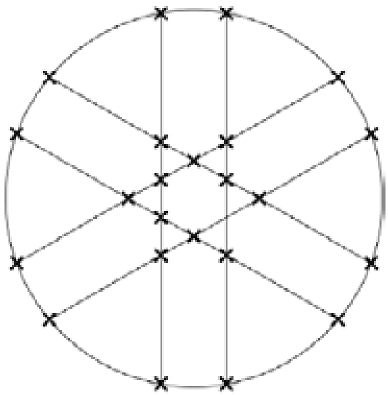


ILLUSTRATION 13. Graph of $\Lambda(1,6,3)$. (Revisited)

There are 24 vertices, 42 edges and 19 faces (excluding the exterior of the unit circle).

Theorem 38 (Face-multiplicity)**.** *The number of faces in the graph of $\Lambda(n(\tau), d(\tau), M)$ is $F = PD(\tau) + I$, where I and P are the number of type I and II knotted tsts in the sum $\Phi(n(\tau), d(\tau), M)$ respectively.*

Proof. Recall that each partial tst comes from a face in $\Lambda(n(\tau), d(\tau), M)$. The number of faces in the graph of $\Lambda(n(\tau), d(\tau), M)$ is equal to the number of partial tsts. Since each of the P knotted tsts $<N(\tau)>_K$ produced from cutting contains $D(\tau)$ partial tsts and $<\tau>_K$ (if produced) contains one partial tst, the total number of partial tsts, and hence that of faces in the graph of $\Lambda(n(\tau), d(\tau), M)$, is $PD(\tau) + I$. □

In the following discussion, we first assumed that $n(\tau) = 1$ for the sake of convenience. We would later drop this assumption and consider the case for other $n(\tau)$'s.

7.1. When are three lines $l_{p1,q1}, l_{p2,q2}, l_{p3,q3}$ in $\Lambda(n(\tau), d(\tau), M)$ concurrent?

Next we consider when three lines $l_{p1,q1}, l_{p2,q2}, l_{p3,q3}$ in $\Lambda(n(\tau), d(\tau), M)$ are concurrent.
For convenience, we introduce the following notation:

Notation 39.

$$|p_1,q_1;p_2,q_2;p_3,q_3| \equiv \begin{vmatrix} [l_{p_1,q_1}(0), \hat{l'}_{p_1,q_1}(s)] & [l_{p_2,q_2}(0), \hat{l'}_{p_2,q_2}(s)] & [l_{p_3,q_3}(0), \hat{l'}_{p_3,q_3}(s)] \\ \hat{l'}_{p_1,q_1}(s) & \hat{l'}_{p_2,q_2}(s) & \hat{l'}_{p_3,q_3}(s) \\ \overline{\hat{l'}_{p_1,q_1}(s)} & \overline{\hat{l'}_{p_2,q_2}(s)} & \overline{\hat{l'}_{p_3,q_3}(s)} \end{vmatrix}$$

where $[a,b] = \frac{1}{2}(\bar{a}b - a\bar{b})$ *for any complex numbers* a *and* b, *and and* $\hat{l}'(s)$ *is the normalized derivative of* $l(s)$ *with respect to* s, *i.e.* $\hat{l}'(s) = \frac{l'(s)}{|l'(s)|}$.

This is the determinant that checks concurrence, in which we have considered the specific three lines $l_{p1,q1}(s_1), l_{p2,q2}(s_2)$ and $l_{p3,q3}(s_3)$ in $\Lambda(n(\tau), d(\tau), M)$ and substituted the suitable values. The reader is referred to the Appendix A for its derivation.

Theorem 40.

$$|p_1, q_1; p_2, q_2; p_3, q_3| = 4[\sin(\pi f_1)\sin(\pi f_2)\sin(\pi f_3) - \sin(\pi f_4)\sin(\pi f_5)\sin(\pi f_6)]$$

, where

$$f_1 = \frac{(p_1 - p_3 + \Delta)M - (M - q_1 - q_3)}{Md(\tau)}, \quad f_4 = \frac{(p_2 - p_1)M + (q_2 - q_1)}{Md(\tau)}$$

$$f_2 = \frac{(p_3 - p_2)M + (q_3 - q_2)}{Md(\tau)}, \quad f_5 = \frac{(p_1 - p_3 + \Delta)M + (M - q_1 - q_3)}{Md(\tau)}$$

$$f_3 = \frac{(p_2 - p_1)M + (q_1 - q_2)}{Md(\tau)}, \quad f_6 = \frac{(p_3 - p_2)M + (q_2 - q_3)}{Md(\tau)}$$

and $\Delta = \frac{d(\tau)}{2}$

Proof. Note that we have $l'_{p,q}(s) = i\big(\exp\frac{2\pi i}{d(\tau)} - 1\big)\exp\frac{2\pi i p}{d(\tau)}$
From the proof of the Lines through Center Lemma, we have:

$$i\Big(\exp\frac{2\pi i}{d(\tau)} - 1\Big)\exp\frac{2\pi i p}{d(\tau)} = -i\Big(1 - \exp\frac{2\pi i}{d(\tau)}\Big)\exp\frac{2\pi i p}{d(\tau)} = -s'\exp\frac{2\pi i p}{d(\tau)}\exp\frac{\pi i}{d(\tau)}$$

where $s' = 2\sin\frac{2\pi}{2d(\tau)} > 0$

Hence, $\hat{l'}_{p,q}(s) = \frac{l'_{p,q}(s)}{|l'_{p,q}(s)|} = -\exp\frac{\pi i(2p+1)}{d(\tau)}$

Consider that $[l_{p,q}(0), \hat{l}'_{p,q}(s)]$

$$= \Big[\exp\frac{2\pi i(pM+q)}{Md(\tau)}, -\exp\frac{\pi i(2p+1)}{d(\tau)}\Big]$$
$$= \Big[\exp\frac{\pi i(2p+1)}{d(\tau)}, \exp\frac{2\pi i(pM+q)}{Md(\tau)}\Big]$$
$$= i\ \mathrm{Im}\Big[\overline{\exp\frac{\pi i(2p+1)}{d(\tau)}}\exp\frac{2\pi i(pM+q)}{Md(\tau)}\Big]$$
$$= i\ \mathrm{Im}\Big[\exp\frac{-\pi i(2p+1)}{d(\tau)}\exp\frac{2\pi i(pM+q)}{Md(\tau)}\Big]$$
$$= i\ \mathrm{Im}\frac{\pi i(2q-M)}{Md(\tau)}$$
$$= i\ \sin\frac{(2q-M)\pi}{Md(\tau)}$$

Hence, $|p_1, q_1; p_2, q_2; p_3, q_3|$

$$= \begin{vmatrix} i\sin\frac{(2q_1-M)\pi}{Md(\tau)} & i\sin\frac{(2q_2-M)\pi}{Md(\tau)} & i\sin\frac{(2q_3-M)\pi}{Md(\tau)} \\ \exp\frac{\pi i(2p_1+1)}{d(\tau)} & \exp\frac{\pi i(2p_2+1)}{d(\tau)} & \exp\frac{\pi i(2p_3+1)}{d(\tau)} \\ \exp\frac{-\pi i(2p_1+1)}{d(\tau)} & \exp\frac{-\pi i(2p_2+1)}{d(\tau)} & \exp\frac{-\pi i(2p_3+1)}{d(\tau)} \end{vmatrix}$$
$$= i\sin\frac{(2q_1-M)\pi}{Md(\tau)}\Big[\exp\frac{2\pi i(p_2-p_3)}{d(\tau)} - \exp\frac{-2\pi i(p_2-p_3)}{d(\tau)}\Big]$$
$$+ i\sin\frac{(2q_2-M)\pi}{Md(\tau)}\Big[\exp\frac{2\pi i(p_3-p_1)}{d(\tau)} - \exp\frac{-2\pi i(p_3-p_1)}{d(\tau)}\Big]$$
$$+ i\sin\frac{(2q_3-M)\pi}{Md(\tau)}\Big[\exp\frac{2\pi i(p_1-p_2)}{d(\tau)} - \exp\frac{-2\pi i(p_1-p_2)}{d(\tau)}\Big]$$
$$= i\sin\frac{(2q_1-M)\pi}{Md(\tau)}\Big[2i\sin\frac{2\pi(p_2-p_3)}{d(\tau)}\Big]$$
$$+ i\sin\frac{(2q_2-M)\pi}{Md(\tau)}\Big[2i\sin\frac{2\pi(p_3-p_1)}{d(\tau)}\Big]$$
$$+ i\sin\frac{(2q_3-M)\pi}{Md(\tau)}\Big[2i\sin\frac{2\pi(p_1-p_2)}{d(\tau)}\Big]$$
$$= -2\sin\frac{(2q_1-M)\pi}{Md(\tau)}\sin\frac{2\pi(p_2-p_3)}{d(\tau)} - 2\sin\frac{(2q_2-M)\pi}{Md(\tau)}\sin\frac{2\pi(p_3-p_1)}{d(\tau)}$$
$$- 2\sin\frac{(2q_3-M)\pi}{Md(\tau)}\sin\frac{2\pi(p_1-p_2)}{d(\tau)}$$

$$
\begin{aligned}
&= \cos\frac{2\pi[(p_2-p_3)M+(q_1-m)]}{Md(\tau)} - \cos\frac{2\pi[(p_2-p_3)M-(q_1-m)]}{Md(\tau)} \\
&\quad + \cos\frac{2\pi[(p_3-p_1)M+(q_2-m)]}{Md(\tau)} - \cos\frac{2\pi[(p_3-p_1)M-(q_2-m)]}{Md(\tau)} \\
&\quad + \cos\frac{2\pi[(p_1-p_2)M+(q_3-m)]}{Md(\tau)} - \cos\frac{2\pi[(p_1-p_2)M-(q_3-m)]}{Md(\tau)}
\end{aligned}
$$

Now $4[\sin(\pi f_1)\sin(\pi f_2)\sin(\pi f_3) - \sin(\pi f_4)\sin(\pi f_5)\sin(\pi f_6)]$

$$
\begin{aligned}
&= 2\{\cos[\pi(f_1-f_2)]\}\sin(\pi f_3) - 2\{\cos[\pi(f_1+f_2)]\}\sin(\pi f_3) \\
&\quad - 2\{\cos[\pi(f_4-f_5)]\}\sin(\pi f_6) + 2\{\cos[\pi(f_4+f_5)]\}\sin(\pi f_6) \\
&= 2\cos\frac{\pi[(p_1+p_2-2p_3+\varDelta)M+q_1+q_2-M]}{Md(\tau)}\sin(\pi f_3) \\
&\quad - 2\cos\frac{\pi[(p_1-p_2+\varDelta)M+q_1-q_2+2q_3-M]}{Md(\tau)}\sin(\pi f_3) \\
&\quad - 2\cos\frac{\pi[(-2p_1+p_2+p_3-\varDelta)M+q_2+q_3-M]}{Md(\tau)}\sin(\pi f_6) \\
&\quad + 2\cos\frac{\pi[(p_2-p_3+\varDelta)M-2q_1+q_2-q_3+M]}{Md(\tau)}\sin(\pi f_6) \\
&= -2\sin\frac{\pi[(p_1+p_2-2p_3)M+q_1+q_2-M]}{Md(\tau)}\sin\frac{\pi[(p_2-p_1)M+(q_1-q_2)]}{Md(\tau)} \\
&\quad + 2\sin\frac{\pi[(p_1-p_2)M+q_1-q_2+2q_3-M]}{Md(\tau)}\sin\frac{\pi[(p_2-p_1)M+(q_1-q_2)]}{Md(\tau)} \\
&\quad - 2\sin\frac{\pi[(-2p_1+p_2+p_3)M+q_2+q_3-M]}{Md(\tau)}\sin\frac{\pi[(p_3-p_2)M+(q_2-q_3)]}{Md(\tau)} \\
&\quad - 2\sin\frac{\pi[(p_2-p_3)M-2q_1+q_2-q_3+M]}{Md(\tau)}\sin\frac{\pi[(p_3-p_2)M+(q_2-q_3)]}{Md(\tau)} \\
&= -\cos\frac{2\pi[(p_3-p_1)M-(q_2-m)]}{Md(\tau)} + \cos\frac{2\pi[(p_2-p_3)M+(q_1-m)]}{Md(\tau)} \\
&\quad + \cos\frac{2\pi[(p_1-p_2)M+(q_3-m)]}{Md(\tau)} - \cos\frac{2\pi(q_1-q_2+q_3-m)}{Md(\tau)} \\
&\quad - \cos\frac{2\pi[(p_1-p_2)M-(q_3-m)]}{Md(\tau)} + \cos\frac{2\pi[(p_3-p_1)M+(q_2-m)]}{Md(\tau)} \\
&\quad - \cos\frac{2\pi[(p_2-p_3)M-(q_1-m)]}{Md(\tau)} + \cos\frac{2\pi(q_1-q_2+q_3-m)}{Md(\tau)} \\
&= \cos\frac{2\pi[(p_2-p_3)M+(q_1-m)]}{Md(\tau)} - \cos\frac{2\pi[(p_2-p_3)M-(q_1-m)]}{Md(\tau)} \\
&\quad + \cos\frac{2\pi[(p_3-p_1)M+(q_2-m)]}{Md(\tau)} - \cos\frac{2\pi[(p_3-p_1)M-(q_2-m)]}{Md(\tau)} \\
&\quad + \cos\frac{2\pi[(p_1-p_2)M+(q_3-m)]}{Md(\tau)} - \cos\frac{2\pi[(p_1-p_2)M-(q_3-m)]}{Md(\tau)}
\end{aligned}
$$

$= |p_1, q_1; p_2, q_2; p_3, q_3|$

□

In the following discussion, we expand use of the notation of $l_{p,q}(s)$ in $\Lambda(n(\tau), d(\tau), M)$ from $0 \leq p \leq D(\tau) - 1$ to all integral values of p. This means that for any integer $k, l_{p+d(\tau)k,q}(s)$ represents the same line.

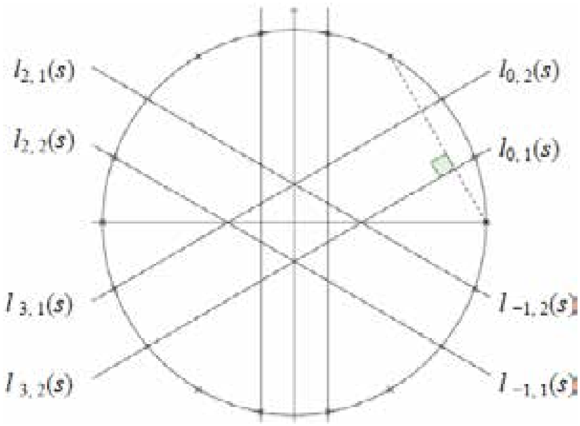


ILLUSTRATION 14. $l_{p,q}$ with various integral values of p

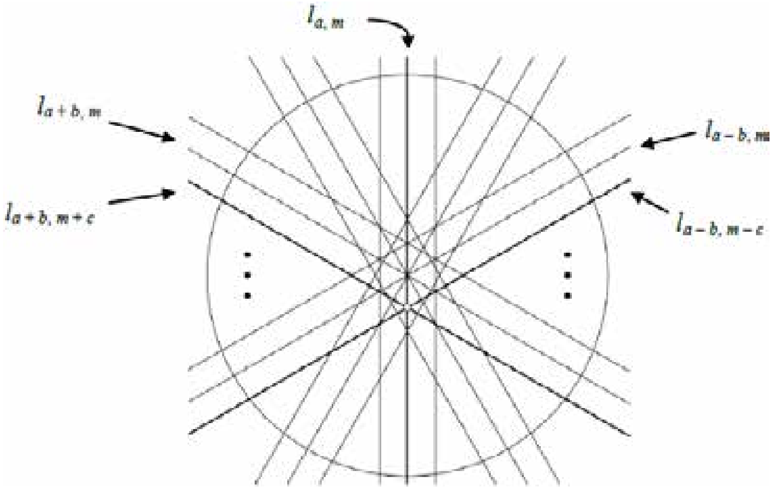


ILLUSTRATION 15. $l_{a+b,m+c}$ and $l_{a-b,m-c}$ are symmetrical about the line $l_{a,m}$

By careful observation, we can find that the lines $l_{a+b,m+c}$ and $l_{a-b,m-c}$ are symmetrical about the line $l_{a,m}$. This is proven below. We call this configuration of three lines "Config I".

Lemma 41. *In $\Lambda(n(\tau), d(\tau), 2m)$, the lines $l_{a+b,m+c}, l_{a,m}$ and $l_{a-b,m-c}$ are concurrent, where $l_{p,q}$ is the line $l_{p,q}(s) = \exp\dfrac{2\pi i(pM+q)}{Md(\tau)} + is[\exp(2\pi i\tau p)]\Big(\exp\dfrac{2\pi i}{d(\tau)} - 1\Big)$.*

Proof. Applying a theorem proven in the Appendix A, the lines $l_{p1,q1}(s_1), l_{p2,q2}(s_2)$ and $l_{p3,q3}(s_3)$ are concurrent

$$\Leftrightarrow |p_1,q_1;p_2,q_2;p_3,q_3| = 0$$

$$\Leftrightarrow \begin{aligned} 0 = &-2\sin\frac{(2q_1-M)\pi}{Md(\tau)}\sin\frac{2\pi(p_2-p_3)}{d(\tau)} - 2\sin\frac{(2q_2-M)\pi}{Md(\tau)}\sin\frac{2\pi(p_3-p_1)}{d(\tau)} \\ &- 2\sin\frac{(2q_3-M)\pi}{Md(\tau)}\sin\frac{2\pi(p_1-p_2)}{d(\tau)} \end{aligned}$$

which is proven in the last Theorem.
Let $d(\tau) = 2\Delta$, so that we have:

$$\begin{aligned} &-2\sin\frac{(2m+2c-2m)\pi}{2m(2\Delta)}\sin\frac{2\pi b}{2\Delta} - 2\sin\frac{(2m-2m)\pi}{2m(2\Delta)}\sin\frac{2\pi(-2b)}{2\Delta} \\ &\quad - 2\sin\frac{(2m-2c-2m)\pi}{2m(2\Delta)}\sin\frac{2\pi b}{2\Delta} \\ &= -2\sin\frac{c\pi}{2m\Delta}\sin\frac{b\pi}{\Delta} - 2\sin\,0\sin\frac{-2\pi b}{\Delta} - 2\sin\frac{-c\pi}{2m\Delta}\sin\frac{b\pi}{\Delta} \\ &= -2\sin\frac{c\pi}{2m\Delta}\sin\frac{b\pi}{\Delta} - 0 + 2\sin\frac{c\pi}{2m\Delta}\sin\frac{b\pi}{\Delta} \\ &= 0 \end{aligned}$$

$\therefore l_{a+b,m+c}, l_{a,m}$ and $l_{a-b,m-c}$ are concurrent.

□

We then go on and figure out other configurations of three lines in $\Lambda(1, d(\tau), M)$ such that they concur. Except the above one, it is obvious that any three lines passing through the origin are concurrent, which can be found only when M is even. We call this configuration of lines "Config II"
In the following discussion, for the sake of simplicity, we write $d(\tau) = 2\Delta$ and $M = 2m$.
We consider the following fact: for $d(\tau) \leq 2$, we cannot find three nonparallel lines in $\Lambda(1, d(\tau), M)$ from its definition. Hence, we assume that $d(\tau) \geq 2$, i.e. $\Delta > 1$. Moreover, we also note that if $d(\tau) = 4$, there are only two families of parallel lines, so it is known that $d(\tau)$ is even, we assume that $d(\tau) \geq 6$, i.e. $\Delta \geq 3$.

Proposition 42. *A line represented by $l_{p+\Delta,M-q}(s)$ is the same as the line represented by $l_{p,q}(s)$.*

Proof. We notice that in fact the line $l_{p+\Delta,M-q}(s)$ takes the equation:

$$\begin{aligned}&\exp\frac{2\pi i[(p+\Delta)M+(M-q)]}{Md(\tau)}+is\exp\frac{2\pi i(p+\Delta)}{d(\tau)}\Big(\exp\frac{2\pi i}{d(\tau)}-1\Big)\\ &=-\exp\frac{2\pi i[(p+1)M-q]}{Md(\tau)}\exp\frac{2\pi i}{d(\tau)}-is\exp\frac{2\pi ip}{d(\tau)}\Big(\exp\frac{2\pi i}{d(\tau)}-1\Big)\end{aligned}$$

But note that, from the proof of the Line through Center Lemma, we know that

$$l'_{p+\Delta,M-q}(s)=-i\exp\frac{2\pi ip}{d(\tau)}\Big(\exp\frac{2\pi i}{d(\tau)}-1\Big)$$

Moreover, since

$$\begin{aligned}&[\exp(2\pi i\zeta_1)+\exp(2\pi i\zeta_2),\ \exp\frac{2\pi i(\zeta_1+\zeta_2)}{2}]\\ =&i\mathrm{Im}\{[\exp(-2\pi i\zeta_1)+\exp(-2\pi i\zeta_2)]\exp\frac{2\pi i(\zeta_1+\zeta_2)}{2}\}\\ =&i\mathrm{Im}\Big[\exp\frac{2\pi i(\zeta_2-\zeta_1)}{2}+\exp\frac{-2\pi i(\zeta_2-\zeta_1)}{2}\Big]\\ =&i\mathrm{Im}\Big[2\cos\frac{2\pi(\zeta_2-\zeta_1)}{2}\Big]\\ =&i(0)=0,\end{aligned}$$

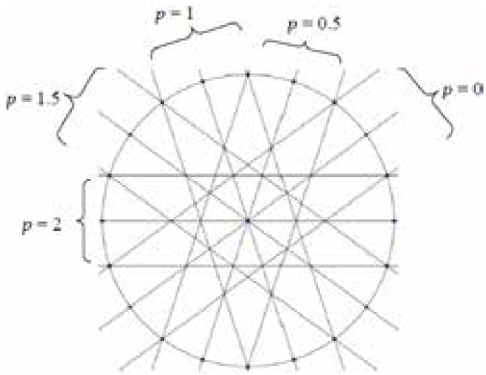
the number $\exp(2\pi i\zeta_1)+\exp(2\pi i\zeta_2)$ is parallel to $\exp\frac{2\pi i(\zeta_1+\zeta_2)}{2}$ and this allows us to write $\exp(2\pi i\zeta_1)+\exp(2\pi i\zeta_2)=r\exp\frac{2\pi i(\zeta_1+\zeta_2)}{2}$ where r is a real number. Therefore we have:

$$\begin{aligned}&\frac{\exp\frac{2\pi i(pM+q)}{Md(\tau)}-l_{p+\Delta,M-q}(0)}{l'_{p+\Delta,M-q}(s)}\\ =&\frac{\exp\frac{2\pi i(pM+q)}{Md(\tau)}-\Big[-\exp\frac{2\pi i[(p+1)M-q]}{Md(\tau)}\Big]}{-i\exp\frac{2\pi ip}{d(\tau)}\Big(\exp\frac{2\pi i}{d(\tau)}-1\Big)}\\ =&\frac{\exp\frac{2\pi ip}{d(\tau)}\Big[\exp\frac{2\pi iq}{Md(\tau)}+\exp\frac{2\pi i(M-q)}{Md(\tau)}\Big]}{-s'\exp\frac{2\pi ip}{d(\tau)}\exp\frac{2\pi i}{2d(\tau)}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\exp \dfrac{2\pi i q}{Md(\tau)} + \exp \dfrac{2\pi i(M-q)}{Md(\tau)}}{-s' \exp \dfrac{2\pi i}{2d(\tau)}} \\
&= \frac{r \exp \dfrac{\dfrac{2\pi i q}{Md(\tau)} + \dfrac{2\pi i(M-q)}{Md(\tau)}}{2}}{-s' \exp \dfrac{2\pi i}{2d(\tau)}} \\
&= \frac{r \exp \dfrac{2\pi i}{2d(\tau)}}{-s' \exp \dfrac{2\pi i}{2d(\tau)}} \\
&= -\frac{r}{s'} \\
&\in \mathbf{R}
\end{aligned}
$$

Hence, the point $\exp \dfrac{2\pi i(pM+q)}{Md(\tau)}$ lies on the line $l_{p+\Delta,M-q}$.
This shows that we can choose another parametrization for the same line:
$l_{p+\Delta,M-q}(s^*) = \exp \dfrac{2\pi i(pM+q)}{Md(\tau)} - is^* \exp \dfrac{2\pi ip}{d(\tau)}\Big(\exp \dfrac{2\pi i}{d(\tau)} - 1\Big)$
And by considering
$l_{p+\Delta,M-q}(-s^*) = \exp \dfrac{2\pi i(pM+q)}{Md(\tau)} + is^* \exp \dfrac{2\pi ip}{d(\tau)}\Big(\exp \dfrac{2\pi i}{d(\tau)} - 1\Big)$
we recover all the points on the line $l_{p,q}(s)$. □

This above proposition allows us to restrict, for any three lines under consideration, the p_1, p_2 and p_3 such that $0 \le p_1, p_2, p_3 < \Delta$.
The following diagram illustrates the use of this proposition on $\Lambda(1,5,4)$ where we have, for some lines $l_{p,q}(s), p$ is not an integer but $2p$ is:



To find out other configurations, we note that the lines $l_{p1,q1}, l_{p2,q2}$ and $l_{p3,q3}$ are concurrent if and only if the following equation holds: $4[\sin(\pi f_1)\sin(\pi f_2)\sin(\pi f_3) - \sin(\pi f_4)\sin(\pi f_5)\sin(\pi f_6)] = 0$,

$$\text{i.e. } \sin(\pi f_1)\sin(\pi f_2)\sin(\pi f_3) = \sin(\pi f_4)\sin(\pi f_5)\sin(\pi f_6), -------- \quad \text{(C)}$$

where:

$$f_1 = \frac{(p_1 - p_3 + \Delta)M - (M - q_1 - q_3)}{Md(\tau)}, \quad f_4 = \frac{(p_2 - p_1)M + (q_2 - q_1)}{Md(\tau)}$$
$$f_2 = \frac{(p_3 - p_2)M + (q_3 - q_2)}{Md(\tau)}, \quad f_5 = \frac{(p_1 - p_3 + \Delta)M + (M - q_1 - q_3)}{Md(\tau)}$$
$$f_3 = \frac{(p_2 - p_1)M + (q_1 - q_2)}{Md(\tau)}, \quad f_6 = \frac{(p_3 - p_2)M + (q_2 - q_3)}{Md(\tau)}$$
$$\text{and } \Delta = \frac{d(\tau)}{2}.$$

We note that $f_1 + f_2 + f_3 + f_4 + f_5 + f_6 = 1$, so that we can follow the paper [6] for all rational solutions to the equation (C), which lists all possible $f_1, f_2, \ldots, f_6$ such that equation (C) is satisfied. We start with the following matrix equation for $f_1, f_2, \ldots, f_6$ in terms of $p_1, p_2, p_3, q_1, q_2, q_3$ as follows:

$$Md(\tau)\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix} = \begin{pmatrix} \Delta M - M \\ 0 \\ 0 \\ 0 \\ \Delta M + M \\ 0 \end{pmatrix} + \begin{pmatrix} M & 0 & -M & 1 & 0 & 1 \\ 0 & -M & M & 0 & -1 & 1 \\ -M & M & 0 & 1 & -1 & 0 \\ -M & M & 0 & -1 & 1 & 0 \\ M & 0 & -M & -1 & 0 & -1 \\ 0 & -M & M & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

We denote the column vector $(f_1 \ \ f_2 \ \ f_3 \ \ f_4 \ \ f_5 \ \ f_6)^{\mathrm{T}}$ by $\mathbf{f}$.
Performing elementary row operation, we obtain:

$$\left(\begin{array}{cccccc|cccccc} M & 0 & -M & 1 & 0 & 1 & 1 & & & & & \\ 0 & -M & M & 0 & -1 & 1 & & 1 & & & & \\ -M & M & 0 & 1 & -1 & 0 & & & 1 & & & \\ -M & M & 0 & -1 & 1 & 0 & & & & 1 & & \\ M & 0 & -M & -1 & 0 & -1 & & & & & 1 & \\ 0 & -M & M & 0 & 1 & -1 & & & & & & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|cccccc} 2M & 0 & -2M & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2M & 2M & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1/4 & -1/4 & 1/4 & -1/4 & -1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1/4 & -1/4 & -1/4 & 1/4 & -1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/4 & 1/4 & -1/4 & 1/4 & -1/4 & -1/4 \end{array}\right)$$

From this result we can then solve for the p's and the q's.
The q's can be solved directly:

$$\begin{aligned}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} &= \frac{Md(\tau)}{4}\begin{pmatrix} 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 \end{pmatrix}\mathbf{f} + \begin{pmatrix} m \\ m \\ m \end{pmatrix} \\ &= m\Delta\begin{pmatrix} 2 & 0 & 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 2 & 0 & 0 \end{pmatrix}\mathbf{f} - m\Delta\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\mathbf{f} + \begin{pmatrix} m \\ m \\ m \end{pmatrix} \\ &= m\Delta\begin{pmatrix} 2 & 0 & 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 2 & 0 & 0 \end{pmatrix}\mathbf{f} + \begin{pmatrix} m - m\Delta \\ m - m\Delta \\ m - m\Delta \end{pmatrix}\end{aligned}$$

Here we have used the fact that $(1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)\mathbf{f} = 1$. Then we consider the p's. Since $\Lambda(n(\tau), d(\tau), M)$ is rotational symmetric, we can first assume that $p_1 = 0$. We also further restrict that $p_2 < p_3$. In combined form, this is $0 = p_1 < p_2 < p_3 < \Delta$. We therefore have:

$$2M\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \Delta\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\mathbf{f} + \begin{pmatrix} -2\Delta M \\ 0 \\ 0 \end{pmatrix}$$

Then we can then solve for p_2 and p_3, which are:

$$\begin{pmatrix} p_2 \\ p_3 \end{pmatrix} = \Delta\begin{pmatrix} -1 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}\mathbf{f} + \begin{pmatrix} \Delta \\ \Delta \end{pmatrix} = \Delta\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}\mathbf{f} + \begin{pmatrix} 0 \\ \Delta \end{pmatrix}$$

Here we have also used $(1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)\mathbf{f} = 1$.
According to the paper [6], there are three types of solutions to the equation (C): the trivial solutions, where f_4, f_5, f_6 is a permutation of f_1, f_2, f_3, four families of one-parameter infinite solutions, and sixty-five sporadic solutions. We can substitute f_1, f_2, f_3 and so on into the solutions listed in the paper [6] and then find whether there are some reasonable configurations of the three lines besides Config. I and II. Details of the consideration can be found in Appendix IV.

<u>Case A</u> The "trivial" solutions.
These are the solutions with $f_1+f_2+f_3 = f_4+f_5+f_6 = \dfrac{1}{2}$ and f_4, f_5, f_6 is a permutation of f_1, f_2, f_3. In other words, we have six subcases: $\begin{cases} f_1 = f_4 \\ f_2 = f_5 \\ f_3 = f_6 \end{cases}$, $\begin{cases} f_1 = f_6 \\ f_2 = f_4 \\ f_3 = f_5 \end{cases}$, $\begin{cases} f_1 = f_5 \\ f_2 = f_6 \\ f_3 = f_4 \end{cases}$, $\begin{cases} f_1 = f_4 \\ f_2 = f_6 \\ f_3 = f_5 \end{cases}$, $\begin{cases} f_1 = f_5 \\ f_2 = f_4 \\ f_3 = f_6 \end{cases}$ and $\begin{cases} f_1 = f_6 \\ f_2 = f_5 \\ f_3 = f_4 \end{cases}$.

<u>Case B</u> One-parameter infinite solutions
In order to reduce the number of cases, by symmetry of $\Lambda(n(\tau), d(\tau), M)$ and substitute the value of U only into f_1 in each of the subcases. We consider twelve

permutation matrices as below (with every unspecified entry zero):

$$P_1 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & & & & & 1 \\ & & & 1 & & \end{pmatrix}, P_3 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & 1 & & \\ & & & & 1 & \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & & 1 \\ & & & & 1 & \end{pmatrix}, P_5 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & & & & 1 \end{pmatrix}, P_6 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \end{pmatrix},$$

$$P_7 = \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, P_8 = \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & 1 & \\ & & & & & 1 \\ & & & 1 & & \end{pmatrix}, P_9 = \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & & 1 \\ & & & 1 & & \\ & & & & 1 & \end{pmatrix},$$

$$P_{10} = \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & & & & 1 \\ & & & & 1 & \end{pmatrix}, P_{11} = \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & & & & 1 \end{pmatrix}, P_{12} = \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \end{pmatrix}$$

From the paper [6], we can form four column vectors for the solutions:

$$s_1 = \begin{pmatrix} \frac{1}{6} \\ t \\ \frac{1}{3} - 2t \\ \frac{1}{3} + t \\ t \\ \frac{1}{6} - t \end{pmatrix}, s_2 = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{2} - 3t \\ t \\ \frac{1}{6} - t \\ 2t \\ \frac{1}{6} + t \end{pmatrix}, \text{both for } 0 < t < \frac{1}{6}; \text{ and}$$

$$s_3 = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} - 2t \\ 2t \\ \frac{1}{6} - 2t \\ t \\ \frac{1}{2} + t \end{pmatrix}, s_4 = \begin{pmatrix} \frac{1}{3} - 4t \\ t \\ \frac{1}{3} + t \\ \frac{1}{6} - 2t \\ 3t \\ \frac{1}{6} + t \end{pmatrix}, \text{both for } 0 < t < \frac{1}{12}.$$

Then in Appendix IV, in the subcase "$B\ i.j$", we substitute $\mathbf{f} = P_j s_i$ where $i = 1, 2, 3, 4$ and $j = 1, 2, 3, \ldots, 12$ and then deduce the range of values of p_1, p_2, p_3, q_1, q_2 and q_3 if the case is not rejected.

Case C Sporadic solutions
The paper [6] provides 65 sets of exact values satisfying the equation (C). We consider this case using a Scilab program. Program codes can be found in Appendix B.

Combining all the logical arguments in the above analysis, we see that the solutions to equation (C) other than those giving Config I and II are as follows: for $k \in \mathbf{Z}^+$,

In $\Lambda(1,5,12k)$, $|0,3k;0.5,k;1,k| = 0$ ($\underline{B1.3}$)

$|0,k;0.5,k;1,3k| = 0$ ($\underline{B1.11}$)

$|0,k;0.5,3k;1.5,9k| = 0$ ($\underline{B2.2}$)

$|0,9k;0.5,6k;1.5,k| = 0$ ($\underline{B2.7}$)

$|0,9k;1,3k;1.5,k| = 0$ ($\underline{B2.10}$)

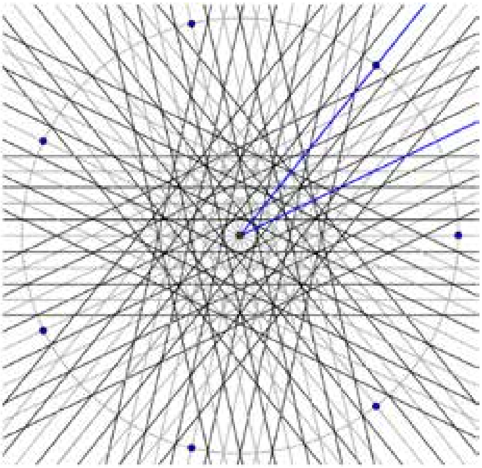
$|0,11k;1.5,6k;2,3k| = 0$ ($\underline{B4.3}$)

$|0,3k;1.5,6k;2,11k| = 0$ ($\underline{B4.11}$)

In $\Lambda(1,7,12k)$, $|0,3k;0.5,5k;2.5,11k| = 0$ ($\underline{B2.5}$)

$|0,11k;2,5k;2.5,3k| = 0$ ($\underline{B2.9}$)

For solutions in $\Lambda(1,5,12k)$, we can know that the lines involved are included by $\Lambda(1,5,12)$ (since the values of 'q's are all multiples of k). As both $\Lambda(1,5,12k)$ and $\Lambda(1,5,12)$ are 10-fold rotational symmetric about the origin, all concurrence in $\Lambda(1,5,12k)$ derived from the solutions also appear in $\Lambda(1,5,12)$. By studying $\Lambda(1,5,12)$ directly, we can know that there are 10 pencils of 5 concurrent lines in $\Lambda(1,5,12k)$. Similarly, there are 28 pencils of 3 concurrent lines in $\Lambda(1,7,12k)$.



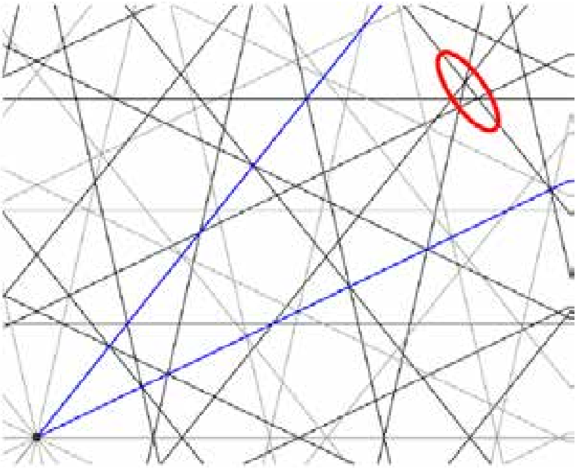


ILLUSTRATION 16. $\Lambda(1,7,12)$ and a part of it

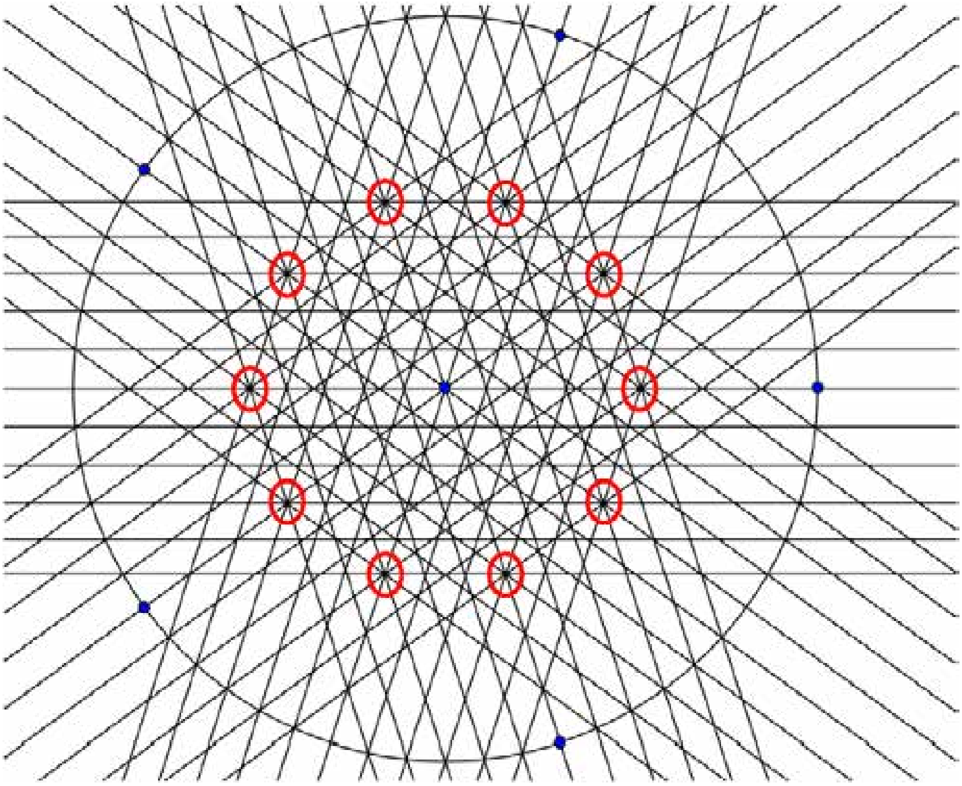


ILLUSTRATION 17. $\Lambda(1,5,12)$

7.2. When do lines $l_{p1,q1}, l_{p2,q2}$ intersect outside the unit circle?

We next consider the problem as in the title and propose the following theorem to answer the above captioned question.

Proposition 43. *If $d(\tau) \geq 3$ is an integer, and p_1 and p_2 satisfy the inequality $\frac{1}{2} \leq |p_1 - p_2| < \Delta = \frac{d(\tau)}{2}$, then $0 < \sin \frac{2\pi|p_1 - p_2|}{d(\tau)} \leq 1$.*

Proof. Through division by $d(\tau) \geq 3$, we have $\frac{1}{2d(\tau)} \leq \frac{|p_1 - p_2|}{d(\tau)} < \frac{1}{2}$.
This yields $0 < \frac{2\pi}{2d(\tau)} \leq \frac{2\pi|p_1 - p_2|}{d(\tau)} < \frac{2\pi}{2}$.
The stated result thus follows. □

Proposition 44. *If $d(\tau)$ is even, then no lines intersect outside the unit circle.*

Proof. Because $d(\tau)$ is even, for any two non-parallel lines $l_{p1,q1}(s)$ and $l_{p2,q2}(s)$ under consideration, we can restrict that $1 \leq p_2 - p_1 \leq \Delta - 1$ and note that we also have $0 \leq |q_1 - q_2| \leq M - 2$. Then by solving $l_{p1,q1}(s_1) = l_{p2,q2}(s_2)$, i.e.

$$\begin{aligned}
&\exp(2\pi i\frac{p_1M + q_1}{Md(\tau)}) + s_1 i \exp\frac{2\pi i p_1}{d(\tau)} (\exp\frac{2\pi i}{d(\tau)} - 1) \\
&\quad = \exp(2\pi i\frac{p_2M + q_2}{Md(\tau)}) + s_2 i \exp\frac{2\pi i p_2}{d(\tau)} (\exp\frac{2\pi i}{d(\tau)} - 1).
\end{aligned}$$

We now find a condition on s such that the point $l_{p,q}(s)$ lies outside the unit circle. In other words, we solve $|l_{p,q}(s)| > 1$, or $|l_{p,q}(s)|^2 > 1$, i.e.

$$\begin{aligned}
&|\exp(2\pi i\frac{pM + q}{Md(\tau)}) - si\exp\frac{2\pi i p}{d(\tau)}(1 - \exp\frac{2\pi i}{d(\tau)})|^2 > 1 \\
&\Big[\exp\Big(2\pi i\frac{pM + q}{Md(\tau)}\Big) - si\exp\frac{2\pi i p}{d(\tau)}\Big(1 - \exp\frac{2\pi i}{d(\tau)}\Big)\Big]\Big[\exp\Big(-2\pi i\frac{pM + q}{Md(\tau)}\Big) \\
&+ si\exp\frac{-2\pi i p}{d(\tau)}\Big(1 - \exp\frac{-2\pi i}{d(\tau)}\Big)\Big] > 1 \\
&s^2\Big[\exp\frac{2\pi i p}{d(\tau)}\Big(1 - \exp\frac{2\pi i}{d(\tau)}\Big)\Big]\Big[\exp\frac{-2\pi i p}{d(\tau)}\Big(1 - \exp\frac{-2\pi i}{d(\tau)}\Big)\Big] \\
&- is\Big[\Big(1 - \exp\frac{2\pi i}{d(\tau)}\Big)\exp\Big(2\pi i\frac{-q}{Md(\tau)}\Big) - (1 - \exp\frac{-2\pi i}{d(\tau)}\Big)\exp\Big(2\pi i\frac{q}{Md(\tau)}\Big)\Big] > 0
\end{aligned}$$

Using the fact that

$$\begin{aligned}
&\Big[\exp\frac{2\pi i p}{d(\tau)}\Big(1 - \exp\frac{2\pi i}{d(\tau)}\Big)\Big]\Big[\exp\frac{-2\pi i p}{d(\tau)}\Big(1 - \exp\frac{-2\pi i}{d(\tau)}\Big)\Big] \\
=&\Big|\exp\frac{2\pi i p}{d(\tau)}\Big(1 - \exp\frac{2\pi i}{d(\tau)}\Big)\Big|^2 = 4\sin^2\frac{\pi}{d(\tau)}
\end{aligned}$$

and that

$$\begin{aligned}
&\Big(1-\exp\frac{2\pi i}{d(\tau)}\Big)\exp\Big(2\pi i\frac{-q}{Md(\tau)}\Big)-\Big(1-\exp\frac{-2\pi i}{d(\tau)}\Big)\exp\Big(2\pi i\frac{q}{Md(\tau)}\Big)\\
&=\exp\Big(2\pi i\frac{-q}{Md(\tau)}\Big)-\exp\frac{2\pi i(M-q)}{Md(\tau)}-\exp\Big(2\pi i\frac{q}{Md(\tau)}\Big)+\exp\frac{-2\pi i(M-q)}{Md(\tau)}\\
&=-2i\sin\frac{2\pi q}{Md(\tau)}-2i\sin\frac{2\pi(M-q)}{Md(\tau)}\\
&=-4i\sin\frac{\pi}{d(\tau)}\cos\frac{2\pi(q-m)}{Md(\tau)},\ \text{where } M=2m,
\end{aligned}$$

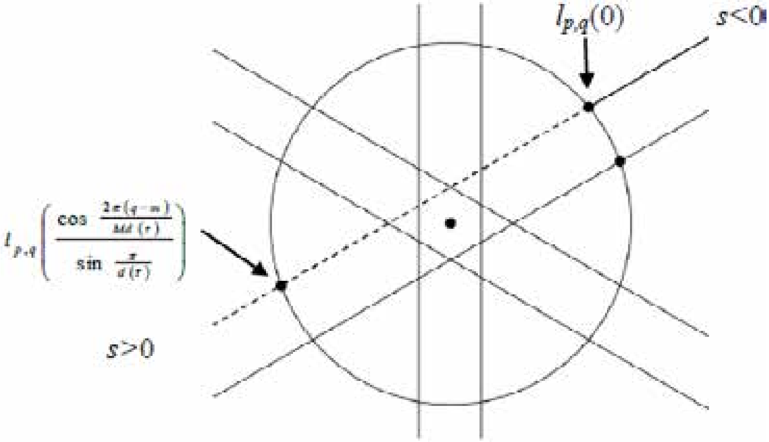
the inequality becomes

$$4s^2\sin^2\frac{\pi}{d(\tau)}-4s\sin\frac{\pi}{d(\tau)}\cos\frac{2\pi(q-m)}{Md(\tau)}>0$$

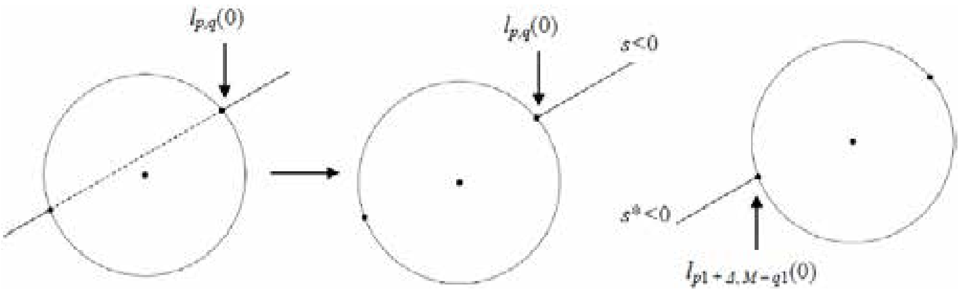
$$s\Big(s\sin\frac{\pi}{d(\tau)}-\cos\frac{2\pi(q-m)}{Md(\tau)}\Big)>0$$

Since $d(\tau)$ is an even integer, $0\le\Big|\frac{q-m}{Md(\tau)}\Big|\le\frac{m-1}{Md(\tau)}<\frac{1}{4}$, $\cos\frac{2\pi(q-m)}{Md(\tau)}>0$. In addition, we have $\sin\frac{\pi}{d(\tau)}>0$. Hence, we have;

$$s<0 \text{ or } s>\frac{\cos\frac{2\pi(q-m)}{Md(\tau)}}{\sin\frac{\pi}{d(\tau)}}>0.$$



But if the intersection point of any two lines $l_{p1,q1}(s_1)$ and $l_{p2,q2}(s_2)$ is outside the unit circle and $s_1<0$ at the intersection point, we can change the parametrization of the line $l_{p1,q1}(s_1)$ to $l_{p1+\Delta,M-q1}(s_1{}^*)$ such that $s_1{}^*$ is now positive.



