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MARKED RULER AS A TOOL FOR GEOMETRIC CONSTRUCTIONS - FROM ANGLE TRISECTION TO N-SIDED POLYGON

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Abstract. The ancient Greeks raised the famous problem of trisecting an arbitrary angle with a compass and an unmarked ruler, which was proved impossible. Such a construction is possible if a marked ruler is used instead. In this article, the possible geometric constructions by a compass and a marked ruler are studied.[2]

1. Introduction

Trisecting an arbitrary angle with a compass and straightedge was one of the famous ancient Greeks unsolved construction problems. Together with duplicating the cube, these problems have been pending to be resolved for more than 2000 years.

Plato (427–347 BC) defined clearly the rules of ruler and compass construction, which implies that the marks or scales in the ruler should not be relevant to the geometric construction. Many learned people tried employing different tools and methods to tackle the problem, in particular, the interesting and simple construction algorithm proposed by Archimedes (287–212 BC) who had employed a marked ruler and compass to solve the trisecting problem, which was very close to the Platos rules.

In 19$^{\text{th}}$ Century, Pierre L. Wantzel (1814–1848) proved in 1837 that based upon Plato's criteria, it is impossible to trisect an arbitrary angle. The

[1]This work is done under the supervision of the author's teacher, Ms. Fei Wong.
[2]The abstract is added by the editor.

problem became even more interesting after it was proved to be possible because of the "magic" marked ruler, which have opened a new area for the study of geometric constructions with marked ruler and compass.

Theorem 1.1. *If we have a marked ruler and a compass, then it is possible to trisect an arbitrary angle.*

Proof. Archimedes proved this theorem by giving a construction algorithm. As shown in figure 1.1 below, let $\angle AOB$ be the angle being trisected and the lengths $|OA| = |OB| = 1$, which is the distance between the two marks on the ruler. Draw a semicircle centered at O from B through A. If we mark C and D such that C is on the semi-circle and D is the intersection of the lines OB and AC with $|CD| = 1$, then $\angle AOB = 3\angle ADB$.

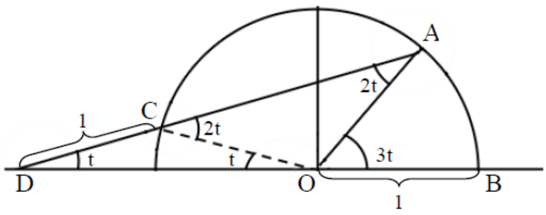


FIGURE 1.1

Let $\angle ADB = t$. Then $\angle COD = t$ and $\angle OCA = \angle OAC = 2t$ (base angles of isosceles triangle).

By the interior angle sum of triangle, $\angle AOC = \pi - 4t$. Hence,

$$\angle AOB = \angle COD.\angle AOC = \pi - t - (\pi - 4t) = 3t = 3\angle ADB. \qquad \square$$

In theorem 1.1, Archimedes made use of the so-called marked ruler instead of a typical straight edge in the construction. Some people criticized that Archimedes did not respect the conventional definition of ruler and his approach was not strict enough hence it was not commonly accepted.

In spite of this, it was quite natural, when compared with using conics, trisectrix or some other strange curves to give a solution to angle trisection, marked ruler was an easy available tool in real life. One should appreciate why adding two marks on a ruler makes the impossible becomes possible. In this project, we try to give some terminology of marked ruler and clarify which types of geometric constructions are possible by using marked ruler and compass.

Definition 1.2. *A ruler or more precisely a straight edge with two notches on it is called a marked ruler. Without loss of generality, the distance between the two notches is taken to be* 1*.*

From definition 1.2, we notice that the marked ruler introduces the concept of unit length into the system of geometric construction. It allows us to cut off equal distance on a straight line in particular, and we will show that the marked ruler is much more useful with the help of compass in the subsequent sections. In order to study geometric construction algebraically, we will introduce a rectangular coordinate system on the two dimensional Euclidean Space. Moreover, by the end of this report, we will study how the problem of constructing regular n-sided polygon is related to the construction by using marked ruler and compass.

Now, let us define the meaning of constructible points and constructible curves.

Definition 1.3. *A constructible curve is a curve constructed from given quantities such as points, lengths, etc, which are provided by given points and constructible points. A constructible point is a point of intersection of two constructible curves.*

Our task is getting clearer that we treat construction as drawing the constructible curves. If we know what curves marked ruler and compass can draw, we will know the properties of the constructible points. Before going deep into our main goal, lets take a brief review on the general construction.

2. Classification of Construction

Up to this moment, our understanding on the term "construction" is too vague for a mathematical theory to build on. In the following sections, we will give a clear definition of "construction", then classify different types of construction and their relative field of extension.

Definition 2.1. *A construction $\mathcal{C}$ is defined to be a finite set of constructible points $\{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_n\}$, where $0 = (0,0)$, $\mathbf{u} = (1,0)$ and $A_0 = (0,1)$, such that A_{n+1} is a point of intersection of any two of the constructible curves γ_i constructed from the points in the sub-construction $\mathcal{C}_k = \{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_k\}$ where $k = 1, 2, \ldots, n$ under specific construction rules.*

Definition 2.2. *Let $\mathcal{C} = \{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_n\}$ be a construction of n steps and $\mathcal{C}_k = \{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_k\}$ where $k \leqslant n$ be a sub-construction* *of C. Also, let $z_1, z_2, \ldots, z_n$ be the complex numbers that represent the points $A_1, A_2, \ldots, A_n$ respectively. Then,* $\mathrm{K}[\mathcal{C}] = \mathbb{Q}(\mathbf{i}, z_1, z_2, \ldots, z_n)$ *is defined to be the field of extension of* $\mathbb{Q}$ *by construction* $\mathcal{C}$*. Note that* $\mathrm{K}[\mathcal{C}]$ *is the smallest field that contains* $\mathbf{i}, z_1, \ldots, z_n$ *and we have*

$$\mathrm{K}[\mathcal{C}_k] = \mathrm{K}[\mathcal{C}_{k-1}](z_k) \text{ for } k = 1, 2, \ldots, n.$$

Remark: Since $0 = (0,0)$, $\mathbf{u} = (1,0)$ and $A_0 = (0,1)$, it is easy to see that $\mathrm{K}[\mathcal{C}_0] = \mathbb{Q}(\mathbf{i})$.

