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DECRYPTING FIBONACCI AND LUCAS SEQUENCES

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Abstract. With the aim of finding new alternatives to resolve large Fibonacci or Lucas numbers, we have immersed ourselves in these two sequences to find that there are other fascinating phenomena about them. We have, in the first part of the report, successfully discovered four new methods to resolve large Fibonacci and Lucas numbers. From the very beginning, we have decided to adopt the normal investigation approach: observe, hypothesize, and then prove. The first two methods were discovered. Then we move on and try to explore these sequences in the two dimensional world. From the tables and triangles thereby created, we have discovered various surprising patterns which then help us generate the third and fourth formula to resolve large Fibonacci and Lucas numbers. In the second part of the report, we have focused on sequences in two dimensions and discovered many amazing properties about them.

1. Introduction

The Fibonacci and Lucas Sequences

To generate the Fibonacci sequence, start with 1 and then another 1. Afterwards, add up the previous two numbers to get the next. So the third term is found by adding 1 to 1: $1 + 1 = 2$; the fourth term $1 + 2 = 3$, the fifth $2 + 3 = 5$. This gives the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... In this report, we denote the Fibonacci sequence with $F(n)$.

The Lucas sequence, likewise, is generated by adding the last two numbers to get the next. The only difference with the Fibonacci sequence is that it starts with 1 and then 3. Hence the sequence is 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ... In this report, we shall denote the Lucas sequence with $L(n)$.

[1]This work is done under the supervision of the authors' teacher, Ms. Yau-Man Sum.

Apart from the above specific sequences, you will also find some recurrence sequences in the project. Recurrence sequences are sequences that satisfy the following relation: $U(n)+U(n+1) = U(n+2)$ in which the two starting terms are denoted by $U(1)$ and $U(2)$. Therefore, $F(n)$ and $L(n)$ are actually examples of $U(n)$.

Note that in this project we would like to focus on $U(n)$ with **positive integers** n. Therefore, the proofs written in this project will only involve $U(n)$ with positive integers n. However, in certain areas, we still have to deal with $U(0)$ and even $U(n)$ with negative integers n.

The History and Background of the Fibonacci and Lucas Sequences

The Fibonacci sequence is a sequence of numbers first created by Leonardo Fibonacci in 1202. He considers the growth of an idealized (biologically unrealistic) rabbit population, assuming that:

(1) in the first month there is just one newly-born pair,
(2) new-born pairs become fertile from their second month on each month,
(3) every fertile pair begets a new pair, and
(4) the rabbits never die.

Let the population at month n be $F(n)$. At this time, only rabbits which were alive in month $(n-2)$ are fertile and produce offspring, so $F(n-2)$ pairs are added to the current population of $F(n-1)$. Thus the total is $F(n) = F(n-1) + F(n-2)$, which gives us the definition of the Fibonacci sequence[1].

Edouard Lucas is best known for his results in number theory: the Lucas sequence is named after him[2].

Known Formulae to Solve Fibonacci and Lucas Numbers

Binet's Formula concerning Fibonacci numbers[3]

$$F(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Successor Formula concerning Fibonacci numbers[3]

$$F(n+1) = \left[\frac{F(n)(1+\sqrt{5})+1}{2} \right]$$

where $[x]$ means the greatest integer smaller than x.

"Analog" of Binet's Formula concerning Lucas numbers[4]

$$L(n) = \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Successor Formula concerning Lucas numbers[4]

$$L(n+1) = \left[\frac{L(n)(1+\sqrt{5})+1}{2} \right] \text{ when } n \geqslant 4$$

where $[x]$ means the greatest integer smaller than x.

Our Aims and Objectives in Doing This Project

The spirit of Mathematics is to try, to believe and to improve. Although there are several methods of finding the general term, there is always an alternative and perhaps simpler way to find the general term. In sight of this, we have tried and successfully found out various other methods in finding large Fibonacci and Lucas numbers. In addition, even large numbers in a sequence following the property of $U(n)+U(n+1) = U(n+2)$ can be found.

Methodology

In this report, the Mathematical Induction, a useful tool for proving hypotheses, is commonly used.

General Organization and a Brief Summary of the Report

We have divided the report into ten sections. Here, we have mainly adopted the normal investigation procedure: **observations** → **generalization** → **hypothesis** → **proof**. In the end we try to **apply** our findings to help us solve large Fibonacci and Lucas numbers.

To facilitate our discussion, we have set up a naming system. It consists of:

(1) the category the discussion belongs to

(2) the numbering of the piece of discussion

The categories of pieces of discussion are **Observation**, **Hypothesis**, **Formula** and **Application**.

The numbering system consists of two parts, namely the section number and the point number.

For example, **Hypothesis 2.19** means that it is a piece of discussion related to generalization of pattern, leading to **Hypothesis**. Also, **2.19** indicates that it is the **19^{th}** piece of discussion in section **2**. There is one more point to note, in order to ensure a smooth presentation, *once a **Hypothesis** is proved, it becomes a **Formula**.* In most of the cases, we have the details of proofs for hypotheses placed in Appendix E.

In sections 2−6, we will discuss various ways to find out large Fibonacci and Lucas numbers. We have managed to put certain patterns or findings into tables or triangles and from these tables we have discovered a number of surprising observations and relations. In section 7, we will discuss more properties about the tables we have constructed in this project.

Most people tend to think that there is nothing special about these two sequences; they are merely about addition and probably lead to exhaustion. However, in this project, we will present to you a brand new view of these sequences and show you the hidden magic behind the two numbers.

Why have we chosen to look into these two sequences?

Why have we chosen to look into the Fibonacci and Lucas sequences? What leads us onto this journey of research and discovery?

It was about 10 years ago. A friend challenged Theodore, one of our teammates, "Let me give you an interesting mathematical problem. Here, we have a sequence: 1, 1, 2, 3, 5, 8, . . . in which the next term is generated by adding the two previous terms. Now, can you tell me the 100^{th} term?"

At that point in time, it goes without saying, Theodore failed to come up with the answer. When he told us about this experience of his, it inspired us to work on this problem.

Throughout the history of Mathematics, many mathematicians have indeed been curiously absorbed in the investigation of special numbers, from the largest prime number to the largest perfect number. Before doing any investigations in these large numbers, we need to evaluate them. Christopher Clavius, an Italian astronomer and mathematician in the 16th century, provided a new way of calculation of product of two enormous numbers in a short time. Can we do something similar? Is there a convenient way to evaluate large Lucas or Fibonacci numbers with pen and paper only?

Immerse yourself in these two sequences and you will soon realize, as we do, how diversified, exciting, special and magical these numbers become.

2. $U(n)$ Formula in 3 unknowns

In this section, we shall develop a formula which can be applied to any recurrence sequence that satisfies the following rule: $U(n)+U(n+1) = U(n+2)$. The Fibonacci and Lucas sequences are two examples of $U(n)$. This new formula generated can help us solve not only the large numbers in the Fibonacci and Lucas sequences, but also those in other $U(n)$ sequences.

Before introducing our discovery, for the sake of convenience, we have put down the first 16 Fibonacci and Lucas numbers (including $F(0)$ and $L(0)$) in the form of a table so that it is easier to refer to. In the Appendices, there are two tables of Fibonacci and Lucas numbers that goes up to the 100^{th} term for your reference as well.

Table 2.1. The first 16 Fibonacci numbers

n	0	1	2	3	4	5	6	7
$F(n)$	0	1	1	2	3	5	8	13

n	8	9	10	11	12	13	14	15
$F(n)$	21	34	55	89	144	233	377	610

Table 2.2. The first 16 Lucas numbers

n	0	1	2	3	4	5	6	7
$L(n)$	2	1	3	4	7	11	18	29

n	8	9	10	11	12	13	14	15
$L(n)$	47	76	123	199	322	521	843	1364

Here $U(n)$ denotes a sequence that always satisfies the following equation: $U(n)+U(n+1)=U(n+2)$. To use a specific case for better understanding and presentation of our observation, we will create a new sequence, $U_1(n)$, with 4 and 5 as $U_1(1)$ and $U_1(2)$ respectively.

Table 2.3. The first 15 $U_1(n)$ numbers(cf. **Reviewer's Comments** 1)

n	1	2	3	4	5	6	7	8
$U_1(n)$	4	5	9	14	23	37	60	107

n	9	10	11	12	13	14	15	
$U_1(n)$	167	274	441	715	1156	1871	3027	

The Lucas sequence will be examined first and $L(11)$ and $L(14)$ will be analysed in detail.

Observation 2.4.

$$\begin{aligned} L(11) &= 199 \\ &= 2\times 123 - 47 && = 3\times 76 - 29 \\ &= 2L(10) - 1L(8) && = 3L(9) - 1L(7) \\ \\ &= 5\times 47 - 2\times 18 && = 8\times 29 - 3\times 11 \\ &= 5L(8) - 2L(6) && = 8L(7) - 3L(5) \end{aligned}$$

$$\begin{aligned} L(14) &= 843 \\ &= 2\times 521 - 199 && = 3\times 322 - 123 \\ &= 2L(13) - 1L(11) && = 3L(12) - 1L(10) \\ \\ &= 5\times 199 - 2\times 76 && = 8\times 123 - 3\times 47 \\ &= 5L(11) - 2L(9) && = 8L(10) - 3L(8) \end{aligned}$$

So far the coefficients of $L(n)$ remind us of the Fibonacci sequence of $\{1,1,2,3,5,\ldots\}$. Is it merely a coincidence or is there more behind the scene?

To generalize the findings, we have

Hypothesis 2.5.

$$L(n) = F(r+2)L(n-r) - F(r)L(n-r-2)$$

where $n > r+2$ and r is any positive integer.

Let us consider the Fibonacci sequence this time. We shall focus on $F(10)$ and $F(13)$.

Observation 2.6.

$$\begin{aligned} F(10) &= 55 \\ &= 2 \times 34 - 13 && = 3 \times 21 - 8 \\ &= 2F(9) - 1F(7) && = 3F(8) - 1F(6) \\ \\ &= 5 \times 13 - 2 \times 5 && = 8 \times 8 - 3 \times 3 \\ &= 5F(7) - 2F(5) && = 8F(6) - 3F(4) \end{aligned}$$

$$\begin{aligned} F(13) &= 233 \\ &= 2 \times 144 - 55 && = 3 \times 89 - 34 \\ &= 2F(12) - 1F(10) && = 3F(11) - 1F(9) \\ \\ &= 5 \times 55 - 2 \times 21 && = 8 \times 34 - 3 \times 13 \\ &= 5F(10) - 2F(8) && = 8F(9) - 3F(7) \end{aligned}$$

To generalize the findings, we have

Hypothesis 2.7.

$$F(n) = F(r+2)F(n-r) - F(r)F(n-r-2)$$

where $n > r+2$ and r is any positive integer.

In the examination of the following sequence, $U_1(n)$, $U_1(5)$ and $U_1(7)$ are used.

Observation 2.8.

$$\begin{aligned} U_1(5) &= 23 \\ &= 2 \times 14 - 5 && = 3 \times 9 - 4 \\ &= 2U_1(4) - 1U_1(2) && = 3U_1(3) - 1U_1(1) \end{aligned}$$

$$\begin{aligned} U_1(7) &= 60 \\ &= 2 \times 37 - 14 && = 3 \times 23 - 9 \\ &= 2U_1(6) - 1U_1(4) && = 3U_1(5) - 1U_1(3) \\ \\ &= 5 \times 14 - 2 \times 5 \\ &= 5U_1(4) - 2U_1(2) \end{aligned}$$

To generalize the findings, we have

Hypothesis 2.9.

$$U_1(n) = F(r+2)U_1(n-r) - F(r)U_1(n-r-2)$$

Have you noticed the similarity of the three hypotheses we have made?

Hypothesis 2.5. $L(n) = F(r+2)L(n-r) - F(r)L(n-r-2)$
Hypothesis 2.7. $F(n) = F(r+2)F(n-r) - F(r)F(n-r-2)$
Hypothesis 2.9. $U_1(n) = F(r+2)U_1(n-r) - F(r)U_1(n-r-2)$

No matter what the generating numbers of the sequence are, ($\{1,1\}$ in Fibonacci sequence, $\{1,3\}$ in Lucas sequence or $\{4,5\}$ in the sequence we have made up $\{U_1(n)\}$) as long as it follows the rule of $U(n)+U(n+1) = U(n+2)$, that is the previous two terms adding up to form the next term, it seems that it satisfies the equation of

Hypothesis 2.10.

$$U(n) = F(r+2)U(n-r) - F(r)U(n-r-2)$$

Details of Proof for Hypothesis 2.10 can be found in Appendix E.

Observation 2.11.

$$
\begin{aligned}
U_1(5) &= 23 \\
&= 14 + 9 && = 2 \times 9 + 5 \\
&= U_1(4) + U_1(3) && = 2U_1(3) + U_1(2)
\end{aligned}
$$

$$
\begin{aligned}
U_1(7) &= 60 \\
&= 37 + 23 && = 2 \times 23 + 14 \\
&= U_1(6) + U_1(5) && = 2U_1(5) + U_1(4) \\
\\
&= 3 \times 14 + 2 \times 9 && = 5 \times 9 + 3 \times 5 \\
&= 3U_1(4) + 2U_1(3) && = 5U_1(3) + 3U_1(2)
\end{aligned}
$$

To generalize the findings, we have

Hypothesis 2.12.

$$U_1(n) = F(r+1)U_1(n-r) + F(r)U_1(n-r-1)$$

From Hypothesis 2.12, another hypothesis is derived, by replacing all $U_1(n)$ with $U(n)$.

Hypothesis 2.13.

$$U(n) = F(r+1)U(n-r) + F(r)U(n-r-1)$$

Details of Proof for Hypothesis 2.13 can be found in Appendix E.

There are only minor differences between Formula 2.10 and Formula 2.13. To have a better understanding of the relationships among $U(n)$, $F(n)$ and r, $U_1(5)$ and $U_1(7)$ are considered again.

Observation 2.14.

$$\begin{aligned} U_1(5) &= 23 \\ &= \frac{46}{2} \\ &= \frac{3\times 14+4}{2} \\ &= \frac{3U_1(4)+U_1(1)}{2} \end{aligned}$$

$$\begin{aligned} U_1(7) &= 60 = \frac{120}{2} \\ &= \frac{3\times 37+9}{2} &&= \frac{5\times 23+5}{2} \\ &= \frac{3U_1(6)+U_1(3)}{2} &&= \frac{5U_1(5)+U_1(2)}{2} \\ \\ &= \frac{8\times 14+2\times 4}{2} \\ &= \frac{8U_1(4)+2U_1(1)}{2} \end{aligned}$$

This seems a bit complicated. However, by reorganizing the terms, another hypothesis can be made.

$$2U_1(5) = 3U_1(4) + 1U_1(1) \quad \begin{aligned} 2U_1(7) &= 3U_1(6) + 1U_1(3) \\ &= 5U_1(5) + 1U_1(2) \\ &= 8U_1(4) + 2U_1(1) \end{aligned}$$

We make a hypothesis that:

Hypothesis 2.15.

$$2U_1(n) = F(r+3)U_1(n-r) + F(r)U_1(n-r-3)$$

But why is it that $U_1(n)$, this time, is multiplied by 2? Perhaps we shall look into $U_1(7)$ again.

Observation 2.16.

$$\begin{aligned} U(7) = 60 &= \frac{180}{3} \\ &= \frac{5 \times 37 - 5}{3} \qquad = \frac{8 \times 23 - 4}{3} \\ &= \frac{5U(6) - U(2)}{3} \qquad = \frac{8U(5) - U(1)}{3} \end{aligned}$$

To generalize the findings, we have

Hypothesis 2.17.

$$3U_1(n) = F(r+4)U_1(n-r) - F(r)U_1(n-r-4)$$

Observation 2.18. Now we have:

$$\begin{aligned} 1U(n) &= F(r+1)U(n-r) + F(r)U(n-r-1) \qquad (1) \\ 1U(n) &= F(r+2)U(n-r) - F(r)U(n-r-2) \qquad (2) \\ 2U(n) &= F(r+3)U(n-r) + F(r)U(n-r-3) \\ 3U(n) &= F(r+4)U(n-r) - F(r)U(n-r-4) \end{aligned}$$

where (1) and (2) are proved.

For the coefficients of $U(n)$, i.e. 1, 1, 2, 3, ..., they remind us of the Fibonacci sequence. We will do some little changes to the coefficients:

$$\begin{aligned} F(1)U(n) &= F(r+1)U(n-r) + F(r)U(n-r-1) \\ F(2)U(n) &= F(r+2)U(n-r) - F(r)U(n-r-2) \\ F(3)U(n) &= F(r+3)U(n-r) + F(r)U(n-r-3) \\ F(4)U(n) &= F(r+4)U(n-r) - F(r)U(n-r-4) \end{aligned}$$

Is it easier for you to observe the pattern now?

$$\begin{aligned} F(1)U(n) &= F(r+1)U(n-r) + F(r)U(n-r-1) \\ F(2)U(n) &= F(r+2)U(n-r) - F(r)U(n-r-2) \\ F(3)U(n) &= F(r+3)U(n-r) + F(r)U(n-r-3) \\ F(4)U(n) &= F(r+4)U(n-r) - F(r)U(n-r-4) \end{aligned}$$

To generalize the findings, we have

Hypothesis 2.19.

$$F(k)U(n) = F(r+k)U(n-r) + (-1)^{k+1}F(r)U(n-r-k)$$

Details of Proof for Hypothesis 2.19 can be found in Appendix E.

By careful observation of the relationship between numbers in the Lucas and Fibonacci sequences, a handful of hypotheses and assumptions have been made. In the end, a new formula is generated and successfully proved. This discovery leads us a lot closer to our aim of resolving large Lucas or Fibonacci numbers. We can now use the formula to help us find out large $U(n)$ (including Lucas and Fibonacci numbers). However, before applying this formula, we must plan carefully as a misuse of this formula will only make things even more complicated.

Let us name this formula the $U(n)$ Formula in three unknowns.

Formula 2.19.

$$F(k)U(n) = F(r+k)U(n-r) + (-1)^{k+1}F(r)U(n-r-k)$$

In this formula, n should be a given number and we should choose appropriate k and r to use.

Replace $U(n)$ by $F(n)$ to generate formula for solving large $F(n)$.

Formula 2.20.

$$F(k)F(n) = F(r+k)F(n-r) + (-1)^{k+1}F(r)F(n-r-k)$$

Replace $U(n)$ by $L(n)$ to generate formula for solving large $L(n)$.

Formula 2.21.

$$F(k)L(n) = F(r+k)L(n-r) + (-1)^{k+1}F(r)L(n-r-k)$$

Having generated Formula 2.20 and Formula 2.21, we will use them to help us prove some other existing formulae as application.

Let us consider all $F(n)$ and $L(n)$ numbers with $n > 25$ large. Hence in all cases below, for $F(1)$ to $F(25)$ and $L(1)$ to $L(25)$, we will take the numbers directly from the tables in Appendix A. For $F(26)$ and $L(26)$ onwards, we will make use of the formulae we have found.

Application 2.22. We will use Formula 2.13 to help us prove two formulae:

$$F(2k) = F(k+1)^2 - F(k-1)^2$$

and

$$L(2k) = F(k+1)L(k+1) - F(k-1)L(k-1)$$

Proof. We substitute $n = 2k$, $r = k$ into Formula 2.13,

$$\begin{aligned} U(2k) &= F(k+1)U(k) + F(k)U(k-1) \\ &= F(k+1)U(k) + [F(k+1) - F(k-1)]U(k-1) \\ &= F(k+1)[U(k) + U(k-1)] - F(k-1)U(k-1) \\ &= F(k+1)U(k+1) - F(k-1)U(k-1) \end{aligned}$$ □

Formula 2.23.

$$U(2k) = F(k+1)U(k+1) - F(k-1)U(k-1)$$

For example, to find $U(50)$,

$$\begin{aligned} U(50) &= U(2 \times 25) \\ &= F(26)U(26) - F(24)U(24) \end{aligned}$$

As $U(n)$ indicates any sequence satisfying $U(n) + U(n+1) = U(n+2)$, we can express $U(n)$ in terms of $F(n)$ and $L(n)$ instead.

Therefore, we have $F(2k) = F(k+1)F(k+1) - F(k-1)F(k-1)$.

Formula 2.24.

$$F(2k) = F(k+1)^2 - F(k-1)^2$$

Consider the following example:

$$\begin{aligned} F(50) &= F(2 \times 25) \\ &= F(26)^2 - F(24)^2 \\ &= F(2 \times 13)^2 - F(2 \times 12)^2 \\ &= [F(14)^2 - F(12)^2]^2 - [F(13)^2 - F(11)^2]^2 \end{aligned}$$

At this point, we can solve $F(50)$ with the table in Appendix A, a calculator and some patience. Hence we will not tire you with the tedious calculation and will carry on with the next formula.

Let us look at Formula 2.24:

$$F(2k) = F(k+1)^2 - F(k-1)^2$$

As we all know, $a^2 - b^2 = (a+b)(a-b)$, $F(2k) = [F(k+1)+F(k-1)][F(k+1) - F(k-1)]$.

$$F(2k) = [F(k+1) + F(k-1)]F(k)$$

or

$$F(2k) = [F(k) + 2F(k-1)]F(k)$$

or

$$F(2k) = [2F(k+1) - F(k)]F(k)$$

For example,

$$F(50) = F(25)[F(26) + F(24)]$$

or

$$F(50) = F(25)[F(25) + 2F(24)]$$

or

$$F(50) = F(25)[2F(26) - F(25)]$$

Note that up to here, we can only further resolve $F(14)$ and $F(12)$, but not $F(13)$ and $F(11)$, since we do not have a formula to resolve $F(2k+1)/U(2k+1)$. We will talk about that in Application 2.26.

Replace $U(n)$ in Formula 2.23 by $L(n)$, we have

Formula 2.25.

$$L(2k) = F(k+1)L(k+1) - F(k-1)L(k-1)$$

Application 2.26. This time, we will use Formula 2.13 to help us prove

$$F(2k+1) = F(k+1)^2 + F(k)^2$$

and

$$L(2k+1) = F(k+1)L(k+1) + F(k)L(k)$$

Proof. Substitute $n = 2k+1$, $r = k$, we have

Formula 2.27.

$$U(2k+1) = F(k+1)U(k+1) + F(k)U(k)$$

Replace $U(n)$ in Formula 2.27 by $F(n)$, we have $F(2k+1) = F(k+1)F(k+1) + F(k)F(k)$.

Formula 2.28.

$$F(2k+1) = F(k+1)^2 + F(k)^2$$

(This was in fact introduced by Lucas in 1876.) □

Back to the previous example in Application 2.22, we can now resolve $F(11)$ and $F(13)$.

$$F(11) = F(2 \times 5 + 1) = F(6)^2 + F(5)^2 \qquad F(13) = F(2 \times 6 + 1) = F(7)^2 + F(6)^2$$

Replace $U(n)$ in Formula 2.27 by $L(n)$, we have

Formula 2.29.

$$L(2k+1) = F(k+1)L(k+1) + F(k)L(k)$$

Consider the following example.

$$\begin{aligned} L(51) &= F(26)L(26) + F(25)L(25) \\ &= [F(14)^2 - F(12)^2][F(14)L(14) - F(12)L(12)] \\ &\quad + [F(13)^2 + F(12)^2][F(13)L(13) + F(12)L(12)] \end{aligned}$$

This is very convenient indeed.

Actually Formula 2.27 can be derived easily from Formula 2.23.

$$\begin{aligned} L.H.S. &= U(2k+1) \\ &= U(2k+2) - U(2k) \quad \text{(by definition)} \\ &= F(k+2)U(k+2) - F(k)U(k) \\ &\quad - F(k+1)U(k+1) + F(k-1)U(k-1) \\ &= [F(k+1) + F(k)]U(k+2) - F(k+1)U(k+1) \\ &\quad - F(k)U(k) + F(k-1)U(k-1) \\ &= F(k+1)U(k+2) - F(k+1)U(k+1) \\ &\quad + F(k)U(k+2) - F(k)U(k) + F(k-1)U(k-1) \\ &= F(k+1)U(k) + F(k)U(k+1) + F(k-1)U(k-1) \\ &= F(k+1)U(k) + F(k)U(k)+ \\ &\quad F(k)U(k-1) + F(k-1)U(k-1) \\ &= F(k+1)U(k) + F(k)U(k) + F(k+1)U(k-1) \\ &= F(k+1)U(k+1) + F(k)U(k) \\ &= R.H.S. \end{aligned}$$

Application 2.30. What is $L(299) - L(113)$?

By the formulae we have known already,

$$\begin{aligned}
&L(299) - L(113) \\
&= F(150)L(150) + F(149)L(149) - F(57)L(57) - F(56)L(56)
\end{aligned}$$

This is the simplest way to solve this problem.

However, we have another approach:

$$\begin{aligned}
&L(299) - L(113) \\
&= [L(299) - L(297)] + [L(297) - L(295)] + [L(295) - L(293)] \\
&\quad + \ldots + [L(117) - L(115)] + [L(115) - L(113)] \\
&= L(298) + L(296) + L(294) + \ldots + L(114) \\
&= \boldsymbol{F(150)L(150)} - F(148)L(148) \\
&\quad + \boldsymbol{F(149)L(149)} - F(147)L(147) + F(148)L(148) \\
&\quad - F(146)L(146) + F(147)L(147) - F(145)L(145) \\
&\quad + \ldots + F(59)L(59) - F(57)L(57) \\
&\quad + F(58)L(58) - \boldsymbol{F(56)L(56)} \\
&= F(150)L(150) + F(149)L(149) - F(57)L(57) - F(56)L(56)
\end{aligned}$$

The answer is the same for both approaches. However, this method shows how we can apply Formula 2.25 to solve this problem.

In this section, the most complicated formula we have got is **Formula 2.19**

$$F(k)U(n) = F(r + k)U(n - r) + (-1)^{k+1}F(r)U(n - r - k)$$

as it has 3 unknowns.

Formula 2.19 is the most complicated and yet perhaps the most useful formula in the section. We should handle the three unknowns in it with great care. If the right numbers are inserted into the unknowns, we can come up with the answer in a few steps. On the other hand, if we insert the numbers randomly, we risk making things even more complicated. To further illustrate our point, consider $U(400)$ and substitute different sets of numbers to it.

Method I

Substitute $n = 400$, $k = 3$, $r = 198$ into Formula 2.19, we have

$$\begin{aligned} F(3)U(400) &= F(198+3)U(400-198) \\ &\quad + (-1)^{3+1}F(198)U(400-198-3) \\ U(400) &= \frac{F(201)U(202)+F(198)U(199)}{2} \end{aligned}$$

Method II

Substitute $n = 400$, $k = 300$, $r = 200$ into Formula 2.19, we have

$$F(300)\,\boldsymbol{U(400)} = F(500)U(200) - F(200)U(-100)$$

Now, as you can see, we have made the problem even more complicated. We have to solve $F(500)$, $F(300)$ and $U(-100)$ in Method II, but in Method I, we have break down $U(400)$ into terms of $U(n)$ and $F(n)$ with n around half of 400, i.e. 200.

In conclusion, there are some tricks in applying Formula 2.19.

To find $U(n_1)$,

(1) put $n = n_1$; (as n should be the greatest value)
(2) let k be the smallest possible non-negative integer; (as we have to divide the whole thing on R.H.S. by it) and
(3) let $(r+k)$, $(n-r)$, r and $(n-r-k)$ be more or less the same as each other.

3. Polynomial Expression of $L(kn)$ in Terms of $L(n)$

In the expansion of $(x+1)^4$, the coefficients of powers of x are 1, 4, 6, 4, 1. In the expression of $L(4n)$ in terms of $L(n)$, the coefficients of powers of $L(n)$ are 1, 5, 9, 7, 2. Why are we drawing comparison between these two strings of numbers that do not seem to have anything in common? In fact, we will show you how these two strings of numbers are closely related in this section.

When we look into the Lucas sequence, we can in fact find out some special relations which can help us express $L(kn)$ in terms of $L(n)$.

First, we shall try to express all $L(2n)$ in terms of $L(n)$.

Observation 3.1.

$$\begin{aligned} L(2) &= 3 & L(4) &= 7 & L(6) &= 18 \\ &= 1+2 & &= 9-2 & &= 16+2 \\ &= 1^2+2 & &= 3^2-2 & &= 4^2+2 \\ &= L(1)^2+2 & &= L(2)^2-2 & &= L(3)^2+2 \end{aligned}$$

To generalize the findings, we have

Hypothesis 3.2.

$$L(2n) = L(n)^2 + (-1)^{n+1}(2)$$

Details of Proof for Hypothesis 3.2 can be found in Application 6.68.

Observation 3.3. Now, we try to see if we can resolve $L(3n)$ into $L(n)$.

$$\begin{aligned} L(3) &= 4 & L(6) &= 18 & L(9) &= 76 \\ &= 1+3 & &= 27-9 & &= 64+12 \\ &= 1^3+3(1) & &= 3^3-3(3) & &= 4^3+3(4) \\ &= L(1)^3+3L(1) & &= L(2)^3-3L(2) & &= L(3)^3+3L(3) \end{aligned}$$

To generalize the findings, we have

Hypothesis 3.4.

$$L(3n) = L(n)^3 + (-1)^{n+1}(3)L(n)$$

Note that $L(4n)$ can be reduced to $L(2n)$ and then to $L(n)$. So, we are going to investigate $L(kn)$ where k is *prime* first.

Observation 3.5.

$$\begin{aligned} L(5) &= 11 \\ &= 1^5 + 10 \\ &= 1^5 + (1)(2)(5) \\ &= L(1)^5 + L(1)[L(2) - 1](5) \end{aligned} \qquad \begin{aligned} L(10) &= 123 \\ &= 3^5 - 120 \\ &= 3^5 - (3)(8)(5) \\ &= L(2)^5 - L(2)[L(4) + 1](5) \end{aligned}$$

$$\begin{aligned} L(15) &= 1364 \\ &= 4^5 + 340 \\ &= 4^5 + (4)(17)(5) \\ &= L(3)^5 + L(3)[L(6) - 1](5) \end{aligned}$$

To generalize the findings, we have

Hypothesis 3.6.

$$\begin{aligned} L(5n) &= L(n)^5 + (-1)^{n+1}(5)L(n)[L(2n) + (-1)^n] \\ &= L(n)^5 + (-1)^{n+1}(5)L(n)[L(n)^2 + (-1)^{n+1}] \end{aligned}$$

(by $L(2n) + (-1)^n = L(n)^2 + (-1)^{n+1}(2) - (-1)^{n+1} = L(n)^2 + (-1)^{n+1}$*)*

Observation 3.7. What about $L(7n)$?

$$\begin{aligned} L(7) &= 29 \\ &= 1^7 + 28 \\ &= 1^7 + +(1)(4)(7) \\ &= L(1)^7 + L(1)[L(2) - 1]^2(7) \end{aligned} \qquad \begin{aligned} L(14) &= 843 \\ &= 3^7 - 1344 \\ &= 3^7 - (3)(64)(7) \\ &= L(2)^7 - L(2)[L(4) + 1]^2(7) \end{aligned}$$

$$\begin{aligned} L(21) &= 24476 \\ &= 4^7 + 8092 \\ &= 4^7 + (4)(289)(7) \\ & L(3)^7 + L(3)[L(6) - 1]^2(7) \end{aligned}$$

To generalize the findings, we have

Hypothesis 3.8.

$$\begin{aligned} L(7n) &= L(n)^7 + (-1)^{n+1}(7)L(n)[L(2n) + (-1)^n]^2 \\ &= L(n)^7 + (-1)^{n+1}(7)L(n)[L(n)^2 + (-1)^{n+1}]^2 \end{aligned}$$

Observation 3.9. However, when it comes to $L(11n)$, we have something different.

$$\begin{aligned} L(11) &= 199 \\ &= 1^{11} + (11)(1)(18) \\ &= 1^{11} + (11)(1)(2)(2^3 + 1^2) \\ &= L(1)^{11} + (11)L(1)[L(2) - 1]\{[L(2) - 1]^3 + L(1)^2\} \end{aligned}$$

$$\begin{aligned} L(22) &= 39603 \\ &= 3^{11} - (11)(3)(4168) \\ &= 3^{11} - (11)(3)(8)[8^3 + 3^2] \\ &= L(2)^{11} - (11)L(2)[L(4) + 1]\{[L(4) + 1]^3 + L(2)^2\} \end{aligned}$$

$$\begin{aligned} L(33) &= 7881196 \\ &= 4^{11} + (11)(4)(83793) \\ &= 4^{11} + (11)(4)(17)[17^3 + 4^2] \\ &= L(3)^{11} + (11)L(3)[L(6) - 1]\{[L(6) - 1]^3 + L(3)^2\} \end{aligned}$$

To generalize the findings, we have

Hypothesis 3.10.

$$\begin{aligned} &L(11n) \\ =\ &L(n)^{11} + (-1)^{n+1}(11)L(n)[L(2n) + (-1)^n]\{[L(2n) + (-1)^n]^3 + L(n)^2\} \\ =\ &L(n)^{11} + (-1)^{n+1}(11)L(n)[L(n)^2 + (-1)^{n+1}] \\ &\{[L(n)^2 + (-1)^{n+1}]^3 + L(n)^2\} \end{aligned}$$

Observation 3.11. What about $L(13n)$?

$$\begin{aligned} L(13) &= 521 \\ &= 1^{13} + (13)(1)(40) \\ &= 1^{13} + (13)(1)(2^2)(2^3 + 2(1)^2) \\ &= L(1)^{13} + (13)L(1)[L(2) - 1]^2\{[L(2) - 1]^3 + 2L(1)^2\} \end{aligned}$$

$$\begin{aligned} L(22) &= 271443 \\ &= 3^{13} - (13)(3)(33920) \\ &= 3^{13} - (13)(3)(8^2)[8^3 + 2(3)^2] \\ &= L(2)^{13} - (13)L(2)[L(4) + 1]^2\{[L(4) + 1]^3 + 2L(2)^2\} \end{aligned}$$

$$\begin{aligned} L(33) &= 141422324 \\ &= 4^{13} + (13)(4)(1429105) \\ &= 4^{13} + (13)(4)(17^2)[17^3 + 2(4)^2] \\ &= L(3)^{13} + (13)L(3)[L(6) - 1]^2\{[L(6) - 1]^3 + 2L(3)^2\} \end{aligned}$$

To generalize the findings, we have

Hypothesis 3.12.

$$\begin{aligned} &L(13n) \\ = \ &L(n)^{13} + (-1)^{n+1}(13)L(n)[L(2n) + (-1)^n]^2 \\ &\{[L(2n) + (-1)^n]^3 + 2L(n)^2\} \\ = \ &L(n)^{13} + (-1)^{n+1}(13)L(n)[L(n)^2 + (-1)^{n+1}]^2 \\ &\{[L(n)^2 + (-1)^{n+1}]^3 + 2L(n)^2\} \end{aligned}$$

Before finding out $L(17n)$ and $L(19n)$ and other $L(kn)$ where k is prime, we have decided to find out the relationship among our previous findings.

Now we shall rearrange our findings and all the hypotheses above and express them in a form which we can observe special patterns among the string of polynomials. We want to express $L(kn)$ in terms of $L(n)$ only.