Hence we can take $s_1 < 0$ as a necessary condition for the two lines to intersect outside the unit circle. Now we deduce the formula for s_1 in terms of the p's and the q's. We introduce two new variables $\tilde{s}_1$ and $\tilde{s}_2$ such that

$$\tilde{s}_1 \exp(\pi i \frac{2p_1+1}{d(\tau)}) = s_1 i \exp\frac{2\pi i p_1}{d(\tau)}(\exp\frac{2\pi i}{d(\tau)} - 1) \text{ and}$$
$$\tilde{s}_2 \exp(\pi i \frac{2p_2+1}{d(\tau)}) = s_2 i \exp\frac{2\pi i p_2}{d(\tau)}(\exp\frac{2\pi i}{d(\tau)} - 1)$$

Here $\tilde{s}_1$ and $\tilde{s}_2$ are apparently real, since

$$\frac{\pi(2p+1)}{d(\tau)} = \arg\Big[i\Big(\exp\frac{2\pi i p}{d(\tau)}\Big)\Big(\exp\frac{2\pi i}{d(\tau)} - 1\Big)\Big]$$

Then we have $s = -\dfrac{1}{2\sin\frac{\pi}{d(\tau)}}\tilde{s}$ and the condition $s < 0$ becomes $\tilde{s} > 0$

Rearrangement from $l_{p1,q1}(\tilde{s}_1) = l_{p2,q2}(\tilde{s}_2)$ gives

$$\tilde{s}_1 \exp(\pi i \frac{2p_1+1}{d(\tau)}) - \tilde{s}_2 \exp(\pi i \frac{2p_2+1}{d(\tau)}) = -\exp(2\pi i \frac{p_1 M + q_1}{M d(\tau)}) + \exp(2\pi i \frac{p_2 M + q_2}{M d(\tau)})$$

We note that for the following equation:
$\tilde{s}_1 z_1 + \tilde{s}_2 z_2 = z_3$ where z_1, z_2 and z_3 are complex numbers, we can have

$$\begin{cases} \tilde{s}_1 \mathrm{Re}(z_1) + \tilde{s}_2 \mathrm{Re}(z_2) = \mathrm{Re}(z_3) \\ \tilde{s}_1 \mathrm{Im}(z_1) + \tilde{s}_2 \mathrm{Im}(z_2) = \mathrm{Im}(z_3) \end{cases}$$

Cramer's Rule yields

$$\tilde{s}_1 = \frac{\begin{vmatrix} \mathrm{Re}(z_3) & \mathrm{Re}(z_2) \\ \mathrm{Im}(z_3) & \mathrm{Im}(z_2) \end{vmatrix}}{\begin{vmatrix} \mathrm{Re}(z_1) & \mathrm{Re}(z_2) \\ \mathrm{Im}(z_1) & \mathrm{Im}(z_2) \end{vmatrix}} = \frac{\begin{vmatrix} \mathrm{Re}(z_3) - \mathrm{Im}(z_3) & \mathrm{Re}(z_2) - \mathrm{Im}(z_2) \\ 2\mathrm{Im}(z_3) & 2\mathrm{Im}(z_2) \end{vmatrix}}{\begin{vmatrix} \mathrm{Re}(z_1) - \mathrm{Im}(z_1) & \mathrm{Re}(z_2) - \mathrm{Im}(z_2) \\ 2\mathrm{Im}(z_1) & 2\mathrm{Im}(z_2) \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \mathrm{Re}(z_3) - \mathrm{Im}(z_3) & \mathrm{Re}(z_2) - \mathrm{Im}(z_2) \\ \mathrm{Re}(z_3) + \mathrm{Im}(z_3) & \mathrm{Re}(z_2) + \mathrm{Im}(z_2) \end{vmatrix}}{\begin{vmatrix} \mathrm{Re}(z_1) - \mathrm{Im}(z_1) & \mathrm{Re}(z_2) - \mathrm{Im}(z_2) \\ \mathrm{Re}(z_1) + \mathrm{Im}(z_1) & \mathrm{Re}(z_2) + \mathrm{Im}(z_2) \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \overline{z_3} & \overline{z_2} \\ z_3 & z_2 \end{vmatrix}}{\begin{vmatrix} \overline{z_1} & \overline{z_2} \\ z_1 & z_2 \end{vmatrix}} = \frac{\overline{z_3}z_2 - z_3\overline{z_2}}{\overline{z_1}z_2 - z_1\overline{z_2}} = \frac{2i\ \mathrm{Im}(\overline{z_3}z_2)}{2i\ \mathrm{Im}(\overline{z_1}z_2)}$$

$$= \frac{[z_3, z_2]}{[z_1, z_2]}$$

where $[a, b] = i\mathrm{Im}(\overline{a}b)$ is the complex bracket for any complex numbers a and b. Therefore, we have

$$\tilde{s}_1 = \frac{\left[-\exp\left(2\pi i \frac{p_1 M + q_1}{M d(\tau)}\right) + \exp\left(2\pi i \frac{p_2 M + q_2}{M d(\tau)}\right), -\exp\frac{\pi i(2p_2+1)}{d(\tau)}\right]}{\left[\exp\frac{\pi i(2p_1+1)}{d(\tau)}, -\exp\frac{\pi i(2p_2+1)}{d(\tau)}\right]}$$

where $[a, b] = i\mathrm{Im}(\overline{a}b)$ is the complex bracket for any complex numbers a and b.

The denominator is:

$$\left[\exp\frac{\pi i(2p_1+1)}{d(\tau)}, -\exp\frac{\pi i(2p_2+1)}{d(\tau)}\right] = \left[\exp\frac{\pi i(2p_2+1)}{d(\tau)}, \exp\frac{\pi i(2p_1+1)}{d(\tau)}\right]$$

$$= i\mathrm{Im}\exp\left[\pi i\left(\frac{2p_2+1}{d(\tau)} - \frac{2p_1+1}{d(\tau)}\right)\right] = -i\sin 2\pi\frac{p_2-p_1}{d(\tau)}$$

Note that $1 \le p_2 - p_1 \le \Delta - 1 < \Delta$ so we can use the previous propositions to give $-1 \le -2\sin 2\pi\frac{p_2-p_1}{d(\tau)} < 0$.

Similarly, we can show that the numerator is equivalent to

$$\left[-\exp\left(2\pi i \frac{p_1 M + q_1}{M d(\tau)}\right) + \exp\left(2\pi i \frac{p_2 M + q_2}{M d(\tau)}\right), -\exp\frac{\pi i(2p_2+1)}{d(\tau)}\right]$$

$$= 2i\sin 2\pi\left(\frac{p_2-p_1}{2d(\tau)} + \frac{q_2-q_1}{2Md(\tau)}\right)\cos 2\pi\left(\frac{p_2-p_1-1}{2d(\tau)} + \frac{q_1+q_2}{2Md(\tau)}\right)$$

Using the fact that $1 \le p_2 - p_1 \le \Delta - 1$ and $0 \le |q_1 - q_2| \le M - 2$, we have

$$\frac{1}{2d(\tau)} + \frac{2-M}{2Md(\tau)} \le \frac{p_2-p_1}{2d(\tau)} + \frac{q_2-q_1}{2Md(\tau)} \le \frac{\Delta}{2d(\tau)} + \frac{M-2}{2Md(\tau)}$$

The boundaries can be loosened to give:

$$\frac{1}{2d(\tau)} - \frac{M}{2Md(\tau)} < \frac{p_2-p_1}{2d(\tau)} + \frac{q_2-q_1}{2Md(\tau)} < \frac{1}{4} + \frac{M}{2Md(\tau)}$$

$$0 < \frac{p_2-p_1}{2d(\tau)} + \frac{q_2-q_1}{2Md(\tau)} < \frac{1}{4} + \frac{1}{2d(\tau)} < \frac{1}{2}$$

Hence $0 < \sin 2\pi\left(\frac{p_2-p_1}{2d(\tau)} + \frac{q_2-q_1}{2Md(\tau)}\right) \le 1$

Using the fact $1 \le p_2 - p_1 \le \Delta - 1$ and $0 \le q_1, q_2 \le M - 1$, we have

$0 \leq \dfrac{p_2 - p_1 - 1}{2d(\tau)} + \dfrac{q_1 + q_2}{2Md(\tau)} \leq \dfrac{\Delta - 2}{2d(\tau)} + \dfrac{2M - 2}{2Md(\tau)} = \dfrac{1}{4} - \dfrac{2}{2Md(\tau)} < \dfrac{1}{4}$

Hence we have $0 < \cos 2\pi\Big(\dfrac{p_2 - p_1 - 1}{2d(\tau)} + \dfrac{q_1 + q_2}{2Md(\tau)}\Big) \leq 1$.

Therefore, in the expression of $\tilde{s}_1$, after canceling the i's, the numerator is positive and the denominator is negative, giving $\tilde{s}_1 < 0$.

The stated result follows. □

Therefore, for the lines in $\Lambda(1, d(\tau), M)$ to intersect outside the unit circle, we must have $d(\tau)$ odd. However, this condition is not sufficient. The following proposition deals with this fact.

Theorem 45 (Intersection outside the unit circle)**.** *In $\Lambda(1, d(\tau), M)$ with $d(\tau)$ an odd integer, the nonparallel lines $l_{p1,q1}(s_1)$ and $l_{p2,q2}(s_2)$ intersect outside the unit circle only if the angle between them is $\dfrac{\pi}{d(\tau)}$.*

Proof. If $d(\tau) = 1$, then there is only one family of parallel lines, so there is nothing to consider. Now suppose $d(\tau) \geq 3$.

For any two lines $l_{p1,q1}(s_1)$ and $l_{p2,q2}(s_2)$ under consideration, choose p_1 and p_2 such that they satisfy $\dfrac{1}{2} \leq |p_1 - p_2| < \Delta = \dfrac{d(\tau)}{2}$, which is always possible for any two lines.

From the proof in the previous proposition, we know that

$$\tilde{s}_1 = -2\sin 2\pi\Big(\frac{p_2 - p_1}{2d(\tau)} + \frac{q_2 - q_1}{2Md(\tau)}\Big)\cos 2\pi\Big(\frac{p_2 - p_1 - 1}{2d(\tau)} + \frac{q_1 + q_2}{2Md(\tau)}\Big)\csc 2\pi\frac{p_2 - p_1}{d(\tau)}.$$

Assume that $p_2 > p_1$, then $\csc 2\pi\dfrac{p_2 - p_1}{d(\tau)} = \csc 2\pi\dfrac{|p_2 - p_1|}{d(\tau)} > 0$ by the previous proposition.

Now we consider the sign of $\cos 2\pi\Big(\dfrac{p_2 - p_1 - 1}{2d(\tau)} + \dfrac{q_1 + q_2}{2Md(\tau)}\Big)$ and $\sin 2\pi\Big(\dfrac{p_2 - p_1}{2d(\tau)} + \dfrac{q_2 - q_1}{2Md(\tau)}\Big)$.

<u>Case I</u> $p_2 - p_1 = \dfrac{1}{2}$

Note that since $0 \leq |q_2 - q_1| \leq (M-1) - 1 = M - 2$, we have:

$\dfrac{1}{Md(\tau)} - \dfrac{1}{4d(\tau)} \leq \dfrac{p_2 - p_1}{2d(\tau)} + \dfrac{q_2 - q_1}{2Md(\tau)} \leq \dfrac{3}{4d(\tau)} - \dfrac{1}{Md(\tau)}$

i.e. $\dfrac{4 - M}{4Md(\tau)} \leq \dfrac{p_2 - p_1}{2d(\tau)} + \dfrac{q_2 - q_1}{2Md(\tau)} \leq \dfrac{3}{4d(\tau)} - \dfrac{1}{Md(\tau)} < \dfrac{3}{4d(\tau)} \leq \dfrac{1}{4}$

and

$-\dfrac{1}{4} < \dfrac{1}{Md(\tau)} - \dfrac{1}{4d(\tau)} \leq \dfrac{p_2 - p_1 - 1}{2d(\tau)} + \dfrac{q_2 + q_1}{2Md(\tau)} \leq \dfrac{-1}{4d(\tau)} + \dfrac{2M - 2}{2Md(\tau)} = \dfrac{3}{4d(\tau)} - \dfrac{1}{Md(\tau)} < \dfrac{1}{4}$

so we have cos $2\pi\Big(\dfrac{p_2-p_1-1}{2d(\tau)}+\dfrac{q_1+q_2}{2Md(\tau)}\Big)>0$

Noting the left hand side of the first inequality may be positive or negative, depending on the value of M, we consider two cases.

(a)$M\leq 4$

$$0\leq\frac{4-M}{4Md(\tau)}\leq\Big(\frac{p_2-p_1}{2d(\tau)}+\frac{q_2-q_1}{2Md(\tau)}\Big)\leq\frac{1}{4},\text{ so }\sin\ 2\pi\Big(\frac{p_2-p_1}{2d(\tau)}+\frac{q_2-q_1}{2Md(\tau)}\Big)\geq 0$$

Hence $\tilde{s}_1<0$.

(b)$M>4$

Let $m=\dfrac{M}{2}$, then for sin $2\pi\Big(\dfrac{p_2-p_1}{2d(\tau)}+\dfrac{q_2-q_1}{2Md(\tau)}\Big)<0$, we have

$0<\dfrac{p_2-p_1-1}{2d(\tau)}+\dfrac{q_2+q_1}{2Md(\tau)}<\dfrac{1}{2}$ and hence $m<q_1-q_2\leq M-2$.

Thus, $\tilde{s}_1<0$ except for $m<q_1-q_2<M$.

<u>Case II</u> $1\leq p_2-p_1<\Delta-1$

$$\frac{M+(2-M)}{2Md(\tau)}\leq\frac{p_2-p_1}{2d(\tau)}+\frac{q_2-q_1}{2Md(\tau)}\leq\frac{\Delta M-M+M-2}{2Md(\tau)}$$
$$0<\frac{p_2-p_1}{2d(\tau)}+\frac{q_2-q_1}{2Md(\tau)}<\frac{1}{2}$$

so sin $2\pi\Big(\dfrac{p_2-p_1}{2d(\tau)}+\dfrac{q_2-q_1}{2Md(\tau)}\Big)>0$.

$$\frac{1}{Md(\tau)}\leq\frac{p_2-p_1-1}{2d(\tau)}+\frac{q_1+q_2}{2Md(\tau)}\leq\frac{\Delta M-2M+2M-2}{2Md(\tau)}$$
$$\frac{1}{15}\leq\frac{p_2-p_1-1}{2d(\tau)}+\frac{q_1+q_2}{2Md(\tau)}<\frac{1}{4}$$

so we have cos $2\pi\Big(\dfrac{p_2-p_1-1}{2d(\tau)}+\dfrac{q_1+q_2}{2Md(\tau)}\Big)>0$ and hence $\tilde{s}_1<0$.

<u>Case III</u>$p_2-p_1=\Delta-\dfrac{1}{2}$

$$\frac{\Delta M-\frac{M}{2}+(2-M)}{2Md(\tau)}\leq\frac{p_2-p_1}{2d(\tau)}+\frac{q_2-q_1}{2Md(\tau)}\leq\frac{\Delta M-\frac{M}{2}+M-2}{2Md(\tau)}$$
$$\frac{\Delta-\frac{3}{2}}{2d(\tau)}+\frac{1}{Md(\tau)}\leq\frac{p_2-p_1}{2d(\tau)}+\frac{q_2-q_1}{2Md(\tau)}<\frac{1}{4}+\frac{1}{4d(\tau)}$$
$$0<\frac{p_2-p_1}{2d(\tau)}+\frac{q_2-q_1}{2Md(\tau)}<\frac{1}{2}$$

so $\sin\ 2\pi\Big(\dfrac{p_2-p_1}{2d(\tau)}+\dfrac{q_2-q_1}{2Md(\tau)}\Big)>0$.
For $\cos\ 2\pi\Big(\dfrac{p_2-p_1-1}{2d(\tau)}+\dfrac{q_1+q_2}{2Md(\tau)}\Big)<0$, we have

$$\frac{p_2-p_1-1}{2d(\tau)}+\frac{q_2+q_1}{2Md(\tau)}<-\frac{1}{4} \text{ or } \frac{p_2-p_1-1}{2d(\tau)}+\frac{q_2+q_1}{2Md(\tau)}>\frac{1}{4}$$

and hence $q_1+q_2<-M\Big(d(\tau)-\dfrac{3}{2}\Big)<0$ (rejected) or $\dfrac{3M}{2}<q_1+q_2$.
Thus, $\tilde{s}_1<0$ except for $3m\leq q_1+q_2\leq 4m-2$.

In conclusion, for any two lines intersecting outside the unit circle, we must have either (A)$p_2-p_1=\dfrac{1}{2}$ and $m<q_1-q_2<M$ or
(B)$p_2-p_1=\Delta-\dfrac{1}{2}$ and $3m\leq q_1+q_2<4m-2$.
In (A), obviously the two lines are from adjacent families of parallel lines and the angle between them is $\dfrac{\pi}{d(\tau)}$. In (B), note that in fact $|p_2-(p_1+\Delta)|=\dfrac{1}{2}$, so the same deduction shows that the stated relation holds. □

7.3. Relations between different $\Lambda(n(\tau),d(\tau),M)$'s

Next, we notice the following relation between $\Lambda(n(\tau),d(\tau),M)$ of two different τ's with the same $D(\tau)$. This allows to compute the multiplicity P of $<N(\tau)>_K$ in $\Phi(n(\tau),d(\tau),M)$ from another sum.

Proposition 46. *$\Lambda(n(\tau),d(\tau),M)$ for any $n(\tau)$ is the same as $\Lambda(1,d(\tau),M)$ if $g(\tau)=1$.*

Proof. The set $\Lambda(n(\tau),d(\tau),M)$ consists of the unit circle and the $l_{p,q}(s)$ which takes the equation

$$l_{p,q}(s)=\exp\frac{2\pi i(pn(\tau)M+q)}{Md(\tau)}+is(\exp\ 2\pi i\tau p)\Big(\exp\frac{2\pi i}{d(\tau)}-1\Big)$$

for $1\leq q\leq M-1$ and $0\leq p\leq d(\tau)-1$.
if $g(\tau)=1$, we can choose a unique integer p' for each p such that
$p'\equiv pn(\tau) \pmod{d(\tau)}$ and $0\leq p'\leq d(\tau)-1$.
Then we can write $\exp\Big(2\pi i\dfrac{pn(\tau)M}{Md(\tau)}\Big)=\exp\Big(2\pi i\dfrac{p'M}{Md(\tau)}\Big)$ and
$\exp\ 2\pi i\tau p=\exp\Big(2\pi i\dfrac{pn(\tau)}{d(\tau)}\Big)=\exp\Big(2\pi i\dfrac{p'}{d(\tau)}\Big)$
Hence we can convert every line $l_{p,q}(s)$ to $l_{p',q}(s')$ which is defined as:

$$l_{p',q}(s')=\exp\frac{2\pi i(p'M+q)}{Md(\tau)}+is'\Big(\exp\frac{2\pi ip'}{d(\tau)}\Big)\Big(\exp\frac{2\pi i}{d(\tau)}-1\Big)$$

which is a line in $\Lambda(1, d(\tau), M)$.
Note that the conversion map $f : \mathbf{Z}_{d(\tau)} \to \mathbf{Z}_{d(\tau)}$ defined by
$p \mapsto p' = pn(\tau) \pmod{d(\tau)}$ is bijective:
Injectivity:$p_1 n(\tau) \equiv p_2 n(\tau) \pmod{d(\tau)} \Leftrightarrow (p_1 - p_2)n(\tau) \equiv 0 \pmod{d(\tau)}$
$\Leftrightarrow d(\tau) \mid (p_1 - p_2)n(\tau)$. Since $g(\tau) = 1, d(\tau) \mid (p_1 - p_2)$.
But since $0 \le p_1, p_2 \le d(\tau)-1, 0 \le |p_1-p_2| \le d(\tau)-1$, so we must have $p_1-p_2 = 0$, which means that $p_1 = p_2$.
Surjectivity: For $p_1 \ne p_2, |p_1-p_2|$ is never a positive multiple of $d(\tau)$, the conversion map is obviously onto.
Because one value of p corresponds to exactly one line, the conversion map $g : \{L_p | p = 0, 1, 2, \ldots, D(\tau) - 1\}$ from $\Lambda(1, d(\tau), M) \to \{L_{p'} | p' = 0, 1, 2, \ldots, D(\tau) - 1\}$ from $\Lambda(n(\tau), d(\tau), M)$ defined by $l_{p,q}(s) \mapsto l_{p',q}(s')$ is bijective, so the stated result follows. □

Now, observe the following pairs of graphs: $\Lambda(1, 3, 3)$ and $\Lambda(3, 9, 3)$:

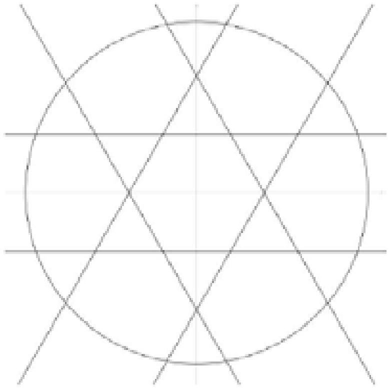


Illustration 18. $\Lambda(1, 3, 3)$.

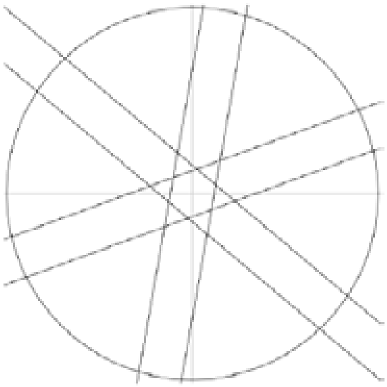


Illustration 19. $\Lambda(3, 9, 3)$.

We note that $\Lambda(1, 3, 3)$ and $\Lambda(3, 9, 3)$ are similar in several aspects. Generalizing the result, we can prove the following proposition.

Proposition 47. *For any given rational τ, $\Lambda(n_1(\tau), d_1(\tau), M)$ and $\Lambda(n_2(\tau), d_2(\tau), M)$ have the same number of lines and the angle between adjacent families of parallel lines is the same.*

Proof. Note that for $\Lambda(n_1(\tau), d_1(\tau), M)$ where $g_1(\tau) \ne 1$, the number of families of parallel lines in the graph is $\begin{cases} \dfrac{D(\tau)}{2} & \text{if } D(\tau) \text{ is even} \\ D(\tau) & \text{if } D(\tau) \text{ is odd} \end{cases}$, and the angle between each adjacent family of parallel lines is therefore $\begin{cases} \dfrac{2\pi}{D(\tau)} & \text{if } D(\tau) \text{ is even} \\ \dfrac{\pi}{D(\tau)} & \text{if } D(\tau) \text{ is odd} \end{cases}$, where $D(\tau) = \dfrac{d(\tau)}{g(\tau)}$. □

Then from our observation of $\Lambda(1,3,3)$ and $\Lambda(1,9,3)$, we notice the following.

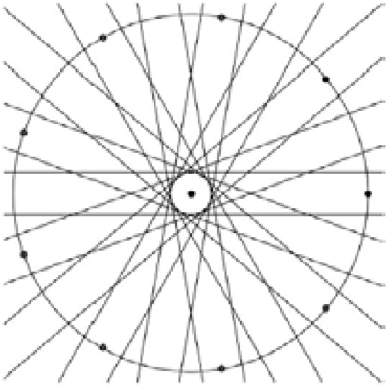


ILLUSTRATION 20. $\Lambda(1,9,3)$.

Proposition 48. *Suppose $g(\tau) > 1$. Then $\Lambda(n(\tau), d(\tau), M)$ is in fact the lines $l_{p,q}(s)$, where $p = 0, g(\tau), 2g(\tau), \ldots, d(\tau) - g(\tau)$, in $\Lambda(1, d(\tau), M)$ together with the unit circle.*

Proof. Note that $l_{p,q}(0) = \exp\ 2\pi i \dfrac{pM+q}{Md(\tau)}$ and $l_{p,q}{}'(s) = \exp \pi i \dfrac{2p+1}{d(\tau)}$ completely determine the line $l_{p,q}(s)$.

In $\Lambda(n(\tau), d(\tau), M), l_{r,q}(0) = \exp 2\pi i \dfrac{rn(\tau)M+q}{Md(\tau)}$ and

$l_{r,q}{}'(s) = \exp \pi i \dfrac{2rn(\tau)+1}{d(\tau)}$. But for $0 \le r \le d(\tau) - 1$,

$$l_{r+D(\tau),q}(0) = \exp 2\pi i \frac{rn(\tau)M + rD(\tau)M + q}{Md(\tau)} = \exp 2\pi i \Big(\frac{rn(\tau)M+q}{Md(\tau)} + rg(\tau) \Big)$$

Since $rg(\tau)$ is an integer, $l_{r+D(\tau),q}(0) = l_{r,q}(0)$.

Similarly, $l_{r+D(\tau),q}{}'(s) = \exp \pi i \Big(\dfrac{2rn(\tau)+1}{d(\tau)} + 2rg(\tau) \Big) = l_{r,q}{}'(s)$.

Therefore, $l_{r+D(\tau),q}(s)$ is in fact the same line as $l_{r,q}(s)$ for all $0 \le r \le d(\tau) - 1 - D(\tau)$.

Hence, we are allowed to restrict $0 \le r \le D(\tau) - 1$ in considering all lines $l_{r,q}(s)$ in $\Lambda(n(\tau), d(\tau), M)$

In $\Lambda(1, d(\tau), M), l_{kg(\tau),q}(0) = \exp 2\pi i \dfrac{g(\tau)kM+q}{Md(\tau)}$,

$l_{kg(\tau),q}{}'(s) = \exp \pi i \dfrac{2kg(\tau)+1}{d(\tau)}$.
Set $k \equiv rN(\tau) \pmod{D(\tau)}$, where $0 \le k \le D(\tau)-1$.
Then $kg(\tau) \equiv rn(\tau) \pmod{D(\tau)}$.
Hence, we can assign a unique k to each r where $0 \le r \le D(\tau)-1$ such that the line $l_{r,q}(s)$ in $\Lambda(n(\tau), d(\tau), M)$ is the same as $l_{kg(\tau),q}(s)$ in $\Lambda(1, d(\tau), M)$.
Hence, the mapping $f : \{L_p | p = 0, g(\tau), 2g(\tau), \ldots, d(\tau) - g(\tau)\}$ in $\Lambda(1, d(\tau), M) \to \{L_p | p = 0, 1, 2, \ldots, D(\tau)-1\}$ defined by
$f : l_{r,q}(s) \mapsto l_{kg(\tau),q}(s)$ is bijective. Therefore, the stated result follows. □

Proposition 49. *In $\Lambda(n(\tau), d(\tau), M)$ with $g(\tau) > 1$, no two lines intersect outside the unit circle.*

Proof. The fact follows from the last proposition. With $g(\tau) > 1$, the difference in the values of p of any two lines is at least $\dfrac{g(\tau)}{2} > \dfrac{1}{2}$, also allowing ourselves to change p to $p + \Delta$ or $p - \Delta$. The angle between any families of parallel lines is therefore at least $\dfrac{2\pi}{D(\tau)}$ if $D(\tau)$ is even and $\dfrac{\pi}{D(\tau)}$ if $D(\tau)$ is odd.
If $D(\tau)$ is even, $d(\tau) = D(\tau)g(\tau)$ must be even, so by a previous proposition, the stated result follows.
If $D(\tau)$ is odd, then for even $g(\tau), d(\tau) = D(\tau)g(\tau)$ is even, so the stated result follows. If $g(\tau)$ is odd, then $g(\tau) \ge 3$, and hence $\dfrac{\pi}{d(\tau)} = \dfrac{\pi}{D(\tau)} g(\tau) \ge \dfrac{3\pi}{D(\tau)} > \dfrac{2\pi}{D(\tau)}$.
Therefore, by the Intersection outside the unit circle Theorem, the stated result follows. □

8. Deducing a Formula for Multiplicity P of $<N(\tau)>$ in Sum $\Phi(n(\tau), d(\tau), M)$

After claiming different propositions in the last section, we are now capable of calculating P. Here we use two main results:

1. Euler's formula for graphs: $V - E + (F + 1) = 2$, where V, E and F are the number of vertices, edges and faces (excluding the exterior of the unit circle) respectively.
   In other words, we have $F = 1 + E - V$.
2. The Face-multiplicity Theorem: $F = PD(\tau) + I$, where $I = \dfrac{1-(-1)^M}{2}$ as proven.
   In other words, we have $P = \dfrac{1 + E - V - I}{D(\tau)}$.

We first consider the case where $M = 2$.

Theorem 50. *The multiplicity P of $<N(\tau)>_K$ in $\Phi(1, d(\tau), 2)$ is* $P = \begin{cases} 1 & \textit{if } D(\tau) \textit{ is even} \\ 2 & \textit{if } D(\tau) \textit{ is odd} \end{cases}$.

Proof. It is easy to see that $F = \begin{cases} D(\tau) & \text{if } D(\tau) \text{ is even} \\ 2D(\tau) & \text{if } D(\tau) \text{ is odd} \end{cases}$ when $M = 2$. Therefore,

$$P = \begin{cases} \dfrac{D(\tau)}{D(\tau)} = 1 & \text{if } D(\tau) \text{ is even} \\ \dfrac{2D(\tau)}{D(\tau)} = 2 & \text{if } D(\tau) \text{ is odd} \end{cases}. \qquad \square$$

Then we calculate the value of P in the tst sum $\Phi(1, D(\tau), M)$, where $D(\tau)$ is an even integer, for which we write $D(\tau) = 2\Delta$ such that Δ is an integer. After that we do the same for odd $d(\tau)$ where $g(\tau) = 1$. Then, the result is generalized to cases of $\Phi(n(\tau), d(\tau), M)$ with $g(\tau) = 1$ and then to cases where $g(\tau) \neq 1$.

Theorem 51. *If $d(\tau)$ is even, the multiplicity P of $<N(\tau)>_K$ in $\Phi(1, 2\Delta, M)$ is*

$P = \frac{1}{4}(M-1)[(\Delta-1)(M-1)+2]$ *if M is odd,*

$$P = \begin{cases} \frac{1}{4}[(M-2)(M\Delta - 3M - \Delta - 2) + 2(M^2 + M - 4)] & \text{if } \Delta \text{ is even} \\ \frac{1}{4}(M-2)(M\Delta + \Delta + 2) & \text{if } \Delta \text{ is odd} \end{cases}$$

if M is even and $M \neq 2$.

Proof. Note that if M is odd, the number of vertices in the graph is

$V = \Delta(M-1)\Big[\frac{1}{2}(\Delta-1)(M-1)+2\Big]$ and that of edges is

$E = \Delta(M-1)[(\Delta-1)(M-1)+3]$.

If M is even and $M \neq 2$, the numbers of vertices (V) and edges (E) are respectively

$$V = [{}_{\Delta}C_2(M-1)^2 - 2\Delta(M-2)(\frac{\Delta}{2} - 1) - {}_{\Delta}C_2 + 1] + 2\Delta(M-1),$$

$$\begin{aligned} E = &\Delta\{[\frac{\Delta-2}{2}(M-2) + (M-1) + 1] + [(\Delta-2)(M-2) + (M-1) + 1](M-2)\} \\ &+ 2\Delta(M-1) \end{aligned}$$

if Δ is even; and

$$V = {}_{\Delta}C_2(M-1)^2 - 2\Delta(\Delta-1)(M-2) - {}_{\Delta}C_2 + 1] + 2\Delta(M-1),$$

$$E = \Delta\{[\frac{\Delta-1}{2}(M-2) + 1 + 1] + [(\Delta-1)(M-2) + 1](M-2)\} + 2\Delta(M-1)$$

if Δ is odd.

Hence the number of faces is $F = 1 + \Delta(M-1)\Big[\frac{1}{2}(\Delta-1)(M-1)+1\Big]$ if M is odd, and

$$F = \begin{cases} \dfrac{\Delta}{2}(M-2)(M\Delta - 3M - \Delta - 2) + \Delta(M^2 + M - 4) & \text{if } \Delta \text{ is even} \\ \dfrac{\Delta}{4}(M-2)(M\Delta - M + \Delta + 1) & \text{if } \Delta \text{ is odd} \end{cases}$$

if M is even and $M \neq 2$, where Δ is an integer.
The result thus follows. □

Now we are calculating P for odd $d(\tau)$. Note that we have assumed $g(\tau) = 1$. For the sake of simplicity, we introduce the following notation:

Notation 52. $\delta_m(n) = \begin{cases} 1 & \textit{if } m \mid n \\ 0 & \textit{otherwise} \end{cases}$

This is also used in the paper [6] to which we have referred in our derivation.

Theorem 53. *If $d(\tau)$ is odd and $d(\tau) \neq 5$ and 7, the multiplicity P of $<N(\tau)>_K$ in $\Phi(1, d(\tau), M)$ is*

$$P = \begin{cases} \dfrac{1}{4}[(2d(\tau) - 3)(M-1) + 6d(\tau)](M-1) & \textit{if } M \textit{ is odd} \\ \dfrac{1}{4}[2(d(\tau) - 1)(M-2)(M+1) - M(M-6)] & \textit{if } M \textit{ is even} \end{cases}$$

where $M \neq 2$.

$$\textit{If } d(\tau) = 5, P = \begin{cases} \dfrac{1}{4}[(2d(\tau) - 3)(M-1) + 6d(\tau)](M-1) & \textit{if } M \textit{ is odd} \\ \dfrac{1}{4}[2(d(\tau) - 1)(M-2)(M+1) - M(M-6)] - 8\delta_{12}(M) & \textit{if } M \textit{ is even} \end{cases}$$

$$\textit{If } d(\tau) = 7, P = \begin{cases} \dfrac{1}{4}[(2d(\tau) - 3)(M-1) + 6d(\tau)](M-1) & \textit{if } M \textit{ is odd} \\ \dfrac{1}{4}[2(d(\tau) - 1)(M-2)(M+1) - M(M-6)] - 4\delta_{12}(M) & \textit{if } M \textit{ is even} \end{cases}$$

Proof. Note that besides that there are some three-line concurrence not of Config I or II in the case of $d(\tau) = 5$ and 7, we also have to consider that there are some lines intersecting at some point outside the unit circle.
Hence, we first consider the entire $\Lambda(n(\tau), d(\tau), M)$, assuming that there are no three-line concurrences other than Config I and II. Having calculated the number of faces in $\Lambda(n(\tau), d(\tau), M)$, we then subtract from this formula the number of faces outside the unit circle. Afterwards, we making the adjustment for $d(\tau) = 5$ and 7 due to the other configurations for 3-line concurrences.
If M is odd, in $\Lambda(n(\tau), d(\tau), M)$, i.e. the graph of $\Lambda(n(\tau), d(\tau), M)$ together with the exterior of the unit circle, the total number of edges is:

$$E = (M-1)[(d(\tau) - 1)(M-1) + 1 - 2d(\tau)]$$

The number of vertices is

$$V = ({}_{d(\tau)}C_2)(M-1)^2 - 2d(\tau)(M-1)$$

But the number of faces outside the unit circle is

$$\frac{1}{2}\Big(\frac{M-1}{2}\Big)\Big(\frac{M-1}{2}-1\Big)2d(\tau) = \frac{1}{4}[(M-1)^2 d(\tau) - (M-1)d(\tau)].$$

Similarly, when M is even and $M \neq 2$, we have

$$V = {}_{d(\tau)}C_2(M-1)^2 - d(\tau)(M-2)(d(\tau)-1) - {}_{d(\tau)}C_2 + 1 + 2d(\tau)$$

The formula for E is as follows. If $M = 4$, we have:

$$E = d(\tau)\{(d(\tau)-1)(M-2)/2 + 1 + 1\} + [(d(\tau)-1)(M-2)-1](M-2)\}$$
$$+ 2(M-2)d(\tau)$$

and for $M \geq 6$, we have:

$$E = d(\tau)\{(d(\tau)-1)(M-2)\Big(\frac{1}{2}\Big) + 1 + 1\} + [(d(\tau)-1)(M-2)-1](M-2)\}$$
$$+ 2(M-2)d(\tau) - 2d(\tau)\Big(\frac{M}{2}-2\Big).$$

However, after we apply Euler's formula for graphs, we can combine the two cases to give one single formula for the number of faces in the graph of $\Lambda(n(\tau), d(\tau), M)$ is:

$$F = \begin{cases} \dfrac{d(\tau)}{4}(M-1)[(M-1)(2d(\tau)-3) + 6d(\tau)] + 1 & \text{if } M \text{ is odd} \\ \dfrac{d(\tau)}{4}[2(d(\tau)-1)(M-2)(M+1) - M(M-6)] & \text{if } M \text{ is even} \end{cases}$$

where $M \neq 2$ and $d(\tau)$ is an odd integer not equal to 5 and 7.
For $d(\tau) = 5$,

$$F = \begin{cases} \dfrac{d(\tau)}{4}(M-1)[(M-1)(2d(\tau)-3) + 6d(\tau)] + 1 & \text{if } M \text{ is odd} \\ \dfrac{d(\tau)}{4}[2(d(\tau)-1)(M-2)(M+1) - M(M-6)] - 40\delta_{12}(M) & \text{if } M \text{ is even} \end{cases}$$

For $d(\tau) = 7$,

$$F = \begin{cases} \dfrac{d(\tau)}{4}(M-1)[(M-1)(2d(\tau)-3) + 6d(\tau)] + 1 & \text{if } M \text{ is odd} \\ \dfrac{d(\tau)}{4}[2(d(\tau)-1)(M-2)(M+1) - M(M-6)] - 28\delta_{12}(M) & \text{if } M \text{ is even} \end{cases}$$

where those δ-terms are to compensate for the pencils of concurrence lines.
The result thus follows. □

Now we consider the case where $g(\tau) > 1$. We first consider the case for $N(\tau) = 1$, i.e. $n(\tau) = g(\tau)$.

Theorem 54. *If $D(\tau)$ is even, the multiplicity P of $<N(\tau)>_K$ in $\Phi(g(\tau), d(\tau), M)$ is $P = \frac{1}{4}\Big[\Big(\frac{D(\tau)}{2} - 1\Big)(M-1) + 2\Big](M-1)$ if M is odd,*

$$P = \begin{cases} \frac{1}{4}[(M-2)(M\mathit{\Delta} - 3M - \mathit{\Delta} - 2) + 2(M^2 + M - 4)] & \text{if } \mathit{\Delta} \text{ is even} \\ \frac{1}{4}(M-2)(M\mathit{\Delta} + \mathit{\Delta} + 2) & \text{if } \mathit{\Delta} \text{ is odd} \end{cases}$$

where $D(\tau) = 2\mathit{\Delta}$.

Proof. From the propositions and theorems from the previous section, we see that if $d(\tau)$ is even, in the graph of $\Phi(1(\tau), D(\tau), M)$, no lines intersect outside the unit circle. Therefore, changing $d(\tau)$ to $D(\tau)$ would give the stated result. □

Theorem 55. *If $D(\tau)$ is odd and $D(\tau) \neq 5, 7$, the multiplicity P of $<N(\tau)>_K$ in $\Phi(g(\tau), d(\tau), M)$, where $g(\tau) > 1$, is*

$$P = \begin{cases} D(\tau)(M-1)(3MD(\tau) - M + D(\tau)) & \text{if } M \text{ is odd} \\ \frac{1}{2}[(D(\tau)-1)(M-2)(M+1) + 2M] & \text{if } M \text{ is even and } M \neq 2 \end{cases}.$$

If $D(\tau) = 5$,

$$P = \begin{cases} D(\tau)(M-1)(3MD(\tau) - M + D(\tau)) & \text{if } M \text{ is odd} \\ \frac{1}{2}[(D(\tau)-1)(M-2)(M+1) + 2M] - 8\delta_{12}(M) & \text{if } M \text{ is even and } M \neq 2 \end{cases}$$

If $D(\tau) = 7$,

$$P = \begin{cases} D(\tau)(M-1)(3MD(\tau) - M + D(\tau)) & \text{if } M \text{ is odd} \\ \frac{1}{2}[(D(\tau)-1)(M-2)(M+1) + 2M] - 4\delta_{12}(M) & \text{if } M \text{ is even and } M \neq 2 \end{cases}$$

Proof. Use the formula of P for $\Phi(1, d(\tau), M)$ where $d(\tau)$ is odd. As no lines intersect outside the unit circle when $g(\tau) \geq 3$, we subtract from the formula the number of regions outside the unit circle divided by $d(\tau)$. Changing $d(\tau)$ to $D(\tau)$ would give the stated result. □

Finally, we consider the case where $N(\tau) \neq 1$.

Theorem 56. $m_{\Phi(n(\tau),d(\tau),M)}(<\tau>) = m_{\Phi(g(\tau),d(\tau),M)}(<\tau>)$.

Proof. Using a proposition from the previous section, we know that the graphs $\mathit{\Lambda}(n(\tau), d(\tau), M)$ is the same as $\mathit{\Lambda}(g(\tau), d(\tau), M)$. Hence, the stated result follows. □

9. Irrational Twist turns

We now extend our discussion to consider tsts with irrational twist turns, i.e. $\tau \in \mathbf{R}\backslash\mathbf{Q}$.

We first consider the former case. As $\tau \in \mathbf{R}\backslash\mathbf{Q}$, it is impossible to calculate $\dfrac{d(\tau)}{g(\tau)}$ directly. However, we have a fact that for any given real number τ, there exists a rational sequence of $\{a_n\}$ such that $\lim_{n\to\infty}\{a_n\} = \tau$. Thus, we have to consider the rational sequence $\left\{\dfrac{d(\tau_n)}{g(\tau_n)}\right\}$ to find an approximation in the deduction of the result for $\Phi(n(\tau)$.

Repeated (or continued) fractions are used to generate the sequence $\{a_n\}$,

i.e. $\tau = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \ddots}}} \equiv [b_0; b_1, b_2, b_3, \ldots]$ where $b_0, b_1, b_2, \ldots$ are integers.

If the representation is periodic, an over-bar is used to denote infinite repetition.

If we define $\tau_n = [b_0; b_1, b_2, \ldots, b_n]$ to be a truncated expression, then τ_n is rational, and we can thus calculate $D(\tau)$.

Notation 57. *Denote $\Phi_M <\tau> \approx_n \Phi(n(\tau_n), d(\tau_n), M)$ where τ_n is the n-th term in the sequence of approximation using continued fractions.*

Example 58.

58.1 Since $\sqrt{2} = [1; \overline{2}]$, *so* $\sqrt{2}_3 = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \dfrac{17}{12}$ *and*

$$\sqrt{2}_4 = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{41}{29},$$

$$\Phi_M \left\langle \sqrt{2} \right\rangle \approx_3 \Phi(17, 12, M) \text{ and } \Phi_M \left\langle \sqrt{2} \right\rangle \approx_4 \Phi(41, 29, M).$$

58.2 $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots]$, *so* $\Phi_M <\pi> \approx_3 \Phi(355, 133, M)$.

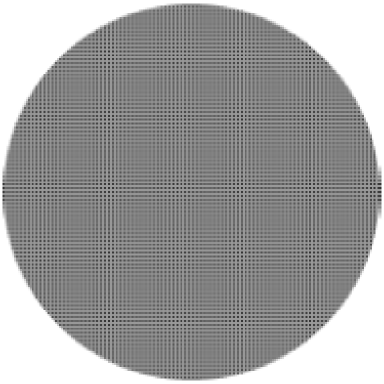
Remarks: Note that in different approximations, the $\dfrac{d(\tau)}{g(\tau)}$ may be different in even-odd property. Hence, we have to use a $\approx$ sign to indicate that only a truncated representation is chosen.