Definition 2.3. *A construction is called plane if it can be solved by using ruler and compass only.*
A construction is called solid if it can be solved by using conic sections only.
A construction is called higher dimensional if it is not plane or solid.

Remark: This classification was introduced by Pappus, but I replace the term "linear" by "higher dimensional" since it will be more appropriate.

Definition 2.4. *For plane constructions, a constructible straight line is a line, which passes through two constructible points; and a constructible circle is a circle centered at a constructible point, which passes through another constructible point.*

Before we state the well-known theorem for ruler and compass construction, we give a lemma, which is used to prove this theorem.

Lemma 2.5. *If two circles intersect or a circle and a straight line intersects, where the coefficients of the equations of the circles and straight lines are in field* K*, then the coordinates of the point of intersection lie in a field of quadratic extension over* K.

Proof. Firstly, let $y = mx + c$ and $x^2 + y^2 + dx + ey + f = 0$ be the equations of a straight line and a circle with m, c, d, e, $f \in \mathrm{K}$ respectively. Then, by solving the two equations, we have

$$x^2 + (mx + c)^2 + dx + e(mx + c) + f = 0$$
$$\Rightarrow (1 + m^2)x^2 + (2mc + d + me)x + c^2 + ce + f = 0$$

Note that the coefficients of the above equation are in K, its roots lie in a field of quadratic extension of K. Also from $y = mx + c$, y is linear to x and so the coordinates of the points of intersection lie in a field of quadratic extension over K.

Secondly, let $x^2 + y^2 + d_1x + e_1y + f_1 = 0$ and $x^2 + y^2 + d_2x + e_2y + f_2 = 0$

be the equations of two distinct circles with d_1, d_2, e_1, e_2, f_1, $f_2 \in \mathrm{K}$. Then by subtracting the two equations, it yields a straight line $(d_2 - d_1)x + (e_2 - e_1)y + (f_2 - f_1) = 0$ with its coefficients in K. Hence, by the above argument, the coordinates of the points of intersection lie in a field of quadratic extension over K. □

Theorem 2.6. *A point (x, y) has a plane construction if and only if $x + y\mathbf{i} \in \mathbb{C}$ lies in a sub-field* K *of* $\mathbb{C}$ *such that* $\exists K_i$, $i = 0, 1, \ldots, n$, *satisfying that*

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}$$

and the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1$ *or* 2 *for* $j = 1, 2, \ldots, n$.

Proof. Let $\mathcal{C} = \{0, \mathbf{u}, A_0, A_1, \ldots, A_{n-1}\}$ be a plane construction with $A_{n-1} = (x, y)$. It is clear that when two constructible lines intersect, no extension of field is needed. Also note that the extension $\mathbb{Q}(\mathbf{i})$ is of degree 2. Then, by lemma 2.5, we must have $[\mathrm{K}[\mathcal{C}_{k+1}] : \mathrm{K}[\mathcal{C}_k]] = 1$ or 2, where $k = 1, 2, \ldots, n-2$.

Hence, by letting $\mathrm{K}_{j+1} = \mathrm{K}[\mathcal{C}_j]$, we have

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}[\mathcal{C}] = \mathrm{K}$$

and the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1$, 2 for $j = 1, 2, \ldots, n$.

Conversely, given a tower of fields

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}$$

where the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1$ or 2 for $j = 1, 2, \ldots, n$. It suffices to verify that there is a plane construction associated to each step of field extension. If $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1$, then $\mathrm{K}_j = \mathrm{K}_{j-1}$ and the result is trivial. If $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 2$, let $z_j = x_j + y_j\mathbf{i}$ such that $\mathrm{K}_j = \mathrm{K}_{j-1}[z_j]$ where $z_j \notin \mathrm{K}_{j-1}$ for $j = 1, 2, \ldots, n$. Since the degree of extension is 2, both x_j and y_j are roots of certain quadratic equations with coefficients in K_{j-1}, say, $x^2 + ax + b = 0$ and $y^2 + py + q = 0$ respectively. Then by constructing the circle $x^2 + y^2 + ax + b = 0$ and the line $y = 0$, we solve x_j as the x-coordinate. Similarly, y_j can be obtained as the y-coordinate of the intersection of the circle $x^2 + y^2 + px + q = 0$ and the line $x = 0$. Hence $z_j = x_j + y_j\mathbf{i}$ has a plane construction. □

Remark: In fact, the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1$ actually means that $\mathrm{K}_j = \mathrm{K}_{j-1}$, so we can omit 1 in the preceding theorem.

Next, we want to show is that every point with solid construction lies in a 2-3-tower over $\mathbb{Q}$. So we should first show that every points are closed under quartic equation and on the other hand every equation of degree at most 4 is solvable by solid construction.

