Hang Lung Mathematics Awards
Vol. 8 (2018), pp. 173–230

OLD AND NEW GENERALIZATIONS OF CLASSICAL TRIANGLE CENTRES TO TETRAHEDRA

TEAM MEMBERS
Trevor Kai Hei CHEUNG, Hon Ching KO

TEACHER
Mr. Pak Leong CHEUNG

SCHOOL
St. Paul's Co-educational College

Abstract. The classical triangle centres, namely centroid, circumcentre, incentre, excentre, orthocentre and Monge point, will be generalized to tetrahedra in a unified approach as points of concurrence of special lines. Our line characterization approach will also enable us to create new tetrahedron centres lying on the Euler lines, which will be a family with nice geometry including Monge point and twelve-point centre.

Two tetrahedron centres generalizing orthocentre of triangles from new perspectives will be constructed through introducing antimedial tetrahedra, tangential tetrahedra and a new kind of orthic tetrahedra. The first one, defined as the circumcentre of the antimedial tetrahedron of a tetrahedron, will be proved to lie on the Euler line. The second one, defined as the incentre or a suitable excentre of the new orthic tetrahedron of a tetrahedron, will be discovered to be collinear with its circumcentre and twenty-fifth Kimberling centre χ_{25}. Surprisingly, these two differently motivated geometric generalizations turn out to have analogous algebraic representations.

A clear definition of *tetrahedron centres*, as a generalization of triangle centres to tetrahedra, will be coined to set up a framework for studying analogies between geometries of triangles and tetrahedra. Fundamental properties of tetrahedron centres will be studied.

1. Introduction

1.1. Generalizing Triangle Centres to Tetrahedra (and Higher-Dimensional Simplices)

The classical triangle centres, namely centroid, circumcentre, incentre, excentre and orthocentre, have been a special part of high school geometry syllabi. The unique feature that they are respectively the points of concurrence of medians, perpendicular bisectors, interior angle bisectors, exterior angle bisectors and altitudes of triangles has fascinated many mathematics lovers. See [14] and [15] for a framework of triangle centres.

Tetrahedra have been the most straightforward generalization of triangles to the three-dimensional space. It is therefore reasonable to investigate how much geometry of triangles carries over or does not carry over to them. Generalizing triangle centres to tetrahedra has been one fruitful aspect of this general theme. See [11] for an excellent survey.

It is imaginable that the first four classical triangle centres can be naturally and satisfactorily generalized to tetrahedra as tetrahedron centres. However, since the altitudes of a tetrahedron may not be concurrent, an orthocentre may not exist; if it exists, the tetrahedron is said to be orthocentric. But then an innovative generalization called Monge point came to rescue such oddity: it exists uniquely in any tetrahedron, and coincides with the orthocentre in any orthocentric tetrahedron. See [3], [6], [11], [13], [16], [17] and [18].

Some non-classical triangle centres have also been satisfactorily generalized to tetrahedra. They include nine-point centre (generalized to twelve-point centre), symmedian point (a.k.a. Lemoine point and Grebe point), Gergonne point, Nagel point, Spieker centre (a.k.a. cleavance centre) and Fermat-Torricelli point (a.k.a. first isogonic centre when no angle exceeds $2\pi/3$). Indeed, all the above tetrahedron centres can be further generalized to higher-dimensional simplices. See [1], [2], [3], [4], [9], [12], [18], [19] and [20]. Nevertheless, the generalized triangle centres are only front members $\chi_n, n = 1, 2, 3, 4, 5, 6, 7, 8, 10, 13$, of Kimberling centres [15], and still a lot more have not been generalized yet.

1.2. Aims, Objectives and Organization

This paper will be devoted to

(a) showing line characterizations of the classical tetrahedron centres,
(b) generalizing Monge point of tetrahedra to a family of tetrahedron centres lying on the Euler line,
(c) generalizing orthocentre of triangles to tetrahedra in two new ways, and
(d) formulating a framework of tetrahedron centres.

We will take a coordinate-free analytic approach which requires only basic linear algebra, whereas high school level synthetic proofs will be provided as well for some of the results.

Traditionally, the classical tetrahedron centres are defined or characterized as points of concurrence of special planes of tetrahedra (cf. [3], [6], [13], [16], [17], [18]). In Section 2, we will characterize them as points of concurrence of special lines of tetrahedra instead. The theory will become better in three aspects because of this line characterization approach: (i) lines are simpler geometric objects than planes, making ideas easier to visualize, (ii) there are fewer special lines than special planes, making pictures less complicated and messy, and (iii) the uniqueness of a common point of lines is usually more obvious than that of planes.

These line characterizations are also main properties of the classical triangle centres that carry over to tetrahedra during the generalizations, and will also motivate and supply useful tools for the latter sections. Although [11] commented that '*The definitions of these three centers [centroid, circumcentre and incentre] and most of their main properties can be carried over to tetrahedra [...] in a very natural manner, and proofs are often routine generalizations.*', we found that a rigourous and meticulous treatment actually requires quite a lot of effort and care. Excentre has been defined in [21], but our description may look clearer.

Among the classical tetrahedron centres, Monge point is a special one. However, in Section 3, we will discover that it is just a member of a vast family of tetrahedron centres lying on the Euler line. This family of tetrahedron centres will be named as *quasi-orthocentres*, since they will be characterized as the points of concurrence of special lines sharing some common properties with altitudes.

While Monge point is the most recognized generalization of orthocentre of triangles, we will present two more generalizations in Section 4, as orthocentre of triangles possesses a range of characterizations. They will be named as *antimedial circumcentre* and *orthic inexcentre*, as they will be constructed through antimedial tetrahedra, tangential tetrahedra and a new kind of orthic tetrahedra. We will discover homothety between a tetrahedron and its antimedial tetrahedron, and between the tangential tetrahedron and the orthic tetrahedron of the tetrahedron. We will then derive geometric and algebraic properties of antimedial circumcentre and orthic inexcentre that are properties of orthocentre of triangles that carry over to tetrahedra during these two generalizations. These results will be closely related to the homotheties. The twenty-fifth Kimberling centre χ_{25} of triangles will also be generalized as a by-product.

Despite we have kept talking about *tetrahedron centre*, it seems that this terminology has been used without a clear definition. Through having generalized various triangle centres to tetrahedra in the preceding sections, we will have come up with the elements needed for defining this general terminology. In Section 5, we will suggest our definition, and show that all the *tetrahedron centres* in this paper fulfill the requirements. Finally, we will prove a couple of simple ways to construct tetrahedron centres from others, and will show how all tetrahedron centres can be expressed in terms of barycentric coordinates.

1.3. Terminologies and Notations

A *tetrahedron* $\Delta(V)/\Delta(V_0, V_1, V_2, V_3)/[V_0, V_1, V_2, V_3]$ in the three-dimensional Euclidean space $\mathbb{R}^3$ is the convex hull

$$[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] := \{\lambda_0\mathbf{v}_0 + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 : \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1 \quad \text{and} \quad 0 \leq \lambda_0, \lambda_1, \lambda_2, \lambda_3 \leq 1\} \tag{1}$$

of its vertex set $V = \{V_0, V_1, V_2, V_3\}$, where $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the position vectors of the *vertices* V_0, V_1, V_2, V_3. The *edge* joining the vertices V_i and V_j is the convex hull $E_{i,j} := [V_i, V_j]$ of them, whose direction vector is $e_{i,j} := \mathbf{v}_i - \mathbf{v}_j$. The *face* opposite to the vertex V_i is the convex hull $F_i := [V \backslash \{V_i\}]$ of all the vertices except V_i.

The vertices V_0, V_1, V_2, V_3 have to be *non-coplanar* in order to form a tetrahedron, which is equivalent to requiring that $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are *affinely independent*, i.e. $\{\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \mathbf{v}_3 - \mathbf{v}_0\}$ are linearly independent.

With this affine independence, the edges $E_{i,j}$ and the faces F_i are guaranteed to be line segments and triangles, and the position vector of every point in the tetrahedron can be expressed uniquely as a convex combination of the form (1).

Throughout this paper, points (as geometric objects) and their position vectors (as algebraic objects on which we can perform operations) will be used interchangeably. For ease of readability, we will adopt the notational convention that while points will be denoted by italic uppercase letters $A, B, C \ldots$ or $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$, their position vectors will be denoted by the corresponding boldface lowercase letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$, and the special lines associated to these points will be denoted by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$.

2. Classical Tetrahedron Centres

First of all, let us recall the classical triangle centres and their main properties through the following figures:

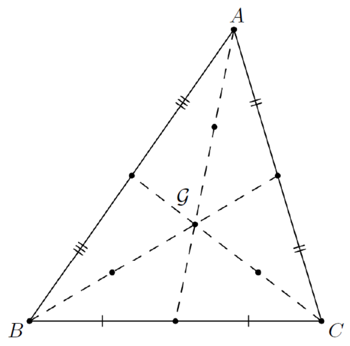

FIGURE 1. Centroid $\mathcal{G}$

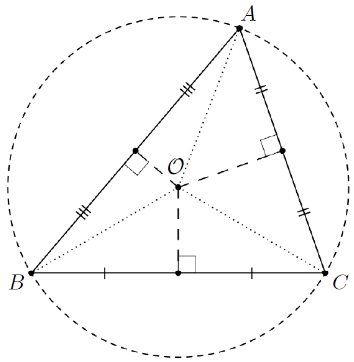

FIGURE 2. Circumcentre $\mathcal{O}$

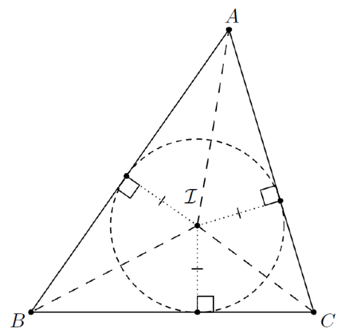

FIGURE 3. Incentre $\mathcal{I}$

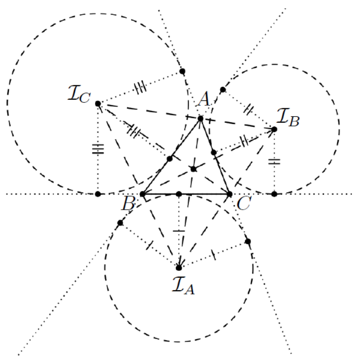

FIGURE 4. Excentres $\mathcal{I}_\mathcal{A}, \mathcal{I}_\mathcal{B}, \mathcal{I}_\mathcal{C}$

In this section, we will use a unified approach — generalize the vertices of a triangle as the vertices of a tetrahedron, and generalize the edges of a triangle as the faces of a tetrahedron — to generalize the classical triangle centres to tetrahedra. We will define centroid, circumcentre, incentre, excentre, orthocentre and Monge point of tetrahedra, and characterize them as the points of concurrence of special lines of tetrahedra. We will also show that the geometric properties of their two-dimensional counterparts as shown in Figures 1, 2, 3, 4 and 5 are retained when generalized to tetrahedra.

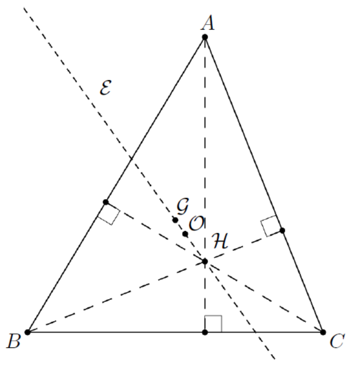

FIGURE 5. Orthocentre $\mathcal{H}$

2.1. Centroid

Definition 1. (Centroid and median) *Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. The centroid $\mathcal{G}$ of Δ is defined as*

$$\mathbf{g} = \frac{1}{4}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \tag{2}$$

A median g_i of Δ is the line passing through V_i and $\mathcal{G}_i$, where $\mathcal{G}_i$ abbreviates the centroid $\mathcal{G}(F_i)$ of the face F_i.

We now re-prove (cf. Commandino's theorem) that centroid can be characterized as the point of concurrence of medians. See Figure 6 for an illustration.

Proposition 2. *(Centroid as point of concurrence of medians) The centroid $\mathcal{G}$ of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ is the point of concurrence of its four medians. It divides each median segment $[V_i, \mathcal{G}_i]$ internally in $V_i\mathcal{G} : \mathcal{G}\mathcal{G}_i = 3 : 1$.*

Proof. Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Then,

$$\mathbf{g} = \frac{1}{4}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \frac{1}{4}\mathbf{v}_i + \frac{1}{4}(\mathbf{v}_j + \mathbf{v}_k + \mathbf{v}_l) = \frac{1}{4}\mathbf{v}_i + \frac{3}{4}\mathbf{g}_i,$$

showing that $\mathcal{G}$ lies on g_i and divides the median segment $[V_i, \mathcal{G}_i]$ internally in the claimed ratio.

Moreover, the medians cannot intersect at more points, otherwise, they cannot be distinct lines since two points determine a line. $\square$

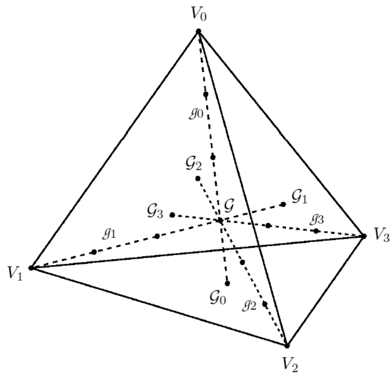

FIGURE 6. Proposition 2

2.2. Circumcentre

Definition 3. *(Circumcentre and perpendicular bisector) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. The circumcentre $\mathcal{O}$ of Δ is the unique point o equidistant to its vertices, i.e.*

$$||\mathbf{o} - \mathbf{v}_0|| = \cdots = ||\mathbf{o} - \mathbf{v}_3|| \tag{3}$$

The common distance in (3) is called the circumradius of Δ, denoted by $R(\Delta)$. The sphere

$$S^{ci}(\Delta) := \{\mathbf{x} : ||\mathbf{x} - \mathbf{o}|| = R(\Delta)\}$$

is called the circumsphere *of Δ.*

A perpendicular bisector o_i of Δ is the line passing through $\mathcal{O}_i$ and perpendicular to F_i, where $\mathcal{O}_i$ abbreviates the circumcentre $\mathcal{O}(F_i)$ of the face F_i.

We now prove the existence and uniqueness of the solution of (3) in order that circumcentre, circumradius and circumsphere are well-defined, as well as that circumcentre can be characterized as the point of concurrence of perpendicular bisectors. See Figure 7 for an illustration.

Proposition 4. *(Circumcentre as point of concurrence of perpendicular bisectors) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. Then, equation 3 has a unique solution* $\mathbf{o}$.

Moreover, the circumcentre $\mathcal{O}$ of Δ is the point of concurrence of its four perpendicular bisectors. Consequently, the projection of $\mathcal{O}$ onto a face F_i is precisely the circumcentre $\mathcal{O}_i$ of the face.

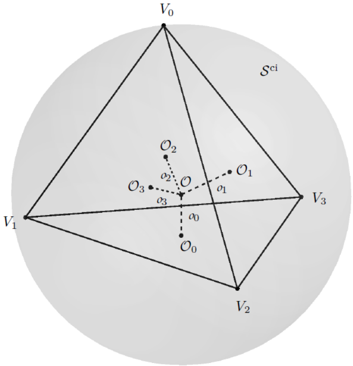

FIGURE 7. Proposition 4

Proof. The equations captured by (3) are

$$\begin{aligned} ||\mathbf{o}-\mathbf{v}_0||^2 &= ||\mathbf{o}-\mathbf{v}_i||^2 \\ ||\mathbf{o}||^2 - 2\mathbf{o}\cdot\mathbf{v}_0 + ||\mathbf{v}_0||^2 &= ||\mathbf{o}||^2 - 2\mathbf{o}\cdot\mathbf{v}_i + ||\mathbf{v}_i||^2 \\ (\mathbf{v}_i-\mathbf{v}_0)\cdot\mathbf{o} &= \frac{1}{2}(||\mathbf{v}_i||^2 - ||\mathbf{v}_0||^2) \quad \text{for } i=1,2,3. \end{aligned} \tag{4}$$

Writing $\mathbf{o}=(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ and $\mathbf{v}_i-\mathbf{v}_0=(a_{i,1},a_{i,2},a_{i,3})$, (4) becomes

$$a_{i,1}x_1 + a_{i,2}x_2 + a_{i,3}x_3 = b_i,$$

where $b_i = \frac{1}{2}(||\mathbf{v}_i||^2 - ||\mathbf{v}_0||^2)$, and we obtain the system of linear equations

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Since $\{\mathbf{v}_1-\mathbf{v}_0, \mathbf{v}_2-\mathbf{v}_0, \mathbf{v}_3-\mathbf{v}_0\}$ are linearly independent, the coefficient matrix will be invertible, and the system has a unique solution, thus proving the required existence and uniqueness.