Theorem 59. *If τ is irrational, $\Phi_M <\tau> = \infty <\infty>_K + I <\tau>_K$*

Proof. Observe that as $n \to \infty, D(\tau) \to \infty$ and $N(\tau) \to \infty$. Regardless of even-odd property, we have $P \to \infty$. ∎

To visualize the result of cutting $<\tau>$, we can consider the terms in the sequence $\{a_n\}$, whose limit is τ, and see how $\Lambda(N(a_n), D(a_n), M)$ looks like. If n is large enough, the graph of $\Lambda(N(a_n), D(a_n), M)$ is generally of the following form:

As n grows larger, the graph of $\Lambda(N(a_n), D(a_n), M)$ becomes:



As n tends to infinity, $D(a_n)$ tends to infinity as well, so does the number of vertices, edges and, thus, faces. Therefore, P tends to infinity.
However, I does not depend on $D(a_n)$, thus not on n; it depends only on the even-odd property of M.

10. Cutting with More Blades

In this section we consider a new way of cutting tsts, called multiple cutting, which we used to define for the sake of discussion in the later sections.

Consider $\Lambda(1,4,3)$ as shown below:

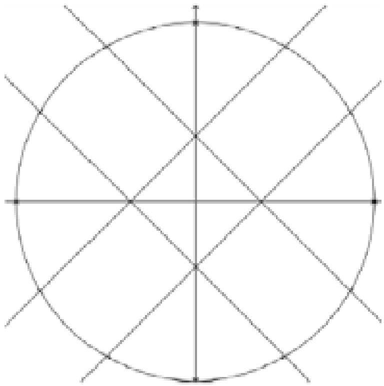


Illustration 21. $\Lambda(1,4,3)$

When we evaluate the twist turn of each of the tsts represented in $\Lambda(n(\tau), d(\tau), M)$, we have considered only the tst being $\left\langle \frac{1}{4} \right\rangle$. However, we can in fact consider this way of cutting to a tst $\left\langle \frac{1}{2} \right\rangle$, by starting with four blades instead of two and then insert the blades at the position such that after insertion, the cross section at $t = 0$ of the tst becomes the one in the above illustration. If we continue the process like the usual cutting method, i.e. allowing the blades to rotate along the tst, and see how this surface formed from rotation intersect the plane $t \to 1-$, keeping on the rotation until the blades return to its original positions, then we can obtain new results from this new method of cutting, in the process of which more blades have to be inserted. This motivates us to the following definition.

Definition 60. *For a positive integer μ, the graph $\Lambda^\mu(n(\tau), d(\tau), M) = \Lambda(n(\tau), \mu d(\tau), M)$ is the cross section at $t = 0$ of the tst $<\tau>$ after $\underline{\Phi^\mu\text{-cutting it into } M}$ $\underline{\text{parts}}$.*

Note that in Φ^μ-cutting a tst into M parts, $\mu(M-1)$ blades are inserted. Besides, we also define:

Notation 61. *For a positive integer μ, the graph $\Lambda^\mu(n(\tau), d(\tau), M) = \Lambda(n(\tau), \mu d(\tau), M)$ is the cross section at $t = 0$ of the tst $<\tau>$ after $\underline{\Phi^\mu\text{-cutting it into } M}$ $\underline{\text{parts}}$.*

The relation of multiple cuts with cuts defined previously is established by the following theorem:

Theorem 62. *$\Phi^\mu(n(\tau), d(\tau), M) = \mu P < N(\tau) >_K + I < \tau >_K$, where $P = m_{\Phi(n(\tau), \mu d(\tau), M)}(< N(\tau) >_K)$ is the multiplicity of $< N(\tau) >_K$ in the knotted tst sum $\Phi(n(\tau), \mu d(\tau), M)$.*

Proof. Consider first $\Phi(n(\tau), \mu d(\tau), M)$ but for the 2 twist turns, evaluate with the twist turn τ instead of $\dfrac{\tau}{\mu}$. Then the number of regions in $\Lambda^{\mu}(n(\tau), d(\tau), M) = \Lambda(n(\tau), \mu d(\tau), M)$ is $\mu D(\tau)P + I$ from the previous section, where P is the multiplicity of $<N(\tau)>_K$ in the sum $\Phi(N(\tau), \mu D(\tau), M)$. If we set the multiplicity of $<N(\tau)>_K$ in the knotted tst sum $\Phi^{\mu}(n(\tau), d(\tau), M)$ to be P', then the number of regions is also $D(\tau)P' + I$. Equating gives $P' = \mu P$. □

Corollary 63. $\Phi^1(n(\tau), d(\tau), M) = \Phi(n(\tau), d(\tau), M)$

Proof. Use the above theorem, and substitute $\mu = 1$. □

This notion of "cutting with more blades" will become very useful when we develop our theory of tst links in later sections.

11. Links from Tst cutting and the General Form of Their Braid Words

After cutting different tsts into M parts, knotted tsts of the same twist turn may be formed, although they may be of different form of links. For example, one may be in the form of a trivial knot, while the other in a trefoil. This motivates us to the think of the knotted tsts as links or knots.

Definition 64. *The resultant knot/link obtained from Φ-cutting a tst $<\tau>$ in the form of a trivial knot into M parts by μ times is called the tst link of $\Phi^{\mu}(n(\tau), d(\tau), M)$ and is denoted by $[\Phi^{\mu}(n(\tau), d(\tau), M)]$.*

We are now constructing the tst link $[\Phi^{\mu}(n(\tau), d(\tau), M)]$.
Consider the tst sum $\Phi^{\mu}(n(\tau), d(\tau), M) = \mu P <N(\tau)> +I <\tau>$ where P is the multiplicity of $<N(\tau)>$ in the tst sum $\Phi(n(\tau), \mu d(\tau), M)$ for $\mu \geq 1$. Using this result, we can deduce the general form of the tst link $[\Phi^{\mu}(n(\tau), d(\tau), M)]$. We first consider the case for $n(\tau) > 0$.

Theorem 65. *If M is even and $n(\tau) > 0$, then the tst link $[\Phi^{\mu}(n(\tau), d(\tau), M)]$ is equivalent to the $(-\mu P N(\tau), \mu P D(\tau))$-torus link.*

Proof. We claim that each of the tsts $<N(\tau)>$ individually forms a $(-N(\tau), D(\tau))$-torus knot. Extract the centers of mass of all the $D(\tau)$ regions which the tst $<N(\tau)>$ intersects the cross section of the original tst at $t = 0$:
Shown below is the example of $\Lambda(1, 4, 4)$, where the faces whose partial tsts contribute to the same knotted tst are shaded in red and have their centers of mass marked with a red cross.

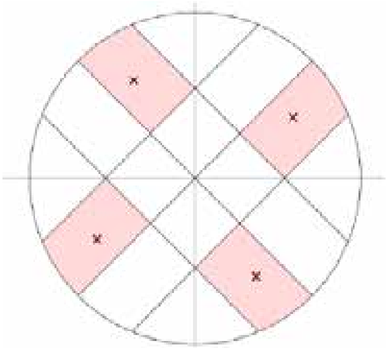


ILLUSTRATION 22

Then dilate the $D(\tau)$ vectors from the origin to the centers of mass, such that it is of length one, and mark the terminal point of the dilated vectors on the circumference, forming $D(\tau)$ points:

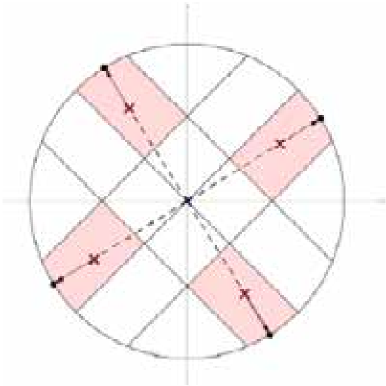


ILLUSTRATION 23

Let these $D(\tau)$ points take coordinates $(\exp 2\pi i \frac{k}{D(\tau)}, 0)$ where $k = 0, 1, 2, \ldots, D(\tau) - 1$. Connect these $D(\tau)$ points to the $D(\tau)$ points with coordinates $(\exp 2\pi i \frac{k}{D(\tau)}, 1-)$ with line segments:

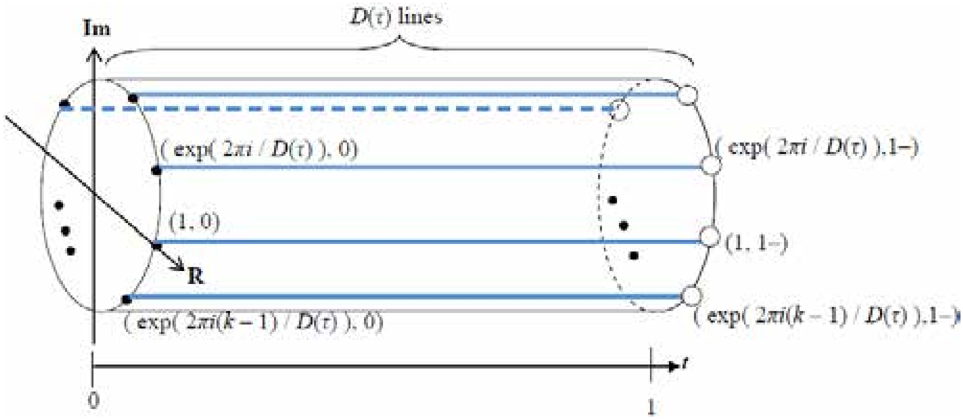


Illustration 24

Now fix the left end and rotate the right end by τ turn(s) as viewed from the point (0,0):

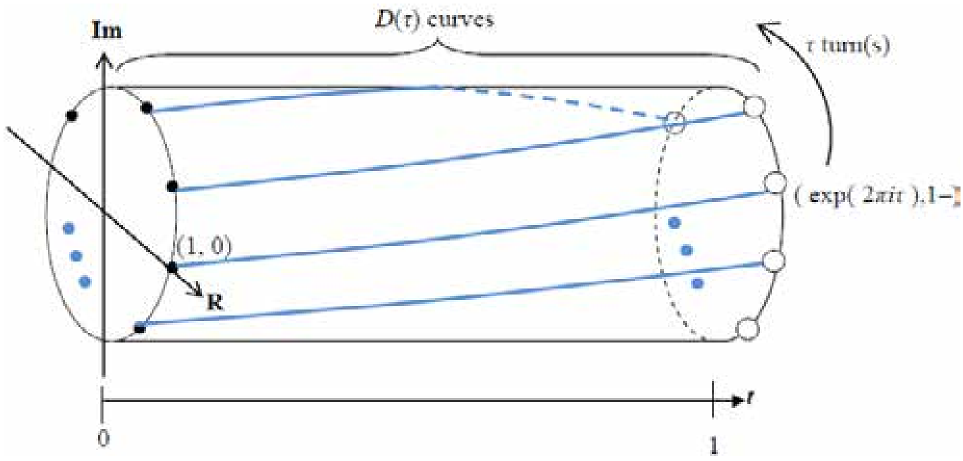


Illustration 25

Next, we identify $(z, 0)$ with $(z, 1-)$. Note that this agrees exactly with the definition of a (q, r)-torus knot. Here $r = D(\tau)$ and q satisfies the equation $\frac{q}{r} = -\tau$ (we are using a left-handed system in our definitions but the standard one is right-handed, so there is a minus sign). Solving gives $q = -N(\tau)$. We can check that this is a knot (or a one-component link) by noticing that $\gcd(-N(\tau), D(\tau))$ must be 1.

Now, consider the μP tsts put on the same cylinder as shown:

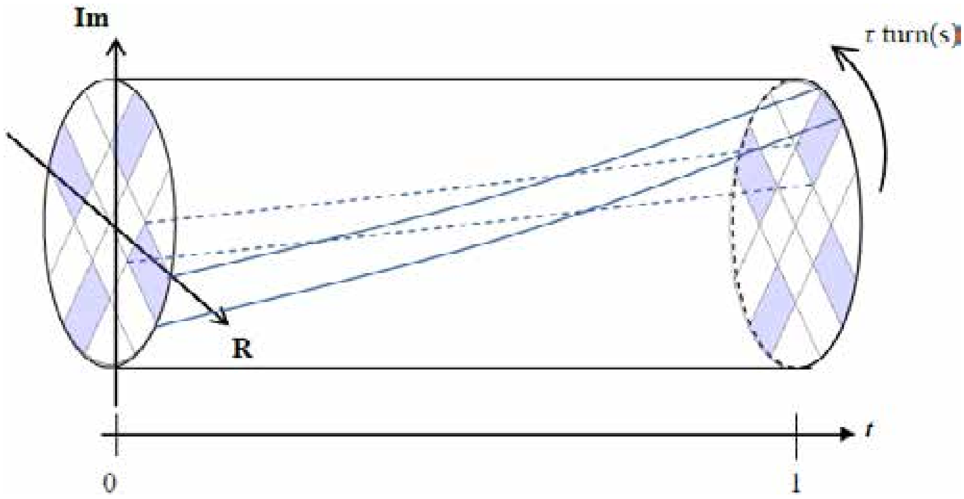


Illustration 26

Extract the centers of mass of all the regions (except the one that contains (0, 0), if any) and project them to the nearest point on the circumference:

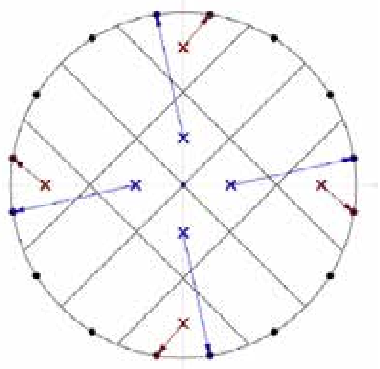


Illustration 27

Some points may coincide after projection. In that case, we allow slight sideway movement of the points in such a way that if one of the projected center of mass of a region from one tst is adjusted clockwisely, the other projected centers of mass of regions from the same tst is adjusted in the same way, and vice versa.

Similar to the argument above, by definition, this is a $(-\mu PN(\tau), \mu PD(\tau))$-torus link. □

From the theory on torus links, we can deduce the following:

(1) The tst link $[\Phi^{\mu}(n(\tau), d(\tau), M)]$ has $\gcd(-\mu PN(\tau), \mu PD(\tau)) = \mu P$ components, which obviously agree with our expectation.
(2) From the theory on torus links, we know that the braid word of the tst link $[\Phi\mu(n(\tau), d(\tau), M)]$ is $({\sigma_1^{-1}}{\sigma_2^{-1}}{\sigma_3^{-1}} \dots {\sigma_{\mu PD(\tau)-1}}^{-1})^{\mu PN(\tau)}$

If M is odd, $I = 1$, so we have to proceed with the tst $<\tau>$. If we individually consider the knot formed by this tst, we get a trivial knot as shown:

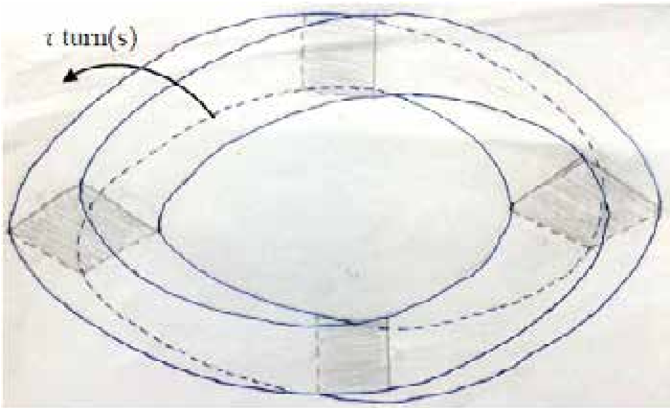


Illustration 28

If we consider this trivial knot together with the torus link formed by the P tsts $<N(\tau)>$, we obtain the following:

Theorem 66. *If M is odd and $n(\tau) > 0$, the tst link $[\Phi^{\mu}(n(\tau), d(\tau), M)]$ has a braid word of:*

$({\sigma_1^{-2}}{\sigma_2^{-1}}{\sigma_3^{-1}}{\sigma_4^{-1}} \dots {\sigma_{\mu PD(\tau)}}^{-1})^{\mu PN(\tau)}$.

Proof. To find the braid word for this link, we first notice that there is a repetitive pattern in it, similar to the case when M is even. The pattern is exactly the braid word obtained from the case with $N(\tau) = 1$, and repeats itself for $\mu PN(\tau)$ times. Put the tst $<\tau>$ together with the P tsts $<N(\tau)>$ to the same cylinder.

We can shrink the tst $<\tau>$ to a line segment passing through $(z,t) = (0,0)$ as shown:

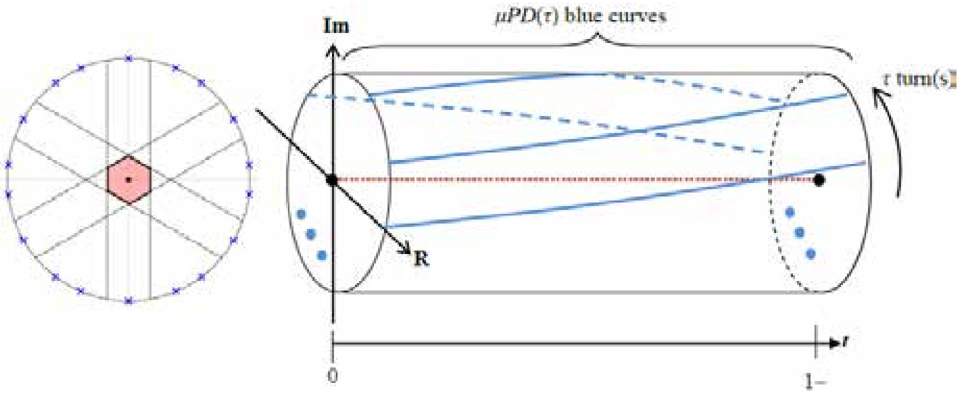


Illustration 29

We have used the tst link $[\Phi(1,6,3)]$ as an example
Then project this line segment of $<\tau>$ onto the circumference in a way shown in the following illustration:

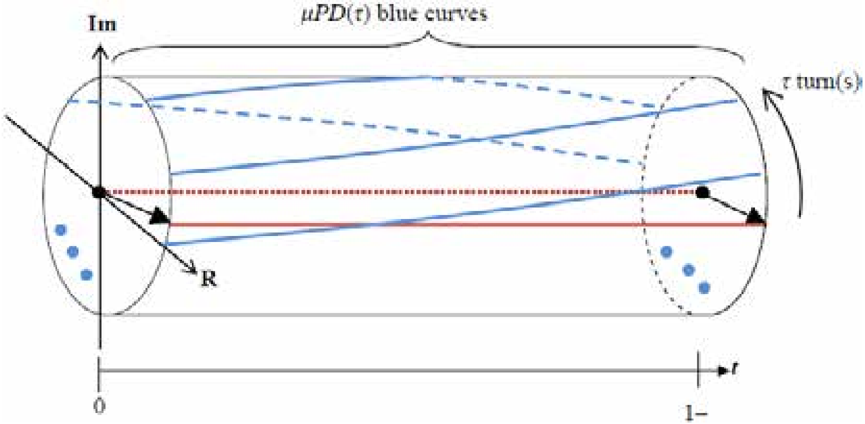
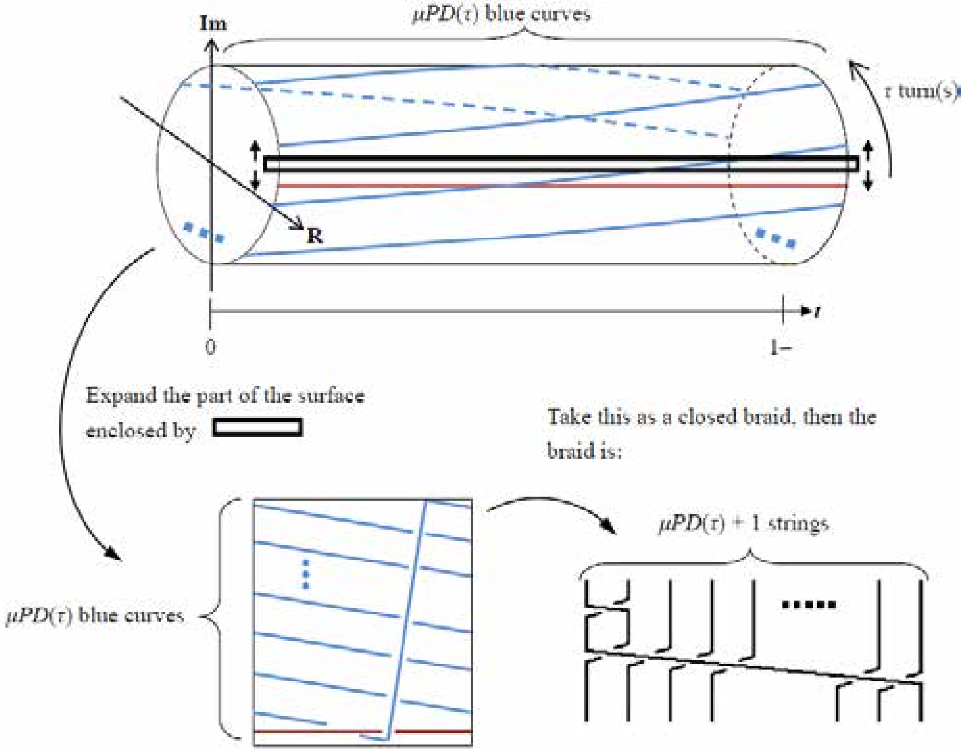


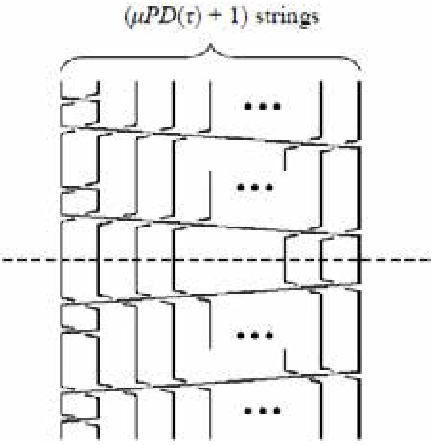
Illustration 30

If we lie the curved surface flat, we obtain the following:



Similar to the example illustrated above, the braid word for the general tst link $[\Phi^{\mu}(n(\tau), d(\tau), M)]$ where $N(\tau) = 1$ is:$\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\ldots\sigma_{\mu PD(\tau)}{}^{-1}$
and that for the general $N(\tau)$ is therefore the above braid word repeated $\mu PN(\tau)$ times: $(\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\ldots\sigma_{\mu PD(\tau)}{}^{-1})^{\mu PN(\tau)}$. □

Next we consider the tst link for $n(\tau) < 0$. Similar to previous construction, we obtain the general form of the braid word as follows:
If M is even and $n(\tau) < 0$, then the tst link $[\Phi^{\mu}(n(\tau), d(\tau), M)]$ is equivalent to the $(\mu PN(\tau), \mu PD(\tau))$-torus link and has the braid word $(\sigma_{\mu PD(\tau)-1}\sigma_{\mu PD(\tau)-2}\cdots\sigma_2\sigma_1)^{\mu PN(\tau)}$.
If M is odd and $n(\tau) < 0$, the braid for the tst link $[\Phi^{\mu}(n(\tau), d(\tau), M)]$ is in fact the mirror image of that for the tst link $[\Phi^{\mu}(|n(\tau)|, d(\tau), M)]$ along the bottom line. Therefore, the braid word will be
$[(\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\ldots\sigma_{\mu PD(\tau)}{}^{-1})^{\mu P|N(\tau)|}]^{-1} = (\sigma_{\mu PD(\tau)}\sigma_{\mu PD(\tau)-1}\ldots\sigma_3\sigma_2\sigma_1{}^2)^{\mu P|N(\tau)|}$



braid word for $[\Phi^\mu(|n(\tau)|, d(\tau), M)]$:
$(\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\ldots\sigma_{\mu PD(\tau)}{}^{-1})^{\mu P|N(\tau)|}$

Reflecting along this line

braid word for $[\Phi^\mu(n(\tau), d(\tau), M)]$:
$\sigma_{\mu PD(\tau)}\sigma_{\mu PD(\tau)-1}\ldots\sigma_3\sigma_2\sigma_1{}^2)^{\mu P|N(\tau)|}$

12. General Form of the Seifert Matrices and Alexander Polynomials of Tst Links

In this section, we establish links between the class of links we have discovered (i.e. tst links) and the common knot theory.
In the first part, we present the result for $[\Phi^\mu(n(\tau), d(\tau), M)]$ where $n(\tau) > 0$. In the second, we would present the part for $n(\tau) \leq 0$.

Notation 67. *The Seifert matrix of the link* $[\Phi^\mu(n(\tau), d(\tau), M)]$ *is denoted by* $V^\mu(n(\tau), d(\tau), M)$.
The Alexander polynomial of the link $[\Phi^\mu(n(\tau), d(\tau), M)]$ *is denoted by* $[\Delta^\mu(n(\tau), d(\tau), M)](t)$.

Using the following notation allows us to express the general form of $V^\mu(n(\tau), d(\tau), M)$, more easily.

Notation 68. $B_r = \begin{bmatrix} -1 & & & & & \\ 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & -1 & \\ & & & & 1 & -1 \end{bmatrix}$ *with zeros in the unspecified entries.*

Theorem 69. *If M is even, then* $V^\mu(n(\tau), d(\tau), M) \underset{S}{=} B_{\mu PD(\tau)-1} \otimes (-B_{\mu PN(\tau)-1}{}^T)$.
If M is odd, then $V^\mu(n(\tau), d(\tau), M) \underset{S}{=} \begin{bmatrix} -B_{2\mu PN(\tau)-1}{}^T & 0 \\ A & B_{\mu PD(\tau)-1} \otimes (-B_{\mu PN(\tau)-1}{}^T) \end{bmatrix}$
, where A is the following matrix block:

$$
\begin{bmatrix}
0 & -1 & 0 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & \ddots & & \vdots & & & 0 \\
 & & & & & 0 & & & \ddots & 0 & & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & 1 & \vdots & 0 & \vdots \\
 & & & & & & & & & -1 & & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \cdots & \cdots & \cdots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & & & \cdots & \cdots & \cdots & & 0 & 0
\end{bmatrix}_{\substack{[(\mu PN(\tau)-1)(\mu PD(\tau)-1)] \\ \times(2\mu PN(\tau)-1)}}
$$

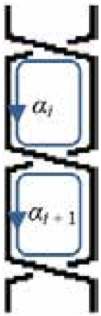
0 *denotes a* $(2\mu PN(\tau)-1)\times(\mu PN(\tau)-1)(\mu PD(\tau)-1)$ *matrix block of zeros. and the* S *below the equality sign denotes "Seifert equivalent to".*

Proof. We use facts from the theory of torus links.
As pointed out in the previous section, if M is even, the tst links $[\Phi^{\mu}(n(\tau),d(\tau),M)]$ is exactly a $(-\mu PN(\tau),\mu PD(\tau))$-torus link. The result follows directly from the theory of torus links.
If M is odd, we note that we can in fact determine the Seifert matrix using the following rules: (Here we have assumed that $n(\tau)$ is positive)

1. The "central-central" configuration:

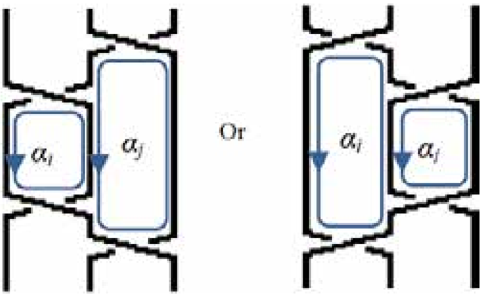
$$\mathrm{lk}(\alpha_i,\alpha_i^{\#})=1.$$

2. The "north-south" configuration:



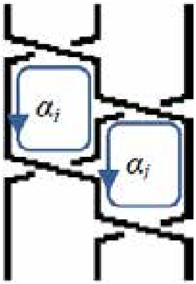
$$\mathrm{lk}(\alpha_i,{\alpha_{i+1}}^{\#})=-1 \text{ and } \mathrm{lk}(\alpha_{i+1},\alpha_i^{\#})=0$$

3. The "east-west" configuration:



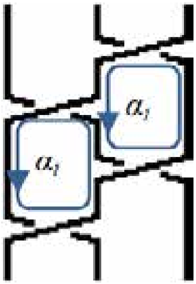
$$\mathrm{lk}(\alpha_i, \alpha_j^{\#}) = \mathrm{lk}(\alpha_j, \alpha_i^{\#}) = 0$$

4. The "northwest-southeast" configuration:



$$\mathrm{lk}(\alpha_i, \alpha_j^{\#}) = 0 \text{ and } \mathrm{lk}(\alpha_j, \alpha_i^{\#}) = -1$$

5. The "northeast-southwest" configuration:



$$\mathrm{lk}(\alpha_i, \alpha_j^{\#}) = 0 \text{ and } \mathrm{lk}(\alpha_j, \alpha_i^{\#}) = 1$$

6. Otherwise

Any other configurations would give $\mathrm{lk}(\alpha_i, \alpha_j^{\#}) = 0$.
Then using the braid word found from the last section, we can obtain the stated result. □

Next, we shift our focus from Seifert matrices to the associated Alexander polynomial.

Theorem 70. *If M is even, then*

$$[\Delta^{\mu}(n(\tau), d(\tau), M)](t) = (-1)^{\mu P-1}\frac{(1-t)(1-t^{\mu PN(\tau)D(\tau)})^{\mu P}}{(1-t^{\mu PN(\tau)})(1-t^{\mu PD(\tau)})}t^{-\frac{(\mu PN(\tau)-1)(\mu PD(\tau)-1)}{2}}$$

Proof. The result follows directly from the formula of Alexander polynomials for torus links that if $K(q, r)$ is a $(-q, r)$-torus link, then from the book [4], its Alexander polynomial is:

$$\Delta(-q, r) = (-1)^{d-1}\frac{(1-t)(1-t^{l})^{d}}{(1-t^{q})(1-t^{r})}t^{-\frac{(q-1)(r-1)}{2}}, \text{ where } d = \gcd(q, r) \text{ and } l = \operatorname{lcm}(q, r).$$

We then substitute $q = \mu PN(\tau)$, and $r = \mu PD(\tau)$.
Then $d = \gcd(\mu PN(\tau), \mu PD(\tau)) = \mu P \gcd(N(\tau), D(\tau)) = \mu P$.
Also, we have $l = \operatorname{lcm}(\mu PN(\tau), \mu PD(\tau)) = \mu PN(\tau)D(\tau)$. □

Next we calculate the Alexander polynomial of tst links where M is odd.
Here we consider some special cases, namely where $\mu PN(\tau)$ and $\mu PD(\tau)$ are 0 or 1.

Case I $\mu PN(\tau) = 0$.
We then have $\mu = 0$ (rejected) or $P = 0$ or $N(\tau) = 0$.

Subcase I.1 $P = 0$.
Here we have $\mu PD(\tau) = 0$ and that $M = 1$. Therefore, the resultant knotted tst after cutting is the same as that before cut. In other words, the resultant knotted tst is in the form of a trivial knot and therefore has an Alexander polynomial of 1.

Subcase I.2 $N(\tau) = 0$.
Here we have $\tau = 0$, with $D(\tau) = 1$. Hence we have $\mu PD(\tau) = \mu P$. The braid word for the link $[\Phi^{\mu}(n(\tau), d(\tau), M)]$ is therefore $(\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\ldots\sigma_{\mu P}{}^{-1})^0$ $= (\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\ldots\sigma_{\mu P}{}^{-1})(\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\ldots\sigma_{\mu P}{}^{-1})^{-1} = e_{\mu P}$, where $e_{\mu P}$ is the trivial braid of μP strings. Hence the link $[\Phi(n(\tau), d(\tau), M)]$ is the μP-component unlink, which has Alexander polynomial 0.

Case II $\mu PD(\tau) = 0$.
Since $D(\tau) > 0$, we must have $\mu P = 0$ and therefore $\mu PN(\tau) = 0$, so the result follows from Case I.

Case III $\mu PN(\tau) = 1$
Since μ, P and $N(\tau)$ are nonnegative integers, we have $\mu = P = N(\tau) = 1$. Hence, we have $\mu PD(\tau) = D(\tau)$. The braid word of the link $[\Phi(n(\tau), d(\tau), M)]$ is σ_1^{-2} which is Markov equivalent to that of a Hopf link. Hence, the Alexander polynomial is $[\Delta^{\mu}(n(\tau), d(\tau), M)](t) = t^{\frac{-1}{2}}(1-t)$.

Case IV $\mu PD(\tau) = 1$
Similar to the above case, we have $\mu = P = D(\tau) = 1$.

The Seifert matrix $V^{\mu}(n(\tau), d(\tau), M)$ is then $-B_{2N(\tau)-1}{}^{\mathrm{T}}$, which is Seifert equivalent to that of a $(2, 2N(\tau)-1)$-torus link, so the Alexander polynomial is $t^{-(N(\tau)-1)}\dfrac{(1-t)[1-t^{2(2N(\tau)-1)}]}{(1-t^2)(1-t^{2N(\tau)-1})}$.
Tabulated in Appendix III is a calculation result by Scilab for $2 \leq \mu PN(\tau) \leq 10$ and $2 \leq \mu PD(\tau) \leq 10$.
Although some of the links may not exist, we put them here so as to guess a general pattern for their Alexander polynomial.
We find that the following statement holds for $2 \leq \mu PN(\tau) \leq 10$ and $2 \leq \mu PD(\tau) \leq 10$, but unfortunately we are unable to prove it in general, so we conjecture that the following result holds for all integers $\mu PN(\tau)$ and $\mu PD(\tau)$.

Conjecture 71. *If M is odd, then the Alexander polynomial of the tst link $[\Delta^{\mu}(n(\tau), d(\tau), M)](t)$ is:*

$$[\Delta^{\mu}(n(\tau), d(\tau), M](t) = t^{\frac{-[(\mu PN(\tau)-1)(\mu PD(\tau)+1)+1]}{2}}\frac{[1-t^{(\mu PD(\tau)+1)N(\tau)}]^{\mu P}(1-t)}{1-t^{\mu PD(\tau)+1}}.$$

13. Tst links and Torus links

In this section we are studying the relationship between the set of tst links and that of torus links.
We denote the <u>set of all links</u> by $\boldsymbol{L}$, <u>set of torus links</u> by $\boldsymbol{T}$ and <u>set of tst links</u> by $\boldsymbol{S}$. We shall state and prove some propositions on $\boldsymbol{T}$ and $\boldsymbol{S}$.

Lemma 72. *There are some links that are both tst links and torus links, i.e. $\boldsymbol{S} \cap \boldsymbol{T} \neq \{\}$.*

Proof. Note that for M is even, the tst link $[\Phi(n(\tau), d(\tau), M)]$ is a $(-\mu PN(\tau), \mu PD(\tau))$-torus link$\in \boldsymbol{T}$. Hence the stated result follows. □

Lemma 73. *Not all tst links are torus links, i.e. $\exists s \in \boldsymbol{S}, s \notin \boldsymbol{T}$.*

Proof. Consider the tst link $[\Phi(1,4,3)]$. Here we have $\mu PN(\tau) = 1(2)(1) = 2$ and $\mu PD(\tau) = 1(2)(4) = 8$. Using the result from a Scilab calculation tabulated in Appendix III, its Alexander polynomial is $t^{\frac{-9}{2}}(1-t)(1-t^8)$ which is obviously not the same as that of any torus link. Hence we have $[\Phi(1,4,3)] \notin \boldsymbol{T}$ and the result therefore follows. □

However, we are not able to deduce whether all torus links are tst links. We can only prove the following proposition:

Proposition 74. *If $D(\tau)$ is odd, then the multiplicity P of $<N(\tau)>$ in the tst sum $[\Phi(n(\tau), d(\tau), M)]$ is even.*

Proof. If $M = 2$, $P = 2$ which is even.
Then we suppose $M > 2$ and consider two cases. In the first, $g(\tau) = 1$, and the other, $g(\tau) > 2$. Write $D(\tau) = 2r + 1$.
For $g(\tau) = 1$, recall the formula for P is:

$$P = \begin{cases} \frac{1}{4}[(2D(\tau) - 3)(M - 1) + 6D(\tau)](M - 1) & \text{if } M \text{ is odd} \\ \frac{1}{4}[2(D(\tau) - 1)(M - 2)(M + 1) - M(M - 6)] & \text{if } M \text{ is even} \end{cases}.$$

where $d(\tau) \neq 5, 7$.
However, even when $d(\tau) = 5$ or 7, adding the term $-4\delta_{12}(M)$ and $-8\delta_{12}(M)$, which must be even, does not affect the even-odd property of P. Therefore we can neglect them in proving our result and write

$$P' = \begin{cases} \frac{1}{4}[(2D(\tau) - 3)(M - 1) + 6D(\tau)](M - 1) & \text{if } M \text{ is odd} \\ \frac{1}{4}[2(D(\tau) - 1)(M - 2)(M + 1) - M(M - 6)] & \text{if } M \text{ is even} \end{cases}$$

for all odd $d(\tau)$.

If M is odd, write $M = 2k + 1$, Then P' is

$$\frac{1}{4}[(4k + 2 - 3)(2r + 1 - 1) + 6(2k + 1)](2r + 1 - 1)$$
$$= r(4kr - r + 6k + 3)$$

For simplicity, we switch to modulus language:

If $r \equiv 1 (\text{mod } 2)$, then $P \equiv P' \equiv 1(4k - 1 + 6k + 3) \equiv 0 \pmod 2$
If $r \equiv 0 (\text{mod } 2)$, then $P \equiv P \equiv 0[4k(0) - 0 + 6k + 3] \equiv 0 \pmod 2$

In both cases we see that P is even.
If M is even, write $M = 2m$, Then P' is

$$\frac{1}{4}[2(2k + 1 - 1)(2m - 2)(2m + 1) - 2m(2m - 6)]$$
$$= 2k(m - 1)(2m + 1) - m(m - 3)$$

If $m \equiv 1 \pmod 2$, then $P \equiv P' \equiv 2k(1 - 1)[2(1) + 1] - 1(1 - 3) \equiv -2 \equiv 0 \pmod 2$
If $m \equiv 0 \pmod 2$, then $P \equiv P' \equiv 2k(0 - 1)[2(0) + 1] - 0(0 - 3) \equiv -2k \equiv 0 \pmod 2$
Therefore, P is even when $D(\tau)$ is odd and $g(\tau) = 1$.

Now for $g(\tau) > 1$, similarly we write:

$$P' = \begin{cases} D(\tau)(M - 1)(3MD(\tau) - M + D(\tau)) & \text{if } M \text{ is odd} \\ \frac{1}{2}[(D(\tau) - 1)(M - 2)(M + 1) + 2M] & \text{if } M \text{ is even and } M \neq 2 \end{cases}$$

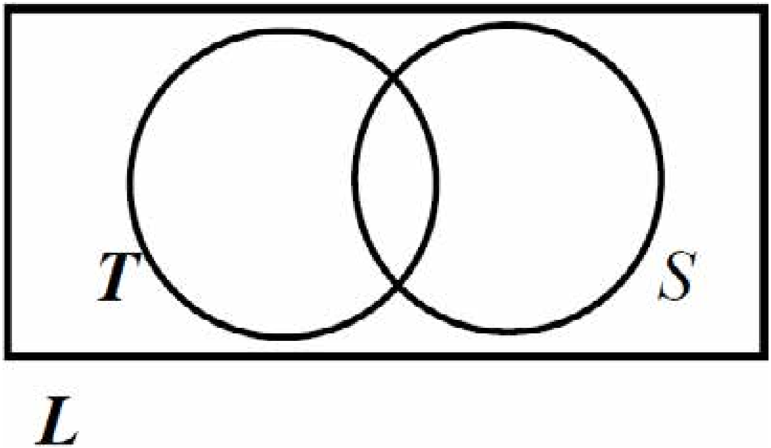
Writing $D(\tau) = 2k+1$ and $M = \begin{cases} 2r+1 & \text{if } M \text{ is odd} \\ 2m & \text{if } M \text{ is even} \end{cases}$, we have:

$$P' = \begin{cases} (2k+1)(2r+1-1)[3(2r+1)(2k+1)-(2r+1)+(2k+1)] & \text{if } M \text{ is odd} \\ \frac{1}{2}[(2k+1-1)(2m-2)(2m+1)+2(2m)] & \text{if } M \text{ is even} \end{cases}$$

$$= \begin{cases} 2r(2k+1)(12rk+4r+8k+3) & \text{if } M \text{ is odd} \\ 2[k(m-1)(2m+1)+m] & \text{if } M \text{ is even} \end{cases}$$

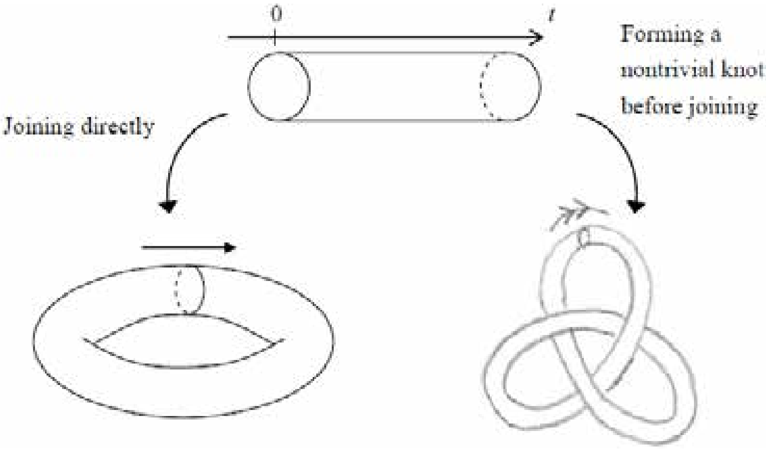
which is obviously even and is thus P.
Hence, the result follows. $\square$

If the conjecture in the previous section is true, then some (q,r)-torus links, where q and r are odd, are not tst links. In that case we conjecture that the following Venn diagram shows the relationship between $\boldsymbol{S}$, $\boldsymbol{T}$ and $\boldsymbol{L}$.

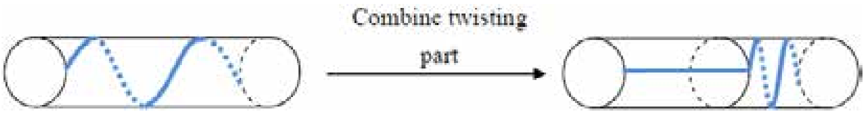


14. Cutting a Tst in the Form of a Nontrivial Knot or Link

Now, let us consider the result of cutting a tst in the form of a nontrivial knot or link. In fact, it can be derived from the result of cutting a tst by taking solid knots as a set of twisted curves. Instead of joining the cross sections at $t = 0$ and $t \to 1^-$ directly, we let the cylinder form a tangle before having its two ends joined.



To apply a specific twist turn on a knot, we combine all twisting part of the curves into a short cylinder and insert it into a solid knot we set to have no twist turn.



When cutting an "untwisted" knot into two parts, we should make two copies of the original knot. It is natural for us to set the two boundaries as two of the "untwisted" curves of the knot.

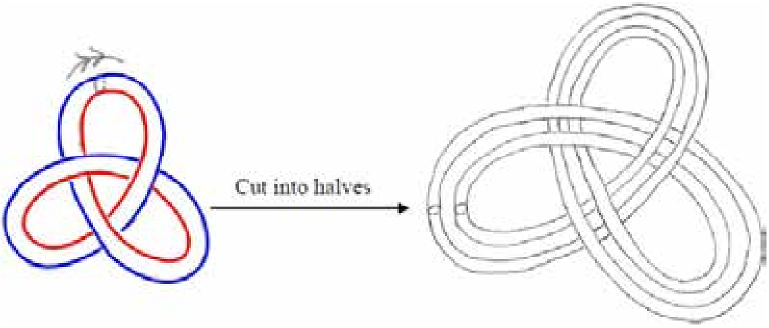
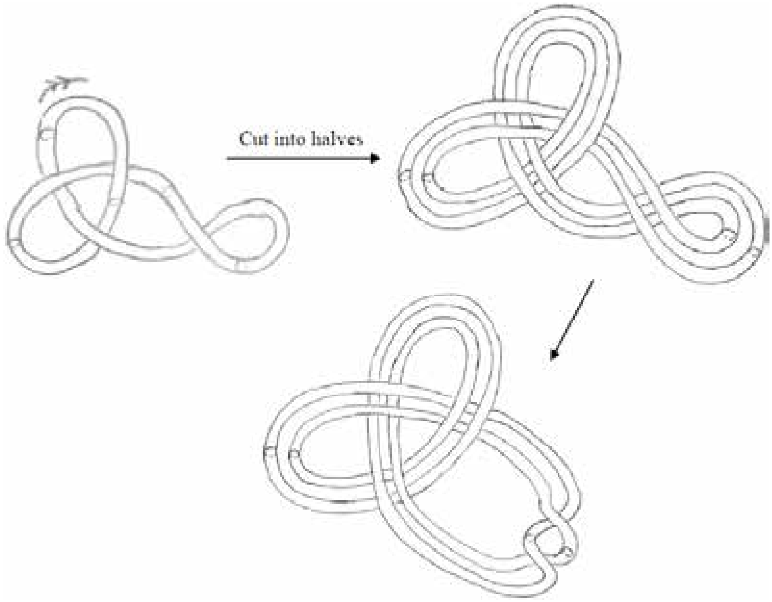


ILLUSTRATION 31. An "untwisted" trefoil cut into halves

However, Reidemeister type 1 moves can produce twist turn without changing the knot. If we perform Reidemeister type 1 moves before we cut the tst, (start from a different knot graph of the same knot,) the cutting result will be different.