Lemma 2.7. *Every points of intersection of any two conic sections are roots of an equation of degree at most four.*

Proof. Let $A_1x^2 + B_1xy + C_1y^2 + D_1x + E_1y + F_1 = 0$ and $A_2x^2 + B_2xy + C_2y^2 + D_2x + E_2y + F_2 = 0$ be two distinct conic sections. Then by Bezout's Theorem, which states that two algebraic plane curves with degree m and n respectively and with no common component have exactly mn points of intersection counting multiplicity and points at infinity, it follows that there are at most mn points of intersection for any two algebraic curves with degree m and n respectively. The points of intersection of the two conic sections, each of degree 2, when solving together, are therfore roots of quartic equations for irreducible cases, and thus yielding four distinct intersections. For reducible cases, the degree of the equation will be even lower. So every points of intersection are roots of an equation of degree at most four.

Practically, one may transform the first equation in the form

$$y^2 = -\frac{A_1}{C_1}x^2 - \frac{B_1}{C_1}xy - \frac{D_1}{C_1}x - \frac{E_1}{C_1}y - \frac{F_1}{C_1} \text{ for } C_1 \neq 0$$

and it is used to reduce the degree of the second equation in y. Eventually, a quartic equation maybe reducible or irreducible is yielded. If $C_1 = C_2 = 0$, then we can eliminate y without much difficulty and also an equation of degree not exceeding four is yielded. □

Lemma 2.8. *Trisecting an arbitrary angle has a solid construction.*

Proof. Whenever we can construct an angle α, it is equivalent to say that we can construct the length $\cos\alpha$, since one can readily construct a right angle triangle with hypotenuse 1 and one side $\cos\alpha$ where the included angle is α from either one condition. Suppose 3θ be the angle to be trisected, then $\cos 3\theta$ is constructible and we aim at showing that $\cos\theta$ is also constructible.

Consider the two conic sections

$$\begin{cases} y = x^2, \\ xy - 3x - 2\cos 3\theta = 0. \end{cases}$$

Since both of them have their coefficients in $\mathbb{Q}(\cos 3\theta)$, they are both constructible.

Solving them together yields a cubic equation

$$x^3 - 3x - 2\cos 3\theta = 0.$$

It is not too difficult to show that $2\cos\theta$, $2\cos(\theta + \frac{2}{3}\pi)$, and $2\cos(\theta + \frac{4}{3}\theta)$ are the roots of it. Therefore $\cos\theta$ is constructible and trisecting an arbitrary angle has a solid construction. $\square$

Lemma 2.9. *Find the cube root of arbitrary length has a solid construction.*

Proof. Suppose the length l is constructible. We are aiming to show that $\sqrt[3]{l}$ is also constructible.

Consider the two conic sections,

$$\begin{cases} y = x^2, \\ xy = l. \end{cases}$$

Both of them have their coefficients in $\mathbb{Q}(l)$, so they are constructible. Then by solving them together, we yield

$$x^3 = l,$$

and so $\sqrt[3]{l}$ is the only real root that satisfies this equation. Hence the cube root of arbitrary length has a solid construction. $\square$

Theorem 2.10. *All equations of degree at most 4 can be solved if and only if one can trisect an angle and find that cube root for arbitrary length in addition to the use of ruler and compass.*

Proof. [$\Rightarrow$] Suppose all equations of degree at most 4 can be solved. Then the equation $x^3 - a = 0$ can be solved and we can find the cube root. As shown in lemma 2.8, arbitrary angle 3α is trisectable if and only if $\cos\alpha$ is constructible length. Since $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$, replacing with $x = \cos\alpha$, it can be written as

$$4x^3 - 3x - \cos 3\alpha = 0,$$

which is solvable, hence we can trisect an arbitrary angle.

[$\Leftarrow$] Now, suppose we can trisect an angle and find the cube root, then both equations

$$\begin{cases} x^3 - k = 0, & (2.1) \\ 4x^3 - 3x - \cos 3\alpha = 0, & (2.2) \end{cases}$$

are solvable.

Case 1: For linear and quadratic equation, we can trivially solve them by ruler and compass only.

Case 2: For a general cubic equation $x^3 + ax^2 + bx + c = 0$, by a suitable change of variable ($x = y - \dfrac{a}{3}$), we always give a principal cubic equation $y^3 + py + q = 0$.

When $p = 0$, by equation (2.1), we can solve it.

When $p \neq 0$, from Cardano Formulas, the solutions of $y^3 + py + q = 0$ are, $A + B$, $A\xi + B\xi^2$, $A\xi^2 + B\xi$, where $\xi = e^{2/3\pi \mathbf{i}}$ and
$A = \sqrt[3]{-1/2q + \sqrt{(1/2q)^2 + (1/3p)^3}}$, $B = \sqrt[3]{-1/2q - \sqrt{(1/2q)^2 + (1/3p)^3}}$.

Since $(1/2q)^2 + (1/3p)^3$ is constructible, if $(1/2q)^2 + (1/3p)^3 \geqslant 0$, A^3 and B^3 are also constructible. Hence, also by equation (2.1), A and B are constructible, and thus $y^3 + py + q = 0$ can be solved.

Now suppose $(1/2q)^2 + (1/3p)^3 < 0$, iff $p^3 < 27/4q^2 < 0$, then $p < 0$ and

$$\left|-1/2q\sqrt{-27/p^3}\right| < 1,$$

hence, there exists α such that

$$\cos 3\alpha = -1/2q\sqrt{-27/p^3}.$$

By a suitable substitution $y = 2\sqrt{1/3p}$, we have

$$4t^3 - 3t - \cos 3\alpha = 0.$$

Then, by equation (2.2), $y^3 + py + q = 0$ is solvable. Therefore, all cubic equations can be solved.

Case 3: Consider a general quartic equation $x^4 + ax^3 + bx^2 + cx + d = 0$.