Let $\{i,j,k,l\} = \{0,1,2,3\}$, and consider the circumcentre $\mathcal{O}_i/o_i$ of the face F_i, which is the point in the plane containing F_i that satisfies

$$\begin{aligned} ||\mathbf{o}_i-\mathbf{v}_j|| &= R_i > 0 \\ ||\mathbf{o}_i||^2 - 2\mathbf{o}_i\cdot\mathbf{v}_j + ||\mathbf{v}_j||^2 &= R_i^2 \end{aligned} \tag{5}$$

To show the required concurrence of the perpendicular bisectors, it suffices to show that

$$(\mathbf{o}-\mathbf{o}_i)\cdot(\mathbf{v}_j-\mathbf{v}_k) = 0 \tag{6}$$

so that the line joining $\mathcal{O}$ and $\mathcal{O}_i$ is perpendicular to F_i and so $\mathcal{O}$ lies on o_i. To this end, use (4) to get

$$\begin{aligned}\mathbf{o}\cdot(\mathbf{v}_j-\mathbf{v}_k) &= \mathbf{o}\cdot(\mathbf{v}_j-\mathbf{v}_0)-\mathbf{o}\cdot(\mathbf{v}_k-\mathbf{v}_0)\\ &= \frac{1}{2}\left(||\mathbf{v}_j||^2-||\mathbf{v}_0||^2\right)-\frac{1}{2}\left(||\mathbf{v}_k||^2-||\mathbf{v}_0||^2\right)\\ &= \frac{1}{2}\left(||\mathbf{v}_j||^2-||\mathbf{v}_k||^2\right) \qquad (7)\end{aligned}$$

and use (5) to get

$$\begin{aligned}\mathbf{o}_i\cdot(\mathbf{v}_j-\mathbf{v}_k) &= \frac{1}{2}\left(||\mathbf{o}_i||^2+||\mathbf{v}_j||^2-R_i^2\right)-\frac{1}{2}\left(||\mathbf{o}_i||^2+||\mathbf{v}_k||^2-R_i^2\right)\\ &= \frac{1}{2}\left(||\mathbf{v}_j||^2-||\mathbf{v}_k||^2\right) \qquad (8)\end{aligned}$$

Then, (6) follows from subtracting (8) from (7). □

2.3. Incentre and Excentre

Definition 5. (Incentre and interior angle bisector) *Let $\Delta=[V_0,V_1,V_2,V_3]$ be a tetrahedron. For $\{i,j,k,l\}=\{0,1,2,3\}$, the inward normal vector of the face F_i is the vector $\mathbf{n}_i$ such that $||\mathbf{n}_i||=1$,*

$$\mathbf{n}_i\cdot\mathbf{e}_{j,k}=0 \quad \textit{and} \quad \mathbf{n}_i\cdot\mathbf{e}_{i,j}>0$$

Then, the inward equation of F_i or the plane containing F_i will be of the form $\mathbf{n}_i\cdot(\mathbf{x}-\mathbf{p}_i)=0$, where $\mathbf{p}_i$ is a point on F_i. See Figure 8 for an illustration.

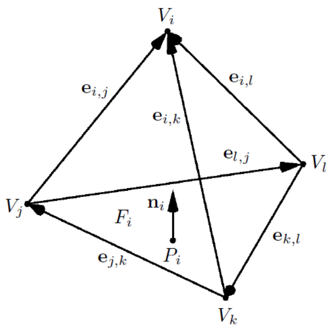

FIGURE 8. Definition (5)

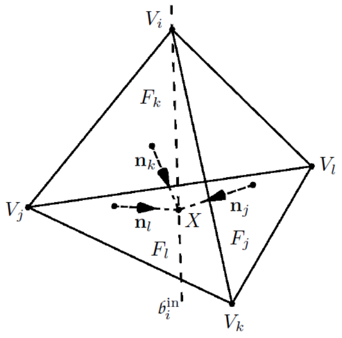

FIGURE 9. Definition (5)

The incentre $\mathcal{I}$ of Δ is the unique point i such that

$$\mathbf{n}_0\cdot(\mathbf{i}-\mathbf{p}_0)=\mathbf{n}_i\cdot(\mathbf{i}-\mathbf{p}_1)=\mathbf{n}_2\cdot(\mathbf{i}-\mathbf{p}_2)=\mathbf{n}_3\cdot(\mathbf{i}-\mathbf{p}_3) \qquad (9)$$

i.e. equidistant to its faces from the interior. The common distance in (9) is called the *inradius* of Δ, denoted by $r(\Delta)$. The sphere

$$S^{in}(\Delta) := \{\mathbf{x} : ||\mathbf{x} - \mathbf{i}|| = r(\Delta)\}$$

is called the *insphere* of Δ.

An *interior angle bisector* $\mathfrak{b}_i^{in}$ of Δ at V_i is the locus of the point χ/x satisfying

$$\mathbf{n}_j \cdot (\mathbf{x} - \mathbf{p}_j) = \mathbf{n}_k \cdot (\mathbf{x} - \mathbf{p}_k) = \mathbf{n}_l \cdot (\mathbf{x} - \mathbf{p}_l), \tag{10}$$

such that χ stays equidistant to the faces F_j, F_k and F_l. See Figure 9 for an illustration.

We now prove the existence and uniqueness of the solution of (9) in order that incentre, inradius and insphere are well-defined, as well as that incentre can be characterized as the point of concurrence of interior angle bisectors. See Figure 10 for an illustration.

Proposition 6. (Incentre as point of concurrence of interior angle bisectors) *Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. Then, equation (9) has a unique solution i.*

Moreover, the incentre $\mathcal{I}$ of Δ is the point of concurrence of its four interior angle bisectors.

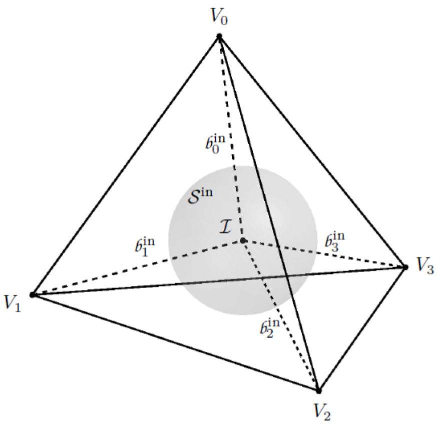

FIGURE 10. Proposition 6

Proof. The equations captured by (9) are

$$(\mathbf{n}_i - \mathbf{n}_0) \cdot \mathbf{i} = \mathbf{n}_i \cdot \mathbf{p}_i - \mathbf{n}_0 \cdot \mathbf{p}_0 \quad \text{for } i = 1, 2, 3.$$

It only suffices to prove that $\{\mathbf{n}_1 - \mathbf{n}_0, \mathbf{n}_2 - \mathbf{n}_0, \mathbf{n}_3 - \mathbf{n}_0\}$ are linearly independent so that this system has a unique solution $\mathbf{i}$ as (4) does by arguing in a similar manner as in the proof of Proposition 4. To this end, suppose there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1(\mathbf{n}_1 - \mathbf{n}_0) + \alpha_2(\mathbf{n}_2 - \mathbf{n}_0) + \alpha_3(\mathbf{n}_3 - \mathbf{n}_0) = 0$$
$$\alpha_1\mathbf{n}_1 + \alpha_2\mathbf{n}_2 + \alpha_3\mathbf{n}_3 = (\alpha_1 + \alpha_2 + \alpha_3)\mathbf{n}_0$$

Taking dot product with $e_{0,i}$ for $i \neq 0$, we have

$$\alpha_i \underbrace{\mathbf{n}_i \cdot \mathbf{e}_{0,i}}_{<0} = (\alpha_1 + \alpha_2 + \alpha_3) \underbrace{\mathbf{n}_0 \cdot \mathbf{e}_{0,i}}_{>0} \tag{11}$$

while taking dot product with $e_{i,j}$ for $i, j \neq 0$, we have

$$\alpha_i\mathbf{n}_i \cdot \mathbf{e}_{i,j} + \alpha_j\mathbf{n}_j \cdot \mathbf{e}_{i,j} = 0$$
$$\alpha_i \underbrace{\mathbf{n}_i \cdot \mathbf{e}_{i,j}}_{>0} = \alpha_j \underbrace{\mathbf{n}_j \cdot \mathbf{e}_{j,i}}_{>0} \tag{12}$$

By considering the signs of $\alpha_1, \alpha_2, \alpha_3$, (11) and (12) are consistent only if

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

and we have proved the claimed linear independence.

We have proved that $\{\mathbf{n}_1 - \mathbf{n}_0, \mathbf{n}_2 - \mathbf{n}_0, \mathbf{n}_3 - \mathbf{n}_0\}$ are linearly independent, so that $\{\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ are affinely independent. Then, for each i, $\{n_j : j \neq i\}$ will also be affinely independent, so that the solutions of (10) form a line. Finally, it is trivial that the unique solution of (9) is the unique common solution of all those (10) for $i = 0, 1, 2, 3$. □

Definition 7. *(Excentre and exterior angle bisector) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. For $\{i, j, k, l\} = \{0, 1, 2, 3\}$, the outward normal vector of the face F_i is the vector*

$$\mathbf{n}'_i = -\mathbf{n}_i$$

Then, the outward equation of F_i or the plane containing F_i will be of the form $\mathbf{n}'_i \cdot (\mathbf{x} - \mathbf{p}_i) = 0$, where $\mathbf{p}_i$ is a point on F_i.

The excentre $\mathcal{I}_i$ of Δ opposite to V_i is the unique point i_i such that

$$\mathbf{n}'_i \cdot (\mathbf{i}_i - \mathbf{p}_i) = \mathbf{n}_j \cdot (\mathbf{i}_i - \mathbf{p}_j) = \mathbf{n}_k \cdot (\mathbf{i}_i - \mathbf{p}_k) = \mathbf{n}_i \cdot (\mathbf{i}_i - \mathbf{p}_i). \tag{13}$$

i.e. equidistant to its faces from the exterior. The common distance in (13) is called the exradius of Δ opposite to V_i, denoted by $r_i(\Delta)$. The sphere

$$S_i^{ex}(\Delta) := \{\mathbf{x} : ||\mathbf{x} - \mathbf{i}_i|| = r_i(\Delta)\}$$

is called the exsphere of Δ opposite to V_i.

The exterior angle bisector $\mathfrak{b}_{i,j}^{ex}$ of Δ at V_i opposite to V_j is the locus of the point χ/x satisfying

$$\mathbf{n}'_j \cdot (\mathbf{x} - \mathbf{p}_j) = \mathbf{n}_k \cdot (\mathbf{x} - \mathbf{p}_k) = \mathbf{n}_i \cdot (\mathbf{x} - \mathbf{p}_i) \tag{14}$$

such that χ stays equidistant to the faces F_j, F_k and F_l. Note that there are three exterior angle bisectors at each vertex. See Figure 11 for an illustration.

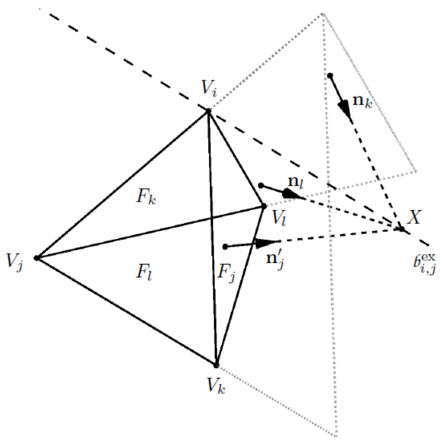

FIGURE 11. Definition 7

We now prove the existence and uniqueness of the solution of (13) in order that excentre, exradius and exsphere are well-defined, as well as that excentre can be characterized as the point of concurrence of interior and exterior angle bisectors. See Figure 12 for an illustration.

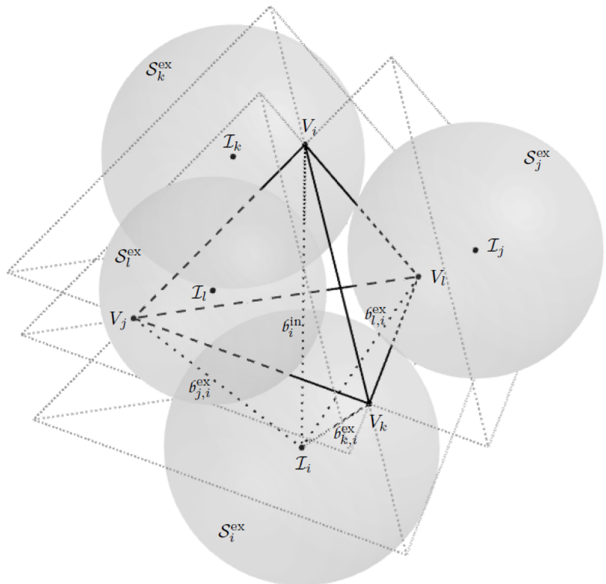

FIGURE 12. Proposition 8

Proposition 8. *(Excentre as point of concurrence of interior and exterior angle bisectors) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. Then, for each $i = 0, 1, 2, 3$, equation (13) has a unique solution i_i.*

Moreover, the excentre I_i of Δ opposite to V_i is the point of concurrence of the interior angle bisector $\mathfrak{b}_i^{in}$ at V_i and the three exterior angle bisectors $\mathfrak{b}_{j,i}^{ex}, \mathfrak{b}_{k,i}^{ex}$ and $\mathfrak{b}_{l,i}^{ex}$ at V_j, V_k and V_l respectively opposite to V_i.