To specify which knot graph we are using, we present the knot in braid words. By smoothly deforming the knot, we can set the left boundary of each string as the curves in an untwisted knot.

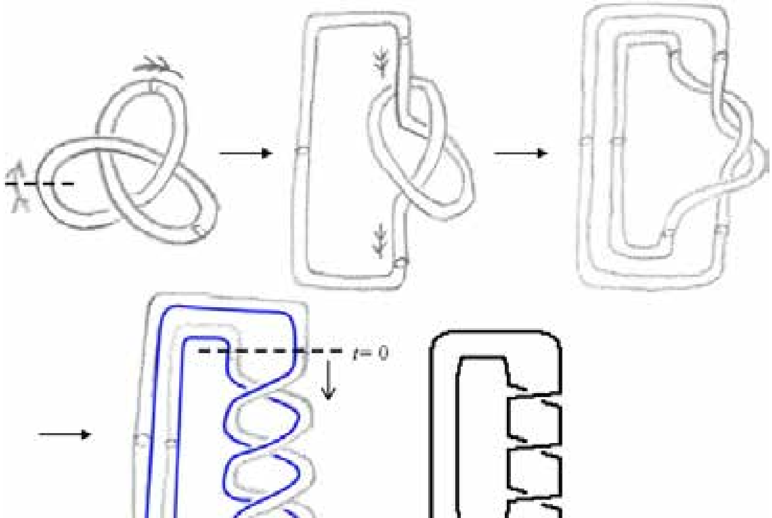


Illustration 32. Conversion from regular diagram of a trefoil to braid

Then we can now insert the twisted short cylinder into the knot. For convenience, we put the cylinder just on top of the leftmost string.

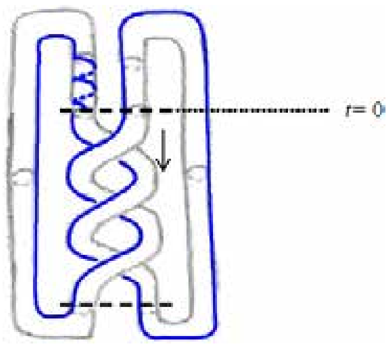


Illustration 33. braid with twist turn inserted

For a tst link, we twist and cut each component once. Braid word can also be used but since we have applied twist turn k times to a k-component link, twisted cylinder should be inserted on top of the leftmost string of each component.

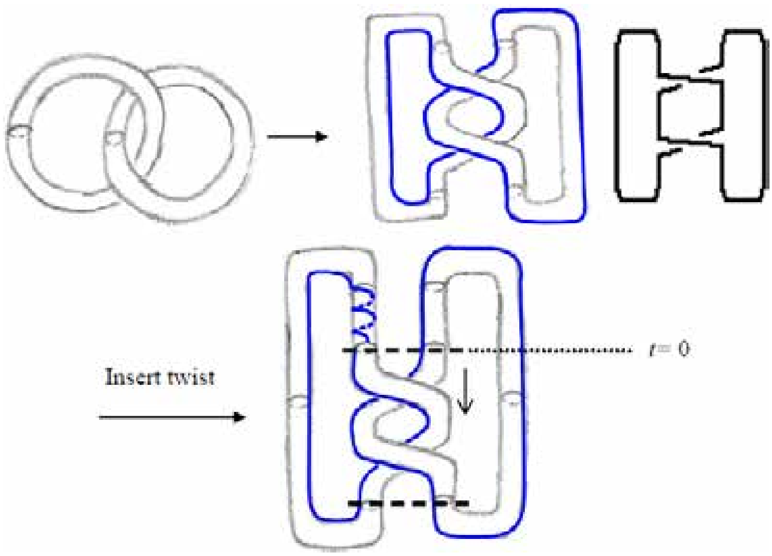
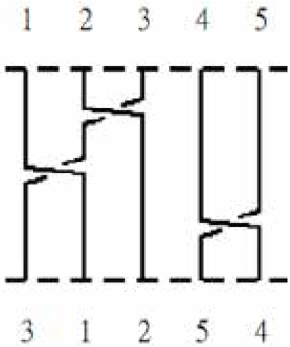


Illustration 34. Conversion from regular diagram of a hopf link to braid with twist turn

We note that for every braid, there is a unique braid permutation. For example, the braid permutation of the braid $\sigma_2\sigma_1\sigma_4$ is permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$, or, in disjoint cycle notation, (1 2 3)(4 5) = (3 1 2)(4 5) = (5 4)(2 3 1), as illustrated below:



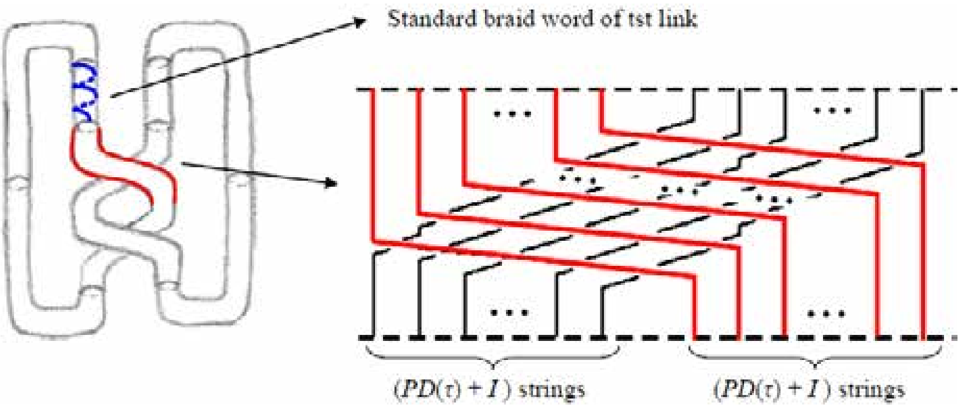
$(1 \to 2, 2 \to 3, 3 \to 1, 4 \to 5, 5 \to 4)$

As the disjoint cycle representation of a permutation is not unique, we choose to use the disjoint cycle notation where the first number of each cycle is the smallest among all numbers in that cycle, and the first numbers in the cycles from left to right are in ascending order. We call this more restricted disjoint cycle representation the "component formula" of the braid.

Definition 75. *The component formula of a r-braid is the its permutation in disjoint cycle form in which every number from 1 to r appears once and the j^{th} number from the left in the i^{th} cycle from the left is denoted by $x_{i,j}$, with $x_{i,1} < x_{i,2} < x_{i,3} < \ldots$ and $x_{1,1} < x_{2,1} < x_{3,1} < \ldots$.*

Apparently $x_{1,1}$ is smaller than every other number in the component formula, so $x_{1,1} = 1$.
Moreover, the number of disjoint cycles in the component formula is exactly the number of components in the link L formed by closing the braid, denoted by $\mu(L)$. To know where the leftmost string of each component is, we consider the "component formula".
The component formula can show which strings in a braid are in the same link component:
$(1, x_{1,2}, x_{1,3}, x_{1,4}, \ldots, x_{1,y_1})$, $(x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, \ldots, x_{2,y_2})$, $(x_{3,1}, x_{3,2}, \ldots, x_{3,y_2})$, ..., $(x_{\mu(L),1}, x_{\mu(L),2}, \ldots, x_{\mu(L),y_{\mu(L)}})$, where the integers $x_{i,j}$ in a bracket represent the positions of strings in the same link component.
When a process similar to cutting the tst $<\tau>$ into M parts is applied on a k-component link as the closed form of a braid, each component is cut into $(PD(\tau)+I)$ partial tsts so there will be k$(PD(\tau)+I)$ partial tsts in total. Then we extract the centre of mass of the cross section at $t = 0$ of each partial tst, project them to $k(PD(\tau)+I)$ distinct points on the circumference with the aid of sideway movement and present the partial tsts with the curves starting from these points. Next, we lies the curved surface flat. The twisted parts will give standard braid words from

cutting a tst. In the tangle part, each crossing in the original braid will become $(PD(\tau)+I)$ strings covering another $(PD(\tau)+I)$ strings.



To clearly present the braid of the resultant link, we develop the idea of braids on braids.

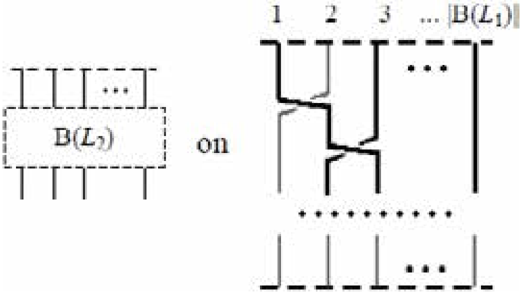
Definition 76 (link on link)**.** *If two links L_1 and L_2 have chosen braid words $B(L_1)$ and $B(L_2)$ respectively, where $B(L_1)=\prod_{s=1}^{k_1}\sigma_{a_s}{}^{m_s}\in B_{p_1}$ and $B(L_2)=\prod_{t=1}^{k_2}\sigma_{b_t}{}^{n_t}\in B_{p_2}$ and the number of components in the link L is $\mu(L)$, then we define the braid word of "$B(L_2)$ on $B(L_1)$" to be:*

$$\Big(\prod_{u=1}^{\mu(L_1)}\prod_{t=1}^{k_2}\sigma_{(x_{u,1}-1)p_2+b_t}{}^{n_t}\Big)\prod_{s=1}^{k_1}\Big(\coprod_{(a_s-1)p_2+1}^{r=a_sp_2}\ \prod_{q=r}^{r+p_2-1}\sigma_q^{sgn(m_s)}\Big)^{|m_s|}$$

where $\prod_{i=1}^{n}\sigma_i=\sigma_1\sigma_2\sigma_3\cdots\sigma_n$ and $\coprod_{1}^{i=n}\sigma_i=\sigma_n\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$.
The closure of this braid is denoted as $L_2L_1(B(L_2),B(L_1))$.

When both L_1 and L_2 are tst links and their standard braids are used, without ambiguity, we can write L_2L_1(B(L_2),B(L_1)) as L_2L_1 only.
Obviously the resultant link depends on the chosen braid words B(L_1) and B(L_2). From this definition, B(L_2L_1(B(L_2),B(L_1))) is actually replacing each string in B(L_1) with p_2 strings and putting B(L_2) on top of the strings originated from the leftmost string of each component in L_1. The first two product signs represent B(L_2) inserted in the braid. From the component formula, the leftmost strings of the link components in B(L_1) are numbered $x_{1,1},x_{2,1},x_{3,1}$ $x_{\mu(L),1}$ respectively. Since the string with number $x_{u,1}$ in B(L_1) will be changed to p_2 strings numbered $(x_{u,1}-1)p_2+1,(x_{u,1}-1)p_2+2,(x_{u,1}-1)p_2+3,\ldots,x_{u,1}p_2$, the B($L_2$) inserted

there will become $\prod_{t=1}^{k_2} \sigma_{(x_{u,1}-1)p_2+b_t}{}^{n_t}$. To present the $\mu(L_1)\mathrm{B}(L_2)$ inserted, another product sign is used.



The leftmost strings of the components in L_1 are darkened.

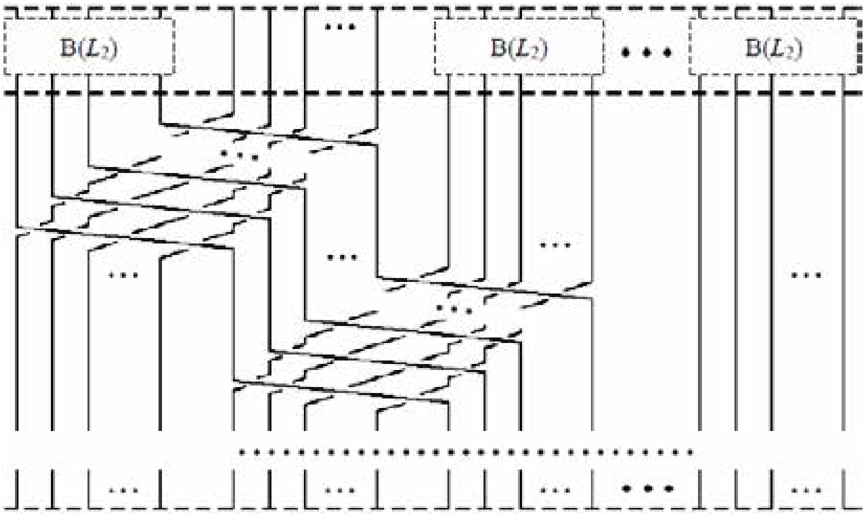
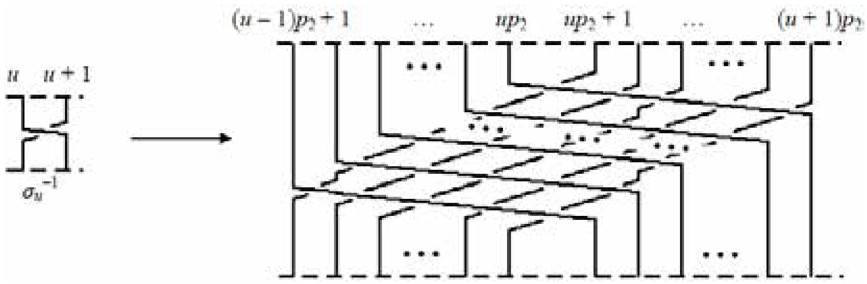
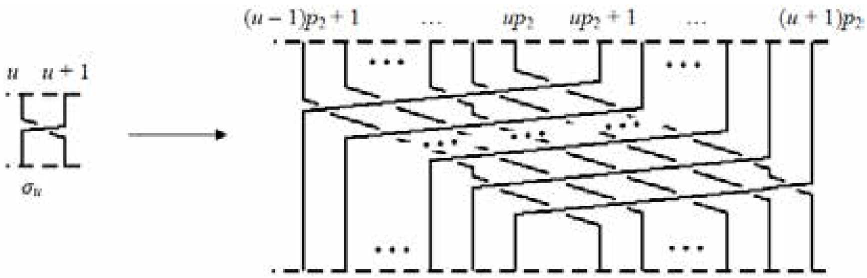


ILLUSTRATION 35. $B(L_2)$ put on p_2 strings coming from $x_{u,1}$

The last two product signs show the crossings in $\mathrm{B}(L_2L_1(\mathrm{B}(L_2),\mathrm{B}(L_1)))$ resulted from one crossing in $\mathrm{B}(L_1)$. The direction of the crossing will be unchanged in the conversion. A crossing in $\mathrm{B}(L_1), \sigma_u^*$ where $* = 1$ or -1, will be changed to $\coprod_{(u-1)p_2+1}^{r=up_2} \prod_{q=r}^{r+p_2-1} \sigma_q^*$ and $\sigma_u^{m_s}$ will be changed to $\left(\coprod_{(u-1)p_2+1}^{r=up_2} \prod_{q=r}^{r+p_2-1} \sigma_q^{\mathrm{sgn}(m_s)} \right)^{m_s}$. To convert the whole $\mathrm{B}(L_1)$, one more product sign is used.



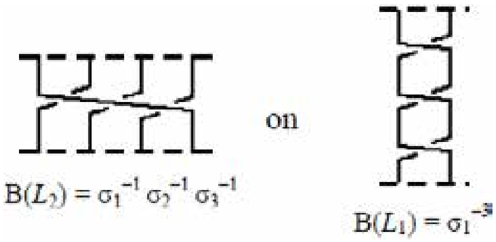
$$
\begin{aligned}
&\sigma_u^{-1} \to \\
&(\sigma_{up_2}^{-1}\sigma_{up_2+1}^{-1}\sigma_{up_2+2}^{-1}\cdots\sigma_{(u+1)p_2-1}^{-1})(\sigma_{up_2-1}^{-1}\sigma_{up_2}^{-1}\sigma_{up_2+1}^{-1}\cdots\sigma_{(u+1)p_2-2}^{-1}) \\
&\cdots(\sigma_{(u-1)p_2+1}^{-1}\sigma_{(u-1)p_2+2}^{-1}\sigma_{(u-1)p_2+3}^{-1}\cdots\sigma_{up_2}^{-1}) \\
&= \coprod_{(u-1)p_2+1}^{r=up_2}(\sigma_r^{-1}\sigma_{r+1}^{-1}\sigma_{r+2}^{-1}\cdots\sigma_{r+p_2-1}^{-1}) = \coprod_{(u-1)p_2+1}^{r=up_2}\prod_{q=r}^{r+p_2-1}\sigma_q^{-1}
\end{aligned}
$$



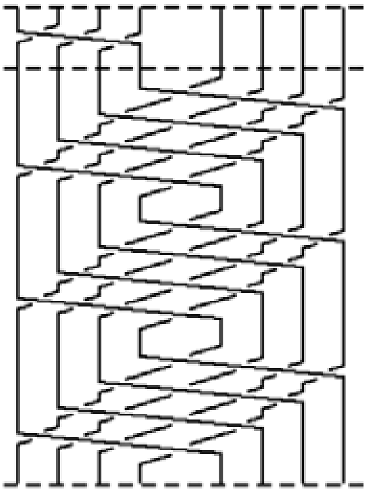
$$
\begin{aligned}
&\sigma_u \to \\
&(\sigma_{up_2}\sigma_{up_2+1}\sigma_{up_2+2}\cdots\sigma_{(u+1)p_2-1})(\sigma_{up_2-1}\sigma_{up_2}\sigma_{up_2+1}\cdots\sigma_{(u+1)p_2-2}) \\
&\cdots(\sigma_{(u-1)p_2+1}\sigma_{(u-1)p_2+2}\sigma_{(u-1)p_2+3}\cdots\sigma_{up_2}^{-1}) \\
&= \coprod_{(u-1)p_2+1}^{r=up_2}(\sigma_r\sigma_{r+1}\sigma_{r+2}\cdots\sigma_{r+p_2-1}) \\
&= \coprod_{(u-1)p_2+1}^{r=up_2}\prod_{q=r}^{r+p_2-1}\sigma_q
\end{aligned}
$$

Illustration 36. A+1 crossing and a-1 crossing converted into a set of crossings

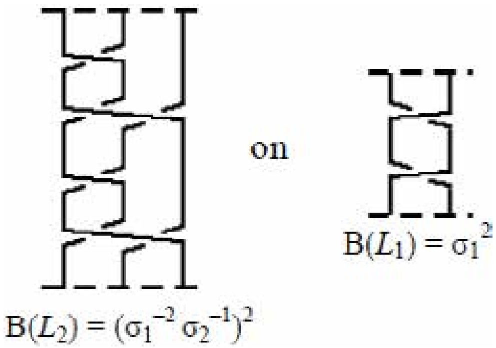
For example, standard braid word of $[\Phi(1,4,2)]$ on trefoil (with braid word σ_1^{-3}) is :



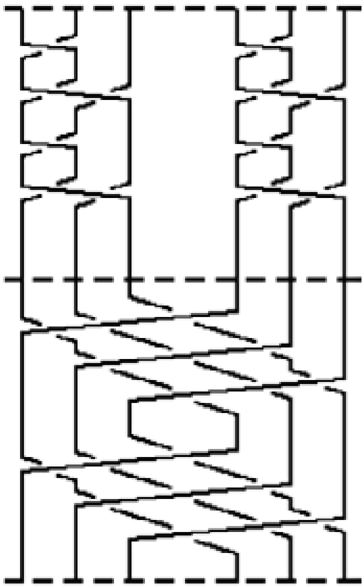
$$\mathrm{B}(L_2L_1(\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1},\sigma_1^{-3}))$$
$$=\Big(\prod_{t=1}^{3}\sigma_t^{-1}\Big)\Big(\coprod_{1}^{r=4}\prod_{q=r}^{r+3}\sigma_q^{-1}\Big)^3$$



standard braid word of $[\Phi(2,2,3)]$ on Hopf link (with braid word $\sigma_1{}^2$) is



$$\mathrm{B}(L_2L_1((\sigma_1^{-2}\sigma_2^{-1})^2,\sigma_1{}^2))$$
$$=\Big(\prod_{u=1}^{2}\big(\sigma_{3u-2}{}^{-2}\sigma_{3u-1}{}^{-1}\big)^2\Big)\Big(\coprod_{1}^{r=3}\prod_{q=r}^{r+2}\sigma_q\Big)^2$$



We also discover some other properties of links on links. The first one is the relation between the numbers of components $\mu(L_1),\mu(L_2)$ and $\mu(L_2L_1(B(L_2),B(L_1)))$.

Theorem 77 (multiplicative property of the number of components)**.** *Let $\mu(L)$ denote the number of components in the link L. Then we have $\mu(L_2L_1(B(L_2),B(L_1)))$ $= \mu(L_1)\mu(L_2)$.*

Proof. Recall that the number of components in a link L is equal to the number of disjoint cycles in the component formula of its braid form $\mathrm{B}(L)$.
Let the component formula of the links L_1 be

$$\begin{pmatrix}1 & x_{1,2} & x_{1,3} & x_{1,4} & \dots & x_{1,m_1}\end{pmatrix}\begin{pmatrix}x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & \dots & x_{2,m_2}\end{pmatrix}$$
$$\begin{pmatrix}x_{3,1} & x_{3,2} & \dots & x_{2,m_2}\end{pmatrix}\dots\begin{pmatrix}x_{\mu(L_1),1} & x_{\mu(L_1),2} & \dots & x_{\mu(L_1),m_{\mu(L)}}\end{pmatrix}$$

and that of L_2 be

$$\begin{pmatrix}1 & y_{1,2} & y_{1,3} & y_{1,4} & \dots & y_{1,n_1}\end{pmatrix}\begin{pmatrix}y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} & \dots & y_{2,n_2}\end{pmatrix}$$
$$\begin{pmatrix}y_{3,1} & y_{3,2} & \dots & y_{2,n_2}\end{pmatrix}\dots\begin{pmatrix}y_{\mu(L_2),1} & y_{\mu(L_2),2} & \dots & y_{\mu(L_2),y_{\mu(L)}}\end{pmatrix}$$

Then we write the component formula of L_1L_2 in terms of the two formulae above. We introduce the notation $|B(L_1)| = \max\{1, x_{1,2}, \dots, x_{1,m_1}, x_{2,1}, \dots, x_{\mu(L_1),m_{\mu(L)}}\}$, which is in fact the number of strings in the braid $\mathrm{B}(L_1)$, and, for the sake of brevity, we use $X_{i,j,k}$ to denote the sequence

$$\begin{array}{llll}|\mathrm{B}(L_1)|(y_{k,2}-1)+x_{i,j} & |\mathrm{B}(L_1)|(y_{k,3}-1)+x_{i,j} & |\mathrm{B}(L_1)|(y_{k,3}-1)+x_{i,j} & \dots \\ |\mathrm{B}(L_1)|(y_{2,n_2}-1)+x_{i,j} & |B(L_1)|(y_{k,1}-1)+x_{i,j}. & & \end{array}$$

Then the component formula of L_1L_2 is composed of all cycles in the form

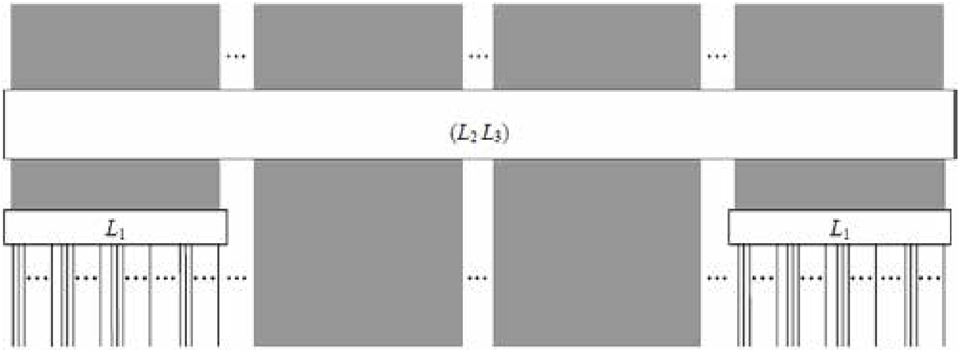
$$\begin{pmatrix}X_{i,1,k} & X_{i,2,k} & X_{i,3,k} & \dots & X_{i,m_i,k}\end{pmatrix}$$

As i ranges from 1 to $\mu(L_1)$ and k ranges from 1 to $\mu(L_2)$, the number of cycles in the component formula is $\mu(L_1)\mu(L_2)$ and the result therefore follows. $\square$

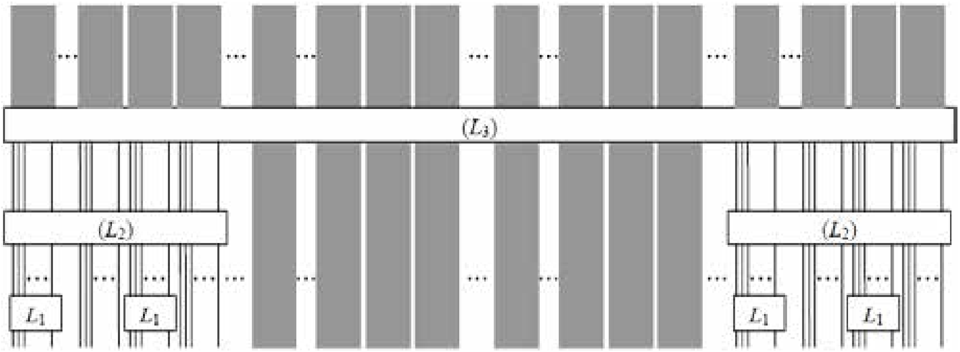
Next, we find that if we take $(L_1, L_2) \mapsto L_1L_2$ as a binary operation on the knots L_1 and L_2, then this operation is associative:

Theorem 78 (associativity). $(L_1L_2)L_3 = L_1(L_2L_3)$ *for the same* $B(L_1)$,$B(L_2)$ *and* $B(L_3)$ *on both sides.*

Proof. We notice that the braid of the link on the right hand side is Markov equivalent to:

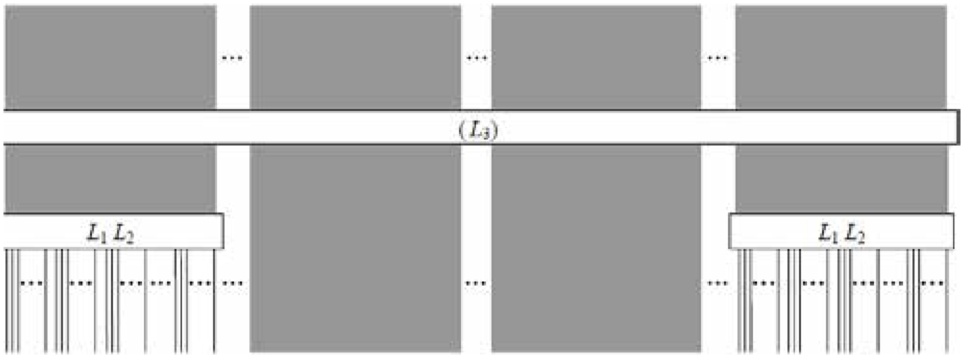


which is equivalent to

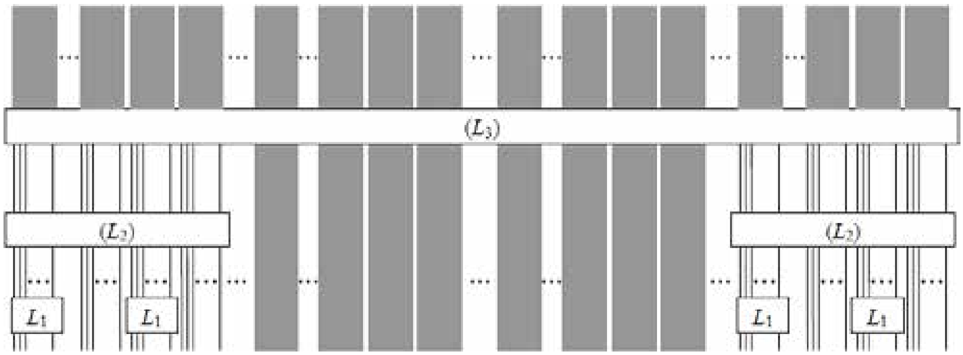


Here we have used a bracket to denote a braid replicated.

That of the one on the left hand side is Markov equivalent to:



which is equivalent to:



As the braid on the two sides on the equation are Markov equivalent to the same braid, by Markov's Theorem, the stated result follows. □

Now, we may consider cutting links that are resulted from cutting a tst. Since we have already obtained the standard braid word of a tst link, we can use the above formula to find the general form of braid words of "tst links on tst links".
Let the standard braid words for $[\Phi^{\mu_1}(n_1(\tau_1), d_1(\tau_1), M_1)]$ and $[\Phi^{\mu_2}(n_2(\tau_2), d_2(\tau_2), M_2)]$ be α_1 and α_2 respectively. The number of strings in α is denoted as $|\alpha|$ so we have $|\alpha| = PD(\tau)+I$. The strings numbered $1, 2, 3, \ldots, (P+I)$ are the leftmost strings of link components of the closed form of α. We consider the braid word of α_2 on α_1 in the following cases.

<u>Case I</u> Both M_1 and M_2 are even
The braid word of "α_2 on α_1" is :

$$\prod_{i=1}^{P_1}\Big(\prod_{j=1}^{|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+j}\Big)^{\mu_2 P_2 N_2(\tau_2)}\Big[\prod_{i=1}^{|\alpha_1|-1}\Big(\coprod_{1}^{j=|\alpha_2|}\prod_{k=j}^{j+|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+k}\Big)\Big]^{\mu_1 P_1 N_1(\tau_1)}$$

Since there is far commutative property in braids, we may simplify the above braid word a bit to have:

$$\prod_{i=1}^{P_1}\Big(\prod_{j=1}^{|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+j}\Big)^{\mu_2 P_2 N_2(\tau_2)}\Big[\coprod_{1}^{j=|\alpha_2|}\Big(\prod_{j=i}^{i+(|\alpha_1|-1)|\alpha_2|+1}\sigma_j^{-1}\Big)\Big]^{\mu_1 P_1 N_1(\tau_1)}$$

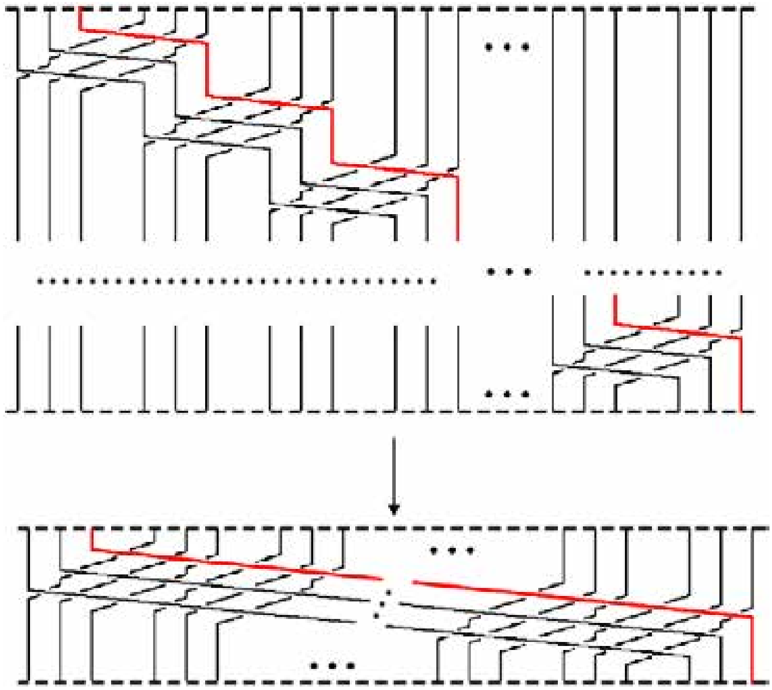


ILLUSTRATION 37. Simplification of braid word

Case II M_1 is odd and M_2 is even

The braid word of "α_2 on α_1" is:

$$\begin{aligned}
&\prod_{i=1}^{P_1+1}\Big(\prod_{j=1}^{|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+j}\Big)^{\mu_2 P_2 N_2(\tau_2)}\\
&\quad\times\Big[\coprod_{1}^{i=|\alpha_2|}\prod_{j=i}^{i+|\alpha_2|-1}\sigma_j^{-1}\prod_{i=1}^{|\alpha_1|-1}\Big(\coprod_{1}^{j=|\alpha_2|}\prod_{k=j}^{j+|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+k}\Big)\Big]^{\mu_1 P_1 N_1(\tau_1)}\\
=&\prod_{i=1}^{P_1+1}\Big(\prod_{j=1}^{|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+j}\Big)^{\mu_2 P_2 N_2(\tau_2)}\\
&\quad\times\Big[\coprod_{1}^{i=|\alpha_2|}\prod_{j=1}^{i+|\alpha_2|-1}\sigma_j^{-1}\coprod_{1}^{j=|\alpha_2|}\Big(\prod_{j=1}^{i+(|\alpha_1|-1)|\alpha_2|+1}\sigma_j^{-1}\Big)\Big]^{\mu_1 P_1 N_1(\tau_1)}
\end{aligned}$$

Case III M_1 is even and M_2 is odd
The braid word of "α_2 on α_1" is:

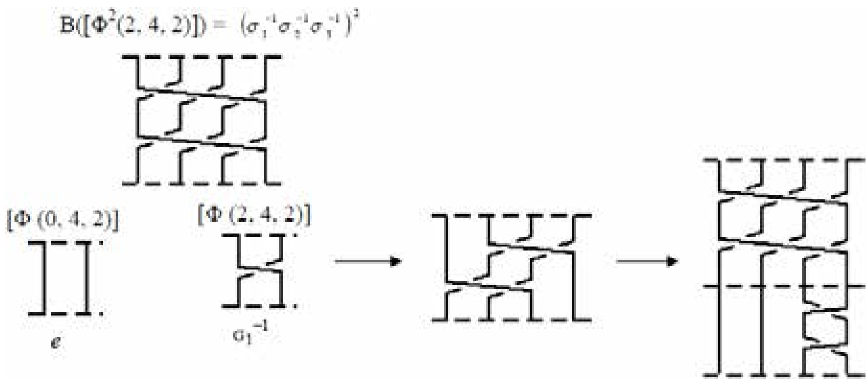
$$\prod_{i=1}^{P_1}\Big(\sigma_{(i-1)|\alpha_2|+1}\prod_{j=1}^{|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+j}\Big)^{\mu_2 P_2 N_2(\tau_2)}\Big[\prod_{i=1}^{|\alpha_1|-1}\Big(\coprod_{1}^{j=|\alpha_2|}\prod_{k=j}^{j+|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+k}\Big)\Big]^{\mu_1 P_1 N_1(\tau_1)}$$

$$=\prod_{i=1}^{P_1}\Big(\sigma_{(i-1)|\alpha_2|+1}\prod_{j=1}^{|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+j}\Big)^{\mu_2 P_2 N_2(\tau_2)}\Big[\coprod_{1}^{j=|\alpha_2|}\Big(\prod_{j=i}^{i+(|\alpha_1|-1)|\alpha_2|+1}\sigma_j^{-1}\Big)\Big]^{\mu_1 P_1 N_1(\tau_1)}$$

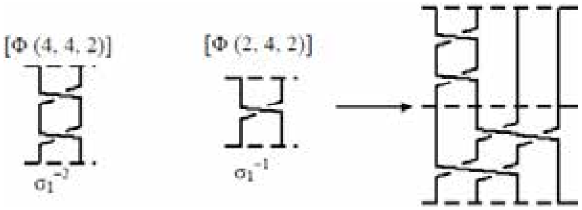
Case IV Both M_1 and M_2 are odd
The braid word of "α_2 on α_1" is:

$$\begin{aligned}&\prod_{i=1}^{P_1+1}\Big(\sigma_{(i-1)|\alpha_2|+1}\prod_{j=1}^{|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+j}\Big)^{\mu_2 P_2 N_2(\tau_2)}\\ &\quad\times\Big[\coprod_{1}^{i=|\alpha_2|}\prod_{j=i}^{i+|\alpha_2|-1}\sigma_j^{-1}\prod_{i=1}^{|\alpha_1|-1}\Big(\coprod_{1}^{j=|\alpha_2|}\prod_{k=j}^{j+|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+k}\Big)\Big]^{\mu_1 P_1 N_1(\tau_1)}\\ &=\prod_{i=1}^{P_1+1}\Big(\sigma_{(i-1)|\alpha_2|+1}\prod_{j=1}^{|\alpha_2|-1}\sigma^{-1}_{(i-1)|\alpha_2|+j}\Big)^{\mu_2 P_2 N_2(\tau_2)}\\ &\quad\times\Big[\coprod_{1}^{i=|\alpha_2|}\prod_{j=i}^{i+|\alpha_2|-1}\sigma_j^{-1}\coprod_{1}^{j=|\alpha_2|}\Big(\prod_{j=i}^{i+(|\alpha_1|-1)|\alpha_2|+1}\sigma_j^{-1}\Big)\Big]^{\mu_1 P_1 N_1(\tau_1)}\end{aligned}$$

Note that the twist turn in the tst link is reset before the link is cut again. Therefore, $\mathrm{B}([\Phi^{\mu}(n(\tau), d(\tau), M)])$ is different from $\mathrm{B}([\Phi(0, d(\tau), \mu)][\Phi(n(\tau), d(\tau), M)])$. For example, $\mathrm{B}([\Phi^2(2,4,2)])$ is the same as $\mathrm{B}([\Phi(4,4,2)][\Phi(2,4,2)])$ instead of $\mathrm{B}([\Phi(0,4,2)][\Phi(2,4,2)])$.



$$\begin{aligned}\mathrm{B}([\Phi(0,4,2)][\Phi(2,4,2)]) &= \sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}\\ &\underset{M}{=}\sigma_1\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_3\\ &\underset{M}{=}(\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})^2\sigma_3^2\end{aligned}$$



$$
\begin{aligned}
&\mathrm{B}([\Phi(4,4,2)][\Phi(2,4,2)])\\
=&(\sigma_1^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1})\\
\underset{M}{=}&(\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})^2
\end{aligned}
$$

In fact, we can cut the resultant link and get an even more complicated link. With the concept of links on links, we can cut a tst again and again. Now, we would like to consider the sets of more and more complicated link after applying a number of cuttings.

15. The Tst Link Hierarchy

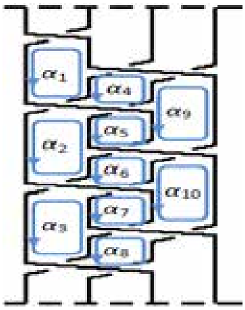
We have considered cutting tst links as "tst links on tst links". Similarly, we can get the result of cutting these new links by considering "links on 'links on links' ". To study different sets of links obtained by cutting a tst different numbers of times, we have the following notation.

Notation 79. *$\boldsymbol{S}^0$ is the singleton containing the trivial knot. Moreover, the set of links that can be written as L_1L_2 where $L_1 \in \boldsymbol{S}$ and $L_2 \in \boldsymbol{S}^{k-1}$ is denoted by $\boldsymbol{S}^k$.*

Obviously, we can write $\boldsymbol{S}^1$ as $\boldsymbol{S}$. Then $\boldsymbol{S}^2$ represents the set of "tst links on tst links" and cutting a tst in the form of a link in $\boldsymbol{S}^2$ gives a link in $\boldsymbol{S}^3$
It is obvious that $\boldsymbol{S}^0$ is a proper subset of $\boldsymbol{S}^1$, since, for example, the tst link $[\Phi(3,2,2)]$ is a trefoil which is different from a trivial knot. Next, we consider the relationship between $\boldsymbol{S}$ and $\boldsymbol{S}^2$.

Theorem 80. *$\boldsymbol{S}$ is a proper subset of $\boldsymbol{S}^2$.*

Proof. First, we will show that $\boldsymbol{S} \subseteq \boldsymbol{S}^2$. We note that if $L_1 = [\Phi(0,1,1)] \in \boldsymbol{S}$, which is a trivial knot, and $L_2 \in \boldsymbol{S}$, then L_1L_2 is by definition contained in $\boldsymbol{S}^2$; however, L_1L_2 is also equivalent to L_2. Then we will find an element in $\boldsymbol{S}^2$ which is not a tst link. Consider the braid word of $[\Phi(1,2,2)]$ on $[\Phi(3,2,2)]$: $\sigma_1^{-1}(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})^3$



Its Seifert matrix is $\left[\begin{array}{ccc:ccccc:cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right]$
and its Alexander polynomial is
$t^{-5}(1-t+t^4-t^5+t^6-t^9+t^{10}) = t^{-5}\dfrac{(1-t)(1-t^{12})(1-t^{14})}{(1-t^4)(1-t^6)(1-t^7)}$ which is not the same as that of any tst link. Therefore, the link $[\Phi(1,2,2)]$ on $[\Phi(3,2,2)]$ belongs to $\boldsymbol{S}^2$, but not to $\boldsymbol{S}$ and hence the result follows. $\square$

The following statement is likely to be true although we are unable to prove it.

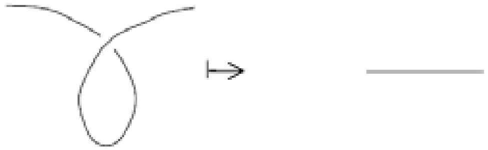
Conjecture 81. *$\boldsymbol{S}^k$ is a proper subset of $\boldsymbol{S}^{k+1}$ for any nonnegative integers k.*

We are able only to prove the case for $k = 0, 1$ and we conjecture that it holds for all nonnegative integers k. If this statement is found true, then we would have a infinitely extending hierarchy of tst links, for which we also conjecture that $\boldsymbol{S}^\infty = \boldsymbol{L}$, where $\boldsymbol{L}$ is the set of all links. We believe that this could open up new directions in researching what hierarchies of links "diverge" to give $\boldsymbol{L}$.
If this statement is found false, then note that $\boldsymbol{S}^k$ must be a subset of $\boldsymbol{S}^{k+1}$: we can consider the link $[\Phi^1(0,1,1)]$ on each link in $\boldsymbol{S}^k$, then by definition $[\Phi^1(0,1,1)]L \in \boldsymbol{S}^{k+1}$, but $[\Phi^1(0,1,1)]L = L$ for every link L in $\boldsymbol{S}^k$. Therefore, there must be some terminating set $\boldsymbol{S}^k$ such that $\boldsymbol{S}^n = \boldsymbol{S}^k$ for every $n \geq k$, i.e. $\boldsymbol{S}^k$ is an improper subset of $\boldsymbol{S}^{k+1}$. This could arouse attention in researching what hierarchies of links "converge" to some set of links.

16. Unknotting a Knotted Tst to Give Additional Twist Turns

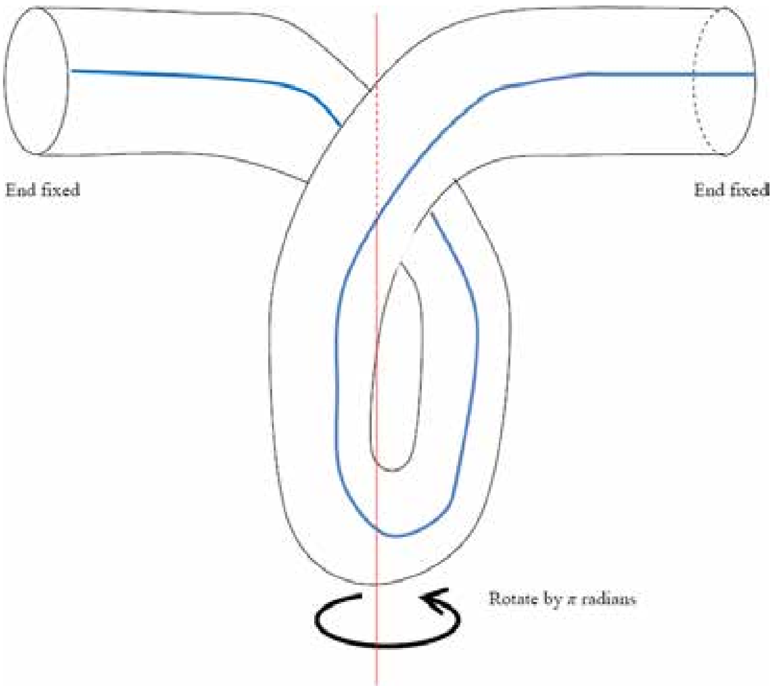
We are motivated to think about this topic because of the difference in the context of the 1937 and 1987 [1] versions of W. Ball's mathematical essays. The earlier version points out that, when switched to our language, if k is odd and positive, then $\Phi(k,2,2) = <k>_K$, which agrees with our result, while the latter claims that $\Phi(k,2,2) = <k+1>_K$. After our observation, we discovered that this one additional twist turn results from unwinding the knotted tst $<k>_K$ using a type I Reidemeister move, as we shall prove below:

Lemma 82. *If a Reidemeister move is applied to a given knotted tst* $<\tau>_K$*, then it becomes* $<\tau + 1>_K$*:*

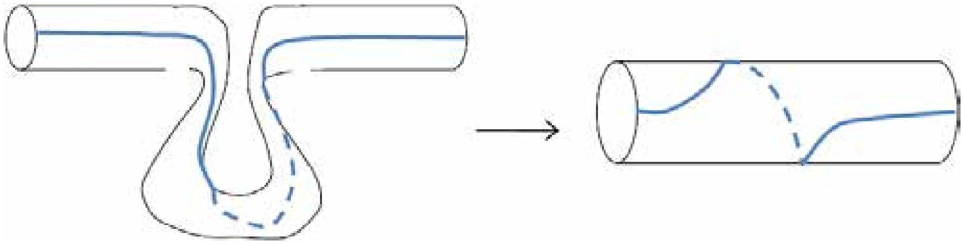


Conversely, if the following Reidemeister move is applied to $<\tau>_K$*, then it becomes* $<\tau - 1>_K$*:*

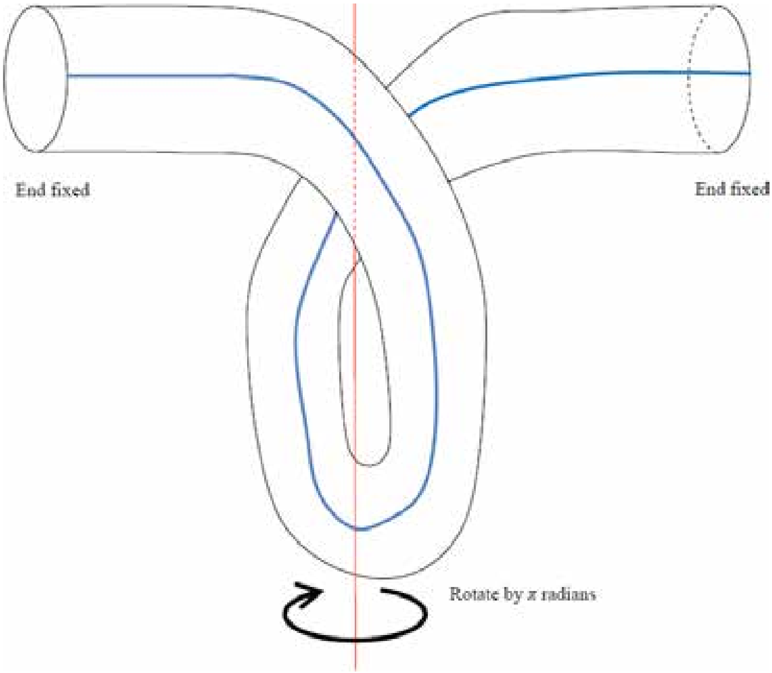
Proof. We consider a portion of the knotted tst that we would apply a Reidemeister move to eliminate the crossing, and then see how this affects the curling of the curve inside a the knotted tst.