By substituting $x = y - a/4$, we obtain the depressed quartic

$$y^4 + py^2 + qy + r = 0. \tag{2.3}$$

If $q = 0$, we solve the quartic by solving the quadratic equation in y^2.

If $q \neq 0$, we rewrite (2.3) as

$$y^4 = -py^2 - qy - r. \tag{2.4}$$

By adding $2zy^2 + z^2$ to both sides of (2.4), we have

$$(y^2 + z)^2 = (2z - p)y^2 - qy + (z^2 - r).$$

Since z is arbitrary depending on our choice, we wish to find z such that

$$(2z - p)y^2 - qy + (z^2 - r) = (gy + h)^2 \tag{2.5}$$

for some constants g, h. Then, we solve (2.3) by $y^2 + z^2 = \pm(gy + h)$ and solve two resulting quadratic equations.

But this situation occurs iff $(2z - p)y^2 - qy + (z^2 - r) = 0$ has a double root, and thus iff

$$q^2 - 4(2z - p)(z^2 - r) = 0. \tag{2.6}$$

Rewrite (2.6) as

$$8z^3 - 4pz^2 - 8rz + 4pr - q^2 = 0 \tag{2.7}$$

which is a cubic equation in z.

Now, from case 2, we can solve (2.7), and then by case 1, we can solve (2.5). Hence (2.3) is solvable.

Therefore, we can solve all equations with degree at most four. □

Theorem 2.11. *A point (x, y) has a solid construction if and only if $x+y\mathbf{i} \in \mathbb{C}$ lies in a sub-field K of $\mathbb{C}$ such that there exists K_i, $i = 0, 1, \dots, n$, which satisfies*

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}$$

and the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 2$ or 3 for $j = 1, 2, \dots, n$.

3. Construction with marked ruler and compass

After classifying different types of construction and their relative field of extension, we should now analyze the constructions made by marked ruler and compass, and thus to show what class of construction it falls in. First, we should introduce the two new curves constructible by marked ruler and compass, then state the rules of constructions by using marked ruler and compass.

Definition 3.1. *Let l be a constructible straight line and P a fixed constructible point. If a straight line is drawn through P and meet l at Q, and if P_1 and P_2 are points on this line such that*

$$|P_1Q| = |QP_2| = k,$$

where k is the distance between two notches of the marked ruler, then the locus of P_1 and P_2 is called a constructible generalized conchoid of l with respect to P.

Definition 3.2. *Let C be a constructible circle and P a fixed constructible point. If a straight line is drawn through P and meet C at Q, and if P_1 and P_2 are points on this line such that*

$$|P_1Q| = |QP_2| = k,$$

where k is the distance between two notches of the marked ruler, then the locus of P_1 and P_2 is called a constructible generalized limacon of C with respect to P.

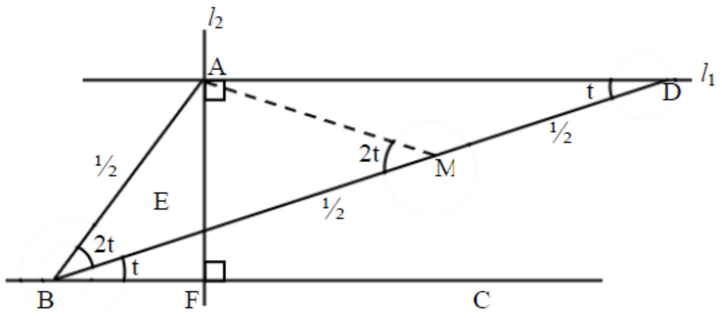
Definition 3.3. *The points constructible by marked ruler and compass are the points of intersection of any two of the following curves.*

(i) *Straight lines passing through two constructible points;*
(ii) *Circles centered at constructible point passing through another constructible point;*
(iii) *Constructible generalized conchoids; and*
(iv) *Constructible generalized limacons.*

Note that one cannot draw any two of the curves (iii) and (iv), hence the intersections between them must be discarded.

From Definition 3.3, we can directly see that all plane constructions are possible. It is, however, worthwhile to show that all solid constructions are also possible, and it would be the main result of this section. Now, we are going to present two algorithms for trisecting an angle and finding cube root, due to Pappus and Nicomedes respectively.

Theorem 3.4. *(Pappus' Trisection Algorithm). Given arbitrary angle $\angle ABC$ with $|AB| = \frac{1}{2}$, where the distance between the two notches of the marked ruler is taken to be 1, draw a line l_1 through A, which is parallel to BC. Also draw a line l_2 through A, which is perpendicular to BC and cut BC at F. Find the intersection D of the line l_1 and the generalized conchoid of l_2 with respect to B. Then the line BD trisects $\angle ABC$ with $\angle ABC = 3\angle CBD$.*



Proof. We will prove only in the case when $\angle ABC$ is acute, other cases can be obtained since we can always trisect right angle. Let $\angle CBD = t$, then $\angle ADB = t$. Since $\angle DAE = \frac{\pi}{2}$, by the converse of the theorem of Thales, A lies on the circle centered at M, which is the midpoint of DE with diameter $|DE|$. Thus $\angle AMB = 2\angle ADB = 2t$ and $\angle ABD = \angle AMB = 2t$ (base angles of isosceles triangle). Hence $\angle ABC = \angle ABD + \angle CBD = 3t = 3\angle CBD$. □

Theorem 3.5. *(Nicomedes' Cube Root Algorithm). Construct $\triangle ABC$ with $|AC| = |BC| = 1$ and $|AB| = \frac{k}{4}$, where the distance between the two notches of the marked ruler is taken to be 1 and k is the length we want to find its cube root with $0 < k < 8$. Let B be the midpoint of C and D and l_1 be AD. If S is the intersection of AB and the generalized conchoid of l_1 with respect to C, then $|AS| = \sqrt[3]{k}$.*