Here are the results. For the steps of calculation, please refer to Appendix C.

$$
\begin{aligned}
L(1n) &= L(n)\\
L(2n) &= L(n)^2 + (-1)^{n+1}(2)\\
L(3n) &= L(n)^3 + (-1)^{n+1}(3)L(n)\\
L(4n) &= L(n)^4 + (-1)^{n+1}(4)L(n)^2 + 2\\
L(5n) &= L(n)^5 + (-1)^{n+1}(5)L(n)^3 + 5L(n)\\
L(6n) &= L(n)^6 + (-1)^{n+1}(6)L(n)^4 + 9L(n)^2 + (-1)^{n+1}(2)\\
L(7n) &= L(n)^7 + (-1)^{n+1}(7)L(n)^5 + 14L(n)^3 + (-1)^{n+1}(7)L(n)\\
L(8n) &= L(n)^8 + (-1)^{n+1}(8)L(n)^6 + 20L(n)^4 + (-1)^{n+1}(16)L(n)^2 + 2\\
L(9n) &= L(n)^9 + (-1)^{n+1}(9)L(n)^7 + 27L(n)^5 + (-1)^{n+1}(30)L(n)^3\\
&\quad + 9L(n)\\
L(10n) &= L(n)^{10} + (-1)^{n+1}(10)L(n)^8 + 35L(n)^6 + (-1)^{n+1}(50)L(n)^4\\
&\quad + 25L(n)^2 + (-1)^{n+1}(2)\\
L(11n) &= L(n)^{11} + (-1)^{n+1}(11)L(n)^9 + 44L(n)^7 + (-1)^{n+1}(77)L(n)^5\\
&\quad + 55L(n)^3 + (-1)^{n+1}(11)L(n)\\
L(12n) &= L(n)^{12} + (-1)^{n+1}(12)L(n)^{10} + 54L(n)^8 + (-1)^{n+1}(112)L(n)^6\\
&\quad + 105L(n)^4 + (-1)^{n+1}(36)L(n)^2 + 2\\
L(13n) &= L(n)^{13} + (-1)^{n+1}(13)L(n)^{11} + 65L(n)^9 + (-1)^{n+1}(156)L(n)^7\\
&\quad + 182L(n)^5 + (-1)^{n+1}(91)L(n)^3 + 13L(n)\\
L(14n) &= L(n)^{14} + (-1)^{n+1}(14)L(n)^{12} + 77L(n)^{10} + (-1)^{n+1}(210)L(n)^8\\
&\quad + 294L(n)^6 + (-1)^{n+1}(196)L(n)^4 + 49L(n)^2 + (-1)^{n+1}(2)\\
L(15n) &= L(n)^{15} + (-1)^{n+1}(15)L(n)^{13} + 90L(n)^{11} + (-1)^{n+1}(275)L(n)^9\\
&\quad + 450L(n)^7 + (-1)^{n+1}(378)L(n)^5 + 140L(n)^3\\
&\quad + (-1)^{n+1}(15)L(n)\\
L(16n) &= L(n)^{16} + (-1)^{n+1}(16)L(n)^{14} + 104L(n)^{12}\\
&\quad + (-1)^{n+1}(352)L(n)^{10} + 660L(n)^8\\
&\quad + (-1)^{n+1}(372)L(n)^6 + 336L(n)^4\\
&\quad + (-1)^{n+1}(64)L(n)^2 + 2
\end{aligned}
$$

Observation 3.13. It seems rather confusing and discouraging when we first get hold of the above equations. But if we compare the coefficients only, it is easier for us to handle and find out special relationships among the coefficients.

(The 1st row represents the terms in the expansion of $L(kn)$, and the 1st column represents k.)

	1st term	2nd term	3rd term	4th term	5th term	6th term	7th term	8th term	9th term
$(-1)^{n+1}$		✔		✔		✔		✔	
1	1								
2	1	2							
3	1	3							
4	1	4	2						
5	1	5	5						
6	1	6	9	2					
7	1	7	14	7					
8	1	8	20	16	2				
9	1	9	27	30	9				
10	1	10	35	50	25	2			
11	1	11	44	77	55	11			
12	1	12	54	112	105	36	2		
13	1	13	65	156	182	91	13		
14	1	14	77	210	294	196	49	2	
15	1	15	90	275	450	378	140	15	
16	1	16	104	**352**	660	672	336	64	2

This is a very interesting table. You will soon find out that it is very similar to the Pascal's Triangle. Before sharing with you how interesting this table is, let us create a naming system for it.

First, let us use 352 as an example. $Lk_{(16,4)}$ refers to 352 where we use Lk as a notation for the above table, 16 as the row number and 4 as the fourth term of the expression arranged in descending power of $L(n)$. Please bear in mind that all the terms on the k-th row are the coefficients of powers of $L(n)$ in the expansion of $L(kn)$.

If we find out the properties of this table, we can find out a way to evaluate the coefficients of powers of $L(n)$ in the expansion of $L(kn)$. Let us find out some special properties of this table.

Property I How to get the numbers on the next row

Observation 3.14. In the Pascal's Triangle, the numbers on the next row can be generated from the previous rows. (In the Pascal's Triangle, ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$) Similarly, we have tried to do this.

For example:

$$\begin{aligned} &Lk_{(4,2)} = 4 && Lk_{(8,3)} = 20 \\ &Lk_{(5,3)} = 5 && Lk_{(9,4)} = 30 \\ &Lk_{(6,3)} = 9 && Lk_{(10,4)} = 50 \\ &Lk_{4,2)} + Lk_{(5,3)} = Lk_{(6,3)} && Lk_{(8,3)} + Lk_{(9,4)} = Lk_{(10,4)} \end{aligned}$$

$$\begin{aligned} &Lk_{(12,6)} = 36 \\ &Lk_{(13,7)} = 13 \\ &Lk_{(14,7)} = 49 \\ &Lk_{(12,6)} + Lk_{(13,7)} = Lk_{(14,7)} \end{aligned}$$

Therefore, we conjecture that:

$$Lk_{(x,y)} + Lk_{(x+1,y+1)} = Lk_{(x+2,y+1)}$$

Property II Special coefficients on odd- and even-number rows

Observation 3.15. When we look at **odd**-number rows, the last coefficient is the same as the degree of the expansion. For instance, the coefficient of the last term of the expansion of $L(9n)$ (arranged in descending power of $L(n)$) in terms of $L(n)$ is 9. Also, the coefficient of the last term of the expansion of $L(11n)$ in terms of $L(n)$s is 11.

Observation 3.16. When we look at **even**-number rows, the constant term is always 2. That also leads to the previous observation in odd-number rows.

For example:

$$\begin{aligned} Lk_{(9,5)} + Lk_{(10,6)} &= 9 + 2 \\ &= 11 \\ &= Lk_{(11,6)} \end{aligned}$$

Observation 3.17. Then why is "+2" or "−2" always the constant term on even rows?

Consider the resolution of L(2kn) in terms of $L(n)$. We can break down $L(2kn)$ into $L(kn)$s by the formula $L(2n) = L(n)^2 + 2(-1)^{n+1}$.

In other words, $L(2kn) = L(kn)^2 + 2(-1)^{kn+1}$.

Thus, depending on whether k is odd or even, $+2$ or -2 is generated.

Property III Usefulness of the Lk Table in tackling prime numbers.

Please refer to Appendix D for details.

Property IV Summation of all the terms on the k-th row

Observation 3.18. Let $S(n)$ denote the summation of all the terms on the k-th row.

$$S(1) = 1$$
$$S(2) = 1 + 2 = 3$$
$$S(3) = 1 + 3 = 4$$
$$S(4) = 1 + 4 + 2 = 7$$
$$S(5) = 1 + 5 + 5 = 11$$

And so on.

$\{1, 3, 4, 7, 11, \ldots\}$ actually form the Lucas sequence.

$$S(n) = L(n)$$

It is in fact very easy to explain.

$S(k)$ is the summation of all the terms on the k-th row in the table (all the coefficients of powers of L(n) in the expression of $L(kn)$ in terms of $L(n)$).

Take $L(7)$ as an example.

Applying $L(7n) = L(n)^7 + (-1)^{n+1}(7)L(n)^5 + (14)L(n)^3 + (-1)^{n+1}(7)L(n)$, in finding $L(7)$ in terms of $L(1)$, substitute $n = 1$.

$$\begin{aligned} L(7) &= L(1)7 + (-1)^{1+1}(7)L(1)^5 + (14)L(1)^3 + (-1)^{1+1}(7)L(1) \\ &= 1 + 7 + 14 + 7 \\ &= \text{summation of all the terms on the } 7^{\text{th}} \text{ row in the table} \\ &= S(7) \end{aligned}$$

In order to have a better representation, we are going to rearrange the terms in the table by rotating the table 45° anticlockwise.

1					
1	2				
1	3				
1	4	2			
1	5	5			
1	6	9	2		
1	7	14	7		
1	8	20	16	2	
1	9	27	30	9	
1	10	35	50	25	2
1	11	44	77	55	11

If we put the numbers of the same colour into a horizontal line, we get the following triangle - the *Lk* Triangle.

1 2
1 3 2
1 4 5 2
1 5 9 7 2
1 6 14 16 9 2
1 7 20 30 25 11 2
1 8 27 50 55 36 13 2
...

Does the *Lk* Triangle remind you of the Pascal's Triangle?

This is actually an altered form of the Pascal's Triangle, only it begins with $\{1, 2\}$, not $\{1, 1\}$.

It is also obvious that, for example, on the 4^{th} row, the coefficients are: 1, 5, 9, 7, 2, which are $\{1, 5, 10, 10, 5, 1\}$ minus $\{0, 0, 1, 3, 3, 1\}$, that is, the 5^{th} row on the Pascal's Triangle minus the 3^{rd} row of the Pascal's Triangle.

Observation 3.19. Now, consider the summation of each row in the triangle above. Let $S(n)$ denote the summation of all the terms on the n-th row.

$$S(1) = 1 + 2 = 3$$
$$S(2) = 1 + 3 + 2 = 6 = 2(3)$$
$$S(3) = 1 + 4 + 5 + 2 = 12 = 2^2(3)$$
$$S(4) = 1 + 5 + 9 + 7 + 2 = 24 = 2^3(3)$$

For $n = k$, $S(k) = 2^{k-1}(3)$.

Explanation for Observation 3.19

Actually, $S(k+1) = 2S(k)$. Since every term on the k-th row will repeat itself 2 times on the next row - $(k+1)^{\text{th}}$ row, the summation is twice. Well, as $S(1) = 3$, $S(k) = 2^{k-1}S(1) = 2^{k-1}(3)$.

The following shows the Lk Triangle.

$$\begin{array}{c}
1 \quad 2 \\
1 \quad 3 \quad 2 \\
1 \quad 4 \quad 5 \quad 2 \\
1 \quad 5 \quad 9 \quad 7 \quad 2 \\
1 \quad 6 \quad 14 \quad 16 \quad 9 \quad 2 \\
1 \quad 7 \quad 20 \quad 30 \quad 25 \quad 11 \quad 2 \\
1 \quad 8 \quad 27 \quad 50 \quad 55 \quad 36 \quad 13 \quad 2 \\
1 \quad 9 \quad 35 \quad 77 \quad 105 \quad 91 \quad 49 \quad 15 \quad 2 \\
1 \quad 10 \quad 44 \quad 112 \quad 182 \quad 196 \quad 140 \quad 64 \quad 17 \quad 2 \\
1 \quad 11 \quad 54 \quad 156 \quad 294 \quad 378 \quad 336 \quad 204 \quad 81 \quad 19 \quad 2 \\
1 \quad 12 \quad 65 \quad 210 \quad 350 \quad 672 \quad 714 \quad 540 \quad 285 \quad 100 \quad 21 \quad 2 \\
1 \quad 13 \quad 77 \quad 275 \quad 560 \quad 1022 \quad 1386 \quad 1254 \quad 825 \quad 385 \quad 121 \quad 23 \quad 2
\end{array}$$

Definition 3.20. *We name terms in the Pascal's Triangle with ${}_nC_r$.*
n = the line the term lies on, $(r+1)$ = the position of the term counting from the left.

$$\boxed{\begin{array}{c}
1 \;\; 1 \\
1 \;\; 2 \;\; 1 \\
1 \;\; 3 \;\; 3 \;\; 1 \\
1 \;\; 4 \;\; 6 \;\; 4 \;\; 1 \\
1 \;\; 5 \;\; 10 \;\; 10 \;\; 5 \;\; 1
\end{array}} \longrightarrow \boxed{\begin{array}{c}
{}_1C_0 \;\; {}_1C_1 \\
{}_2C_0 \;\; {}_2C_1 \;\; {}_2C_2 \\
{}_3C_0 \;\; {}_3C_1 \;\; {}_3C_2 \;\; {}_3C_3 \\
{}_4C_0 \;\; {}_4C_1 \;\; {}_4C_2 \;\; {}_4C_3 \;\; {}_4C_4 \\
{}_5C_0 \;\; {}_5C_1 \;\; {}_5C_2 \;\; {}_5C_3 \;\; {}_5C_4 \;\; {}_5C_5
\end{array}}$$

In a similar manner, we name the Lk Triangle below with ${}_nL_r$.
n = *the line the term lies on,* $(r+1)$ = *the position of the term counting from the left.*

1 2 1 3 2 1 4 5 2 1 5 9 7 2 1 6 14 16 9 2	$\longrightarrow$	${}_1L_0$ ${}_1L_1$ ${}_2L_0$ ${}_2L_1$ ${}_2L_2$ ${}_3L_0$ ${}_3L_1$ ${}_3L_2$ ${}_3L_3$ ${}_4L_0$ ${}_4L_1$ ${}_4L_2$ ${}_4L_3$ ${}_4L_4$ ${}_5L_0$ ${}_5L_1$ ${}_5L_2$ ${}_5L_3$ ${}_5L_4$ ${}_5L_5$

Now we have created a naming system of this Triangle, we can look more in depth into the relationship between the Pascal's Triangle and the Lk Triangle.

Observation 3.21. Let us consider ${}_3L_3$ and ${}_4L_2$.

$$\begin{aligned}{}_3L_3 &= 2 = 4-2 = {}_4C_3 - {}_2C_1\\ {}_4L_2 &= 9 = 10-1 = {}_5C_2 - {}_3C_0\\ {}_4L_3 &= 7 = 10-3 = {}_5C_3 - {}_3C_1\end{aligned}$$

Hypothesis 3.22. *We can make an assumption that*

$${}_nL_r = {}_{n+1}C_r - {}_{n-1}C_{r-2}$$

Details of Proof for Hypothesis 3.22 can be found in Appendix E.

As we all know that ${}_nC_r$ has another representation

$${}_nC_r = \frac{n!}{r!(n-r)!}.$$

$$\begin{aligned}{}_nL_r &= {}_{n+1}C_r - {}_{n-1}C_{r-2}\\ &= \frac{(n+1)!}{(n+1-r)!r!} - \frac{(n-1)!}{[n-1-(r-2)]!(r-2)!}\\ &= \frac{(n+1)!}{(n+1-r)!r!} - \frac{(n-1)!}{(n+1-r)!(r-2)!}\\ &= \frac{(n+1)!-(n-1)!r(r-1)}{(n+1-r)!r!}\\ &= \frac{(n-1)![n(n+1)-r(r-1)]}{(n+1-r)!r!}\\ &= \frac{(n-1)!(n^2+n-r^2+r)}{(n+1-r)!r!}\end{aligned}$$

Formula 3.23.

$$ {}_nL_r = \frac{(n-1)!(n^2+n-r^2+r)}{(n+1-r)!r!} $$

There is a lot that we have discovered up to this point. The most important thing we need to do is to resolve $L(kn)$. Since we have established the relationship between ${}_nL_r$ and ${}_nC_r$, we can use this to find out the coefficients of $L(kn)$.

Table 3.24. Let us express all the terms in the Lk Table in terms of ${}_nL_r$.

k \ Expansion of $L(kn)$	1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	…	$(r+1)^{\text{th}}$ term	…	$(2p-1)^{\text{th}}$ term	$(2p)^{\text{th}}$ term	$(2p+1)^{\text{th}}$ term
$(-1)^{n+1}$		✔		✔		N/A			✔	
1	${}_1L_0$	N/A								
2	${}_2L_0$	${}_1L_1$	N/A							
3	${}_3L_0$	${}_2L_1$	N/A							
4	${}_4L_0$	${}_3L_1$	${}_2L_2$	N/A						
5	${}_5L_0$	${}_4L_1$	${}_3L_2$	N/A						
6	${}_6L_0$	${}_5L_1$	${}_4L_2$	${}_3L_3$	N/A					
7	${}_7L_0$	${}_6L_1$	${}_5L_2$	${}_4L_3$	…	N/A				
8	${}_8L_0$	${}_7L_1$	${}_6L_2$	${}_5L_3$	…	N/A				
⋮										
$4p-3$	${}_{4p-3}L_0$	${}_{4p-4}L_1$	${}_{4p-5}L_2$	…		${}_{4p-3-r}L_r$	…	${}_{2p-1}L_{2p-2}$	N/A	
$4p-2$	${}_{4p-2}L_0$	${}_{4p-3}L_1$	${}_{4p-4}L_2$	…		${}_{4p-2-r}L_r$	…	${}_{2p}L_{2p-2}$	${}_{2p-1}L_{2p-1}$	N/A
$4p-1$	${}_{4p-1}L_0$	${}_{4p-2}L_1$	${}_{4p-3}L_2$	…		${}_{4p-1-r}L_r$	…	${}_{2p+1}L_{2p-2}$	${}_{2p}L_{2p-1}$	N/A
$4p$	${}_{4p}L_0$	${}_{4p-1}L_1$	${}_{4p-2}L_2$	…		${}_{4p-r}L_r$	…	${}_{2p+2}L_{2p-2}$	${}_{2p+1}L_{2p-1}$	${}_{2p}L_{2p}$

Hypothesis 3.25. *Let $k = 4p$, where p is an integer.*

$$\begin{aligned} L(kn) &= L(4pn) \\ &= {}_{4p}L_0 L(n)^{4p} + (-1)^{n+1}{}_{4p-1}L_1 L(n)^{4p-2} + {}_{4p-2}L_2 L(n)^{4p-4} \\ &\quad + (-1)^{n+1}{}_{4p-3}L_3 L(n)^{4p-6} + \ldots + {}_{4p-r}L_r L(n)^{4p-2r+2} \\ &\quad + \ldots + (-1)^{n+1}{}_{2p+1}L_{2p-1} L(n)^2 + {}_{2p}L_{2p} \end{aligned}$$

Note: $(-1)^{n+1}$ *occurs in the* 2^{nd}, 4^{th}, 6^{th} *and other even-number terms,* ${}_{4p}L_0 = 1$; ${}_{4p-1}L_1 = 4p$; ${}_{2p}L_{2p} = 2$.

Hypothesis 3.26. *Let $k = 4p - 1$, where p is an integer.*

$$\begin{aligned} L(kn) &= L((4p-1)n) \\ &= {}_{4p-1}L_0 L(n)^{4p-1} + (-1)^{n+1}{}_{4p-2}L_1 L(n)^{4p-3} + {}_{4p-3}L_2 L(n)^{4p-5} \\ &\quad + (-1)^{n+1}{}_{4p-4}L_3 L(n)^{4p-7} + \ldots + {}_{4p-1-r}L_r L(n)^{4p-2r+1} \\ &\quad + \ldots + (-1)^{n+1}{}_{2p}L_{2p-1} L(n) \end{aligned}$$

Note: $(-1)^{n+1}$ *occurs in the* 2^{nd}, 4^{th}, 6^{th} *and other even-number terms,* ${}_{4p-1}L_0 = 1$; ${}_{4p-2}L_1 = 4p - 1$.

Hypothesis 3.27. *Let $k = 4p - 2$, where p is an integer.*

$$\begin{aligned} L(kn) &= L((4p-2)n) \\ &= {}_{4p-2}L_0 L(n)^{4p-2} + (-1)^{n+1}{}_{4p-3}L_1 L(n)^{4p-4} + {}_{4p-4}L_2 L(n)^{4p-6} \\ &\quad + (-1)^{n+1}{}_{4p-5}L_3 L(n)^{4p-8} + \ldots + {}_{4p-2-r}L_r L(n)^{4p-2r} \\ &\quad + \ldots + (-1)^{n+1}{}_{2p-1}L_{2p-1} \end{aligned}$$

Note: $(-1)^{n+1}$ *occurs in the* 2^{nd}, 4^{th}, 6^{th} *and other even-number terms,* ${}_{4p-2}L_0 = 1$; ${}_{4p-3}L_1 = 4p - 2$; ${}_{2p-1}L_{2p-1} = 2$.

Hypothesis 3.28. *Let $k = 4p - 3$, where p is an integer.*

$$\begin{aligned} L(kn) &= L((4p-3)n) \\ &= {}_{4p-3}L_0 L(n)^{4p-3} + (-1)^{n+1}{}_{4p-4}L_1 L(n)^{4p-5} + {}_{4p-5}L_2 L(n)^{4p-7} \\ &\quad + (-1)^{n+1}{}_{4p-6}L_3 L(n)^{4p-9} + \ldots + {}_{4p-3-r}L_r L(n)^{4p-2r-1} \\ &\quad + \ldots + (-1)^{n+1}{}_{2p-1}L_{2p-2} L(n) \end{aligned}$$

Note: $(-1)^{n+1}$ *occurs in the* 2^{nd}, 4^{th}, 6^{th} *and other even-number terms,* ${}_{4p-3}L_0 = 1$; ${}_{4p-4}L_1 = 4p-3$.

Application 3.29. Suppose we want to compute L(98).

$$\begin{aligned}
L(98) &= L(14 \times 7)\\
&= L((4 \times 4 - 2) \times 7) \quad \text{(here } p = 4)\\
&= 1L(7)^{14} + (-1)^{7+1}14L(7)^{12} + {}_{12}L_2L(7)^{10} + (-1)^{7+1}{}_{11}L_3L(7)^8\\
&\quad + {}_{10}L_4L(7)^6 + (-1)^{7+1}{}_9L_5L(7)^4 + {}_8L_6L(7)^2 + (-1)^{7+1}(2)\\
&= 1L(7)^{14} + 14L(7)^{12} + {}_{12}L_2L(7)^{10} + {}_{11}L_3L(7)^8\\
&\quad +_{10} L_4L(7)^6 +_9 L_5L(7)^4 + {}_8L_6L(7)^2 + 2\\
&= 29^{14} + 14 \times 29^{12} + ({}_{13}C_2 - {}_{11}C_0)29^{10} + ({}_{12}C_3 - {}_{10}C_1)29^8\\
&\quad + ({}_{11}C_4 - {}_9C_2)29^6 + ({}_{10}C_5 - {}_8C_3)29^4 + ({}_9C_6 - {}_7C_4)29^2 + 2\\
&= 29^{14} + 14 \times 29^{12} + 77 \times 29^{10} + 210 \times 29^8\\
&\quad + 294 \times 29^6 + 196 \times 29^4 + 49 \times 29^2 + 2\\
&= 2975582326757994 63481 + 4953406964876566574\\
&\quad + 32394456964115477 + 105051746721810\\
&\quad + 174878056374 + 138627076 + 41209 + 2\\
&= 3025441393244035920 03
\end{aligned}$$

Note: Recall that ${}_nL_r = {}_{n+1}C_r - {}_{n-1}C_{r-2}$.

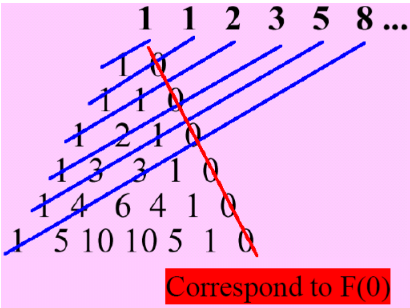
Application 3.30.

$$\begin{aligned}
L(104) &= L(13 \times 8) \\
&= L((4 \times 4 - 3) \times 8) \\
&= {}_{13}L_0 L(8)^{13} + (-1)^{8+1}{}_{12}L_1 L(8)^{11} + {}_{11}L_2 L(8)^9 \\
&\quad + (-1)^{8+1}{}_{10}L_3 L(8)^7 + {}_9L_4 L(8)^5 + (-1)^{8+1}{}_8L_5 L(8)^3 + {}_7L_6 L(8) \\
&= ({}_{14}C_0 - {}_{12}C_{-2})L(8)^{13} + ({}_{13}C_1 - {}_{11}C_{-1})L(8)^{11} \\
&\quad + ({}_{12}C_2 - {}_{10}C_0)L(8)^9 + ({}_{11}C_3 - {}_9C_1)L(8)^7 \\
&\quad + ({}_{10}C_4 - {}_8C_2)L(8)^5 + ({}_9C_5 - {}_7C_3)L(8)^3 + ({}_8C_6 - {}_6C_4)L(8) \\
&= (1-0)47^{13} - (13-0)47^{11} + (66-1)47^9 - (165-9)47^7 \\
&\quad + (210-28)47^5 - (126-35)47^3 + (28-15)47 \\
&= 1 \times 47^{13} - 13 \times 47^{11} + 65 \times 47^9 - 156 \times 47^7 \\
&\quad + 182 \times 47^5 - 91 \times 47^3 + 13 \times 47 \\
&= 5460999706120583177327 - 32138069796092159939 \\
&\quad + 72743480751679855 - 79033206792228 \\
&\quad + 41740791274 - 9447893 + 611 \\
&= 5428934300813767249007
\end{aligned}$$

The Pascal's Triangle and the *Lk* Triangle

As the coefficients in binomial expansion and the coefficients in polynomial expression of $L(kn)$ in terms of $L(n)$ have some similar properties, we try to compare the Pascal's Triangle and the Lk Triangle.

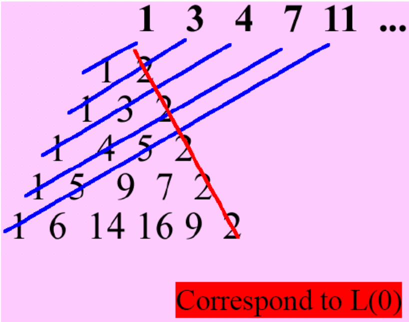
Observation 3.31. In Pascal's Triangle, the numbers are as follow.



The first two numbers on the Pascal's Triangle, 1 and 0, remind us of $F(1)$ and $F(0)$. Let us draw some inclined lines (from the bottom left to the top right, a.k.a. "shallow diagonals") across the Pascal's Triangle. Along each shallow diagonal, we will find out the sum of the numbers that pass through it. The sums are 1, 1, 2, 3, 5, ... and this is the Fibonacci sequence.

Compare the above situation with the Lk Triangle.

Observation 3.32. Now let us look at the Lk Triangle.



As we draw shallow diagonals on the Lk Triangle and find out the sum of the numbers that pass through them, we obtained the Lucas sequence.

In the above examples, it is observed that a sequence with the property $U(n) + U(n+1) = U(n+2)$ can be obtained by the sum of numbers lying on the shallow diagonals.

In fact, we have every reason to believe that any triangle that starts with $U(1)$ and $U(0)$ will have this property.

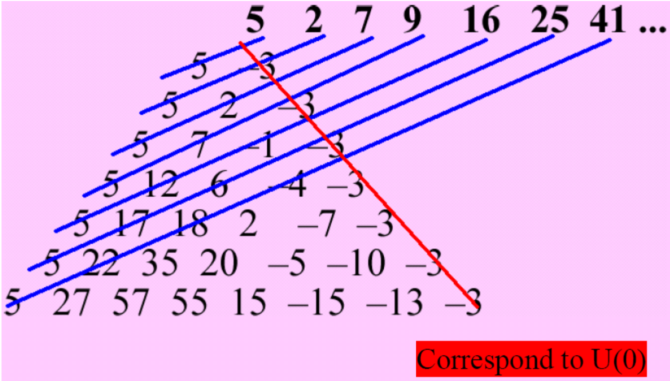
Let us consider the Pascal's Triangle again. Now we try to analyze the numbers in the Triangle. On the first row of the Triangle, the two starting numbers are 1 and 0, which correspond to $F(1)$ and $F(0)$ respectively. And what we get in the sequence are 1, 1, 2, 3, 5, 8, ... which correspond to $F(1)$, $F(2)$, $F(3)$, $F(4)$, $F(5)$, $F(6)$, ... respectively.

Now let us consider the Lk Triangle again. We try to analyze the numbers in the Triangle. On the first row of the Triangle, the two starting numbers are 1 and 2, which correspond to $L(1)$ and $L(0)$ respectively. And what we get in the sequence are 1, 3, 4, 7, 11, ... which correspond to $L(1)$, $L(2)$, $L(3)$, $L(4)$, $L(5)$, ... respectively.

Therefore, we have this hypothesis:

Is the sequence obtained $U(1)$, $U(2)$, $U(3)$, ... if we try to create a triangle in a similar way starting with $U(1)$ and $U(0)$ on the first row?

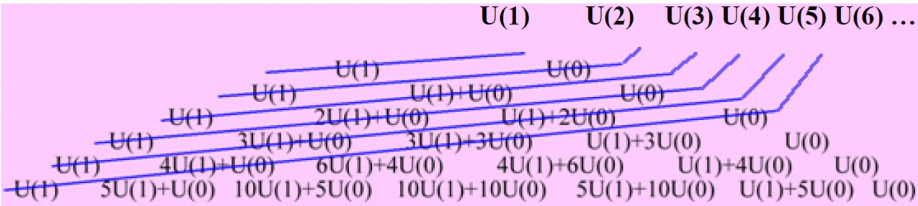
We are going to set up an example. The numbers on the first row of the following triangle are $\{5, -3\}$, i.e. $U(1) = 5$ and $U(0) = 3$.



Again, we obtain a sequence $\{5, 2, 7, 9, 16, 25, 41, \ldots\}$ obeying $U(n) + U(n+1) = U(n+2)$.

We are going to prove this by constructing the required triangle and representing each term in terms of $U(1)$ and $U(0)$ only.

Figure 3.33.



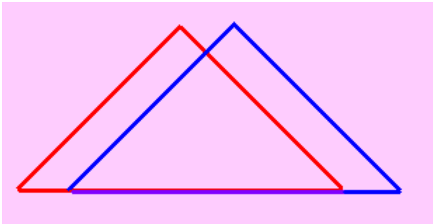
We denote the sum of numbers lying on the n-th shallow diagonal by $D(n)$.

$$D(1) = U(1)$$
$$D(2) = U(1) + U(0) = U(2)$$
$$D(3) = U(1) + U(1) + U(0) = U(3)$$
$$D(4) = U(1) + 2U(1) + U(0) + U(0) = U(4)$$
$$D(5) = U(1) + 3U(1) + U(0) + U(1) + 2U(0) = U(5)$$
$$\ldots$$

Therefore, we conjecture that $D(n) = U(n)$.

To prove this, first, we are going to find out how Figure 3.33 is formed.

Figure 3.34.



In fact, Figure 3.33 consists of two Pascal's Triangles, as illustrated by Figure 3.34. Every term in the triangle on the left is multiplied by $U(1)$ and every term in the triangle on the right is multiplied by $U(0)$. Then the two triangles are merged by adding up the terms in the overlapped area, resulting in the Triangle in Figure 3.33.

$$D(1) = {}_0C_1U(1) = U(1)$$
$$D(2) = {}_1C_0U(1) + {}_1C_1U(0) = U(2)$$
$$D(3) = ({}_2C_0 + {}_1C_1)U(1) + {}_1C_0U(0) = U(3)$$
$$D(4) = ({}_3C_0 + {}_2C_1)U(1) + ({}_2C_0 + {}_1C_1)U(0) = U(4)$$
$$D(5) = ({}_4C_0 + {}_3C_1 + {}_2C_2)U(1) + ({}_3C_0 + {}_2C_1)U(0) = U(5)$$
$$D(6) = ({}_5C_0 + {}_4C_1 + {}_3C_2)U(1) + ({}_4C_0 + {}_3C_1 + {}_2C_2)U(0) = U(6)$$
$$\ldots$$

From the above observations, we conjecture that

Hypothesis 3.35.

$$\begin{aligned} D(2p+1) &= ({}_{2p}C_0 + {}_{2p-1}C_1 + \ldots + {}_{p+1}C_{p-1} + {}_pC_p)U(1) \\ &\quad + ({}_{2p-1}C_0 + {}_{2p-2}C_1 + \ldots + {}_{p+1}C_{p-2} + {}_pC_{p-1})U(0) \\ D(2p+2) &= ({}_{2p+1}C_0 + {}_{2p}C_1 + \ldots + {}_{p+2}C_{p-1} + {}_{p+1}C_p)U(1) \\ &\quad + ({}_{2p}C_0 + {}_{2p-1}C_1 + \ldots + {}_{p+1}C_{p-1} + {}_pC_p)U(0) \end{aligned}$$

In short, we are going to prove $D(n) = U(n)$.

Details of Proof for Hypothesis 3.35 can be found in Appendix E.

Is it amazing? We can form a sequence with the property $U(n)+U(n+1) =$

$U(n+2)$ in this way in the Triangle starting with $U(1)$ and $U(0)$.

In this section, we have found out how to express $L(kn)$ in terms of $L(n)$ only and the relation is shown in the Lk Triangle. You will be interested in knowing if the Pascal's Triangle can help us express $F(kn)$ in terms of $F(n)$. We will have a more detailed discussion in section 5.

4. Introduction of the Tables (Fibonacci Table, Lucas-Fibonacci Table, Lucas Table)

Why do we introduce the Tables?

In geometry, we have point, line, plane and solid, which represent 0, 1, 2, and 3 dimensions respectively. It is these definitions in geometry that inspire us to investigate Fibonacci and Lucas numbers in two dimensions.

Constructing a table helps us observe patterns and present our findings. In section 5 and 6, we will use tables to illustrate our discoveries. In these two sections, we have come up with formulae that can be used to resolve large $F(n)$ and $L(n)$; at the same time, we have spotted a lot of special patterns and interesting phenomena that contribute to the second part of our report.

Definitions Concerning the Tables

Before you begin to read the contents of the following sectionss, there is a need to define notations that we will use for the Tables.

Row \ Column	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	1	2	3	5	8	13	21	34	55
2	1	1	2	3	5	8	13	21	34	55
3	2	2	4	6	10	16	26	42	68	110
4	3	3	6	9	15	24	39	63	102	165
5	5	5	10	15	25	40	65	105	170	275
6	8	8	**16**	24	40	64	104	168	272	440
7	13	13	26	39	65	104	169	273	442	715
8	21	21	42	63	105	168	273	441	714	1155
9	34	34	68	102	170	272	442	714	1156	1870
10	55	55	110	165	275	440	715	1155	1870	3025

In order to facilitate the reading of the Table and locating terms on it, we have created a naming system. Under this naming system, $(F3, F6)$ refers to the number on the third column, sixth row; that is 16.

Procedure of creating the Fibonacci Table

(1) The 1st number of the sequence, 1, is placed in the first row and the first column.
(2) Afterwards, in the horizontal and vertical direction, the Fibonacci sequence is generated.
(3) On the second row, another Fibonacci sequence is generated and the same occurs to the second column.
(4) The third row is created by adding row 1 and row 2. The fourth by adding row 2 and row 3. Similarly, column 3 is generated by adding column 1 and 2; while column 4 is a sum of column 2 and 3.
(5) Repeat the addition of rows and columns to get the Fibonacci Table.

Row \ Column	**1**	**2**	**3**	**4**	**5**	**6**
1	1					
2						
3						
4						
5						
6						

⟶

Row \ Column	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1	2	3	5	8
2	1					
3	2					
4	3					
5	5					
6	8					

↓

Row \ Column	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1	2	3	5	8
2	1	1	2	3	5	8
3	2	2	4	6	10	16
4	3	3	8	9	15	24
5	5	5	10	15	25	40
6	8	8	16	24	40	64

⟵

Row \ Column	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1	2	3	5	8
2	1	1	2	3	5	8
3	2	2				
4	3	3				
5	5	5				
6	8	8				

The following table illustrates more clearly what happens after the generation of Fibonacci Table.

Row \ Column	**1**	**2**	**3**	**4**	**5**	**6**
1	$F(1)F(1)$	$F(2)F(1)$	$F(3)F(1)$	$F(4)F(1)$	$F(5)F(1)$	$F(6)F(1)$
2	$F(1)F(2)$	$F(2)F(2)$	$F(3)F(2)$	$F(4)F(2)$	$F(5)F(2)$	$F(6)F(2)$
3	$F(1)F(3)$	$F(2)F(3)$	$F(3)F(3)$	$F(4)F(3)$	$F(5)F(3)$	$F(6)F(3)$
4	$F(1)F(4)$	$F(2)F(4)$	$F(3)F(4)$	$F(4)F(4)$	$F(5)F(4)$	$F(6)F(4)$
5	$F(1)F(5)$	$F(2)F(5)$	$F(3)F(5)$	$F(4)F(5)$	$F(5)F(5)$	$F(6)F(5)$
6	$F(1)F(6)$	$F(2)F(6)$	$F(3)F(6)$	$F(4)F(6)$	$F(5)F(6)$	$F(6)F(6)$

The Fibonacci Triangle

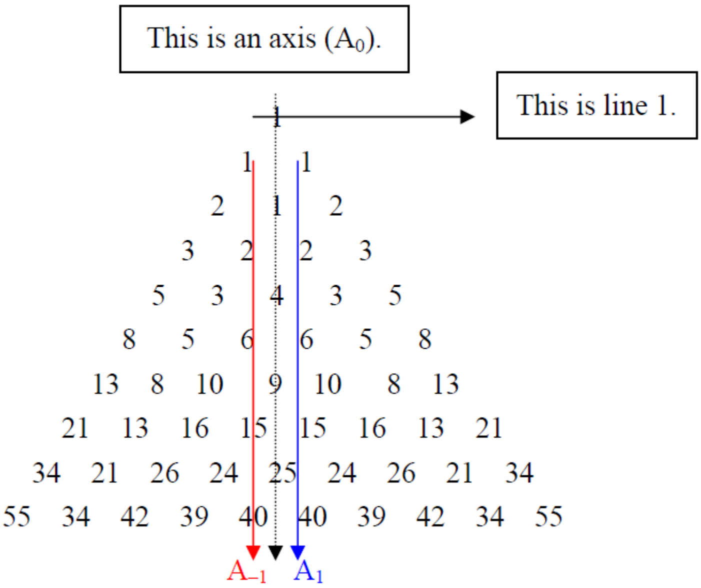
If we try to rotate the Fibonacci Table 45° clockwise, a triangle below is formed. Let us name this triangle the Fibonacci Triangle.