After rotation, this becomes:



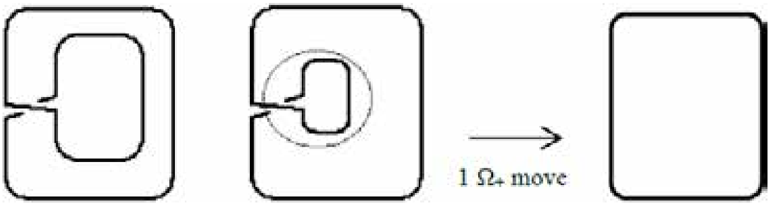
Hence, the twist turn of the knotted tst increases by 1 after the application of such a Reidemeister move.

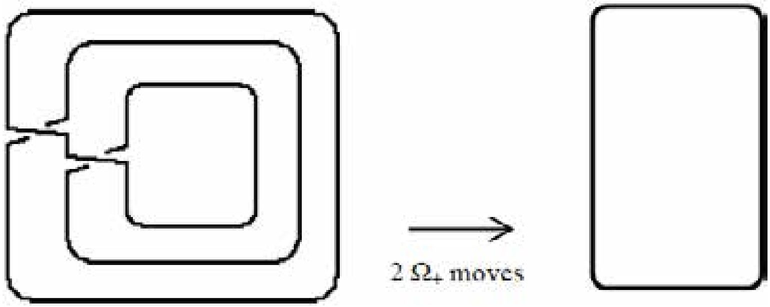


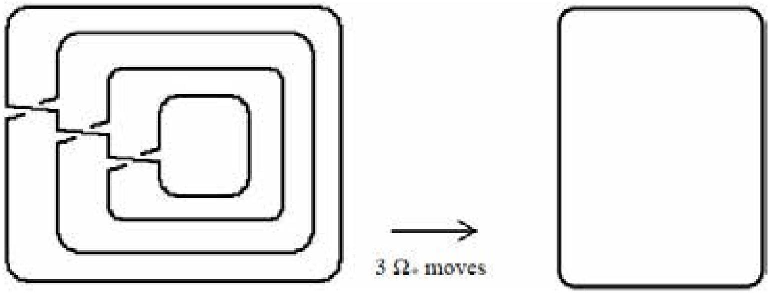
A similar operation on the knotted tst gives:

Hence, the twist turn of the knotted tst decreases by 1. □

For this reason, we denote the move that adds one twist turn to the knotted tst by Ω_+ and the other by Ω_-. In the light of this observation, we search for similar situation for other tst links. For the link $[\Phi(1, d(\tau), 2)]$, where $d(\tau)$ is even, we found that using some number of Ω_+ moves can "unknot" it to give a circle, i.e. a knot with no crossings, as shown below:

$d(\tau) = 2$:


$d(\tau) = 4$:


$d(\tau) = 6$:


This situation is the same for any link $[\Phi(g(\tau), g(\tau)D(\tau), 2)]$ for any $g(\tau) \in \mathbf{N}$, since for any such $g(\tau)$, the standard braid word for the link $[\Phi(g(\tau), g(\tau)D(\tau), 2)]$ is determined by $\mu = 1$, $P = 1, N(\tau) = 1$ and the same $D(\tau)$ and given by:
$(\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\dots\sigma_{D(\tau)-1}{}^{-1})^1 = \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\dots\sigma_{D(\tau)-1}{}^{-1} \underset{M}{\sim} \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\dots\sigma_{D(\tau)-2}{}^{-1}$
$\underset{M}{\sim} \dots \underset{M}{\sim} \sigma_1^{-1} \underset{M}{\sim} e_1$, where e_1 is the identity braid with only one string.
Since the closure of e_1 is a trivial knot, the links $[\Phi(g(\tau), g(\tau)D(\tau), 2)]$ are equivalent to the trivial knot. Here each cancelling of the σ_i^{-1}'s requires one Ω_+ move. As each such move produces one additional positive twist turn in the knotted tst, we can hence state the following theorem:

Theorem 83. *If k is even, the additional twist turn produced by eliminating all the crossings of the closure of the standard braid word of the link $[\Phi(g(\tau), g(\tau)k, 2)]$ is $k-1$.*

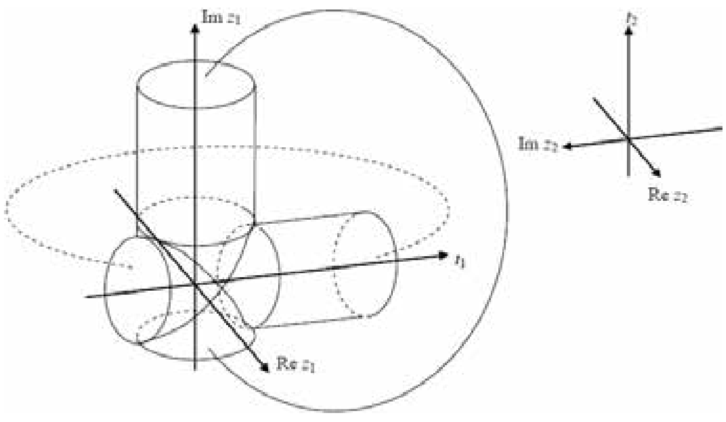
Proof. See above derivation. □

Similarly, $(k-1)\Omega_-$ moves are used to eliminate the crossings in the link $[\Phi(-g(\tau), g(\tau)k, 2)]$ to give an unknotted circle, so the resultant twist turn produced on the knotted tst is $-(k-1)$.

Hence the result of the later version of W. Ball's essay is correct only when $k=1$, since for larger positive odd k's, the link $[\Phi(k, 2, 2)]$ is not equivalent to the trivial knot and applying type I Reidemeister moves does not give any particularly pleasant form of the link.

17. Cutting Tst Products

We are motivated by an online source [2] to think of the cutting of combinations of tsts.
We call such a combination "tst product", which is defined below:
In previous section, we define a tst in $\mathbf{C} \times [0,1)$, where $[0,1)$ is a representative class of elements in the group R/Z so that the two ends of the tst are joined.
To construct a "tst product", i.e. two tsts merged together, we first lengthen the tsts so that they now have height 3, leaving room to insert another tst. The heights are identified with the t_1 and t_2 coordinates, both of which in the interval $[-1,2)$, our chosen representative class of elements in the group $R/3Z$.
We now define a tst product in $\mathbf{R}^3$, using two coordinate systems: (z_1, t_1) and (z_2, t_2), where in 3-space, they are depicted with (Re z_1, Im z_1, t_1) = (Re z_2, t_2, $-$Im z_2)
Note that both systems are left-handed.



Now we can define a "tst product" as follows.

Definition 84. $<\tau_1><\tau_2>= \{(\gamma_1(u), 3u-1), (\gamma_2(u), 3u-1)$*:*

$\gamma_k(u) = \begin{cases} z_{k,0} & 0 \le u < \frac{2}{3} \\ z_{k,0} \exp\left[2\pi i \theta_k (3u-2)\right] & \frac{2}{3} \le u < 1 \end{cases}$ *for* $u \in [0,1)$ *and* $|z_{k,0}| \le 1$ *for* $k = 1, 2\}$*,*
where $\theta_k(t) : [0,1) \to \boldsymbol{R}$ *is a function which has a continuous derivative with* $\theta_k(0) = 0$ *and* $\theta_k(1-) = \tau_k$ *for* $k = 1, 2$*, and* $(Re\, z_1, Im\, z_1, t_1) = (Re\, z_2, t_2, -Im\, z_2)$*, is called a* *twisted solid torus product (tst product)* *or* $<\tau_1>$ *multiplied by* $<\tau_2>$*.*

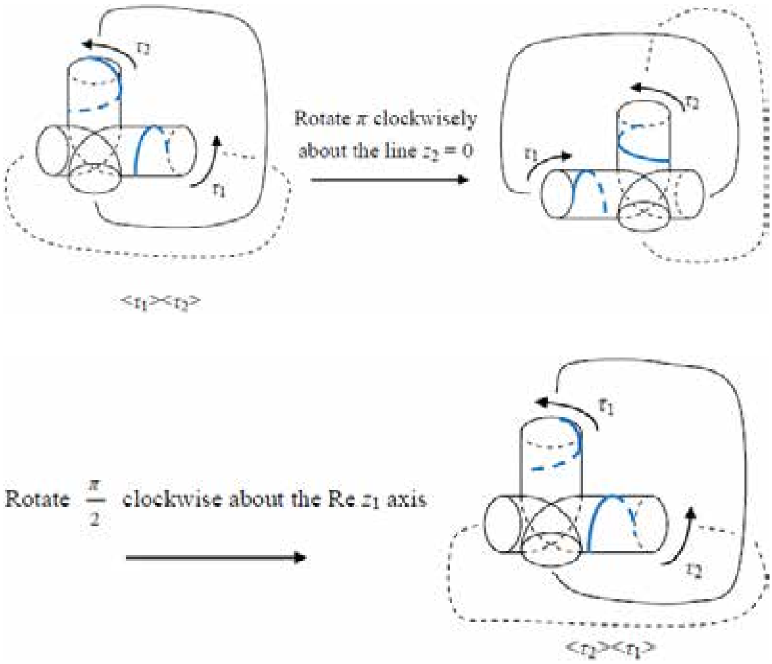
In this case, we call $<\tau_1>$ tst1 and $<\tau_2>$ tst 2. The two ends of tst 1 are joined above the t_1-t_2 plane while those of tst 2 are joined below.

Definition 85. *Two tst products* $<\tau_1><\tau_2>$ *and* $<\tau_3><\tau_4>$ *are* *equal* $\Leftrightarrow (\tau_1, \tau_2) = (\tau_3, \tau_4)$ *or* $<\tau_1><\tau_2>$ *can be obtained from a series of orientation-preserving isometries on* $<\tau_3><\tau_4>$*, We denote equality by the usual "=" sign.*

We next discover the fact that the order of such tst multiplication does not matter, as the same object is given when the order is exchanged.

Theorem 86. $<\tau_1><\tau_2>=<\tau_2><\tau_1>$*.*

Proof. We will show that $<\tau_2><\tau_1>$ can be obtained by rotating $<\tau_1><\tau_2>$ (which is obviously orientation-preserving).



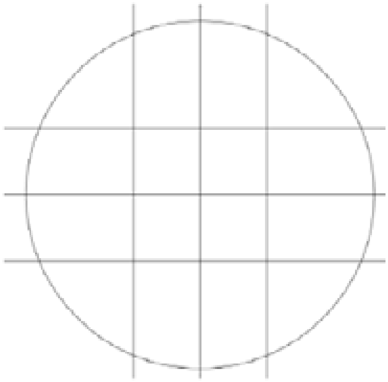
The stated result thus follows. □

Next we cut the tst product. We would restrict ourselves to cut the tst product only into an even number of parts, i.e. M, the number of parts, must be even. We also use the notation $M = 2m$.

Like the procedure of cutting ordinary tst, we first define where we insert the blades. Note, however, that this insertion is different from the ordinary tsts for a more pleasant result. We insert the blades at $t_k = -1$ such that they would create the following lines on the plane $t_k = -1$:

$$L_{k,0} = \{l_{k,0,q}(s) = \exp \frac{2\pi i(q-m)}{Md(\tau)} + s \mid q = 1, 2, \ldots, M-1\}$$

Then we let the blades run through the heights, letting them rotate along the way just as what we have done for ordinary tsts. Then we would have the recursive relation $L_{k,p+1} = L_{k,p} \exp(2\pi i\tau)$. The graph of the sets of lines $L_{k,0}, L_{k,1}, L_{k,2}, \ldots,$ $L_{k,D(\tau k)-1}$ together with the unit circle would be the same as the ordinary $\Lambda(n(\tau_k), d(\tau_k), M)$ rotated clockwise by $\dfrac{\pi}{d(\tau_k)}$ radians about the center of the unit circle. For example, here is $\Lambda(1, 4, 4)$ rotated by $\dfrac{\pi}{4}$ radians about the center of the unit circle:

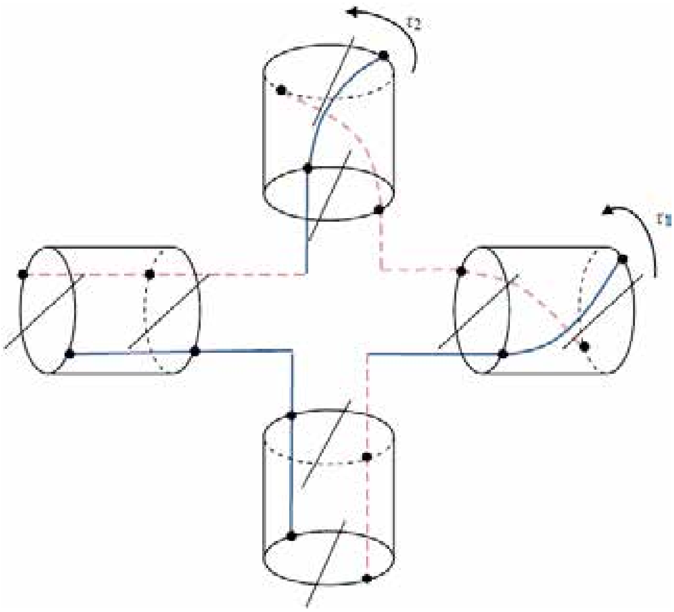


Then we obtain a general picture of the result after cutting the tst products $<\tau_1><\tau_2>$.
Now we consider the link resulting from the cutting first.
For simplicity, we shrink strips to 'polygonal arcs' (by regarding it as a knot / link and then taking its regular diagram) for simplicity, just as we have done for ordinary tst links. But first we introduce the following notation:

Notation 87. *The link resulting from cutting the tst product* $<\tau_1><\tau_2>$ *into* M *parts, with the chosen denominators* $d(\tau_1)$ *and* $d(\tau_2)$ *respectively, is denoted by* $[\Phi(n(\tau_1), d(\tau_1), M)(n(\tau_2), d(\tau_2), M)]$. *If* $d(\tau_1) = d(\tau_2)$, *we write it as* $[\Phi((n(\tau_1), n(\tau_2)), d(\tau_1), M)]$.

17.1. $d(\tau) = 2$

We would start with the simplest case, where $M = 2$, and $d(\tau_1) = d(\tau_2) = 2$ and then deduce the general form for $[\Phi((a, b), 2, 2)]$ by employing the same algorithmic for ordinary tst links. This is illustrated below with the example of $[\Phi((1, 1), 2, 2)]$:



It results in the link with the general form shown below:

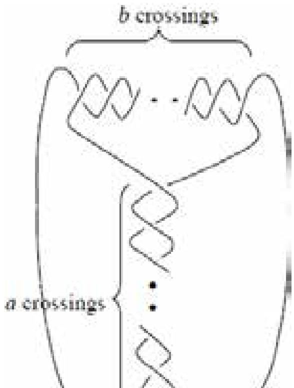
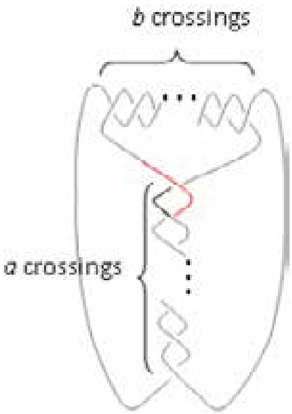


ILLUSTRATION 38. General form of $[\Phi((a, b), 2, 2)]$

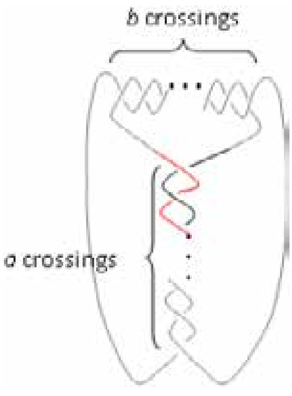
Here we have shortened "negative crossings" to "crossings" and used the convention that a crossings to mean $|a|$ positive crossings if a is a negative integer.

Lemma 88. *The number of components of* $[\Phi((a,b),2,2)]$ *is* $\begin{cases} 1 & \textit{if both } a \textit{ and } b \textit{ are odd} \\ 2 & \textit{otherwise} \end{cases}$

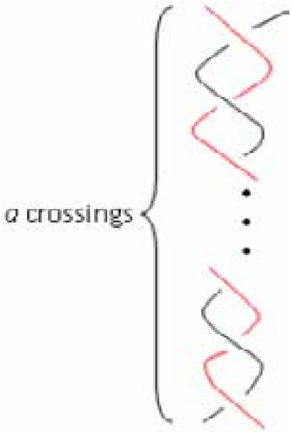
Proof. <u>Case I</u> Both a and b are odd.
To compute the number of components, we fill the whole link with the smallest number of different colors, each for one component. We start with the portion with b crossings, red and black on the left and right respectively.



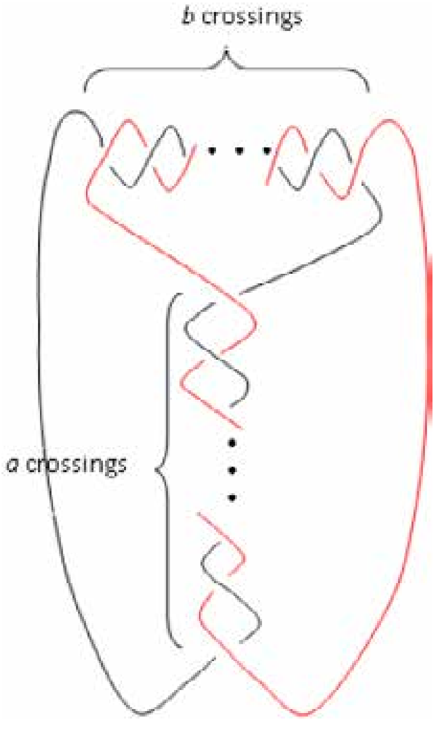
After that, we observe that when we pass through a crossing, we swap the colors on the left and right:



Since a is odd, as each crossing swaps the colors on the left and right once, we can see that the coloring of the lower portion would result in:

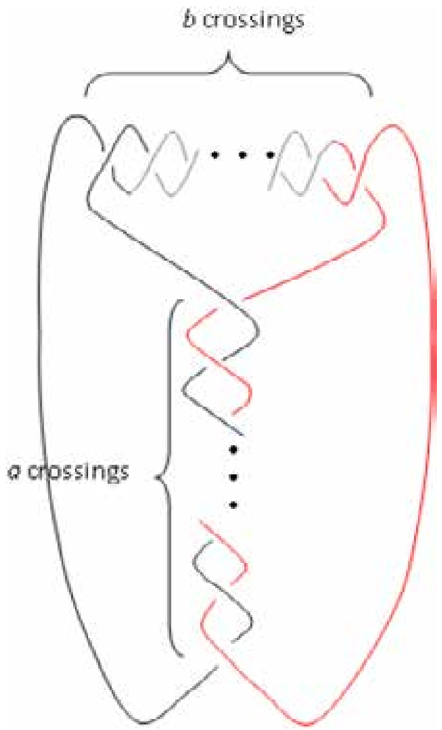


We go on and then color the portion with b crossings:
The final result becomes:

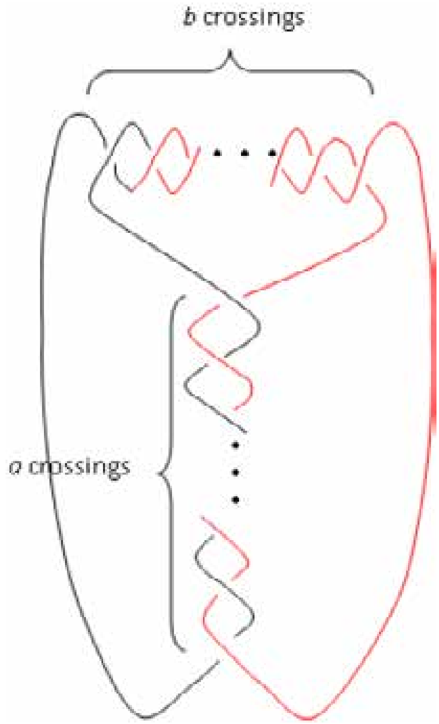


Now we have filled the whole link with two colors, so this shows that the link has two components.

<u>Case II</u> Otherwise, i.e. at least one of a and b is even.
Note that $\left\langle \frac{a}{2} \right\rangle \left\langle \frac{b}{2} \right\rangle = \left\langle \frac{b}{2} \right\rangle \left\langle \frac{a}{2} \right\rangle$, so we exchanges a and b if necessary and assume a is even.
In a similar manner we try to fill in the link with two colors. We start with the portion with a crossings.
This results in the following:



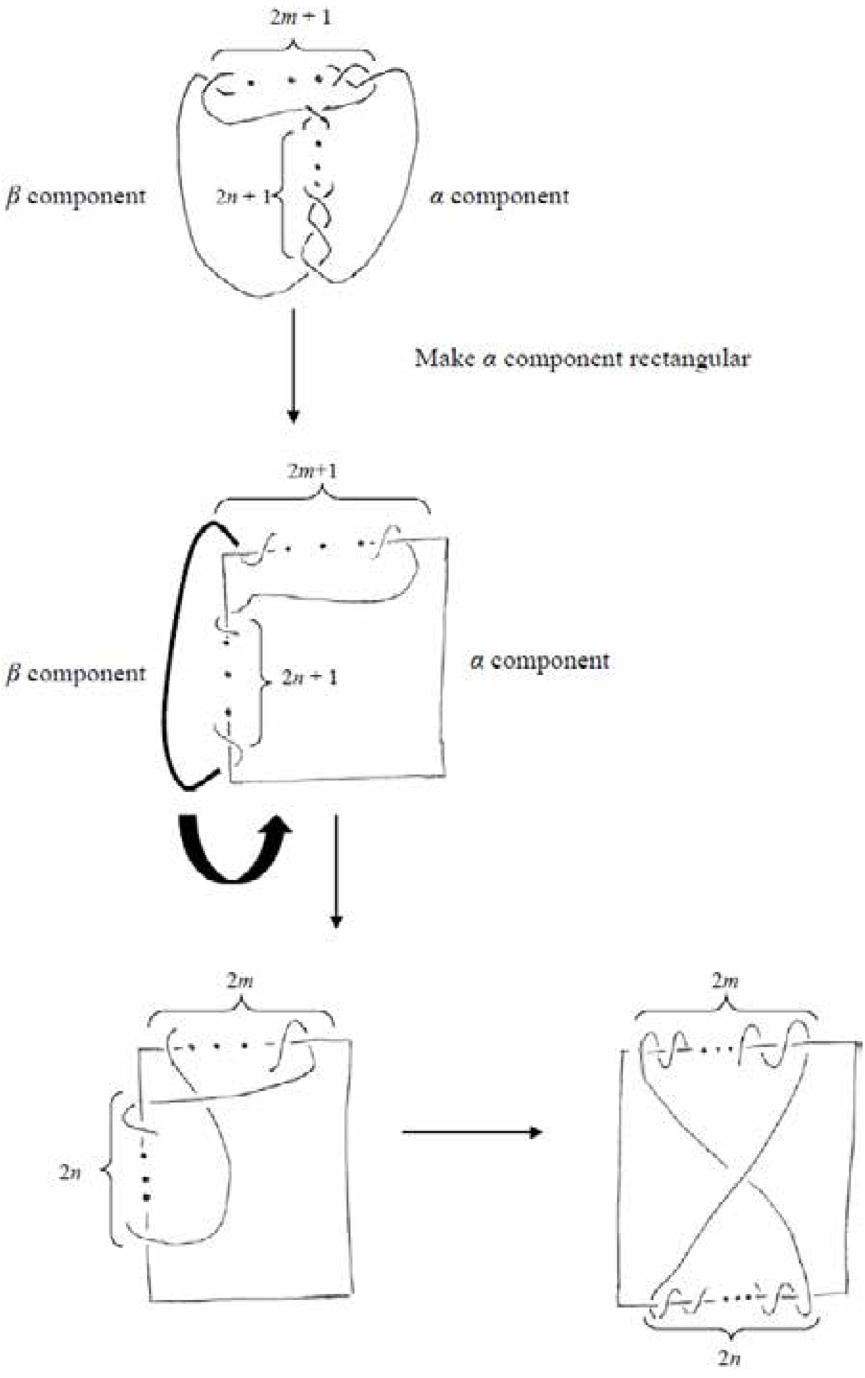
For any integer b, odd or even, we would obtain:



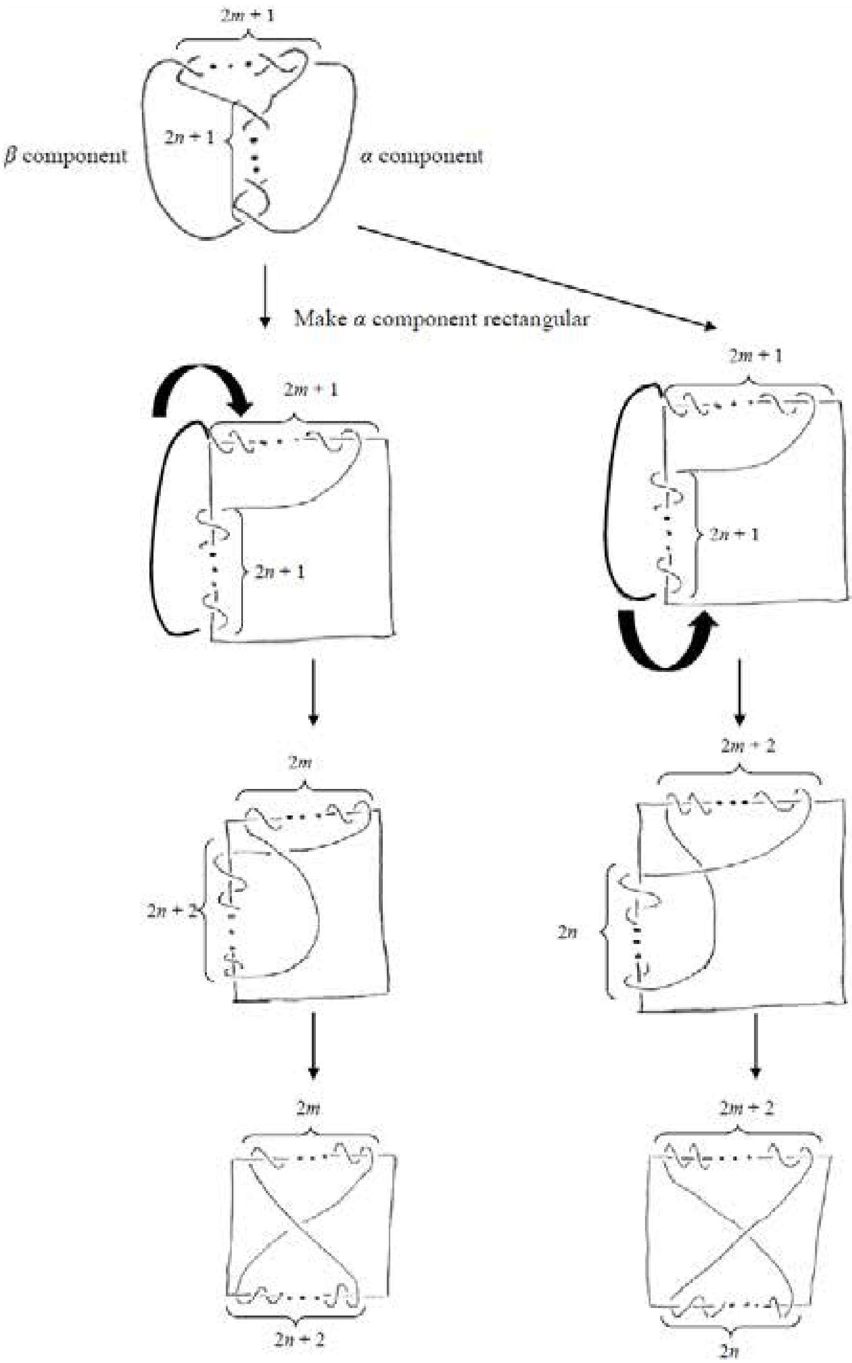
As the component filled in red is in fact the same as that in black, we can see that this link has in fact only one component. □

Afterwards, we go on and consider some other alternative forms of $[\Phi((a, b), 2, 2)]$. In the following, we have assumed that both m and n are positive integers. Besides, if two components are formed, we call them α and β components respectively. The matching is completely arbitrary, i.e. one can call any one of them α and the other β. The reader is reminded of the fact that $<\tau_1><\tau_2>=<\tau_2><\tau_1>$, which allows us to exchange τ_1 and τ_2 if necessary, so that the tst product under consideration fits into any one of the following seven cases.

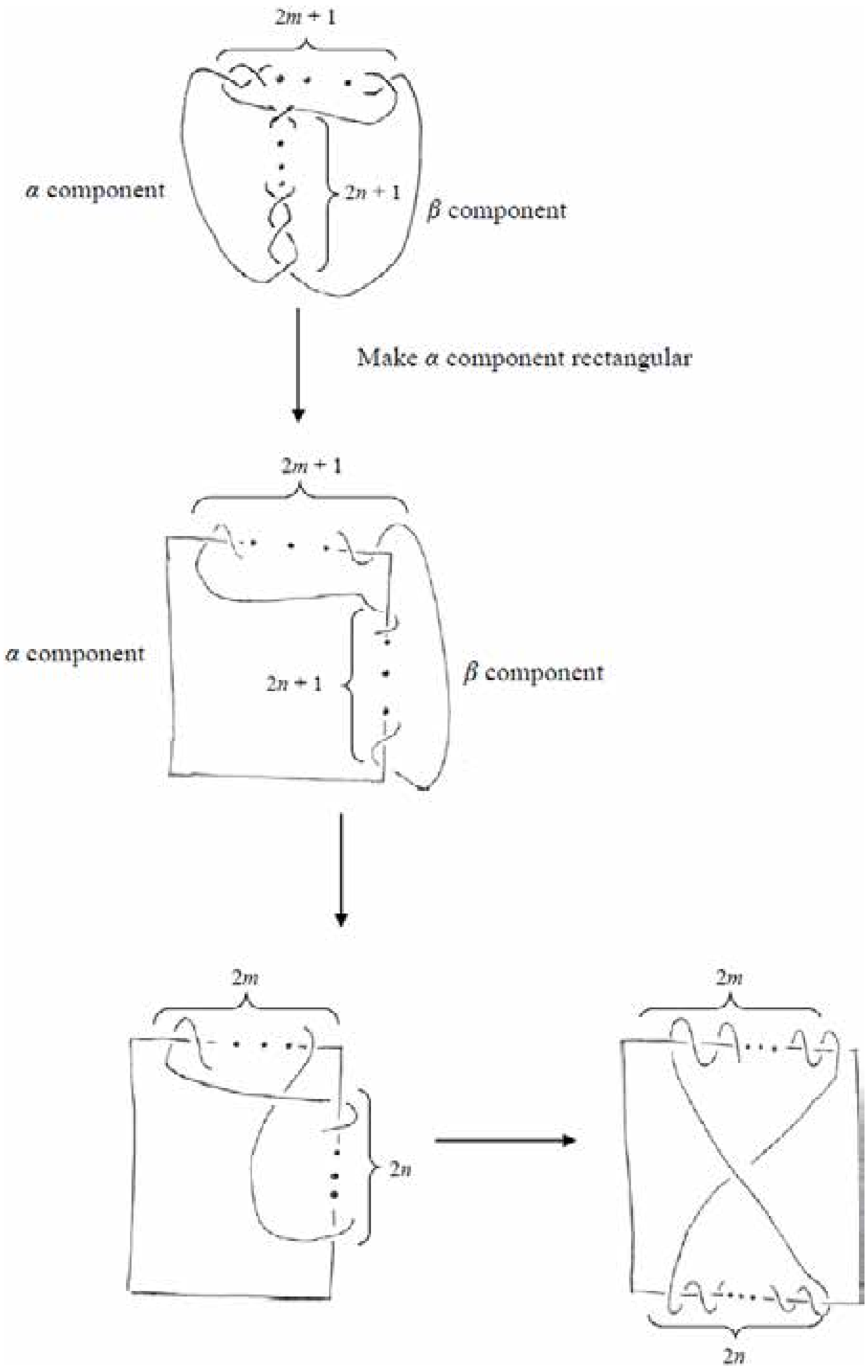
Case I $a = 2m + 1$ and $b = 2n + 1$.



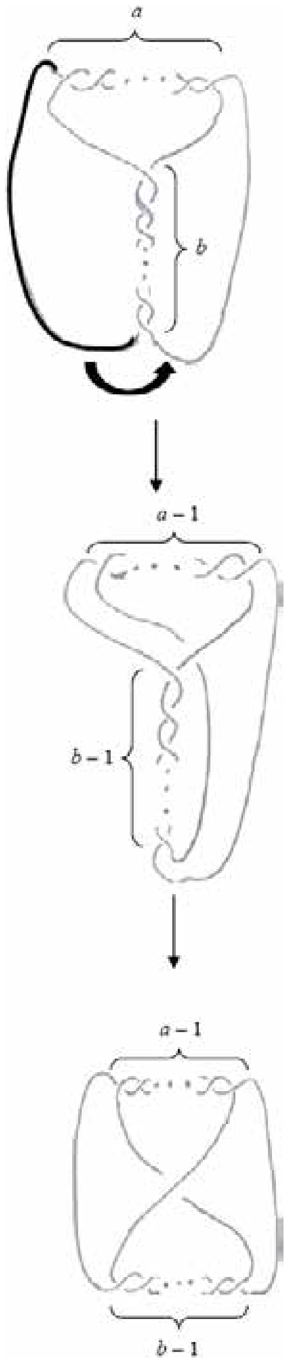
<u>Case II</u> $a = 2m + 1$ and $b = -(2n + 1)$.



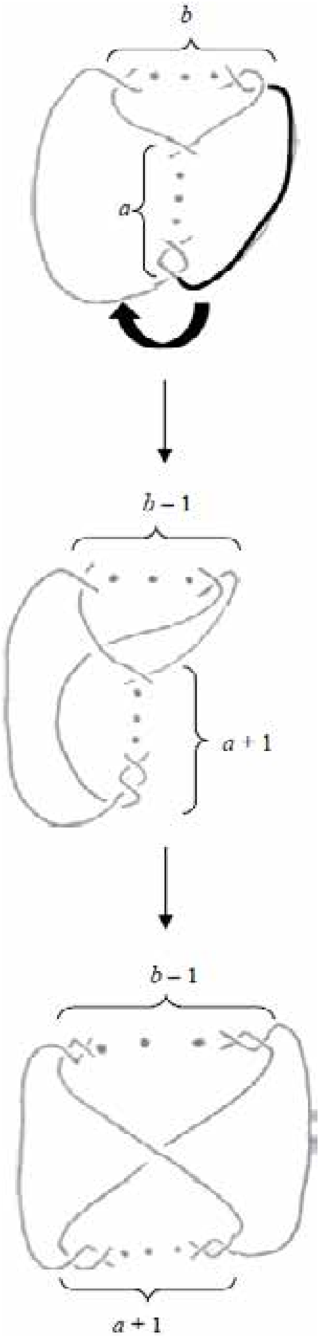
Case III $a = -(2m+1)$ and $b = -(2n+1)$



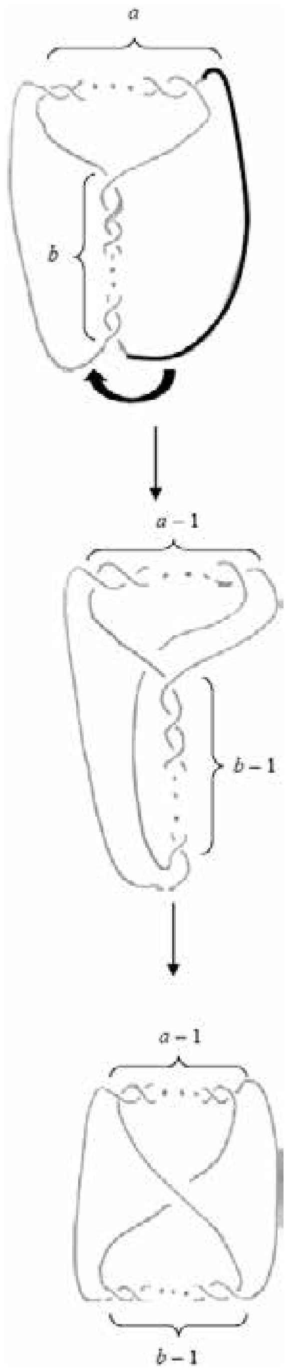
Case IV ($a = 2m$ and $b = 2n + 1$) or ($a = 2m$ and $b = 2n$)



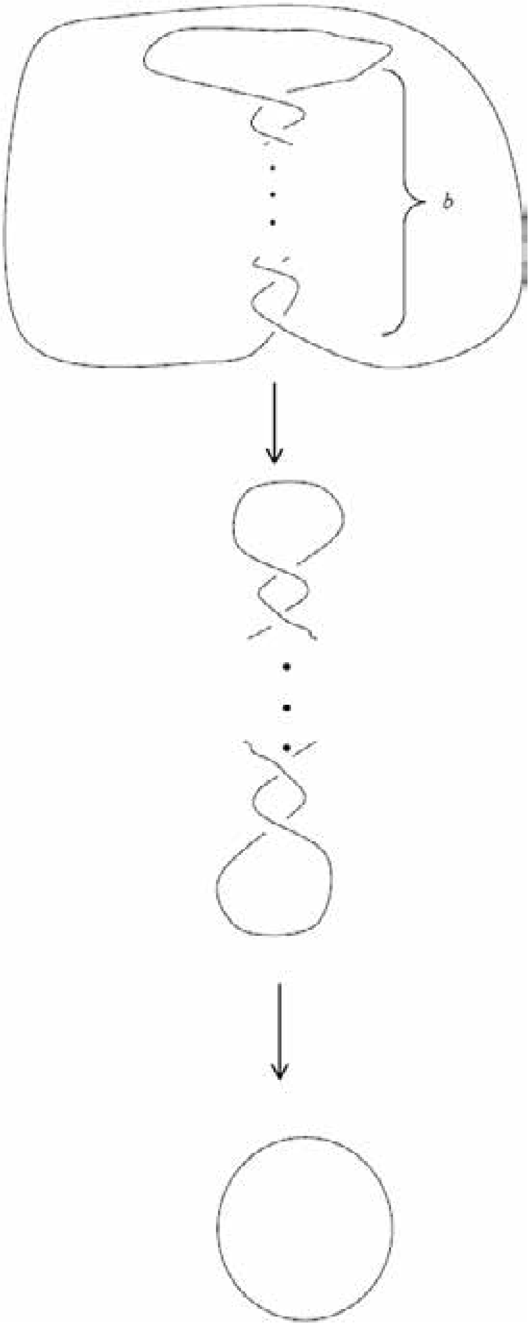
<u>Case V</u> ($a = 2m$ and $b = -(2n+1)$) or ($a = 2m$ and $b = -2n$)



<u>Case VI</u> $a = -2m$ and ($b = -(2n+1)$ or $b = -2n$)



Case VII $a = 0$



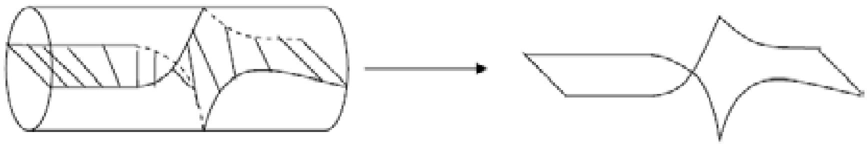
After that, we consider the twist turn of the resultant object after we cut $\left\langle \frac{a}{2} \right\rangle \left\langle \frac{b}{2} \right\rangle$.
We denote the object formed from cutting $\left\langle \frac{a}{2} \right\rangle \left\langle \frac{b}{2} \right\rangle$ by $\Phi((a,b),2,2)$.
If we define the twist turn of each component of $\Phi((a,b),2,2)$ just as we have done for knotted tsts, then the result would be as follows:

Theorem 89. *If both a and b are odd, the twist turn of each of the two components in $\Phi((a,b),2,2)$ is $\frac{a+b}{2}$. Otherwise, the one component in $\Phi((a,b),2,2)$ has a twist turn of $a+b$.*

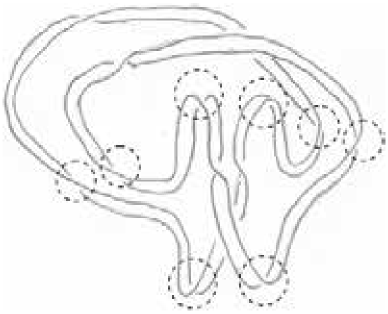
Proof. If a and b are not both odd, two components are formed. We note that each of the component comes from one-half of the original tst product before cut, and hence the twist turn of each of the component is $\frac{a}{2}+\frac{b}{2}$. Otherwise, $\Phi((a,b),2,2)$ has only one component. As each of the tst in the tst product is split into two, the twist turn of that component is double the sum of the original, i.e. $2\left(\frac{a}{2}+\frac{b}{2}\right)$. The stated result hence follows. $\square$

However, as we have done in Section 16, we can take the effect of type I Reidemeister move and other moves that preserve ambient isotopy into consideration. We note that, the object formed from cutting a tst product can be made into a Hopf link or unlink constituting two trivial knots. In doing so, we would change the twist turn of the resultant object as discussed below.

We can take, for any tst , its "strip projection", which is defined to be the set of curves $\gamma(t)$ in it, where Im $\gamma(0)=0$, as illustrated by an example of $\left\langle \frac{1}{2} \right\rangle$ below:



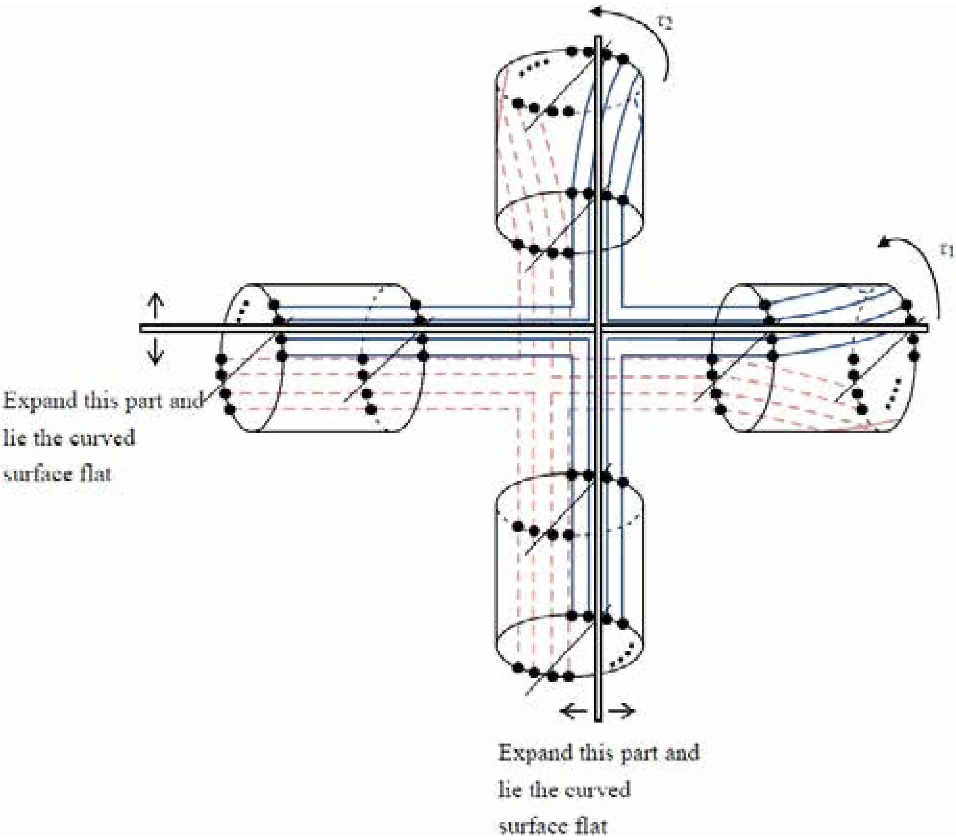
So after performing this operation, we obtain a surface (or "strip"). The "strip projection" of $\Phi((a,b),2,2)$ is two strips. Changing it to a Hopf link or unlink of two unknots using knot moves will change its twist turn, and such change can derived as follows. Consider the strip projection of $\Phi((0,0),2,2)$:



To see the twist turn of each component, we can in fact take the linking number of its two boundary curves. As such, it is easy to see that each of the components form a unknot, although, in contrary to our expectation, the twist turn of each of them is not zero. One component has a twist turn of 2 and the other has a twist turn of -2.
Since the total twist turn does not change, if a and b are not both odd, the twist turn of $\Phi((a,b),2,2)$, which has one component only, remains the same after we use knot moves to convert it into an unknotted circle. However, in the case where both a and b are odd, as two components are obtained, there is an increase in twist turn by 2 in one component and a decrease by 2 in the other.