Proof. Construct CE such that it is parallel to AB and E is on AD. Then $\triangle ABD$ is similar to $\triangle ECD$. Then, since B bisects CD, we have $|CE| = 2|BA| = \frac{k}{2}$. Also, since $\triangle ECR$ is similar to $\triangle ASR$, we have:

$$\frac{|CE|}{|CR|} = \frac{|AS|}{|RS|}.$$

Let $x = |AS|$, then $|CR| = \frac{k}{2x}$. with M the midpoint of A and B, by the Pythagorean Theorem, we have

$$[1 + \frac{k}{2x}]^2 = [CS]^2 = |CM|^2 + |MS|^2$$
$$= [|CB|^2 - |BM|^2] + |MS|^2 = [1^2 - (\frac{k}{8})^2] + [x + \frac{k}{8}]^2.$$

A quartic equation $4x^4+5x^3-4kx-k^2=0$ is derived and it is factorized to $(4x+k)(x^3-k)=0$. Since $4x+k>0$, we have $x^3-k=0$. Hence $x=\sqrt[3]{k}$ is the only real root as desired. □

Remarks: Theorem 3.5 only provides an algorithm of finding cube root for $0<k<8$, we express k as $8^n m$ where $0<m<8$. Then the cube root of m can be found and $\sqrt[3]{8^n m}=2^n$. $\sqrt[3]{m}$ can also be found by doubling the length $\sqrt[3]{m}$ by n times.

Theorem 3.6. *All solid constructions can be solved by using marked ruler and compass.*

Proof. According to theorem 3.4 and 3.5, we can trisect an angle and find cube root by using marked ruler and compass. It follows from theorem 2.10 that all equations of degree at most four can be solved and thus all solid constructions can be solved. □

4. Further study on generalized conchoid and generalized limacon

As elaborated in Section 3 that all plane and solid constructions can be solved by using marked ruler and compass. The subsequent question is how about the higher dimensional constructions? Apparently, some but not all of them can be constructed. In order to further our study, it is convenient to know the exact equations of generalized conchoids and generalized limacons in rectangular coordinates.

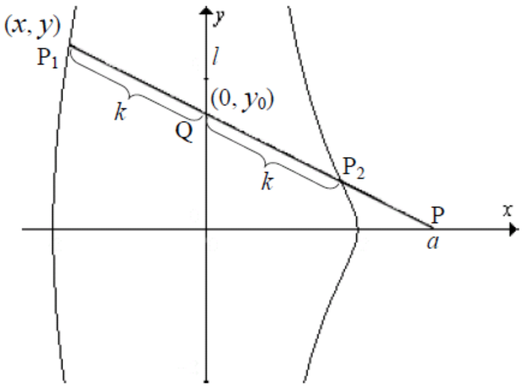
Theorem 4.1. *The equation of a generalized conchoid is of the form*

$$x^2y^2=(k^2-x^2)(x-a)^2$$

where a is a constant and k is the distance between the two notches of the marked ruler.

Proof. According to definition 3.1, without loss of generality, we set l be the line $x=0$ and P be a point $(a,0)$ on the x-axis. Then for any point $Q(0,y_0)$

on l, we have



$$(x, y) = (0, y_0) \pm k\sqrt{y_0^2 + a^2}(-a, y_0)$$

$$x = \mp ak/\sqrt{y_0^2 + a^2} \text{ and } y = \left(1 \pm k/\sqrt{y_0^2 + a^2}\right) y_0 \tag{4.1}$$

$$\Rightarrow \quad y = (1 - x/a)y_0$$

$$\Rightarrow y_0 = ay/(a - x) \quad (x \neq 0) \tag{4.2}$$

Substitute (4.2) into (4.1), we have

$$x = \mp ak/\sqrt{a^2y^2/(a - x)^2 + a^2}$$

$$x^2[a^2y^2/(a - x)^2 + a^2] = a^2k^2$$

$$x^2[(x - a)^2 + y^2] = k^2(x - a)^2 \tag{4.3}$$

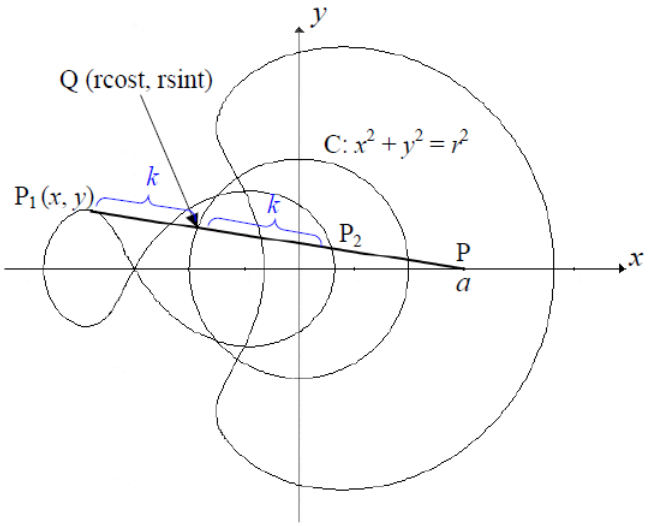
Therefore,

$$x^2y^2 = (k^2 - x^2)(x - a)^2. \tag{4.4}$$

□

Theorem 4.2. *The equation of a generalized limacon is of the form*

$$(x^2 + y^2 + k^2 - r^2)^2((x - a)^2 + y^2) = 4k^2(x^2 + y^2 - ax)^2.$$



Proof. According to definition 3.2, without loss of generality, we set C be the circle $x^2+y^2=r^2$ and P be a point $(a,0)$ on the x-axis. Then for any point $Q(r\cos t, r\sin t)$ on C, we have:

$$(x,y)=(r\cos t, r\sin t)\pm k/\sqrt{(r\cos t-a)^2+r^2\sin^2 t}(r\cos t-a, r\sin t).$$