1
1 1
2 1 2
3 2 2 3
5 3 4 3 5
8 5 6 6 5 8
13 8 10 9 10 8 13
21 13 16 15 15 16 13 21
34 21 26 24 25 24 26 21 34
55 34 42 39 40 40 39 42 34 55

This is the Fibonacci Triangle. For the sake of convenience, we have created a naming system.

The axis $\{1, 1, 4, 9, 25\}$ is named as A_0, vertical lines next to the axis are named as A_1 (to the right of the axis) and A_{-1} (to the left of the axis), as shown in the figure.

To name the 3[rd] term on line 9, i.e. 26, we first locate the term that lies on the axis on line 9, in this case, 25. Then look for 26, which is located on axis A_{-4} and on line 9. Hence the 3[rd] term on line 9 is named as $A_{-4}L_9$.



The Lucas-Fibonacci Table

Before introducing the Lucas-Fibonacci Triangle, it is necessary to introduce the Lucas-Fibonacci Table. While the method of generation of the table is the same, **the top horizontal sequence is the Lucas sequence while the leftmost vertical sequence is the Fibonacci sequence**.

Column Row	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	1	3	4	7	11	18	29	47	76
2	1	3	4	7	11	18	29	47	76
3	2	6	8	14	22	36	58	94	152
4	3	9	12	21	33	54	87	141	228
5	5	15	20	35	55	90	145	235	380
6	8	24	32	56	88	144	232	376	608
7	13	39	52	91	143	234	377	611	988
8	21	63	64	147	231	378	609	987	1596
9	34	102	136	238	374	612	986	1598	2584

The Lucas-Fibonacci Triangle

The Lucas-Fibonacci Triangle is formed by rotating the Lucas-Fibonacci Table 45° clockwise. The following shows part of the Lucas-Fibonacci Triangle.

1
1 3
2 3 4
3 6 4 7
5 9 8 7 11
8 15 12 14 11 18
13 24 20 21 22 18 29
21 39 32 35 33 36 29 47
34 63 52 56 55 54 58 47 76
55 102 84 91 88 90 87 94 76 123

The Lucas Table

Since we are going to use the Lucas Table in the following sections, there is also a need to introduce the Lucas Table. It is formed by the Lucas sequence. The method of generation of the Lucas Table is just the same as that of the Fibonacci Table.

Row \ Column	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	1	3	4	7	11	18	29	47	76
2	3	9	12	21	33	54	87	141	228
3	4	12	16	28	44	72	116	188	304
4	7	21	28	49	77	126	203	329	532
5	11	33	44	77	121	198	319	517	836
6	18	54	72	126	198	324	522	846	1368
7	29	87	116	203	319	522	841	1363	2204
8	47	141	188	329	517	846	1363	2209	3572
9	76	228	304	532	836	1368	2204	3572	5776

The Lucas Triangle

With the method of generation the same as that of the Fibonacci Triangle, the Lucas Triangle is generated.

1
3 3
4 9 4
7 12 12 7
11 21 16 21 11
18 33 28 28 33 18
29 54 44 49 44 54 29
47 87 72 77 77 72 87 47
76 141 116 126 121 126 116 141 76

5. Relationships between Fibonacci and Lucas Sequences

Have you ever thought that the Fibonacci sequence and the Lucas sequence are inter-related?

One of the main reasons for us to insert the tables (Fibonacci Table, Lucas-Fibonacci Table and Lucas Table) is because it is good for observing patterns. Some of these patterns can even help us break down big $F(n)$ and $L(n)$.

5.1. Expressing $L(n)$ in terms of $F(n)$

Here, we are going to introduce some special patterns in the **Lucas-Fibonacci Table**. Note that every term in the Table represents the product of a Lucas number and a Fibonacci number.

Observations		Hypothesis	
5.1	$(L1, F2) = 1 \times 1 = \mathbf{1}$ $= 2 - 1$ $= F(3) - F(1)$ $(L1, F3) = 1 \times 2 = \mathbf{2}$ $= 3 - 1$ $= F(4) - F(2)$ $(L1, F4) = 1 \times 3 = \mathbf{3}$ $= 5 - 2$ $= F(5) - F(3)$	**5.2**	$L(1)F(k) = F(k+1) - F(k-1)$
5.3	$(L2, F3) = 3 \times 2 = \mathbf{6}$ $= 5 + 1$ $= F(5) + F(1)$ $(L2, F4) = 3 \times 3 = \mathbf{9}$ $= 8 + 1$ $= F(6) + F(2)$ $(L2, F5) = 3 \times 5 = \mathbf{15}$ $= 13 + 2$ $= F(7) + F(3)$	**5.4**	$L(2)F(k) = F(k+2) + F(k-2)$

5.5	$(L3, F4) = 4 \times 3 = \mathbf{12}$ $= 13 - 1$ $= F(7) - F(1)$ $(L3, F5) = 4 \times 5 = \mathbf{20}$ $= 21 - 1$ $= F(8) - F(2)$ $(L3, F6) = 4 \times 8 = \mathbf{32}$ $= 34 - 2$ $= F(9) - F(3)$	**5.6**	$L(3)F(k) = F(k{+}3){+}F(k{-}3)$
5.7	$(L4, F5) = 7 \times 5 = \mathbf{35}$ $= 34 + 1$ $= F(9) + F(1)$ $(L4, F6) = 7 \times 8 = \mathbf{56}$ $= 55 + 1$ $= F(10) + F(2)$ $(L4, F7) = 7 \times 13 = \mathbf{91}$ $= 89 + 2$ $= F(11) + F(3)$	**5.8**	$L(4)F(k) = F(k{+}4){+}F(k{-}4)$
5.9	$(L5, F6) = 11 \times 8 = \mathbf{88}$ $= 89 - 1$ $= F(11) - F(1)$ $(L5, F7) = 11 \times 13 = \mathbf{143}$ $= 144 - 1$ $= F(12) - F(2)$ $(L5, F8) = 11 \times 21 = \mathbf{231}$ $= 233 - 2$ $= F(13) - F(3)$	**5.10**	$L(5)F(k) = F(k{+}5){+}F(k{-}5)$

5.11	$(L6, F7) = 18 \times 13 = \mathbf{234}$ $= 233 + 1$ $= F(13) + F(1)$ $(L6, F8) = 18 \times 21 = \mathbf{378}$ $= 377 + 1$ $= F(14) + F(2)$ $(L6, F9) = 18 \times 34 = \mathbf{612}$ $= 610 + 2$ $= F(15) + F(3)$	**5.12**	$L(6)F(k) = F(k{+}6){+}F(k{-}6)$
5.13	$(L2, F1) = 3 \times 1 = \mathbf{3}$ $= 2 + 1$ $= F(3) + F(1)$ $(L3, F1) = 4 \times 1 = \mathbf{4}$ $= 3 + 1$ $= F(4) + F(2)$ $(L4, F1) = 7 \times 1 = \mathbf{7}$ $= 5 + 2$ $= F(5) + F(3)$	**5.14**	$F(1)L(k) = F(k{+}1){+}F(k{-}1)$
5.15	$(L3, F2) = 4 \times 1 = \mathbf{4}$ $= 5 - 1$ $= F(5) - F(1)$ $(L4, F2) = 7 \times 1 = \mathbf{7}$ $= 8 - 1$ $= F(6) - F(2)$ $(L5, F2) = 11 \times 1 = \mathbf{11}$ $= 13 - 2$ $= F(7) - F(3)$	**5.16**	$F(2)L(k) = F(k{+}2){-}F(k{-}2)$

5.17	$(L4, F3) = 7 \times 2 = \mathbf{14}$ $= 13 + 1$ $= F(7) + F(1)$ $(L5, F3) = 11 \times 2 = \mathbf{22}$ $= 21 + 1$ $= F(8) + F(2)$ $(L6, F3) = 18 \times 2 = \mathbf{36}$ $= 34 + 2$ $= F(9) + F(3)$	**5.18**	$F(3)L(k) = F(k+3)+F(k-3)$
5.19	$(L5, F4) = 11 \times 3 = \mathbf{33}$ $= 34 - 1$ $= F(9) - F(1)$ $(L6, F4) = 18 \times 3 = \mathbf{54}$ $= 55 - 1$ $= F(10) - F(2)$ $(L7, F4) = 29 \times 3 = \mathbf{87}$ $= 89 - 2$ $= F(11) - F(3)$	**5.20**	$F(4)L(k) = F(k+4)-F(k-4)$
5.21	$(L6, F5) = 18 \times 5 = \mathbf{90}$ $= 89 + 1$ $= F(11) + F(1)$ $(L7, F5) = 29 \times 5 = \mathbf{145}$ $= 144 + 1$ $= F(12) + F(2)$ $(L8, F5) = 47 \times 5 = \mathbf{235}$ $= 233 + 2$ $= F(13) + F(3)$	**5.22**	$F(5)L(k) = F(k+5)+F(k-5)$

5.23	$(L7, F6) = 29 \times 8 = \mathbf{232}$ $= 233 - 1$ $= F(13) - F(1)$ $(L8, F6) = 47 \times 8 = \mathbf{376}$ $= 377 - 1$ $= F(14) - F(2)$ $(L9, F6) = 76 \times 8 = \mathbf{608}$ $= 610 - 2$ $= F(15) - F(3)$	**5.24**	$F(6)L(k) = F(k+6) - F(k-6)$

Let us recap what we have found related to $F(k)$.

Hypothesis 5.2. $L(1)F(k) = F(k+1) - F(k-1)$
Hypothesis 5.4. $L(2)F(k) = F(k+2) + F(k-2)$
Hypothesis 5.6. $L(3)F(k) = F(k+3) - F(k-3)$
Hypothesis 5.8. $L(4)F(k) = F(k+4) + F(k-4)$
Hypothesis 5.10. $L(5)F(k) = F(k+5) - F(k-5)$
Hypothesis 5.12. $L(6)F(k) = F(k+6) + F(k-6)$

To generalize the findings, we have

Hypothesis 5.25.

$$L(r)F(k) = F(k+r) + (-1)^r F(k-r)$$

Details of Proof for Hypothesis 5.25 can be found in Appendix E.

Let us recap what we have found related to $L(k)$.

Hypothesis 5.14. $F(1)L(k) = F(k+1) + F(k-1)$
Hypothesis 5.16. $F(2)L(k) = F(k+2) - F(k-2)$
Hypothesis 5.18. $F(3)L(k) = F(k+3) + F(k-3)$
Hypothesis 5.20. $F(4)L(k) = F(k+4) - F(k-4)$
Hypothesis 5.22. $F(5)L(k) = F(k+5) + F(k-5)$
Hypothesis 5.24. $F(6)L(k) = F(k+6) - F(k-6)$

To generalize the findings, we have

Hypothesis 5.26.

$$F(r)L(k) = F(k+r) + (-1)^{r+1} F(k-r)$$

Details of Proof for Hypothesis 5.26 can be found in Appendix E.

Application 5.27. When we look into the Fibonacci sequence, we do not just look at $F(n)$ with positive n. We sometimes may have to encounter $F(n)$ with negative n, for example, in doing some proofs. How can we find them? The answer is easy and simple. Using the property $F(n+2) = F(n) + F(n+1)$, we can find out the negative part of the sequence.

$F(0) = 0$	$F(0) = 0$
Positive side	**Negative side**
$F(1) = 1$	$F(-1) = 1$
$F(2) = 1$	$F(-2) = -1$
$F(3) = 2$	$F(-3) = 2$
$F(4) = 3$	$F(-4) = -3$
$F(5) = 5$	$F(-5) = 5$
$F(6) = 8$	$F(-6) = -8$
$F(7) = 13$	$F(-7) = 13$
$F(8) = 21$	$F(-8) = -21$
$F(9) = 34$	$F(-9) = 34$
$F(10) = 55$	$F(-10) = -55$

You may have noticed that $F(k) = (-1)^{k+1}F(-k)$. Now, we are going to prove this simple property of the Fibonacci sequence by Formula 5.25 and Formula 5.26.

Proof. First, consider Formula 5.25,

$$L(r)F(k) = F(k+r) + (-1)^r F(k-r)$$

Putting $r = a$, $k = b$, we have

$$L(a)F(b) = F(a+b) + (-1)^a F(b-a) \tag{3}$$

Then, consider Formula 5.26,

$$F(r)L(k) = F(k+r) + (-1)^{r+1} F(k-r)$$

Putting $r = b$, $k = a$, we have

$$F(b)L(a) = F(a+b) + (-1)^{b+1} F(a-b) \tag{4}$$

Now, sub (3) into (4).

$$\begin{aligned} F(a+b) + (-1)^a F(b-a) &= F(a+b) + (-1)^{b+1} F(a-b) \\ (-1)^a F(b-a) &= (-1)^{b+1} F(a-b) \\ (-1)^{a-(b+1)} F(b-a) &= F(a-b) \\ (-1)^{a-b-1} F(b-a) &= F(a-b) \end{aligned}$$

Substitute $k = a - b$,

$$\begin{aligned} (-1)^{k-1} F(-k) &= F(k) \\ F(k) &= (-1)^{k+1} F(-k) \end{aligned}$$

□

Although our project does not focus on the negative part of the Fibonacci sequence, sometimes we do encounter large $F(-n)$. For instance, we want to find $F(-100)$, we can just find $F(100)$ first and then apply $F(k) = (-1)^{k+1} F(-k)$ to obtain $F(-100)$.

Application 5.28. As we have two formulae:

$$L(a)F(b) = F(b+a) + (-1)^a F(b-a) \tag{3}$$

$$F(a)L(b) = F(b+a) + (-1)^{a+1} F(b-a) \tag{4}$$

Now, we can generate some useful formulae to resolve large $F(n)$ from them.

(3) + (4):

$$\begin{aligned} L(a)F(b) + F(a)L(b) &= F(b+a) + F(b+a) + (-1)^a F(b-a) \\ &\quad - (-1)^a F(b-a) \\ L(a)F(b) + F(a)L(b) &= 2F(b+a) \end{aligned}$$

By this method, we can in fact reduce $F(n)$ conveniently.

For example,

$$\begin{aligned} F(80) &= F(30+50) \\ &= \frac{L(30F(50) + F(30)L(50)}{2} \end{aligned}$$

Actually, if we want to reduce $F(b+a)$ quickly, a and b should be more or less the same. That is, we should substitute $a = 40$ and $b = 40$ in the previous example.

(3) − (4):

$$\begin{aligned} L(a)F(b) - F(a)L(b) &= F(b+a) - F(b+a) + (-1)^a F(b-a) \\ &\quad + (-1)^a F(b-a) \\ L(a)F(b) + F(a)L(b) &= (-1)^a 2F(b-a) \\ L(a)F(b) &= (-1)^a 2F(b-a) + F(a)L(b) \end{aligned}$$

Actually this formula cannot help us much on the breakdown of large $F(n)$ or $L(n)$. However, please look at the formula again.

$$(La, Fb) - (Lb, Fa) = (-1)^a 2F(b-a)$$

In the Fibonacci Triangle, (Fa, Fb) actually equals (Fb, Fa), because they both represents $F(a)F(b)$. Therefore, A_0 is actually the axis of symmetry in the Fibonacci Triangle. The same thing also occurs in the Lucas Triangle.

Now, let us look at the Lucas-Fibonacci Triangle. Although A_0 is not the axis of symmetry of the Triangle, can we find out the relation between (La, Fb) and (Lb, Fa)? In fact, this relation is given by the above formula.

Let us consider an example.

Refer to line 12.

$$\begin{aligned} (L5, F8) &= 231 \\ (L8, F5) &= 235 \\ 231 - 235 &= -4 \\ (L5, F8) - (L8, F5) &= (-1)^5 2F(3) \end{aligned}$$

This is how we can use the formula in the Tables.

Application 5.29. We shall try to use Formula 5.26 to compute large $F(n)$.

From Formula 5.26,

$$\begin{aligned} F(k)L(n) &= F(n+k) + (-1)^{k+1} F(n-k) \\ F(n+k) &= F(k)L(n) + (-1)^k F(n-k) \end{aligned}$$

Note that, in the formula, the largest term is $F(n+k)$.

How useful is this formula? Consider the following example.

$$\begin{aligned} F(41) &= F(21+20) \\ &= F(20)L(21) + (-1)^{20}F(1) \quad \text{(by Formula 5.25)} \\ &= F(20)L(21) + 1 \\ &= 6765 \times 24476 + 1 \\ &= 165580141 \end{aligned}$$

Application 5.30. In section 3, we have investigated how to express $L(kn)$ in terms of $L(n)$. Can we do the same on $F(kn)$ by applying Formula 5.25 $F(k+r) = F(k)L(r) + (-1)^{r+1}F(k-r)$?

Now, substitute $k = r = n$,

$$\begin{aligned} F(n+n) &= F(n)L(n) + (-1)^{n+1}F(n-n) \\ F(2n) &= F(n)L(n) + (-1)^{n+1}F(0) \end{aligned}$$

Formula 5.31.

$$F(2n) = F(n)L(n)$$

Application 5.32. If we substitute $k = 2n$, $r = n$ into Formula 5.25, we have

$$\begin{aligned} F(2n+n) &= F(2n)L(n) + (-1)^{n+1}F(2n-n) \\ F(3n) &= F(n)L(n)L(n) + (-1)n + 1F(n) \quad \text{(by Formula 5.31)} \end{aligned}$$

Formula 5.33.

$$F(3n) = F(n)L(n)^2 + (-1)^{n+1}F(n)$$

Application 5.34. If we substitute $k = 3n$, $r = n$ into Formula 5.25, we have

$$\begin{aligned} F(3n+n) &= F(3n)L(n) + (-1)^{n+1}F(3n-n) \\ F(4n) &= F(n)L(n)^3 + (-1)^{n+1}F(n)L(n) + (-1)^{n+1}F(n)L(n) \\ &\quad \text{(by Formula 5.31 and Formula 5.33)} \end{aligned}$$

Formula 5.35.

$$F(4n) = F(n)L(n)^3 + (-1)^{n+1}2F(n)L(n)$$

Application 5.36. If we substitute $k = 4n$, $r = n$ into Formula 5.25, we have

$$\begin{aligned} F(4n+n) &= F(4n)L(n) + (-1)^{n+1}F(4n-n) \\ F(5n) &= F(n)L(n)^4 + (-1)^{n+1}2F(n)L(n)^2 \\ &\quad + (-1)^{n+1}F(n)L(n)^2 + F(n) \\ &\quad \text{(by Formula 5.33 and Formula 5.35)} \end{aligned}$$

Formula 5.37.

$$F(5n) = F(n)L(n)^4 + (-1)^{n+1}3F(n)L(n)^2 + F(n)$$

Application 5.38. If we substitute $k = 5n$, $r = n$ into Formula 5.25, we have

$$\begin{aligned} F(5n+n) &= F(5n)L(n) + (-1)^{n+1}F(5n-n) \\ F(6n) &= F(n)L(n)^5 + (-1)^{n+1}3F(n)L(n)^3 \\ &\quad + (-1)^{n+1}F(n)L(n)^3 + 2F(n)L(n) \\ &\quad \text{(by Formula 5.35 and Formula 5.37)} \end{aligned}$$

Formula 5.39.

$$F(6n) = F(n)L(n)^5 + (-1)^{n+1}4F(n)L(n)^3 + 3F(n)L(n)$$

Application 5.40. If we substitute $k = 6n$, $r = n$ into Formula 5.25, we have

$$\begin{aligned} F(6n+n) &= F(6n)L(n) + (-1)^{n+1}F(6n-n) \\ F(7n) &= F(n)L(n)^6 + (-1)^{n+1}4F(n)L(n)^4 + 3F(n)L(n)^2 \\ &\quad + (-1)^{n+1}F(n)L(n)^4 + 3F(n)L(n)^2 + (-1)^{n+1}F(n) \\ &\quad \text{(by Formula 5.37 and Formula 5.39)} \end{aligned}$$

Formula 5.41.

$$F(7n) = F(n)L(n)^6 + (-1)^{n+1}5F(n)L(n)^4 + 6F(n)L(n)^2 + (-1)^{n+1}F(n)$$

Application 5.42. If we substitute $k = 7n$, $r = n$ into Formula 5.25, we have

$$\begin{aligned} F(7n+n) &= F(7n)L(n) + (-1)^{n+1}F(7n-n) \\ F(8n) &= F(n)L(n)^7 + (-1)^{n+1}5F(n)L(n)^5 + 6F(n)L(n)^3 \\ &\quad + (-1)^{n+1}F(n)L(n)^5 + 4F(n)L(n)^3 + (-1)^{n+1}3F(n)L(n) \\ &\quad \text{(by Formula 5.39 and Formula 5.41)} \end{aligned}$$

Formula 5.43.

$$
\begin{aligned}
F(8n) = {} & F(n)L(n)^7 + (-1)^{n+1}6F(n)L(n)^5 \\
& + 10F(n)L(n)^3 + (-1)^{n+1}4F(n)L(n)
\end{aligned}
$$

Now, let us recap what we have found.

	$F(1n) = F(n)$
Formula 5.31.	$F(2n) = F(n)L(n)$
Formula 5.33.	$F(3n) = F(n)[L(n)^2 + (-1)^{n+1}]$
Formula 5.35.	$F(4n) = F(n)[L(n)^3 + (-1)^{n+1}2L(n)]$
Formula 5.37.	$F(5n) = F(n)[L(n)^4 + (-1)^{n+1}3L(n)^2 + 1]$
Formula 5.39.	$F(6n) = F(n)[L(n)^5 + (-1)^{n+1}4L(n)^3 + 3L(n)]$
Formula 5.41.	$F(7n) = F(n)[L(n)^6 + (-1)^{n+1}5L(n)^4 + 6L(n)^2 + (-1)^{n+1}]$
Formula 5.43.	$F(8n) = F(n)[L(n)^7 + (-1)^{n+1}6L(n)^5 + 10L(n)^3 + (-1)^{n+1}4L(n)]$

Please look at the coefficients.

1			
1			
1	1		
1	2		
1	3	1	
1	4	3	
1	5	6	1
1	6	10	4

$\longrightarrow$

${}_0C_0$			
${}_1C_0$			
${}_2C_0$	${}_1C_1$		
${}_3C_0$	${}_2C_1$		
${}_4C_0$	${}_3C_1$	${}_2C_2$	
${}_5C_0$	${}_4C_1$	${}_3C_2$	
${}_6C_0$	${}_5C_1$	${}_4C_2$	${}_3C_3$
${}_7C_0$	${}_6C_1$	${}_5C_2$	${}_4C_3$

Note: $(-1)^{n+1}$ appears on even number terms when arranged in descending power of $L(n)$.

We have now found out the relationship between $F(n)$ and Pascal's Triangle. Now, by observation, we generalize that

Hypothesis 5.44.

$$\begin{aligned}
F(4pn) &= F(n)[{}_{4p-1}C_0L(n)^{4p-1} + (-1)^{n+1}{}_{4p-2}C_1L(n)^{4p-3} \\
&\quad + {}_{4p-3}C_2L(n)^{4p-5} + \ldots + {}_{2p+1}C_{2p-2}L(n)^3 \\
&\quad + (-1)^{n+1}{}_{2p}C_{2p-1}L(n)] \\
F((4p+1)n) &= F(n)[{}_{4p}C_0L(n)^{4p} + (-1)^{n+1}{}_{4p-1}C_1L(n)^{4p-2} \\
&\quad + {}_{4p-2}C_2L(n)^{4p-4} + \ldots + (-1)^{n+1}{}_{2p+1}C_{2p-1}L(n)^2 \\
&\quad + {}_{2p}C_{2p}] \\
F((4p+2)n) &= F(n)[{}_{4p+1}C_0L(n)^{4p+1} + (-1)^{n+1}{}_{4p}C_1L(n)^{4p-1} \\
&\quad + {}_{4p-1}C_2L(n)^{4p-3} + \ldots + (-1)^{n+1}{}_{2p+2}C_{2p-1}L(n)^3 \\
&\quad {}_{2p+1}C_{2p}L(n)] \\
F((4p+3)n) &= F(n)[{}_{4p+2}C_0L(n)^{4p+2} + (-1)^{n+1}{}_{4p+1}C_1L(n)^{4p} \\
&\quad + {}_{4p}C_2L(n)^{4p-2} + \ldots + {}_{2p+2}C_{2p}L(n)^2 \\
&\quad + (-1)^{n+1}{}_{2p+1}C_{p+1}]
\end{aligned}$$

Details of Proof for Hypothesis 5.44 can be found Appendix E.

5.2. Expressing $F(n)$ in terms of $L(n)$

In section 5.1, we have found out how to express $L(n)$ in terms of $F(n)$. Now, we want to find out how we can express $F(n)$ in terms of $L(n)$. To find out this relationship, we can use the **Fibonacci Table**, together with the **Lucas sequence**.

Observation 5.45. Let us concentrate on the 1st column. (which is interchangeable with the 1st row, same in the following examples)

$$\begin{aligned} L(4)+L(2) &= 7+3=\mathbf{10}=5\times 2\\ &= 5\times(1\times 2)=5\times(F1,F3)\\ \text{Therefore,}\quad 5F(1)F(3) &= L(4)+L(2). \end{aligned}$$

$$\begin{aligned} L(5)+L(3) &= 11+4=\mathbf{15}=5\times 3\\ &= 5\times(1\times 3)=5\times(F1,F4)\\ \text{Therefore,}\quad 5F(1)F(4) &= L(5)+L(3). \end{aligned}$$

$$\begin{aligned} L(6)+L(4) &= 18+7=\mathbf{25}=5\times 5\\ &= 5\times(1\times 5)=5\times(F1,F5)\\ \text{Therefore,}\quad 5F(1)F(5) &= L(6)+L(4). \end{aligned}$$

And so on.

To generalize the findings, we have

Hypothesis 5.46.

$$5F(1)F(n)=L(n+1)+L(n-1)$$

Observation 5.47. Let us concentrate on the 2nd column.

$$\begin{aligned} L(5)-L(1) &= 11-1=\mathbf{10}=5\times 2\\ &= 5\times(1\times 2)=5\times(F2,F3)\\ \text{Therefore,}\quad 5F(2)F(3) &= L(5)-L(1). \end{aligned}$$

$$\begin{aligned} L(6)-L(2) &= 18-3=\mathbf{15}=5\times 3\\ &= 5\times(1\times 3)=5\times(F2,F4)\\ \text{Therefore,}\quad 5F(2)F(4) &= L(6)-L(2). \end{aligned}$$

$$\begin{aligned} L(7)-L(3) &= 29-4=\mathbf{25}=5\times 5\\ &= 5\times(1\times 5)=5\times(F2,F5)\\ \text{Therefore,}\quad 5F(2)F(5) &= L(7)-L(3). \end{aligned}$$

And so on.

To generalize the findings, we have

Hypothesis 5.48.

$$5F(2)F(n) = L(n+2) - L(n-2)$$

Observation 5.49. Let us concentrate on the 3rd column.

$$\begin{aligned} L(7) + L(1) &= 29 + 1 = \mathbf{30} = 5 \times 6 \\ &= 5 \times (2 \times 3) = 5 \times (F3, F4) \\ \text{Therefore,} \quad 5F(3)F(4) &= L(7) + L(1). \end{aligned}$$

$$\begin{aligned} L(8) + L(2) &= 47 + 3 = \mathbf{50} = 5 \times 10 \\ &= 5 \times (2 \times 5) = 5 \times (F3, F5) \\ \text{Therefore,} \quad 5F(3)F(5) &= L(8) + L(2). \end{aligned}$$

$$\begin{aligned} L(9) + L(3) &= 76 + 4 = \mathbf{25} = 5 \times 16 \\ &= 5 \times (2 \times 8) = 5 \times (F3, F6) \\ \text{Therefore,} \quad 5F(3)F(6) &= L(9) + L(3). \end{aligned}$$

And so on.

To generalize the findings, we have

Hypothesis 5.50.

$$5F(3)F(n) = L(n+3) + L(n-3)$$

Observation 5.51. What about expressing $F(4)F(n)$ in terms of $L(n+4)$ and $L(n-4)$? Let us concentrate on the 4th column.

$$\begin{aligned} L(9) - L(1) &= 76 - 1 = \mathbf{75} = 5 \times 15 \\ &= 5 \times (3 \times 5) = 5 \times (F4, F5) \\ \text{Therefore,} \quad 5F(4)F(5) &= L(9) - L(1). \end{aligned}$$

$$\begin{aligned} L(10) - L(2) &= 123 - 3 = \mathbf{120} = 5 \times 24 \\ &= 5 \times (3 \times 8) = 5 \times (F4, F6) \\ \text{Therefore,} \quad 5F(4)F(6) &= L(10) - L(2). \end{aligned}$$

$$\begin{aligned} L(11) - L(3) &= 199 - 4 = \mathbf{195} = 5 \times 39 \\ &= 5 \times (3 \times 13) = 5 \times (F4, F7) \\ \text{Therefore,} \quad 5F(4)F(7) &= L(11) - L(3). \end{aligned}$$

And so on.

To generalize the findings, we have

Hypothesis 5.52.

$$5F(4)F(n) = L(n+4) - L(n-4)$$

Observation 5.53. Let us concentrate on the 5^{th} column.

$$\begin{aligned} L(11) + L(1) &= 199 + 1 = \mathbf{200} = 5 \times 40 \\ &= 5 \times (5 \times 8) = 5 \times (F5, F6) \\ \text{Therefore,} \quad 5F(5)F(6) &= L(11) + L(1). \end{aligned}$$

$$\begin{aligned} L(12) + L(2) &= 322 + 3 = \mathbf{325} = 5 \times 65 \\ &= 5 \times (5 \times 13) = 5 \times (F5, F7) \\ \text{Therefore,} \quad 5F(5)F(7) &= L(12) + L(2). \end{aligned}$$

$$\begin{aligned} L(13) + L(3) &= 521 + 4 = \mathbf{525} = 5 \times 105 \\ &= 5 \times (5 \times 21) = 5 \times (F5, F8) \\ \text{Therefore,} \quad 5F(5)F(8) &= L(13) + L(3). \end{aligned}$$

And so on.

To generalize the findings, we have

Hypothesis 5.54.

$$5F(5)F(n) = L(n+5) + L(n-5)$$

Up to this point, let us recap what we have found.

Hypothesis 5.46. $5F(1)F(n) = L(n+1) + L(n-1)$
Hypothesis 5.48. $5F(2)F(n) = L(n+2) - L(n-2)$
Hypothesis 5.50. $5F(3)F(n) = L(n+3) + L(n-3)$
Hypothesis 5.52. $5F(4)F(n) = L(n+4) - L(n-4)$
Hypothesis 5.54. $5F(5)F(n) = L(n+5) + L(n-5)$

To generalize the findings, we have

Hypothesis 5.55.

$$5F(k)F(n) = L(n+k) + (-1)^{k+1}L(n-k)$$

Details of Proof for Hypothesis 5.55 can be found in Appendix E.

You may wonder, why we can generate 2 formulae, Formula 5.25 and Formula 5.26, from the observations in section 5.1 but can only generate one formula, Formula 5.55, in section 5.2.

Actually, in section 5.1, since we use the Lucas-Fibonacci Table, which is formed by 2 different sequences (the Lucas sequence and the Fibonacci sequence), we can generate 2 formulae, Formula 5.25 from $L(k)$ and Formula 5.26 from $F(k)$. However, in section 5.2, the Fibonacci Table is in fact formed by the product of two identical sequences, the Fibonacci sequence. Therefore, we can only get Formula 5.55.

Application 5.56. When we look into the Lucas sequence, we do not just look at $L(n)$ with positive n. We sometimes may have to encounter $L(n)$ with negative n, for example, in doing some proofs.

How can we find them?

The answer is easy and simple. By the property $L(n+2) = L(n) + L(n+1)$, we can find out the negative part of the sequence.

$L(0) = 2$	$L(0) = 2$
Positive side	**Negative side**
$L(1) = 1$	$L(-1) = 1$
$L(2) = 3$	$L(-2) = 3$
$L(3) = 4$	$L(-3) = -4$
$L(4) = 7$	$L(-4) = 7$
$L(5) = 11$	$L(-5) = -11$
$L(6) = 18$	$L(-6) = 18$
$L(7) = 29$	$L(-7) = -29$
$L(8) = 47$	$L(-8) = 47$
$L(9) = 76$	$L(-9) = -76$
$L(10) = 123$	$L(-10) = 123$

You may have noticed that $L(k) = (-1)^k L(-k)$. Now, we are going to prove this simple property of the Lucas sequence by Formula 5.55.

Proof. Subsitute $n = a$, $k = b$ into Formula 5.55,

$$L(a+b) = 5F(b)F(a) + (-1)^b L(a-b) \tag{5}$$

Substitute $n = b$, $k = a$ into Formula 5.55,

$$L(b + a) = 5F(a)F(b) + (-1)^a L(b - a) \tag{6}$$

(6) − (5).

$$\begin{aligned} 0 &= (-1)^a L(b-a) - (-1)^b L(a-b) \\ (-1)^b L(a-b) &= (-1)^a L(b-a) \\ L(a-b) &= (-1)^{a-b} L(b-a) \end{aligned}$$

Substitute $p = a - b$, we have

$$L(p) = (-1)^p L(-p).$$

□

Application 5.57. By Formula 5.55, we have

$$L(n + k) = 5F(k)F(n) + (-1)^k L(n - k).$$

Replace k by n,

$$L(2n) = 5F(n)^2 + (-1)^n L(0)$$

Formula 5.58.

$$L(2n) = 5F(n)^2 + (-1)^n (2)$$

This formula will help us prove Hypothesis 3.2. We will discuss the proof in detail in Application 6.68.

6. Squares of Fibonacci and Lucas Numbers

We have introduced various Tables to you. Now, can we make good use of the Tables to prove the relationship between squares of $F(n)$ and $L(n)$ in terms of $F(n \pm k)$ and $L(n \pm k)$ respectively?

6.1. Fibonacci Numbers

For Fibonacci numbers, we have discovered some interesting pattern in squaring Fibonacci numbers:

Observation 6.1.

$$(F4, F4) = F(4)F(4) = 3^2 = 9$$
$$(F3, F5) = F(3)F(5) = 2 \times 5 = 10$$
$$\text{Therefore,} \quad F(4)^2 = F(3)F(5) - 1.$$

$$(F5, F5) = F(5)F(5) = 5^2 = 25$$
$$(F4, F6) = F(4)F(6) = 3 \times 8 = 24$$
$$\text{Therefore,} \quad F(5)^2 = F(4)F(6) + 1.$$

$$(F6, F6) = F(6)F(6) = 8^2 = 64$$
$$(F5, F7) = F(5)F(7) = 5 \times 13 = 65$$
$$\text{Therefore,} \quad F(6)^2 = F(5)F(7) - 1.$$

To generalize the above findings, we have

Hypothesis 6.2.

$$F(n)^2 = F(n-1)F(n+1) + (-1)^{n+1}$$

Details of Proof for Hypothesis 6.2 can be found in Appendix E.

Observation 6.3. This time, instead of using $F(n-1)$ and $F(n+1)$ to compare with $F(n)^2$, we consider $F(n-2)$ and $F(n+2)$.

$$(F5, F5) = F(5)F(5) = 5^2 = 25$$
$$(F3, F7) = F(3)F(7) = 2 \times 13 = 26$$
$$\text{Therefore,} \quad F(5)^2 = F(3)F(7) - 1.$$

$$(F6, F6) = F(6)F(6) = 8^2 = 64$$
$$(F4, F8) = F(4)F(8) = 3 \times 21 = 63$$
$$\text{Therefore,} \quad F(6)^2 = F(4)F(8) + 1.$$

$$(F7, F7) = F(7)F(7) = 13^2 = 169$$
$$(F5, F9) = F(5)F(9) = 5 \times 34 = 170$$
$$\text{Therefore,} \quad F(7)^2 = F(5)F(9) - 1.$$

To generalize the above findings, we have

Hypothesis 6.4.

$$F(n)^2 = F(n-2)F(n+2) + (-1)^n$$

Details of Proof for Hypothesis 6.4 can be found in Appendix E.