Proof. Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. The equations captured by (13) are

$$\begin{aligned}(\mathbf{n}_j + \mathbf{n}_i) \cdot \mathbf{i}_i &= \mathbf{n}_j \cdot \mathbf{p}_j + \mathbf{n}_i \cdot \mathbf{p}_i,\\ (\mathbf{n}_k + \mathbf{n}_i) \cdot \mathbf{i}_i &= \mathbf{n}_k \cdot \mathbf{p}_k + \mathbf{n}_i \cdot \mathbf{p}_i \quad \text{and}\\ (\mathbf{n}_l + \mathbf{n}_i) \cdot \mathbf{i}_i &= \mathbf{n}_l \cdot \mathbf{p}_l + \mathbf{n}_i \cdot \mathbf{p}_i,\end{aligned}$$

since $\mathbf{n}_i' = -\mathbf{n}_i$. As with the case of incentre, it only suffices to prove that $\{\mathbf{n}_j + \mathbf{n}_i, \mathbf{n}_k + \mathbf{n}_i, \mathbf{n}_l + \mathbf{n}_i\}$ are linearly independent so that this system has a unique solution i_i. To this end, suppose there are $\alpha_j, \alpha_k, \alpha_l \in \mathbb{R}$ such that

$$\alpha_j(\mathbf{n}_j + \mathbf{n}_i) + \alpha_k(\mathbf{n}_k + \mathbf{n}_i) + \alpha_l(\mathbf{n}_l + \mathbf{n}_i) = 0$$

$$(\alpha_j + \alpha_k + \alpha_j)\mathbf{n}_i = -\alpha_j\mathbf{n}_j - \alpha_k\mathbf{n}_k - \alpha_l\mathbf{n}_l \tag{15}$$

Taking dot product with $\mathbf{e}_{i,j}, \mathbf{e}_{i,k}$ and $\mathbf{e}_{i,l}$ respectively, we have

$$\begin{aligned}(\alpha_j + \alpha_k + \alpha_l)\underbrace{\mathbf{n}_i \cdot \mathbf{e}_{i,j}}_{>0} &= -\alpha_j \underbrace{\mathbf{n}_j \cdot \mathbf{e}_{i,j}}_{>0},\\ (\alpha_j + \alpha_k + \alpha_l)\underbrace{\mathbf{n}_i \cdot \mathbf{e}_{i,k}}_{>0} &= -\alpha_k \underbrace{\mathbf{n}_k \cdot \mathbf{e}_{i,k}}_{>0} \quad \text{and}\\ (\alpha_j + \alpha_k + \alpha_l)\underbrace{\mathbf{n}_i \cdot \mathbf{e}_{i,l}}_{>0} &= -\alpha_l \underbrace{\mathbf{n}_l \cdot \mathbf{e}_{i,l}}_{>0},\end{aligned}$$

which imply that either

$$\alpha_j, \alpha_k, \alpha_l > 0 \quad \text{or} \quad \alpha_j, \alpha_k, \alpha_l < 0 \quad \text{or} \quad \alpha_j, \alpha_k, \alpha_l = 0$$

But taking dot product with $\mathbf{n}_i$ in (15), with $||\mathbf{n}_i|| = 1$, we have

$$\alpha_j + \alpha_k + \alpha_l = -\alpha_j\mathbf{n}_j \cdot \mathbf{n}_i - \alpha_k\mathbf{n}_k \cdot \mathbf{n}_i - \alpha_l\mathbf{n}_l \cdot \mathbf{n}_i \tag{16}$$

If $\alpha_j, \alpha_k, \alpha_l > 0$, using the Cauchy-Schwarz inequality, with $||\mathbf{n}_j|| = ||\mathbf{n}_k|| = ||\mathbf{n}_l|| = 1$, then (16) would become

$$\begin{aligned}\alpha_j, \alpha_k, \alpha_l &= \alpha_j(-\mathbf{n}_j \cdot \mathbf{n}_i) + \alpha_k(-\mathbf{n}_k \cdot \mathbf{n}_i) + \alpha_l(-\mathbf{n}_l \cdot \mathbf{n}_i)\\ &\le \alpha_j||\mathbf{n}_j||||\mathbf{n}_i|| + \alpha_k||\mathbf{n}_k||||\mathbf{n}_i|| + \alpha_l||\mathbf{n}_l||||\mathbf{n}_i||\\ &= \alpha_j + \alpha_k + \alpha_l \end{aligned} \tag{17}$$

Similarly, if $\alpha_j, \alpha_k, \alpha_l < 0$, then (16) would become

$$\begin{aligned}\alpha_j, \alpha_k, \alpha_l &= (-\alpha_j)\mathbf{n}_j \cdot \mathbf{n}_i + (-\alpha_k)\mathbf{n}_k \cdot \mathbf{n}_i + (-\alpha_l)\mathbf{n}_l \cdot \mathbf{n}_i\\ &\ge (-\alpha_j)(-||\mathbf{n}_j||||\mathbf{n}_i||) + (-\alpha_k)(-||\mathbf{n}_k||||\mathbf{n}_i||) + (-\alpha_l)(-||\mathbf{n}_l||||\mathbf{n}_i||)\\ &= \alpha_j + \alpha_k + \alpha_l \end{aligned} \tag{18}$$

However, both (17) and (18) forces $\mathbf{n}_j \cdot \mathbf{n}_i = \mathbf{n}_k \cdot \mathbf{n}_i = \mathbf{n}_l \cdot \mathbf{n}_i = -1$, which is impossible as it implies that $\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k, \mathbf{n}_l$ are parallel. Therefore, $\alpha_j, \alpha_k, \alpha_l = 0$, and we have proved the claimed linear independence.

We have proved that for $\{i, j, k, l\} = \{0, 1, 2, 3\}$, $\{\mathbf{n}_j + \mathbf{n}_i, \mathbf{n}_k + \mathbf{n}_i, \mathbf{n}_l + \mathbf{n}_i\}$ are linearly independent, so that $\{-\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k, \mathbf{n}_l\}$ are affinely independent. Then, for each i, $\{\mathbf{n}_j, \mathbf{n}_k, \mathbf{n}_l\}$ will also be affinely independent, so that the solutions of (10) form a line. Also, for each $i, j, \{-\mathbf{n}_j, \mathbf{n}_k, \mathbf{n}_l\}$ will also be affinely independent, so that the solutions of (14) form a line. Finally, by listing the following equations

$$\begin{aligned}
\mathfrak{b}_i^{in} &: \mathbf{n}_j \cdot (\mathbf{x} - \mathbf{p}_j) = \mathbf{n}_k \cdot (\mathbf{x} - \mathbf{p}_k) = \mathbf{n}_l \cdot (\mathbf{x} - \mathbf{p}_l) \\
\mathfrak{b}_{j,i}^{ex} &: \mathbf{n}_i' \cdot (\mathbf{x} - \mathbf{p}_i) = \mathbf{n}_k \cdot (\mathbf{x} - \mathbf{p}_k) = \mathbf{n}_l \cdot (\mathbf{x} - \mathbf{p}_l) \\
\mathfrak{b}_{k,i}^{ex} &: \mathbf{n}_i' \cdot (\mathbf{x} - \mathbf{p}_i) = \mathbf{n}_k \cdot (\mathbf{x} - \mathbf{p}_j) = \mathbf{n}_l \cdot (\mathbf{x} - \mathbf{p}_l) \\
\mathfrak{b}_{l,i}^{ex} &: \mathbf{n}_i' \cdot (\mathbf{x} - \mathbf{p}_i) = \mathbf{n}_k \cdot (\mathbf{x} - \mathbf{p}_j) = \mathbf{n}_l \cdot (\mathbf{x} - \mathbf{p}_k)
\end{aligned}$$

it is trivial that the unique solution of (13) is the unique common solution of them. □

2.4. Orthocentre and Monge Point

Definition 9. *(Orthocentre, altitude and orthocentric tetrahedron)*
Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. For $i = 0, 1, 2, 3$, the altitude $\mathfrak{h}_i$ of Δ from V_i is the line joining V_i and its projection H_i onto the face F_i. H_i is also called the foot of altitude from V_i.

If the four altitudes are concurrent, then the point of concurrence is called the orthocentre $\mathcal{H}$ of Δ, and Δ is called an orthocentric tetrahedron.

Definition 10. *(Monge point and Monge line) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. The Monge point M of Δ is defined as*

$$\mathbf{m} = \mathbf{o} + 2(\mathbf{g} - \mathbf{o}) = 2\mathbf{g} - \mathbf{o} \tag{19}$$

where $\mathcal{G}$ and $\mathcal{O}$ are the centroid and the circumcentre of Δ respectively.

For $i = 0, 1, 2, 3$, let P_i be the mid-point of $[H_i, \mathcal{M}_i]$, where M_i abbreviates the orthocentre $\mathcal{M}(F_i)$ of the face F_i. Then, a *Monge line* m_i of Δ is the line passing through P_i and perpendicular to F_i (cf. [13] where the same line has also been introduced without this name). See Figure 13 for an illustration.

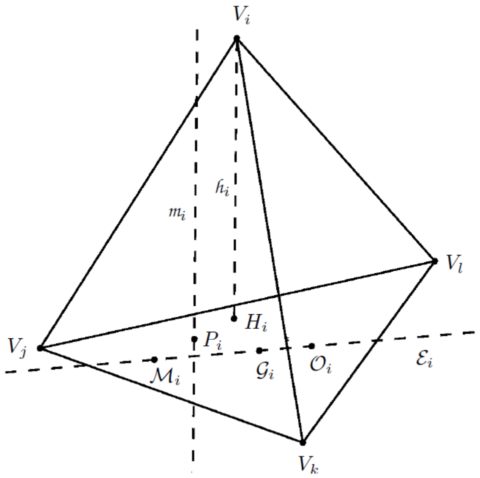

FIGURE 13. Definition 10

We now prove that Monge point can be characterized as the point of concurrence of Monge lines. See Figure 14 for an illustration.

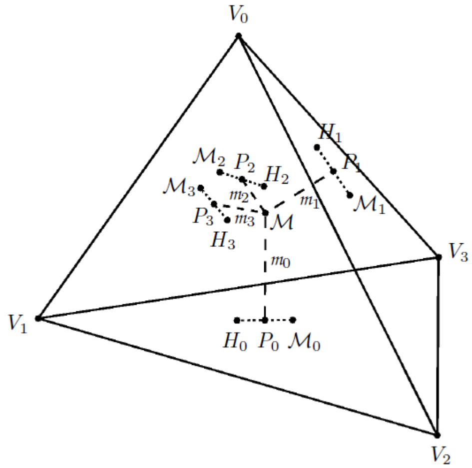

FIGURE 14. Proposition ??

Proposition 11. *(Monge point as point of concurrence of Monge lines) The Monge point $\mathcal{M}$ of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ is the point of concurrence of its four Monge lines.*

Proof. Let $i = 0, 1, 2, 3$. It is well-known that the orthocentre of F_i is given as

$$\mathbf{m}_i = 3\mathbf{g}_i - 2\mathbf{o}_i \tag{20}$$

so that the Monge point of Δ

$$\mathbf{m} = 2\mathbf{g} - \mathbf{o} = \frac{1}{2}(\mathbf{v}_i + 3\mathbf{g}_i) - \mathbf{o} = \frac{1}{2}(\mathbf{v}_i + \mathbf{m}_i + 2\mathbf{o}_i) - o = \frac{1}{2}(\mathbf{v}_i + \mathbf{m}_i) + (\mathbf{o}_i - \mathbf{o}) \tag{21}$$

Let P_i' be the midpoint of $[V_i, M_i]$. By Proposition 4, the last term $\mathbf{o}_i - \mathbf{o}$ in (21) is a vector perpendicular to F_i, thus (21) shows that $\mathcal{M}$ lies on the line l_i passing through P_i' and perpendicular to F_i. But by applying the intercept theorem in $\Delta\mathcal{M}_i V_i H_i$, l_i actually hits P_i, showing that $\mathcal{M}$ lies on $\mathbf{m}_i$.

Moreover, the Monge lines cannot intersect at more points, otherwise, they cannot be distinct lines since two points determine a line. □

Corollary 12. *(Monge point of orthocentric tetrahedron) In an orthocentric tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$, the foot of altitude $\mathcal{H}_i$ coincides with the orthocentre $\mathcal{M}_i$ of the face F_i, and the Monge point $\mathcal{M}$ of Δ coincides with the orthocentre $\mathcal{H}$ of Δ.*

Proof. Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Since the orthocentre $\mathcal{H}$ of the orthocentric tetrahedron Δ is the intersection of the altitudes h_i and h_j, we have

$$\begin{aligned} t_i\mathbf{v}_i + (1 - t_i)\mathbf{h}_i = h = t_j\mathbf{v}_j + (1 - t_j)\mathbf{h}_j \\ \mathbf{v}_j - \mathbf{h}_i = t_i \underbrace{(\mathbf{v}_i - \mathbf{h}_i)}_{\perp F_i} + (1 - t_j) \underbrace{(\mathbf{v}_j - \mathbf{h}_j)}_{\perp F_j} \end{aligned}$$

for some $t_i, t_j \in \mathbb{R}$. But as $(\mathbf{v}_i - \mathbf{h}_i) \cdot \mathbf{e}_{kl} = 0$ and $(\mathbf{v}_j - \mathbf{h}_j) \cdot \mathbf{e}_{kl} = 0$, we get

$$(\mathbf{v}_j - \mathbf{h}_i) \cdot \mathbf{e}_{kl} = t_i \cdot 0 + (1 - t_j) \cdot 0 = 0$$

which is saying that the line joining V_j and H_i is perpendicular to the edge $E_{k,l}$ of the face F_i. Hence, H_i is the orthocentre $\mathcal{M}_i$ of F_i.

Once $H_i = \mathcal{M}_i$, from Figure 13, the Monge line m_i will coincide with the altitude h_i. As a result, the point of concurrence of the Monge lines (i.e. the Monge point according to Proposition 11) coincides with the point of concurrence of the altitudes (i.e. orthocentre). □

In view of Corollary 12, Monge point is a perfect generalization of orthocentre. And as the altitudes of a triangle must be concurrent so that every triangle can be regarded as being orthocentric, the orthocentre of a triangle can be regarded as the Monge point of the triangle. This justifies denoting the orthocentre of the face F_i by $\mathcal{M}_i$ in Definition 10.

3. Quasi-orthocentres and Euler Line

This section will show one of the major discoveries of this paper: the *quasi-orthocentres*, which was inspired by the line characterization of Monge point in Section 2.4.

It is immediate from (19) in Definition 10 that the Monge point M of a tetrahedron lies on the Euler line $\mathcal{E}$ of the tetrahedron (i.e. the line joining the centroid $\mathcal{G}$ and the circumcentre $\mathcal{O}$, which is well-defined only when $\mathcal{G} \neq \mathcal{O}$). Similarly, (20) also shows that the orthocentre / Monge point $\mathcal{M}_i$ of a face lies on the Euler line $\mathcal{E}_i$ of the face (i.e. the line joining the centroid $\mathcal{G}_i$ and the circumcentre O_i of the face, which is well-defined only when $\mathcal{G}_i \neq \mathcal{O}_i$).

Proposition 11 in Section 2.4 is therefore demonstrating the following nice geometric feature of the Monge point of a tetrahedron: it is the point of concurrence of lines (i) parallel to the altitudes and (ii) emerging from specific points of division of the segments joining the feet of altitude and certain facial centres lying on the Euler lines of the faces. From the proof, it seems that the scaling factor 2 in (19) was so carefully chosen to faciliate such feature, but we will discover a family of tetrahedron centres that possess the same feature.

Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron, and let $i = 0, 1, 2, 3$. Let χ_i be a point on the Euler line $\mathcal{E}_i$ of the face F_i of Δ given as

$$\mathbf{x}_i = \mathbf{o}_i + r(\mathbf{g}_i - \mathbf{o}_i) = r\mathbf{g}_i + (1-r)\mathbf{o}_i,$$

where $r \in \mathbb{R}\backslash\{0\}$ is a constant. Also let $\mathcal{Y}$ be a point on the Euler line $\mathcal{E}$ of Δ given as

$$\mathbf{y} = s\mathbf{g} + (1-s)\mathbf{o},$$

where $s \in \mathbb{R}$ is a constant. Then,

$$\begin{aligned} \mathbf{y} &= \frac{s}{4}(\mathbf{v}_i + 3g_i) + (1-s)\mathbf{o} \\ &= \frac{s}{4}\left(\mathbf{v}_i + \frac{3}{r}\mathbf{x}_i - \frac{3(1-r)}{r}\mathbf{o}_i\right) + (1-s)\mathbf{o} \\ &= \frac{s}{4}\mathbf{v}_i + \frac{3s}{4r}\mathbf{x}_i - \frac{3s(1-r)}{4r}\mathbf{o}_i + (1-s)\mathbf{o} \end{aligned} \tag{22}$$

For (22) to mean that Y lies on a line parallel to the altitude h_i and passing through a point on the line joining V_i and χ_i, it requires first of all that

$$\begin{aligned} \frac{s}{4} + \frac{3s}{4r} &= 1 \\ s = \frac{4r}{r+3} \quad &\text{and} \quad r \neq -3 \end{aligned} \tag{23}$$

Letting $r = \frac{3}{2+k}$ in (23), where $k \in \mathbb{R}\backslash\{-2,-3\}$, we have

$$s = \frac{4 \cdot \frac{3}{2+k}}{\frac{3}{2+k} + 3} = \frac{4}{3+k} \tag{24}$$

Substituting (24) into (22), we have

$$\mathbf{y} = \frac{1}{3+k}\mathbf{v}_i + \frac{2+k}{3+k}\mathbf{x}_i + \frac{1-k}{3+k}(\mathbf{o}_i - \mathbf{o}) \tag{25}$$

where the last term is just a vector parallel to F_i according to Proposition 4.