By eliminating t from the system of the equation

$$x=\left(1\pm k/\sqrt{(\cos t-a)^2+(r\sin t)^2}\right)(r\cos t-a)+a,$$
$$y=\left(1\pm k/\sqrt{(\cos t-a)^2+(r\sin t)^2}\right)(r\sin t),$$

we have

$$(x^2+y^2+k^2-r^2)^2((x-a)^2+y^2)=4k^2(x^2+y^2-ax)^2. \tag{4.5}$$

After the equations of generalized conchoid and generalized limacon are clarified, our interests are focused on the phenomena of the points of intersection when these curves intersect with straight lines or circles.

Firstly, consider the intersections of a generalized conchoid and a straight line. Since the equation of a generalized conchoid is of degree 4, when it intersects with a straight line, the x-coordinates of the points of intersection are roots of quartic polynomial equations. Recall from section 3 that all quartic polynomial equations can be solved, so no new types of construction yield.

Secondly, when we intersect a generalized conchoid with a circle, we expect to have something new. Consider a circle in standard form

$$(x-p)^2 + (y-q)^2 = s^2. \tag{4.6}$$

Then,

$$(x-a)^2 + y^2 = 2(p-a)x + 2qy + a^2 + s^2 - p^2 - q^2. \tag{4.7}$$

By substituting (4.7) into (4.3), we have

$$x^2[2(p-a)x + 2qy + a^2 + s^2 - p^2 - q^2] = k^2(x-a)^2.$$

So, there exists two polynomials $f(x)$ and $g(x)$ of degree 2 and 3 respectively such that

$$f(x) = g(x).$$

Then,

$$f(x)(y-q) = g(x) - qf(x)$$
$$f^2(x)(y-a)^2 = [g(x) - qf(x)]^2.$$

From (4.6), we substitute $(y-q)^2$ by $s^2 - (x-p)^2$,

$$f^2(x)[s^2 - (x-p)^2] = [g(x) - qf(x)]^2. \tag{4.8}$$

Equation (4.8) is clearly a sextic polynomial, with the coefficients of this polynomial lie in an entire field K; its roots belong to a field extension of degree at most 6 over K. Besides, the y-coordinates of the intersections correspond to a field of quadratic extension over the x-coordinates.

Similarly, from (4.5), the equation of a generalized limacon is of degree 6. When it intersects with a straight line, there is no doubt that the resulting equation in x is also of degree 6. Hence, the points of intersection involve a field of extension of degree at most 6 also.

Finally, we concern the case when a generalized limacon intersects with a circle. Consider again the circle $(x-p)^2 + (y-q)^2 = s^2$

$$x^2 + y^2 = 2px + 2qy + s^2 - p^2 - q^2. \tag{4.9}$$

By substituting (4.7) and (4.9) into (4.5), we have:

$$\begin{aligned}&(2px + 2qy + s^2 - p^2 - q^2 + k^2 - r^2)\\&[2(p-a)x + 2qy + a^2 + s^2 - p^2 - q^2]\\= {}&4k^2(2px + 2qy + s^2 - p^2 - q^2 - ax)^2.\end{aligned} \tag{4.10}$$

Notice that (4.10) is a curve of degree 3 polynomial. By Bezout's theorem, it predicts that this curve has at most six points of intersection with the circle and thus is the roots of a sextic polynomial equation in x. Again the

y-coordinates are 2 degree over the x-coordinates, since it involves solving a quadratic equation. $\square$

To summarize the above results, we give the following theorem that shows construction by using marked ruler and compass lies in a 2-3-5-6 tower.

Theorem 4.3. *If a point (x, y) is constructible by marked ruler and compass, then $x + y\mathbf{i} \in \mathbb{C}$ lies in a sub-field K of $\mathbb{C}$ such that there exists K_i with $i = 0, 1, 2, \ldots, n$ satisfying that*

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}$$

and the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 2,\ 3,\ 5$ or 6 for $j = 1, 2, \ldots, n$.

Proof. The intersections of straight lines and circles, as shown in theorem 2.6, contribute to the field extension of 2 only. Then from the above results, intersecting any two curves under the construction rules for marked ruler and compass yields a polynomial equation of at most degree 6. It implies that the field extension of 2, 3, 4, 5 & 6 should be enough for marked ruler and compass. The remaining problem is why 4 is not necessary? The reason follows from the fact that any quartic polynomial can be resolved by solving a cubic and a quadratic one instead. It means that if $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 4$ for some j, then we can insert a field F such that $\mathrm{K}_{j-1} \subset \mathrm{F} \subset \mathrm{K}_j$ with $[\mathrm{K}_i : \mathrm{F}] = 3$ and $[\mathrm{F} : \mathrm{K}_{j-1}] = 2$. So the theorem follows. $\square$

Corollary 4.4. *For a marked ruler and compass construction, if K is the field of extension of the construction over $\mathbb{Q}$, then 2, 3 and 5 are the only primes that divide the index $[\mathrm{K} : \mathbb{Q}]$.*

Proof. Since it is a marked ruler and compass construction by theorem 4.3, there exists K_i with $i = 0, 1, 2, \ldots, n$ satisfying that

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}$$

and the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 2,\ 3,\ 5,\ 6$ for $j = 1, 2, \ldots, n$. Then, by the Tower Theorem, we have

$$[\mathrm{K} : \mathbb{Q}] = \prod_{j=1}^{n} [\mathrm{K}_j : \mathrm{K}_{j-1}].$$

Since 2, 3 and 5 are the only primes that divide $[\mathrm{K}_j : \mathrm{K}_{j-1}]$ for $j = 1, 2, \ldots, n$, it follows that 2, 3 and 5 are also the only primes that divide $[\mathrm{K} : \mathbb{Q}]$. $\square$

5. Construction of regular n-gon with marked ruler and compass

In this section, we are going to study the possibilities of constructing regular n-gon by using marked ruler and compass. In particular, we mainly concern with regular n-gon with $n \leqslant 100$.