Observation 6.5. In this observation, we choose $F(n-3)$ and $F(n+3)$ to compare with $F(n)$.

$$\begin{aligned}
(F5, F5) &= F(5)F(5) = 5^2 = 25 \\
(F2, F8) &= F(2)F(8) = 1 \times 21 = 21 \\
\text{Therefore,} \quad F(5)^2 &= F(2)F(8) + 4.
\end{aligned}$$

$$\begin{aligned}
(F6, F6) &= F(6)F(6) = 8^2 = 64 \\
(F3, F9) &= F(3)F(9) = 2 \times 34 = 68 \\
\text{Therefore,} \quad F(6)^2 &= F(3)F(9) - 4.
\end{aligned}$$

$$\begin{aligned}
(F7, F7) &= F(7)F(7) = 13^2 = 169 \\
(F4, F10) &= F(4)F(10) = 3 \times 55 = 165 \\
\text{Therefore,} \quad F(7)^2 &= F(4)F(10) + 4.
\end{aligned}$$

To generalize the above findings, we have

Hypothesis 6.6.

$$F(n)^2 = F(n-3)F(n+3) + (-1)^{n+1}(4)$$

Observation 6.7. It is expected that we shall use $F(n-4)$ and $F(n+4)$ for comparison this time.

$$(F5, F5) = F(5)F(5) = 5^2 = 25$$
$$(F1, F9) = F(1)F(9) = 1 \times 34 = 34$$
$$\text{Therefore,} \quad F(5)^2 = F(1)F(9) - 9.$$

$$(F6, F6) = F(6)F(6) = 8^2 = 64$$
$$(F2, F10) = F(2)F(10) = 1 \times 55 = 55$$
$$\text{Therefore,} \quad F(6)^2 = F(2)F(10) + 9$$

$$(F7, F7) = F(7)F(7) = 13^2 = 169$$
$$(F3, F11) = F(3)F(11) = 2 \times 89 = 178$$
$$\text{Therefore,} \quad F(7)^2 = F(3)F(11) - 9.$$

To generalize the above findings, we have

Hypothesis 6.8.

$$F(n)^2 = F(n-4)F(n+4) + (-1)^n(9)$$

Observation 6.9. A further investigation of Fibonacci numbers with $F(n-5)$ and $F(n+5)$ and $F(n)^2$ is conducted as follows:

$$(F8, F8) = F(8)F(8) = 21^2 = 441$$
$$(F3, F13) = F(3)F(13) = 2 \times 233 = 466$$
$$\text{Therefore,} \quad F(8)^2 = F(3)F(13) - 25.$$

$$(F9, F9) = F(9)F(9) = 34^2 = 1156$$
$$(F4, F14) = F(4)F(14) = 3 \times 377 = 1131$$
$$\text{Therefore,} \quad F(9)^2 = F(4)F(14) + 25.$$

To generalize the above findings, we have

Hypothesis 6.10.

$$F(n)^2 = F(n-5)F(n+5) + (-1)^{n+1}(25)$$

Up to this point, let us recap what we have found.

Hypothesis 6.2. $F(n)^2 = F(n-1)F(n+1) + (-1)^{n+1}(1)$
Hypothesis 6.4. $F(n)^2 = F(n-2)F(n+2) + (-1)^{n}(1)$
Hypothesis 6.6. $F(n)^2 = F(n-3)F(n+3) + (-1)^{n+1}(4)$
Hypothesis 6.8. $F(n)^2 = F(n-4)F(n+4) + (-1)^{n}(9)$
Hypothesis 6.10. $F(n)^2 = F(n-5)F(n+5) + (-1)^{n+1}(25)$

In other words,

$$F(n)^2 = F(n-1)F(n+1) + (-1)^{n+1}F(1)^2$$
$$F(n)^2 = F(n-2)F(n+2) + (-1)^{n+2}F(2)^2$$
$$F(n)^2 = F(n-3)F(n+3) + (-1)^{n+3}F(3)^2$$
$$F(n)^2 = F(n-4)F(n+4) + (-1)^{n+4}F(4)^2$$
$$F(n)^2 = F(n-5)F(n+5) + (-1)^{n+5}F(5)^2$$

From the above table, generally speaking,

Hypothesis 6.11.

$$F(n)^2 = F(n-k)F(n+k) + (-1)^{n+k}F(k)^2$$

Proof for Hypothesis 6.11.

To do this proof, we can actually follow what we have done in the proofs of Hypothesis 6.2 and Hypothesis 6.4. However, in that way, the proof will be very complicated and may even lead to a dead end. In sight of this, we are going to get them by another approach. It is definitely amazing that, this time, we are actually doing the proof by the Fibonacci Table itself!

Before doing the proof, we have to introduce a special and interesting technique to use the Fibonacci Table.

Observation 6.12. Let us make some observations in the Fibonacci Table.

1	1	2	3	5	8	13	21
1	**1**	2	3	5	8	13	21
2	2	**4**	6	10	16	26	42
3	3	6	**9**	15	24	39	63
5	5	10	15	**25**	40	65	105
8	8	16	24	40	**64**	104	168
13	13	26	39	65	104	**169**	273

What pattern can you observe between the numbers in bold (i.e. numbers that lie on A_0 in the Fibonacci Triangle) and their neighbours?

In fact, across a row or down a column, $U(n) + U(n+1) = U(n+2)$ applies everywhere. With this property of the Fibonacci Table, we can find the sum of numbers on an axis of any length, and the following is the method of summation that we are going to introduce.

First we consider A_0,

Case I: Summing up numbers in bold starting with horizontal summation

0	**1**	1							$0 + \mathbf{1} = 1$
		1							$1 + \mathbf{1} = 2$
		2	**4**	6					$2 + \mathbf{4} = 6$
				9					$6 + \mathbf{9} = 15$
				15	**25**	40			$15 + \mathbf{25} = 40$
						64			$40 + \mathbf{64} = 104$
						104	**169**	$\underline{273}$	$104 + \mathbf{169} = \underline{273}$

You might have already noticed that:

$$\begin{aligned}&\mathbf{1} + 1 + \mathbf{1} + 2 + \mathbf{4} + 6 + \mathbf{9} + 15 + \mathbf{25} + 40 + \mathbf{64} + 104 + \mathbf{169}\\ =\ &1 + 2 + 6 + 15 + 40 + 104 + 273\end{aligned}$$

$$\begin{aligned}&\mathbf{1} + \mathbf{1} + \mathbf{4} + \mathbf{9} + \mathbf{25} + \mathbf{64} + \mathbf{169} + (1 + 2 + 6 + 15 + 40 + 104)\\ =\ &273 + (1 + 2 + 6 + 15 + 40 + 104)\end{aligned}$$

$$\underline{1 + 1 + 4 + 9 + 25 + 64 + 169 = 273}$$

Case II: Summing up numbers in bold starting with vertical summation

0								
1								$0 + \mathbf{1} = 1$
1	**1**	2						$1 + \mathbf{1} = 2$
		4						$2 + \mathbf{4} = 6$
		6	**9**	15				$6 + \mathbf{9} = 15$
				25				$15 + \mathbf{25} = 40$
				40	**64**	104		$40 + \mathbf{64} = 104$
						169		$104 + \mathbf{169} = 273$
						$\underline{273}$		

In fact, we can see that we can find the sum of number on an axis of any length by looking for the number on the right or immediately below the number at the end of the summation.

In case I, since we start with horizontal summation, we get $0 + \mathbf{1} = 1$. After the first horizontal summation, we go downwards to do the vertical summation, obtaining $1 + \mathbf{1} = 2$. After that, we go to the right to do the horizontal summation, obtaining $2 + \mathbf{4} = \underline{6}$. This $\underline{6}$ is the sum of the first three numbers in the A_0. Also, this number 6 is found to the right of the final number in the summation, that is, 4.

In case II, since we start with vertical summation, we get $0 + \mathbf{1} = 1$. After the first vertical summation, we go to the right to do the horizontal summation, obtaining $1 + \mathbf{1} = 2$. After that, we go downwards to do the vertical summation, obtaining $2 + \mathbf{4} = \underline{6}$. This $\underline{6}$ is the sum of the first three numbers in the A_0. Also, this number 6 is found immediately below the final number in the summation, that is, 4.

It is interesting to know that this method of summation is just like playing Chinese checker ("Chinese-checker-like method of summation"). This is what we would like to use in the proof.

Then, what if we want to add up the numbers from 4 to 169 in the A_0 instead of starting from the beginning? The mechanism is just the same. We start by horizontal summation in the context of case I. So we take the number 2 to 'trigger' the series of summation. After a series of horizontal and vertical summation, we get 273 in the end. And the sum from 4 to 169 on A_0 should be $273 - 2 = 271$.

Application 6.13. With this method, we can find out the summation of the terms lying on A_k. For example, as shown above, all the numbers that lie on A_0 in the Fibonacci Triangle can be illustrated by $F(n) \times F(n)$, i.e. $F(n)^2$.

From the above observation and since $F(0)$ is defined as 0, we get the following special property:

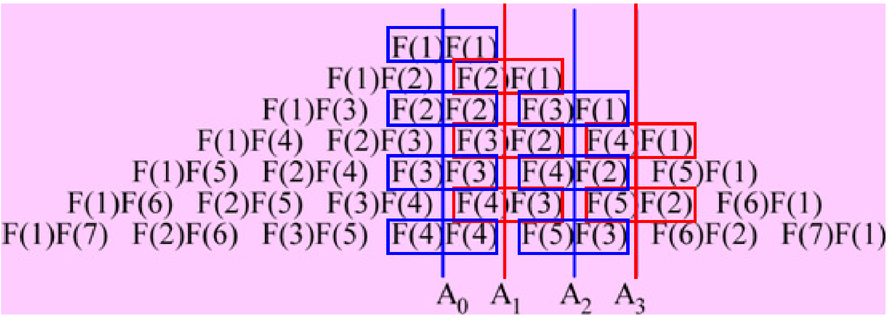
$$\sum_{n=1}^{7} F(n)^2 = 273$$
$$\sum_{n=1}^{7} F(n)^2 = F(7)F(8)$$

From this, we have

Formula 6.14.

$$\sum_{k=1}^{n} F(k)^2 = F(n)F(n+1)$$

Observation 6.15.



A_0 represents $F(n)F(n)$, A_1 represents the terms $F(n)F(n-1)$, ... and A_k represents the terms $F(n)F(n-k)$. By the Chinese-checker-like method of summation, we can find out the sum of terms on A_k i.e. summation of $F(n)F(n-k)$.

Now we have investigated the special property of the summation of axis A_0. So does the same property apply to the axis A_1 or A_{-1}?

Consider A_1,

Case I: Summing up numbers in bold starting with horizontal summation

0	**1**	1							$0 + \mathbf{1} = 1$
		2							$1 + \mathbf{2} = 3$
		3	**6**	9					$3 + \mathbf{6} = 9$
				15					$9 + \mathbf{15} = 24$
				24	**40**	64			$24 + \mathbf{40} = 64$
						104			$64 + \mathbf{104} = \underline{168}$
						$\underline{168}$			

Case II: Summing up numbers in bold starting with vertical summation

1								
1								$1 + \mathbf{1} = 2$
2	**2**	4						$2 + \mathbf{2} = 4$
		6						$4 + \mathbf{6} = 10$
		10	**15**	25				$10 + \mathbf{15} = 25$
				40				$25 + \mathbf{40} = 65$
				65	**104**	$\underline{169}$		$65 + \mathbf{104} = \underline{169}$

We discovered that using the same method of summation, there is a difference of 1 between the two numbers what we get in the underlined. Let us investigate a longer series of numbers on A_1.

Case I: Summing the numbers starting with horizontal summation

0	**1**	1							$0 + \mathbf{1} = 1$
		2							$1 + \mathbf{2} = 3$
		3	**6**	9					$3 + \mathbf{6} = 9$
				15					$9 + \mathbf{15} = 24$
				24	**40**	64			$24 + \mathbf{40} = 64$
						104			$64 + \mathbf{104} = 168$
						168	**273**	$\underline{441}$	$168 + \mathbf{273} = \underline{441}$

Case II: Summing up numbers in bold starting with vertical summation

1								
1								$1+\mathbf{1}=2$
2	**2**	4						$2+\mathbf{2}=4$
		6						$4+\mathbf{6}=10$
		10	**15**	25				$10+\mathbf{15}=25$
				40				$25+\mathbf{40}=65$
				65	**104**	<u>169</u>		$65+\mathbf{104}=169$
						273		$169+\mathbf{273}=\underline{442}$
						<u>442</u>		

We still get the same result using a longer series of numbers. That leads us to the following observations:

$$\sum_{k=1}^{6} F(k)F(k+1) = 169-1 = F(7)F(7) - F(1)F(1)$$

or

$$= 168 - 0 = F(6)F(8)$$

$$\sum_{k=1}^{7} F(k)F(k+1) = 442-1 = F(7)F(9) - F(1)F(1)$$

or

$$= 441 - 0 = F(8)F(8)$$

Application 6.16. Here is a question that can be solved easily with the help of the Fibonacci Table. Compute $F(1)F(6)+F(2)F(7)+F(3)F(8)+\ldots+F(r)F(r+5)+\ldots+F(10)F(15)$.

By using the table, there are two approaches to solve this problem.

Method I

Consider $F(11)F(15)$. By reversing the process of Chinese-checker-like method of summation, we will go through the numbers $F(9)F(15)$, $F(9)F(13)$, $F(7)F(13)$, $F(7)F(11)$, $F(5)F(11)$, $F(5)F(9)$, $F(3)F(9)$, $F(3)F(7)$, $F(3)F(5)$ and finally $F(1)F(5)$.

Therefore,

$$
\begin{aligned}
& F(1)F(6) + F(2)F(7) + F(3)F(8) + \ldots + F(r)F(r+5) \\
& \quad + \ldots + F(10)F(15) \\
= {} & F(11)F(15) - F(1)F(5) \\
= {} & (89)(610) - (1)(5) \\
= {} & 54290 - 5 \\
= {} & 54285
\end{aligned}
$$

Method II

Consider $F(10)F(16)$. By reversing the process of Chinese-checker-like method of summation, we will go through the numbers $F(10)F(14)$, $F(8)F(14)$, $F(8)F(12)$, $F(6)F(12)$, $F(6)F(10)$, $F(4)F(10)$, $F(4)F(8)$, $F(2)F(8)$, $F(2)F(6)$ and finally $F(0)F(6)$.

Therefore,

$$
\begin{aligned}
& F(1)F(6) + F(2)F(7) + F(3)F(8) + \ldots + F(r)F(r+5) \\
& \quad + \ldots + F(10)F(15) \\
= {} & F(10)F(16) - F(0)F(6) \\
= {} & (55)(987) - (0)(8) \\
= {} & 54285 - 0 \\
= {} & 54285
\end{aligned}
$$

Here is a little trick in solving this problem. Consider the final term $F(a)F(a+k)$ which lies on A_k. If k is odd, it means either a or $a+k$ is odd, say, a is odd and $a+k$ is even, then the answer is directly given by $F(a+1)F(a+k)$. This is due to the reverse process of Chinese-checker-like method of summation, we finally come to $F(0)F(k-1)$, which is 0. It should be noted that this trick is applied only when the summation is done from the beginning of the axis, i.e. $F(1)F(n)$ or $F(n)F(1)$.

Therefore, the answer to the above question is $F(10)F(16)$ as the following conditions are given: (1) the last term in the summation is $F(10)F(15)$; (2) the summation is done from the beginning of the axis A_5; and (3) The number 15 in $F(15)$ is odd.

Application 6.17.

Here, we would like you to observe some special things in the table. (For the sake of convenience, we rotate the table so that it stands upright like a triangle.)

1
1 1
2 1 2
3 2 2 3
5 3 4 3 5
8 5 6 6 5 8
13 8 10 9 10 8 13
21 13 16 15 15 16 13 21
34 21 26 24 25 24 26 21 34

Please keep in mind that, each line in the Triangle represents a diagonal in the Table.

Now focus on the differences between neighbouring numbers on each line.

Line
4   3 2 2 3
+1 0 +1
5   5 3 4 3 5
+2 −1 −1 +2
6   8 5 6 6 5 8
+3 −1 0 −1 +3
7   13 8 10 9 10 8 13
+5 −2 +1 +1 −2 +5
8   21 13 16 15 15 16 13 21
+8 −3 +1 0 +1 −3 +8
9   34 21 26 24 25 24 26 21 34
+13 −5 +2 −1 −1 +2 −5 +13

From the above observation, we can generalize them as follows:

(Note: For the sake of formatting, F_n is used to indicate $F(n)$ in Application 6.17 only)

Line

$\underline{4k}$

$$F_{2k-2}F_{2k+3} \quad F_{2k-1}F_{2k+2} \quad F_{2k}F_{2k+1} \quad F_{2k+1}F_{2k} \quad F_{2k+2}F_{2k-1} \quad F_{2k+3}F_{2k-2}$$
$$+8 \quad -3 \quad +1 \quad 0 \quad +1 \quad -3 \quad +8$$
$$+F(6) \quad -F(4) \quad +F(2) \quad F(0) \quad +F(2) \quad -F(4) \quad +F(6)$$

$\underline{4k+1}$

$$F_{2k-1}F_{2k+3} \quad F_{2k}F_{2k+2} \quad F_{2k+1}F_{2k+1} \quad F_{2k+2}F_{2k} \quad F_{2k+3}F_{2k-1}$$
$$-5 \quad +2 \quad -1 \quad -1 \quad +2 \quad -5$$
$$-F(5) \quad +F(3) \quad -F(1) \quad -F(1) \quad +F(3) \quad -F(5)$$

$\underline{4k+2}$

$$F_{2k-1}F_{2k+4} \quad F_{2k}F_{2k+3} \quad F_{2k+1}F_{2k+2} \quad F_{2k+2}F_{2k+1} \quad F_{2k+3}F_{2k} \quad F_{2k+4}F_{2k-1}$$
$$-8 \quad +3 \quad -1 \quad 0 \quad -1 \quad +3 \quad -8$$
$$-F(6) \quad +F(4) \quad -F(2) \quad F(0) \quad -F(2) \quad +F(4) \quad -F(6)$$

$\underline{4k+3}$

$$F_{2k}F_{2k+4} \quad F_{2k+1}F_{2k+3} \quad F_{2k+2}F_{2k+2} \quad F_{2k+3}F_{2k+1} \quad F_{2k+4}F_{2k}$$
$$+5 \quad -2 \quad +1 \quad +1 \quad -2 \quad +5$$
$$+F(5) \quad -F(3) \quad +F(1) \quad +F(1) \quad -F(3) \quad +F(5)$$

You may think that if we want to prove this relationship in a purely mathematical way, the proof will be very complicated. In fact, this Chinese-checker-like method of summation itself serves as an elegant proof!

Consider the following cases.

Case I

Consider line 5.

1
1 1
2 1 2
3 2 2 3
5 **3** **4** 3 5

How can we explain that $4 = 1 + 3$, i.e. $F(3)F(3) = F(1) + F(2)F(4)$?

We can convert the Triangle back to the Table again.

0	0	0	0	0	0
0	**1**	1	2	3	5
0	(1)	1	2	3	
0	2	(2)	**4**		
0	3	**3**			
0	5				

By the Chinese-checker-like method of summation:
If we use 4,

$$(1) + (2) = 4 - 1$$

If we use 3,

$$(1) + (2) = 3 - 0$$

Therefore,

$$4 - 1 = 3 - 0$$
$$4 = 1 + 3$$

This approach proves the difference between neighbouring numbers on each line.

Case II

To further our explanation, let us consider one more case.

Row \ Column	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	1	2	3	5	8	13	21	34	55
2	1	1	2	3	5	8	13	21	34	55
3	2	2	4	6	10	16	26	42	68	110
4	3	3	6	9	15	24	39	63	102	165
5	5	5	10	15	25	40	65	105	170	275
6	<u>8</u>	8	16	24	40	64	104	168	272	440
7	13	<u>13</u>	26	39	65	104	169	273	442	715
8	21	21	<u>42</u>	63	105	168	273	441	714	1155
9	34	34	68	<u>102</u>	**170**	272	442	714	1156	1870
10	55	55	110	**165**	275	440	715	1155	1870	3025

To prove: $170 = 165 + F(5)$.

Applying the Chinese-checker-like method of summation, we have:

$$8 + 13 + 42 + 102 = 170 - 5 = 170 - F(5) \tag{7}$$
$$8 + 13 + 42 + 102 = 165 - 0 = 165 - F(0) \tag{8}$$

Combining (7) and (8), we have:

$$170 - F(5) = 165 - F(0)$$
$$170 = 165 + F(5)$$

Up to this point, by the two examples above, you should be able to understand how the Chinese-checker-like method of summation can prove **the differences between neighbouring numbers on each line** in the Fibonacci Triangle.

Now, we are going to use it to find the difference between the middle term (as the middle term represents the square of $F(n)$) and other terms (not neighbouring to each other) on the same line in the Fibonacci Triangle.

Observation 6.18. On line $4k$, middle term $= F(2k)F(2k+1) = M_1$. Consider the n^{th} term from M on the same line,

n	n^{th} term	
1	$M_1 + 1$	$= M_1 + 1$
2	$M_1 + 1 - 3$	$= M_1 - 2$
3	$M_1 + 1 - 3 + 8$	$= M_1 + 6$
4	$M_1 + 1 - 3 + 8 - 21$	$= M_1 - 15$
5	$M_1 + 1 - 3 + 8 - 21 + 55$	$= M_1 + 40$
6	$M_1 + 1 - 3 + 8 - 21 + 55 - 144$	$= M_1 - 104$
...		

Hypothesis 6.19.

p	$M_1 + 1 - 3 + 8 - 21 + \dots + (-1)^{p+1}F(2p)$	$= M_1 + (-1)^{p+1}F(p)F(p+1)$

Note: 1, 2, 6, 15, 40, 104 lie on $A_{\pm 1}$.

An illustration of line $4k$ would be as follows:

On line 12, the middle term $= F(6)F(7) = 104$

$$\begin{aligned}\text{The term on } A_3 &= M_1 + (-1)^4 F(3)F(4)\\ &= 104 + 2 \times 3\\ &= 110\\ &= 2 \times 55\\ &= F(3)F(10)\end{aligned}$$

Details of Proof for Hypothesis 6.19 can be found in Appendix E.

Observation 6.20. On line $(4k+1)$, middle term $= F(2k+1)F(2k+1) = M_2$

n	n^{th} term	
1	$M_2 - 1$	$= M_2 - 1$
2	$M_2 - 1 + 2$	$= M_2 + 1$
3	$M_2 - 1 + 2 - 5$	$= M_2 - 4$
4	$M_2 - 1 + 2 - 5 + 13$	$= M_2 + 9$
5	$M_2 - 1 + 2 - 5 + 13 - 34$	$= M_2 - 25$
6	$M_2 - 1 + 2 - 5 + 13 - 34 + 89$	$= M_2 + 64$
...		

Hypothesis 6.21.

p	$M_2 - 1 + 2 - 5 + 13 - 34 + \dots + (-1)^p F(2p-1)$	$= M_2 + (-1)^p F(p)F(p)$

An illustration of line $(4k+1)$ would be as follows:

On line 13, the middle term $= F(7)F(7) = 169$

$$\begin{aligned}\text{The term on } A_4 &= M_2 + (-1)^4 F(4)F(4)\\ &= 169 + 3 \times 3\\ &= 178\\ &= 2 \times 89\\ &= F(3)F(11)\end{aligned}$$

Details of Proof for Hypothesis 6.21 can be found in Appendix E.

Observation 6.22. On line $(4k+2)$, middle term $= F(2k+1)F(2k+2) =$

M_3

n	n^{th} term	
1	$M_3 - 1$	$= M_3 - 1$
2	$M_3 - 1 + 3$	$= M_3 + 2$
3	$M_3 - 1 + 3 - 8$	$= M_3 - 6$
4	$M_3 - 1 + 3 - 8 + 21$	$= M_3 + 15$
5	$M_3 - 1 + 3 - 8 + 21 - 55$	$= M_3 - 40$
6	$M_3 - 1 + 3 - 8 + 21 - 55 + 144$	$= M_3 + 104$
...		

Hypothesis 6.23.

$p \quad M_3 - 1 + 3 - 8 + \ldots + (-1)^p F(2p) = M_3 + (-1)^p F(p)F(p+1)$

An illustration of line $(4k+2)$ would be as follows:

On line 14, the middle term $= F(7)F(8) = 273$

$$\begin{aligned}\text{The term on } A_3 &= M_3 + (-1)^3 F(3)F(4)\\ &= 273 - 2 \times 3\\ &= 267\\ &= 3 \times 89\\ &= F(4)F(11)\end{aligned}$$

Details of Proof for Hypothesis 6.23 can be found in Appendix E.

Observation 6.24. On line $(4k+3)$, middle term $= F(2k+2)F(2k+2) = M_4$.

n	n^{th} term	
1	$M_4 + 1$	$= M_4 + 1$
2	$M_4 + 1 - 2$	$= M_4 - 1$
3	$M_4 + 1 - 2 + 5$	$= M_4 + 4$
4	$M_4 + 1 - 2 + 5 - 13$	$= M_4 - 9$
5	$M_4 + 1 - 2 + 5 - 13 + 34$	$= M_4 + 25$
6	$M_4 + 1 - 2 + 5 - 13 + 34 - 89$	$= M_4 - 64$
...		

Hypothesis 6.25.

$p \quad M_4 + 1 - 2 + 5 + \ldots + (-1)^{p+1} F(2p-1) = M_4 + (-1)^{p+1} F(p)F(p)$

An illustration of line $(4k+3)$ would be as follows:

On line 15, the middle term $= F(8)F(8) = 441$

$$\begin{aligned}\text{The term on } A_5 &= M_4 + (-1)^6 F(5)F(5)\\ &= 441 + 25\\ &= 466\\ &= 2 \times 233\\ &= F(3)F(13)\end{aligned}$$

Details of Proof for Hypothesis 6.25 can be found in Appendix E.

On the whole, 2 new formulae are formed.

$$F(2) - F(4) + F(6) + \ldots + (-1)^{n+1}F(2n) = (-1)^{n+1}F(n)F(n+1)$$

i.e.

Formula 6.26.

$$\sum_{k=1}^{n} (-1)^{k+1} F(2k) = (-1)^{n+1} F(n)F(n+1)$$

$$F(1) - F(3) + F(5) + \ldots + (-1)^{n+1}F(2n-1) = (-1)^{n+1}F(n)^2$$

i.e.

Formula 6.27.

$$\sum_{k=1}^{n} (-1)^{k+1} F(2k-1) = (-1)^{n+1} F(n)F(n)$$

Application 6.28. By adding up Formula 6.26 and Formula 6.27, we have

$$\begin{aligned}\sum_{k=1}^{n} (-1)^{k+1}[F(2k) + F(2k-1)] &= (-1)^{n+1}F(n)[F(n) + F(n+1)]\\ &= (-1)^{n+1}F(n)F(n+2)\end{aligned}$$

In order to explain more clearly, we are going to use the previous example. Here is the Fibonacci Table.

1	1	2	3	5	8	13	21	34
1	**1**	2	3	5	8	13	21	34
2	2	**4**	6	10	16	26	42	68
3	3	6	**9**	15	24	39	63	102
5	5	10	15	**25**	40	65	105	170
8	8	16	24	40	**64**	104	168	272
13	13	26	39	65	104	**169**	273	442
21	21	42	63	105	168	273	**441**	714
34	34	68	102	170	272	442	714	**1156**

Please look at the bold-faced numbers in the table. They all represent $F(n)^2$. For instance, $(F5, F5) = 25 = 5 \times 5 = F(5)^2$ and $169 = 13 \times 13 = F(7)^2$. All these numbers lie on A_0 and odd-number lines. For instance, 25 lies on line 9 and 169 lies on line 13.

Therefore, we have underlined all the terms on odd number lines for easy visualization. As mentioned above, every underlined term differs from its neighbouring term by $F(2k-1)$.

For example, on line 11,

$$\begin{aligned}
65 &= 64 + 1 = 64 + F(1) \\
63 &= 65 - 2 = 65 - F(3) \\
68 &= 63 + 5 = 63 + F(5), \text{ and so on.}
\end{aligned}$$

Also, look at line 9,

$$\begin{aligned}
24 &= 25 - 1 = 25 - F(1) \\
26 &= 24 + 2 = 24 + F(3) \\
21 &= 26 - 5 = 26 - F(5) \\
34 &= 21 + 13 = 21 + F(7), \text{ and so on.}
\end{aligned}$$

In fact, the pattern repeats itself every 4 lines.

Now, we are going to express all the underlined terms in terms of $F(n)^2$ on the same line. Consider the previous examples.

On line 11,

$$65 = 64 + 1 = F(6)^2 + F(1)^2$$
$$63 = 64 + 1 - 2 = 64 - 1 = F(6)^2 - F(2)^2$$
$$68 = 64 + 1 - 2 + 5 = 64 + 4 = F(6)^2 + F(3)^2, \text{ and so on.}$$

Note: This is actually $F(1) - F(3) + F(5) + \ldots + (-1)^{n+1}F(2n-1) = (-1)^{n+1}F(n)^2$.

Also, on line 9,

$$24 = 25 - 1 = F(5)^2 - F(1)^2$$
$$26 = 25 - 1 + 2 = 25 + 1 = F(5)^2 + F(2)^2$$
$$21 = 25 - 1 + 2 - 5 = 25 - 4 = F(5)^2 - F(3)^2$$
$$34 = 25 - 1 + 2 - 5 + 13 = 25 + 9 = F(5)^2 + F(4)^2, \text{ and so on.}$$

Note: This is actually $-F(1) + F(3) + F(5) + \ldots + (-1)^n F(2n-1) = (-1)^n F(n)^2$.

Note that in the Table, we only have a term representing $F(n)^2$ on every **odd**-number lines. We have used line 9 and 11 to demonstrate this.

In general, we have two cases.

Case I

On line $(4r+1)$, as represented by line 9,

$$\begin{aligned} & \text{The } k\text{-th term from } F(2r+1)^2 \\ = {} & F(2r+1+k)F(2r+1-k) \\ = {} & F(2r+1)^2 - F(1) + F(3) - F(5) + \ldots + (-1)^k F(2k-1) \\ = {} & F(2r+1)^2 + (-1)^k F(k)F(k) \quad \text{(by Formula 6.27)} \\ = {} & F(2r+1)^2 + (-1)^k F(k)^2 \end{aligned}$$

Case II

On line $(4r + 3)$, as represented by line 11,

$$\begin{aligned} &\text{The } k\text{-th term from } F(2r+2)^2 \\ &= F(2r+2+k)F(2r+2-k) \\ &= F(2r+2)^2 + F(1) - F(3) + F(5) - \ldots + (-1)^{k+1}F(2k-1) \\ &= F(2r+2)^2 + (-1)^{k+1}F(k)F(k) \quad \text{(by Formula 6.27)} \\ &= F(2r+2)^2 + (-1)^{k+1}F(k)^2 \end{aligned}$$

Now we have two formulae:

From case I,

$$F(2r+1+k)F(2r+1-k) = F(2r+1)^2 + (-1)^k F(k)^2$$

or

Formula 6.29.

$$F(2r+1)^2 = F(2r+1+k)F(2r+1-k) + (-1)^{k+1}F(k)^2$$

From case II,

$$F(2r+2+k)F(2r+2-k) = F(2r+2)^2 + (-1)^{k+1}F(k)^2$$

or

Formula 6.30.

$$F(2r+2)^2 = F(2r+2+k)F(2r+2-k) + (-1)^k F(k)^2$$

From Formula 6.29, we have

$$F(2r+1)^2 = F(2r+1+k)F(2r+1-k) + (-1)^{k+2r+1}F(k)^2$$

From Formula 6.30, we have

$$F(2r+2)^2 = F(2r+2+k)F(2r+2-k) + (-1)^{k+2r+2}F(k)^2$$

Now, replace $(2r + 1)$ by n, we have
(a) $F(n)^2 = F(n+k)F(n-k) + (-1)^{n+k}F(k)^2$
(b) $F(n+1)^2 = F(n+1+k)F(n+1-k) + (-1)^{n+k+1}F(k)^2$

So we have exactly proved Formula 6.11. Isn't that miraculous?

Application 6.31. Compute $F(10) - F(11) - F(12) + F(13) + \ldots - F(43) - F(44)$.

Compare this question to Application 6.16.

$$\begin{aligned}\text{Answer} &= [(-1)^{23}F(22)F(24) - (-1)^5F(4)F(6)] - F(9)\\ &= [-17711 \times 46368 + 3 \times 8] - 34\\ &= -821223658\end{aligned}$$

Application 6.32. If we want to resolve large $F(n)$s in terms of small $F(n)$s only, we can always use Formula 6.11

$$F(n+k) = \frac{F(n)^2 + (-1)^{n+k+1}F(k)^2}{F(n-k)}$$

because $F(n+k)$ is the largest among all the terms.

Note that $(n-k)$ is not equal to 0.

For example, we want to find $F(80)$.

Substitute $n = 50$, $k = 30$ into Formula 6.11,

$$\begin{aligned}F(80) &= F(50+30)\\ &= \frac{F(50)^2 + (-1)^{81}F(30)^2}{F(20)}\end{aligned}$$

$$\begin{aligned}F(30) &= F(20+10)\\ &= \frac{F(20)^2 + (-1)^{31}F(10)^2}{F(10)}\\ &= \frac{6765^2 - 55^2}{55}\\ &= 832040\end{aligned}$$

$$\begin{aligned}F(50) &= F(30+20)\\ &= \frac{F(30)^2 + (-1)^{51}F(20)^2}{F(10)}\\ &= \frac{832040^2 - 6765^2}{55}\\ &= 12586269025\end{aligned}$$

Hence,

$$\begin{aligned} F(80) &= F(50+30) \\ &= \frac{F(50)^2 + (-1)^{81}F(30)^2}{F(20)} \\ &= \frac{12586269025^2 - 832040^2}{6765} \\ &= 23416728348467685 \end{aligned}$$

If we want to apply this formula, however, when we choose n and k, n and k should not be too far away. Take $F(50)$ as an example. If we choose $n = 48$, $k = 2$, when we break down $F(48+2)$, we will get $F(48)^2$, $F(2)^2$ and $F(46)$. That, in fact, makes things more complicated as there are two large Fibonacci numbers to be resolved.

Application 6.33. If $(n+k)$ is odd, we substitute $n = k+1$ into Formula 6.11, we have

$$F(2k+1) = \frac{F(k+1)^2 + (-1)^{2k+2}F(k)^2}{F(1)}$$

Therefore, we have

Formula 2.28.

$$F(2k+1) = F(k+1)^2 + F(k)^2$$

We come back to Lucas' discovery in 1876.

Application 6.34. If $(n+k)$ is even, we cannot substitute $n = k$ because $F(n-k)$ will become $F(0) = 0$ which cannot be the denominator.

We substitute $n = k+2$ into Formula 6.11, we have

$$F(2k+2) = \frac{F(k+2)^2 + (-1)^{2k+3}F(k)^2}{F(2)}$$

$$F(2k+2) = F(k+2)^2 - F(k)^2$$

Therefore, we have

Formula 2.24.

$$F(2k) = F(k+1)^2 - F(k-1)^2$$

6.2. Lucas Numbers

Observation 6.35.

$$
\begin{aligned}
(L3, L3) &= L(3)^2 = 16 \\
(L2, L4) &= L(2)L(4) = 21 \\
\text{Therefore,} \quad L(3)^2 &= L(2)L(4) - 5.
\end{aligned}
$$

$$
\begin{aligned}
(L4, L4) &= L(4)^2 = 49 \\
(L3, L5) &= L(3)L(5) = 44 \\
\text{Therefore,} \quad L(4)^2 &= L(3)L(5) + 5.
\end{aligned}
$$

$$
\begin{aligned}
(L5, L5) &= L(5)^2 = 121 \\
(L4, L6) &= L(4)L(6) = 126 \\
\text{Therefore,} \quad L(5)^2 &= L(4)L(6) - 5.
\end{aligned}
$$

To generalize the above findings, we have

Hypothesis 6.36.

$$L(n)^2 = L(n-1)L(n+1) + (-1)^n(5)$$

Details of Proof for Hypothesis 6.36 can be found in Appendix E.