Hence, we can now define:

Definition 13. *(Quasi-orthocentre and quasi-altitude) Fix any $k \neq -2, -3$. Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. The k-quasi-orthocentre Q_k of Δ is defined as*

$$\mathbf{q}_k = \mathbf{o} + \frac{4}{3+k}(\mathbf{g} - \mathbf{o}) = \frac{4}{3+k}\mathbf{g} - \frac{1-k}{3+k}\mathbf{o}, \tag{26}$$

where $\mathcal{G}$ and $\mathcal{O}$ are the centroid and the circumcentre of Δ respecitively.

The k-quasi-orthocentre $\mathcal{Q}_{k,i}$ of the face F_i of Δ is defined as

$$\mathbf{q}_{k,i} = \mathbf{o}_i + \frac{3}{2+k}(\mathbf{g}_i - \mathbf{o}_i) = \frac{3}{2+k}\mathbf{g}_i - \frac{1-k}{2+k}\mathbf{o}_i$$

where $\mathcal{G}_i$ and $\mathcal{O}_i$ are the centroid and the circumcentre of F_i respectively.

For $i = 0, 1, 2, 3$, let P_i be the point on the line joining H_i and $\mathcal{Q}_{k,i}$ such that $H_iP_i : P_i\mathcal{Q}_{k,i} = (2+k) : 1$. Then, a *$k$-quasi-altitude* $q_{k,i}$ of Δ is the line passing through P_i and perpendicular to F_i. See Figure 15 for an illustration.

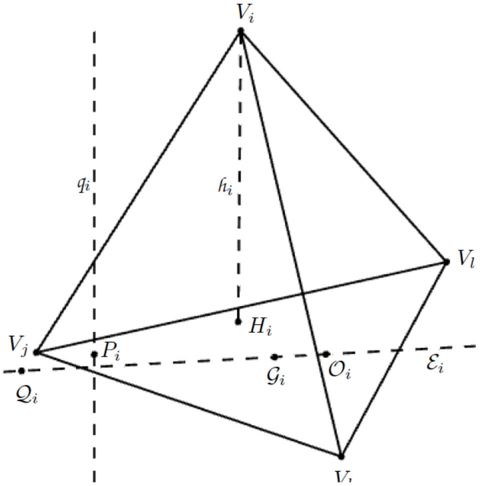

FIGURE 15. Definition 13

By (25), we have actually proved that the k-quasi-orthocentre of a tetrahedron is the point of concurrence of its four k-quasi-altitudes. See Figure 16 for an illustration.

Proposition 14. *(Quasi-orthocentre as point of concurrence of quasi-altitudes) Fix any $k \neq -2, -3$. The k-quasi-orthocentre Q_k of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ is the point of concurrence of its four k-quasialtitudes.*

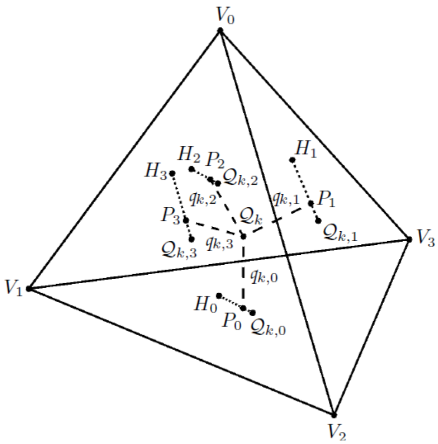

FIGURE 16. Proposition 14

Note that Monge point $\mathcal{M}$ and *twelve-point centre* $\mathcal{N}$ of tetrahedra are members of the family of quasiorthocentres $\mathcal{Q}_k$:

$$\mathbf{m} = 2\mathbf{g} - \mathbf{o} = q_{-1} \quad \text{and} \quad \mathbf{n} = \frac{4}{3}\mathbf{g} - \frac{1}{3}\mathbf{o} = \mathbf{q}_0$$

For twelve-point centre, see [2], [3], [4] and [18]. Also, their triangle counterparts — orthocentre $\mathcal{M}/\mathcal{H}$ and *nine-point centre* $\mathcal{N}$ — are also quasi-orthocentres $\mathcal{Q}_{k,i}$ of faces of tetrahedra:

$$\mathbf{m}/\mathbf{h} = 3\mathbf{g} - 2\mathbf{o} = q_{-1} \quad \text{and} \quad \mathbf{n} = \frac{3}{2}\mathbf{g} - \frac{1}{2}\mathbf{o} = \mathbf{q}_0$$

Hence, Monge point and twelve-point centre of tetrahedra share the common geometric feature of being the point of concurrence of altitude-like special lines derived from their triangle counterparts.

4. Antimedial Circumcentre and Orthic Inexcentre

This section will show the other two major discoveries of this paper: the *antimedial circumcentre* and the *orthic inexcentre*, as two new generalizations of orthocentre of triangles. As shown in Section 2.4, Monge point of tetrahedra generalizes orthocentre of triangles as the latter is regarded as the point of concurrence of altitudes. But using altitudes is not the only way to characterize orthocentre of triangles (cf. [10]).

Indeed, a quick proof for high school students of the fact that the three altitudes of a triangle concur at the orthocentre of the triangle is often through considering its antimedial triangle, whereby the altitudes of the former become the perpendicular bisectors of the latter. Figures 17 and 18 show that the orthocentre $\mathcal{H}(\Delta)$ of $\Delta = \Delta ABC$ is exactly the circumcentre $\mathcal{O}(\Delta')$ of its antimedial triangle $\Delta' = \Delta' A'B'C'$. This motivates us to introduce antimedial tetrahedron in a natural manner in Section 4.2, and its circumcentre will be one generalization of orthocentre. In fact, [18] has proved that the antimedial circumcentre, without using this name, lies on the Euler line. Our work will be re-proving its tetrahedron version with simpler presentation.

The characterization of the orthocentre of a triangle as the incentre or an excentre of its orthic triangle may be less well-known to high school students. Figures 19 and 20 show that the orthocentre $\mathcal{H}(\Delta)$ of $\Delta = \Delta ABC$ is exactly the incentre $I(\Delta')$ (when Δ is acute-angled) or the excentre $I_A(\Delta')$ (when Δ is obtuse-angled at A) of its orthic triangle $\Delta' = \Delta A'B'C'$. This inspires us to construct *orthic tetrahedron* in Section 4.3, and its incentre or excentre will be another generalization of orthocentre.

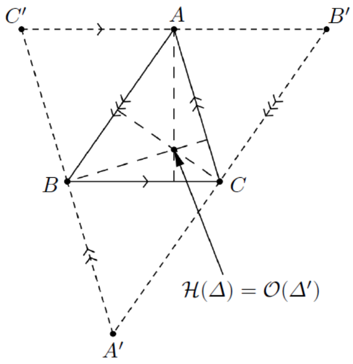

FIGURE 17. Antimedial triangle

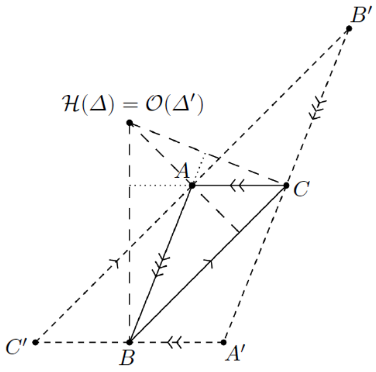

FIGURE 18. Antimedial triangle

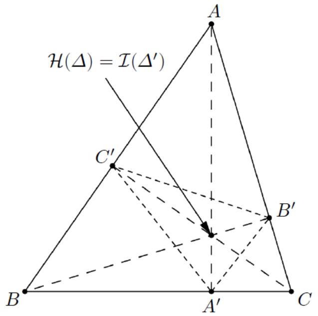

FIGURE 19. Orthic triangle

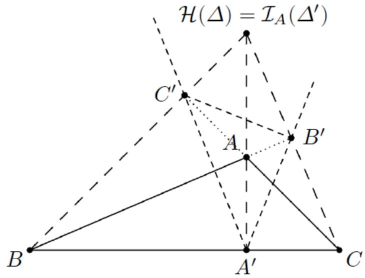

FIGURE 20. Orthic triangle

The two most natural kinds of *orthic tetrahedron* have already been studied in [5], but their vertices are confined to the planes containing the faces of the original tetrahedron. One could definitely consider their incentres or excentres as analogues of orthocentre of triangles. But we found that none of these could carry any good properties of orthic triangles or orthocentre of triangles over to tetrahedra. We will think out of the box to construct a new kind of *orthic tetrahedron*, whose vertices need not be restricted as suffered by the ordinary orthic tetrahedra.

Actually, we will prove that the following well-known properties of orthocentre of triangles can carry over to tetrahedra through antimedial circumcentre:

i The orthocentre $\mathcal{H}$ of a triangle is collinear with its centroid $\mathcal{G}$ and circumcentre $\mathcal{O}$.
ii The orthocentre h of a triangle $[V_0, V_1, V_2]$ can be expressed as

$$\mathbf{o} + (\mathbf{g}_0 - \mathbf{o}) + (\mathbf{g}_1 - \mathbf{o}) + (\mathbf{g}_2 - \mathbf{o})$$

where $\mathbf{o}$ is its circumcentre and $\mathbf{g}_i := \frac{1}{2}(v_j + v_k)$ for $\{i, j, k\} = \{0, 1, 2\}$ is treated as the centroid of the edge $E_{j,k}$.

We will also prove that the following well-known properties of orthocentre of triangles can carry over to tetrahedra through orthic inexcentre:

i The orthocentre $\mathcal{H}$ of a triangle is collinear with its circumcentre $\mathcal{O}$ and twenty-fifth Kimberling centre χ_{25}.

ii The orthocentre h of a triangle $[V_0, V_1, V_2]$ can be expressed as

$$\mathbf{o} + (\mathbf{o}_0 - \mathbf{o}) + (\mathbf{o}_1 - \mathbf{o}) + (\mathbf{o}_2 - \mathbf{o})$$

where $\mathbf{o}$ is its circumcentre and $o_i := \frac{1}{2}(v_j + v_k)$ for $\{i, j, k\} = \{0, 1, 2\}$ is treated as the circumcentre of the edge $E_{j,k}$.

Both Sections 4.2 and 4.3 will be highly related to homothety. Therefore, before them, we will prove a necessary and sufficient condition for homothety between two tetrahedra in Section 4.1.

4.1. Homothetic Tetrahedra

Definition 15. *(Homothety) A homothetic transformation T is a function $T : \mathbb{R}^3 \to \mathbb{R}^3$ of the form*

$$T(\mathbf{u}) = \mathbf{z} + t(\mathbf{u} - \mathbf{z}) = t\mathbf{u} + (1 - t)\mathbf{z} \quad \text{for } \mathbf{u} \in \mathbb{R}^3 \tag{27}$$

where $\mathbf{z} \in \mathbb{R}^3$ is called the homothetic centre and $t \in \mathbb{R}\backslash\{0\}$ is called the homothetic ratio.

Two tetrahedra Δ and Δ' are said to be homothetic if one of them can be obtained from the other through a homothetic transformation, i.e. there exists a homothetic transformation T such that $T(\Delta) = \Delta'$. The homothetic centre will be denoted by $Z(\Delta, \Delta')$.

There is no harm to say that a translation transformation

$$\mathbf{u} \mapsto \mathbf{u} + \mathbf{b} \tag{28}$$

is a homothetic transformation with homothetic centre $z = \infty$ and homothetic ratio $t = 1$.

Note that from (27) and (28) that $\mathbf{z}, \mathbf{u}$ and $T(\mathbf{u})$ must be collinear.

To justify that homothety between two tetrahedra is well-defined, we need the following lemma:

Lemma 16. *(Triangle and tetrahedron under homothetic transformation) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron and T be a homothetic transformation. Then,*

(a) $\{T(\mathbf{v}_0), T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are affinely independent,

(b) $T(\Delta) = [T(V_0), T(V_1), T(V_2), T(V_3)]$,
(c) $T(E_{i,j}) = [T(V_i), T(V_j)]$, *where* $i \neq j$, *and*
(d) $T(F_i) = [T(V_j), T(V_k), T(V_l)]$, *where* $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

Proof. Under homothetic transformation T of the form (27) or (28), we have

$$T(\mathbf{v}_i) - T(\mathbf{v}_0) = t(\mathbf{v}_i - \mathbf{v}_0) \quad \text{for } i = 1, 2, 3.$$

Suppose there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1(T(\mathbf{v}_1) - T(\mathbf{v}_0)) + \alpha_2(T(\mathbf{v}_2) - T(\mathbf{v}_0)) + \alpha_3(T(\mathbf{v}_3) - T(\mathbf{v}_0)) &= 0 \\ t\alpha_1(\mathbf{v}_1 - \mathbf{v}_0) + t\alpha_2(\mathbf{v}_2 - \mathbf{v}_0) + t\alpha_3(\mathbf{v}_3 - \mathbf{v}_0) &= 0 \\ \alpha_1(\mathbf{v}_1 - \mathbf{v}_0) + \alpha_2(\mathbf{v}_2 - \mathbf{v}_0) + \alpha_3(\mathbf{v}_3 - \mathbf{v}_0) &= 0 \end{aligned}$$

Since $\{\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \mathbf{v}_3 - \mathbf{v}_0\}$ are linearly independent, we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, the transformed vertices $\{T(\mathbf{v}_0), T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are affinely independent too, hence proving (a).

As mentioned in Section 1.3, any point $\mathbf{x}$ in the tetrahedron can be represented as a convex combination as in (1). If T takes the form of (27), where $t \neq 1$, then

$$\begin{aligned} T(\mathbf{x}) &= t(\lambda_0\mathbf{v}_0 + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3) + (1-t)\mathbf{z} \\ &= \lambda_0(t\mathbf{v}_0 + (1-t)\mathbf{z}) + \lambda_1(t\mathbf{v}_1 + (1-t)\mathbf{z}) + \lambda_2(t\mathbf{v}_2 + (1-t)\mathbf{z}) \\ &\quad + \lambda_3(t\mathbf{v}_3 + (1-t)\mathbf{z}) \\ &= \lambda_0 T(\mathbf{v}_0) + \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2) + \lambda_3 T(\mathbf{v}_3). \end{aligned}$$

If T takes the form of (28), then

$$\begin{aligned} T(\mathbf{x}) &= \lambda_0\mathbf{v}_0 + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \mathbf{b} \\ &= \lambda_0(\mathbf{v}_0 + \mathbf{b}) + \lambda_1(\mathbf{v}_1 + \mathbf{b}) + \lambda_2(\mathbf{v}_2 + \mathbf{b}) + \lambda_3(\mathbf{v}_3 + \mathbf{b}) \\ &= \lambda_0 T(\mathbf{v}_0) + \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2) + \lambda_3 T(\mathbf{v}_3). \end{aligned}$$

In both cases, $T(\mathbf{x}) = \lambda_0 T(\mathbf{v}_0) + \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2) + \lambda_3 T(\mathbf{v}_3)$, which shows that as $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ run through the condition in (1) so that $\mathbf{x}$ runs through Δ, $T(\mathbf{x})$ will run through every point the tetrahedron $[T(\mathbf{v}_0), T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)]$ in a one-to-one correspondence manner. More precisely, homothetic transformations preserve convex combination. Hence, (b), (c) and (d) follow. □

From Lemma 16, a tetrahedron is transformed to another tetrahedron through a homothetic transformation, so homothety between two tetrahedra is well-defined.