First, we will show in the subsequent study how it helps in determining whether or not a regular n-gon is constructible.

Theorem 5.1. *Given a marked ruler and a compass, we cannot construct a regular* 23*-gon or* 29*-gon.*

Proof. If p is an odd prime, whenever we can construct a regular p-gon, it means that we can construct

$$\xi_p = e^{2\pi i/p}.$$

That is, the primitive p-th root of unity. It turns out that it is a root of the cyclotomic polynomial Φ_p of degree $p-1$. But then by corollary 4.4, ξ_p lies in a field K of degree $p-1$ over $\mathbb{Q}$, such that 2, 3 and 5 are the only primes that divides $p-1$. In particular, when $p = 23$, then 11 is a prime that divides $p-1$, hence a regular 23-gon is not constructible by marked ruler and compass. On the other hand, when $p = 29$, thus 7 is a prime that divides $p-1$, so a regular 29-gon is not constructible by marked ruler and compass too. In fact, regular 43-gon, 47-gon, 53-gon, etc are also non-constructible by marked ruler and compass due to the same reason. □

From the above theorem, we show that some regular n-gons are strictly inconstructible by marked ruler and compass, but how about the remaining n-gon? Trisecting an angle is one of the distinguished features of the marked ruler and compass construction, and we will show in the following theorem which regular n-gons are constructible by ruler and compass together with a trisector. When we look back to the plane construction, Gauss had already given a perfect theorem that a regular n-gon has a plane construction if and only if $n = 2^s p_1 p_2 \dots p_k$ where p_i is a Fermat prime for $i = 1, 2, \dots, k$ and s is a non-negative integer. So without much difficulty, we should also find an analog theorem for a solid construction. It was Pierpont who discovered this theorem.

Theorem 5.2. *A regular n-gon has a solid construction if and only if*

$$n = 2^s 3^t p_1 p_2 \dots p_k$$

where p_i are primes of the form $2^a 3^b + 1$ and a, b, s, t are non-negative integers.

Proof. Suppose n is an integer greater than 2. Let $\xi_n = e^{2\pi i/n}$ be the primitive nth root of unity, and $\eta = \xi_n + \xi_{n-1} = 2\cos 2\pi/n$. Then the Galois group over $\mathbb{Q}$ of the cyclotomic field $\mathbb{Q}(\xi_n)$ is abelian with $\varphi(n)$ elements, where φ is Euler's phi function. Consequently, every field between $\mathbb{Q}$ and $\mathbb{Q}(\xi_n)$ is normal over $\mathbb{Q}$. Since ξ_n has 2 degree over $\mathbb{Q}(\eta)$,

$$[\mathbb{Q}(\eta) : \mathbb{Q}] = [\mathbb{Q}(\xi_n) : \mathbb{Q}]/[\mathbb{Q}(\xi_n) : \mathbb{Q}(\eta)] = \frac{\varphi(n)}{2}.$$

Now suppose $n = 2^s 3^t p_1 p_2 \dots p_k$ as stated in theorem 5.2, then

$$\begin{aligned}\varphi(n) &= \varphi(2^s 3^t p_1 p_2 \dots p_k)\\ &= \varphi(2)^s \varphi(3)^k \varphi(p_1)\varphi(p_2)\dots\varphi(p_k)\\ &= 2^t(p_1-1)(p_2-1)\dots(p_k-1) = 2^u 3^v\end{aligned}$$

for some non-negative integers u, v.

So the Galois group of $\mathbb{Q}(\eta)$ has $2^{u-1}3^v$ elements, and therefore has a composition series of length $u+v-1$ with all quotients isomorphic either to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Correspondingly, there is a tower of fields

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \dots \subset K_{u+v-1} = \mathbb{Q}(\eta) \subset \mathbb{Q}(\xi_n)$$

such that $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 2$ or 3 for $j = 1, 2, \dots, u+v-1$ and $[\mathbb{Q}(\xi_n) : \mathbb{Q}(\eta)] = 2$. Then by theorem 2.11, ξ_n has a solid construction and thus a regular n-gon has solid construction.

Conversely, suppose a regular n-gon has a solid construction. Then ξ_n can be constructed. Again by theorem 2.11, there exists fields K_i with $i = 0, 1, 2, \dots, m$ such that

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \dots \subset \mathrm{K}_m = \mathbb{Q}(\xi_n) \text{ and}$$

$$[\mathrm{K}_i : \mathrm{K}_{j-1}] = 2 \text{ or } 3 \text{ where } j = 1, 2, \dots, m.$$

From Tower Theorem, $[\mathbb{Q}(\xi_n) : \mathbb{Q}] = \prod_{j=1}^{m}[\mathrm{K}_j : \mathrm{K}_{j-1}] = 2^c 3^d$ for some non-negative integers c, d. It follows that $\varphi(n) = 2^c 3^d$ and it implies that n is of the form stated in theorem 5.2. □

Based upon this theorem, we are able to show that regular 7-gon, 13-gon, 19-gon, etc are constructible by marked ruler and compass. However, between the possible and impossible, there still have some uncertainties lies in between. Especially, we are interested in the constructibility of regular 11-gon, 31-gon, 41-gon and 61-gon. From the inspiration given by the above theorem by the above theorem, we should appreciate the role that $\varphi(n)$ plays in the proof. It is natural to think that if a prime p divides $\varphi(n)$, and if we can p-sect an angle, we can in some sense solve the problem. It turns out to be true, and the following theorem will be the best description of this result.