Observation 6.37.

$$
\begin{aligned}
(L4, L4) &= L(4)^2 = 49 \\
(L2, L6) &= L(2)L(6) = 54 \\
\text{Therefore,} \quad L(4)^2 &= L(2)L(6) - 5.
\end{aligned}
$$

$$
\begin{aligned}
(L5, L5) &= L(5)^2 = 121 \\
(L3, L7) &= L(3)L(7) = 116 \\
\text{Therefore,} \quad L(5)^2 &= L(3)L(7) + 5.
\end{aligned}
$$

$$
\begin{aligned}
(L6, L6) &= L(6)^2 = 324 \\
(L4, L8) &= L(4)L(8) = 329 \\
\text{Therefore,} \quad L(6)^2 &= L(4)L(8) - 5.
\end{aligned}
$$

To generalize the above findings, we have

Hypothesis 6.38.

$$L(n)^2 = L(n-2)L(n+2) + (-1)^{n+1}(5)$$

Details of Proof for Hypothesis 6.38 can be found in Appendix E.

Observation 6.39.

$$\begin{aligned} (L4, L4) &= L(4)^2 = 49 \\ (L1, L7) &= L(1)L(7) = 29 \\ \text{Therefore,} \quad L(4)^2 &= L(1)L(7) + 20. \end{aligned}$$

$$\begin{aligned} (L5, L5) &= L(5)^2 = 121 \\ (L2, L8) &= L(2)L(8) = 141 \\ \text{Therefore,} \quad L(5)^2 &= L(2)L(8) - 20. \end{aligned}$$

$$\begin{aligned} (L6, L6) &= L(6)^2 = 324 \\ (L3, L9) &= L(3)L(9) = 304 \\ \text{Therefore,} \quad L(6)^2 &= L(3)L(9) + 20. \end{aligned}$$

To generalize the above findings, we have

Hypothesis 6.40.

$$L(n)^2 = L(n-3)L(n+3) + (-1)^n(20)$$

or

$$L(n)^2 = L(n-3)L(n+3) + (-1)^n(5)(4)$$

Observation 6.41.

$$(L5, L5) = L(5)^2 = 121$$
$$(L1, L9) = L(1)L(9) = 76$$
$$\text{Therefore,} \quad L(5)^2 = L(1)L(9) + 45.$$

$$(L6, L6) = L(6)^2 = 324$$
$$(L2, L10) = L(2)L(10) = 369$$
$$\text{Therefore,} \quad L(6)^2 = L(2)L(10) - 45.$$

$$(L7, L7) = L(7)^2 = 841$$
$$(L3, L11) = L(3)L(11) = 796$$
$$\text{Therefore,} \quad L(7)^2 = L(3)L(11) + 45.$$

To generalize the above findings, we have

Hypothesis 6.42.

$$L(n)^2 = L(n-4)L(n+4) + (-1)^{n+1}(45)$$

or

$$L(n)^2 = L(n-4)L(n+4) + (-1)^{n+1}(5)(9)$$

Observation 6.43.

$$(L6, L6) = L(6)^2 = 324$$
$$(L1, L11) = L(1)L(11) = 199$$
$$\text{Therefore,} \quad L(6)^2 = L(1)L(11) + 125.$$

$$(L7, L7) = L(7)^2 = 841$$
$$(L2, L12) = L(2)L(12) = 966$$
$$\text{Therefore,} \quad L(7)^2 = L(2)L(12) - 125.$$

To generalize the above findings, we have

Hypothesis 6.44.

$$L(n)^2 = L(n-5)L(n+5) + (-1)^n(125)$$

or

$$L(n)^2 = L(n-5)L(n+5) + (-1)^n(5)(25)$$

Up to this point, let us recap what we have found.

Hypothesis 6.36. $L(n)^2 = L(n-1)L(n+1) + (-1)^n(5)$
Hypothesis 6.38. $L(n)^2 = L(n-2)L(n+2) + (-1)^{n+1}(5)$
Hypothesis 6.40. $L(n)^2 = L(n-3)L(n+3) + (-1)^n(5)(4)$
Hypothesis 6.42. $L(n)^2 = L(n-4)L(n+4) + (-1)^{n+1}(5)(9)$
Hypothesis 6.44. $L(n)^2 = L(n-5)L(n+5) + (-1)^n(5)(25)$

In other words,

$$L(n)^2 = L(n-1)L(n+1) + (-1)^{n+2}(5)F(1)^2$$
$$L(n)^2 = L(n-2)L(n+2) + (-1)^{n+3}(5)F(2)^2$$
$$L(n)^2 = L(n-3)L(n+3) + (-1)^{n+4}(5)F(3)^2$$
$$L(n)^2 = L(n-4)L(n+4) + (-1)^{n+5}(5)F(4)^2$$
$$L(n)^2 = L(n-5)L(n+5) + (-1)^{n+6}(5)F(5)^2$$

To generalize the findings, we have

Hypothesis 6.45.

$$L(n)^2 = L(n-k)L(n+k) + (-1)^{n+k+1}(5)F(k)^2$$

Proof for Hypothesis 6.45.

Again, to do this proof, we can actually follow what we have done in the proofs for Hypothesis 6.36 and Hypothesis 6.38. However, in that way, the proof is going to be very complicated and may even come to a dead end. In sight of this, we are going to get them by the Chinese-checker-like method of summation we have introduced before.

This time, we are going to do the proof by using the Lucas Table.

With the Chinese-checker-like method, we can find out the summation of the terms lying on A_k. For example, all the numbers that lie on A_0 in the Lucas Triangle can be illustrated by $L(n) \times L(n)$, i.e. $L(n)^2$.

From the above observation and since $L(0)$ is defined as 2, we get the following special property:

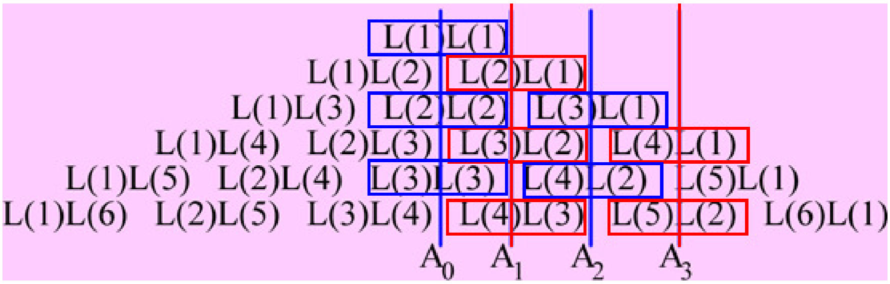
$$\sum_{n=1}^{7} L(n)^2 = 1361$$
$$\sum_{n=1}^{7} L(n)^2 = 29 \times 47 - 2 = L(7)L(8) - 2$$

From this, we have

Formula 6.46.

$$\sum_{k=1}^{n} L(k)^2 = L(n)L(n+1) - 2$$

Observation 6.47.



Similar to the Fibonacci Triangle, in the Lucas Triangle, A_0 represents $L(n)L(n)$, A_1 represents the terms $L(n)L(n-1)$, ... and A_k represents the terms $L(n)L(n-k)$. By the Chinese-checker-like method of summation, we can find out the sum of terms on A_k i.e. summation of $L(n)L(n-k)$. As we have mentioned this previously in section 6.1, we have decided not to do this again here.

Application 6.48. Here, we would like to invite you to observe some special things in the table. (For the sake of convenience, we rotate the table so that it stands upright like a triangle.)

1
3 3
4 9 4
7 12 12 7
11 21 16 21 11
18 33 28 28 33 18
29 54 44 49 44 54 29
47 87 72 77 77 72 87 47
76 141 116 126 121 126 116 141 76

Please keep in mind that, each line in the Triangle represents a diagonal in the Table.

Now let us focus on the differences between neighbouring numbers on each line.

Line	
4	7 12 12 7
	−5 0 −5
5	11 21 16 21 11
	−10 +5 +5 −10
6	18 33 28 28 33 18
	−15 +5 0 +5 −15
7	29 54 44 49 44 54 29
	−25 +10 −5 −5 +10 −25
8	47 87 72 77 77 72 87 47
	−40 +15 −5 0 −5 +15 −40
9	76 141 116 126 121 126 116 141 76
	−65 +25 −10 +5 +5 −10 +25 −65

From the above observation, we can generalize them as follows:

(Note: For the sake of formatting, L_n is used to indicate $L(n)$ in Application 6.48 only)

Line

$\underline{4k}$

$L_{2k-2}L_{2k+3} \quad L_{2k-1}L_{2k+2} \quad L_{2k}L_{2k+1} \quad L_{2k+1}L_{2k} \quad L_{2k+2}L_{2k-1} \quad L_{2k+3}L_{2k-2}$

$-40 \quad +15 \quad -5 \quad 0 \quad -5 \quad +15 \quad -40$

$-5F(6) \quad +5F(4) \quad -5F(2) \quad 5F(0) \quad -5F(2) \quad +5F(4) \quad -5F(6)$

$\underline{4k+1}$

$L_{2k-1}L_{2k+3} \quad L_{2k}L_{2k+2} \quad L_{2k+1}L_{2k+1} \quad L_{2k+2}L_{2k} \quad L_{2k+3}L_{2k-1}$

$+25 \quad -10 \quad +5 \quad +5 \quad -10 \quad +25$

$+5F(5) \quad -5F(3) \quad +5F(1) \quad +5F(1) \quad -5F(3) \quad +5F(5)$

$\underline{4k+2}$

$L_{2k-1}L_{2k+4} \quad L_{2k}L_{2k+3} \quad L_{2k+1}L_{2k+2} \quad L_{2k+2}L_{2k+1} \quad L_{2k+3}L_{2k} \quad L_{2k+4}L_{2k-1}$

$+40 \quad -15 \quad +5 \quad 0 \quad +5 \quad -15 \quad +40$

$+5F(6) \quad -5F(4) \quad +5F(2) \quad 5F(0) \quad -5F(2) \quad +5F(4) \quad +5F(6)$

$\underline{4k+3}$

$L_{2k}L_{2k+4} \quad L_{2k+1}L_{2k+3} \quad L_{2k+2}L_{2k+2} \quad L_{2k+3}L_{2k+1} \quad L_{2k+4}L_{2k}$

$-25 \quad +10 \quad -5 \quad -5 \quad +10 \quad -25$

$-5F(5) \quad +5F(3) \quad -5F(1) \quad -5F(1) \quad +5F(3) \quad -5F(5)$

Again, we are going use the Chinese-checker-like method of summation as an elegant proof.

Consider the following cases:

Please look at line 5.

1
3 3
4 9 4
7 12 12 7
11 **21** **16** 21 11

How can we explain that $16 = 21 - 5$, i.e. $L(3)L(3) = F(2)F(4) - 5$?

We can convert the Triangle back to the Table again.

2	**1**	3	4	7	11
6	(3)	9	12	21	
8	4	(12)	**16**		
	7	**21**			
	11				

By the Chinese-checker-like method of summation:
If we use 16,

$$(3) + (12) = 16 - 1$$

If we use 3,

$$(3) + (12) = 21 - 6$$

Therefore,

$$16 - 1 = 21 - 6$$
$$16 = 21 - 5$$

The Chinese-checker-like method of summation has proved **the differences between neighbouring numbers on each line** in the Lucas Triangle.

Now, we are going to use it to find the difference between the middle term (as the middle term itself is representing the square of $L(n)$) and other terms (not neighbouring to each other) on the same line in the Lucas Triangle.

Observation 6.49. On line $4k$, middle term $= L(2k)L(2k+1) = M_1$. Consider the n^{th} term from M on the same line,

n	n^{th} term	
1	$M_1 - 5$	$= M_1 - 5$
2	$M_1 - 5 + 15$	$= M_1 + 10$
3	$M_1 - 5 + 15 - 40$	$= M_1 - 30$
4	$M_1 - 5 + 15 - 40 + 105$	$= M_1 + 75$
...		

Hypothesis 6.50.

p	$M_1 - 5(1) + 5(3) - 5(8) + \dots + (-1)^p 5F(2p)$	$= M_1 + (-1)^p 5F(p)F(p+1)$

Note: $F(p)F(p+1)$ are terms lying on A_1 in the Fibonacci Triangle.

An illustration of line $4k$ would be as follows:

On line 12, the middle term $= L(6)L(7) = 522$

$$\begin{aligned}\text{The term on } A_3 &= M_1 + (-1)^3 5F(3)F(4)\\ &= 522 - 5 \times 2 \times 3\\ &= 492\\ &= 4 \times 123\\ &= L(3)L(10)\end{aligned}$$

Details of Proof for Hypothesis 6.50 can be found in Appendix E.

Observation 6.51. On line $(4k+1)$, middle term $= L(2k+1)L(2k+1) = M_2$.

n	n^{th} term	
1	$M_2 + 5$	$= M_2 + 5$
2	$M_2 + 5 - 10$	$= M_2 - 5$
3	$M_2 + 5 - 10 + 25$	$= M_2 + 20$
4	$M_2 + 5 - 10 + 25 - 65$	$= M_2 - 45$
...		

Hypothesis 6.52.

$$p \quad \begin{array}{l} M_2 + 5(1) - 5(2) + 5(5) - 5(13) \\ + 5(34) + \ldots + (-1)^{p+1} 5F(2p-1) \end{array} = M_2 + (-1)^{p+1} 5F(p)F(p)$$

An illustration of line $(4k+1)$ would be as follows:

On line 13, the middle term $= L(7)L(7) = 841$

$$\begin{aligned} \text{The term on } A_4 &= M_2 + (-1)^5 5F(4)F(4) \\ &= 841 - 5 \times 3 \times 3 \\ &= 796 \\ &= 4 \times 199 \\ &= L(3)L(11) \end{aligned}$$

Details of Proof for Hypothesis 6.52 can be found in Appendix E.

Observation 6.53. On line $(4k+2)$, middle term $= L(2k+1)L(2k+2) = M_3$.

n	n^{th} term	
1	$M_3 + 5$	$= M_3 + 5$
2	$M_3 + 5 - 15$	$= M_3 - 10$
3	$M_3 + 5 - 15 + 40$	$= M_3 + 30$
4	$M_3 + 5 - 15 + 40 - 105$	$= M_3 - 75$
...		

Hypothesis 6.54.

$$p \quad \begin{array}{l} M_3 + 5(1) - 5(3) + 5(8) + \ldots \\ + (-1)^{p+1} 5F(2p) \end{array} = M_3 + (-1)^{p+1} 5F(p)F(p+1)$$

An illustration of line $(4k+2)$ would be as follows:

On line 14, the middle term $= L(7)L(8) = 1363$

$$\begin{aligned}\text{The term on } A_4 &= M_3 + (-1)^4 5F(3)F(4)\\ &= 1363 + 5 \times 2 \times 3\\ &= 1393\\ &= 7 \times 199\\ &= L(4)L(11)\end{aligned}$$

Details of Proof for Hypothesis 6.54 can be found in Appendix E.

Observation 6.55. On line $(4k+3)$, middle term $= L(2k+2)L(2k+2) = M_4$.

n	n^{th} term	
1	$M_4 - 5$	$= M_4 - 5$
2	$M_4 - 5 + 10$	$= M_4 + 5$
3	$M_4 - 5 + 10 - 25$	$= M_4 - 20$
4	$M_4 - 5 + 10 - 25 + 65$	$= M_4 + 45$
...		

Hypothesis 6.56.

p	$M_4 - 5(1) + 5(2) - 5(5) + \ldots + (-1)^p 5F(2p-1)$	$= M_4 + (-1)^p 5F(p)F(p)$

An illustration of line $(4k+3)$ would be as follows:

On line 15, the middle term $= L(8)L(8) = 2209$

$$\begin{aligned}\text{The term on } A_3 &= M_4 + (-1)^5 5F(5)F(5)\\ &= 2209 - 5 \times 5 \times 5\\ &= 2084\\ &= 4 \times 521\\ &= L(3)L(13)\end{aligned}$$

Details of Proof for Hypothesis 6.56 can be found in Appendix E.

On the whole, 2 new formulae are formed by cancelling M from both sides of the equation:

$$5F(2) - 5F(4) + 5F(6) + \ldots + (-1)^{n+1} 5F(2n) = (-1)^{n+1} 5F(n)F(n+1)$$

i.e.

Formula 6.57.

$$\sum_{k=1}^{n}(-1)^{k+1}5F(2k) = (-1)^{n+1}5F(n)F(n+1)$$

$$5F(1) - 5F(3) + 5F(5) + \ldots + (-1)^{n+1}5F(2n-1) = (-1)^{n+1}5F(n)F(n)$$

i.e.

Formula 6.58.

$$\sum_{k=1}^{n}(-1)^{k+1}5F(2k-1) = (-1)^{n+1}5F(n)F(n)$$

This is actually what we have got in section 6.1, only that both sides are multiplied by 5.

(Note that in the Table, we only have a term representing $L(n)^2$ on all the **odd**-number lines.)

In general, we have two cases.

Case I

On line $(4r+1)$,

$$\begin{aligned}
&\text{The } k\text{-th term from } L(2r+1)^2 \\
=\;& L(2r+1+k)L(2r+1-k) \\
=\;& L(2r+1)^2 - 5F(1) + 5F(3) - 5F(5) + \ldots + (-1)^{k+1}5F(2k-1) \\
=\;& L(2r+1)^2 + (-1)^{k+1}5F(k)F(k) \quad \text{(by Formula 6.58)} \\
=\;& L(2r+1)^2 + (-1)^{k+1}5F(k)^2
\end{aligned}$$

Case II

On line $(4r+3)$,

$$\begin{aligned}
&\text{The } k\text{-th term from } L(2r+2)^2 \\
=\;& L(2r+2+k)L(2r+2-k) \\
=\;& L(2r+2)^2 + 5F(1) - 5F(3) + 5F(5) - \ldots + (-1)^{k}5F(2k-1) \\
=\;& L(2r+2)^2 + (-1)^{k}5F(k)F(k) \quad \text{(by Formula 6.58)} \\
=\;& L(2r+2)^2 + (-1)^{k}5F(k)^2
\end{aligned}$$

Now we have two formulae:

From case I,

$$L(2r+1+k)L(2r+1-k) = L(2r+1)^2 + (-1)^{k+1}5F(k)^2$$

or

Formula 6.59.

$$L(2r+1)^2 = L(2r+1+k)L(2r+1-k) + (-1)^k F(k)^2$$

From case II,

$$L(2r+2+k)L(2r+2-k) = L(2r+2)^2 + (-1)^k 5F(k)^2$$

or

Formula 6.60.

$$L(2r+2)^2 = L(2r+2+k)L(2r+2-k) + (-1)^{k+1}5F(k)^2$$

From Formula 6.59, we have

$$L(2r+1)^2 = L(2r+1+k)L(2r+1-k) + (-1)^{k+2r+2}5F(k)^2$$

From Formula 6.30, we have

$$L(2r+2)^2 = L(2r+2+k)L(2r+2-k) + (-1)^{k+2r+3}5F(k)^2$$

Now, replace $(2r+1)$ by n, we have
(a) $L(n)^2 = L(n+k)L(n-k) + (-1)^{n+k+1}F(k)^2$
(b) $L(n+1)^2 = L(n+1+k)L(n+1-k) + (-1)^{n+k+2}5F(k)^2$

Note that (a) and (b) represent the same thing and (a) is essentially the same as Formula 6.45.

Application 6.61. If we want to break down large L(n)s, we can always use Formula 6.45.

$$L(n+k) = \frac{L(n)^2 + (-1)^{n+k}(5)F(k)^2}{L(n-k)}$$

Note that, $(n+k)$ is the largest among all.

For example, we want to find $L(80)$.

$$\begin{aligned} L(80) &= L(60+20) \\ &= \frac{L(60)^2 + (-1)^{80}(5)F(20)^2}{L(40)} \end{aligned}$$

$$\begin{aligned} L(40) &= L(21+19) \\ &= \frac{L(21)^2 + (-1)^{40}(5)F(19)^2}{L(2)} \\ &= \frac{24476^2 - 5 \times 4181^2}{3} \\ &= 228826127 \end{aligned}$$

$$\begin{aligned} L(60) &= L(40+20) \\ &= \frac{L(40)^2 + (-1)^{60}(5)F(20)^2}{L(20)} \\ &= \frac{228826127^2 - 5 \times 6765^2}{15127} \\ &= 3461452808002 \end{aligned}$$

Hence,

$$\begin{aligned} L(80) &= \frac{L(60)^2 + (-1)^{80}(5)F(20)^2}{L(40)} \\ &= \frac{3461452808002^2 - 5 \times 6765^2}{228826127} \\ &= 523613963978201 27 \end{aligned}$$

Application 6.62. If $(n+k)$ is odd, we substitute $n = k+1$ into Formula 6.45,

$$L(2k+1) = \frac{L(k+1)^2 + (-1)^{2k+1}(5)F(k)^2}{L(1)}$$

Formula 6.63.

$$L(2k+1) = L(k+1)^2 - 5F(k)^2$$

Application 6.64. If $(n+k)$ is even, we substitute $n = k+2$ into Formula 6.45,

$$L(2k+2) = \frac{L(k+2)^2 + (-1)^{2k+2}(5)F(k)^2}{L(2)}$$
$$L(2k+2) = \frac{L(k+2)^2 + (5)F(k)^2}{3}$$

In other words,

Formula 6.65.

$$L(2k) = \frac{L(k+1)^2 + 5F(k-1)^2}{3}$$

Let us look at an example.

$$\begin{aligned}
L(40) &= \frac{L(21)^2 + 5F(19)^2}{3} \quad \text{(by Formula 6.65)} \\
&= \frac{[L(11)^2 - 5F(10)^2]^2 + 5[F(10)^2 + F(9)^2]^2}{3} \\
&\quad \text{(by Formula 6.63 and Formula 2.28)} \\
&= \frac{[199^2 - (5)(55)^2]^2 + 5(55^2 + 34^2)^2}{3} \\
&= \frac{(39601 - 15125)^2 + 5(3025 + 1156)^2}{3} \\
&= \frac{24476^2 + (5)(4181)^2}{3} \\
&= \frac{599074576 + 87403805}{3} \\
&= 228826127
\end{aligned}$$

Application 6.66. However, unlike $F(0)$, $L(0) = 2$ can be the denominator. If $(n+k)$ is even, we can substitute $n = k$ into Formula 6.45,

$$L(2k) = \frac{L(k)^2 + (-1)^{2k}(5)F(k)^2}{L(0)}$$

Formula 6.67.

$$L(2k) = \frac{L(k)^2 + 5F(k)^2}{2}$$

Let us find L(40) again.

$$
\begin{aligned}
L(40) &= \frac{L(20)^2 + 5F(20)^2}{2} \quad \text{(by Formula 6.67)} \\
&= \frac{\left[\frac{L(10)^2 + 5F(10)^2}{2}\right]^2 + 5[F(11)^2 - F(9)^2]^2}{3} \\
&\quad \text{(by Formula 6.67 and Formula 2.24)} \\
&= \frac{\left[\frac{123^2 + (5)(55)^2}{2}\right]^2 + 5(89^2 - 34^2)^2}{2} \\
&= \frac{\left(\frac{(15129 + 15125)^2}{2}\right)^2 + 5(7921 - 1156)^2}{3} \\
&= \frac{15127^2 + 5(6765)^2}{2} \\
&= 228826127
\end{aligned}
$$

Application 6.68. Actually the above findings can help us complete an important proof.

By Formula 5.58, we have

$$L(2n) = 5F(n)^2 + (-1)^n(2) \tag{9}$$

By Formula 6.67, we have

$$2L(2n) = L(n)^2 + 5F(n)^2 \tag{10}$$

(9) – (10) :

$$L(2n) = L(n)^2 + (-1)^{n+1}(2)$$

Hypothesis 3.2 is proved here.

Application 6.69. Can we work out some direct relations between $F(k)^2$ and $L(k)^2$? Back to the example in Application 6.66, in the resolution of $L(40)$, have you noticed that

$$L(10)^2 = 15129 \quad 5F(10)^2 = 15125$$

and

$$L(20)^2 = 228826129 \quad 5F(20)^2 = 228826125?$$

Furthermore,

$$L(5)^2 = 121 \quad 5F(5)^2 = 125$$

It seems that

$$L(k)^2 = 5(k)^2 \pm 4.$$

Why is it like that?

Let us look at Formula 6.67.

$$\begin{aligned} L(2k) &= \frac{L(k)^2 + 5F(k)^2}{2} \\ L(k)^2 + (-1)^{k+1}(2) &= \frac{L(k)^2 + 5F(k)^2}{2} \\ &\quad \text{(by Formula 3.2 } L(2n) = L(n)^2 + (-1)^{n+1}(2)) \\ 2L(k)^2 + (-1)^{k+1}(4) &= L(k)^2 + 5(k)^2 \end{aligned}$$

Formula 6.70.

$$L(k)^2 = 5F(k)^2 + (-1)^k(4)$$

For instance, we have $F(8) = 21$. How can we find $L(8)$ directly?

Applying Formula 6.70,

$$\begin{aligned} L(8)^2 &= 5(21)^2 + 4 \\ &= 2209 \end{aligned}$$

Therefore, $L(8) = 47$.

7. More About the Tables (Fibonacci Table, Lucas-Fibonacci Table and Lucas Table)

Usually Fibonacci and Lucas sequences are investigated in one dimension only. Now we are going to investigate them in two dimensions, that is, in a table form.

Now that we have introduced to you the Fibonacci Table (“F-Table”), we are going to investigate a number of relationships in the F-Table. As we are investigating into a two-dimensional plane, we will find out relationships on diagonals, lines parallel to diagonals, rows and columns. Do you notice the relationship among the sums of numbers on each row of the F-Table?

We have found something in the Fibonacci Table, but how about the Fibonacci Triangle? Can you find out a specific term in the Fibonacci triangle with the help of the numbers on the axis of symmetry of the Triangle?

7.1. Summation of all terms on n-th Line in the Triangles

Observation 7.1. Refer to the Fibonacci Triangle. Let $S_F(n)$ denote the sum of all the terms on the n-th line in the Fibonacci Triangle.

For example, on the 5^{th} line, there are five terms: 5, 3, 4, 3, 5. Therefore, $S_F(5) = 5 + 3 + 4 + 3 + 5 = 20$.

Observe carefully the following,

$$\begin{aligned} S_F(1) &= 1 \\ S_F(2) &= 2 \\ S_F(3) &= 5 \\ S_F(4) &= 10 \\ S_F(5) &= 20 \\ S_F(6) &= 38 \\ S_F(7) &= 71 \\ &\cdots \end{aligned}$$

It seems that $S(n)$ does not take a general pattern. However, in fact, there are some relationships among these numbers:

$$\begin{aligned} S_F(3) &= 5 = 1 + 2 + 2 = S_F(1) + S_F(2) + F(3) \\ S_F(4) &= 10 = 2 + 5 + 3 = S_F(2) + S_F(3) + F(4) \\ S_F(5) &= 20 = 5 + 10 + 5 = S_F(3) + S_F(4) + F(5) \\ S_F(6) &= 38 = 10 + 20 + 8 = S_F(4) + S_F(5) + F(6) \\ S_F(7) &= 71 = 20 + 38 + 13 = S_F(5) + S_F(6) + F(7) \\ &\cdots \end{aligned}$$

From the above observations, we conjecture that:

Hypothesis 7.2.

$$S_F(n) + S_F(n+1) + F(n+2) = S_F(n+2)$$

Details of Proof for Hypothesis 7.2 can be found in Appendix E.

Observation 7.3. Refer to the Lucas-Fibonacci Triangle. It can be expected that the above special property should also be found in the Lucas-Fibonacci Triangle. Let $S_{LF}(n)$ denote the sum of all the terms on the n-th line in the Lucas-Fibonacci Triangle.

For example, on the 5^{th} line, there are five terms: 5, 9, 8, 7, 11. Therefore, $S_{LF}(5) = 5 + 9 + 8 + 7 + 11 = 40$.

Observe carefully,

$$\begin{aligned} S_{LF}(1) &= 1 \\ S_{LF}(2) &= 4 \\ S_{LF}(3) &= 9 \\ S_{LF}(4) &= 20 \\ S_{LF}(5) &= 40 \\ S_{LF}(6) &= 78 \\ S_{LF}(7) &= 147 \\ &\dots \end{aligned}$$

Note that:

$$\begin{aligned} S_{LF}(3) &= 9 = 1 + 4 + 4 = S_{LF}(1) + S_{LF}(2) + L(3) \\ S_{LF}(4) &= 20 = 4 + 9 + 7 = S_{LF}(2) + S_{LF}(3) + L(4) \\ S_{LF}(5) &= 40 = 9 + 20 + 11 = S_{LF}(3) + S_{LF}(4) + L(5) \\ S_{LF}(6) &= 78 = 20 + 40 + 18 = S_{LF}(4) + S_{LF}(5) + L(6) \\ S_{LF}(7) &= 147 = 40 + 78 + 29 = S_{LF}(5) + S_{LF}(6) + L(7) \\ &\dots \end{aligned}$$

From the above relationships, we can generalize them into the following formula:

Hypothesis 7.4.

$$S_{LF}(n) + S_{LF}(n+1) + L(n+2) = S_{LF}(n+2)$$

It is interesting to note that although the Lucas-Fibonacci Triangle we are focusing here is formed by both the Fibonacci and the Lucas sequences, each $S_{LF}(n+2)$ only involves $S_{LF}(n)$, $S_{LF}(n+1)$ and $L(n+2)$, but does not involve any $F(k)$. This observation is unlike the one found in the Fibonacci Triangle.

Details of Proof for Hypothesis 7.4 can be found in Appendix E.

Observation 7.5. Refer to the Lucas Triangle. It can be expected that the above special property should also be found in the Lucas Triangle. Let $S_L(n)$ denote the sum of all the terms on the n-th line in the Lucas Triangle.

For example, on the 5^{th} line, there are five terms: 11, 21, 16, 21, 11. Therefore, $S_L(5) = 11 + 21 + 16 + 21 + 11 = 80$.

Observe carefully,

$$\begin{aligned}
S_L(1) &= 1 \\
S_L(2) &= 6 \\
S_L(3) &= 17 \\
S_L(4) &= 38 \\
S_L(5) &= 80 \\
S_L(6) &= 158 \\
&\cdots
\end{aligned}$$

Note that:

$$\begin{aligned}
S_L(3) &= 17 = 1 + 6 + 10 = S_L(1) + S_L(2) + 5F(3) \\
S_L(4) &= 38 = 6 + 17 + 15 = S_L(2) + S_L(3) + 5F(4) \\
S_L(5) &= 80 = 17 + 38 + 25 = S_L(3) + S_L(4) + 5F(5) \\
S_L(6) &= 158 = 38 + 80 + 40 = S_L(4) + S_L(5) + 5F(6) \\
&\cdots
\end{aligned}$$

From the above relationships, we can generalize them into the following formula:

Hypothesis 7.6.

$$S_L(n) + S_L(n+1) + 5F(n+2) = S_L(n+2)$$

It is interesting to note that although the Lucas Triangle we are focusing here is formed by Lucas sequence only, however, we have the term $5F(n+2)$ in the hypothesis.

Details of Proof for Hypothesis 7.6 can be found in Appendix E.

7.2. Location of terms in the Fibonacci Triangle

Observation 7.7. Now we reorganize the Fibonacci Triangle so that the terms are arranged to form a sequence $\{1, \mathbf{1}, \mathbf{1}, 2, 1, 2, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{3}, 5, 3, 4, 3, 5, \ldots\}$.

$$\begin{array}{ccccccccccccccccccc}
&&&&&&&&&1&&&&&&&&&\\
&&&&&&&&\mathbf{1}&&\mathbf{1}&&&&&&&&\\
&&&&&&&2&&1&&2&&&&&&&\\
&&&&&&\mathbf{3}&&\mathbf{2}&&\mathbf{2}&&\mathbf{3}&&&&&&\\
&&&&&5&&3&&4&&3&&5&&&&&\\
&&&&8&&5&&6&&6&&5&&8&&&&\\
&&&13&&8&&10&&9&&10&&8&&13&&&\\
&&21&&13&&16&&15&&15&&16&&13&&21&&\\
&34&&21&&26&&24&&25&&24&&26&&21&&34&\\
55&&34&&42&&39&&40&&40&&39&&42&&34&&55
\end{array}$$

This Fibonacci Triangle can also be represented by:

$$\begin{array}{ccccccc}
&&&F(1)F(1)&&&\\
&&F(1)F(2)&&F(2)F(1)&&\\
&F(1)F(3)&&F(2)F(2)&&F(3)F(1)&\\
F(1)F(4)&&F(2)F(3)&&F(3)F(2)&&F(4)F(1)\\
\iddots&&&\vdots&&&\ddots
\end{array}$$

$$F(1)F(n)\ F(2)F(n-1)\ F(3)F(n-2)\ \ldots\ F(n-2)F(3)\ F(n-1)F(2)\ F(n)F(1)$$

Hence the general form of a term is given by

$$F(a^{\text{th}} \text{ term } \underline{\text{of the line}}) \times F(\text{line number } -a+1)$$

Application 7.8. For example,

$$\begin{aligned}
&\text{the } 3^{\text{rd}} \text{ term on line 9 (we know that is 26)}\\
&= F(3) \times F(9-3+1)\\
&= F(3) \times F(7)\\
&= 2 \times 13\\
&= 26
\end{aligned}$$

Application 7.9. If we are asked to find the 10000^{th} term of the sequence, there are actually two methods.

Method I

First, we aim at looking for the line the 10000^{th} term belongs to, in this case n, on the Fibonacci triangle. This can be found using the inequality:

$$(1+2+3+4+5+\ldots+n) \geqslant 10000$$

where n is the smallest integer possible.

Solving the inequality gives $n = 141$. Hence the 10000^{th} term lies on the 141^{st} line.

The last term on the 140^{th} line the $(1+2+3+\ldots+140) = 9870^{\text{th}}$ term of the sequence. Hence the 10000^{th} term of the sequence is the $(10000-9870) = 130^{\text{th}}$ term on the 141^{st} line.

Hence

$$\begin{aligned}
&\text{the } 10000^{\text{th}} \text{ term} \\
&= F(130) \times F(141-130+1) \\
&= F(130) \times F(12) \\
&= 659034621587630041982498215 \times 144.
\end{aligned}$$

By applying
Formula 2.24 $F(2k) = F(k+1)^2 - F(k-1)^2$,
Formula 2.28 $F(2k+1) = F(k+1)^2 + F(k)^2$
and
Formula 6.11 $F(n+k) = \dfrac{F(n)^2 + (-1)^{n+k+1}F(k)^2}{F(n-k)}$,
the 10000^{th} term is 94 900 985 508 618 726 045 479 742 960.

Method II

We can also use the axis in the Fibonacci Triangle to help us evaluate the 10000^{th} term.

The term on line 141 that lies on $A_0 = F(71)F(71)$. It is the 71^{st} term of the line from the left. It is $(130-71) = 59^{\text{th}}$ term away from the 130^{th} term on the 141^{st} line. (i.e. the 10000^{th} term)

Hence by Formula 6.21

$$M_2 - 1 + 2 - 5 + 13 - 34 + \ldots + (-1)^p F(2p-1) = M_2 + (-1)^p F(p)F(p),$$

$$\begin{aligned} &\text{the } 60^{\text{th}} \text{ term from the term on } A_0 L_{141} \\ &= F(71)F(71) + (-1)^{59} F(59)F(59) \\ &= F(71)F(71) - F(59)F(59) \\ &= F(71)^2 - F(59)^2 \\ &= 308061521170129^2 - 956722026041^2 \\ &= 94\ 900\ 985\ 508\ 618\ 726\ 045\ 479\ 742\ 960. \end{aligned}$$

8. Conclusion

Ever since the invention of the Fibonacci and Lucas numbers, people have been trying to figure out ways to solve these numbers. Many formulae have been generated. In this report, we have discovered and presented to you four different approaches to find large $F(n)$ and $L(n)$.

These formulae can be applied to find large Fibonacci and Lucas numbers. In different situations, we should use the appropriate formula to simplify the problem.

Moreover, as we work on the Fibonacci and Lucas sequences, the idea of developing them in two dimensions strikes us. Fibonacci and Lucas sequences contain numbers that show special relationships when we put them in different order or arrange them in different layouts. As we put these sequences into tables and triangles, we are all thrilled to observe many fascinating patterns.