It is easy to check that homothetic tetrahedra have parallel edges and faces:

Lemma 17. *(Edges and faces of homothetic tetrahedra) Two homothetic tetrahedra have parallel corresponding edges and faces.*

Proof. Let T be as in (27) or (28), $\Delta = [V_0, V_1, V_2, V_3]$ and $\Delta' = [V_0', V_1', V_2', V_3'] = T(\Delta) = \Delta'$. Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$, then $e_{i,j}' = \mathbf{v}_i' - \mathbf{v}_j' = T(\mathbf{v}_i) - T(\mathbf{v}_j) =$

$t\mathbf{v}_i - t\mathbf{v}_j = t\mathbf{e}_{i,j}$, showing that $E_{i,j}||E'_{i,j}$. Thus, corresponding edges of Δ and Δ' are parallel.

If $\mathbf{n}_k$ is a normal vector of the face F_k, then $\mathbf{n}_k \cdot \mathbf{e}_{i,j} = 0$, and then $\mathbf{n}_k \cdot \mathbf{e}'_{i,j} = \mathbf{n}_k \cdot t\mathbf{e}_{i,j} = t\mathbf{n}_k \cdot \mathbf{e}_{i,j} = 0$, showing that $\mathbf{n}_k$ is also a normal vector of the face F'_k, and $F_k||F'_k$. Thus, corresponding faces of Δ and Δ' are parallel. □

How about the converse of Lemma 17? The next two lemmas will prove it:

Lemma 18. *(Tetrahedra with parallel corresponding faces) Two tetrahedra with parallel corresponding faces have parallel corresponding edges.*

Proof. Consider $\Delta = [V_0, V_1, V_2, V_3]$ and $\Delta' = [V'_0, V'_1, V'_2, V'_3]$ where $F_i||F'_i$ for $i = 0, 1, 2, 3$. Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$, and consider the direction vectors $e_{i,j}$ and $e'_{i,j}$. If n_k and n_l are the respective common normal vectors of F_k and F'_k and of F_l and F'_l, then we have

$$\mathbf{n}_k \cdot \mathbf{e}_{i,j} = 0$$
$$\mathbf{n}_i \cdot \mathbf{e}_{i,j} = 0$$

and

$$\mathbf{n}_k \cdot \mathbf{e}'_{i,j} = 0$$
$$\mathbf{n}_i \cdot \mathbf{e}'_{i,j} = 0$$

as $E_{i,j}$ and $E'_{i,j}$ are the respective intersections of F_k and F_l and of F'_k and F'_l. This means that both $e_{i,j}$ and $e'_{i,j}$ are solutions to the system of linear equations

$$\begin{pmatrix}\mathbf{n}_k & \mathbf{n}_l\end{pmatrix}^T x = 0 \tag{29}$$

But $\mathbf{n}_k$ and $\mathbf{n}_l$ have to be linearly independent, because otherwise F_k and F_l would be parallel, therefore, the solution space of (29) is one-dimensional, i.e. $e_{i,j}||e'_{i,j}$. Hence, corresponding edges of Δ and Δ' are parallel. □

Lemma 19. *(Tetrahedra with parallel edges) Two tetrahedra with parallel corresponding edges are homothetic.*

Proof. Consider $\Delta = [V_0, V_1, V_2, V_3]$ and $\Delta' = [V'_0, V'_1, V'_2, V'_3]$ where $E_{i,j}||E'_{i,j}$ for $i, j = 0, 1, 2, 3$ and $i \neq j$. Let $\mathbf{e}'_{0,1} = t\mathbf{e}_{0,1}$, where $t \neq 0$.

If $t \neq 1$, then from

$$\mathbf{e}'_{0,1} = t\mathbf{e}_{0,1}$$
$$\mathbf{v}'_0 - \mathbf{v}'_1 = t(\mathbf{v}_0 - \mathbf{v}_1)$$
$$\frac{1}{1-t}\mathbf{v}'_0 + \frac{-t}{1-t}\mathbf{v}_0 = \frac{1}{1-t}\mathbf{v}'_1 + \frac{-t}{1-t}\mathbf{v}_1$$

we see that the lines V_0V_0' and V_1V_1' intersect at $Z/\mathbf{z}$

$$\mathbf{z} := \frac{1}{1-t}\mathbf{v}_0' + \frac{-t}{1-t}\mathbf{v}_0 = \frac{1}{1-t}\mathbf{v}_1' + \frac{-t}{1-t}\mathbf{v}_1 \tag{30}$$

We will show that the required homothetic transformation is

$$T(\mathbf{u}) = \mathbf{z} + t(\mathbf{u}-\mathbf{z}) \quad \text{for } \mathbf{u} \in \mathbb{R}^3$$

Indeed, by Lemma 33, it only suffices to show that it satisfies $T(\mathbf{v}_i) = \mathbf{v}_i'$ for $i = 0, 1, 2, 3$. To this end, let $i = 0, 1$ first, then from (30),

$$\mathbf{z} = \frac{1}{1-t}\mathbf{v}_i' + \frac{-t}{1-t}\mathbf{v}_i$$

$$\mathbf{v}_i' = (1-t)\mathbf{z} + t\mathbf{v}_i = T(\mathbf{v}_i) \tag{31}$$

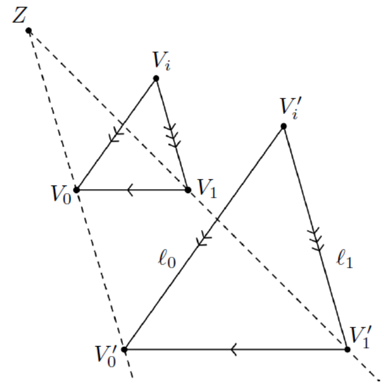

FIGURE 21. Lemma 19

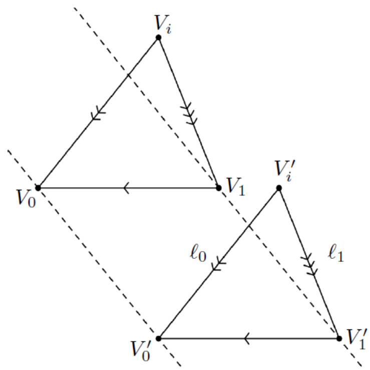

FIGURE 22. Lemma 19

For $i = 2, 3$, consider the line

$$l_0 : \mathbf{x}_0(r) = \mathbf{v}_0' + r(\mathbf{v}_0 - \mathbf{v}_i) \quad \text{for } r \in \mathbb{R} \tag{32}$$

through V_0' and parallel to $[V_0, V_i]$, and the line

$$l_1 : \mathbf{x}_1(s) = \mathbf{v}_1' + s(\mathbf{v}_1 - \mathbf{v}_0) \quad \text{for } s \in \mathbb{R} \tag{33}$$

through V_1' and parallel to $[V_1, V_i]$. Refer to Figure 21. Since $E_{0,i}||E_{0,i}'$ and $E_{1,i}||E_{1,i}'$, l_0 and l_1 intersect at V_i'. Setting $\mathbf{x}_0(r) = x_1(s)$, with (31), we have

$$(1-t)\mathbf{z} + t\mathbf{v}_0 + r(\mathbf{v}_0 - \mathbf{v}_1) = (1-t)\mathbf{z} + t\mathbf{v}_1 + s(\mathbf{v}_1 - \mathbf{v}_i)$$

$$(t+r)(\mathbf{v}_0 - \mathbf{v}_i) = (t+s)(\mathbf{v}_1 - \mathbf{v}_i)$$

But $\{\mathbf{v}_0 - \mathbf{v}_i, \mathbf{v}_1 - \mathbf{v}_i\}$ are linearly independent, so $t + r = t + s = 0$ or $r = s = -t$. Substituting into (32) and using (31), we have

$$\mathbf{v}_i' = (1-t)\mathbf{z} + t\mathbf{v}_0 - t(\mathbf{v}_0 - \mathbf{v}_i) = T(\mathbf{v}_i),$$

and we are done.

If $t = 1$, then from

$$\mathbf{e}'_{0,1} = \mathbf{e}_{0,1}$$
$$\mathbf{v}'_0 - \mathbf{v}_0 = \mathbf{v}'_1 - \mathbf{v}_1,$$

we can construct a homothetic (actually translation) transformation

$$T(\mathbf{u}) = \mathbf{u} + \mathbf{b} \quad \text{for } \mathbf{u} \in \mathbb{R}^3,$$

where

$$\mathbf{b} := \mathbf{v}'_0 - \mathbf{v}_0 = \mathbf{v}'_1 - \mathbf{v}_1, \tag{34}$$

and show that this is the required one. To this end, let $i = 0, 1$ first, then from (34),

$$\mathbf{b} = \mathbf{v}'_i - \mathbf{v}_i$$
$$\mathbf{v}'_i = \mathbf{v}_i + \mathbf{b} = T(\mathbf{v}_i) \tag{35}$$

For $i = 2, 3$, consider the lines (32) and (33). Refer to Figure 22. Again, since $E_{0,i}||E'_{0,i}$ and $E_{1,i}||E'_{1,i}$, l_0 and l_1 intersect at V'_i. Setting $\mathbf{x}_0(r) = \mathbf{x}_1(s)$, with (31), we have

$$\mathbf{v}_0 + \mathbf{b} + r(\mathbf{v}_0 - \mathbf{v}_i) = \mathbf{v}_1 + \mathbf{b} + s(\mathbf{v}_1 - \mathbf{v}_i)$$
$$(1 + r)(\mathbf{v}_0 - \mathbf{v}_i) = (1 + s)(\mathbf{v}_1 - \mathbf{v}_i),$$

But $\{\mathbf{v}_0 - \mathbf{v}_i, \mathbf{v}_1 - \mathbf{v}_i\}$ are linearly independent, so $1+r = 1+s = 0$ or $r = s = -1$. Substituting into (32) and using (35), we have

$$\mathbf{v}'_i = \mathbf{v}_0 + \mathbf{b} - (\mathbf{v}_0 - \mathbf{v}_i) = T(\mathbf{v}_i)$$

and we are done. □

Be aware that Lemma 19 does not work for polyhedra other than tetrahedra as the above proof relies on that in a tetrahedron any two vertices are connected by an edge. One counterexample is triangular prism in which changing the height produces parallel corresponding faces and edges but does not produce a homothetic triangular prism.

Combining Lemma 17, 18 and 19, we obtain:

Theorem 20. *(Necessary and sufficient condition for homothetic tetrahedra) Two tetrahedra are homothetic if and only if they have parallel corresponding faces.*

4.2. Circumcentre of Antimedial Tetrahedron

Recall from Proposition 4 that circumcentre must uniquely exist in any tetrahedron. For the sake of brevity, from this point on, we assume, without any loss of generality, that the circumcentre $\mathcal{O}$ of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ is the origin. Thus, in particular,

$$||v_0|| = ||v_1|| = ||v_2|| = ||v_3|| = R,$$

where R is the circumradius of Δ.

Definition 21. *(Antimedial tetrahedron and antimedial circumcentre) The antimedial tetrahedron $\Delta_{am} = [V_0', V_1', V_2', V_3']$ of a tetrehedron $\Delta = [V_0, V_1, V_2, V_3]$ is the tetrahedron whose faces F_0', F_1', F_2', F_3' satisfy*

$$V_i \in F_i' \quad \textit{and} \quad F_i' || F_i \quad \textit{for } i = 0, 1, 2, 3.$$

The antimedial circumcentre $\mathcal{J}/j$ of a tetrahedron Δ is the circumcentre of its antimedial tetrahedron Δ_{am}, i.e.

$$\mathcal{J}(\Delta) = O(\Delta_{am}).$$

We now prove the important properties, one algebraic and one geometric, of orthocentre of triangles that are carried over through generalizing as antimedial circumcentre of tetrahedra. See Figure 23 for an illustration.

Theorem 22. *(Antimedial circumcentre) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. If its circumcentre $\mathcal{O}$ is the origin, then its antimedial circumcentre $\mathcal{J}$ is given by*

$$\mathbf{j} = 4\mathbf{g} = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{g}_0 + \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3 \tag{36}$$

where $\mathcal{G}_i$ denotes the centroid of the face F_i.

Moreover, $\mathcal{J}$ lies on the Euler line $\mathcal{E}$ of Δ.

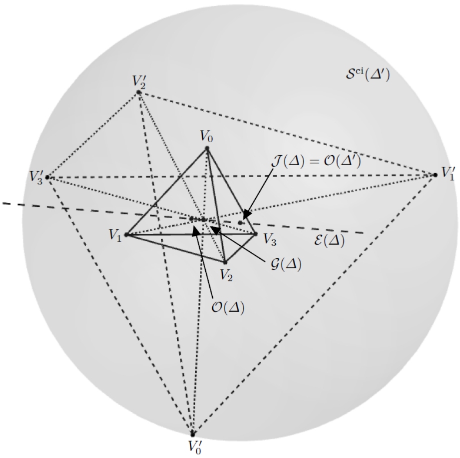

FIGURE 23. Theorem 22

Proof. Let $V_i'/\mathbf{v}_i'$ be a point such that

$$\mathbf{g} = \frac{3}{4}\mathbf{v}_i + \frac{1}{4}\mathbf{v}_i' \tag{37}$$

i.e. the centroid $\mathcal{G}$ of Δ divides $[V_i, V_i']$ internally in $V_i\mathcal{G} : \mathcal{G}V_i' = 1 : 3$, or

$$\mathbf{v}_i' = 4\mathbf{g} - 3\mathbf{v}_i = j - 3\mathbf{v}_i.$$

Recalling that $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are affinely independent, then by noting that $\mathbf{v}_i' - \mathbf{v}_j' = -3(\mathbf{v}_i - \mathbf{v}_j)$, so are $\{\mathbf{v}_0', \mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3'\}$. We shall show that

$$\Delta' := [V_0', V_1', V_2', V_3'] = \Delta_{am}$$

Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Note that

$$\begin{aligned}\mathbf{v}_j' + \mathbf{v}_k' + \mathbf{v}_l' &= 12\mathbf{g} - 3(\mathbf{v}_j + \mathbf{v}_k + \mathbf{v}_l)\\ &= 12\mathbf{g} - 3(4\mathbf{g} - \mathbf{v}_i)\\ \mathbf{v}_i &= \frac{1}{3}(\mathbf{v}_j' + \mathbf{v}_k' + \mathbf{v}_l')\end{aligned}$$

which shows that V_i is the centroid $\mathcal{G}(F_i')$ of F_i' and lies on F_i'. Moreover, if $\mathbf{n}_i$ is a normal vector of F_i, i.e. $\mathbf{n}_i \cdot (\mathbf{v}_j - \mathbf{v}_k) = 0$, then

$$\mathbf{n}_i \cdot (\mathbf{v}_j' - \mathbf{v}_k') = -3\mathbf{n}_i \cdot (\mathbf{v}_j - \mathbf{v}_k) = 0,$$

i.e. $\mathbf{n}_i$ is also a normal vector of F_i', so $F_i || F_i'$. Hence $\Delta' = \Delta_{am}$.

Checking that

$$||\mathbf{j} - \mathbf{v}_i'|| = ||3\mathbf{v}_i|| = 3R,$$

where R is the circumradius of Δ, so (36) provides $J(\Delta) = O(\Delta_{am})$.