Theorem 5.3. *Let n be an integer greater than 2. A regular n-gon can be constructed if, in addition to ruler and compass, tools are available to p-sect any given angle for each odd prime p that divides $\varphi(n)$.*

This theorem is actually a direct generalization of a corollary from Gauss's Disquistiones Arithmeticae (Article 55). It states that for a prime p, we can construct the primitive p^{th} root of unity if we are provided with tools that can construct all p_i^{th} root of unity for some p_i which divides $p-1$. To prove that, we should use the method of the Lagrange resolvent and show that there exists intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\xi_p)$ such that the field extension of each step is exactly one of p_i which divides $p-1$. However, since the proof is very lengthy and involves much more background knowledge in Galois Theory and cyclotomic equations, which are out of our scope of interest, we will not include the proof here.

Now we will show an application of this theorem.

Corollary 5.4. *If one can quinquesect an angle with marked ruler and compass, then the regular 11-gon, 31-gon, 41-gon and 61-gon are constructible by marked ruler and compass.*

Proof. Since 11, 31, 41 and 61 are primes, $\varphi(11) = 10 = (2)(5)$, $\varphi(31) = 30 = (2)(3)(5)$, $\varphi(41) = 40 = (2)^3(5)$ and $\varphi(61) = (2)^2(3)(5)$. As shown in theorem 3.6, marked ruler and compass can act as a trisector. So it follows from theorem 5.3 that if marked ruler and compass can quinquesect an angle, then regular 11-gon, 31-gon, 41-gon and 61-gon are constructible by marked ruler and compass. $\square$

From this corollary, we show that the regular n-gons for which their constructibility are not certain is closely related to the possibility of solving some quintics by marked ruler and compass. The situation is getting clear that the verification of the solvability of different types of quintic equations are of primary importance in knowing what classification of constructions are associated with marked ruler and compass. In particular, quinquesecting an angle is associated with a quintic equation which is solvable by radicals. In the next section, we will show that there does exist irreducible equation that is solvable by marked ruler and compass.

6. Solvability of irreducible quintic equation (not solvable by radical)

As shown in section 4, the points of intersection of two curves are associated with sextic equations, but none of them are quintic equation at all. To construct an irreducible quintic equation over $\mathbb{Q}$, we should first fix one point of intersection so that exactly one root is in $\mathbb{Q}$. We will give an example generated by intersecting a generalized conchoid with a circle.

Consider the generalized conchoid

$$x^2y^2 = (1-x^2)(x+2)^2 \tag{6.1}$$

and the circle

$$(x+1)^2 + (y-1)^2 = 5. \tag{6.2}$$

Obviously, $(1,0)$ is a point of intersection.

By solving (6.1) and (6.2) together according to section 4, we have a sextic polynomial

$$x^6 + 5x^5 + 4x^4 - 10x^3 - 6x^2 + 4x + 2 = 0. \tag{6.3}$$

Since 1 is a root of (6.3), we factorize it to:

$$(x-1)(x^5 + 6x^4 + 10x^3 - 6x^2 - 2) = 0$$

which yields a quintic equation

$$x^5 + 6x^4 + 10x^3 - 6x^2 - 2 = 0. \tag{6.4}$$

It is easy to see that equation (6.4) is irreducible over $\mathbb{Q}$ according to Eisenstein's Irreducibility Criterion. So marked ruler and compass really can solve certain problems associated with an irreducible quintic equation.

Moreover, if we solve (6.1) and (6.2) geometrically, these two curves actually have four points of intersection. It implies that (6.4) has three real roots and two complex roots. As a result, the complex conjugation is an automorphism of the splitting field for (6.4) that fixes three real roots and transposes the two complex roots, which means that the Galois Group for (6.4) includes a transposition, and also a five-cycle. It follows that the group must be S_5. Since S_5 is not solvable, from Abel's result, (6.4) is not solvable by radical indeed.

7. Conclusion

Throughout the project, we have shown that marked ruler is a very powerful tool in geometric constructions. With two notches more than an unmarked straight edge, a great variety of new constructions have been accomplished. We know now that the full strength of a marked ruler together with compass would not exceed the algebraic closure of equations of at most degree 6. On the other hand, it is also certain that all equations of at most degree 4 can be solved. And the uncertainty is limited to the solvability of quintic and sextic equations. The remaining unsolved construction problems by marked ruler and compass are getting clearer. First, can we solve all quintic and sextic equations? If we can solve it, all the problems are solved. However, the problem is very complicated since it involves solving some exotic algebraic curve (4.4) and (4.5) with little regularities. So we come up with the second question. Are all quintics solvable by radicals also solvable by marked ruler and compass? The significance of this question is shown in corollary 5.4 that it opens the possibility of constructing regular 11-gon, 31-gon, etc, since quinquesecting an angle involves only a quintic equation, which is solvable by radicals. Although I do not give a proof, we incline to believe that it should be true because in the course of our study we find that some irreducible quintic equations which are not solvable by radicals can be solved by marked ruler and compass. I think that these two questions alone would provide interesting topics for further study in similar mathematics project competition to be held in the future.

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