17.2. Even $d(\tau) \neq 2$

Next, we consider the case for even $d(\tau) \neq 2$, starting with the link $[\Phi((1,1), d(\tau), 2)]$. Here we use the notation $\Delta = \dfrac{d(\tau)}{2}$. We use the method for deducing the link as in the previous sections to derive the general form of the link $[\Phi((1,1), d(\tau), 2)]$:



Eventually we would be able to derive that the general form of $[\Phi((1,1), d(\tau), 2)]$ is equivalent to:

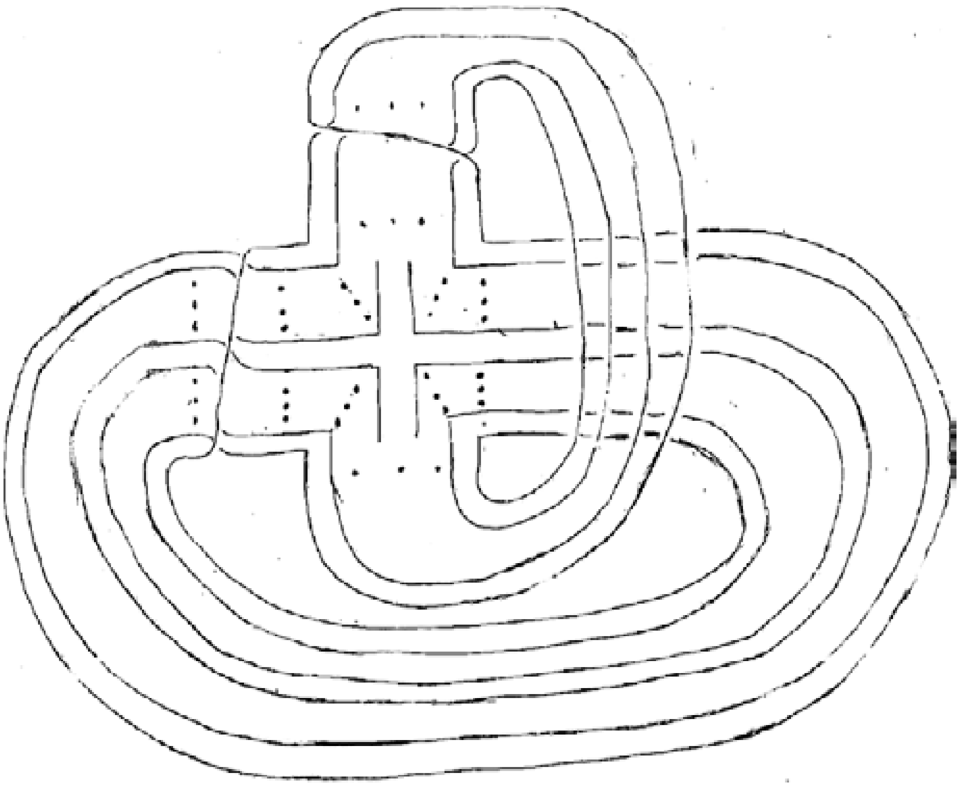
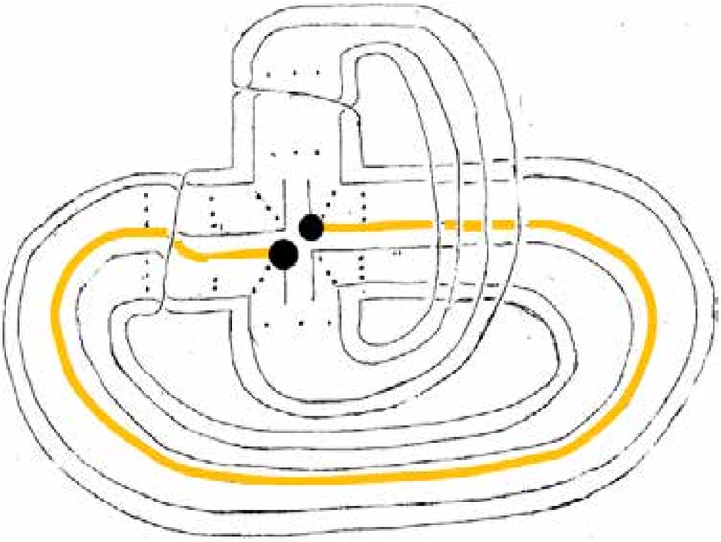


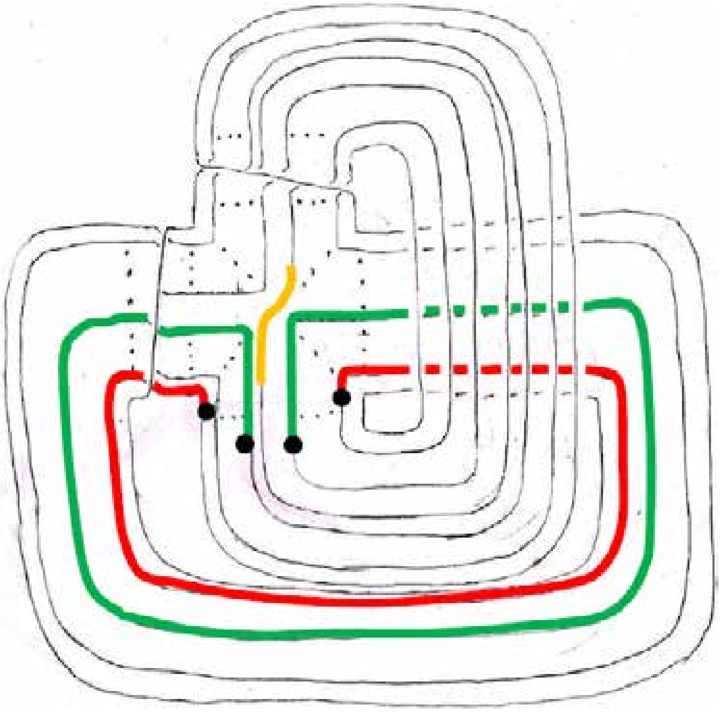
ILLUSTRATION 39. General form of $\Phi((1,1), d(\tau), 2)$.

We found that we can also use knot moves to make it into some alternative (and more pleasant) form, following the series of knot moves below:

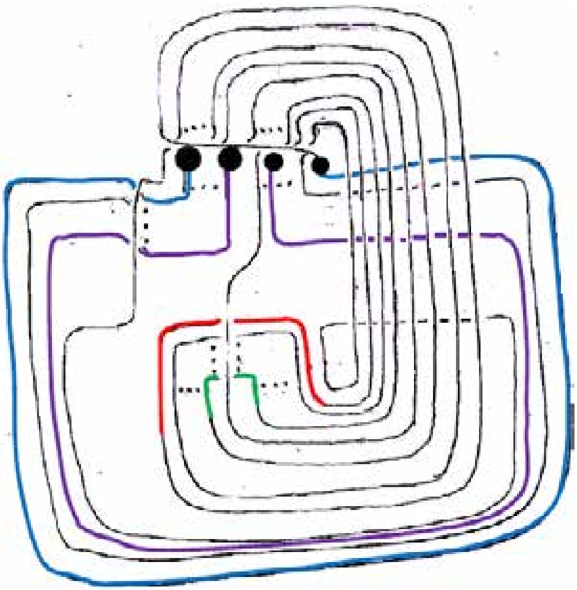
Step 1 Shorten the colored strands:



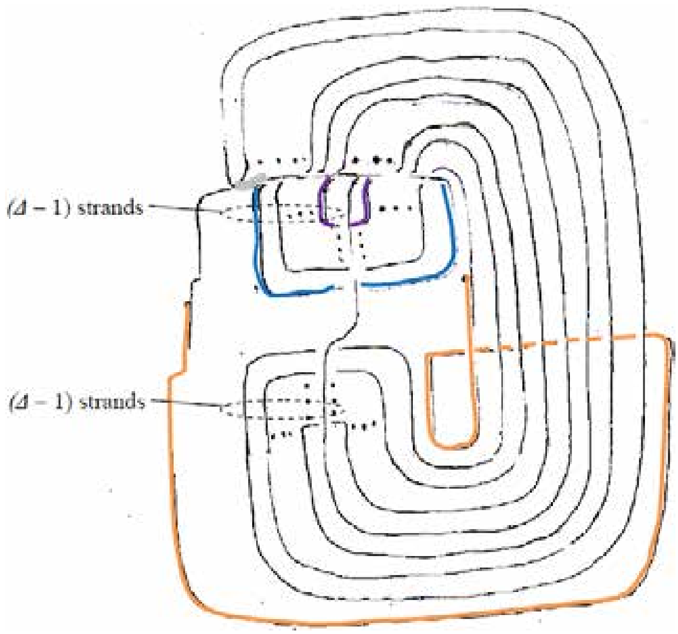
Step 2 Shorten the colored strands:



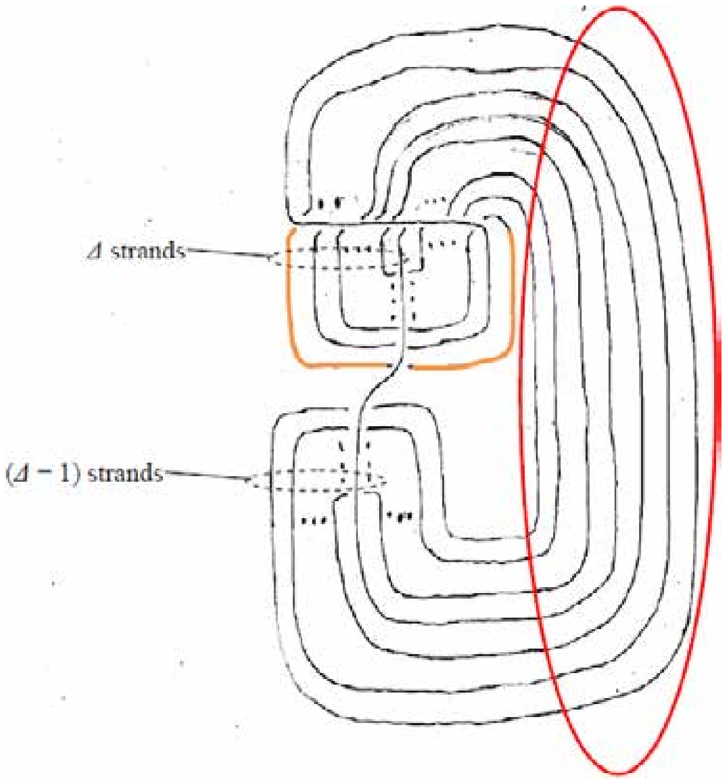
Step 3 Shorten the colored strands:



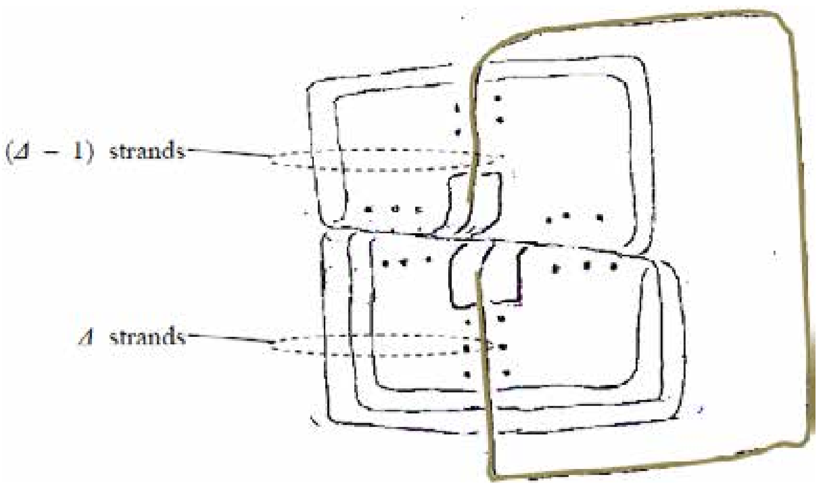
Step 4 Shorten the colored strands:



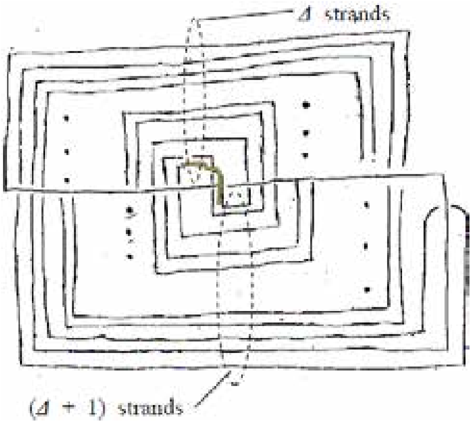
Step 5 Shorten the strands enclosed in the red ellipse:



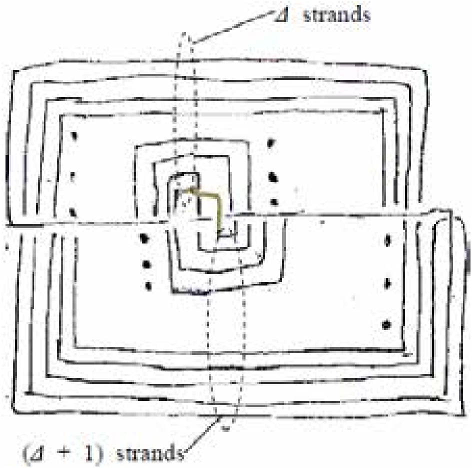
Step 6 Shorten the colored strands enclosed in the red ellipse:



We now obtain the following form of the link:



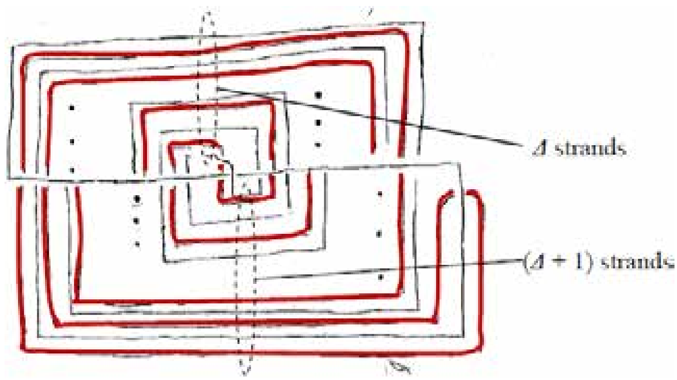
Δ is odd



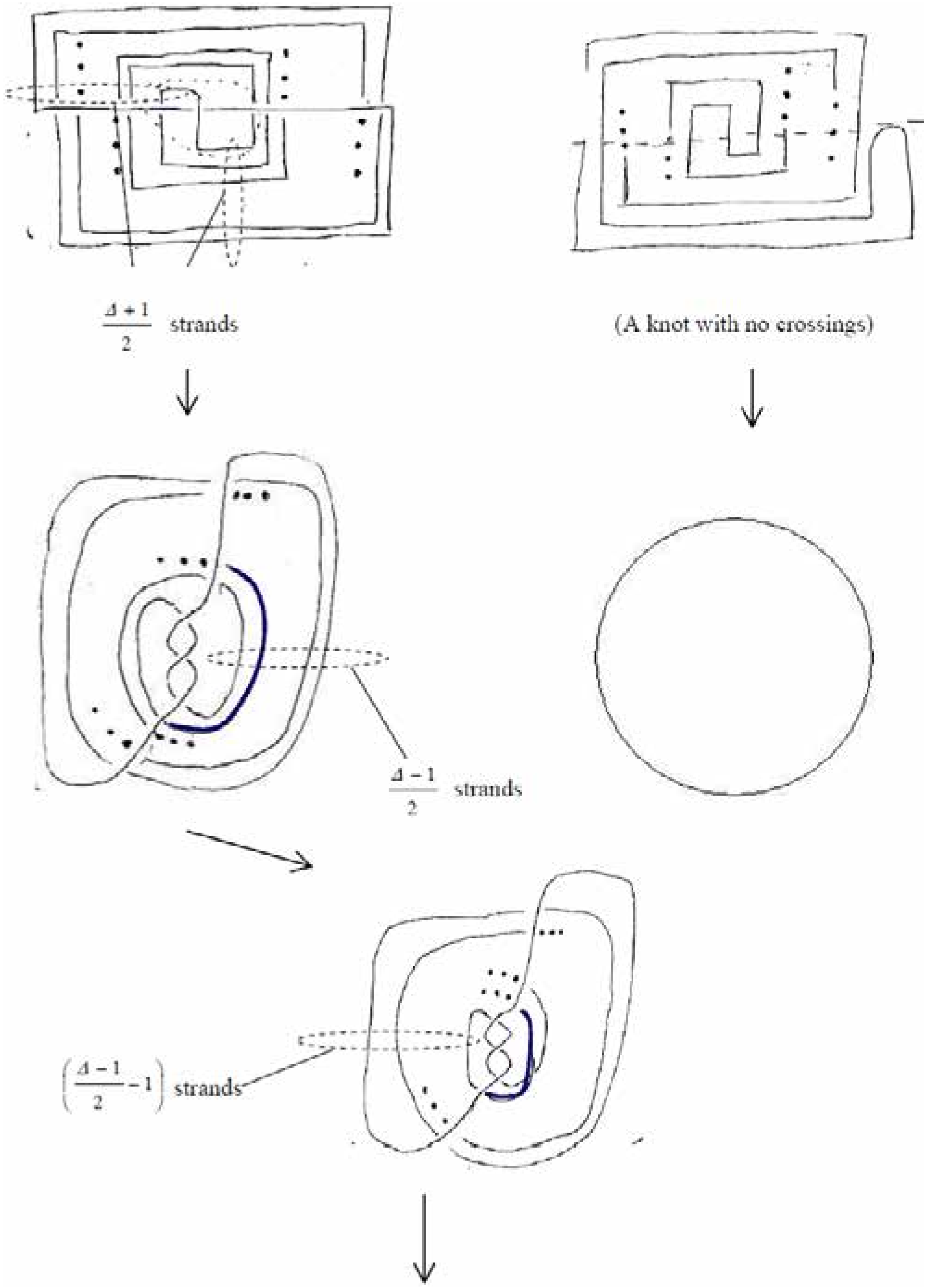
Δ is even

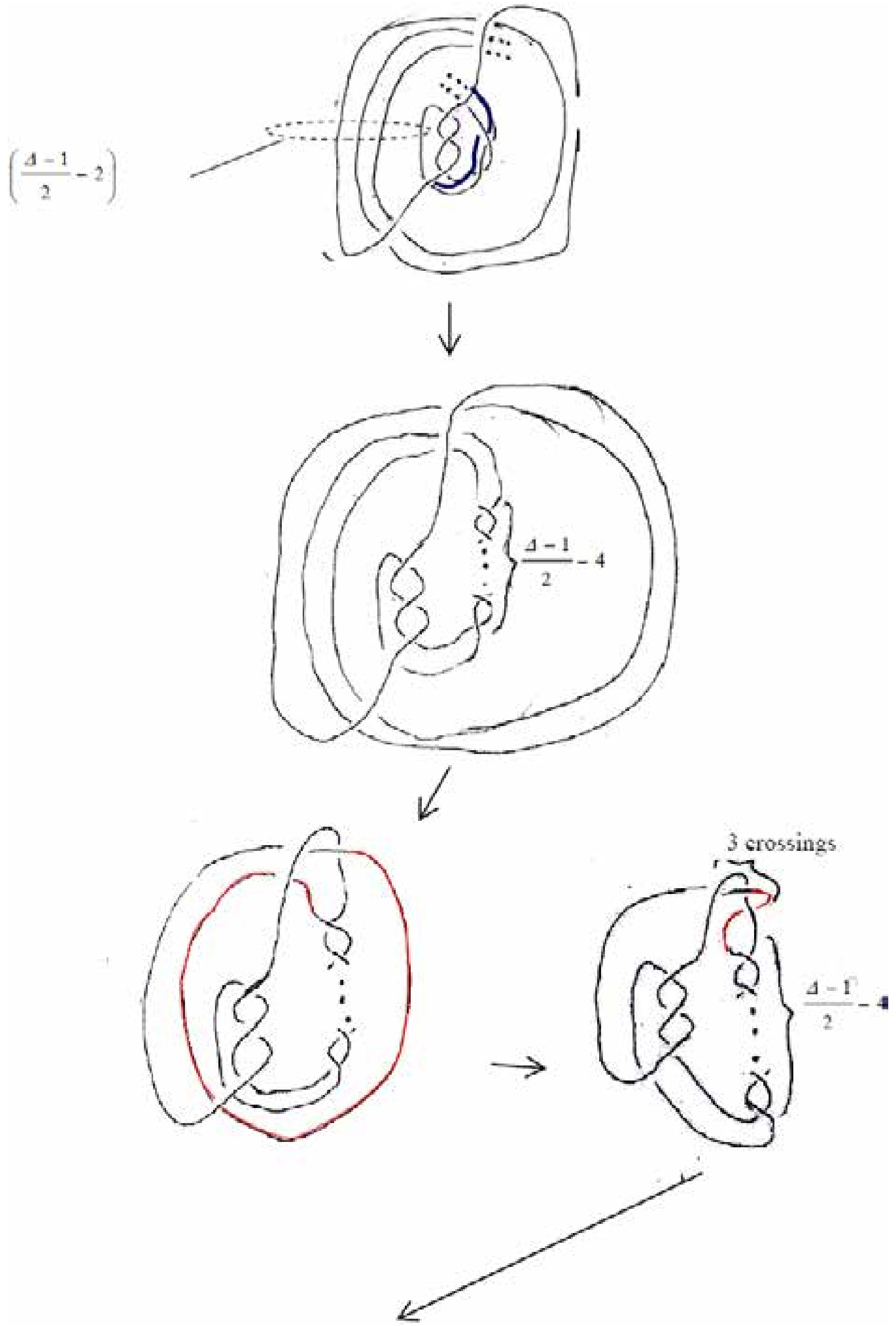
Here, we consider the two cases separately:

<u>Step 7</u> (<u>Case I</u> Δ is odd) Split the two unlinked components in the link.

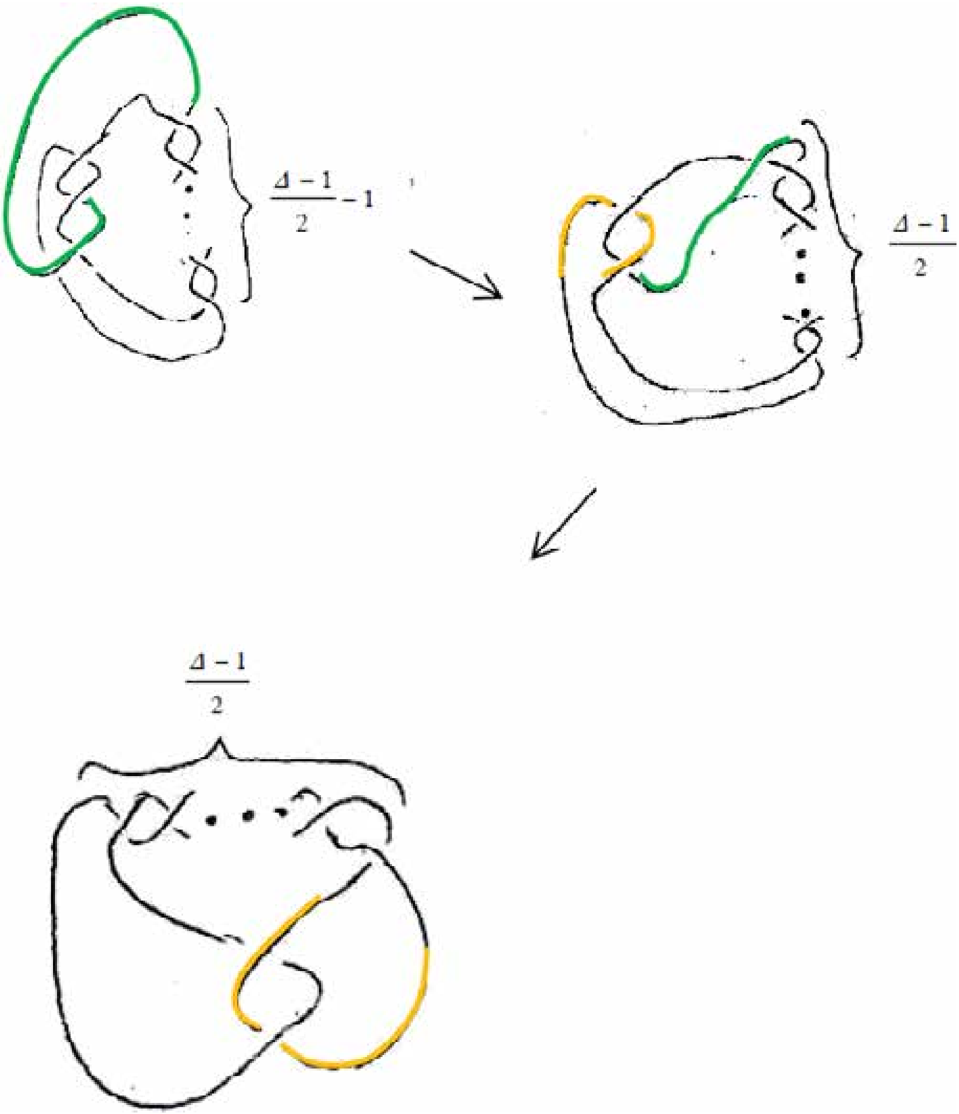


We call the crimson strands α component and the black strands β component, and by observation, we found that they can be separated using knot moves:


$\frac{\Delta+1}{2}$ strands
(A knot with no crossings)
$\frac{\Delta-1}{2}$ strands
$\left(\frac{\Delta-1}{2}-1\right)$ strands

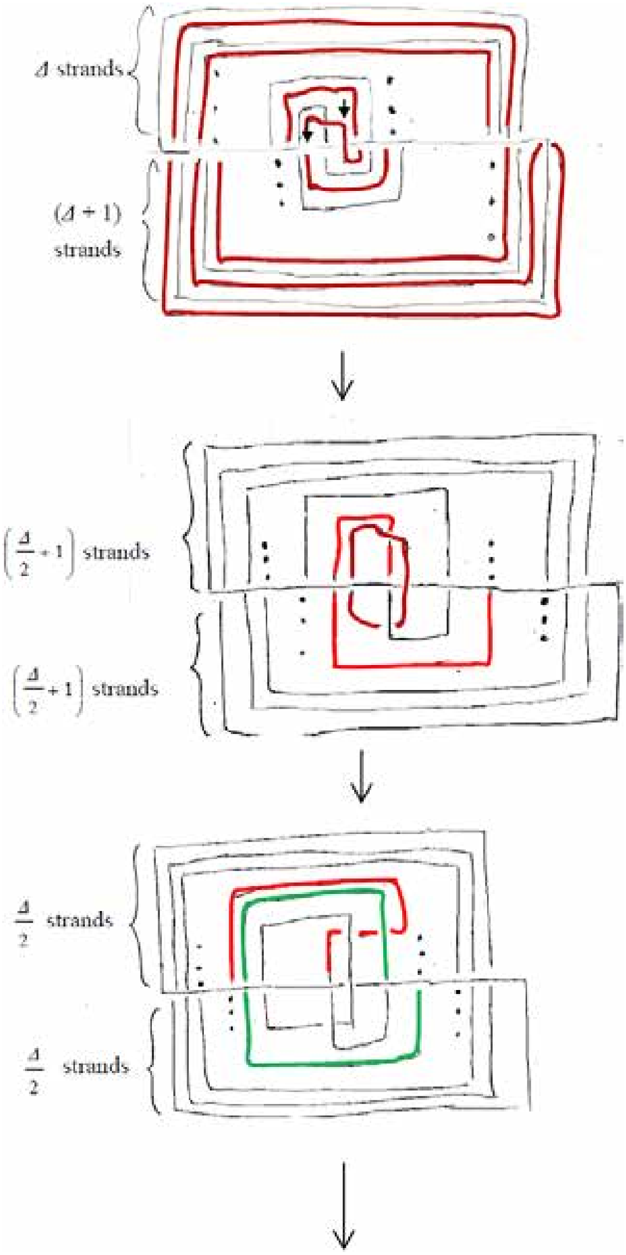


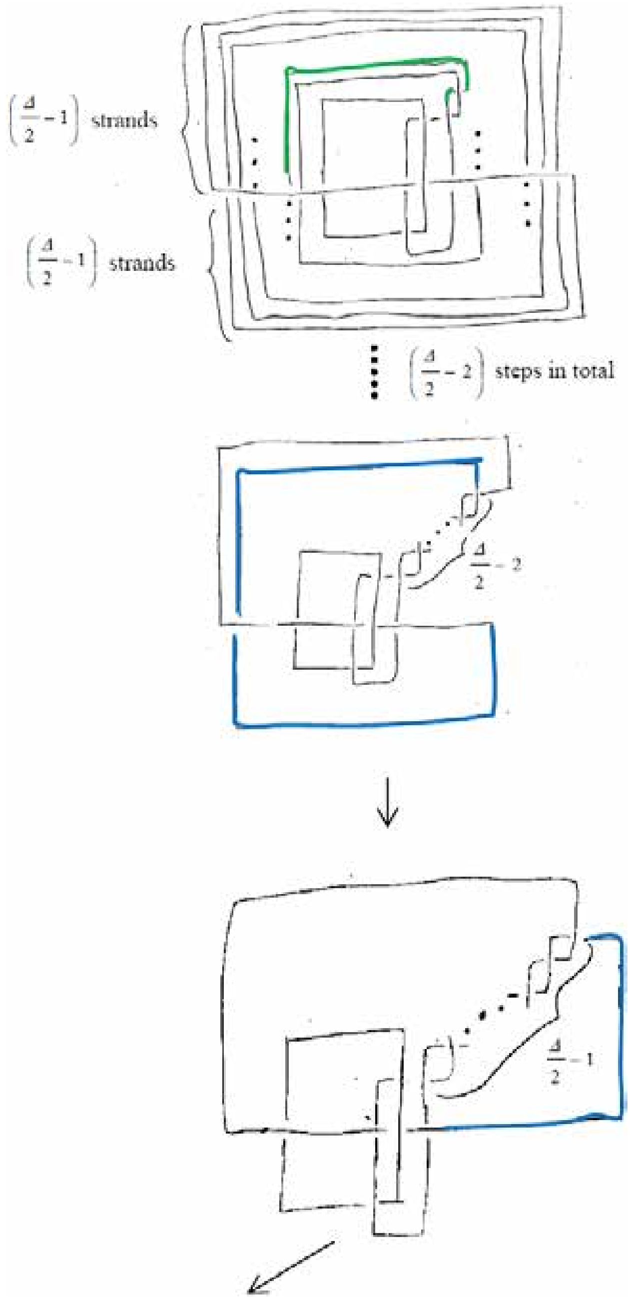
$\left(\frac{\Delta-1}{2}-2\right)$
$\frac{\Delta-1}{2}-4$
3 crossings
$\frac{\Delta-1}{2}-4$



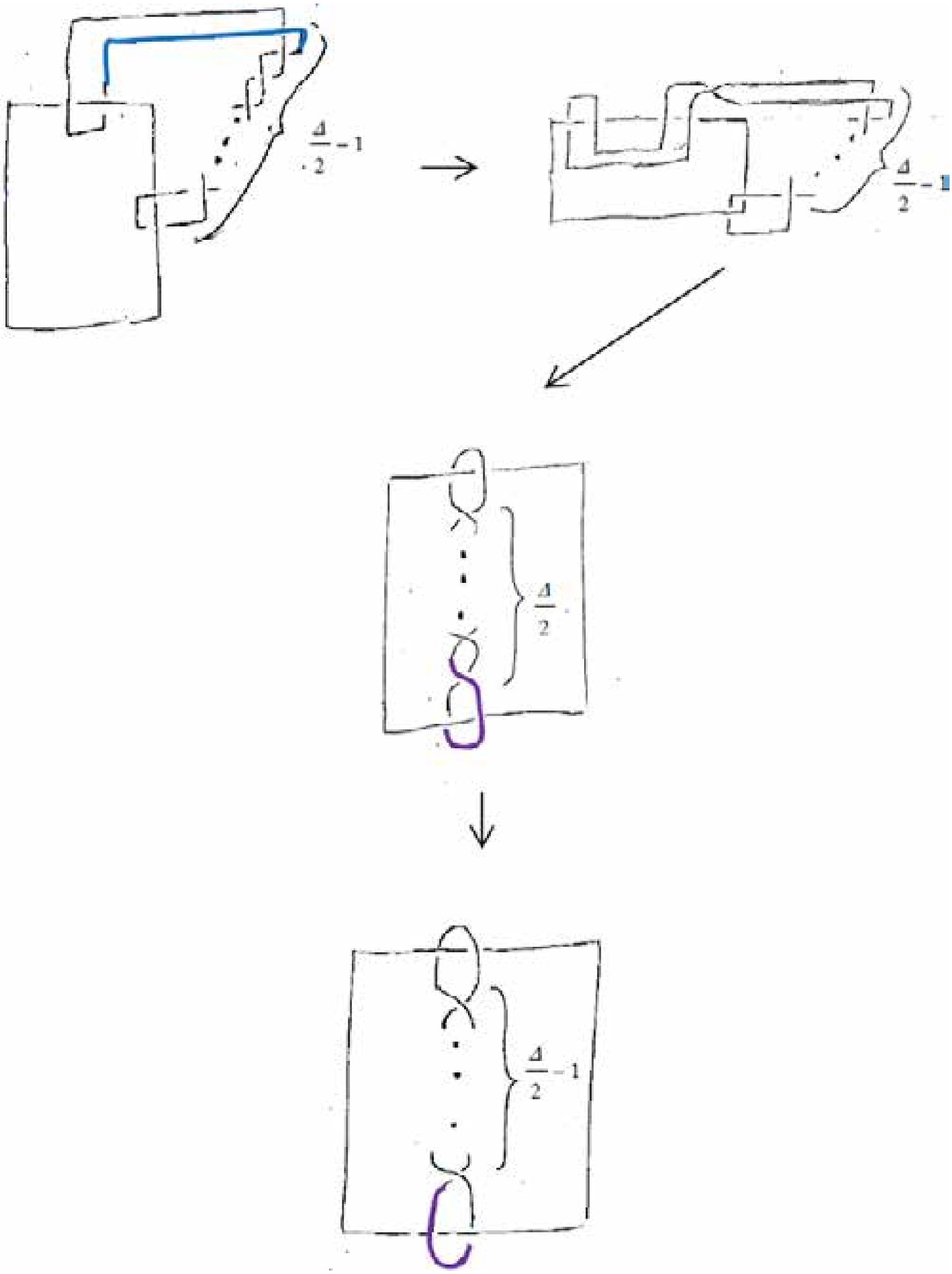
$\frac{\Delta - 1}{2} - 1$
$\frac{\Delta - 1}{2}$
$\frac{\Delta - 1}{2}$

Step 7 (Case II $\varDelta$ is even) In the following diagram, call the black strands α component and the crimson strands β component. As the part of β component between the two arrows are completely under α component, it can be shortened.





$\left(\frac{\Delta}{2}-1\right)$ strands
$\left(\frac{\Delta}{2}-1\right)$ strands
$\left(\frac{\Delta}{2}-2\right)$ steps in total
$\frac{\Delta}{2}-2$
$\frac{\Delta}{2}-1$



As seen in the above figures, we can conclude that the number of components of the link $[\Phi((1,1),d(\tau),2)]$ is 2 for even integers $d(\tau) \geq 4$.

Conclusions

In this paper, we have considered the cutting of a tst, our generalized version for strips. We have studied the object formed after we cut it, which we call a "knotted tst sum". We have considered its twist turn and the number of each type of knotted tsts in it. Then we consider the resultant link formed from tst cutting, and derive the general form of their braid word. We also consider the resultant link formed from cutting a tst in the form of a nontrivial knot. After that, we consider the effect of applying type I Reidemeister moves to the twist turn of the knotted tst. Finally, we consider the cutting of tst product, which is a combination of two tsts, for some specific cases.
In this paper, we have laid down several conjectures about the properties of tst links and their Alexander polynomials, which may help open up new research directions in knot theory. Possible further investigations also include the combination of more-than-two tsts.
Moreover, our result can in fact be generalized to higher dimensions, by simply extending 2-dimensional rotations modeled by complex numbers to 3-dimensions that can be depicted by quaternions. Research in this direction may also give new insights to the higher-dimensional generalization of knots, i.e. knotted surfaces.

Appendix A.

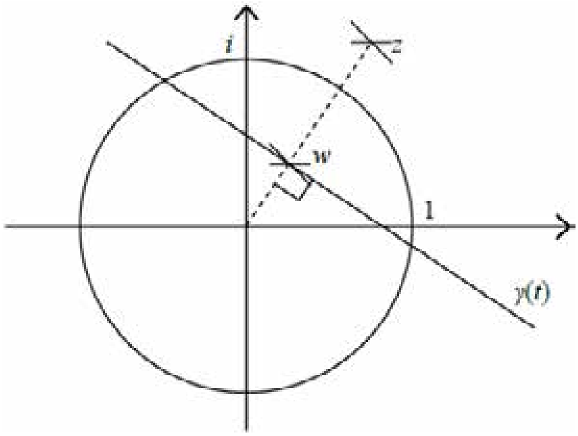
We note that the checking of collinearity using complex numbers is very easy, since the points z_1, z_2, z_3 are collinear $\Leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \bar{z_1} & \bar{z_2} & \bar{z_3} \end{vmatrix} = 0$. However, the condition that three lines are concurrent is not easy to check. Therefore, we employ the notion of pole-polar duality with respect to the unit circle from inversive geometry.

Theorem 90. *The dual of a line $\gamma(t)$ is* $\dfrac{\gamma'(t)}{[\gamma(0), \gamma'(t)]}$,
where $[a,b] = \dfrac{1}{2}(\overline{a}b - a\overline{b})$ *for any complex numbers a and b.*

Proof. The dual of a point z is a line that can be parametrized by

$$\gamma(t) = \frac{z}{|z|^2} + izt = \frac{1}{\bar{z}} + izt.$$

Let the dual point (or pole) of the line $\gamma(t)$ be z. Then we note that by definition the segment joining the origin and z is perpendicular to the line γ. We let the intersection of the segment Oz and the line $\gamma(t)$ be w such that $z = kw$ for some real $k > 0$ and $\begin{cases} iw = r\gamma'(t) \\ w = \gamma(0) + s\gamma'(t) \end{cases}$ where r and s are real numbers.



Rearrangement gives $\begin{cases} w = -ir\gamma'(t) \\ w = \gamma(0) + s\gamma'(t) \end{cases}$.

Subtraction gives $\gamma(0) + (s + ir)\gamma'(t) = 0$.

Rearrangement yields: $\begin{cases} s \operatorname{Re} \gamma'(t) - r \operatorname{Im} \gamma'(t) = -\operatorname{Re} \gamma(0) \\ s \operatorname{Im} \gamma'(t) + r \operatorname{Re} \gamma'(t) = -\operatorname{Im} \gamma(0) \end{cases}$

Solving gives: $r = \dfrac{\begin{vmatrix} \operatorname{Re} \gamma'(t) & -\operatorname{Re} \gamma(0) \\ \operatorname{Im} \gamma'(t) & -\operatorname{Im} \gamma(0) \end{vmatrix}}{\begin{vmatrix} \operatorname{Re} \gamma'(t) & -\operatorname{Im} \gamma'(t) \\ \operatorname{Im} \gamma'(t) & \operatorname{Re} \gamma'(t) \end{vmatrix}} = \dfrac{-[\gamma'(0), \gamma(0)]}{i|\gamma'(t)|^2} = \dfrac{i[\gamma'(t), \gamma(0)]}{|\gamma'(t)|^2}$

Hence $w = -ir\gamma'(t) = \dfrac{[\gamma'(t), \gamma(0)]}{|\gamma'(t)|^2}\gamma'(t) = \dfrac{[\gamma'(t), \gamma(0)]}{\overline{\gamma'(t)}}$.

By definition of pole-polar duality it follows that $|z||w| = (\text{radius of circle})^2 = 1$.

Hence, $|k||w|^2 = 1$ and $k = \dfrac{1}{|w|^2}$.

The result follows from $z = \dfrac{1}{|w|^2}w = \dfrac{1}{\overline{w}} = \dfrac{\gamma'(t)}{-[\gamma'(t), \gamma(0)]}$. □

Theorem 91. *Three lines represented by $\gamma_1(t), \gamma_2(t), \gamma_3(t)$ are concurrent*

$$\Leftrightarrow \begin{vmatrix} [\hat{\gamma}_1'(t), \gamma_1(0)] & [\hat{\gamma}_2'(t), \gamma_2(0)] & [\hat{\gamma}_3'(t), \gamma_3(0)] \\ \dfrac{\hat{\gamma}_1'(t)}{\overline{\hat{\gamma}_1'(t)}} & \dfrac{\hat{\gamma}_2'(t)}{\overline{\hat{\gamma}_2'(t)}} & \dfrac{\hat{\gamma}_3'(t)}{\overline{\hat{\gamma}_3'(t)}} \end{vmatrix} = 0.$$

where $\hat{\gamma}_1'(t)$, $\hat{\gamma}_2'(t)$, $\hat{\gamma}_3'(t)$ are unit complex numbers in the same direction as $\gamma_1'(t)$, $\gamma_2'(t)$ and $\gamma_3'(t)$ respectively.

Proof. Three lines $\gamma_1, \gamma_2, \gamma_3$ are concurrent

$\Leftrightarrow$ Duals of $\gamma_1, \gamma_2, \gamma_3$ are collinear

$\Leftrightarrow \dfrac{\gamma_1'(t)}{[\gamma_1(0), \gamma_1'(t)]}, \dfrac{\gamma_2'(t)}{[\gamma_2(0), \gamma_2'(t)]}, \dfrac{\gamma_3'(t)}{[\gamma_3(0), \gamma_3'(t)]}$ are collinear

$$\Leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ \frac{\gamma_1'(t)}{-[\gamma_1'(t),\gamma_1(0)]} & \frac{\gamma_2'(t)}{-[\gamma_2'(t),\gamma_2(0)]} & \frac{\gamma_3'(t)}{-[\gamma_3'(t),\gamma_3(0)]} \\ \frac{\overline{\gamma_1'(t)}}{-[\gamma_1'(t),\gamma_1(0)]} & \frac{\overline{\gamma_2'(t)}}{[\gamma_2'(t),\gamma_2(0)]} & \frac{\overline{\gamma_3'(t)}}{[\gamma_3'(t),\gamma_3(0)]} \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} [\gamma_1'(t),\gamma_1(0)] & [\gamma_2'(t),\gamma_2(0)] & [\gamma_3'(t),\gamma_3(0)] \\ \gamma_1'(t) & \gamma_2'(t) & \gamma_3'(t) \\ \overline{\gamma_1'(t)} & \overline{\gamma_2'(t)} & \overline{\gamma_3'(t)} \end{vmatrix} = 0 \text{ if all of } [\gamma_k'(t),\gamma_k(0)] \text{ are nonzero.}$$

$$\Leftrightarrow |\gamma_1'(t)||\gamma_2'(t)||\gamma_3'(t)| \begin{vmatrix} [\hat\gamma_1'(t),\gamma_1(0)] & [\hat\gamma_2'(t),\gamma_2(0)] & [\hat\gamma_3'(t),\gamma_3(0)] \\ \hat\gamma_1'(t) & \hat\gamma_2'(t) & \hat\gamma_3'(t) \\ \overline{\hat\gamma_1'(t)} & \overline{\hat\gamma_2'(t)} & \overline{\hat\gamma_3'(t)} \end{vmatrix} = 0.$$

If one of $[\gamma_k'(t),\gamma_k(0)]$ is zero, then we could have worked in a complex projective line and the above condition still holds. Note also that $\gamma_k'(t)$ must be nonzero, or else $\gamma_k(t)$ is not a line but a point. This verifies the validity of working in the projective complex line. The stated result thus follows. □

Appendix B.

We first define $m\mathbf{Z} = \{mx \mid x \in \mathbf{Z}\}$ for $m \in \mathbf{R}^+$.
For a reasonable configuration, the values of $d(\tau), p_i$ and q_i for $i = 1,2,3$ should satisfy the following conditions:

- $d(\tau) \geq 3$ if $d(\tau)$ is odd and $d(\tau) \geq 6$ if $d(\tau)$ is even so that three non-parallel lines can be found. Therefore, $\Delta = \dfrac{d(\tau)}{2} = \dfrac{3}{2}, \dfrac{5}{2}, 3, \dfrac{7}{2}, 4, \dfrac{9}{2}, \ldots$
- If $d(\tau)$ is odd, $p \in \frac{1}{2}\mathbf{Z}$ and $0 \leq p \leq \Delta - \frac{1}{2} < \Delta$
  If $d(\tau)$ is even, $p \in \mathbf{Z}$ and $0 \leq p \leq \Delta - 1 < \Delta$
- $q \in \mathbf{Z}$ and $1 \leq q \leq M - 1 < M$

The solutions to the equation (C) in Section 7 are considered below.