The whole team benefited from learning to put thoughts into words. After all, mathematicians need language.

We hope that this project will bring about a new path for research into sequences in mathematics.

9. Evaluation on the Major Formulae

The objective and purpose of this project is to find out formulae that can help us resolve large Fibonacci and Lucas numbers. At this point, we have already generated four methods to help us with this task. Now it is time to evaluate these formulae and compare them with one another to understand their limitations and usefulness. This allows us to learn to apply them appropriately and wisely in different situations.

The major formulae in sections 2, 3, 5 and 6

Section 2

Formula 2.19.

$$F(k)U(n) = F(r+k)U(n-r) + (-1)^{k+1}F(r)U(n-r-k)$$

Section 3

Hypotheses 3.25–3.28.

$$\begin{aligned} L(4pn) = {}& {}_{4p}L_0 L(n)^{4p} + (-1)^{n+1}{}_{4p-1}L_1 L(n)^{4p-2} + {}_{4p-2}L_2 L(n)^{4p-4} \\ & + (-1)^{n+1}{}_{4p-3}L_3 L(n)^{4p-6} + \ldots + {}_{4p-r}L_r L(n)^{4p-2r+2} \\ & + \ldots + (-1)^{n+1}{}_{2p+1}L_{2p-1} L(n)^{2} + {}_{2p}L_{2p} \end{aligned}$$

$$\begin{aligned} L((4p-1)n) = {}& {}_{4p-1}L_0 L(n)^{4p-1} + (-1)^{n+1}{}_{4p-2}L_1 L(n)^{4p-3} \\ & + {}_{4p-3}L_2 L(n)^{4p-5} + (-1)^{n+1}{}_{4p-4}L_3 L(n)^{4p-7} \\ & + \ldots + {}_{4p-1-r}L_r L(n)^{4p-2r+1} \\ & + \ldots + (-1)^{n+1}{}_{2p}L_{2p-1} L(n) \end{aligned}$$

$$\begin{aligned} L((4p-2)n) = {}& {}_{4p-2}L_0 L(n)^{4p-2} + (-1)^{n+1}{}_{4p-3}L_1 L(n)^{4p-4} \\ & + {}_{4p-4}L_2 L(n)^{4p-6} + (-1)^{n+1}{}_{4p-5}L_3 L(n)^{4p-8} \\ & + \ldots + {}_{4p-2-r}L_r L(n)^{4p-2r} \\ & + \ldots + (-1)^{n+1}{}_{2p-1}L_{2p-1} \end{aligned}$$

$$\begin{aligned} L((4p-3)n) = {}& {}_{4p-3}L_0 L(n)^{4p-3} + (-1)^{n+1}{}_{4p-4}L_1 L(n)^{4p-5} \\ & + {}_{4p-5}L_2 L(n)^{4p-7} + (-1)^{n+1}{}_{4p-6}L_3 L(n)^{4p-9} \\ & + \ldots + {}_{4p-3-r}L_r L(n)^{4p-2r-1} \\ & + \ldots + (-1)^{n+1}{}_{2p-1}L_{2p-2} L(n) \end{aligned}$$

Note: These hypotheses are to be proved.

Section 5

Formula 5.25.

$$L(r)F(k) = F(k+r) + (-1)^r F(k-r)$$

Formula 5.26.

$$F(r)L(k) = F(k+r) + (-1)^{r+1} F(k-r)$$

Formula 5.55.

$$5F(k)F(n) = L(n+k) + (-1)^{k+1} L(n-k)$$

Formula 5.44.

$$\begin{aligned} F(4pn) = \; & F(n)[{}_{4p-1}C_0 L(n)^{4p-1} + (-1)^{n+1}{}_{4p-2}C_1 L(n)^{4p-3} \\ & + {}_{4p-3}C_2 L(n)^{4p-5} + \ldots + {}_{2p+1}C_{2p-2} L(n)^3 \\ & + (-1)^{n+1}{}_{2p}C_{2p-1} L(n)] \\ F((4p+1)n) = \; & F(n)[{}_{4p}C_0 L(n)^{4p} + (-1)^{n+1}{}_{4p-1}C_1 L(n)^{4p-2} \\ & + {}_{4p-2}C_2 L(n)^{4p-4} + \ldots + (-1)^{n+1}{}_{2p+1}C_{2p-1} L(n)^2 \\ & + {}_{2p}C_{2p}] \\ F((4p+2)n) = \; & F(n)[{}_{4p+1}C_0 L(n)^{4p+1} + (-1)^{n+1}{}_{4p}C_1 L(n)^{4p-1} \\ & + {}_{4p-1}C_2 L(n)^{4p-3} + \ldots + (-1)^{n+1}{}_{2p+2}C_{2p-1} L(n)^3 \\ & {}_{2p+1}C_{2p} L(n)] \\ F((4p+3)n) = \; & F(n)[{}_{4p+2}C_0 L(n)^{4p+2} + (-1)^{n+1}{}_{4p+1}C_1 L(n)^{4p} \\ & + {}_{4p}C_2 L(n)^{4p-2} + \ldots + {}_{2p+2}C_{2p} L(n)^2 \\ & + (-1)^{n+1}{}_{2p+1}C_{p+1}] \end{aligned}$$

Section 6

Formula 6.11.

$$F(n)^2 = F(n-k)F(n+k) + (-1)^{n+k} F(k)^2$$

Formula 6.45.

$$L(n)^2 = L(n-k)L(n+k) + 5(-1)^{n+k+1} F(k)^2$$

Formulae	Usefulness	Limitations	Remarks
2.19	1. It can resolve large $F(n)$ to small $F(n)$ when we choose large r and k. 2. It does not involve powers. Hence, it is possible to calculate manually. 3. It can be used to solve $U(n)$, but not the methods in other 3 sections.	1. It consists of 3 unknowns. To find suitable values of k and r is difficult and requires much practice. 2. It is necessary to find $F(n)$ to resolve $L(n)$ or any $U(n)$. 3. For very large $F(n)$ or $L(n)$, with $n > 100$, we may need to apply the formula several times.	1. Put n the greatest value. 2. Let k be the smallest possible non-negative integer. 3. Let $(r+k)$, $(n-r)$, r and $(n-r-k)$ be more or less the same as each other.
(Hypothesis) 3.25–3.28	1. If we have the function ${}_nL_r$ built in the calculator or computer (computer modelling), the equation is convenient to use, just like coefficients in binomial expansion. 2. We can have different combinations in resolving large $L(n)$. If we are able to choose the best combination, we can come to the answer quickly. 3. After applying the formulae, $L(kn)$ will be reduced to a polynomial expression with $L(n)$ only, so that we can focus on $L(n)$ only in the next resolution.	1. There are 4 cases. 2. The whole expression is very long and tedious. 3. We need to convert all ${}_nL_r$ to ${}_nC_r$. It is easy to make mistakes in the meantime. 4. The form of polynomial expression may involve high powers and eventually lead to calculation mistakes. 5. Choosing the proper combination requires much practice.	For example, $L(105) = L(1\times 105)$ $= L(3\times 35)$ $= L(5\times 21)$ $= L(7\times 15)$. We actually have 7 ways to resolve $L(105)$.

5.25, 5.26, 5.55	1. Little involves multiplication. Hence it is easy to calculate manually. 2. For substitution of proper values into the unknowns, we can generate many useful formulae. 3. These formulae enable us to resolve $F(kn)$ into $F(n)$ or $L(n)$.	1. To resolve $L(n)$, we also need to deal with $F(n)$. 2. Involves division. 3. For very large $F(n)$ or $L(n)$, with $n > 100$, we may need to apply the formula several times.	For Formulae 5.25 and 5.26, $F(k + r)$ should be the largest term among all. For Formula 5.55, $L(n + k)$ should be the largest term among all.
5.44	1. Built-in functions of ${}_nC_r$ are present in the calculators and computers. The equation is very convenient to use. 2. We can have different combinations in resolving large $F(n)$. If we are able to choose the best combination, we can come to the answer quickly. 3. After applying the formulae, $F(kn)$ will be reduced to a polynomial expression with $F(n)$ and $L(n)$ only, so that we can focus on $F(n)$ and $L(n)$ only in the next resolution.	1. There are 4 cases. 2. The whole expression is very long and tedious. 3. The form of polynomials expression may involve high powers and eventually lead to calculation mistakes. 4. Choosing the proper combination requires much practice.	For example, $F(105) = F(1 \times 105)$ $= F(3 \times 35)$ $= F(5 \times 21)$ $= F(7 \times 15)$. We actually have 7 ways to resolve $F(105)$.

6.11, 6.55	1. The formulae in fact indicate the relationships among terms lying on the same line. They may help us understand the Tables better and also benefit our future investigation on the Tables.	1. They require multiplication. 2. To resolve $L(n)$, we also need to deal with $F(n)$. 3. For very large $F(n)$ or $L(n)$, with $n > 100$, we may need to apply the formula several times.	Note that $(n + k)$ should be the largest term among all By the formulae, we can locate the terms in the tables easily (please refer to Application 7.9).
Binet's Formulae	1. If we insert the Binet's Formulae into the computer, we can find $F(n)$ easily–with only one step. 2. They inspire us to drill into the general expression for any recurrence sequence $U(n)$.	1. They involve $\sqrt{5}$ which is an irrational number. 2. They involve the n-th power, which means it is almost impossible to calculate manually if n is large.	It serves as a general formula to solve $F(n)$.
Successor Formulae	1. They can show us the relationships between consecutive Fibonacci or Lucas numbers.	1. They involve $\sqrt{5}$ which is an irrational number. 2. They involve the n-th power, which means it is almost impossible to calculate manually if n is large. 3. We have to round down the result to the greatest integer smaller than it. 4. We cannot use them to resolve large $F(n)$ and $L(n)$. It is because before finding $F(n + 1)$ or $L(n + 1)$, we have to find $F(n)$ or $L(n)$ respectively.	

10. Suggestions for Future Investigation

In the process of our work, we have actually come up with many ideas worthy of further examination. However, time does not allow us to do too much research. They have to be left and dealt with when the opportunities arise in future. We would like to list some of these ideas for any preliminary interest.

(1) Although our project only focuses on the Fibonacci and Lucas numbers, we have also successfully generated formulae that can be used to find large recurrence sequences, $U(n)$ in section 2. Had we been able to extend the scope of investigation in sections 5 and 6 to $U(n)$, we believe that a fuller picture and a better understanding of the topic could be achieved.

(2) In this project, we have constructed three Tables, the Fibonacci Table, the Lucas-Fibonacci Table and the Lucas Table. These Tables have helped us observe the relationships between and within sequences. In section 6, we can even produce the proofs using these Tables. If we can construct the Tables with different recurrence sequences, we can observe more patterns and generate more useful formulae to deal with various problems in this topic. Please note that, the choice of $U(1)$ and $U(2)$ will also lead to completely different results.

(3) In the past, people tend to relate these recurrence sequences (mainly $F(n)$ and $L(n)$) to 1 dimension only. (That is, the sequence itself) Our effort here of relating it to 2 dimensions gives the subject a greater depth. As mentioned before, in geometry, we have points, lines, planes and solids. Inspired by this, we generate the Tables. In the Tables, the recurrence property is actually going in 2 directions, giving justifications to our relating it to 2 dimensions.

Is it possible to put the sequences in three dimensions? In the project, we have introduced the Chinese-checker-like method. Can we still apply this method in 3 dimensions?

Despite the rigid definition of dimensions in physics, in mathematics, we can extend our scope of investigation of dimensions to n-th power freely. Hence, is it possible to put the sequences in n-th dimension? These questions have yet to be answered.

(4) In Hypotheses 3.25−3.28, we have invented a method to resolve $L(kn)$ into a polynomial expression consisting of powers of L(n) only. In Formulae 5.44, we have invented another method to resolve $F(kn)$ into $F(n)$ and a polynomial expression consisting of powers of $L(n)$ only. By considering the coefficients in the polynomial expressions, we can actually obtain triangles similar to the

Pascal's Triangle. We have already talked about these Tables in 3.31−3.34. By inventing and using the Uk Tables, we may discover how we can resolve $U(kn)$ into $U(n)$ and a polynomial expression containing powers of $L(n)$ only. The expression should be similar. By resolving $U(kn)$ into $U(n)$ and a polynomial expression containing powers of $L(n)$ only, we can have many combinations to solve large $U(kn)$. We can even apply computer modelling to help us solve these large $U(kn)$.

(5) In Appendix C, we have talked about the prime-number rows in the Lk Table. We conjecture that all the numbers on the prime-number rows (except the first term) are divisible by the prime numbers themselves, that is,

$$n \mid {}_{n-k}L_k \quad \text{where } 0 < k \leqslant \frac{n}{2}$$

or

$$n \mid {}_{n-k+1}C_k - {}_{n-k-1}C_{k-2} \quad \text{where } 0 < k \leqslant \frac{n}{2}$$

We have already proved that this hypothesis holds up to the 41^{st} row in the Lk Table. However, we have not completed the proof yet. Is this one of the special properties of the Lucas numbers? Can this property be used to derive a new formula for prime numbers?

(6) In Formula 5.55, Formula 5.58, Formula 6.45 (including the proof of Formula 6.45) and Formula 7.6, we discovered that the Lucas sequence has a very special relation with the integer "5". How can we explain this phenomenon? Is there a special integer for every recurrence sequence? Can we explain this by referring to the Binet's Formula, which involves $\sqrt{5}$?

There is still a long way to go before anybody can fully decrypt these recurrence sequences. Our scrutiny so far in the subject will hopefully spark off some interest and act as a catalyst for other studies on these fascinating numbers which might prove to be of greater magnitude.

Appendix A. The first 100 Fibonacci numbers

n	$F(n)$	n	$F(n)$
1	1	51	20365011074
2	1	52	32951280099
3	2	53	53316291173
4	3	54	86267571272
5	5	55	139583862445
6	8	56	225851433717
7	13	57	365435296162
8	21	58	591286729879
9	34	59	956722026041

10	55
11	89
12	144
13	233
14	377
15	610
16	987
17	1597
18	2584
19	4181
20	6765
21	10946
22	17711
23	28657
24	46368
25	75025
26	121393
27	196418
28	317811
29	514229
30	832040
31	1346269
32	2178309
33	3524578
34	5702887
35	9227465
36	14930352
37	24157817
38	39088169
39	63245986
40	102334155
41	165580141
42	267914296
43	433494437
44	701408733
45	1134903170
46	1836311903
47	2971215073
48	4807526976
49	7778742049
50	12586269025
60	1548008755920
61	2504730781961
62	4052739537881
63	6557470319842
64	10610209857723
65	17167680177565
66	27777890035288
67	44945570212853
68	72723460248141
69	117669030460994
70	190392490709135
71	308061521170129
72	498454011879264
73	806515533049393
74	1304969544928657
75	2111485077978050
76	3416454622906707
77	5527939700884757
78	8944394323791464
79	14472334024676221
80	23416728348467685
81	37889062373143906
82	61305790721611591
83	99194853094755497
84	160500643816367088
85	259695496911122585
86	420196140727489673
87	679891637638612258
88	1100087778366101931
89	1779979416004714189
90	2880067194370816120
91	4660046610375530309
92	7540113804746346429
93	12200160415121876738
94	19740274219868223167
95	31940434634990099905
96	51680708854858323072
97	83621143489848422977
98	135301852344706746049
99	218922995834555169026
100	354224848179261915075

Appendix B. The first 100 lucas numbers

n	$L(n)$
1	1
2	3
3	4
4	7
5	11
6	18
7	29
8	47
9	76
10	123
11	199
12	322
13	521
14	843
15	1364
16	2207
17	3571
18	5778
19	9349
20	15127
21	24476
22	39603
23	64079
24	103682
25	167761
26	271443
27	439204
28	710647
29	1149851
30	1860498
31	3010349
32	4870847
33	7881196
34	12752043
35	20633239
36	33385282
37	54018521
38	87403803
39	141422324
40	228826127
41	370248451
42	599074578
43	969323029
44	1568397607
51	45537549124
52	73681302247
53	119218851371
54	192900153618
55	312119004989
56	505019158607
57	817138163596
58	1322157322203
59	2139295485799
60	3461452808002
61	5600748293801
62	9062201101803
63	14662949395604
64	23725150497407
65	38388099893011
66	62113250390418
67	100501350283429
68	162614600673847
69	263115950957276
70	425730551631123
71	688846502588399
72	1114577054219522
73	1803423556807921
74	2918000611027443
75	4721424167835364
76	7639424778862807
77	12360848946698171
78	20000273725560978
79	32361122672259149
80	52361396397820127
81	84722519070079276
82	137083915467899403
83	221806434537978679
84	358890350005878082
85	580696784543856761
86	939587134549734843
87	1520283919093591604
88	2459871053643326447
89	3980154972736918051
90	6440026026380244498
91	10420180999117162549
92	16860207025497407047
93	27280388024614569596
94	44140595050111976643

45	2537720636	95	71420983074726546239
46	4106118243	96	115561578124838522882
47	6643838879	97	186982561199565069121
48	10749957122	98	302544139324403592003
49	17393796001	99	489526700523968661124
50	28143753123	100	792070839848372253127

Appendix C. Steps of Calculation for expressing $L(kn)$ in terms of $L(n)$ only

$$L(1n) = L(n)$$

$$L(2n) = L(n)^2 + 2(-1)^{n+1}$$

$$L(3n) = L(n)^3 + 3L(n)(-1)^{n+1}$$

$$\begin{aligned} L(4n) &= L(2 \times 2n) \\ &= L(2n)^2 + 2(-1)^{2n+1} \\ &= [L(n)^2 + 2(-1)^{n+1}]^2 - 2 \\ &= L(n)^4 + 4L(n)^2(-1)^{n+1} + 4(-1)^{2n+2} - 2 \\ &= L(n)^4 + 4L(n)^2(-1)^{n+1} + 4 - 2 \\ &= L(n)^4 + 4L(n)^2(-1)^{n+1} + 2 \end{aligned}$$

$$\begin{aligned} L(5n) &= L(n)^5 + 5L(n)(-1)^{n+1}[L(n)^2 + (-1)^{n+1}] \\ &= L(n)^5 + 5L(n)^3(-1)^{n+1} + 5L(n)(-1)^{2n+2} \\ &= L(n)^5 + 5L(n)^3(-1)^{n+1} + 5L(n) \end{aligned}$$

Method I for $L(6n)$

$$\begin{aligned} L(6n) &= L(2n)^3 + 3L(2n)(-1)^{2n+1} \\ &= [L(n)^2 + 2(-1)^{n+1}]^3 - 3[L(n)^2 + 2(-1)^{n+1}] \\ &= L(n)^6 + 6L(n)^4(-1)^{n+1} + 12L(n)^2(-1)^{2n+2} + 8(-1)^{3n+3} \\ &\quad - 3L(n)^2 + 6(-1)^n \\ &= L(n)^6 + 6L(n)^4(-1)^{n+1} + 9L(n)^2 + 2(-1)^{n+1} \\ &\quad (\text{Note that: } (-1)^{3n+3} = (-1)^{3n+1} = (-1)^{n+1}) \end{aligned}$$

Method II for $L(6n)$

$$\begin{aligned} L(6n) &= L(3n)^2 + 2(-1)^{3n+1} \\ &= [L(n)^3 + 3L(n)(-1)^{n+1}]^2 + 2(-1)^{n+1} \\ &= L(n)^6 + 6L(n)^4(-1)^{n+1} + 9L(n)^2 + 2(-1)^{n+1} \end{aligned}$$

$$\begin{aligned} L(7n) &= L(n)^7 + 7L(n)(-1)^{n+1}[L(n)^2 + (-1)^{n+1}]^2 \\ &= L(n)^7 + 7L(n)(-1)^{n+1}[L(n)^4 + 2L(n)^2(-1)^{n+1} + 1] \\ &= L(n)^7 + 7L(n)^5(-1)^{n+1} + 14L(n)^3 + 7L(n)(-1)^{n+1} \end{aligned}$$

$$\begin{aligned} L(8n) &= L(2n)^4 + 4L(2n)^2(-1)^{2n+1} + 2 \\ &= [L(n)^2 + 2(-1)^{n+1}]^4 + 4(-1)^{2n+1}[L(n)^2 + 2(-1)^{n+1}]^2 + 2 \\ &= L(n)^8 + 8L(n)^6(-1)^{n+1} + 24L(n)^4 + 32L(n)^2(-1)^{n+1} \\ &\quad + 16 - 4L(n)^4 - 16L(n)^2(-1)^{n+1} - 16 + 2 \\ &= L(n)^8 + 8L(n)^6(-1)^{n+1} + 20L(n)^4 + 16L(n)^2(-1)^{n+1} + 2 \end{aligned}$$

$$\begin{aligned} L(9n) &= L(3n)^3 + 3L(3n)(-1)^{3n+1} \\ &= [L(n)^3 + 3L(n)(-1)^{n+1}]^3 + 3(-1)^{n+1}[L(n)^3 + 3L(n)(-1)^{n+1}] \\ &= L(n)^9 + 9L(n)^7(-1)^{n+1} + 27L(n)^5 + 30L(n)^3(-1)^{n+1} + 9L(n) \end{aligned}$$

$$\begin{aligned}
L(10n) &= L(2(5n)) \\
&= L(5n)^2 + 2(-1)^{5n+1} \\
&= [L(n)^5 + 5L(n)^3(-1)^{n+1} + 5L(n)]^2 + 2(-1)^{n+1} \\
&= L(n)^{10} + 10L(n)^8(-1)^{n+1} + 35L(n)^6 + 50L(n)^4(-1)^{n+1} \\
&\quad + 25L(n)^2 + 2(-1)^{n+1}
\end{aligned}$$

$$\begin{aligned}
L(11n) &= L(n)^{11} + (-1)^{n+1}(11)L(n)[L(n)^2 + (-1)^{n+1}] \\
&\quad \{[L(n)^2 + (-1)^{n+1}]^3 + L(n)^2\} \\
&= L(n)^{11} + (-1)^{n+1}(11)L(n)[L(n)^2 + (-1)^{n+1}] \\
&\quad \{L(n)^6 + 3L(n)^4(-1)^{n+1} + 3L(n)^2 + [(-1)^{n+1}]^3 + L(n)^2\} \\
&= L(n)^{11} + 11L(n)(-1)^{n+1}[L(n)^8 + 4L(n)^6(-1)^{n+1} \\
&\quad + 7L(n)^4 + 5L(n)^2 + L(n)^2(-1)^{n+1}] \\
&= L(n)^{11} + 11L(n)^9(-1)^{n+1} + 44L(n)^7 + 77L(n)^5(-1)^{n+1} \\
&\quad + 55L(n)^3 + 11L(n)(-1)^{n+1}
\end{aligned}$$

$$\begin{aligned}
L(12n) &= L(2(6n)) \\
&= L(6n)^2 + 2(-1)^{6n+1} \\
&= [L(n)^6 + 6L(n)^4(-1)^{n+1} + 9L(n)^2 + 2]^2 - 2 \\
&= L(n)^{12} + 12L(n)^{10}(-1)^{n+1} + 54L(n)^8 + 112L(n)^6(-1)^{n+1} \\
&\quad + 105L(n)^4 + 36L(n)^2(-1)^{n+1} + 4 - 2 \\
&= L(n)^{12} + 12L(n)^{10}(-1)^{n+1} + 54L(n)^8 + 112L(n)^6(-1)^{n+1} \\
&\quad + 105L(n)^4 + 36L(n)^2(-1)^{n+1} + 2
\end{aligned}$$

$$\begin{aligned}
L(13n) &= L(n)^{13} + (-1)^{n+1}(13)L(n)[L(n)^2 + (-1)^{n+1}]^2 \\
&\quad \{[L(n)^2 + (-1)^{n+1}]^3 + 2L(n)^2\} \\
&= L(n)^{13} + 13L(n)(-1)^{n+1}[L(n)^4 + 2L(n)^2(-1)^{n+1} + 1] \\
&\quad \{[L(n)^2 + (-1)^{n+1}]^3 + 2L(n)^2\} \\
&= L(n)^{13} + [13L(n)^5(-1)^{n+1} + 26L(n)^3 + 13L(n)(-1)^{n+1}] \\
&\quad [L(n)^6 + 3L(n)^4(-1)^{n+1} + 5L(n)^2 + (-1)^{n+1}] \\
&= L(n)^{13} + 13L(n)^{11}(-1)^{n+1} + 65L(n)^9 + 156L(n)^7(-1)^{n+1} \\
&\quad + 182L(n)^5 + 91L(n)^3(-1)^{n+1} + 13L(n)
\end{aligned}$$

$$\begin{aligned}
L(14n) &= L(2(7n)) \\
&= L(7n)^2 + 2(-1)^{7n+1} \\
&= [L(n)^7 + (-1)^{n+1}(7)L(n)^5 + (14)L(n)^3 + (-1)^{n+1}(7)L(n)]^2 \\
&\quad + 2(-1)^{n+1} \\
&= L(n)^{14} + 14L(n)^{12}(-1)^{n+1} + 77L(n)^{10} + 210L(n)^8(-1)^{n+1} \\
&\quad + 294L(n)^6 + 196L(n)^4(-1)^{n+1} + 49L(n)^2 + 2(-1)^{n+1}
\end{aligned}$$

$$\begin{aligned}
L(15n) &= L(3(5n)) \\
&= L(5n)^3 + (-1)^{5n+1}(3)L(5n) \\
&= [L(n)^5 + (-1)^{n+1}(5)L(n)^3 + (5)L(n)]^3 \\
&\quad + (-1)^{n+1}(3)[L(n)^5 + (-1)^{n+1}(5)L(n)^3 + 5L(n)] \\
&= L(n)^{15} + 15L(n)^{13}(-1)^{n+1} + 90L(n)^{11} + 275L(n)^9(-1)^{n+1} \\
&\quad + 450L(n)^7 + 378L(n)^5(-1)^{n+1} + 140L(n)^3 + 15L(n)(-1)^{n+1}
\end{aligned}$$

$$\begin{aligned}
L(16n) &= L(2(8n)) \\
&= L(8n)^2 + (-1)^{8n+1}(2) \\
&= [L(n)^8 + (-1)^{n+1}(8)L(n)^6 + 24L(n)^4 + (-1)^{n+1}(32)L(n)^2 \\
&\quad + 16 - 4L(n)^4 - (-1)^{n+1}(16)L(n)^2 - 6 + 2]^2 - 2 \\
&= L(n)^{16} + 16L(n)^{14}(-1)^{n+1} + 104L(n)^{12} \\
&\quad + 352L(n)^{10}(-1)^{n+1} + 660L(n)^8 + 372L(n)^6(-1)^{n+1} \\
&\quad + 336L(n)^4 + 64L(n)^2(-1)^{n+1} + 2
\end{aligned}$$

Appendix D. Usefulness of the *Lk* Table in tackling prime numbers

When using the Excel to continue to evaluate this table up to the 41st row, it turns out that when k is a prime number, all the coefficients (except the leading term) on the k-th row are all divisible by k.

Take the coefficients on the 11th row as an example: 11, 44, 77, 55, 11 are all divisible by 11.

Also look at the 13th row, 13, 65, 156, 182, 91, 13 are all divisible by 13.

This is interesting. Perhaps we can try to use this property to help us determine whether a number is prime. If we are not sure if an integer n is a prime number or not, look at the n-th row. If all the terms on the n-th row are all divisible by n, then n is a prime number. Otherwise, n is a composite number.

Let us look at the prime number rows to verify this observation:

On the 17th row,
17, 119, 442, 935, 1122, 714, 204, 17 are all divisible by 17.

On the 19th row,
19, 152, 665, 1729, 2717, 2508, 1254, 285, 19 are all divisible by 19.

On the 23rd row,
23, 230, 1311, 4692, 10948, 16744, 16445, 9867, 3289, 506, 23 are all divisible by 23.

On the 29th row,
29, 377, 2900, 14674, 51359, 127281, 224808, 281010, 243542, 140998, 51272, 10556, 1015, 29 are all divisible by 29.

On the 31st row,
31, 434, 3627, 20150, 78430, 219604, 447051, 660858, 700910, 520676, 260338, 82212, 14756, 1240, 31 are all divisible by 31.

On the 37th row,
37, 629, 6512, 45880, 232841, 878787, 2510820, 5476185, 9126975, 11560835, 10994920, 7696444, 3848222, 1314610, 286824, 35853, 2109, 37 are all divisible by 37.

On the 41st row,
41, 779, 9102, 73185, 429352, 1901416, 6487184, 17250012, 35937525, 58659315, 74657310, 73370115, 54826020, 30458900, 12183560, 3350479, 591261, 59983, 2870, 41 are all divisible by 41.

Up to the 41^{st} row, the hypothesis still holds for the prime number.

We will set up a counter example for each composite number row to show that the divisibility does not hold for all numbers on composite number row.

Row number	Counter example	Row number	Counter example
4	2	25	19380
6	9	26	299
8	20	27	2277
9	30	28	350
10	35	30	405
12	54	32	464
14	77	33	4466
15	275	34	527
16	104	35	166257
18	135	36	594
20	170	38	665
21	952	39	7735
22	209	40	740
24	252		

As we have found that on prime-number row, every number except the leading one is divisible by the prime number. If we express it mathematically, by referring to [Table 2.24], we have the following hypothesis:

$$n \mid {}_{n-k}L_k \quad \text{where } 0 < k \leqslant \frac{n}{2}$$

That is,

$$n \mid {}_{n-k+1}C_k - {}_{n-k-1}C_{k-2} \quad \text{where } 0 < k \leqslant \frac{n}{2}$$

Appendix E. Proofs

Proofs for **Hypothesis 2.10** and **Hypothesis 2.13**.

First, we express $U(n)$ in terms of $U(k)$ and $U(k-2)$.

Suppose $U(n) = AU(k) + BU(k-2)$ where A and B are real.

$$\begin{aligned} U(n) &= AU(k) + BU(k-2) \\ &= A[U(k-1) + U(k-2)] + BU(k-2) \\ &= AU(k-1) + (A+B)U(k-2) \\ &= AU(k-1) + (A+B)[U(k-1) - U(k-3)] \\ &= (2A+B)U(k-1) - (A+B)U(k-3) \end{aligned} \tag{11}$$

We use the above expression for this special recurrence relation that can facilitate our proof below.

Using Mathematical Induction, prove that
Hypothesis 2.10.

$$U(n) = F(r+2)U(n-r) - F(r)U(n-r-2)$$

Proof. Let $P(r)$ denote the statement "$U(n) = F(r+2)U(n-r) - F(r)U(n-r-2)$" for all positive integers r.

Consider $P(1)$.

$$U(n) = 2U(n-1) - U(n-3) \quad \text{(proved)}$$

Therefore, $P(1)$ is true.

Consider $P(2)$.
Substitute $k = n-1$, $A = 2$, $B = -1$ into (11), we have

$$U(n) = 3U(n-2) - U(n-4)$$

Therefore, $P(2)$ is true.

Consider $P(3)$.
Substitute $k = n-2$, $A = 3$, $B = -1$ into (11), we have

$$U(n) = 5U(n-3) - 2U(n-5).$$

Therefore, $P(3)$ is true.

Assume $P(k)$ is true, that is,

$$U(n) = F(k+2)U(n-k) - F(k)U(n-k-2).$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. &= U(n) \\ &= F(k+2)U(n-k) - F(k)U(n-k-2) \quad (by P(k)) \\ &= [2F(k+2) - F(k)]U(n-k-1) \\ &\quad - [F(k+2) - F(k)]U(n-k-3) \quad \text{(by (11))} \\ &= [F(k+2) + F(k+1)]U(n-k-1) - [F(k+1)]U(n-k-3) \\ &= F(k+3)U(n-k-1) - F(k+1)U(n-k-3) \\ &= F((k+1)+2)U(n-(k+1)) - F(k+1)U(n-(k+1)-2) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(r)$ is true for all positive integers r. □

Using Mathematical Induction, prove that
Hypothesis 2.13.

$$U(n) = F(r+1)U(n-r) + F(r)U(n-r-1)$$

$$\begin{aligned} U(n) &= AU(k) + BU(k-1) \\ &= A[U(k-1) + U(k-2)] + BU(k-1) \\ &= (A+B)U(k-1) + AU(k-2) \end{aligned} \tag{12}$$

We use the above expression for this special recurrence relation that can facilitate our proof below.

Proof. Let $P(r)$ denote the statement "$U(n) = F(r+1)U(n-r)+F(r)U(n-r-1)$" for all positive integers r.

Consider $P(1)$.

$$\begin{aligned} L.H.S. &= U(n) \\ R.H.S. &= F(2)U(n-1) + F(1)U(n-2) = U(n-1) + U(n-2) \\ L.H.S. &= R.H.S. \quad \text{(definition)} \end{aligned}$$

Therefore, $P(1)$ is true.

Suppose $P(k)$ is true, i.e.

$$U(n) = F(k+1)U(n-k) + F(k)U(n-k-1).$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. &= U(n) \\ &= F(k+1)U(n-k) + F(k)U(n-k-1) \quad (\text{by } P(k)) \\ &= [F(k+1) + F(k)]U(n-k-1) \\ &\quad + F(k+1)U(n-k-2) \quad (\text{by } (12)) \\ &= F(k+2)U(n-k-1) + F(k+1)U(n-k-2) \\ &= F((k+1)+1)U(n-(k+1)) + F(k+1)U(n-(k+1)-1) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(r)$ is true for all positive integers r. □

Proof for **Hypothesis 2.21**.

Proof. Let $P(k)$ denote the statement "$F(k)U(n) = F(r+k)U(n-r) + (-1)^{k+1} F(r)U(n-r-k)$" for all positive integers k.

Consider $P(1)$.

$$\begin{aligned} L.H.S. &= F(1)U(n) \\ &= U(n) \\ R.H.S. &= F(r+1)U(n-r) + (-1)^2 F(r)U(n-r-1) \\ &= F(r+1)U(n-r) + F(r)U(n-r-1) \\ L.H.S. &= R.H.S. \quad (\text{by (1) in Observation 2.18}) \end{aligned}$$

Therefore, $P(1)$ is true.

Consider $P(2)$.

$$\begin{aligned} L.H.S. &= F(2)U(n) \\ &= U(n) \\ R.H.S. &= F(r+2)U(n-r) + (-1)^3 F(r)U(n-r-2) \\ &= F(r+2)U(n-r) + F(r)U(n-r-2) \\ L.H.S. &= R.H.S. \quad (\text{by (2) in Observation 2.18}) \end{aligned}$$

Therefore, $P(2)$ is also true.

Assume $P(k')$ is true, i.e.

$$F(k')U(n) = F(r+k')U(n-r) + (-1)^{k'+1} F(r)U(n-r-k')$$

and P(k' + 1) is also true, i.e.

$$F(k'+1)U(n) = F(r+k'+1)U(n-r) + (-1)^{k'+2}F(r)U(n-r-k'-1).$$

Consider $P(k'+2)$.

$$\begin{aligned} L.H.S. &= F(k'+2)U(n) \\ &= [F(k') + F(k'+1)]U(n) \\ &= F(k')U(n) + F(k'+1)U(n) \\ &= F(r+k')U(n-r) + (-1)^{k'+1}F(r)U(n-r-k') \\ &\quad + F(r+k'+1)U(n-r) + (-1)^{k'+2}F(r)U(n-r-k'-1) \\ &\qquad \text{(by } P(k') \text{ and } P(k'+1)) \\ &= [F(r+k') + F(r+k'+1)]U(n-r) + (-1)^{k'+1}F(r)U(n-r-k') \\ &\quad - (-1)^{k'+1}F(r)U(n-r-k'-1) \\ &= F(r+k'+2)U(n-r) + (-1)^{k'+3}F(r)U(n-r-k'-2) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k'+2)$ is also true.

By Mathematical Induction, $P(k)$ is true for all positive integers k. □

Proof for **Hypothesis 3.22**.

Proof. Given

$$_nL_r + {_nL_{r-1}} = {_{n+1}L_{r+1}}, \tag{13}$$
$$_nC_r + {_nC_{r-1}} = {_{n+1}C_{r+1}}. \tag{14}$$

Let $P(n, r)$ denote the statement "$_nL_r = {_{n+1}C_r} - {_{n-1}C_{r-2}}$" where $n \geqslant r \geqslant 0$.

Consider $P(1, r)$.

Consider $P(1, 0)$.

$$\begin{aligned} L.H.S. &= {_1L_0} = 1 \\ R.H.S. &= {_2C_0} - {_0C_{-2}} = 1 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(1, 0)$ is true.