Moreover, (37) also means that all $[V_i, W_i]$ concur at $\mathcal{G}$, so according to the proof of Lemma 19, $\mathcal{G} = Z(\Delta, \Delta_{am})$. Therefore, $\mathcal{J}(\Delta) = O(\Delta_{am})$ lies on the line joining $\mathcal{O}(\Delta)$ and $\mathcal{G}$, i.e. the Euler line $\mathcal{E}$ of Δ. □

Note that if the circumcentre $\mathcal{O}$ is not assumed to be the origin, then (36) should be modified as

$$\begin{aligned}\mathbf{j} - \mathbf{o} = 4(\mathbf{g} - \mathbf{o}) &= (\mathbf{v}_0 - \mathbf{o}) + (\mathbf{v}_1 - \mathbf{o}) + (\mathbf{v}_2 - \mathbf{o}) + (\mathbf{v}_3 - \mathbf{o})\\ &= (\mathbf{g}_0 - \mathbf{o}) + (\mathbf{g}_1 - \mathbf{o}) + (\mathbf{g}_2 - \mathbf{o}) + (\mathbf{g}_3 - \mathbf{o})\\ \mathbf{j} = 4\mathbf{g} - 3\mathbf{o} &= \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 - 3\mathbf{o} = \mathbf{g}_0 + \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3 - 3\mathbf{o}\end{aligned} \tag{38}$$

4.3. Inexcentre of Orthic Tetrahedron

Again, for the sake of brevity, the circumcentre $\mathcal{O}$ of a tetrahedron

$$\Delta = [V_0, V_1, V_2, V_3]$$

will be assumed to be at the origin without any loss of generality. Thus, in particular,

$$||\mathbf{v}_0|| = ||\mathbf{v}_1|| = ||\mathbf{v}_2|| = ||\mathbf{v}_3|| = R,$$

where R is the circumradius of Δ.

Figures 24 (when ΔABC is acute-angled) and 25 (when ΔABC is obtuse-angled) recall how an edge of an orthic triangle may be obtained by a semi-circle. This is the underlying idea of our construction of orthic tetrahedron.

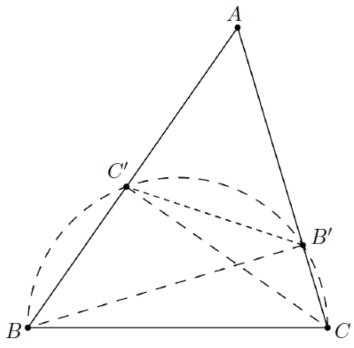

FIGURE 24. Edge of orthic triangle

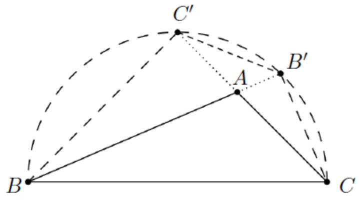

FIGURE 25. Edge of orthic triangle

It is known that the orthic triangle and the tangential triangle of a triangle have parallel edges, and in Lemma 25, we show, both analytically and synthetically, that tangential planes in Definition 23 and the orthic planes in Definition 24 of a tetrahedron are parallel. We will then define orthic tetrahedron in Definition 26 and orthic inexcentre in Definition 27, and prove some properties due to the homothety between the orthic and tangential tetrahedra. Theorem 29 will show a vector representation of the orthic inexcentre the which is analogous to that of the antimedial circumcentre in (36).

Definition 23. *(Tangent plane and tangential tetrahedron) The tangent plane T_i of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ at V_i is the plane tangent to the circumsphere S^{ci} of Δ at V_i. Its equation is given by*

$$\mathbf{v}_i \cdot (\mathbf{x} - \mathbf{v}_i) = 0 \quad or \quad \mathbf{v}_i \cdot \mathbf{x} = R^2 \tag{39}$$

where R is the circumradius of Δ. See Figure 26 for an illustration.

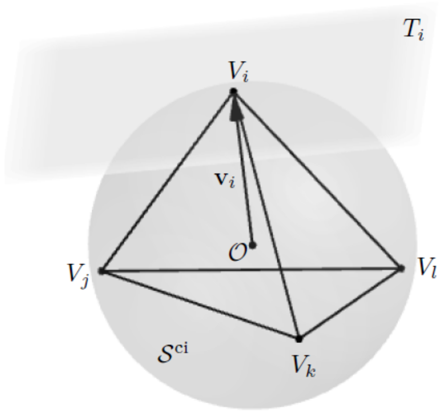

FIGURE 26. Definition 23

The tangential tetrahedron Δ_{tg} of Δ is the tetrahedron enclosed by its four tangential planes. Note that the position of the circumcentre $\mathcal{O}$ of Δ determines one of the following configurations:

(i) *Acute-angled case: If $\mathcal{O}(\Delta)$ lies inside Δ, then Δ_{ig} touches Δ at all the vertices of Δ, and*

$$\mathcal{O}(\Delta) = \mathcal{T}(\Delta_{tg})$$

as shown in Figure 27.

(ii) *Obtuse-angled case: If $\mathcal{O}(\Delta)$ lies outside Δ and not on any of (the planes containining) a face of Δ, then Δ_{tg} touches Δ only at the vertex V_i opposite to $\mathcal{O}(\Delta)$, and*

$$\mathcal{O}(\Delta) = \mathcal{I}_i(\Delta_{tg})$$

as shown in Figure 28. The vertex V_i is called the obtuse vertex of Δ.

(iii) *Right-angled case: If $\mathcal{O}(\Delta)$ lies on (the plane containining) a face of Δ, then Δ_{tg} cannot be formed as shown in Figure 29, and tangential tetrahedron would be undefined.*

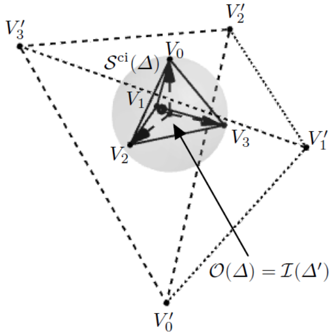

FIGURE 27. Definition 23

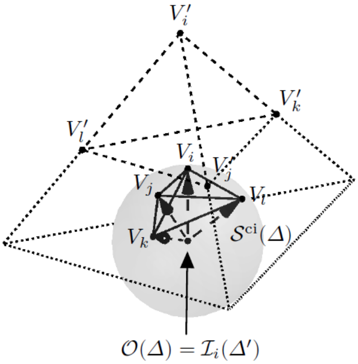

FIGURE 28. Definition 23

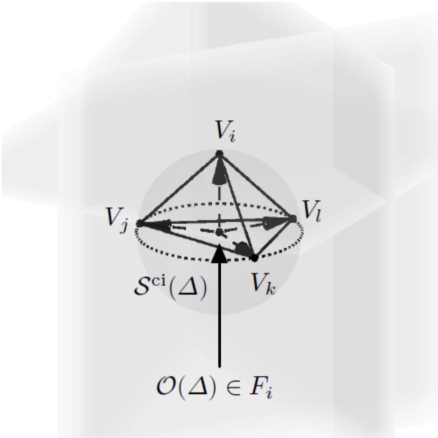

FIGURE 29. Definition 23

Beware the terminologies *acute-angled*, *obtuse-angled* and *right-angled*, which classify triangles by angle size, may not mean anything about angle size in a tetrahedron. They are borrowed from the two-dimensional scenario only to mean those in Definition 23.

Definition 24. *(Orthic plane) Consider a tetrahedron* $\Delta = [V_0, V_1, V_2, V_3]$. *Let* $\{i, j, k, l\} = \{0, 1, 2, 3\}$. *Construct a sphere* $\mathcal{S}_i$ *with centre* $\mathcal{O}_i$ *and radius* $R_i = \mathcal{O}_i V_j = \mathcal{O}_i V_k = \mathcal{O}_i V_l$, *where* $\mathcal{O}_i$ *is the circumcentre of face* F_i. *Denote by* W_j^i *the reflection of* V_j *across the projection of* O_i *onto the (extended) edge* $E_{i,j}$. *Note that* W_j^i *is simply the intersection of the sphere* $\mathcal{S}_i$ *and the (extended) edge* $E_{i,j}$ *other* *than V_j, unless $E_{i,j} \perp [V_j, O_i]$ in which $W^i_j = V_j$. Likewise, the points W^i_k and W^i_l are defined.*

The orthic plane U_i of Δ (corresponding to V_i) is the plane containing W^i_j, W^i_k and W^i_l. See Figures 30 (acute-angled case) and 31 (obtuse-angled case) for illustrations. Note that orthic plane is undefined if $\mathcal{O}_i$ coincides with the circumcentre $\mathcal{O}$ of Δ as $W^i_j = W^i_k = W^i_l$ (right-angled case).

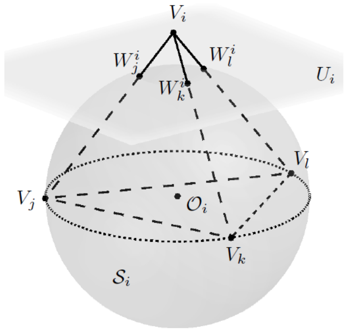

FIGURE 30. Definition 24

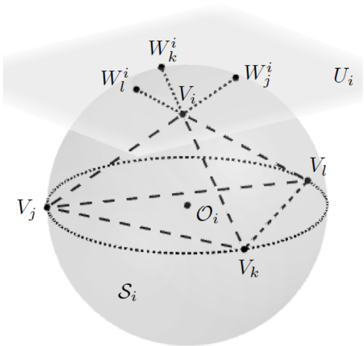

FIGURE 31. Definition 24

Lemma 25. *(Orthic plane parallel to tangent plane)*
For a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$, the tangential plane T_i is parallel to the orthic plane U_i.

Proof. (Analytic approach) Refer to Figure 32. Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Since $\mathbf{v}_i \perp T_i$ according to Definition 23, the statement is equivalent to proving that $\mathbf{v}_i \perp U_i$, then one needs to find out why

$$(\mathbf{w}^i_j - \mathbf{w}^i_k) \cdot \mathbf{v}_i = 0 \quad \text{and} \quad (\mathbf{w}^i_j - \mathbf{w}^i_l) \cdot \mathbf{v}_i = 0 \tag{40}$$

To this end, we shall compute

$$\mathbf{w}^i_j \cdot \mathbf{v}_i \cdot \mathbf{w}^i_k \cdot \mathbf{v}_i \quad \text{and} \quad \mathbf{w}^i_l \cdot \mathbf{v}_i \tag{41}$$

Recall that the circumradius R of Δ satisfies $R = ||\mathbf{v}_i|| = ||\mathbf{v}_j|| = ||\mathbf{v}_k|| = ||\mathbf{v}_l||$. Also recall that the circumradius R_i of F_i satisfies

$$R_i = ||\mathbf{v}_j - \mathbf{o}_i|| = ||\mathbf{v}_k - \mathbf{o}_i|| = ||\mathbf{v}_l - \mathbf{o}_i||, \tag{42}$$

from which we get

$$\begin{aligned} R_i^2 &= ||\mathbf{v}_j||^2 - 2\mathbf{v}_j \cdot \mathbf{o}_i + ||\mathbf{o}_i||^2 \\ &= R^2 - 2\mathbf{v}_j \cdot \mathbf{o}_i + R^2 - R_i^2 \quad (\because \mathbf{o}_i \perp F_i \quad \text{from Proposition 4}) \end{aligned}$$

$$R^2 - \mathbf{v}_j \cdot \mathbf{o}_i = R_i^2 \tag{43}$$

Since $\mathbf{w}_j^i$ lies on the (extended) edge $E_{i,j}$, we let

$$\mathbf{w}_j^i = t\mathbf{v}_i + (1-t)\mathbf{v}_j \quad \text{for some } t \in \mathbb{R}\backslash\{0\} \tag{44}$$

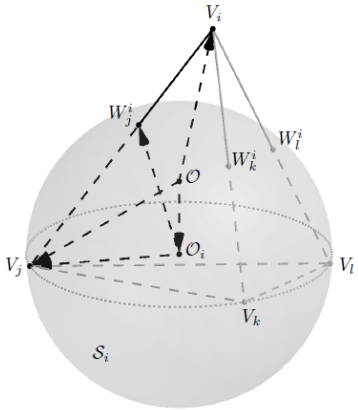

FIGURE 32. Lemma 25

By the definition of W_j^i,

$$\begin{aligned} ||\mathbf{w}_j^i - \mathbf{o}_i|| &= R_i \\ ||t(\mathbf{v}_i - \mathbf{v}_j) + (\mathbf{v}_j - \mathbf{o}_i)|| &= R_i \quad \text{(by (44))} \\ t^2||\mathbf{v}_i - \mathbf{v}_j||^2 + 2t(\mathbf{v}_i - \mathbf{v}_j)\cdot(\mathbf{v}_j - \mathbf{o}_i) + ||\mathbf{v}_j - \mathbf{o}_i||^2 &= R_i^2 \\ t^2||\mathbf{v}_i - \mathbf{v}_j||^2 + 2(\mathbf{v}_i - \mathbf{v}_j)\cdot(\mathbf{v}_j - \mathbf{o}_i) &= 0 \quad \text{(by (42) and } t \neq 0) \end{aligned}$$

$$\begin{aligned} t &= \frac{2(\mathbf{v}_j - \mathbf{v}_i)\cdot(\mathbf{v}_j - \mathbf{o}_i)}{||\mathbf{v}_i - \mathbf{v}_j||^2} \\ &= \frac{2(R^2 - \mathbf{v}_j\cdot\mathbf{o}_i - \mathbf{v}_i\cdot\mathbf{v}_j + \mathbf{v}_i\cdot\mathbf{o}_i)}{2R^2 - 2\mathbf{v}_i\cdot\mathbf{v}_j} \\ &= \frac{R_i^2 - \mathbf{v}_i\cdot\mathbf{v}_j + \mathbf{v}_i\cdot\mathbf{o}_i}{R^2 - \mathbf{v}_i\cdot\mathbf{v}_j} \quad \text{(by (43))} \end{aligned}$$

Substituting into (44),

$$
\begin{aligned}
\mathbf{w}^i_j &= \frac{R_i^2 - \mathbf{v}_i \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{o}_i}{R^2 - \mathbf{v}_i \cdot \mathbf{v}_j}\mathbf{v}_i + \frac{R^2 - R_i^2 - \mathbf{v}_i \cdot \mathbf{o}_i}{R^2 - \mathbf{v}_i \cdot \mathbf{v}_j}\mathbf{v}_j \\
\mathbf{w}^i_j \cdot \mathbf{v}_i &= \frac{R_i^2 - \mathbf{v}_i \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{o}_i}{R^2 - \mathbf{v}_i \cdot \mathbf{v}_j}R^2 + \frac{R^2 - R_i^2 - \mathbf{v}_i \cdot \mathbf{o}_i}{R^2 - \mathbf{v}_i \cdot \mathbf{v}_j}\mathbf{v}_i \cdot \mathbf{v}_j \\
&= \frac{R^2 R_i^2 + R^2 \mathbf{v}_i \cdot \mathbf{o}_i - R_i^2 \mathbf{v}_i \cdot \mathbf{v}_j - (\mathbf{v}_i \cdot \mathbf{o}_i)\mathbf{v}_i \cdot \mathbf{v}_j}{R^2 - \mathbf{v}_i \cdot \mathbf{v}_j} \\
&= \frac{(R_i^2 + \mathbf{v}_i \cdot \mathbf{o}_i)(R^2 - \mathbf{v}_i \cdot \mathbf{v}_j)}{R^2 - \mathbf{v}_i \cdot \mathbf{v}_j} \\
&= R_i^2 + \mathbf{v}_i \cdot \mathbf{o}_i
\end{aligned} \tag{45}
$$

Likewise, we have $\mathbf{w}^i_k = \mathbf{w}^i_l = R_i^2 + \mathbf{v}_i \cdot \mathbf{o}_i$ in (41), and then (40) is established.