Case A The "trivial" solutions.
Note that $\frac{(p_1 - p_3 + \Delta)M - (M - q_1 - q_3)}{Md(\tau)} + \frac{(p_3 - p_2)M + (q_3 - q_2)}{Md(\tau)} + \frac{(p_2 - p_1)M + (q_1 - q_2)}{Md(\tau)} = \frac{1}{2}$ implies that $\dfrac{Md(\tau) - 2M + 4q_1 - 4q_2 + 4q_3}{2Md(\tau)} = \dfrac{1}{2}$ or $M - 2q_1 + 2q_2 - 2q_3 = 0$, i.e. $q_1 + q_3 = m + q_2$.

Subcase A1

$$\begin{cases} f_1 = f_4 \\ f_2 = f_5 \\ f_3 = f_6 \end{cases} \Rightarrow \begin{pmatrix} \Delta M - M \\ -\Delta M - M \\ 0 \end{pmatrix} + \begin{pmatrix} 2M & -M & -M & 2 & -1 & 1 \\ M & M & -2M & -1 & 1 & -2 \\ M & -2M & M & -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Performing row reductions gives $\begin{pmatrix} M & 0 & -M & 0 & 1/3 & -2/3 \\ 0 & M & -M & 0 & -1/3 & -1/3 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{-Md(\tau)-m}{3} \\ \frac{-M\Delta - M}{3} \\ m \end{pmatrix}$

Further adding the first two rows gives $M(p_1 + p_2 - 2p_3) - q_3 = -M\Delta - m$, hence we have:
$q_3 = m(2p_1 + 2p_2 - 4p_3 + d(\tau) + 1) \in m\mathbf{Z}$, so since $0 < q_3 < M = 2m$, we must have $q_3 = m$
Similarly, the second row gives $M(p_2 - p_3) - \frac{1}{3}q_2 - \frac{1}{3}m = \frac{-M\Delta - M}{3}$. We therefore have:
$q_2 = m(-6p_1 + 6p_2 - d(\tau) + 1) \in m\mathbf{Z}$, and hence $q_2 = m$.
The last row gives $q_1 - m + m = q_1 = m$, so we have $q_1 = q_2 = q_3 = m$ (Config II).

<u>*Subcase A2*</u> $\left\{ \begin{array}{l} f_1 = f_6 \\ f_2 = f_4 \\ f_3 = f_5 \end{array} \right. \Rightarrow \begin{pmatrix} \Delta M - M \\ 0 \\ -\Delta M - M \end{pmatrix} + \begin{pmatrix} M & M & -2M & 1 & -1 & 2 \\ M & -2M & M & 1 & -2 & 1 \\ -2M & M & M & 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Performing row reduction gives $\begin{pmatrix} M & M & -2M & 0 & 0 & 1 \\ 0 & M & -M & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} m - M\Delta \\ \frac{M - M\Delta}{3} \\ m \end{pmatrix}$

The first row gives $q_3 = m(-2p_1 - 2p_2 + 4p_3 - d(\tau) + 1) \in m\mathbf{Z}$, so $q_3 = m$.
Similarly, the second row gives $M(p_2 - p_3) + \frac{1}{3}q_2 + \frac{m}{3} = \frac{M - M\Delta}{3}$,
i.e. $q_2 = m(-6p_2 + 6p_3 - d(\tau) + 1) \in m\mathbf{Z}$, so $q_2 = m$. The third equation gives $q_1 - m + m = q_1 = m$.
Hence, $q_1 = q_2 = q_3 = m$ (Config II).

<u>*Subcase A3*</u> $\left\{ \begin{array}{l} f_1 = f_5 \\ f_2 = f_6 \\ f_3 = f_4 \end{array} \right. \Rightarrow \begin{pmatrix} -2M \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Performing row reduction gives $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix}$

The second row gives $q_2 = q_3$ and the first row gives $q_1 + q_3 = M$. Note that we have $q_1 + q_3 = m + q_2 = M$, so $q_1 = q_2 = q_3 = m$ (Config II).

<u>*Subcase A4*</u> $\left\{ \begin{array}{l} f_1 = f_4 \\ f_2 = f_6 \\ f_3 = f_5 \end{array} \right. \Rightarrow \begin{pmatrix} \Delta M - M \\ 0 \\ -\Delta M - M \end{pmatrix} + \begin{pmatrix} 2M & -M & -M & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ -2M & M & M & 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Performing row reduction gives $\begin{pmatrix} 2M & -M & -M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} -\Delta M \\ 0 \\ m \end{pmatrix}$

The second row gives $q_2 = q_3$ and the last row gives $q_1 = m$. The first row gives $(2p_1 - p_2 - p_3) = -\Delta$. Using the assumption $p_1 = 0, p_2 + p_3 = \Delta$.
Hence $p_2 + p_3 = \Delta$ and $q_2 = q_3$ (Config I).

<u>*Subcase A5*</u> $\left\{ \begin{array}{l} f_1 = f_5 \\ f_2 = f_4 \\ f_3 = f_6 \end{array} \right. \Rightarrow \begin{pmatrix} -2M \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 2 \\ M & -2M & M & 1 & -2 & 1 \\ -M & 2M & -M & 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Performing row reduction gives $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} M \\ 0 \\ m \end{pmatrix}$

This shows that $p_1 + p_3 = 2p_2, q_2 = m$, and $q_1 + q_3 = M$ (Config I)

<u>*Subcase A6*</u> $\left\{ \begin{array}{l} f_1 = f_6 \\ f_2 = f_5 \\ f_3 = f_4 \end{array} \right. \Rightarrow \begin{pmatrix} \Delta M - M \\ -\Delta M - M \\ 0 \end{pmatrix} + \begin{pmatrix} M & M & -2M & 1 & -1 & 2 \\ -M & -M & 2M & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Performing row reduction gives $\begin{pmatrix} 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} -\Delta \\ m \\ 0 \end{pmatrix}$

The second row gives $q_3 = m$ and the last row gives $q_1 = q_2$.
The first row gives $p_1 + p_2 - 2p_3 = -\Delta$. (Confin I)

<u>Case B</u> One-parameter infinite solutions

<u>*Subcase B1.1*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{3} - 2t & \frac{1}{3} + t & t & \frac{1}{6} - t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{2}{3} - t\Big) \qquad q_1 = m + m\Delta\Big(\frac{1}{3} - 6t\Big)$$
$$p_3 = \Delta\Big(\frac{5}{6} - t\Big) \qquad q_2 = m + m\frac{\Delta}{3}$$
$$q_3 = m + m\Delta(4t)$$

$p_3 - p_2 = \dfrac{\Delta}{6} \Rightarrow \dfrac{d(\tau)}{6} = 2(p_3 - p_2) \in \mathbf{Z}^+$
$\therefore d(\tau)$ is even and $\Delta \geq 3$
But $q_2 \geq m + m = M$ (rejected)

Subcase B1.2 $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{3} - 2t & t & \frac{1}{6} - t & \frac{1}{3} + t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{1}{3} - t\Big) \qquad q_1 = m + m\Delta\Big(\frac{2}{3} - 2t\Big)$$
$$p_3 = \Delta\Big(\frac{2}{3} + t\Big) \qquad q_2 = m + m\Delta(4t)$$
$$q_3 = m + m\Delta\Big(-\frac{2}{3} + 4t\Big)$$

$2\Delta\Big(\frac{1}{3} - t\Big) = 2p_2 \in \mathbf{Z}^+$

$q_3 = m + m\Big[2\Delta\Big(\frac{1}{3} - t\Big)\Big] \geq m + m = M$ (rejected)

Subcase B1.3 $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{3} - 2t & \frac{1}{6} - t & \frac{1}{3} + t & t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{1}{2} - 3t\Big) \qquad q_1 = m + m\Delta(-2t)$$
$$p_3 = \Delta\Big(\frac{1}{2} - t\Big) \qquad q_2 = m + m\Delta\Big(-\frac{1}{3}\Big)$$
$$q_3 = m + m\Delta\Big(-\frac{1}{3}\Big)$$

$p_3 - p_2 = 2\Delta t \Rightarrow 4\Delta t = 2(p_3 - p_2) \in \mathbf{Z}^+$

$2q_1 = M - m(4\Delta t) \in m\mathbf{Z}$

$\because 0 < 2q_1 < M$ and $m \mid 2q_1$

$\therefore 2q_1 = m$ and $4\Delta t = 1$

$0 < q_2 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = 1.5$ or $\Delta = 2.5$

If $\Delta = 1.5$, then $t = \frac{1}{6}$ (rejected)

If $\Delta = 2.5$, then $t = \frac{1}{10}$, which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{4}M \\ \frac{1}{12}M \\ \frac{1}{12}M \end{pmatrix}$

$\therefore$ In $\Lambda(1, 5, 12k)$, where $k \in \mathbf{Z}^+$, $|0, 3k; 0.5, k; 1, k| = 0$

Subcase B1.4 $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{3} - 2t & \frac{1}{3} + t & \frac{1}{6} - t & t \end{pmatrix}^T$

$p_2 = \Delta\Big(\frac{2}{3} - t\Big) \qquad q_1 = m + m\Delta(-2t)$

$p_3 = \Delta\Big(\frac{2}{3} + t\Big) \qquad q_2 = m + m\Delta(4t)$

$q_3 = m + m\Delta(4t)$

$p_3 - p_2 = 2\Delta t \Rightarrow 4\Delta t = 2(p_3 - p_2) \in \mathbf{Z}^+$
$q_2 = m + m(4\Delta t) \geq m + m = M$ (rejected)

Subcase B1.5 $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{3} - 2t & t & \frac{1}{3} + t & \frac{1}{6} - t \end{pmatrix}^T$

$p_2 = \Delta\Big(\frac{1}{3} - t\Big) \qquad q_1 = m + m\Delta\Big(\frac{1}{3} - 6t\Big)$

$p_3 = \Delta\Big(\frac{1}{2} - t\Big) \qquad q_2 = m + m\Delta\Big(-\frac{1}{3}\Big)$

$q_3 = m + m\Delta\Big(-\frac{2}{3} + 4t\Big)$

$p_3 - p_2 = \dfrac{\Delta}{6} \Rightarrow \dfrac{d(\tau)}{6} = 2(p_3 - p_2) \in \mathbf{Z}^+$
$\therefore d(\tau)$ is even and $\Delta \geq 3$
However, $q_2 \leq m - m = 0$ (rejected)

Subcase B1.6 $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{3} - 2t & \frac{1}{6} - t & t & \frac{1}{3} + t \end{pmatrix}^T$

$p_2 = \Delta\Big(\frac{1}{2} - 3t\Big) \qquad q_1 = m + m\Delta\Big(\frac{2}{3} - 2t\Big)$

$p_3 = \Delta\Big(\frac{5}{6} - t\Big) \qquad q_2 = m + m\Delta\Big(\frac{1}{3}\Big)$

$q_3 = m + m\Delta\Big(-\frac{1}{3}\Big)$

$0 < q_2 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = 1.5$ or $\Delta = 2.5$
If $\Delta = 1.5$, then $p_2 = 0.5$ and $p_3 = 1 \Rightarrow t = \dfrac{1}{18}$ and $t = \dfrac{1}{6}$ (rejected)
If $\Delta = 2.5$, then
$m < q_1 < M \Rightarrow 0 < 2.5(\dfrac{2}{3} - 2t) < 1 \Rightarrow \dfrac{1}{6} < \dfrac{1}{3} < t < \dfrac{5}{6}$ (rejected)

<u>*Subcase B1.7*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} - 2t & t & \frac{1}{3} + t & t & \frac{1}{6} - t \end{pmatrix}^T$

$p_2 = \Delta(\frac{1}{3} + 2t)$ $\qquad q_1 = m + m\Delta(-\frac{1}{3})$

$p_3 = \Delta(\frac{5}{6} - t)$ $\qquad q_2 = m + m\Delta(\frac{1}{3})$

$\qquad q_3 = m + m\Delta(\frac{2}{3} - 2t)$

$0 < q_2 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = 1.5$ or $\Delta = 2.5$

If $\Delta = 1.5$, then $p_2 = 0.5$ and $p_3 = 1 \Rightarrow t = \frac{1}{3}$ and $t = \frac{1}{6}$ (rejected)

If $\Delta = 2.5$, then

$m < q_1 < M \Rightarrow 0 < 2.5(\frac{2}{3} - 2t) < 1 \Rightarrow \frac{1}{6} < \frac{1}{3} < t < \frac{5}{6}$ (rejected)

<u>*Subcase B1.8*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} - 2t & t & t & \frac{1}{6} - t & \frac{1}{3} + t \end{pmatrix}^T$ T

$p_2 = 2\Delta t$ $\qquad q_1 = m + m\Delta(4t)$

$p_3 = \mathit{\Delta}\Big(\frac{2}{3} + t\Big)$ $\qquad q_2 = m + m\mathit{\Delta}(4t)$

$\qquad q_3 = m + m\mathit{\Delta}(-2t)$

$4\mathit{\Delta}t = 2p_2 \in \mathbf{Z}^+$

$q_2 = m + m(4\mathit{\Delta}t) \geq m + m = M$ (rejected)

<u>*Subcase B1.9*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} - 2t & t & \frac{1}{6} - t & \frac{1}{3} + t & t \end{pmatrix}^T$

$p_2 = \frac{\mathit{\Delta}}{6}$ $\qquad q_1 = m + m\mathit{\Delta}\Big(-\frac{2}{3} + 4t\Big)$

$p_3 = \Delta(\frac{1}{2} - t)$ $\qquad q_2 = m + m\mathit{\Delta}\Big(-\frac{1}{3}\Big)$

$\qquad q_3 = m + m\mathit{\Delta}\Big(\frac{1}{3} - 4t\Big)$

$\frac{d(\tau)}{6} = 2p_2 \in \mathbf{Z}^+$

$\therefore d(\tau)$ is even and $\mathit{\Delta} \geq 3$

However, $q_2 \leq m - m = 0$ (rejected)

<u>*Subcase B1.10*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} - 2t & t & \frac{1}{3} + t & \frac{1}{6} - t & t \end{pmatrix}^T$

$$p_2 = \Delta(\frac{1}{3} + 2t) \qquad q_1 = m + m\Delta(-\frac{2}{3} + 4t)$$

$$p_3 = \Delta(\frac{2}{3} + t) \qquad q_2 = m + m\Delta(4t)$$

$$q_3 = m + m\Delta(\frac{2}{3} - 2t)$$

$$2\Delta\Big(\frac{1}{3} - t\Big) = 2(p_3 - p_2) \in \mathbf{Z}^+$$

$$q_3 = m + m\Big[2\Delta\Big(\frac{1}{3} - t\Big)\Big] \geq m + m = M \text{ (rejected)}$$

<u>*Subcase B1.11*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} - 2t & t & t & \frac{1}{3} + t & \frac{1}{6} - t \end{pmatrix}^T$

$$p_2 = 2\mathit{\Delta}t \qquad q_1 = m + m\mathit{\Delta}\Big(-\frac{1}{3}\Big)$$

$$p_3 = \mathit{\Delta}\Big(\frac{1}{2} - t\Big) \qquad q_2 = m + m\mathit{\Delta}\Big(-\frac{1}{3}\Big)$$

$$q_3 = m + m\mathit{\Delta}(-2t)$$

$4\mathit{\Delta}t = 2p_2 \in \mathbf{Z}^+$
$2q_3 = M - m(4\mathit{\Delta}t) \in m\mathbf{Z}$
$\because 0 < 2q_3 < M$ and $m \mid 2q_3$
$\therefore q_3 = m$ and $4\mathit{\Delta}t = 1$
$0 < q_2 < M \Rightarrow \mathit{\Delta} < 3 \Rightarrow \mathit{\Delta} = 1.5$ or $\mathit{\Delta} = 2.5$
If $\mathit{\Delta} = 1.5$, then $t = \frac{1}{6}$ (rejected)

If $\mathit{\Delta} = 2.5$, then $t = \frac{1}{10}$, which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{12}M \\ \frac{1}{12}M \\ \frac{1}{4}M \end{pmatrix}$

$\therefore$ In $\mathit{\Lambda}(1, 5, 12k)$, where $k \in \mathbf{Z}^+$, $|0, k; 0.5, k; 1, 3k| = 0$

<u>*Subcase B1.12*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} - 2t & t & \frac{1}{6} - t & t & \frac{1}{3} + t \end{pmatrix}^T$

$$p_2 = \frac{\Delta}{6} \qquad q_1 = m + m\Delta(4t)$$

$$p_3 = \Delta(\frac{2}{3} + t) \qquad q_2 = m + m\Delta(\frac{1}{3})$$

$$q_3 = m + m\Delta(\frac{1}{3} - 6t)$$

$\dfrac{d(\tau)}{6} = 2p_2 \in \mathbf{Z}^+$
$\therefore d(\tau)$ is even and $\Delta \geq 3$
However, $q_2 \geq m + m = M$ (rejected)

<u>*Subcase B2.1*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} - 3t & t & \frac{1}{6} - t & 2t & \frac{1}{6} + t \end{pmatrix}^T$

$$p_2 = \frac{\varDelta}{6} \qquad q_1 = m + m\varDelta\Big(-\frac{1}{3} + 4t\Big)$$
$$p_3 = \varDelta\Big(\frac{5}{6} - 2t\Big) \qquad q_2 = m$$
$$q_3 = m + m\varDelta\Big(\frac{2}{3} - 8t\Big)$$

$\dfrac{d(\tau)}{6} = 2p_2 \in \mathbf{Z}^+$
$\therefore d(\tau)$ is even and $\varDelta \geq 3$
$p_3 = 5p_2 - 12p_2t \in \mathbf{Z}^+ \Rightarrow 12p_2t \in \mathbf{Z}$
$q_1 = m(1 - 2p_2 + 24p_2t) \in m\mathbf{Z}$
$\because 0 < q_1 < M$
$\therefore q_1 = m$ and $1 - 2p_2 + 24p_2t = 1 \Rightarrow t = \dfrac{1}{12}$
$\therefore q_1 = q_2 = q_3 = m$ (Config II)

<u>*Subcase B2.2*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} - 3t & t & 2t & \frac{1}{6} + t & \frac{1}{6} - t \end{pmatrix}^T$

$$p_2 = 3\varDelta t \qquad q_1 = m + m\varDelta\Big(-\frac{1}{3}\Big)$$
$$p_3 = \varDelta\Big(\frac{2}{3} - t\Big) \qquad q_2 = m + m\varDelta\Big(-\frac{1}{3} + 2t\Big)$$
$$q_3 = m + m\varDelta\Big(\frac{1}{3} - 2t\Big)$$

$0 < q_1 < M \Rightarrow \varDelta < 3 \Rightarrow \varDelta = \dfrac{3}{2}$ or $\varDelta = \dfrac{5}{2}$
If $\varDelta = \dfrac{3}{2}$, then $p_2 = \dfrac{1}{2}$ and $p_3 = 1 \Rightarrow t = \dfrac{1}{9}$ and $t = 0$ (rejected)
If $\varDelta = \dfrac{5}{2}$, then $p_2 = \dfrac{15}{2}t < \dfrac{15}{2}\Big(\dfrac{1}{6}\Big) = \dfrac{5}{4}$
$\therefore p_2 = \dfrac{1}{2}$ or $p_2 = 1$

When $p_2 = \dfrac{1}{2}$, $t = \dfrac{1}{15}$ which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ \frac{1}{12}M \\ \frac{1}{4}M \\ \frac{3}{4}M \end{pmatrix}$

$\therefore$ In $\Lambda(1,5,12k)$, where $k \in \mathbf{Z}^+, |0,k;,0.5,3k;1.5,9k| = 0$
When $p_2 = 1, t = \dfrac{2}{15} \Rightarrow p_3 = \dfrac{4}{3}$ (rejected)

<u>*Subcase B2.3*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} - 3t & t & \frac{1}{6} + t & \frac{1}{6} - t & 2t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{1}{6} + 2t\Big) \qquad q_1 = m + m\Delta\Big(-\frac{2}{3} + 6t\Big)$$
$$p_3 = \Delta\Big(\frac{2}{3} + t\Big) \qquad q_2 = m + m\Delta\Big(-\frac{1}{3} + 6t\Big)$$
$$q_3 = m + m\Delta\Big(\frac{2}{3} - 4t\Big)$$

$2p_3 - p_2 = \dfrac{7\Delta}{6} \Rightarrow \dfrac{7d(\tau)}{6} = 2(2p_3 - p_2) \in \mathbf{Z}^+$
$\therefore d(\tau)$ is even $\Rightarrow \Delta \geq 3$ and $\Delta, p_1, p_2, p_3 \in \mathbf{Z}$
$4p_2 - 2p_3 = \Delta\Big(-\dfrac{2}{3} + 6t\Big) \Rightarrow \Delta\Big(-\dfrac{2}{3} + 6t\Big) = 2(2p_2 - p_3) \in \mathbf{Z}$
$q_1 = m\Big[1 + \Delta\Big(-\dfrac{2}{3} + 6t\Big)\Big] \in m\mathbf{Z}^+$
$\because 0 < q_1 < M$
$\therefore q_1 = m$ and $1 + \Delta\Big(-\dfrac{2}{3} + 6t\Big) = 1 \Rightarrow t = \dfrac{1}{9}$
$\therefore q_2 = m + m\Delta\Big(\dfrac{1}{3}\Big) \geq m + m = M$ (rejected)

<u>*Subcase B2.4*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} - 3t & t & \frac{1}{6} - t & \frac{1}{6} + t & 2t \end{pmatrix}^T$

$$p_2 = \frac{\Delta}{6} \qquad q_1 = m + m\Delta\Big(-\frac{2}{3} + 6t\Big)$$
$$p_3 = \Delta\Big(\frac{2}{3} - t\Big) \qquad q_2 = m + m\Delta\Big(-\frac{1}{3} + 2t\Big)$$
$$q_3 = m + m\Delta\Big(\frac{2}{3} - 8t\Big)$$

$\dfrac{d(\tau)}{6} = 2p_2 \in \mathbf{Z}^+$
$\therefore d(\tau)$ is even and $\Delta \geq 3$
$p_3 = 4p_2 - 6p_2 t \in \mathbf{Z}^+ \Rightarrow 6p_2 t \in \mathbf{Z}^+$
$q_2 = m(1 - 2p_2 + 12p_2 t) \in m\mathbf{Z}$
$\because 0 < q_2 < M$
$\therefore q_1 = m$ and $1 - 2p_2 + 12p_2 t = 1 \Rightarrow t = \dfrac{1}{6}$ (rejected)

<u>*Subcase B2.5*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} - 3t & t & 2t & \frac{1}{6} - t & \frac{1}{6} + t \end{pmatrix}^T$

$p_2 = 3\Delta t \qquad q_1 = m + m\Delta\Big(-\frac{1}{3} + 4t\Big)$

$p_3 = \Delta\Big(\frac{2}{3} + t\Big) \qquad q_2 = m + m\Delta\Big(-\frac{1}{3} + 6t\Big)$

$q_3 = m + m\Delta\Big(\frac{1}{3} - 2t\Big)$

Consider $0 < q_2 < M$ and $m < q_3 < M$

$-1 < \Delta\Big(-\frac{1}{3} + 6t\Big) < 1$ and $0 < \Delta\Big(\frac{1}{3} - 2t\Big) < 1$

$\frac{1}{18} - \frac{1}{6\Delta} < t < \frac{1}{18} + \frac{1}{6\Delta}$ and $\frac{1}{6} - \frac{1}{2\Delta} < t < \frac{1}{6}$

$\frac{1}{18} - \frac{1}{6\Delta} < \frac{p_2}{3\Delta} < \frac{1}{18} + \frac{1}{6\Delta}$ and $\frac{1}{6} - \frac{1}{2\Delta} < \frac{p_2}{3\Delta} < \frac{1}{6}$

$\frac{\Delta}{6} - \frac{1}{2} < p_2 < \frac{\Delta}{6} + \frac{1}{2}$ and $\frac{\Delta}{2} - \frac{3}{2} < p_2 < \frac{\Delta}{2}$

$\therefore \frac{\Delta}{2} - \frac{3}{2} < \frac{\Delta}{6} + \frac{1}{2} \Rightarrow \Delta < 6$

If $d(\tau)$ is even, then $\Delta \geq 3$ and $\Delta, p_1, p_2, p_3 \in \mathbf{Z}$

When $\Delta = 3$, $0 < p_2 < 1$ and $0 < p_2 < \frac{3}{2} \Rightarrow 0 < p_2 < 1$ (rejected)

When $\Delta = 4$, $\frac{1}{6} < p_2 < \frac{7}{6}$ and $\frac{1}{2} < p_2 < 2 \Rightarrow \frac{1}{2} < p_2 < \frac{7}{6}$

$\therefore p_2 = 1$ and $t = \frac{1}{12}$ which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ m \\ \frac{5}{3}m \\ \frac{5}{3}m \end{pmatrix}$ (Config I)

When $\Delta = 5$, $\frac{1}{3} < p_2 < \frac{4}{3}$ and $1 < p_2 < \frac{5}{2} \Rightarrow 1 < p_2 < \frac{4}{3}$ (rejected)

If $d(\tau)$ is odd, then $\Delta \geq 1.5$ and $\Delta, p_1, p_2, p_3 \in \frac{1}{2}\mathbf{Z}$

When $\Delta = \frac{3}{2}, 1 < p_2 < \frac{3}{4} \Rightarrow p_2 = \frac{1}{2} \Rightarrow t = \frac{1}{9} \Rightarrow p_3 = \frac{7}{6}$ (rejected)

When $\Delta = \frac{5}{2}, 1 < p_2 < \frac{11}{12} \Rightarrow p_2 = \frac{1}{2} \Rightarrow t = \frac{1}{15} \Rightarrow p_3 = \frac{11}{6}$ (rejected)

When $\Delta = \frac{7}{2}, \frac{1}{4} < p_2 < \frac{13}{12} \Rightarrow p_2 = \frac{1}{2}$ or $p_2 = 1$

If $p_2 = \dfrac{1}{2}$, then $t = \dfrac{1}{21}$ which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{5}{2} \\ \frac{1}{4}M \\ \frac{5}{12}M \\ \frac{11}{12}M \end{pmatrix}$

$\therefore$ In $\Lambda(1, 7, 12k)$, where $k \in \mathbf{Z}^+$, $|0, 3k; 0.5, 5k; 2.5, 11k| = 0$

If $p_2 = 1$, then $t = \dfrac{2}{21} \Rightarrow p_3 = \dfrac{8}{3}$ (rejected)

When $\Delta = \dfrac{9}{2}, \dfrac{3}{4} < p_2 < \dfrac{5}{4} \Rightarrow p_2 = 1 \Rightarrow t = \dfrac{2}{27} \Rightarrow p_3 = \dfrac{10}{3}$ (rejected)

When $\Delta = \dfrac{11}{2}, 1 < \dfrac{5}{4} < p_2 < \dfrac{17}{12} < \dfrac{3}{2}$ (rejected)

Subcase B2.6 $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} - 3t & t & \frac{1}{6} + t & 2t & \frac{1}{6} - t \end{pmatrix}^T$

$p_2 = \Delta\Big(\dfrac{1}{6} + 2t\Big)$ $\qquad q_1 = m + m\Delta\Big(-\dfrac{1}{3}\Big)$

$p_3 = \Delta\Big(\dfrac{5}{6} - 2t\Big)$ $\qquad q_2 = m$

$q_3 = m + m\Delta\Big(\dfrac{2}{3} - 4t\Big)$

$2\Delta\Big(\dfrac{2}{3} - 4t\Big) = 2(p_3 - p_2) \in \mathbf{Z}^+$

$2q_3 = m\Big[2 + 2\Delta\Big(\dfrac{2}{3} - 4t\Big)\Big] \in m\mathbf{Z}^+$

$\because M < 2q_3 < 2M$

$\therefore q_3 = \dfrac{3}{2}m$ and $2 + 2\Delta\Big(\dfrac{2}{3} - 4t\Big) = 3 \Rightarrow t = \dfrac{1}{6} - \dfrac{1}{8\Delta}$

$0 < q_3 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = \dfrac{3}{2}$ or $\Delta = \dfrac{5}{2}$

If $\Delta = \dfrac{3}{2}$, then $t = \dfrac{7}{60} \Rightarrow p_2 = -\dfrac{1}{6}$ (rejected)

If $\Delta = \dfrac{5}{2}$, then $t = \dfrac{1}{12}$, which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2}m \\ m \\ \frac{3}{2}m \end{pmatrix}$ (Config I)

<u>*Subcase B2.7*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{2} - 3t & \frac{1}{6} - t & 2t & \frac{1}{6} + t \end{pmatrix}^T$

$p_2 = \Delta\left(\frac{2}{3} - 4t\right) \qquad q_1 = m + m\Delta\left(\frac{2}{3} - 4t\right)$

$p_3 = \Delta\left(\frac{5}{6} - 2t\right) \qquad q_2 = m$

$q_3 = m + m\Delta\left(-\frac{1}{3}\right)$

$0 < q_3 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = \frac{3}{2}$ or $\Delta = \frac{5}{2}$

If $\Delta = \frac{3}{2}$, then $p_2 = 1 - 6t$

$\therefore 0 < p_2 < 1$

But $p_2 \in \frac{1}{2}\mathbf{Z}^+$, $\therefore p_2 = \frac{1}{2}$ and $t = \frac{1}{12}$

$$\therefore \begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{3}{2}m \\ m \\ \frac{1}{2}m \end{pmatrix} \text{ (Config I)}$$

If $\Delta = \frac{5}{2}$, then $p_2 = \frac{5}{3} - 10t$

$\therefore 0 < p_2 < \frac{5}{3} = 1\frac{2}{3}$

$\therefore p_2 = \frac{1}{2}$ or $p_2 = 1$ or $p_2 = \frac{3}{2}$

When $p_2 = \frac{1}{2}$, $t = \frac{7}{60}$, which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ \frac{3}{4}M \\ \frac{1}{2}M \\ \frac{1}{12}M \end{pmatrix}$.

$\therefore$ In $\Lambda(1, 5, 12k)$, where $k \in \mathbf{Z}^+$, $|0, 9k; 0.5, 6k; 1.5, k| = 0$

When $p_2 = 1$, $t = \frac{1}{15} \Rightarrow p_3 = \frac{7}{4}$ (rejected)

When $p_2 = \frac{3}{2}$, $t = \frac{1}{60} \Rightarrow q_1 = \frac{5}{2}m > M$ (rejected)

<u>*Subcase B2.8*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{2} - 3t & 2t & \frac{1}{6} + t & \frac{1}{6} - t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{1}{2} - t\Big) \qquad q_1 = m + m\Delta\Big(\frac{2}{3} - 8t\Big)$$
$$p_3 = \Delta\Big(\frac{2}{3} - t\Big) \qquad q_2 = m + m\Delta\Big(-\frac{1}{3} + 2t\Big)$$
$$q_3 = m + m\Delta\Big(-\frac{2}{3} + 6t\Big)$$

$p_3 - p_2 = \dfrac{\Delta}{6} \Rightarrow \dfrac{d(\tau)}{6} = 2(p_3 - p_2) \in \mathbf{Z}^+$
$\therefore d(\tau)$ is even $\Rightarrow \Delta \geq 3$ and $\Delta, p_1, p_2, p_3 \in \mathbf{Z}^+$
$\Delta = 6(p_3 - p_2)$ and $\Delta t = 3p_3 - 4p_2 \in \mathbf{Z}^+$
$q_2 = m[1 - 2(p_3 - p_2) + 2\Delta t] \in m\mathbf{Z}$
$\because 0 < q_2 < M$
$\therefore q_2 = m$ and $\Delta\Big(-\dfrac{1}{3} + 2t\Big) = 0 \Rightarrow t = \dfrac{1}{6}$ (rejected)

<u>*Subcase B2.9*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{2} - 3t & \frac{1}{6} + t & \frac{1}{6} - t & 2t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{2}{3} - 2t\Big) \qquad q_1 = m + m\Delta\Big(\frac{1}{3} - 2t\Big)$$
$$p_3 = \Delta\Big(\frac{2}{3} + t\Big) \qquad q_2 = m + m\Delta\Big(-\frac{1}{3} + 6t\Big)$$
$$q_3 = m + m\Delta\Big(-\frac{1}{3} + 4t\Big)$$

$p_3 - p_2 = 3\Delta t$ and $6\Delta t = 2(p_3 - p_2) \in \mathbf{Z}^+$
Let $6\Delta t = k_1 \in \mathbf{Z}^+$
Consider $m < q_1 < M$ and $0 < q_2 < M$

$$0 < \Delta\Big(\frac{1}{3} - 2t\Big) < 1 \text{ and } -1 < \Delta\Big(-\frac{1}{3} + 6t\Big) < 1$$
$$\frac{1}{6} - \frac{1}{2\Delta} < t < \frac{1}{6} \text{ and } \frac{1}{18} - \frac{1}{6\Delta} < t < \frac{1}{18} + \frac{1}{6\Delta}$$
$$\frac{1}{6} - \frac{1}{2\Delta} < \frac{k_1}{6\Delta} < \frac{1}{6} \text{ and } \frac{1}{18} - \frac{1}{6\Delta} < \frac{k_1}{6\Delta} < \frac{1}{18} + \frac{1}{6\Delta}$$
$$\Delta - 3 < k_1 < \Delta \text{ and } \frac{\Delta}{3} - 1 < k_1 < \frac{\Delta}{3} + 1$$
$$\therefore \Delta - 3 < \frac{\Delta}{3} + 1 \Rightarrow \Delta < 6$$

If $d(\tau)$ is even, then $\Delta \geq 3$ and $\Delta, p_1, p_2, p_3 \in \mathbf{Z}$
$k_1 = 2(p_3 - p_2) \in 2\mathbf{Z}$
When $\Delta = 3$, $0 < k_1 < 3$ and $0 < k_1 < 2 \Rightarrow 0 < k_1 < 2$ (rejected)
When $\Delta = 4$, $\dfrac{1}{3} < k_1 < \dfrac{7}{3}$ and $1 < k_1 < 4 \Rightarrow 1 < k_1 < \dfrac{7}{3}$

$\therefore k_1 = 2$ and $t = \dfrac{1}{12}$, which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ \frac{5}{3}m \\ \frac{5}{3}m \\ m \end{pmatrix}$ (Config I)

When $\Delta = 5$, $\dfrac{2}{3} < k_1 < \dfrac{8}{3}$ and $2 < k_1 < 5 \Rightarrow 2 < k_1 < \dfrac{8}{3} < 4$ (rejected)

If $d(\tau)$ is odd, then $\Delta \geq \dfrac{3}{2}$ and $\Delta, p_1, p_2, p_3 \in \dfrac{1}{2}\mathbf{Z}$

When $\Delta = \dfrac{3}{2}$, $0 < k_1 < \dfrac{3}{2} \Rightarrow k_1 = 1 \Rightarrow t = \dfrac{1}{9} \Rightarrow p_2 = \dfrac{2}{3}$ (rejected)

When $\Delta = \dfrac{5}{2}$, $0 < k_1 < \dfrac{11}{6} \Rightarrow k_1 = 1 \Rightarrow t = \dfrac{1}{15} \Rightarrow p_2 = \dfrac{4}{3}$ (rejected)

When $\Delta = \dfrac{7}{2}$, $\dfrac{1}{2} < k_1 < \dfrac{13}{6} < 3 \Rightarrow k_1 = 1$ or $k_1 = 2$

If $k_1 = 1$, then $t = \dfrac{1}{21}$, which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{5}{2} \\ \frac{11}{12}M \\ \frac{5}{12}M \\ \frac{1}{4}M \end{pmatrix}$

$\therefore$ In $\Lambda(1, 7, 12k)$, where $k \in \mathbf{Z}^+$, $|0, 11k; 2, 5k; 2.5, 3k| = 0$

If $k_1 = 2$, then $t = \dfrac{2}{21} \Rightarrow p_2 = \dfrac{5}{3}$ (rejected)

When $\Delta = \dfrac{9}{2}$, $\dfrac{3}{2} < k_1 < \dfrac{5}{2} \Rightarrow k_1 = 2 \Rightarrow t = \dfrac{2}{27} \Rightarrow p_2 = \dfrac{7}{3}$ (rejected)

When $\Delta = \dfrac{11}{2}$, $2 < \dfrac{5}{2} < k_1 < \dfrac{17}{6} < 3$ (rejected)

<u>*Subcase B2.10*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & t & \frac{1}{2} - 3t & \frac{1}{6} - t & \frac{1}{6} + t & 2t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{2}{3} - 4t\Big) \qquad q_1 = m + m\Delta\Big(\frac{1}{3} - 2t\Big)$$

$$p_3 = \Delta\Big(\frac{2}{3} - t\Big) \qquad q_2 = m + m\Delta\Big(-\frac{1}{3} + 2t\Big)$$

$$q_3 = m + m\Delta\Big(-\frac{1}{3}\Big)$$

$0 < q_3 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = \dfrac{3}{2}$ or $\Delta = \dfrac{5}{2}$

If $\Delta = \dfrac{3}{2}$, then $p_2 = 0.5$ and $p_3 = 1 \Rightarrow t = \dfrac{1}{12}$ and $t = 0$ (rejected)

If $\Delta = \dfrac{5}{2}$, then $p_3 = \dfrac{5}{3} - \dfrac{5}{2}t > \dfrac{5}{3} - \dfrac{5}{2}\Big(\dfrac{1}{6}\Big) = \dfrac{5}{4} > 1$

$\therefore p_3 = \dfrac{3}{2}$ or $p_3 = 2$

When $p_3 = \dfrac{3}{2}$, $t = \dfrac{1}{15}$, which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{3}{2} \\ \dfrac{3}{4}M \\ \dfrac{1}{4}M \\ \dfrac{1}{12}M \end{pmatrix}$

$\therefore$ In $\Lambda(1, 5, 12k)$, where $k \in \mathbf{Z}^+$, $|0, 9k; , 1, 3k; 1.5, k| = 0$

When $p_3 = 2$, $t = \dfrac{2}{15} \Rightarrow p_2 = \dfrac{1}{3}$ (rejected)

<u>*Subcase B2.11*</u> $\mathbf{f} = \begin{pmatrix} \dfrac{1}{6} & t & \dfrac{1}{2} - 3t & 2t & \dfrac{1}{6} - t & \dfrac{1}{6} + t \end{pmatrix}^T$

$p_2 = \Delta\Big(\dfrac{1}{2} - t\Big)$ $\qquad$ $q_1 = m + m\Delta\Big(\dfrac{2}{3} - 4t\Big)$

$p_3 = \Delta\Big(\dfrac{2}{3} + t\Big)$ $\qquad$ $q_2 = m + m\Delta\Big(-\dfrac{1}{3} + 6t\Big)$

$q_3 = m + m\Delta\Big(-\dfrac{2}{3} + 6t\Big)$

$2p_3 - 4p_2 = \Delta\Big(\dfrac{2}{3} - 4t\Big) \Rightarrow \Delta\Big(\dfrac{2}{3} - 4t\Big) = 2(p_3 - 2p_2) \in \mathbf{Z}$

$q_1 = m\Big[1 + \Delta\Big(\dfrac{2}{3} - 4t\Big)\Big] \in m\mathbf{Z}$

$\because 0 < q_1 < M$

$\therefore q_1 = m$ and $1 + \Delta\Big(\dfrac{2}{3} - 4t\Big) = 1 \Rightarrow t = \dfrac{1}{6}$ (rejected)

<u>*Subcase B2.12*</u> $\mathbf{f} = \begin{pmatrix} \dfrac{1}{6} & t & \dfrac{1}{2} - 3t & \dfrac{1}{6} + t & 2t & \dfrac{1}{6} - t \end{pmatrix}^T$

$p_2 = \Delta\Big(\dfrac{2}{3} - 2t\Big)$ $\qquad$ $q_1 = m + m\Delta\Big(\dfrac{2}{3} - 8t\Big)$

$p_3 = \Delta\Big(\dfrac{5}{6} - 2t\Big)$ $\qquad$ $q_2 = m$

$q_3 = m + m\Delta\Big(-\dfrac{1}{3} + 4t\Big)$

$6p_3 - 8p_2 = \Delta\Big(-\dfrac{1}{3} + 4t\Big) \Rightarrow \Delta\Big(-\dfrac{1}{3} + 4t\Big) = 2(3p_3 - 4p_2) \in \mathbf{Z}$

$q_3 = m\Big[1 + \Delta\Big(-\dfrac{1}{3} + 4t\Big)\Big] \in m\mathbf{Z}$

$\because 0 < q_3 < M$

$\therefore q_3 = m$ and $1 + \varDelta\Big(-\dfrac{1}{3} + 4t\Big) = 1 \Rightarrow t = \dfrac{1}{12}$
$\therefore q_1 = q_2 = q_3 = m$ (Config II)

<u>*Subcase B3.1*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} - 2t & 2t & \frac{1}{6} - 2t & t & \frac{1}{2} + t \end{pmatrix}^T$

$$p_2 = \frac{\varDelta}{6} \qquad q_1 = m + m\varDelta\Big(\frac{1}{3} + 6t\Big)$$
$$p_3 = \varDelta\Big(\frac{5}{6} - t\Big) \qquad q_2 = m + m\varDelta\Big(\frac{2}{3} - 2t\Big)$$
$$q_3 = m + m\varDelta(-8t)$$

$2p_2 = \dfrac{d(\tau)}{6} \in \mathbf{Z}^+$, so $d(\tau)$ is even and $\varDelta \geq 3$.
Hence $q_2 \geq m + 3m\Big(\dfrac{2}{3} - 2t\Big) \geq m + m = M$ (rejected)

<u>*Subcase B3.2*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} - 2t & 2t & \frac{1}{2} + t & \frac{1}{6} - 2t & t \end{pmatrix}^T$

$$p_2 = \varDelta\Big(\frac{1}{2} + 3t\Big) \qquad q_1 = m + \varDelta m\Big(-\frac{2}{3} + 6t\Big)$$
$$p_3 = \varDelta\Big(\frac{2}{3} + 2t\Big) \qquad q_2 = m + \varDelta m\Big(\frac{1}{3} + 4t\Big)$$
$$q_3 = m + \varDelta m\Big(\frac{2}{3} - 2t\Big)$$

$q_3 < M = 2m \Rightarrow \varDelta\Big(\dfrac{2}{3} - \dfrac{1}{3}\Big) < \varDelta\Big(\dfrac{2}{3} - 2t\Big) < 1 \Rightarrow \varDelta < 3 \Rightarrow \varDelta = 1.5$ or 2.5
If $\varDelta = 1.5$, $p_2 = 0.5$ and $p_3 = 1$, so $t = -\dfrac{1}{18}$ and $t = 0$ (contradiction).
If $\varDelta = 2.5$, $q_3 > m + \dfrac{5}{2}m\Big(\dfrac{2}{3} - \dfrac{1}{6}\Big) = \dfrac{9}{4}m > 2m = M$ (rejected).

<u>*Subcase B3.3*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} - 2t & 2t & t & \frac{1}{2} + t & \frac{1}{6} - 2t \end{pmatrix}^T$

$$p_2 = 3\varDelta t \qquad q_1 = m + \varDelta m\Big(-\frac{1}{3}\Big)$$
$$p_3 = \varDelta\Big(\frac{1}{3} - t\Big) \qquad q_2 = m + \varDelta m\Big(-\frac{1}{3} - 2t\Big)$$
$$q_3 = m + \varDelta m\Big(-\frac{1}{3} - 2t\Big)$$

$2(p_2+p_3) = d(\tau)\Big(\dfrac{1}{3}+2t\Big) \in \mathbf{Z}$, hence $2q_2 = M - md(\tau)\Big(\dfrac{1}{3}+2t\Big) = M - M(p_2+p_3) < M(1 - 0.5 - 1) < -m$ (rejected)

<u>*Subcase B3.4*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} - 2t & 2t & \frac{1}{6} - 2t & \frac{1}{2} + t & t \end{pmatrix}^T$

$$p_2 = \frac{\mathit{\Delta}}{6} \qquad q_1 = m + \mathit{\Delta} m\Big(-\frac{2}{3} + 6t\Big)$$

$$p_3 = \mathit{\Delta}\Big(\frac{5}{6} - t\Big) \qquad q_2 = m + \mathit{\Delta} m\Big(-\frac{2}{3} - 2t\Big)$$

$$q_3 = m + \mathit{\Delta} m(-8t)$$

$2p_2 = \dfrac{d(\tau)}{6} \in \mathbf{Z}^+$, so $d(\tau)$ is even and $\mathit{\Delta} \geq 3$

$q_2 < m + 3m\Big(-\dfrac{2}{3}\Big) = -m$ (rejected)

<u>*Subcase B3.5*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} - 2t & 2t & t & \frac{1}{6} - 2t & \frac{1}{2} + t \end{pmatrix}^T$

$$p_2 = 3\mathit{\Delta} t \qquad q_1 = m + \mathit{\Delta} m\Big(\frac{1}{3} + 6t\Big)$$

$$p_3 = \mathit{\Delta}\Big(\frac{2}{3} + 3t\Big) \qquad q_2 = m + \mathit{\Delta} m\Big(\frac{1}{3} + 4t\Big)$$

$$q_3 = m + \mathit{\Delta} m\Big(-\frac{1}{3} - 2t\Big)$$

$q_1 < M = 2m$, $\therefore m + \mathit{\Delta} m\Big(\dfrac{1}{3}\Big) < q_1 < 2m \Rightarrow \mathit{\Delta} < 3 \Rightarrow \mathit{\Delta} = 1.5$ or 2.5.