Consider $P(1,1)$.

$$\begin{aligned} L.H.S. &= {}_1L_1 = 2 \\ R.H.S. &= {}_2C_1 - {}_0C_{-1} = 2 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(1,1)$ is true.

Thus, $P(1,r)$ is true.

Assume $P(k,r)$ is true, that is ${}_kL_r = {}_{k+1}C_r - {}_{k-1}C_{r-2}$.

Consider $P(k+1,r)$.

$$\begin{aligned} L.H.S. &= {}_{k+1}L_r \\ &= {}_kL_{r-1} + {}_kL_r \quad \text{(by (13))} \\ &= {}_{k+1}C_{r-1} - {}_{k-1}C_{r-3} + {}_{k+1}C_r - {}_{k-1}C_{r-2} \quad \text{(by } P(k,r)) \\ &= {}_{k+1}C_{r-1} + {}_{k+1}C_r - ({}_{k-1}C_{r-3} + {}_{k-1}C_{r-2}) \\ &= {}_{k+2}Cr - kC_{r-2} \quad \text{(by (14))} \\ &= {}_{[(k+1)+1]}C_r - {}_{[(k+1)-1]}C_{r-2} \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1,r)$ is true.

By Mathematical Induction, $P(k,r)$ is true for all non-negative integers n and r satisfying $n \geqslant r$. □

Proof for **Hypothesis 3.35**.

Proof. First,

$$\begin{aligned} D(1) &= ({}_0C_0)U(1) = U(1), \\ D(2) &= ({}_1C_0)U(1) + ({}_1C_1)U(0) = U(1) + U(0) = U(2). \end{aligned}$$

As $U(1) = D(1)$, $U(2) = D(2)$ and with the property $U(n) + U(n+1) = U(n+2)$, we have to prove that, when $D(k) = U(k)$ and $D(k+1) = U(k+1)$,

$$\begin{aligned} D(k+2) &= U(k+2) \\ &= U(k) + U(k+1) \\ &= D(k) + D(k+1). \end{aligned}$$

In other words, we want to prove $D(k) + D(k+1) = D(k+2)$.

However, there are two cases since k can be odd or even.

Case I: Let $k = 2p + 1$.

$$\begin{aligned}
& D(2p+1) \\
= & ({}_{2p}C_0 + {}_{2p-1}C_1 + \ldots + {}_{p+1}C_{p-1} + {}_{p}C_p)U(1) \\
& + ({}_{2p-1}C_0 + {}_{2p-2}C_1 + \ldots + {}_{p+1}C_{p-2} + {}_{p}C_{p-1})U(0) \\
& D(2p+2) \\
= & ({}_{2p+1}C_0 + {}_{2p}C_1 + \ldots + {}_{p}C_{p-1} + {}_{p+1}C_p)U(1) \\
& + ({}_{2p}C_0 + {}_{2p-1}C_1 + \ldots + {}_{p+1}C_{p-1} + {}_{p}C_p)U(0) \\
& D(2p+1) + D(2p+2) \\
= & [({}_{2p}C_0 + {}_{2p-1}C_1 + \ldots + {}_{p+1}C_{p-1} + {}_{p}C_p)U(1) \\
& + ({}_{2p-1}C_0 + {}_{2p-2}C_1 + \ldots + {}_{p+1}C_{p-2} + {}_{p}C_{p-1})U(0)] \\
& + [({}_{2p+1}C_0 + {}_{2p}C_1 + \ldots + {}_{p}C_{p-1} + {}_{p+1}C_p)U(1) \\
& + ({}_{2p}C_0 + {}_{2p-1}C_1 + \ldots + {}_{p+1}C_{p-1} + {}_{p}C_p)U(0)] \\
= & [{}_{2p+1}C_0 + ({}_{2p}C_1 + {}_{2p}C_0) + ({}_{2p-1}C_2 + {}_{2p-1}C_1) \\
& + \ldots + ({}_{p+1}C_p + {}_{p+1}C_{p-1}) + {}_{p}C_p]U(1) \\
& + [{}_{2p}C_0 + ({}_{2p-1}C_1 + {}_{2p-1}C_0) + ({}_{2p-2}C_2 + {}_{2p-2}C_1) \\
& + \ldots + ({}_{p+1}C_{p-1} + {}_{p+1}C_p) + ({}_{p}C_p + {}_{p}C_{p-1})]U(0) \\
= & ({}_{2p+2}C_0 + {}_{2p+1}C_1 + {}_{2p}C_2 + \ldots + {}_{p+2}C_p + {}_{p+1}C_{p+1})U(1) \\
& + ({}_{2p+1}C_0 + {}_{2p}C_1 + {}_{2p-1}C_2 + \ldots +_{p+2} C_p + {}_{p+1}C_p)U(0) \\
& (\text{by } (1)\ {}_{n}C_r + {}_{n}C_{r+1} =_{n+1} C_{r+1},\ (2)\ {}_{a}C_0 = {}_{a}C_a \\
& = {}_{b}C_0 = {}_{b}C_b = 1 \text{ in the Pascal's Trianlge}) \\
= & D(2p+3)
\end{aligned}$$

Therefore, Case I is true.

Case II: Let $k = 2p + 2$.

$$\begin{aligned}
D(2p+2) + D(2p+3) &= [({}_{2p+1}C_0 + {}_{2p}C_1 + \ldots + {}_{p}C_{p-1} + {}_{p+1}C_p)U(1) \\
&\quad + ({}_{2p}C_0 + {}_{2p-1}C_1 + \ldots + {}_{p+1}C_{p-1} + {}_{p}C_p)U(0)] \\
&\quad + [({}_{2p+2}C_0 + {}_{2p+1}C_1 + {}_{2p}C_2 \\
&\quad + \ldots + {}_{p+2}C_p + {}_{p+1}C_{p+1})U(1) \\
&\quad + ({}_{2p+1}C_0 + {}_{2p}C_1 + {}_{2p-1}C_2 \\
&\quad + \ldots + {}_{p+2}C_p + {}_{p+1}C_p)U(0)] \\
&= [{}_{2p+2}C_0 + ({}_{2p+1}C_1 + {}_{2p+1}C_0) + \ldots \\
&\quad + ({}_{p+2}C_p + {}_{p+2}C_{p-1}) + ({}_{p+1}C_{p+1} + {}_{p+1}C_p)]U(1) \\
&\quad + [{}_{2p+1}C_0 + ({}_{2p}C_1 + {}_{2p}C_0) + \ldots \\
&\quad + ({}_{p+1}C_p + {}_{p+1}C_{p-1}) + {}_{p}C_p]U(0) \\
&= ({}_{2p+3}C_0 + {}_{2p+2}C_1 + \ldots + {}_{p+3}C_p + {}_{p+2}C_{p+1})U(1) \\
&\quad + ({}_{2p+2}C_0 + {}_{2p+1}C_1 + \ldots + {}_{p+2}C_p + {}_{p+1}C_{p+1})U(0) \\
&\quad \text{(by (1) } {}_{n}C_r + {}_{n}C_{r+1} = {}_{n+1}C_{r+1},\ \text{(2) } {}_{a}C_0 = {}_{a}C_a \\
&\quad = {}_{b}C_0 = {}_{b}C_b = 1 \text{ in the Pascal's Trianlge)} \\
&= D(2p+4)
\end{aligned}$$

Therefore, Case II is true.

Considering both cases, $D(n) + D(n+1) = D(n+2)$ is true for all positive integers n. As $D(1) = U(1)$, $D(2) = U(2)$ and $U(n) + U(n+1) = U(n+2)$, $D(n) = U(n)$. ☐

Proof for **Hypothesis 5.25**.

Proof. Let $P(n)$ denote the statement "$L(1)F(n) = F(n+1) - F(n-1)$" for all positive integers $n \geqslant 2$.

Consider $P(2)$.

$$\begin{aligned}
L.H.S. &= L(1)F(2) = 1 \\
R.H.S. &= F(3) - F(1) = 2 - 1 = 1 \\
L.H.S. &= R.H.S.
\end{aligned}$$

Therefore, $P(2)$ is true.

Consider $P(3)$.

$$L.H.S. = L(1)F(3) = 2$$
$$R.H.S. = F(4) - F(2) = 3 - 1 = 2$$
$$L.H.S. = R.H.S.$$

Therefore, $P(3)$ is also true.

Assume both $P(k)$ and $P(k+1)$ are true, i.e.

$$L(1)F(k) = F(k+1) - F(k-1)$$

and

$$L(1)F(k+1) = F(k+2) - F(k).$$

Consider $P(k+2)$.

$$\begin{aligned} L.H.S. &= L(1)F(k+2) \\ &= L(1)[F(k) + F(k+1)] \\ &= L(1)F(k) + L(1)F(k+1) \\ &= F(k+1) - F(k-1) + F(k+2) - F(k) \quad \text{(by } P(k) \text{ and } P(k+1)) \\ &= F(k+1) + F(k+2) - [F(k-1) + F(k)] \\ &= F(k+3) - F(k+1) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+2)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers $n \geqslant 2$.

Let $Q(n)$ denote the statement "$L(2)F(n) = F(n+2) + F(n-2)$" for all positive integers $n \geqslant 3$.

Consider $Q(3)$.

$$L.H.S. = L(2)F(3) = 6$$
$$R.H.S. = F(5) + F(1) = 5 + 1 = 6$$
$$L.H.S. = R.H.S.$$

Therefore, $Q(3)$ is true.

Consider $Q(4)$.

$$L.H.S. = L(2)F(4) = 9$$
$$R.H.S. = F(6) + F(2) = 8 + 1 = 9$$
$$L.H.S. = R.H.S.$$

Therefore, Q(4) is also true.

Assume both $Q(k)$ and $Q(k+1)$ are true, i.e.

$$L(2)F(k) = F(k+2) + F(k-2)$$

and

$$L(2)F(k+1) = F(k+3) + F(k-1).$$

Consider $Q(k+2)$.

$$\begin{aligned} L.H.S. &= L(2)F(k+2) \\ &= L(2)[F(k) + (k+1)] \\ &= L(2)F(k) + L(2)F(k+1) \\ &= F(k+2) + F(k-2) + F(k+3) + F(k-1) \quad \text{(by } Q(k) \text{ and } Q(k+1)\text{)} \\ &= F(k+2) + F(k+3) + F(k-2) + F(k-1) \\ &= F(k+4) + F(k) \\ &= R.H.S. \end{aligned}$$

Therefore, $Q(k+2)$ is also true.

By Mathematical Induction, $Q(n)$ is true for all positive integers $n \geqslant 3$.

Let $R(k)$ denote the statement "$L(k)F(n) = F(n+k) + (-1)^k F(n-k)$" for all positive integers k.

Consider $R(1)$.

$$\begin{aligned} L.H.S. &= L(1)F(n) \\ R.H.S. &= F(n+1) - F(n-1) \\ L.H.S. &= R.H.S. \quad \text{(proved in } P(n)\text{)} \end{aligned}$$

Therefore, $R(1)$ is true.

Consider $R(2)$.

$$\begin{aligned} L.H.S. &= L(2)F(n) \\ R.H.S. &= F(n+2) + F(n-2) \\ L.H.S. &= R.H.S. \quad \text{(proved in } Q(n)\text{)} \end{aligned}$$

Therefore, $R(2)$ is true.

Suppose both $R(a)$ and $R(a+1)$ are true, i.e.

$$L(a)F(n) = F(n+a) + (-1)^a F(n-a)$$

and

$$L(a+1)F(n) = F(n+a+1) + (-1)^{a+1} F(n-a-1).$$

Consider $R(a+2)$.

$$\begin{aligned} L.H.S. &= L(a+2)F(n) \\ &= [L(a)+L(a+1)]F(n) \\ &= L(a)F(n)+L(a+1)F(n) \\ &= F(n+a)+(-1)^a F(n-a)+F(n+a+1)+(-1)^{a+1}F(n-a-1) \\ &\quad \text{(by } R(a) \text{ and } R(a+1)) \\ &= F(n+a)+F(n+a+1)+(-1)^{a+2}F(n-a)-(-1)^{a+2}F(n-a-1) \\ &= F(n+a+2)+(-1)^{a+2}F(n-a-2) \\ &= R.H.S. \end{aligned}$$

Therefore, $R(a+2)$ is also true.

By Mathematical Induction, $R(k)$ is true for all positive integers k. □

Proof for **Hypothesis 5.26**.

Proof. Let $P(n)$ denote the statement "$L(n) = F(n+1)+F(n-1)$" for all positive integers $n \geqslant 2$.

Consider $P(2)$.

$$\begin{aligned} L.H.S. &= L(2) = 3 \\ R.H.S. &= F(3)+F(1) = 2+1 = 3 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(2)$ is true.

Consider $P(3)$.

$$\begin{aligned} L.H.S. &= L(3) = 4 \\ R.H.S. &= F(4)+F(2) = 3+1 = 4 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(3)$ is also true.

Suppose $P(k)$ and $P(k+1)$ are true, that is,

$$L(k) = F(k+1)+F(k-1),$$

$$L(k+1) = F(k+2)+F(k).$$

Consider $P(k+2)$.

$$\begin{aligned} L.H.S. &= L(k+2) \\ &= L(k) + L(k+1) \\ &= F(k+1) + F(k-1) + F(k+2) + F(k) \quad \text{(by } P(k) \text{ and } P(k+1)) \\ &= F(k+1) + F(k+2) + F(k-1) + F(k) \\ &= F(k+3) + F(k+1) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+2)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers $n \geqslant 2$.

Let $Q(n)$ denote the statement "$L(n) = F(n+2) - F(n-2)$" for all positive integers $n \geqslant 3$.

Consider $Q(3)$.

$$\begin{aligned} L.H.S. &= L(3) = 4 \\ R.H.S. &= F(5) - F(1) = 5 - 1 = 4 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $Q(3)$ is true.

Consider $Q(4)$.

$$\begin{aligned} L.H.S. &= L(4) = 7 \\ R.H.S. &= F(6) - F(2) = 7 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $Q(4)$ is also true.

Suppose $Q(k)$ and $Q(k+1)$ are true, that is,

$$L(k) = F(k+2) - F(k-2),$$

$$L(k+1) = F(k+3) - F(k-1).$$

Consider $Q(k+2)$.

$$\begin{aligned} L.H.S. &= L(k+2) \\ &= L(k) + L(k+1) \\ &= F(k+2) - F(k-2) + F(k+3) - F(k-1) \quad \text{(by } Q(k) \text{ and } Q(k+1)) \\ &= F(k+2) + F(k+3) - [F(k-2) + F(k-1)] \\ &= F(k+4) - F(k) \\ &= R.H.S. \end{aligned}$$

Therefore, $Q(k+2)$ is also true.

By Mathematical Induction, $Q(n)$ is true for all positive integers $n \geqslant 3$.

Let $R(k)$ denote the statement "$F(k)L(n) = F(n+k) + (-1)^{k+1}F(n-k)$" for all positive integers k.

Consider $R(1)$.

$$\begin{aligned} L.H.S. &= F(1)L(n) = L(n) \\ R.H.S. &= F(n+1) + F(n-1) = L(n) \quad \text{(proved)} \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $R(1)$ is true.

Consider $R(2)$.

$$\begin{aligned} L.H.S. &= F(2)L(n) = L(n) \\ R.H.S. &= F(n+2) - F(n-2) = L(n) \quad \text{(proved)} \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $R(2)$ is true.

Suppose $R(r)$ and $R(r+1)$ are true, that is,

$$F(r)L(n) = F(n+r) + (-1)^{r+1}F(n-r),$$

$$F(r+1)L(n) = F(n+r+1) + (-1)^{r+2}F(n-r-1).$$

Consider $R(r+2)$.

$$\begin{aligned} L.H.S. &= F(r+2)L(n) \\ &= F(r)L(n) + F(r+1)L(n) \\ &= F(n+r) + (-1)^{r+1}F(n-r) + F(n+r+1) + (-1)^{r+2}F(n-r-1) \\ &\quad \text{(by } R(r) \text{ and } R(r+1)) \\ &= F(n+r+2) + (-1)^{r+3}[F(n-r) - F(n-r-1)] \\ &= F(n+r+2) + (-1)^{r+3}F(n-r-2) \\ &= R.H.S. \end{aligned}$$

Therefore, $R(r+2)$ is also true.

By Mathematical Induction, $R(k)$ is true for all positive integers k. □

Proof for **Hypothesis 5.44**.

Proof. Let $P(k)$ denote the statement "$F(kn)$ satisfies Hypothesis 5.44" for all positive integers n.

Case I: Given that $P(4p)$ and $P(4p+1)$ are true, Consider $P(4p+2)$.

$$\begin{aligned} L.H.S. &= F((4p+2)n) \\ &= F((4p+1)n)L(n) + (-1)^{n+1}F(4pn) \quad \text{(by Formula 5.25)} \\ &= F(n)[{}_{4p}C_0 L(n)^{4p+1} + (-1)^{n+1}{}_{4p-1}C_1 L(n)^{4p-1} + {}_{4p-2}C_2 L(n)^{4p-3} \\ &\quad + \ldots + (-1)^{n+1}{}_{2p+1}C_{2p-1}L(n)^3 + {}_{2p}C_{2p}L(n)] \\ &\quad + F(n)[(-1)^{n+1}{}_{4p-1}C_0 L(n)^{4p-1} + {}_{4p-2}C_1 L(n)^{4p-3} \\ &\quad + (-1)^{n+1}{}_{4p-3}C_2 L(n)^{4p-5} + \ldots + (-1)^{n+1}{}_{2p+1}C_{2p-2}L(n)^3 \\ &\quad + {}_{2p}C_{2p-1}L(n)] \quad \text{(by } P(4p) \text{ and } P(4p+1)) \\ &= F(n)[{}_{4p}C_0 L(n)^{4p+1} + (-1)^{n+1}({}_{4p-1}C_1 + {}_{4p-1}C_0)L(n)^{4p-1} \\ &\quad + ({}_{4p-2}C_2 + {}_{4p-2}C_1)L(n)^{4p-3} + \ldots + (-1)^{n+1}({}_{2p+1}C_{2p-1} \\ &\quad + {}_{2p+1}C_{2p-2})L(n)^3 + ({}_{2p}C_{2p} + {}_{2p}C_{2p-1})L(n)] \\ &= F(n)[{}_{4p+1}C_0 L(n)^{4p+1} + (-1)^{n+1}{}_{4p}C_1 L(n)^{4p-1} + {}_{4p-1}C_2 L(n)^{4p-3} \\ &\quad + \ldots + (-1)^{n+1}{}_{2p+2}C_{2p-1}L(n)^3 + {}_{2p+1}C_{2p}L(n)] \\ &= R.H.S. \end{aligned}$$

Therefore, $P(4p+2)$ is also true.

Note that (1) ${}_aC_0 = {}_aC_a = {}_bC_0 = {}_bC_b = 1$; (2) ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$; and (3) $(-1)^{2n+2} = 1$.

Case II: Given that $P(4p+1)$ and $P(4p+2)$ are true, Consider $P(4p+3)$.

$$\begin{aligned}
L.H.S. &= F((4p+3)n) \\
&= F((4p+2)n)L(n) + (-1)^{n+1}F((4p+1)n) \quad \text{(by Formula 5.25)} \\
&= F(n)[{}_{4p+1}C_0 L(n)^{4p+2} + (-1)^{n+1}{}_{4p}C_1 L(n)^{4p} + {}_{4p-1}C_2 L(n)^{4p-2} \\
&\quad + \ldots + (-1)^{n+1}{}_{2p+2}C_{2p-1}L(n)^4 + {}_{2p+1}C_{2p}L(n)^2] \\
&\quad + F(n)[(-1)^{n+1}{}_{4p}C_0 L(n)^{4p} + {}_{4p-1}C_1 L(n)^{4p-2} \\
&\quad + (-1)^{n+1}{}_{4p-2}C_2 L(n)^{4p-4} + \ldots + {}_{2p+1}C_{2p-1}L(n)^2 \\
&\quad + (-1)^{n+1}{}_{2p}C_{2p}] \quad \text{(by } P(4p+1) \text{ and } P(4p+2)) \\
&= F(n)[{}_{4p+2}C_0 L(n)^{4p+2} + (-1)^{n+1}({}_{4p}C_1 + {}_{4p}C_0)L(n)^{4p} \\
&\quad + ({}_{4p-1}C_2 + {}_{4p-1}C_1)L(n)^{4p-2} + \ldots + ({}_{2p+1}C_{2p} + {}_{2p+1}C_{2p-1})L(n)^2 \\
&\quad + (-1)^{n+1}{}_{2p+1}C_{2p+1}] \\
&= F(n)[{}_{4p+2}C_0 L(n)^{4p+2} + (-1)^{n+1}{}_{4p+1}C_1 L(n)^{4p} + {}_{4p}C_2 L(n)^{4p-2} \\
&\quad + \ldots + (-1)^{n+1}{}_{2p+1}C_{2p+1}] \\
&= R.H.S.
\end{aligned}$$

Therefore, $P(4p+3)$ is also true.

Note that (1) ${}_aC_0 = {}_aC_a = {}_bC_0 = {}_bC_b = 1$; (2) ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$; and (3) $(-1)^{2n+2} = 1$.

Case III: Given that $P(4p+2)$ and $P(4p+3)$ are true, Consider $P(4p+4)$.

$$\begin{aligned}
L.H.S. &= F((4p+4)n) \\
&= F((4p+3)n)L(n) + (-1)^{n+1}F((4p+2)n) \quad \text{(by Formula 5.25)} \\
&= F(n)[{}_{4p+2}C_0 L(n)^{4p+3} + (-1)^{n+1}{}_{4p+1}C_1 L(n)^{4p+1} + {}_{4p}C_2 L(n)^{4p} \\
&\quad + \ldots + {}_{2p+2}C_{2p}L(n)^3 + (-1)^{n+1}{}_{2p+1}C_{2p+1}L(n)] \\
&\quad + F(n)[(-1)^{n+1}{}_{4p+1}C_0 L(n)^{4p+1} + {}_{4p}C_1 L(n)^{4p-1} \\
&\quad + (-1)^{n+1}{}_{4p-1}C_2 L(n)^{4p-3} + \ldots + {}_{2p+2}C_{2p-1}L(n)^3 \\
&\quad + (-1)^{n+1}{}_{2p+1}C_{2p}L(n)] \quad \text{(by } P(4p+2) \text{ and } P(4p+3)) \\
&= F(n)[{}_{4p+3}C_0 L(n)^{4p+3} + (-1)^{n+1}({}_{4p+1}C_1 + {}_{4p+1}C_0)L(n)^{4p+1} \\
&\quad + ({}_{4p}C_2 + {}_{4p}C_1)L(n)^{4p-1} + \ldots + ({}_{2p+2}C_{2p} + {}_{2p+2}C_{2p-1})L(n)^3 \\
&\quad + (-1)^{n+1}({}_{2p+1}C_{2p-1} + {}_{2p+1}C_{2p})L(n)] \\
&= F(n)[{}_{4p+3}C_0 L(n)^{4p+3} + (-1)^{n+1}{}_{4p+2}C_1 L(n)^{4p+1} \\
&\quad + \ldots + (-1)^{n+1}{}_{2p+2}C_{2p+1}L(n)] \\
&= R.H.S.
\end{aligned}$$

Therefore, $P(4p+4)$ is also true.

Note that (1) ${}_aC_0 = {}_aC_a = {}_bC_0 = {}_bC_b = 1$; (2) ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$; and (3) $(-1)^{2n+2} = 1$.

Case IV: Given that $P(4p+3)$ and $P(4p+4)$ are true, Consider $P(4p+5)$.

$$\begin{aligned}
L.H.S. &= F((4p+5)n) \\
&= F((4p+4)n)L(n) + (-1)^{n+1}F((4p+3)n) \quad \text{(by Formula 5.25)} \\
&= F(n)[{}_{4p+3}C_0L(n)^{4p+4} + (-1)^{n+1}{}_{4p+2}C_1L(n)^{4p+2} \\
&\quad + \ldots + {}_{2p+3}C_{2p}L(n)^4 + (-1)^{n+1}{}_{2p+2}C_{2p+1}L(n)^2] \\
&\quad + F(n)[(-1)^{n+1}{}_{4p+2}C_0L(n)^{4p+2} + {}_{4p+1}C_1L(n)^{4p} \\
&\quad + \ldots + (-1)^{n+1}{}_{2p+2}C_{2p}L(n)^2 + {}_{2p+1}C_{2p+1}] \\
&\quad \text{(by } P(4p+3) \text{ and } P(4p+4)\text{)} \\
&= F(n)[{}_{4p+4}C_0L(n)^{4p+4} + (-1)^{n+1}({}_{4p+2}C_1 + {}_{4p+2}C_0)L(n)^{4p+2} \\
&\quad + \ldots + (-1)^{n+1}({}_{2p+2}C_{2p+1} + {}_{2p+2}C_{2p})L(n)^2 \\
&\quad + {}_{2p+2}C_{2p+2}] \\
&= F(n)[{}_{4p+4}C_0L(n)^{4p+4} + (-1)^{n+1}{}_{4p+3}C_1L(n)^{4p+2} \\
&\quad + \ldots + (-1)^{n+1}{}_{2p+3}C_{2p+1}L(n)^2 + {}_{2p+2}C_{2p+2}] \\
&= R.H.S.
\end{aligned}$$

Therefore, $P(4p+5)$ is also true.

Note that (1) ${}_aC_0 = {}_aC_a = {}_bC_0 = {}_bC_b = 1$; (2) ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$; and (3) $(-1)^{2n+2} = 1$.

As $P(1)$ and $P(2)$ are true, by Mathematical Induction, $P(k)$ is true for all positive integers k. □

Proof for **Formula 5.55**.

Proof. Let $P(n)$ denote the statement "$5F(1)F(n) = L(n+1) + L(n-1)$" for all positive integers $n \geqslant 2$.

Consider $P(2)$.

$$\begin{aligned}
L.H.S. &= 5F(1)F(2) = 5 \\
R.H.S. &= L(3) + L(1) = 4 + 1 = 5 \\
L.H.S. &= R.H.S.
\end{aligned}$$

Therefore, $P(2)$ is true.

Consider $P(3)$.

$$\begin{aligned} L.H.S. &= 5F(1)F(3) = 10 \\ R.H.S. &= L(4) + L(2) = 7 + 3 = 10 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(3)$ is also true.

Suppose $P(k)$ and $P(k+1)$ are true, that is,

$$\begin{aligned} 5F(1)F(k) &= L(k+1) + L(k-1), \\ 5F(1)F(k+1) &= L(k+2) + L(k). \end{aligned}$$

Consider $P(k+2)$.

$$\begin{aligned} L.H.S. &= 5F(1)F(k+2) \\ &= 5F(1)F(k) + 5F(1)F(k+1) \\ &= L(k+1) + L(k-1) + L(k+2) + L(k) \quad \text{(by } P(k) \text{ and } P(k+1)\text{)} \\ &= L(k+1) + L(k+2) + L(k-1) + L(k) \\ &= L(k+3) + L(k+1) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+2)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers $n \geqslant 2$.

Let $Q(n)$ denote the statement "$5F(2)F(n) = L(n+2) - L(n-2)$" for all positive integers $n \geqslant 3$.

Consider $Q(3)$.

$$\begin{aligned} L.H.S. &= 5F(2)F(3) = 10 \\ R.H.S. &= L(5) - L(1) = 11 - 1 = 10 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $Q(3)$ is true.

Consider $Q(4)$.

$$\begin{aligned} L.H.S. &= 5F(2)F(4) = 15 \\ R.H.S. &= L(6) - L(2) = 18 - 3 = 15 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $Q(4)$ is also true.

Suppose $Q(k)$ and $Q(k+1)$ are true, that is,

$$5F(2)F(k) = L(k+2) - L(k-2),$$

$$5F(2)F(k+1) = L(k+3) - L(k-1).$$

Consider $Q(k+2)$.

$$\begin{aligned} L.H.S. &= 5F(2)F(k+2) \\ &= 5F(2)F(k) + 5F(2)F(k+1) \\ &= L(k+2) - L(k-2) + L(k+3) - L(k-1) \quad \text{(by } Q(k) \text{ and } Q(k+1)\text{)} \\ &= L(k+2) + L(k+3) - [L(k-2) + L(k-1)] \\ &= L(k+4) - L(k) \\ &= R.H.S. \end{aligned}$$

Therefore, $Q(k+2)$ is also true.

By Mathematical Induction, $Q(n)$ is true for all positive integers $n \geqslant 3$.

Let $R(k)$ denote the statement "$5F(k)F(n) = L(n+k) + (-1)^{k+1}L(n-k)$" for all positive integers k.

Consider $R(1)$.

$$\begin{aligned} L.H.S. &= 5F(1)F(n) = 5F(n) \\ R.H.S. &= L(n+1) + L(n-1) = 5F(n) \quad \text{(by } P(n)\text{)} \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $R(1)$ is true.

Consider $R(2)$.

$$\begin{aligned} L.H.S. &= 5F(2)F(n) = 5F(n) \\ R.H.S. &= L(n+2) - L(n-2) = 5F(n) \quad \text{(by } Q(n)\text{)} \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $R(2)$ is true.

Suppose $R(r)$ and $R(r+1)$ are true, that is,

$$5F(r)F(n) = L(n+r) + (-1)^{r+1}L(n-r),$$

$$5F(r+1)F(n) = L(n+r+1) + (-1)^{r+2}L(n-r-1).$$

Consider $R(r+2)$.

$$\begin{aligned} L.H.S. &= 5F(r+2)F(n) \\ &= 5F(r)F(n) + 5F(r+1)F(n) \\ &= L(n+r) + (-1)^{r+1}L(n-r) + L(n+r+1) + (-1)^{r+2}L(n-r-1) \\ &\quad \text{(by } R(r) \text{ and } R(r+1)) \\ &= L(n+r+2) + (-1)^{r+3}[L(n-r) - L(n-r-1)] \\ &= L(n+r+2) + (-1)^{r+3}L(n-r-2) \\ &= R.H.S. \end{aligned}$$

Therefore, $R(r+2)$ is also true.

By Mathematical Induction, $R(k)$ is true for all positive integers k. □

Proof for **Hypothesis 6.2**.

Proof. Let $P(n)$ denote the statement "$F(n)^2 = F(n-1)F(n+1) + (-1)^{n+1}$" for all positive integers $n \geqslant 2$.

Consider $P(2)$.

$$\begin{aligned} L.H.S. &= F(2)^2 = 1 \\ R.H.S. &= F(1)F(3) + (-1)^3 = 1 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(2)$ is true.

Suppose $P(k)$ is true, i.e.

$$F(k)^2 = F(k-1)F(k+1) + (-1)^{k+1},$$

i.e.

$$F(k-1)F(k+1) = F(k)^2 - (-1)^{k+1},$$

i.e.

$$F(k-1)F(k+1) = F(k)^2 + (-1)^{k+2}.$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. &= F(k+1)^2 \\ &= [F(k+1)^2 - F(k)F(k+1)] + F(k)F(k+1) \\ &= F(k+1)F(k-1) + F(k)F(k+1) \\ &= F(k)^2 + (-1)^{k+2} + F(k)F(k+1) \quad \text{(by } P(k)) \\ &= F(k)F(k+2) + (-1)^{k+2} \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers $n \geqslant 2$. $\square$

Proof for **Hypothesis 6.4**.

Proof. Let $P(n)$ denote the statement "$F(n)^2 = F(n-2)F(n+2) + (-1)^n$" for all positive integers $n \geqslant 3$.

Consider $P(3)$.

$$\begin{aligned} L.H.S. &= F(3)^2 = 4 \\ R.H.S. &= F(1)F(5) - 1 = 1 \times 5 = 4 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(3)$ is true.

Assume $P(k)$ is true, i.e.

$$F(k)^2 = F(k-2)F(k+2) + (-1)^k,$$

i.e.

$$F(k-2)F(k+2) = F(k)^2 + (-1)^{k+1}.$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. &= F(k+1)^2 \\ &= [F(k+1)F(k+1) + F(k)F(k+1)] - F(k)F(k+1) \\ &= F(k+2)F(k+1) - F(k)F(k+1) \\ &= [F(k+2)F(k+1) - F(k+2)F(k-1)] - F(k)F(k+1) \\ &\quad + F(k+2)F(k-1) \\ &= [F(k+2)F(k) - F(k+2)F(k-1)] - F(k)F(k+1) \\ &\quad + 2F(k+2)F(k-1) \\ &= F(k+2)F(k-2) - F(k)F(k+1) + 2F(k+2)F(k-1) \\ &= [F(k)F(k) + (-1)^{k+1}] - F(k)F(k+1) + 2F(k+2)F(k-1) \\ &\quad (\text{by } P(k)) \\ &= -F(k)[F(k+1) - F(k)] + 2F(k+2)F(k-1) + (-1)^{k+1} \\ &= F(k+2)F(k-1) + F(k+2)F(k-1) - F(k)F(k-1) + (-1)^{k+1} \\ &= F(k+2)F(k-1) + F(k+1)F(k-1) + (-1)^{k+1} \\ &= F(k+3)F(k-1) + (-1)^{k+1} \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers $n \geqslant 3$. □

Proof for **Hypothesis 6.19**.

Proof. Let $P(n)$ denote the statement "$F(2)-F(4)+F(6)+\ldots+(-1)^{n+1}F(2n) = (-1)^{n+1}F(n)F(n+1)$" for all positive integers n.

Consider $P(1)$.

$$\begin{aligned} L.H.S. &= F(2) = 1 \\ R.H.S. &= (-1)^2F(1)F(2) = 1 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(1)$ is true.

Suppose $P(k)$ is true, i.e.

$$F(2) - F(4) + F(6) + \ldots + (-1)^{k+1}F(2k) = (-1)^{k+1}F(k)F(k+1).$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. &= F(2) - F(4) + F(6) + \ldots + (-1)^{k+1}F(2k) + (-1)^{k+2}F(2k+2) \\ &= (-1)^{k+1}F(k)F(k+1) + (-1)^{k+2}F(2k+2) \\ &= (-1)^{k+2}[F(2k+2) - F(k)F(k+1)] \\ &= (-1)^{k+2}[F(k+2)^2 - F(k)^2 - F(k)F(k-1)] \\ &\quad (\text{apply } F(2k) = F(k+1)^2 - F(k-1)^2) \\ &= (-1)^{k+2}\{[F(k+2)^2 - F(k)[F(k) + F(k+1)]\} \\ &= (-1)^{k+2}[F(k+2)^2 - F(k)F(k+2)] \\ &= (-1)^{k+2}F(k+1)F(k+2) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers n. □

Proof for **Hypothesis 6.21**.

Proof. Let $P(n)$ denote the statement "$-F(1)+F(3)-F(5)+\ldots+(-1)^nF(2n-1) = (-1)^nF(n)F(n)$" for all positive integers n.

When $n = 1$,

$$L.H.S. = -F(1) = -1$$
$$R.H.S. = (-1)^1 F(1)F(1) = -1$$
$$L.H.S. = R.H.S.$$

Therefore, $P(1)$ is true.

Suppose $P(k)$ is true, i.e.

$$-F(1) + F(3) - F(5) + \ldots + (-1)^k F(2k-1) = (-1)^k F(k)F(k).$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. = & -F(1) + F(3) - F(5) + \ldots + (-1)^k F(2k-1) + (-1)^{k+1} F(2k+1) \\ &= (-1)^k F(k)F(k) + (-1)^{k+1} F(2k+1) \\ &= (-1)^{k+1}[F(2k+1) - F(k)^2] \\ &= (-1)^{k+1}[F(k+1)^2 + F(k)^2 - F(k)^2] \\ &\quad (\text{apply } F(2k+1) = F(k+1)^2 + F(k)^2) \\ &= (-1)^{k+1} F(k+1)F(k+1) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers n. □

Proof for **Hypothesis 6.23**.

Proof. Let $P(n)$ denote the statement "$-F(2) + F(4) - F(6) + \ldots + (-1)^n F(2n) = (-1)^n F(n)F(n+1)$" for all positive integers n.

When $n = 1$,

$$L.H.S. = -F(2) = -1$$
$$R.H.S. = (-1)^1 F(1)F(2) = -1$$
$$L.H.S. = R.H.S.$$

Therefore, $P(1)$ is true.