Finally, since the vectors $\mathbf{w}^i_j - \mathbf{w}^i_k \nparallel \mathbf{w}^i_j - \mathbf{w}^i_l$ as $F_k \nparallel F_l$, by (40), we can conclude that $\mathbf{v}_i \perp U_i$, and hence $T_i \parallel U_i$. $\square$

Actually, looking back at (45), we can gain more insight about the situation:

$$
\begin{aligned}
\mathbf{w}^i_j \cdot \mathbf{v}_i &= R_i^2 + \mathbf{v}_i \cdot \mathbf{o}_i \\
(\mathbf{w}^i_j - \mathbf{o}_i) \cdot \mathbf{v}_i &= R_i^2 \\
&= (\mathbf{w}^i_j - \mathbf{o}_i) \cdot (\mathbf{w}^i_j - \mathbf{o}_i) \\
(\mathbf{w}^i_j - \mathbf{o}_i) \cdot (\mathbf{v}_i + \mathbf{o}_i - \mathbf{w}^i_j) &= 0
\end{aligned}
$$

which means that the vector from $\mathbf{w}^i_j$ to $\mathbf{v}_i + \mathbf{o}_i$ is perpendicular to $\mathbf{w}^i_j - \mathbf{o}_i$. This simple geometric interpretation of (45) motivates us to seek an elementary synthetic proof:

Proof. (Synthetic approach, rectilinear geometry) Refer to Figures 33, 34, 35, 36, 37 and 38.

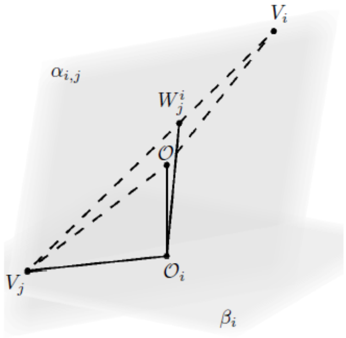

FIGURE 33. Lemma 25

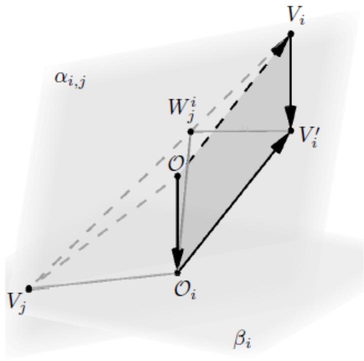

FIGURE 34. Lemma 25

In Figure 33, $\alpha_{i,j}$ denotes the plane passing through V_i, V_j and $\mathcal{O}$, while β_i denotes the plane containing F_i. The dashed segments lie on $\alpha_{i,j}$, while the solid segments lie off $\alpha_{i,j}$.

In Figure 34, V_i is translated by the vector $\mathbf{o}_i$ to V_i', forming a parallelogram $\mathcal{O}V_iV_i'\mathcal{O}_i$, and

$$V_i'\mathcal{O}_i \parallel \alpha_{i,j}, V_i\mathcal{O} \parallel V_i'\mathcal{O}_i \quad \text{and} \quad V_i\mathcal{O} = V_i'\mathcal{O}_i \tag{46}$$

In Figure 35, $V_i'\mathcal{O}$ is projected onto $\alpha_{i,j}$ to $V_i''\mathcal{O}_i'$, forming a rectangle $\mathcal{O}_iV_i'V_i''\mathcal{O}_i'$, and

$$\mathcal{O}_i\mathcal{O}_i' \perp \alpha_{i,j}, V_i'V_i'' \perp \alpha_{i,j}, \mathcal{O}_i\mathcal{O}_i' = V_i'V_i'', V_i'\mathcal{O}_i||V_i''\mathcal{O}_i' \quad \text{and} \quad V_i'\mathcal{O}_i = V_i''\mathcal{O}_i' \tag{47}$$

Also note that $\Delta V_j\mathcal{O}_i\mathcal{O}_i' \cong \Delta W_j^i\mathcal{O}_i\mathcal{O}_i'$, both $\perp_{\alpha_{i,j}}$, as $V_j\mathcal{O}_i = W_j^i\mathcal{O}_i$. As a result,

$$V_j\mathcal{O}_i' = W_j^i\mathcal{O}_i' \tag{48}$$

Moreover, we can prove that

$$\Delta V_j\mathcal{O}_i'\mathcal{O} \cong \Delta\mathcal{O}_i'W_j^iV_i'' \tag{49}$$

(the shaded triangles), as the next paragraph will explain. Figure 36 shows $\alpha_{i,j}$. First of all, $V_j\mathcal{O}_i' = W_j^i\mathcal{O}_i'$ (marked by '=') from (48), so that

$$\angle W_j^iV_j\mathcal{O}_i' = \angle V_jW_j^i\mathcal{O}_i' \quad \text{(marked by } \measuredangle \text{ with ' = ')} \tag{50}$$

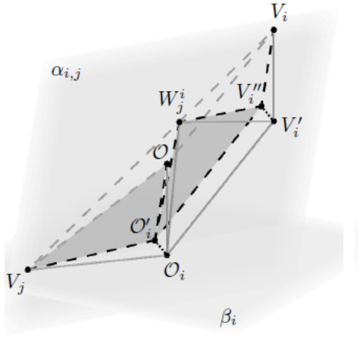

FIGURE 35. Lemma 25

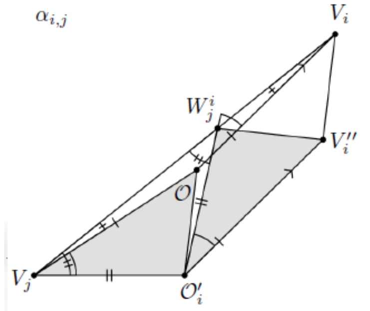

FIGURE 36. Lemma 25

Next,

$$\begin{aligned} V_j\mathcal{O} &= V_i\mathcal{O} && (51)\\ &= V_i'\mathcal{O}_i \quad \text{(by (46))} \\ &= V_i''\mathcal{O}_i' \quad \text{(by (47))} && (52)\end{aligned}$$

and from (51) again, we have

$$\angle V_iV_j\mathcal{O} = \angle V_jV_i\mathcal{O} \quad \text{(marked by } \measuredangle \text{ with ‘ – ‘)} \tag{53}$$

Since $V_i\mathcal{O} \parallel V_i'\mathcal{O}_i \parallel V_i''\mathcal{O}_i'$ by (46) and (47), we have

$$\begin{aligned} \angle W_j^i\mathcal{O}_i'V_i'' &= \angle W_j^i\chi V_i, \text{where } \chi \text{ is the intersection of } W_j^i\mathcal{O}_i' \text{ and } V_i\mathcal{O} \\ &= \angle V_jW_j^i\mathcal{O}_i' - \angle V_jV_i\mathcal{O} \\ &= \angle W_j^iV_j\mathcal{O}_i' - \angle V_iV_j\mathcal{O} \quad \text{(by (\textbf{??}) and (53))} \\ &= \angle \mathcal{O}V_j\mathcal{O}_i' && (54)\end{aligned}$$

Hence, (49) follows from (48), (52) and (54).

Back to Figure 35. From (49), we have $\mathcal{O}\mathcal{O}_i' = W_j^iV_i''$. Plus $\mathcal{O}_i\mathcal{O}_i' \perp \alpha_{i,j}$ and $V_i'V_i'' \perp \alpha_{i,j}$ from (47), we can then prove that

$$\Delta V_j\mathcal{O}_i\mathcal{O} \cong \Delta\mathcal{O}_iW_j^iV_i' \tag{55}$$

Note that as $\mathcal{O}\mathcal{O}_i \perp V_j$, (55) will imply that $W_j^i\mathcal{O}_i \perp W_j^iV`_i$ — the simple geometric interpretation of (45) which has motivated the present proof. But now we have proved something stronger. Similar to (55), we will also have

$$\Delta V_k\mathcal{O}_i\mathcal{O} \cong \Delta\mathcal{O}_iW_k^iV_i' \quad \text{and} \quad \Delta V_l\mathcal{O}_i\mathcal{O} \cong \Delta\mathcal{O}_iW_l^iV_i' \tag{56}$$

In Figure 37, the three shaded triangles $\Delta V_j\mathcal{O}_i\mathcal{O}, \Delta V_k\mathcal{O}_i\mathcal{O}$ and $\Delta V_l\mathcal{O}_i\mathcal{O}$ are congruent as $\mathcal{O}\mathcal{O}_i \perp F_i$ and $\mathcal{O}$ is equidistant to V_j, V_k, V_l. Therefore, by (55) and (56),

$$\Delta\mathcal{O}_iW^i_jV'_i \cong \Delta\mathcal{O}_iW^i_kV'_i \cong \Delta\mathcal{O}_iW^i_lV'_i \tag{57}$$

as shown as the shaded triangles in Figure 38. Because of (57), W^i_j, W^i_k and W^i_l will project onto the line $V'_i\mathcal{O}_i$ to the same point P as shown. This shows that the orthic plane U_i, i.e. the plane containing W^i_j, W^i_k and W^i_l, intersects the line $V'_i\mathcal{O}_i$ perpendicularly at P.

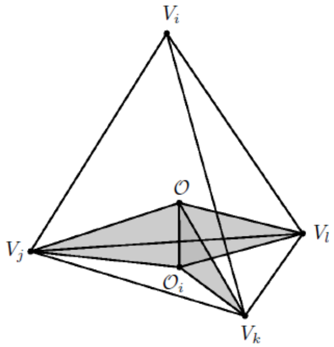

FIGURE 37. Lemma 25

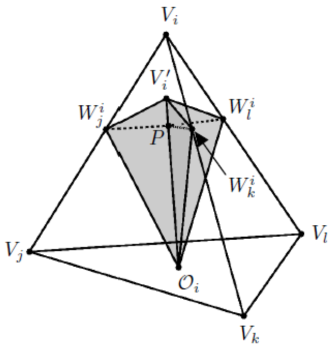

FIGURE 38. Lemma 25

Finally, recalling from (46) that $V'_i\mathcal{O}_i||V_i\mathcal{O}$, we have actually proved that the orthic plane U_i is parallel to the tangent plane T_i. □

We have just proved more than what Lemma 25 has claimed. However, the final conclusion that U_i intersects the line $V'_i\mathcal{O}_i$ perpendicularly at P will play a crucial role when proving Theorem 28. If only the parallelism betweeen T_i and U_i is concerned, Lemma 25 actually admits a more elegant synthetic proof, through some slightly higher geometry:

Proof. (Synthetic approach, circle geometry) Refer to Figure 39, 40, 41, 42 and 43.

In Figure 39, π_l is the plane containing the face F_l. Its intersection with the sphere $\mathcal{S}_i$ (centred at $\mathcal{O}_i$ with radius $V_j\mathcal{O}_i$) is the circle denoted by $C_{i,l}$. Then, V_j, V_k, W^i_j and W^i_k are concyclic on $C_{i,l}$, so that $\alpha = \beta$.

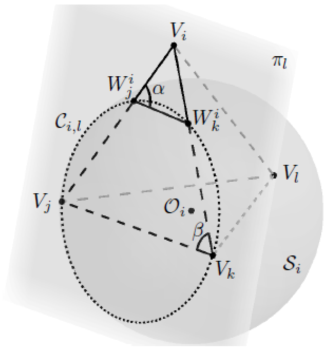

FIGURE 39. Lemma 25

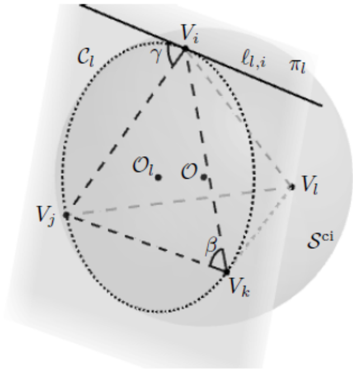

FIGURE 40. Lemma 25

In Figure 40, C_l denotes the circle of intersection of the plane π_l and the circumsphere S^{ci} of Δ. The tangent line to C_l at V_i on π_l is denoted by $l_{l,i}$. By the tangency, $\beta = \gamma$. As a result, $\alpha = \gamma$.

In Figure 41, recall from Proposition 4 that $\mathcal{O}\mathcal{O}_l \perp \pi_l$. But as $l_{l,i}$ is tangent to C_l on π_l, we also have $l_{l,i} \perp V_i\mathcal{O}_l$. As a result, $l_{1,i} \perp V_i\mathcal{O}$.

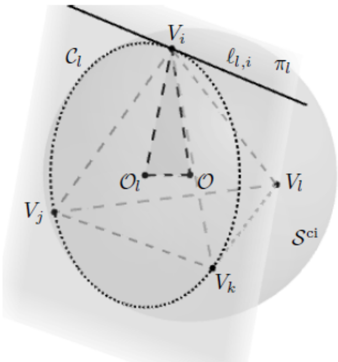

FIGURE 41. Lemma 25

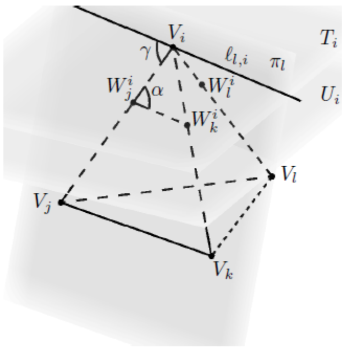

FIGURE 42. Lemma 25

In Figure 42, since $l_{l,i} \perp V_i\mathcal{O}$ as we have just proved, $l_{l,i}$ is indeed tangent to the circumsphere S^{ci} of Δ, so $l_{l,i}$ actually resides on the tangential plane T_i of Δ. Then by $\alpha = \gamma$, we have $l_{l,i} \parallel W^i_j W^i_k$, which is a pair of parallel lines lying on T_i and U_i respectively.

In Figure 43, we run the same argument for the plane π_j containing the face F_j to get $l_{j,i} \parallel W_k^i W_l^i$, which is another pair of parallel lines lying on T_i and U_i respectively, and we can conclude that $T_i \parallel U_i$. $\square$

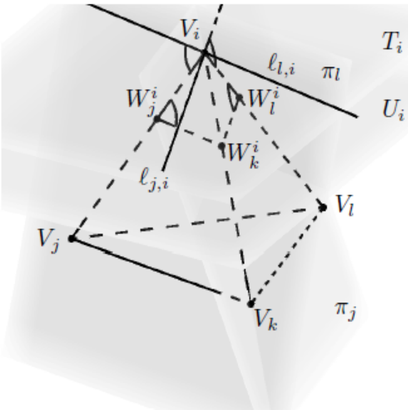

FIGURE 43. Lemma 25

As a consequence of Lemma 25, we can define:

Definition 26. *(Orthic tetrahedron) The orthic tetrahedron Δ_{ot} of a tetrehadron Δ is the tetrahedron enclosed by its four orthic planes. Similar to tangential tetrahedron in Definition 23, it is well-defined only if Δ is acute-angled or obtuse-angled, and is undefined if Δ is right-angled. See Figures 44 (acute-angled case) and 45 (obtuse-angled case) for illustrations.*

By Definition 26, Lemma 25 and Theorem 20, Δ_{tg} and Δ_{ot} are homothetic, and we define the following:

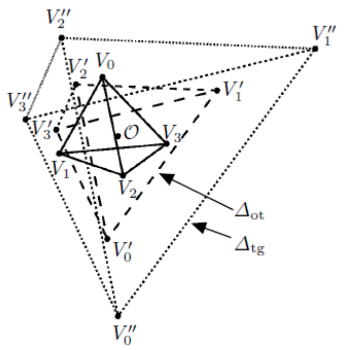

FIGURE 44. Definition 26

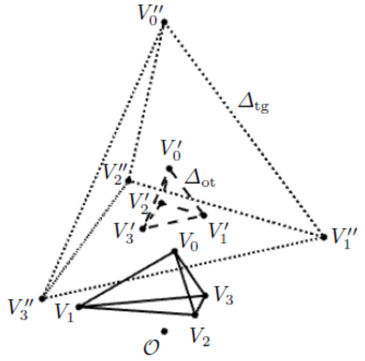

FIGURE 45. Definition 26

Definition 27. *(χ_{25} and orthic inexcentre) The point χ_{25} of a tetrahedron Δ is defined as the homothetic centre $\mathcal{Z}(\Delta_{tg}, \Delta_{ot})$ between its tangential tetrahedron Δ_{tg} and its orthic tetrahedron Δ_{ot}. This definition is completely analogous to that of χ_{25} of triangles* [15].