If $\mathit{\Delta} = 1.5$, then $p_2 = 0.5$ and $p_3 = 1$, $\therefore t = \dfrac{1}{9}$ and $t = 0$ (contradiction)

If $\mathit{\Delta} = 2.5$, then $2p_2 = 15t \in \mathbf{Z}^+$, $\therefore 15t \geq 1$, i.e. $t \geq \dfrac{1}{15}$

$\therefore q_1 \geq m + m(2.5)\Big(\dfrac{1}{3} + 6\Big(\dfrac{1}{15}\Big)\Big) = \dfrac{17}{6}m > 2m = M$ (rejected)

<u>*Subcase B3.6*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} - 2t & 2t & \frac{1}{2} + t & t & \frac{1}{6} - 2t \end{pmatrix}^T$

$$p_2 = \mathit{\Delta}\Big(\frac{1}{2} + 3t\Big) \qquad q_1 = m + \mathit{\Delta} m\Big(\frac{-1}{3}\Big)$$

$$p_3 = \mathit{\Delta}\Big(\frac{5}{6} - t\Big) \qquad q_2 = m + \mathit{\Delta} m\Big(\frac{2}{3} - 2t\Big)$$

$$q_3 = m + \mathit{\Delta} m\Big(\frac{2}{3} - 2t\Big)$$

$q_2 < M = 2m$, $\therefore 1 > \mathit{\Delta}\Big(\dfrac{2}{3} - 2t\Big) > \mathit{\Delta}\Big(\dfrac{2}{3}\Big) \Rightarrow \mathit{\Delta} < 1.5$ (rejected)

<u>*Subcase B3.7*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & 2t & \frac{1}{6} - 2t & \frac{1}{6} - 2t & t & \frac{1}{2} + t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{1}{3} - 4t\Big) \qquad q_1 = m + \Delta m\Big(\frac{2}{3} - 2t\Big)$$

$$p_3 = \Delta\Big(\frac{2}{3} + 2t\Big) \qquad q_2 = m\Delta m\Big(\frac{2}{3} - 2t\Big)$$

$$q_3 = m + \Delta m\Big(- \frac{1}{3}\Big)$$

$q_1 < M = 2m$, $\therefore 1 > \Delta\Big(\frac{2}{3} - 2t\Big) > \Delta\Big(\frac{2}{3}\Big) \Rightarrow \Delta < 1.5$ (rejected)

<u>*Subcase B3.8*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & 2t & \frac{1}{6} - 2t & \frac{1}{2} + t & \frac{1}{6} - 2t & t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{1}{3} - t\Big) \qquad q_1 = m + \Delta m\Big(- \frac{1}{3} - 2t\Big)$$

$$p_3 = \Delta\Big(\frac{2}{3} + 2t\Big) \qquad q_2 = m + \Delta m\Big(\frac{1}{3} + 4t\Big)$$

$$q_3 = m + \Delta m\Big(\frac{1}{3} + 3t\Big)$$

$2(p_3 - p_2) = d(\tau)\Big(\frac{1}{3} + 3t\Big) \in \mathbf{Z}^+$, and $2q_3 = M + M(p_3 - p_2) \in M\mathbf{Z}^+$, and since $0 < 2q_3 < 2M$, we must have $2q_3 = M$, i.e. $q_3 = m$.
But then $\Big(\frac{1}{3} + 4t\Big) = 0$, or $t = -\frac{1}{12}$ (rejected)

<u>*Subcase B3.9*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & 2t & \frac{1}{6} - 2t & t & \frac{1}{2} + t & \frac{1}{6} - 2t \end{pmatrix}^T$

$$p_2 = \Delta\Big(\frac{1}{6} - t\Big) \qquad q_1 = m + \Delta m(-8t)$$

$$p_3 = \Delta\Big(\frac{1}{3} - t\Big) \qquad q_2 = m + \Delta m\Big(- \frac{1}{3} - 2t\Big)$$

$$q_3 = m + \Delta m\Big(- \frac{2}{3} - 6t\Big)$$

$q_2 > 0$, $\therefore 1 > \Delta\Big(\frac{1}{3} + 2t\Big) > \Delta\Big(\frac{1}{3}\Big)$, or $\Delta < 3$, i.e. $\Delta = 1.5$ or 2.5.
But $2(p_3 - p_2) = d(\tau)\Big(\frac{1}{6}\Big) \in \mathbf{Z}^+$ and $\frac{1.5}{6} \notin \mathbf{Z}^+$ and $\frac{2.5}{6} \notin \mathbf{Z}^+$ (rejected).

<u>*Subcase B3.10*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & 2t & \frac{1}{6} - 2t & \frac{1}{6} - 2t & \frac{1}{2} + t & t \end{pmatrix}^T$

$p_2 = \mathit{\Delta}\Big(\frac{1}{3} - 4t\Big)$ $\qquad q_1 = m + \mathit{\Delta} m\Big(-\frac{1}{3} - 2t\Big)$

$p_3 = \mathit{\Delta}\Big(\frac{1}{3} - t\Big)$ $\qquad q_2 = m + \mathit{\Delta} m\Big(-\frac{1}{3} - 2t\Big)$

$\qquad q_3 = m + \mathit{\Delta} m\Big(-\frac{1}{3}\Big)$

$2(p_2 - 2p_3) = d(\tau)\Big(-\frac{1}{3} - 2t\Big) \in -\mathbf{Z}^+$, $\therefore 2q_1 = M - M(p_2 - 2p_3) \in M\mathbf{Z}$ and since $0 < 2q_1 < 2M$, we must have $2q_1 = M$, i.e. $q_1 = m$. But then $-\frac{1}{3} - 2t = 0$, or $t = -\frac{1}{6}$ (rejected).

<u>*Subcase B3.11*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & 2t & \frac{1}{6} - 2t & t & \frac{1}{6} - 2t & \frac{1}{2} + t \end{pmatrix}^T$

$p_2 = \mathit{\Delta}\Big(\frac{1}{6} - t\Big)$ $\qquad q_1 = m + \mathit{\Delta} m\Big(\frac{2}{3} - 2t\Big)$

$p_3 = \mathit{\Delta}\Big(\frac{2}{3} + 2t\Big)$ $\qquad q_2 = m + \mathit{\Delta} m\Big(\frac{1}{3} + 4t\Big)$

$\qquad q_3 = m + \mathit{\Delta} m\Big(-\frac{2}{3} + 6t\Big)$

$q_1 < M = 2m$, $\therefore 1 > \mathit{\Delta}\Big(\frac{2}{3} - 2t\Big) > \Big(\frac{3}{2}\Big)\mathit{\Delta}$, i.e. $\mathit{\Delta} < \frac{2}{3}$ (rejected).

<u>*Subcase B3.12*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{6} & 2t & \frac{1}{6} - 2t & \frac{1}{2} + t & t & \frac{1}{6} - 2t \end{pmatrix}^T$

$p_2 = \mathit{\Delta}\Big(\frac{2}{3} - t\Big)$ $\qquad q_1 = m + \mathit{\Delta} m(-8t)$

$p_3 = \mathit{\Delta}\Big(\frac{5}{6} - t\Big)$ $\qquad q_2 = m + \mathit{\Delta} m\Big(\frac{2}{3} - 2t\Big)$

$\qquad q_3 = m + \mathit{\Delta} m\Big(\frac{1}{3} + 4t\Big)$

$2(p_3 - p_2) = \dfrac{d(\tau)}{6} \in \mathbf{Z}^+$, hence $d(\tau)$ is even and $\mathit{\Delta} \geq 3$.

But $q_2 < M = 2m$, $\therefore 1 > \mathit{\Delta}\Big(\frac{2}{3} - 2t\Big) > \mathit{\Delta}\Big(\frac{2}{3}\Big)$, or $\mathit{\Delta} < 1.5$ (contradiction)

<u>*Subcase B4.1*</u> $\mathbf{f} = \left(\frac{1}{3} - 4t \quad t \quad \frac{1}{3} + t \quad \frac{1}{6} - 2t \quad 3t \quad \frac{1}{6} + t\right)^T$

$$p_2 = \Delta\Big(\frac{1}{2} - t\Big) \qquad q_1 = m + m\Delta\Big(\frac{2}{3} - 4t\Big)$$
$$p_3 = \Delta\Big(\frac{2}{3} + t\Big) \qquad q_2 = m + m\Delta\Big(\frac{1}{3} - 10t\Big)$$
$$q_3 = m + m\Delta(-10t)$$

$p_2 + p_3 = \frac{7\Delta}{6} \Rightarrow \frac{7d(\tau)}{6} = 2(p_2 + p_3) \in \mathbf{Z}^+$
$\therefore d(\tau)$ is even and hence $\Delta \geq 3$
$\therefore q_2 \geq m + m(2 - 12t) > m + m(2 - 1) = M$ (rejected)

<u>*Subcase B4.2*</u> $\mathbf{f} = \left(\frac{1}{3} - 4t \quad t \quad \frac{1}{3} + t \quad 3t \quad \frac{1}{6} + t \quad \frac{1}{6} - 2t\right)^T$

$$p_2 = \Delta\Big(\frac{1}{3} + 4t\Big) \qquad q_1 = m + m\Delta\Big(\frac{2}{3} - 10t\Big)$$
$$p_3 = \Delta\Big(\frac{1}{2} + 3t\Big) \qquad q_2 = m + m\Delta(-6t)$$
$$q_3 = m + m\Delta\Big(-\frac{1}{3}\Big)$$

$3p_2 - 2p_3 = 6\Delta t \Rightarrow 12\Delta t = 2(3p_2 - 2p_3) \in \mathbf{Z}^+$
$2q_2 = m(2 - 12\Delta t) \in m\mathbf{Z}$
$\because 0 < 2q_2 < M$
$\therefore q_2 = m$ and $2 - 12\Delta t = 1 \Rightarrow t = \frac{1}{12\Delta}$
$0 < q_3 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = \frac{3}{2}$ or $\Delta = \frac{5}{2}$
If $\Delta = \frac{3}{2}$, then $t = \frac{1}{18} \Rightarrow p_2 = \frac{5}{6}$ (rejected)
If $\Delta = \frac{5}{2}$, then $t = \frac{1}{30} \Rightarrow p_2 = \frac{7}{6}$ (rejected)

<u>*Subcase B4.3*</u> $\mathbf{f} = \left(\frac{1}{3} - 4t \quad t \quad \frac{1}{3} + t \quad \frac{1}{6} + t \quad \frac{1}{6} - 2t \quad 3t\right)^T$

$$p_2 = \Delta\Big(\frac{1}{2} + 2t\Big) \qquad q_1 = m + m\Delta\Big(\frac{1}{3}\Big)$$
$$p_3 = \Delta\Big(\frac{1}{2} + 6t\Big) \qquad q_2 = m$$
$$q_3 = m + m\Delta(-4t)$$

$p_3 - p_2 = 4\Delta t \Rightarrow 8\Delta t = 2(p_3 - p_2) \in \mathbf{Z}^+$
$2q_3 = m(2 - 8\Delta t) \in m\mathbf{Z}$
$\because 0 < 2q_3 < M$
$\therefore q_3 = m$ and $2 - 8\Delta t = 1 \Rightarrow t = \frac{1}{8\Delta}$

$0 < q_2 < M \Rightarrow \mathit{\Delta} < 3 \Rightarrow \mathit{\Delta} = \dfrac{3}{2}$ or $\mathit{\Delta} = \dfrac{5}{2}$

If $\mathit{\Delta} = \dfrac{3}{2}$, then $t = \dfrac{1}{12}$ (rejected)

If $\mathit{\Delta} = \dfrac{5}{2}$, then $t = \dfrac{1}{20}$, which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 2 \\ \frac{11}{12}M \\ \frac{1}{2}M \\ \frac{1}{4}M \end{pmatrix}$

$\therefore$ In $\mathit{\Lambda}(1, 5, 12k)$, where $k \in \mathbf{Z}^+$, $|0, 11k; 1.5, 6k; 2, 3k| = 0$

<u>*Subcase B4.4*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{3} - 4t & t & \frac{1}{3} + t & \frac{1}{6} - 2t & \frac{1}{6} + t & 3t \end{pmatrix}^T$

$p_2 = \mathit{\Delta}\Big(\dfrac{1}{2} - t\Big) \qquad q_1 = m + m\mathit{\Delta}\Big(\dfrac{1}{3}\Big)$

$p_3 = \mathit{\Delta}\Big(\dfrac{1}{2} + 3t\Big) \qquad q_2 = m + m\mathit{\Delta}(-6t)$

$q_3 = m + m\mathit{\Delta}(-10t)$

$p_3 - p_2 = 4\mathit{\Delta}t \Rightarrow 8\mathit{\Delta}t = 2(p_3 - p_2) \in \mathbf{Z}^+ \Rightarrow 8\mathit{\Delta}t \geq 1$

$\therefore q_3 = m - m(10\mathit{\Delta}t) < m - m(8\mathit{\Delta}t) \leq m - m = 0$ (rejected)

<u>*Subcase B4.5*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{3} - 4t & t & \frac{1}{3} + t & 3t & \frac{1}{6} - 2t & \frac{1}{6} + t \end{pmatrix}^T$

$p_2 = \mathit{\Delta}\Big(\dfrac{1}{3} + 4t\Big) \qquad q_1 = m + m\mathit{\Delta}\Big(\dfrac{2}{3} - 4t\Big)$

$p_3 = \mathit{\Delta}\Big(\dfrac{1}{2} + 6t\Big) \qquad q_2 = m$

$q_3 = m + m\mathit{\Delta}\Big(-\dfrac{1}{3}\Big)$

$0 < q_3 < M \Rightarrow \mathit{\Delta} < 3 \Rightarrow \mathit{\Delta} = \dfrac{3}{2}$ or $\mathit{\Delta} = \dfrac{5}{2}$

If $\mathit{\Delta} = \dfrac{3}{2}$, then $p_2 < \dfrac{3}{2}\Big(\dfrac{1}{3} + 4\Big(\dfrac{1}{12}\Big)\Big) = 1 \Rightarrow p_2 = \dfrac{1}{2} \Rightarrow t = 0$ (rejected)

If $\mathit{\Delta} = \dfrac{5}{2}$, then $p_2 < \dfrac{5}{2}\Big(\dfrac{1}{3} + 4\Big(\dfrac{1}{12}\Big)\Big) = \dfrac{5}{3} \Rightarrow p_2 = \dfrac{1}{2}$ or $p_2 = 1$ or $p_2 = \dfrac{3}{2}$

When $p_2 = \dfrac{1}{2}$, $t = -\dfrac{1}{30} < 0$ (rejected)

When $p_2 = 1$, $t = \dfrac{1}{60} \Rightarrow q_1 = \dfrac{5}{2}m > M$ (rejected)

When $p_2 = \dfrac{3}{2}$, $t = \dfrac{1}{15} \Rightarrow p_3 = \dfrac{9}{4}$ (rejected)

<u>*Subcase B4.6*</u> $\mathbf{f} = \left(\frac{1}{3} - 4t \quad t \quad \frac{1}{3} + t \quad \frac{1}{6} + t \quad 3t \quad \frac{1}{6} - 2t\right)^T$

$$p_2 = \Delta\Big(\frac{1}{2} + 2t\Big) \qquad q_1 = m + m\Delta\Big(\frac{2}{3} - 4t\Big)$$
$$p_3 = \Delta\Big(\frac{2}{3} + t\Big) \qquad q_2 = m + m\Delta\Big(\frac{1}{3} - 10t\Big)$$
$$q_3 = m + m\Delta(-4t)$$

$4p_3 - 4p_2 = \Delta\Big(\frac{2}{3} - 4t\Big) \Rightarrow \Delta\Big(\frac{2}{3} - 4t\Big) = 2(2p_3 - 2p_2) \in \mathbf{Z}$
$q_1 = m\Big[1 + \Delta\Big(\frac{2}{3} - 4t\Big)\Big] \geq m(1+1) = M$ (rejected)

<u>*Subcase B4.7*</u> $\mathbf{f} = \left(\frac{1}{3} - 4t \quad \frac{1}{3} + t \quad t \quad \frac{1}{6} - 2t \quad 3t \quad \frac{1}{6} + t\right)^T$

$$p_2 = \Delta\Big(\frac{1}{6} - t\Big) \qquad q_1 = m + m\Delta(-4t)$$
$$p_3 = \Delta\Big(\frac{2}{3} + t\Big) \qquad q_2 = m + m\Delta\Big(\frac{1}{3} - 10t\Big)$$
$$q_3 = m + m\Delta\Big(\frac{2}{3} - 10t\Big)$$

$p_2 + p_3 = \frac{5\Delta}{6} \Rightarrow \frac{5d(\tau)}{6} = 2(p_2 + p_3) \in \mathbf{Z}^+$
$\therefore d(\tau)$ is even $\Rightarrow \Delta \geq 3$ and $\Delta, p_1, p_2, p_3 \in \mathbf{Z}$
Let $\Delta = 6k_1$, where $k_1 \in \mathbf{Z}^+$
$p_2 = k_1 - \Delta t \in \mathbf{Z}^+ \Rightarrow \Delta t \in \mathbf{Z}^+$
$q_1 = m(1 - 4\Delta t) \leq m(1-4) = -3m < 0$ (rejected)

<u>*Subcase B4.8*</u> $\mathbf{f} = \left(\frac{1}{3} - 4t \quad \frac{1}{3} + t \quad t \quad 3t \quad \frac{1}{6} + t \quad \frac{1}{6} - 2t\right)^T$

$$p_2 = 4\Delta t \qquad q_1 = m + m\Delta(-10t)$$
$$p_3 = \Delta\Big(\frac{1}{2} + 3t\Big) \qquad q_2 = m + m\Delta(-6t)$$
$$q_3 = m + m\Delta\Big(\frac{1}{3}\Big)$$

$0 < q_3 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = \frac{3}{2}$ or $\Delta = \frac{5}{2}$
If $\Delta = \frac{3}{2}$, then $p_2 = 6t < \frac{1}{2}$ (rejected)
If $\Delta = \frac{5}{2}$, then $p_2 = 10t < \frac{5}{6} \Rightarrow p_2 = \frac{1}{2}$
$\therefore t = \frac{1}{20} \Rightarrow p_3 = \frac{13}{8}$ (rejected)

<u>*Subcase B4.9*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{3} - 4t & \frac{1}{3} + t & t & \frac{1}{6} + t & \frac{1}{6} - 2t & 3t \end{pmatrix}^T$

$p_2 = \Delta\Big(\frac{1}{6} + 2t\Big)$ $\qquad q_1 = m + m\Delta\Big(-\frac{1}{3}\Big)$

$p_3 = \Delta\Big(\frac{1}{2} + 2t\Big)$ $\qquad q_2 = m$

$\qquad q_3 = m + m\Delta\Big(\frac{2}{3} - 4t\Big)$

$0 < q_1 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = \frac{3}{2}$ or $\Delta = \frac{5}{2}$

If $\Delta = \frac{3}{2}$, then $p_2 = \frac{1}{4} + 3t < \frac{1}{2}$ (rejected)

If $\Delta = \frac{5}{2}$, then $p_2 = \frac{5}{12} + 5t < \frac{5}{6} \Rightarrow p_2 = \frac{1}{2}$

$\therefore t = \frac{1}{60} \Rightarrow p_3 = \frac{4}{3}$ (rejected)

<u>*Subcase B4.10*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{3} - 4t & \frac{1}{3} + t & t & \frac{1}{6} - 2t & \frac{1}{6} + t & 3t \end{pmatrix}^T$

$p_2 = \Delta\Big(\frac{1}{6} - t\Big)$ $\qquad q_1 = m + m\Delta\Big(-\frac{1}{3}\Big)$

$p_3 = \Delta\Big(\frac{1}{2} + 3t\Big)$ $\qquad q_2 = m + m\Delta(-6t)$

$\qquad q_3 = m + m\Delta\Big(\frac{2}{3} - 10t\Big)$

$p_3 - 3p_2 = 6\Delta t \Rightarrow 12\Delta t = 2(p_3 - 3p_2) \in \mathbf{Z}^+$

$2q_2 = m(2 - 12\Delta t) \in m\mathbf{Z}$

$\because 0 < 2q_3 < M$

$\therefore q_3 = m$ and $2 - 12\Delta t = 1 \Rightarrow t = \frac{1}{12\Delta}$

$0 < q_1 < M \Rightarrow \Delta < 3 \Rightarrow \Delta = \frac{3}{2}$ or $\Delta = \frac{5}{2}$

If $\Delta = \frac{3}{2}$, then $t = \frac{1}{18} \Rightarrow p_2 = \frac{1}{6} < \frac{1}{2}$ (rejected)

If $\Delta = \frac{5}{2}$, then $t = \frac{1}{30} \Rightarrow p_2 = \frac{1}{3} < \frac{1}{2}$ (rejected)

<u>*Subcase B4.11*</u> $\mathbf{f} = \begin{pmatrix} \frac{1}{3} - 4t & \frac{1}{3} + t & t & 3t & \frac{1}{6} - 2t & \frac{1}{6} + t \end{pmatrix}^T$

$p_2 = 4\Delta t$ $\qquad q_1 = m + m\Delta(-4t)$

$p_3 = \Delta\Big(\frac{1}{2} + 6t\Big)$ $\qquad q_2 = m$

$\qquad q_3 = m + m\Delta\Big(\frac{1}{3}\Big)$

$2q_1 = m(2 - 8\Delta t) = m(2 - 2p_2) \in m\mathbf{Z}$

$\because 0 < 2q_3 < M$

$\therefore q_1 = m$ and $2 - 8\mathit{\Delta} t = 1 \Rightarrow t = \dfrac{1}{8\mathit{\Delta}}$

$0 < q_2 < M \Rightarrow \mathit{\Delta} < 3 \Rightarrow \mathit{\Delta} = \dfrac{3}{2}$ or $\mathit{\Delta} = \dfrac{5}{2}$

If $\mathit{\Delta} = \dfrac{3}{2}$, then $t = \dfrac{1}{12}$ (rejected)

If $\mathit{\Delta} = \dfrac{5}{2}$, then $t = \dfrac{1}{20}$, which gives $\begin{pmatrix} p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 2 \\ \frac{1}{4}M \\ \frac{1}{2}M \\ \frac{11}{12}M \end{pmatrix}$

$\therefore$ In $\mathit{\Lambda}(1, 5, 12k)$, where $k \in \mathbf{Z}^+$, $|0, 3k; 1.5, 6k; 2, 11k| = 0$

Subcase B4.12 $\mathbf{f} = \begin{pmatrix} \frac{1}{3} - 4t & \frac{1}{3} + t & t & \frac{1}{6} + t & 3t & \frac{1}{6} - 2t \end{pmatrix}^T$

$$p_2 = \mathit{\Delta}\Big(\frac{1}{6} + 2t\Big) \qquad q_1 = m + m\mathit{\Delta}(-10t)$$

$$p_3 = \mathit{\Delta}\Big(\frac{2}{3} + t\Big) \qquad q_2 = m + m\mathit{\Delta}\Big(\frac{1}{3} - 10t\Big)$$

$$q_3 = m + m\mathit{\Delta}\Big(\frac{2}{3} - 4t\Big)$$

$2p_3 - p_2 = \dfrac{7\mathit{\Delta}}{6} \Rightarrow \dfrac{7d(\tau)}{6} = 2(2p_3 - p_2) \in \mathbf{Z}^+$

$\therefore d(\tau)$ is even and hence $\mathit{\Delta} \geq 3$

$\therefore q_2 \geq m + m(2 - 8t) > m + m(2 - 1) = M$ (rejected)

Case C Sporadic solutions

We wrote a Scilab program to check whether any of q_1, q_2 and q_3 is a multiple of m when a set of exact values is substituted. If they are, then since $1 \leq q_i \leq M - 1$, for $i = 1, 2, 3$, we must have $q_i = m$. If q_1, q_2 and q_3 are all equal to m in a certain case, then either the lines must be in Config II, or the p's take unreasonable values. Hence this checking program can greatly reduce the number of cases under consideration. After we run the program, we found that in all of the cases we must have $q_1 = q_2 = q_3 = m$ and hence the sporadic solutions do not give rise to new configurations.

Here are the source codes of the Scilab program:

```
clc;
clear;
//sporadic soln matrix
M=[1/10 2/15 3/10 2/15 1/6 1/6;
1/15 1/15 7/15 1/15 1/10 7/30;
```

```
1/30 7/30 4/15 1/15 1/10 3/10;
1/30 1/10 7/15 1/15 1/15 4/15;
1/30 1/15 19/30 1/15 1/10 1/10;
1/15 1/6 4/15 1/10 1/10 3/10;
1/15 2/15 11/30 1/10 1/6 1/6;
1/30 1/6 13/30 1/10 2/15 2/15;
1/30 1/30 7/10 1/30 1/15 2/15;
1/30 7/30 3/10 1/15 2/15 7/30;
1/30 1/6 11/30 1/15 1/10 4/15;
1/30 1/10 13/30 1/30 2/15 4/15;
1/30 1/15 8/15 1/30 1/10 7/30;
1/14 5/42 5/14 2/21 5/42 5/21;
1/21 4/21 13/42 1/14 1/6 3/14;
1/42 3/14 5/14 1/21 1/6 4/21;
1/42 1/6 19/42 1/14 2/21 4/21;
1/42 1/6 13/42 1/21 1/14 8/21;
1/42 1/21 13/21 1/42 1/14 3/14;
1/20 1/12 29/60 1/15 1/10 13/60;
1/20 1/12 9/20 1/15 1/12 4/15;
1/20 1/12 5/12 1/20 1/10 3/10;
1/60 4/15 3/10 1/20 1/12 17/60;
1/60 13/60 9/20 1/12 1/10 2/15;
1/60 13/60 5/12 1/20 2/15 1/6;
1/12 1/6 17/60 2/15 3/20 11/60;
1/12 2/15 19/60 1/10 3/20 13/60;
1/15 11/60 13/60 1/12 1/10 7/20;
1/20 11/60 3/10 1/12 7/60 4/15;
1/20 1/10 23/60 1/15 1/12 19/60;
1/30 7/60 19/60 1/20 1/15 5/12;
1/30 1/12 7/12 1/15 1/10 2/15;
1/30 1/20 11/20 1/30 1/15 4/15;
1/60 3/10 7/20 1/12 7/60 2/15;
1/60 4/15 23/60 1/12 1/10 3/20;
1/60 7/30 5/12 1/15 7/60 3/20;
1/60 13/60 11/30 1/20 1/12 4/15;
1/60 1/6 31/60 1/15 1/10 2/15;
1/60 1/6 5/12 1/20 1/15 17/60;
1/60 2/15 9/20 1/30 1/12 17/60;
1/60 1/10 31/60 1/30 1/15 4/15;
1/12 3/14 19/84 11/84 13/84 4/21;
1/14 11/84 23/84 1/12 2/21 29/84;
1/21 13/84 23/84 1/14 1/12 31/84;
1/42 1/12 7/12 1/21 1/14 4/21;
1/84 25/84 5/14 5/84 1/12 4/21;
1/84 5/21 5/12 5/84 1/14 17/84;
1/84 3/14 37/84 1/21 1/12 17/84;
1/84 1/6 43/84 1/21 1/14 4/21;
1/18 13/90 7/18 11/90 2/15 7/45;
1/45 19/90 16/45 1/18 1/10 23/90;
1/90 23/90 31/90 2/45 1/15 5/18;
1/90 17/90 47/90 1/18 4/45 2/15;
13/120 3/20 31/120 2/15 19/120 23/120;
1/12 19/120 29/120 1/10 13/120 37/120;
1/20 23/120 29/120 1/15 13/120 41/120;
1/60 13/120 73/120 1/20 1/12 2/15;
1/120 7/20 43/120 7/120 11/120 2/15;
1/120 3/10 49/120 7/120 1/12 17/120;
1/120 4/15 53/120 1/20 11/120 17/120;
1/120 13/60 61/120 1/20 1/12 2/15;
```

```
1/15 41/210 8/35 1/14 31/210 61/210;
13/210 1/10 83/210 1/14 4/35 9/35;
1/35 2/15 97/210 1/14 17/210 47/210;
1/210 3/14 121/210 11/210 1/15 3/35];
//M = 65x6 matrix
//Add and find denominator matrix
B1 = [0 1 0 1 0 0;
0 1 0 0 1 0;
0 1 0 0 0 1;
0 0 1 1 0 0;
0 0 1 0 1 0;
0 0 1 0 0 1];
 [n1,d1] = rat(M*B1');
B2 = [1 0 0 1 0 0;
1 0 0 0 1 0;
1 0 0 0 0 1];
 [n2,d2] = rat(M*B2');
V1 = int32(d1);
V2 = int32(d2);
// Make lcm matrix
for i =1:65
   C(i,1) = lcm([V1(i,2), V2(i,1)]);
   C(i,2) = lcm([V1(i,3), V2(i,1)]);
   C(i,3) = lcm([V1(i,5), V2(i,1)]);
   C(i,4) = lcm([V1(i,6), V2(i,1)]);
   C(i,5) = lcm([V1(i,1), V2(i,2)]);
   C(i,6) = lcm([V1(i,3), V2(i,2)]);
   C(i,7) = lcm([V1(i,4), V2(i,2)]);
   C(i,8) = lcm([V1(i,6), V2(i,2)]);
   C(i,9) = lcm([V1(i,1), V2(i,3)]);
   C(i,10) = lcm([V1(i,2), V2(i,3)]);
   C(i,11) = lcm([V1(i,4), V2(i,3)]);
   C(i,12) = lcm([V1(i,5), V2(i,3)]);
end
//c = 65 x 12 matrix
//Make 3 copies of denominator matrix
for r = 1:3:34
   for s = 1:65
      Denom(s,r) = C(s, (r+2)/3)
      Denom(s, r+1) = C(s, (r+2)/3)
      Denom(s, r+2) = C(s, (r+2)/3)
   end
end
//Denom = 65x36 matrix
//Choose and add suitable f's
B3 = [2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
2 2 2;
2 0 0 2 0 0 0 0 2 0 0 2 2 0 0 2 0 0 0 0 2 0 0 2 2 0 0 2 0 0 0 0 2 0 0 2;
0 0 2 0 0 2 2 0 0 2 0 0 0 0 2 0 0 2 2 0 0 2 0 0 0 0 2 0 0 2 2 0 0 2 0 0;
0 0 0 0 0 0 0 0 0 0 0 0 0 2 2 2 2 0 0 2 2 2 2 0 0 2 2 2 2 0 0 2 2 2 2 0;
0 2 2 2 2 0 0 2 2 2 2 0 0 0 0 0 0 0 0 0 0 0 0 0 2 2 0 0 2 2 2 2 0 0 2 2;
2 2 0 0 2 2 2 2 0 0 2 2 2 2 0 0 2 2 2 2 0 0 2 2 0 0 0 0 0 0 0 0 0 0 0 0]
X = M*B3
//X = 65x36 matrix
//Make product matrix of Denom and X
for k = 1:65
   for l = 1:36
      Prod(k, l) = X(k, l) * Denom(k, l)
   end
```

```
end
//Prod = 65x36 matrix
//Check integer by making floor matrix
Res = Prod − floor(Prod);
for a = 1:65
   for b = 1:36
      if (Res(a,b) == 0) then ChecIn(a,b) = 1
         else ChecIn(a,b) = 0
      end
   end
end
//ChecIn = 65x36 matrix
for c = 1:65
   for d = 1:36
      if(ChecIn(c,d) == 0) then printf("%d, %d\n", c, d)
      end
   end
end
printf(''end of check\n")
```

Appendix C.

Tabulated below is the output of a Scilab program which calculates the determinant $|V - tV^T|$ where V is the Seifert matrix of the (hypothetical) tst link where $\mu PN(\tau)$ and $\mu PD(\tau)$ are as given and M is odd. The program codes are as the follows:

```
t=poly(0, 't')
function [A]=block(r)
 A=zeros(r,r)
  for i=1:r
   for j = 1:r
    if i==j then A(i,j)=1
    elseif j==i+1 then A(i,j)=-1
    end
   end
  end
endfunction
function [B]=altzeros(b,a)
 C=-block((a-1)/2)
 B=zeros(b,a)
 for k=1:(a-1)/2
  for l=2:2:(a-1)
   B(k,l)=C(k, (1/2))
  end
 end
endfuction
function [M]=neg_tsmat(a,b)
 D=block(a-1)
 M=zeros((a-1)*(b-1), (a-1)*(b-1))
 for i=1:(b-1)
M((a-1)*(i-1)+1:(a-1)*i, (a-1)*(i-1)+1:(a-1)*i)=D
 end
 for i=1:(b-2)
M((a-1)*i+1:(a-1)*(i+1), (a-1)*(i-1)+1:(a-1)*i)=-D
 end
 //printf(''(-%d,%d)_torus knot has Seifert
matrix:\n",a,b)
endfunction
function [E]=alexpoly(u,s)
 printf(''mil P N(tau)=%d mil P D(tau)=%d\n",u,s)
 D=[block(2*u-1) zeros((2*u-1),(u-1)*(s-1));
 altzeros((u-1)*(s-1),(2*u-1)) neg_tsmat(u,s)]
 printf(''The Seifert matrix is:\n")
 disp(D)
 E=det(D-t*D')
endfunction
```

This table has been used to guess the formula for the Alexander polynomial of the link $[\Phi(n(\tau), d(\tau), M)]$ where M is odd.

$\mu PN(\tau)$ / $\mu PD(\tau)$	2	3	4	5	6
2	$1-t-t^3+t^4$	$1-t+t^3-t^4+t^6-t^7$	$1-t+t^3-t^4-t^6+t^7-t^9+t^{10}$	$1-t+t^3-t^4+t^6-t^7+t^9-t^{10}+t^{12}-t^{13}$	$1-t+t^3-t^4+t^6-t^7-t^9+t^{10}-t^{12}+t^{13}-t^{15}+t^{16}$
3	$1-t+t^4-t^5$	$1-t-2t^4+2t^5+t^8-t^9$	$1-t+t^4-t^5+t^8-t^9+t^{12}-t^{13}$	$1-t+t^4-t^5+t^8-t^9+t^{12}-t^{13}+t^{16}-t^{17}$	$1-t+t^4-t^5-2t^8+2t^9-2t^{12}+2t^{13}+t^{16}-t^{17}+t^{20}-t^{21}$
4	$1-t-t^5+t^6$	$1-t+t^5-t^6+t^{10}-t^{11}$	$1-t-3t^5+3t^6+3t^{10}-3t^{11}-t^{15}+t^{16}$	$1-t+t^5-t^6+t^{10}-t^{11}+t^{15}-t^{16}+t^{20}-t^{21}$	$1-t+t^5-t^6+t^{10}-t^{11}-t^{15}+t^{16}-t^{20}+t^{21}-t^{25}+t^{26}$
5	$1-t+t^6-t^7$	$1-t+t^6-t^7+t^{12}-t^{13}$	$1-t+t^6-t^7+t^{12}-t^{13}+t^{18}-t^{19}$	$1-t-4t^6+4t^7+6t^{12}-6t^{13}-4t^{18}+4t^{19}+t^{24}-t^{25}$	$1-t+t^6-t^7+t^{12}-t^{13}+t^{18}-t^{19}+t^{24}-t^{25}+t^{30}-t^{31}$
6	$1-t-t^7+t^8$	$1-t-2t^7+2t^8+t^{14}-t^{15}$	$1-t+t^7-t^8-t^{14}+t^{15}-t^{21}+t^{22}$	$1-t+t^7-t^8+t^{14}-t^{15}+t^{21}-t^{22}+t^{28}-t^{29}$	$1-t-5t^7+5t^8+10t^{14}-10t^{15}-10t^{21}+10t^{22}+5t^{28}-5t^{29}-t^{35}+t^{36}$
7	$1-t+t^8-t^9$	$1-t+t^8-t^9+t^{16}-t^{17}$	$1-t+t^8-t^9+t^{16}-t^{17}+t^{24}-t^{25}$	$1-t+t^8-t^9+t^{16}-t^{17}+t^{24}-t^{25}+t^{32}-t^{33}$	$1-t+t^8-t^9+t^{16}-t^{17}+t^{24}-t^{25}+t^{32}-t^{33}+t^{40}-t^{41}$
8	$1-t-t^9+t^{10}$	$1-t+t^9-t^{10}+t^{18}-t^{19}$	$1-t-3t^9+3t^{10}+3t^{18}-3t^{19}-t^{27}+t^{28}$	$1-t+t^9-t^{10}+t^{18}-t^{19}+t^{27}-t^{28}+t^{36}-t^{37}$	$1-t+t^9-t^{10}+t^{18}-t^{19}-t^{27}+t^{28}-t^{36}+t^{37}-t^{45}+t^{46}$
9	$1-t+t^{10}-t^{11}$	$1-t-2t^{10}+2t^{11}+t^{20}-t^{21}$	$1-t+t^{10}-t^{11}+t^{20}-t^{21}+t^{30}-t^{31}$	$1-t+t^{10}-t^{11}+t^{20}-t^{21}+t^{30}-t^{31}+t^{40}-t^{41}$	$1-t+t^{10}-t^{11}-2t^{20}+2t^{21}-2t^{30}+2t^{31}+t^{40}-t^{41}+t^{50}-t^{51}$
10	$1-t-t^{11}+t^{12}$	$1-t+t^{11}-t^{12}+t^{22}-t^{23}$	$1-t+t^{11}-t^{12}-t^{22}+t^{23}-t^{33}+t^{34}$	$1-t-4t^{11}+4t^{12}+6t^{22}-6t^{23}-4t^{33}+4t^{34}+t^{44}-t^{45}$	$1-t+t^{11}-t^{12}+t^{22}-t^{23}-t^{33}+t^{34}-t^{44}+t^{45}-t^{55}+t^{56}$

$\mu PN(\tau)$ / $\mu PD(\tau)$	7	8	9	10
2	$1-t+t^3-t^4+t^6-t^7+t^9-t^{10}+t^{12}-t^{13}+t^{15}-t^{16}+t^{18}-t^{19}$	$1-t+t^3-t^4+t^6-t^7+t^9-t^{10}-t^{12}+t^{13}-t^{15}+t^{16}-t^{18}+t^{19}-t^{21}+t^{22}$	$1-t+t^3-t^4+t^6-t^7+t^9-t^{10}+t^{12}-t^{13}+t^{15}-t^{16}+t^{18}-t^{19}+t^{21}-t^{22}+t^{24}-t^{25}$	$1-t+t^3-t^4+t^6-t^7+t^9-t^{10}+t^{12}-t^{13}-t^{15}+t^{16}-t^{18}+t^{19}-t^{21}+t^{22}-t^{24}+t^{25}-t^{27}+t^{28}$
3	$1-t+t^4-t^5+t^8-t^9+t^{12}-t^{13}+t^{16}-t^{17}+t^{20}-t^{21}+t^{24}-t^{25}$	$1-t+t^4-t^5+t^8-t^9+t^{12}-t^{13}+t^{16}-t^{17}+t^{20}-t^{21}+t^{24}-t^{25}+t^{28}-t^{29}$	$1-t+t^4-t^5+t^8-t^9-2t^{12}+2t^{13}-2t^{16}+2t^{17}-2t^{20}+2t^{21}+t^{24}-t^{25}+t^{28}-t^{29}+t^{32}-t^{33}$	$1-t+t^4-t^5+t^8-t^9+t^{12}-t^{13}+t^{16}-t^{17}+t^{20}-t^{21}+t^{24}-t^{25}+t^{28}-t^{29}+t^{32}-t^{33}+t^{36}-t^{37}$
4	$1-t+t^5-t^6+t^{10}-t^{11}+t^{15}-t^{16}+t^{20}-t^{21}+t^{25}-t^{26}+t^{30}-t^{31}$	$1-t+t^5-t^6-3t^{10}+3t^{11}-3t^{15}+3t^{16}+3t^{20}-3t^{21}+3t^{25}-3t^{26}-t^{30}+t^{31}-t^{35}+t^{36}$	$1-t+t^5-t^6+t^{10}-t^{11}+t^{15}-t^{16}+t^{20}-t^{21}+t^{25}-t^{26}+t^{30}-t^{31}+t^{35}-t^{36}+t^{40}-t^{41}$	$1-t+t^5-t^6+t^{10}-t^{11}+t^{15}-t^{16}+t^{20}-t^{21}-t^{25}+t^{26}-t^{30}+t^{31}-t^{35}+t^{36}-t^{40}+t^{41}-t^{45}+t^{46}$
5	$1-t+t^6-t^7+t^{12}-t^{13}+t^{18}-t^{19}+t^{24}-t^{25}+t^{30}-t^{31}+t^{36}-t^{37}$	$1-t+t^6-t^7+t^{12}-t^{13}+{}^{18}t-t^{19}+t^{24}-t^{25}+t^{30}-t^{31}+t^{36}-t^{37}+t^{42}-t^{43}$	$1-t+t^6-t^7+t^{12}-t^{13}+t^{18}-t^{19}+t^{24}-t^{25}+t^{30}-t^{31}+t^{36}-t^{37}+t^{42}-t^{43}+t^{48}-t^{49}$	$1-t+t^6-t^7-4t^{12}+4t^{13}-4t^{18}+4t^{19}+6t^{24}-6t^{25}+6t^{30}-6t^{31}-4t^{36}+4t^{37}-4t^{42}+4t^{43}+t^{48}-t^{49}+t^{54}-t^{55}$
6	$1-t+t^7-t^8+t^{14}-t^{15}+t^{21}-t^{22}+t^{28}-t^{29}+t^{35}-t^{36}+t^{42}-t^{43}$	$1-t+t^7-t^8+t^{14}-t^{15}+t^{21}-t^{22}-t^{28}+t^{29}-t^{35}+t^{36}-t^{42}+t^{43}-t^{49}+t^{50}$	$1-t+t^7-t^8+t^{14}-t^{15}-2t^{21}+2t^{22}-2t^{28}+2t^{29}-2t^{35}+2t^{36}+t^{42}-t^{43}+t^{49}-t^{50}+t^{56}-t^{57}$	$1-t+t^7-t^8+t^{14}-t^{15}+t^{21}-t^{22}+t^{28}-t^{29}-t^{35}+t^{36}-t^{42}+t^{43}-t^{49}+t^{50}-t^{56}+t^{57}-t^{63}+t^{64}$
7	$1-t-6t^8+6t^9+15t^{16}-15t^{17}-20t^{24}+20t^{25}+15t^{32}-15t^{33}-6t^{40}+6t^{41}+t^{48}-t^{49}$	$1-t+t^8-t^9+t^{16}-t^{17}+t^{24}-t^{25}+t^{32}-t^{33}+t^{40}-t^{41}+t^{48}-t^{49}+t^{56}-t^{57}$	$1-t+t^8-t^9+t^{16}-t^{17}+t^{24}-t^{25}+t^{32}-t^{33}+t^{40}-t^{41}+t^{48}-t^{49}+t^{56}-t^{57}+t^{64}-t^{65}$	$1-t+t^8-t^9+t^{16}-t^{17}+t^{24}-t^{25}+t^{32}-t^{33}+t^{40}-t^{41}+t^{48}-t^{49}+t^{56}-t^{57}+t^{64}-t^{65}+t^{72}-t^{73}$
8	$1-t+t^9-t^{10}+t^{18}-t^{19}+t^{27}-t^{28}+t^{36}-t^{37}+t^{45}-t^{46}+t^{54}-t^{55}$	$1-t-7t^9+7t^{10}+21t^{18}-21t^{19}-35t^{27}+35t^{28}+35t^{36}-35t^{37}-21t^{45}+21t^{46}+7t^{54}-7t^{55}-t^{63}+t^{64}$	$1-t+t^9-t^{10}+t^{18}-t^{19}+t^{27}-t^{28}+t^{36}-t^{37}+t^{45}-t^{46}+t^{54}-t^{55}+t^{63}-t^{64}+t^{72}-t^{73}$	$1-t+t^9-t^{10}+t^{18}-t^{19}+t^{27}-t^{28}+t^{36}-t^{37}-t^{45}+t^{46}-t^{54}+t^{55}-t^{63}+t^{64}-t^{72}+t^{73}-t^{81}+t^{82}$

9	$1-t+t^{10}-t^{11}+t^{20}-t^{21}+t^{30}-t^{31}+t^{40}-t^{41}+t^{50}-t^{51}+t^{60}-t^{61}$	$1-t+t^{10}-t^{11}+t^{20}-t^{21}+t^{30}-t^{31}+t^{40}-t^{41}+t^{50}-t^{51}+t^{60}-t^{61}+t^{70}-t^{71}$	$1-t-8t^{10}+8t^{11}+28t^{20}-28t^{21}-56t^{30}+56t^{31}+70t^{40}-70t^{41}-56t^{50}+56t^{51}+28t^{60}-28t^{61}-8t^{70}+8t^{71}+t^{80}-t^{81}$	$1-t+t^{10}-t^{11}+t^{20}-t^{21}+t^{30}-t^{31}+t^{40}-t^{41}+t^{50}-t^{51}+t^{60}-t^{61}+t^{70}-t^{71}+t^{80}-t^{81}+t^{90}-t^{91}$
10	$1-t+t^{11}-t^{12}+t^{22}-t^{23}+t^{33}-t^{34}+t^{44}-t^{45}+t^{55}-t^{56}+t^{66}-t^{67}$	$1-t+t^{11}-t^{12}+t^{22}-t^{23}+t^{33}-t^{34}-t^{44}+t^{45}-t^{55}+t^{56}-t^{66}+t^{67}-t^{77}+t^{78}$	$1-t+t^{11}-t^{12}+t^{22}-t^{23}+t^{33}-t^{34}+t^{44}-t^{45}+t^{55}-t^{56}+t^{66}-t^{67}+t^{77}-t^{78}+t^{88}-t^{89}$	$1-t-9t^{11}+9t^{12}+36t^{22}-36t^{23}-84t^{33}+84t^{34}+126t^{44}-126t^{45}-126t^{55}+126t^{56}+84t^{66}-84t^{67}-36t^{77}+36t^{78}+9t^{88}-9t^{89}-t^{99}+t^{100}$

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Reviewer's Comments

In this paper, the authors generalized the well-known question of cutting a Mobius strip in half to that of cutting a twisted solid torus in different ways. The resultant links were studied. General forms of their braid words, Seifert matrices, Alexander polynomials and some other results were deduced.

The paper is very rich in content and contains a lot of details. With the length of the paper, the authors may consider stating some of the main results in the introduction. This should help the readers to understand the paper more easily.