Suppose $P(k)$ is true, i.e.

$$-F(2) + F(4) - F(6) + \ldots + (-1)^k F(2k) = (-1)^k F(k)F(k+1).$$

Consider $P(k+1)$.

$$\begin{aligned}
L.H.S. =& -F(2)+F(4)-F(6)+\ldots+(-1)^kF(2k)+(-1)^{k+1}F(2k+2)\\
=& (-1)^kF(k)F(k+1)+(-1)^{k+1}F(2k+2)\\
=& (-1)^{k+1}[F(2k+2)-F(k)F(k+1)]\\
=& (-1)^{k+1}[F(k+2)^2-F(k)^2-F(k)F(k+1)]\\
& (\text{apply } F(2k)=F(k+1)^2-F(k-1)^2)\\
=& (-1)^{k+1}[F(k+2)^2-F(k)F(k+2)]\\
=& (-1)^{k+1}F(k+1)F(k+2)\\
=& R.H.S.
\end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers n. □

Proof for **Hypothesis 6.25**.

Proof. Let $P(n)$ denote the statement "$F(1)-F(3)+F(5)+\ldots+(-1)^{n+1}F(2n-1)=(-1)^{n+1}F(n)F(n)$" for all positive integers n.

When $n=1$,

$$\begin{aligned}
L.H.S. &= F(1)=1\\
R.H.S. &= (-1)^2F(1)F(1)=1\\
L.H.S. &= R.H.S.
\end{aligned}$$

Therefore, $P(1)$ is true.

Suppose $P(k)$ is true, i.e.

$$F(1)-F(3)+F(5)+\ldots+(-1)^{k+1}F(2k-1)=(-1)^{k+1}F(k)F(k).$$

Consider $P(k+1)$.

$$\begin{aligned}
L.H.S. =& F(1)-F(3)+F(5)+\ldots+(-1)^{k+1}F(2k-1)+(-1)^{k+2}F(2k+1)\\
=& (-1)^{k+1}F(k)F(k)+(-1)^{k+2}F(2k+1)\\
=& (-1)^{k+2}[F(2k+1)-F(k)^2]\\
=& (-1)^{k+2}[F(k+1)^2+F(k)^2-F(k)^2]\\
& (\text{apply } F(2k+1)=F(k+1)^2+F(k)^2)\\
=& (-1)^{k+2}F(k+1)F(k+1)\\
=& R.H.S.
\end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers n. □

Proof for **Hypothesis 6.36**.

Proof. Let $P(n)$ denote the statement "$L(n)^2 = L(n-1)L(n+1) + (-1)^{n+2}$ $(5)F(1)^2$" for all positive integers $n \geqslant 2$.

Consider $P(2)$,

$$\begin{aligned}L.H.S. &= L(2)^2 = 9\\ R.H.S. &= (L(1)L(3) + 5(1) = 1 \times 4 + 5 = 9\\ L.H.S. &= R.H.S.\end{aligned}$$

Therefore, $P(2)$ is true.

Suppose $P(k)$ is true, i.e.

$$L(k)^2 = L(k-1)L(k+1) + (-1)^{k+2}(5)F(1)^2,$$

i.e.

$$L(k)^2 = L(k-1)L(k+1) + 5,$$

i.e.

$$L(k-1)L(k+1) = L(k)^2 - 5.$$

Consider $P(k+1)$.

$$\begin{aligned}L.H.S. &= L(k+1)^2\\ &= [L(k+1)^2 - L(k)L(k+1)] + L(k)L(k+1)\\ &= L(k-1)L(k+1) + L(k)L(k+1)\\ &= L(k)^2 - 5 + L(k)L(k+1) \quad \text{(by } P(k))\\ &= L(k)L(k+2) - 5\\ &= L(k+1-1)L(k+1+1) + (-1)^{k+3}(5)\\ &= R.H.S.\end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers $n \geqslant 2$. □

Proof for **Hypothesis 6.38**.

Proof. Let $P(n)$ denote the statement "$L(n)^2 = L(n-2)L(n+2) + (-1)^{n+1}(5)$" for all positive integers $n \geqslant 3$.

Consider $P(3)$,

$$\begin{aligned} L.H.S. &= L(3)^2 = 16 \\ R.H.S. &= L(1)L(5) + 5 = 1 \times 11 + 5 = 16 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(3)$ is true.

Suppose $P(k)$ is true, i.e.

$$L(k)^2 = L(k-2)L(k+2) + (-1)^{k+1}(5).$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. &= L(k+1)^2 \\ &= [L(k+1)L(k+1) + L(k)L(k+1)] - L(k)L(k+1) \\ &= L(k+2)L(k+1) - L(k)L(k+1) \\ &= [L(k+2)L(k+1) - L(k+2)L(k-1)] - L(k)L(k+1) \\ &\quad + L(k+2)L(k-1) \\ &= [L(k+2)L(k) - L(k+2)L(k-1)] - L(k)L(k+1) \\ &\quad + 2L(k+2)L(k-1) \\ &= L(k+2)L(k-2) - L(k)L(k+1) + 2L(k+2)L(k-1) \\ &= [L(k)L(k) + (-1)^{k+2}(5)] - L(k)L(k+1) + 2L(k+2)L(k-1) \\ &\quad \text{(by } P(k)) \\ &= -L(k)[L(k+1) - L(k)] + 2L(k+2)L(k-1) + (-1)^{k+2}(5) \\ &= L(k+2)L(k-1) + L(k+2)L(k-1) - L(k)L(k-1) + (-1)^{k+2}(5) \\ &= L(k+2)L(k-1) + L(k+1)L(k-1) + (-1)^{k+2}(5) \\ &= L(k+3)L(k-1) + (-1)^{k+1+1}(5) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers $n \geqslant 3$. □

Proof for **Hypothesis 6.50**.

Proof. Let $P(n)$ denote the statement "$-5F(2) + 5F(4) - 5F(6) + \ldots + (-1)^n$ $5F(2n) = (-1)^n 5F(n)F(n+1)$" for all positive integers n.

Consider $P(1)$,

$$L.H.S. = -5F(2) = -5$$
$$R.H.S. = (-1)^1 5F(1)F(2) = -5$$
$$L.H.S. = R.H.S.$$

Therefore, $P(1)$ is true.

Suppose $P(k)$ is true, i.e.

$$-5F(2) + 5F(4) - 5F(6) + \ldots + (-1)^k 5F(2k) = (-1)^k 5F(k)F(k+1).$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. =& -5F(2) + 5F(4) - 5F(6) + \ldots + (-1)^k 5F(2k) + (-1)^{k+1} 5F(2k+2) \\ =& (-1)^k 5F(k)F(k+1) + (-1)^{k+1} 5F(2k+2) \\ =& (-1)^{k+1} 5[F(2k+2) - F(k)F(k+1)] \\ =& (-1)^{k+1} 5[F(k+2)^2 - F(k)^2 - F(k)F(k-1)] \\ & (\text{apply } F(2k) = F(k+1)^2 - F(k-1)^2) \\ =& (-1)^{k+1} 5\{F(k+2)^2 - F(k)[F(k) + F(k+1)]\} \\ =& (-1)^{k+1} 5[F(k+2)^2 - F(k)F(k+2)] \\ =& (-1)^{k+1} 5F(k+1)F(k+2) \\ =& R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers n. □

Proof for **Hypothesis 6.52**.

Proof. Let $P(n)$ denote the statement "$5F(1) - 5F(3) + 5F(5) - \ldots + (-1)^{n+1} 5F(2n-1) = (-1)^{n+1} 5F(n)F(n)$" for all positive integers n.

Consider $P(1)$,

$$L.H.S. = 5F(1) = 5$$
$$R.H.S. = (-1)^2 5F(1)F(1) = 5$$
$$L.H.S. = R.H.S.$$

Therefore, $P(1)$ is true.

Suppose $P(k)$ is true, i.e.

$$5F(1) - 5F(3) + 5F(5) - \ldots + (-1)^{k+1} 5F(2k-1) = (-1)^{k+1} 5F(k)F(k).$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. &= 5F(1) - 5F(3) + 5F(5) + \ldots + (-1)^{k+1}5F(2k-1) \\ &\quad + (-1)^{k+2}5F(2k+1) \\ &= (-1)^{k+1}5F(k)F(k) + (-1)^{k+2}5F(2k+1) \\ &= (-1)^{k+2}5[F(2k+1) - F(k)^2] \\ &= (-1)^{k+2}5[F(k+2)^2 + F(k)^2 - F(k)^2] \\ &\quad (\text{apply } F(2k+1) = F(k+1)^2 + F(k)^2) \\ &= (-1)^{k+2}5F(k+1)F(k+1) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers n. □

Proof for **Hypothesis 6.54**.

Proof. Let $P(n)$ denote the statement "$5F(2) - 5F(4) + 5F(6) + \ldots + (-1)^{n+1}$ $5F(2n) = (-1)^{n+1}5F(n)F(n+1)$" for all positive integers n.

Consider $P(1)$,

$$\begin{aligned} L.H.S. &= 5F(2) = 5 \\ R.H.S. &= (-1)^2 5F(1)F(2) = 5 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(1)$ is true.

Suppose $P(k)$ is true, i.e.

$$5F(2) - 5F(4) + 5F(6) + \ldots + (-1)^{k+1}5F(2k) = (-1)^{k+1}5F(k)F(k+1).$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. &= 5F(2) - 5F(4) + 5F(6) + \dots (-1)^{k+1} 5F(2k) + (-1)^{k+2} 5F(2k+2) \\ &= (-1)^{k+1} 5F(k)F(k+1) + (-1)^{k+2} 5F(2k+2) \\ &= (-1)^{k+2} 5[F(2k+2) - F(k)F(k+1)] \\ &= (-1)^{k+2} 5[F(k+2)^2 - F(k)^2 - F(k)F(k+1)] \\ &\quad (\text{apply } F(2k) = F(k+1)^2 - F(k-1)^2) \\ &= (-1)^{k+2} 5[F(k+2)^2 - F(k)F(k+2)] \\ &= (-1)^{k+2} F(k+1)F(k+2) \\ &= R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers n. □

Proof for **Hypothesis 6.56**.

Proof. Let $P(n)$ denote the statement "$-5F(1) + 5F(3) - 5F(5) + \dots + (-1)^n$ $5F(2n-1) = (-1)^n 5F(n)F(n)$" for all positive integers n.

Consider $P(1)$,

$$\begin{aligned} L.H.S. &= -5F(1) = -5 \\ R.H.S. &= (-1)^1 5F(1)F(1) = -5 \\ L.H.S. &= R.H.S. \end{aligned}$$

Therefore, $P(1)$ is true.

Suppose $P(k)$ is true, i.e.

$$-5F(1) + 5F(3) - 5F(5) + \dots + (-1)^k 5F(2k-1) = (-1)^k 5F(k)F(k).$$

Consider $P(k+1)$.

$$\begin{aligned} L.H.S. = & -5F(1)+5F(3)-5F(5)+\ldots+(-1)^k 5F(2k-1) \\ & +(-1)^{k+1}5F(2k+1) \\ = & (-1)^k 5F(k)F(k)+(-1)^{k+1}5F(2k+1) \\ = & (-1)^{k+1}5[F(2k+1)-F(k)^2] \\ = & (-1)^{k+1}5[F(k+1)^2+F(k)^2-F(k)^2] \\ & (\text{apply } F(2k+1)=F(k+1)^2+F(k)^2) \\ = & (-1)^{k+1}F(k+1)F(k+1) \\ = & R.H.S. \end{aligned}$$

Therefore, $P(k+1)$ is also true.

By Mathematical Induction, $P(n)$ is true for all positive integers n. □

Proof for **Formula 7.2**

Proof.

$$\begin{aligned} S_F(1) &= F(1)F(1) \\ S_F(2) &= F(1)F(2)+F(2)F(1) \\ S_F(3) &= F(1)F(3)+F(2)F(2)+F(3)F(1) \\ S_F(4) &= F(1)F(4)+F(2)F(3)+F(3)F(2)+F(4)F(1) \\ S_F(5) &= F(1)F(5)+F(2)F(4)+F(3)F(3)+F(4)F(2)+F(5)F(1) \\ &\ldots \\ S_F(k) &= F(1)F(k)+F(2)F(k-1)+\ldots+F(r)F(k-r+1) \\ &\quad +\ldots+F(k)F(1) \\ S_F(k+1) &= F(1)F(k+1)+F(2)F(k)+\ldots+F(r)F(k-r+2) \\ &\quad +\ldots+F(k)F(2)+F(k+1)F(1) \\ S_F(k+2) &= F(1)F(k+2)+F(2)F(k+1)+\ldots+F(r)F(k-r+3) \\ &\quad +\ldots+F(k)F(3)+F(k+1)F(2)+F(k+2)F(1) \\ &\ldots \end{aligned}$$

$$\begin{aligned}
S_F(k) + S_F(k+1) &= [F(1)F(k) + F(2)F(k-1) + \ldots + F(r)F(k-r+1) \\
&\quad + \ldots + F(k)F(1)] + [F(1)F(k+1) + F(2)F(k) \\
&\quad + \ldots + F(r)F(k-r+2) \\
&\quad + \ldots + F(k)F(2) + F(k+1)F(1)] \\
&= [F(1)F(k) + F(1)F(k+1)] + [F(2)F(k-1) + F(2)F(k)] \\
&\quad + \ldots + [F(r)F(k-r+1) + F(r)F(k-r+2)] \\
&\quad + \ldots + [F(k)F(1) + F(k)F(2)] + F(k+1)F(1) \\
&= F(1)[F(k) + F(k+1)] + F(2)[F(k-1) + F(k)] \\
&\quad + \ldots + F(r)[F(k-r+1) + F(k-r+2)] \\
&\quad + \ldots + F(k)[F(1) + F(2)] + F(k+1)F(1) \\
&= F(1)F(k+2) + F(2)F(k+1) + \ldots + F(r)F(k-r+3) \\
&\quad + \ldots + F(k)F(3) + F(k+1)F(2) \\
&\quad \text{(Note that } F(1) = F(2) = 1\text{)} \\
&= S_F(k+2) - F(k+2)F(1) \\
&= S_F(k+2) - F(k+2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_F(k) + S_F(k+1) &= S_F(k+2) - F(k+2), \\
S_F(k) + S_F(k+1) + F(k+2) &= S_F(k+2).
\end{aligned}$$

In other words,

$$S_F(n) + S_F(n+1) + F(n+2) = S_F(n+2). \qquad \square$$

Proof for **Formula 7.4**.

Proof.

$$\begin{aligned} S_{LF}(1) &= L(1)F(1) \\ S_{LF}(2) &= L(1)F(2) + L(2)F(1) \\ S_{LF}(3) &= L(1)F(3) + L(2)F(2) + L(3)F(1) \\ &\dots \\ S_{LF}(k) &= L(1)F(k) + L(2)F(k-1) + \ldots + L(r)F(k-r+1) \\ &\quad + \ldots + L(k)F(1) \\ S_{LF}(k+1) &= L(1)F(k+1) + L(2)F(k) + \ldots + L(r)F(k-r+2) \\ &\quad + \ldots + L(k)F(2) + L(k+1)F(1) \\ S_{LF}(k+2) &= L(1)F(k+2) + L(2)F(k+1) + \ldots + L(r)F(k-r+3) \\ &\quad + \ldots + L(k)F(3) + L(k+1)F(2) + L(k+2)F(1) \\ &\dots \end{aligned}$$

$$\begin{aligned} S_{LF}(k) + S_{LF}(k+1) &= [L(1)F(k) + L(2)F(k-1) + \ldots + L(r)F(k-r+1) \\ &\quad + \ldots + L(k)F(1)] + [L(1)F(k+1) + L(2)F(k) \\ &\quad + \ldots + L(r)F(k-r+2) \\ &\quad + \ldots + L(k)F(2) + L(k+1)F(1)] \\ &= [L(1)F(k) + L(1)F(k+1)] \\ &\quad + [L(2)F(k-1) + L(2)F(k)] \\ &\quad + \ldots + [L(r)F(k-r+1) + L(r)F(k-r+2)] \\ &\quad + \ldots + [L(k)F(1) + L(k)F(2)] + L(k+1)F(1) \\ &= L(1)[F(k) + F(k+1)] + L(2)[F(k-1) + F(k)] \\ &\quad + \ldots + L(r)[F(k-r+1) + F(k-r+2)] \\ &\quad + \ldots + L(k)[F(1) + F(2)] + L(k+1)F(1) \\ &= L(1)F(k+2) + L(2)F(k+1) + \ldots + L(r)F(k-r+3) \\ &\quad + \ldots + L(k)F(3) + L(k+1)F(2) \\ &\quad \text{(Note that } F(1) = F(2) = 1) \\ &= S_{LF}(k+2) - L(k+2)F(1) \\ &= S_{LF}(k+2) - L(k+2) \end{aligned}$$

Therefore,

$$\begin{aligned} S_{LF}(k) + S_{LF}(k+1) &= S_{LF}(k+2) - L(k+2), \\ S_{LF}(k) + S_{LF}(k+1) + L(k+2) &= S_{LF}(k+2). \end{aligned}$$

In other words,

$$S_{LF}(n) + S_{LF}(n+1) + L(n+2) = S_{LF}(n+2). \qquad \square$$

Previously, we start the proof by considering the leftmost term on the k-th line, that is, $L(1)F(k)$. Now, we are going to do the proof again by considering the rightmost term on the k-th line, that is, $F(1)L(k)$.

Proof.

$$\begin{aligned}
S_{LF}(1) &= F(1)L(1)\\
S_{LF}(2) &= F(1)L(2) + F(2)L(1)\\
S_{LF}(3) &= F(1)L(3) + F(2)L(2) + F(3)L(1)\\
S_{LF}(4) &= F(1)L(4) + F(2)L(3) + F(3)L(2) + F(4)L(1)\\
S_{LF}(5) &= F(1)L(5) + F(2)L(4) + F(3)L(3) + F(4)L(2) + F(5)L(1)\\
S_{LF}(6) &= F(1)L(6) + F(2)L(2) + F(3)L(4) + F(4)L(3) + F(5)L(2)\\
&\quad + F(6)L(1)\\
&\quad \dots\\
S_{LF}(k) &= F(1)L(k) + F(2)L(k-1) + \dots + F(r)L(k-r+1)\\
&\quad + \dots + F(k)L(1)\\
S_{LF}(k+1) &= F(1)L(k+1) + F(2)L(k) + \dots + F(r)L(k-r+2)\\
&\quad + \dots + F(k)L(2) + F(k+1)L(1)\\
S_{LF}(k+2) &= F(1)L(k+2) + F(2)L(k+1) + \dots + F(r)L(k-r+3)\\
&\quad + \dots + F(k)L(3) + F(k+1)L(2) + F(k+2)L(1)\\
&\quad \dots
\end{aligned}$$

$$\begin{aligned}
S_{LF}(k) + S_{LF}(k+1) &= [F(1)L(k) + F(2)L(k-1) + \dots + F(r)L(k-r+1)\\
&\quad + \dots + F(k)L(1)] + [F(1)L(k+1) + F(2)L(k)\\
&\quad + \dots + F(r)L(k-r+2)\\
&\quad + \dots + F(k)L(2) + F(k+1)L(1)]\\
&= [F(1)L(k) + F(1)L(k+1)]\\
&\quad + [F(2)L(k-1) + F(2)L(k)]\\
&\quad + \dots + [F(r)L(k-r+1) + F(r)L(k-r+2)]\\
&\quad + \dots + [F(k)L(1) + F(k)L(2)] + F(k+1)L(1)\\
&= F(1)[L(k) + L(k+1)] + F(2)[L(k-1) + L(k)]\\
&\quad + \dots + F(r)[L(k-r+1) + L(k-r+2)]\\
&\quad + \dots + F(k)[L(1) + L(2)] + F(k+1)L(1)\\
&= F(1)L(k+2) + F(2)L(k+1) + \dots + F(r)L(k-r+3)\\
&\quad + \dots + F(k)L(3) + F(k+1)L(1)
\end{aligned}$$

Since

$$\begin{aligned} S_{LF}(k+2) = &\ F(1)L(k+2) + F(2)L(k+1) + \ldots + F(r)L(k-r+3) \\ &+ \ldots + F(k)L(3) + F(k+1)L(2) + F(k+2)L(1) \end{aligned}$$

Therefore,

$$\begin{aligned} S_{LF}(k) + S_{LF}(k+1) - F(k+1)L(1) &= S_{LF}(k+2) - F(k+1)L(2) \\ &\quad - F(k+2)L(1) \\ S_{LF}(k) + S_{LF}(k+1) - F(k+1) &= S_{LF}(k+2) - 3F(k+1) \\ &\quad - F(k+2) \\ S_{LF}(k) + S_{LF}(k+1) + 2F(k+1) + F(k+2) &= S_{LF}(k+2) \\ S_{LF}(k) + S_{LF}(k+1) + F(k+1) + F(k+3) &= S_{LF}(k+2) \\ (\text{apply } L(n) = F(n+1) + F(n-1) &\text{ where } n = k+2) \\ S_{LF}(k) + S_{LF}(k+1) + L(k+2) &= S_{LF}(k+2) \end{aligned}$$

In other words,

$$S_{LF}(n) + S_{LF}(n+1) + F(n+2) = S_{LF}(n+2). \qquad \square$$

Proof for **Formula 7.6**.

Proof.

$$\begin{aligned} S_L(1) &= L(1)L(1) \\ S_L(2) &= L(1)L(2) + L(2)L(1) \\ S_L(3) &= L(1)L(3) + L(2)L(2) + L(3)L(1) \\ &\ldots \\ S_L(k) &= L(1)L(k) + L(2)L(k-1) + \ldots + L(r)L(k-r+1) \\ &\quad + \ldots + L(k)L(1) \\ S_L(k+1) &= L(1)L(k+1) + L(2)L(k) + \ldots + L(r)L(k-r+2) \\ &\quad + \ldots + L(k)L(2) + L(k+1)L(1) \\ S_L(k+2) &= L(1)L(k+2) + L(2)L(k+1) + \ldots + L(r)L(k-r+3) \\ &\quad + \ldots + L(k)L(3) + L(k+1)L(2) + L(k+2)L(1) \\ &\ldots \end{aligned}$$

$$
\begin{aligned}
S_L(k) + S_L(k+1) &= [L(1)L(k) + L(2)L(k-1) + \ldots + L(r)L(k-r+1) \\
&\quad + \ldots + L(k)L(1)] + [L(1)L(k+1) + L(2)L(k) \\
&\quad + \ldots + L(r)L(k-r+2) \\
&\quad + \ldots + L(k)L(2) + L(k+1)L(1)] \\
&= [L(1)L(k) + L(1)L(k+1)] + [L(2)L(k-1) + L(2)L(k)] \\
&\quad + \ldots + [L(r)L(k-r+1) + L(r)L(k-r+2)] \\
&\quad + \ldots + [L(k)L(1) + L(k)L(2)] + L(k+1)L(1) \\
&= L(1)[L(k) + L(k+1)] + L(2)[L(k-1) + L(k)] \\
&\quad + \ldots + L(r)[L(k-r+1) + L(k-r+2)] \\
&\quad + \ldots + L(k)[L(1) + L(2)] + L(k+1)L(1) \\
&= L(1)L(k+2) + L(2)L(k+1) + \ldots + L(r)L(k-r+3) \\
&\quad + \ldots + L(k)L(3) + L(k+1)L(1) \\
&= S_L(k+2) - L(k+2)L(1) - L(k+1)L(2) + L(k+1)L(1) \\
&= S_L(k+2) - L(k+2) - 2L(k+1)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S_L(k) + S_L(k+1) &= S_L(k+2) - L(k+2) - 2L(k+1) \\
S_L(k) + S_L(k+1) + L(k+2) + 2L(k+1) &= S_L(k+2) \\
S_L(k) + S_L(k+1) + [L(k+3) + L(k+1)] &= S_L(k+2)
\end{aligned}
$$

Applying Formula 5.35

$$5F(1)F(n) = L(n+1) + L(n-1),$$

we have

$$S_L(k) + S_L(k+1) + 5F(k+2) = S_L(k+2).$$

In other words,

$$S_L(n) + S_L(n+1) + 5F(n+2) = S_L(n+2). \qquad \square$$

Appendix F. Formulae

Formula 2.10.

$$U(n) = F(r+2)U(n-r) - F(r)U(n-r-2)$$

Formula 2.13.

$$U(n) = F(r+1)U(n-r) + F(r)U(n-r-1)$$

Formula 2.19.

$$F(k)U(n) = F(r+k)U(n-r) + (-1)^{k+1}F(r)U(n-r-k)$$

Formula 2.20.

$$F(k)F(n) = F(r+k)F(n-r) + (-1)^{k+1}F(r)F(n-r-k)$$

Formula 2.21.

$$F(k)L(n) = F(r+k)L(n-r) + (-1)^{k+1}F(r)L(n-r-k)$$

Formula 2.23.

$$U(2k) = F(k+1)U(k+1) - F(k-1)U(k-1)$$

Formula 2.24.

$$F(2k) = F(k+1)^2 - F(k-1)^2$$

Formula 2.25.

$$L(2k) = F(k+1)L(k+1) - F(k-1)L(k-1)$$

Formula 2.27.

$$U(2k+1) = F(k+1)U(k+1) + F(k)U(k)$$

Formula 2.28.

$$F(2k+1) = F(k+1)^2 + F(k)^2$$

Formula 2.29.

$$L(2k+1) = F(k+1)L(k+1) + F(k)L(k)$$

Formula 2.31.

$$F(2k) = [F(k+1) + F(k-1)]F(k)$$

or

$$F(2k) = [F(k) + 2F(k-1)]F(k)$$

or

$$F(2k) = [2F(k+1) - F(k)]F(k)$$

Formula 3.2.

$$L(2n) = L(n)^2 + (-1)^{n+1}(2)$$

Hypothesis 3.4.

$$L(3n) = L(n)^3 + (-1)^{n+1}(3)L(n)$$

Hypothesis 3.6.

$$L(5n) = L(n)^5 + (-1)^{n+1}(5)L(n)[L(n)^2 + (-1)^{n+1}]$$

Hypothesis 3.8.

$$L(7n) = L(n)^7 + (-1)^{n+1}(7)L(n)[L(n)^2 + (-1)^{n+1}]^2$$

Hypothesis 3.10.

$$\begin{aligned} &L(11n) \\ &= L(n)^{11} + (-1)^{n+1}(11)L(n)[L(n)^2 + (-1)^{n+1}]\{[L(n)^2 + (-1)^{n+1}]^3 + L(n)^2\} \end{aligned}$$

Hypothesis 3.12.

$$\begin{aligned} &L(13n) \\ &= L(n)^{13} + (-1)^{n+1}(13)L(n)[L(n)^2 + (-1)^{n+1}]^2\{[L(n)^2 + (-1)^{n+1}]^3 + 2L(n)^2\} \end{aligned}$$

Formula 3.22.

$${}_nL_r = {}_{n+1}C_r - {}_{n-1}C_{r-2}$$

Formula 3.23.

$${}_nL_r = \frac{(n-1)!(n^2+n-r^2+r)}{(n+1-r)!r!}$$

Hypothesis 3.25.

$$\begin{aligned} L(4pn) = {} & {}_{4p}L_0L(n)^{4p} + (-1)^{n+1}{}_{4p-1}L_1L(n)^{4p-2} + {}_{4p-2}L_2L(n)^{4p-4} \\ & + (-1)^{n+1}{}_{4p-3}L_3L(n)^{4p-6} + \ldots + {}_{4p-r}L_rL(n)^{4p-2r+2} \\ & + \ldots + (-1)^{n+1}{}_{2p+1}L_{2p-1}L(n)^2 + {}_{2p}L_{2p} \end{aligned}$$

Note: $(-1)^{n+1}$ occurs in the 2nd, 4th, 6th and other even-number terms, ${}_{4p}L_0 = 1$; ${}_{4p-1}L_1 = 4p$; ${}_{2p}L_{2p} = 2$.

Hypothesis 3.26.

$$\begin{aligned} L((4p-1)n) = {} & {}_{4p-1}L_0L(n)^{4p-1} + (-1)^{n+1}{}_{4p-2}L_1L(n)^{4p-3} + {}_{4p-3}L_2L(n)^{4p-5} \\ & + (-1)^{n+1}{}_{4p-4}L_3L(n)^{4p-7} + \ldots + {}_{4p-1-r}L_rL(n)^{4p-2r+1} \\ & + \ldots + (-1)^{n+1}{}_{2p}L_{2p-1}L(n) \end{aligned}$$

Note: $(-1)^{n+1}$ occurs in the 2nd, 4th, 6th and other even-number terms, ${}_{4p-1}L_0 = 1$; ${}_{4p-2}L_1 = 4p-1$.

Hypothesis 3.27.

$$L((4p-2)n) = {}_{4p-2}L_0 L(n)^{4p-2} + (-1)^{n+1}{}_{4p-3}L_1 L(n)^{4p-4} + {}_{4p-4}L_2 L(n)^{4p-6} + (-1)^{n+1}{}_{4p-5}L_3 L(n)^{4p-8} + \ldots + {}_{4p-2-r}L_r L(n)^{4p-2r} + \ldots + (-1)^{n+1}{}_{2p-1}L_{2p-1}$$

Note: $(-1)^{n+1}$ occurs in the 2^{nd}, 4^{th}, 6^{th} and other even-number terms, ${}_{4p-2}L_0 = 1$; ${}_{4p-3}L_1 = 4p-2$; ${}_{2p-1}L_{2p-1} = 2$.

Hypothesis 3.28.

$$L((4p-3)n) = {}_{4p-3}L_0 L(n)^{4p-3} + (-1)^{n+1}{}_{4p-4}L_1 L(n)^{4p-5} + {}_{4p-5}L_2 L(n)^{4p-7} + (-1)^{n+1}{}_{4p-6}L_3 L(n)^{4p-9} + \ldots + {}_{4p-3-r}L_r L(n)^{4p-2r-1} + \ldots + (-1)^{n+1}{}_{2p-1}L_{2p-2} L(n)$$

Note: $(-1)^{n+1}$ occurs in the 2^{nd}, 4^{th}, 6^{th} and other even-number terms, ${}_{4p-3}L_0 = 1$; ${}_{4p-4}L_1 = 4p-3$.

Formula 3.35.

$$D(n) = U(n)$$

Formula 5.25.

$$L(r)F(k) = F(k+r) + (-1)^r F(k-r)$$

Formula 5.26.

$$F(r)L(k) = F(k+r) + (-1)^{r+1} F(k-r)$$

Formula 5.31.

$$F(2n) = F(n)L(n)$$

Formula 5.33.

$$F(3n) = F(n)L(n)^2 + (-1)^{n+1}F(n)$$

Formula 5.35.

$$F(4n) = F(n)L(n)^3 + (-1)^{n+1}2F(n)L(n)$$

Formula 5.37.

$$F(5n) = F(n)L(n)^4 + (-1)^{n+1}3F(n)L(n)^2 + F(n)$$

Formula 5.39.

$$F(6n) = F(n)L(n)^5 + (-1)^{n+1}4F(n)L(n)^3 + 3F(n)L(n)$$

Formula 5.41.

$$F(7n) = F(n)L(n)^6 + (-1)^{n+1}5F(n)L(n)^4 + 6F(n)L(n)^2 + (-1)^{n+1}F(n)$$

Formula 5.43.

$$F(8n) = F(n)L(n)^7 + (-1)^{n+1}6F(n)L(n)^5 + 10F(n)L(n)^3 + (-1)^{n+1}4F(n)L(n)$$

Formula 5.44.

$$\begin{aligned}
F(4pn) &= F(n)[{}_{4p-1}C_0L(n)^{4p-1} + (-1)^{n+1}{}_{4p-2}C_1L(n)^{4p-3} \\
&\quad + {}_{4p-3}C_2L(n)^{4p-5} + \ldots + {}_{2p+1}C_{2p-2}L(n)^3 \\
&\quad + (-1)^{n+1}{}_{2p}C_{2p-1}L(n)] \\
F((4p+1)n) &= F(n)[{}_{4p}C_0L(n)^{4p} + (-1)^{n+1}{}_{4p-1}C_1L(n)^{4p-2} \\
&\quad + {}_{4p-2}C_2L(n)^{4p-4} + \ldots + (-1)^{n+1}{}_{2p+1}C_{2p-1}L(n)^2 \\
&\quad + {}_{2p}C_{2p}] \\
F((4p+2)n) &= F(n)[{}_{4p+1}C_0L(n)^{4p+1} + (-1)^{n+1}{}_{4p}C_1L(n)^{4p-1} \\
&\quad + {}_{4p-1}C_2L(n)^{4p-3} + \ldots + (-1)^{n+1}{}_{2p+2}C_{2p-1}L(n)^3 \\
&\quad {}_{2p+1}C_{2p}L(n)] \\
F((4p+3)n) &= F(n)[{}_{4p+2}C_0L(n)^{4p+2} + (-1)^{n+1}{}_{4p+1}C_1L(n)^{4p} \\
&\quad + {}_{4p}C_2L(n)^{4p-2} + \ldots + {}_{2p+2}C_{2p}L(n)^2 \\
&\quad + (-1)^{n+1}{}_{2p+1}C_{p+1}]
\end{aligned}$$

Formula 5.55.

$$5F(k)F(n) = L(n+k) + (-1)^{k+1}L(n-k)$$

Formula 5.58.

$$L(2n) = 5F(n)^2 + (-1)^n(2)$$

Formula 6.2.

$$F(n)^2 = F(n-1)F(n+1) + (-1)^{n+1}$$

Formula 6.4.

$$F(n)^2 = F(n-2)F(n+2) + (-1)^n$$

Formula 6.11.

$$F(n)^2 = F(n-k)F(n+k) + (-1)^{n+k}F(k)^2$$

Formula 6.14.

$$\sum_{k=1}^{n} F(k)^2 = F(n)F(n+1)$$

Formula 6.26.

$$\sum_{k=1}^{n}(-1)^{k+1}F(2k) = (-1)^{n+1}F(n)F(n+1)$$

Formula 6.27.

$$\sum_{k=1}^{n}(-1)^{k+1}F(2k-1) = (-1)^{n+1}F(n)F(n)$$

Formula 6.29.

$$F(2r+1)^2 = F(2r+1+k)F(2r+1-k) + (-1)^{k+1}F(k)^2$$

Formula 6.30.

$$F(2r+2)^2 = F(2r+2+k)F(2r+2-k) + (-1)^{k}F(k)^2$$

Formula 6.36.

$$L(n)^2 = L(n-1)L(n+1) + (-1)^{n}(5)$$

Formula 6.38.

$$L(n)^2 = L(n-2)L(n+2) + (-1)^{n+1}(5)$$

Formula 6.45.

$$L(n)^2 = L(n-k)L(n+k) + (-1)^{n+k+1}(5)F(k)^2$$

Formula 6.46.

$$\sum_{k=1}^{n} L(k)^2 = L(n)L(n+1) - 2$$

Formula 6.57.

$$\sum_{k=1}^{n}(-1)^{k+1}5F(2k) = (-1)^{n+1}5F(n)F(n+1)$$

Formula 6.58.

$$\sum_{k=1}^{n}(-1)^{k+1}5F(2k-1) = (-1)^{n+1}5F(n)F(n)$$

Formula 6.59.

$$L(2r+1)^2 = L(2r+1+k)L(2r+1-k) + (-1)^{k}5F(k)^2$$

Formula 6.60.

$$L(2r+2)^2 = L(2r+2+k)L(2r+2-k) + (-1)^{k+1}5F(k)^2$$

Formula 6.63.

$$L(2k+1) = L(k+1)^2 - 5F(k)^2$$

Formula 6.65.

$$L(2k) = \frac{L(k+1)^2 + 5F(k-1)^2}{3}$$

Formula 6.67.

$$L(2k) = \frac{L(k)^2 + 5F(k)^2}{2}$$

Formula 6.70.

$$L(k)^2 = 5F(k)^2 + (-1)^k(4)$$

Formula 7.2.

$$S_F(n) + S_F(n+1) + F(n+2) = S_F(n+2)$$

Formula 7.4.

$$S_{LF}(n) + S_{LF}(n+1) + L(n+2) = S_{LF}(n+2)$$

Formula 7.6.

$$S_L(n) + S_L(n+1) + 5F(n+2) = S_L(n+2)$$

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Reviewer's Comments

1. On Page 5, Table 2.3, the first 15 $U_1(n)$ numbers should be as follows.

n	1	2	3	4	5	6	7	8
$U_1(n)$	4	5	9	14	23	37	60	97

n	9	10	11	12	13	14	15	
$U_1(n)$	157	254	411	665	1076	1741	2817	