The orthic inexcentre K of Δ is defined as

1. *the incentre $T(\Delta_{ot})$ of Δ_{ot} if Δ is acute-angled,*
2. *the excentre $T_i(\Delta_{ot})$ of Δ_{ot} if Δ is obtuse-angled, where V_i is the obtuse vertex of Δ.*

See Figure 46 (*acute-angled case*) and 47 (*obtuse-angled case*) for illustrations.

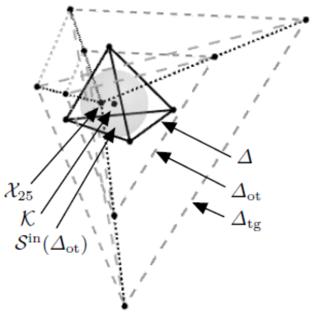

FIGURE 46. Definition 27

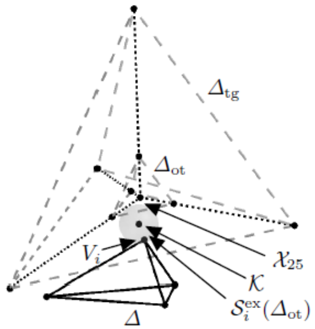

FIGURE 47. Definition 27

We now prove the important properties, one algebraic and one geometric, of orthocentre of triangles that are carried over through generalizing as orthic inexcentre of tetrahedra.

Theorem 28. *(Collinearity of orthic inexcentre, circumcentre and χ_{25}) The orthic inexcentre K, the circumcentre $\mathcal{O}$ and χ_{25} of a tetrahedron are collinear.*

Proof. It only suffices to show that $\mathcal{K}$ and $\mathcal{O}$ of a tetrahedron Δ are corresponding points under the homothety between Δ_{tg} and Δ_{ot}. Recalling from Definition 23 that

$$\mathcal{O}(\Delta) = \mathcal{I}(\Delta_{tg}) \quad \text{or} \quad \mathcal{O}(\Delta) = \mathcal{I}_i(\Delta_{tg})$$

respectively in the *acute-angled* or *obtuse-angled* cases, as well as from Definition 27 that

$$K(\Delta) = \mathcal{I}(\Delta_{ot}) \quad \text{or} \quad \mathcal{K}(\Delta) = \mathcal{I}_i(\Delta_{ot})$$

accordingly, we may indeed try to show that incentre and excentre are preserved under homothetic transformations.

In fact, if i satisfies (9) to become the incentre $\mathcal{I}(\Delta(W))$ of a tetrahedron $\Delta(W) := [W_0, W_1, W_2, W_3]$, where $\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are the inward normal vectors of $\Delta(W)$, then under any homothetic transformation T of the form (27) or (28), by Theorem 20, the inward normal vectors of $T(\Delta(W)) = [T(W_0), T(W_1), T(W_2), T(W_3)]$ will also be $\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, and

$$\mathbf{n}_i \cdot (T(\mathbf{i}) - T(\mathbf{p}_i)) = \mathbf{n}_i \cdot (t\mathbf{i} - t\mathbf{p}_i) = t\mathbf{n}_i \cdot (\mathbf{i} - \mathbf{p}_i)$$

As a result, $T(\mathbf{i})$ will satisfy

$$\mathbf{n}_0 \cdot (T(\mathbf{i}) - T(\mathbf{p}_0)) = \mathbf{n}_1 \cdot (T(\mathbf{i}) - T(\mathbf{p}_1)) = \mathbf{n}_2 \cdot (T(\mathbf{i}) - T(\mathbf{p}_2)) = \mathbf{n}_3 \cdot (T(\mathbf{i}) - T(\mathbf{p}_3))$$

to become the incentre $\mathcal{I}(T(\Delta(W)))$ of $T(\Delta(W))$. Therefore, incentre is preserved under homothetic transformations. Similarly, letting $\{i, j, k, l\} = \{0, 1, 2, 3\}$, if ii satisfies (13) to become the excentre $I_i(\Delta(W))$ of $\Delta(W)$, then $T(\mathbf{i})$ will satisfy

$$\mathbf{n}_i' \cdot (T(\mathbf{i}_i) - T(\mathbf{p}_0)) = \mathbf{n}_j \cdot (T(\mathbf{i}_i) - T(\mathbf{p}_j)) = \mathbf{n}_k \cdot (T(\mathbf{i}_i) - T(\mathbf{p}_k)) = \mathbf{n}_l \cdot (T(\mathbf{i}_i) - T(\mathbf{p}_l))$$

to become the excentre $\mathcal{I}_i(T(\Delta(W)))$ of $T(\Delta(W))$. Therefore, excentre is preserved under homothetic transformations.

Hence, incentre and excentre are preserved under homothetic transformations, and $\mathcal{K}, \mathcal{O}$ and χ_{25} are collinear. □

We have proved the geometric property of $\mathcal{K}$, now we turn to prove the algebraic property of $\mathcal{K}$.

Theorem 29. *(Vector representation of orthic inexcentre) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. If its circumcentre $\mathcal{O}$ is the origin, then its orthic inexcentre $\mathcal{K}$ of Δ can be expressed as*

$$\mathbf{k} = \mathbf{o}_0 + \mathbf{o}_1 + \mathbf{o}_2 + \mathbf{o}_3 \tag{58}$$

where $\mathcal{O}_i$ denotes the circumcentre of the face F_i.

Proof. We shall compute the equations of the orthic planes U_0, U_1, U_2, U_3, and then verify that the point represented by the right-hand side of (58) satisfies the incentre or excentre requirement for $\mathcal{K}$ in Definition 27.

Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Recall from (39) that the equation of T_i is given by $(-\mathbf{v}_i/R) \cdot \mathbf{x} = -R$. Note that we have multiplied $-1/||\mathbf{v}_i|| = -1/R$ to both sides, which will make the normal vector $-v_i/R$ an inward one if $\mathcal{O}$ and V_i lie on the same side of F_i, and an outward one if $\mathcal{O}$ and V_i lie on opposite sides of F_i. Therefore, $-\mathbf{v}_i/R$ will be the required normal vector of U_i for both cases in Definition 27.

To compute the equation of U_i, we may compute the position vector p of P in Figure 38. By (55) in the proof of Lemma 25 and referring to Figure 35, if Q is the projection of O_i onto V_jO, then we have

$$O_iP = V_jQ = \frac{(\mathbf{o}_i - \mathbf{v}_j) \cdot (-\mathbf{v}_j)}{||\mathbf{v}_j||} = \frac{R^2 - \mathbf{v}_j \cdot \mathbf{o}_i}{R} = \frac{R_i^2}{R}$$

with the help of (43), and then

$$p = \mathbf{o}_i + \frac{R_i^2}{R} \frac{\mathbf{v}_i}{||\mathbf{v}_i||} = \mathbf{o}_i + \frac{R_i^2}{R^2} \mathbf{v}_i$$

as shown in Figure 34. Therefore, the required equation of U_i is

$$\frac{-\mathbf{v}_i}{R} \cdot \left(\mathbf{x} - \mathbf{o}_i - \frac{R_i^2}{R^2} \mathbf{v}_i \right) = 0$$

By substituting

$$\mathbf{x} = \mathbf{o}_0 + \mathbf{o}_1 + \mathbf{o}_2 + \mathbf{o}_3 \tag{59}$$

into the left-hand side, its distance from

$$\begin{aligned} U_i &= \frac{-\mathbf{v}_i}{R} \cdot \left(\mathbf{o}_j + \mathbf{o}_k + \mathbf{o}_l - \frac{R_i^2}{R^2} \mathbf{v}_i \right) \\ &= \frac{-\mathbf{v}_i \cdot \mathbf{o}_j - \mathbf{v}_i \cdot \mathbf{o}_k - \mathbf{v}_i \cdot \mathbf{o}_k}{R} + \frac{R_i^2}{R} \\ &= \frac{R_i^2 + R_j^2 + R_k^2 + R_l^2 - 3R^2}{R} \quad \text{(by (43))} \end{aligned} \tag{60}$$

Note that (60) is the same for U_i, U_j, U_k, U_l, so (59) is equidistant from all U_i, U_j, U_k, U_l, and (58) is verified. □

Note that (58) is well-defined even for *right-angled* tetrahedra, thus Theorem 29 enables us define orthic inexcentre by (58) for any tetrahedron. Also, if the circumcentre $\mathcal{O}$ is not assumed to be the origin, then (58) should be modified as

$$\begin{aligned} k - \mathbf{o} &= (\mathbf{o}_0 - \mathbf{o}) + (\mathbf{o}_1 - \mathbf{o}) + (\mathbf{o}_2 - \mathbf{o}) + (\mathbf{o}_3 - \mathbf{o}) \\ k &= \mathbf{o}_0 + \mathbf{o}_1 + \mathbf{o}_2 + \mathbf{o}_3 - 3\mathbf{o} \end{aligned} \tag{61}$$

Corollary 30. *(Vector representation of χ_{25}) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. If its circumcentre $\mathcal{O}$ is the origin, then its χ_{25} of Δ can be expressed as*

$$\chi_{25} = \frac{R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} k, \tag{62}$$

where O_i and R_i denote the circumcentre and the circumradius of the face F_i respectively.

Proof. While the common distance from $O(\Delta)$ to the faces T_0, T_1, T_2, T_3 of Δ_{tg} is R, that from $\mathcal{K}(\Delta)$ to the faces U_0, U_1, U_2, U_3 of Δ_{ot} is as in (60). Therefore,

$$\chi_{25} = \frac{R}{R + \frac{R_0^2+R_1^2+R_2^2+R_3^2-3R^2}{R}} k,$$

and we obtain (62). □

Note that if the circumcentre $\mathcal{O}$ is not assumed to be the origin, then (62) should be modified as

$$\begin{aligned} \chi_{25} - \mathbf{o} &= \frac{R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} (\mathbf{k} - \mathbf{o}) \\ \chi_{25} &= \frac{R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} \mathbf{k} + \frac{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 3R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} \mathbf{o} \end{aligned} \tag{63}$$

5. Tetrahedron Centres and Barycentric Coordinates

We have been talking about tetrahedron centres, but unlike triangle centres which have already been clearly defined in [14], apparently no precise definition of tetrahedron centre has ever been written down yet. In [8], the author did define and use the terminology when formulating and investigating the so-called *centre conjecture*, which has a completely different purpose.

Here we will express our own perception of this seemingly immediate generalization of triangle centres to tetrahedra, and will formulate a preliminary framework of tetrahedron centres. Moreover, the notion of (homogeneous) barycentric coordinates have provided a powerful tool for analyzing triangle centres problems, as shown in [22]. This inspired us to introduce barycentric function to construct tetrhedron centres.

In Section 5.1, we will lay down precise definitions of triangle centres and tetrahedron centres in the space from our perspective in Definition **??**. Then, we will prove Lemma 33 about behaviours of tetrahedra under similarity transformations, which will also justify that Definition **??** is well-defined. Proposition 35 will verify that all the aforementioned tetrahedron centres satisfy Definition **??**. Propositions 36 and 36 will provide simple ways to generate tetrahedron centres from others.

In Section 5.2, a general analytic form for tetrahedron centres will be constructed through the concept of barycentric function which will be introduced in Definition 37. Proposition 38 and Theorem **??** will prove that such construction completely characterizes all tetrahedron centres.

5.1. Defining Tetrahedron Centres as Functions

In [14], functions with certain homogeneity and symmetry were used to define triangle centres in the plane. Inspired by that approach, here we express our own perception of triangles centres and tetrahedra centres in the space, which could be described in a more succinct manner.

Definition 31. *(Similarity transformation) A similarity transformation T of $\mathbb{R}^3$ is a function $T : \mathbb{R}^3 \to \mathbb{R}^3$ of the form*

$$T(\mathbf{u}) = tA\mathbf{u} + \mathbf{b} \quad \textit{for } u \in \mathbb{R}^3 \tag{64}$$

where $t \in \mathbb{R}\backslash\{0\}$, A is a 3×3 orthogonal matrix (i.e. $A^TA = I$ or $A^{-1} = A^T$) and b is a vector in $\mathbb{R}^3$.

Note that similarity transformations T of the form (64) have the very useful properties

$$\begin{aligned}(T(\mathbf{u}) - T(\mathbf{v})) \cdot (T(\mathbf{u}') - T(\mathbf{v}')) &= tA(\mathbf{u} - \mathbf{v}) \cdot tA(\mathbf{u}' - \mathbf{v}') \\ &= t^2(A(\mathbf{u} - \mathbf{v}))^T(A(\mathbf{u}' - \mathbf{v}')) \\ &= t^2(\mathbf{u} - \mathbf{v})^T A^T A(\mathbf{u}' - \mathbf{v}') \\ &= t^2(\mathbf{u} - \mathbf{v})^T(u' - \mathbf{v}') \\ &= t^2(\mathbf{u} - \mathbf{v}) \cdot (u' - \mathbf{v}') \end{aligned} \tag{65}$$

and

$$\begin{aligned}A\mathbf{w} \cdot (T(\mathbf{u}) - T(\mathbf{v})) &= A\mathbf{w} \cdot tA(\mathbf{u} - \mathbf{v}) \\ &= t(A\mathbf{w})^T(A(\mathbf{u} - \mathbf{v})) \\ &= t\mathbf{w}^T A^T A(\mathbf{u} - \mathbf{v}) \\ &= t\mathbf{w}^T(\mathbf{u} - \mathbf{v}) \\ &= t\mathbf{w} \cdot (\mathbf{u} - \mathbf{v}) \end{aligned} \tag{66}$$

Definition 32. *(Triangle centre and tetrahedron centre) Let $\mathbb{S}^2$ denote the set of all triangles in $\mathbb{R}^3$. Then, a triangle centre $\chi^2/\mathbf{x}^2$ is a function $\chi^2 : \mathbb{S}^2 \to \mathbb{R}^3$ assigning to each $\Delta^2 \in \mathbb{S}^2$ a point in the plane containing Δ^2, such that it is equivariant under similarity transformations T of $\mathbb{R}^3$, i.e.*

$$T(\mathbf{x}^2(\Delta^2)) = \mathbf{x}^2(T(\Delta^2)) \quad \textit{for } \Delta^2 \in \mathbb{S}^2 \tag{67}$$

Let $\mathbb{S}^3$ denote the set of all tetrahedra in $\mathbb{R}^3$. Then, a tetrahedron centre $\chi^3/\mathbf{x}^3$ is a function $\chi^3 : \mathbb{S}^3 \to \mathbb{R}^3$ such that it is equivariant under similarity transformations T of $\mathbb{R}^3$, i.e.

$$T(\mathbf{x}^3(\Delta^3)) = \mathbf{x}^3(T(\Delta^3)) \quad \textit{for } \Delta^2 \in \mathbb{S}^3 \tag{68}$$

Let $\mathbb{S} = \mathbb{S}^2 \cup \mathbb{S}^3$ denote the set of all triangles and tetrahedra in $\mathbb{R}^3$. Then, joining the above triangle and tetrahedron centres by

$$\chi|_{\mathbb{S}^2} = \chi^2 \quad \textit{and} \quad \chi|_{\mathbb{S}^3} = \chi^3$$

defines a centre $\chi/\mathbf{x}$ as a function $\chi : \mathbb{S} \to \mathbb{R}^3$.

The requirement that $\chi^2(\Delta^2)$ lies in the plane containing Δ^2 can, in particular, avoid the circumcentre of a triangle from becoming an arbitrary point in a line when it is described as "a point equidistant from the vertices of the triangle". For some non-classical centres such as the Fermat-Torricelli point which is defined only for a subclass of triangles, the domains may need modifications.

The equivariances (67) and (68) require that for any triangle or tetrahedron and any similarity transformation of the space, the centre of the transformed triangle or tetrahedron is exactly the transformed centre of the original triangle or tetrahedron. Moreover, the right-hand sides of these two equations require that $T(\Delta^2)$ and $T(\Delta^3)$ are also a triangle and a tetrahedron respectively – they are, as we now prove:

Lemma 33. *(Triangle and tetrahedron under similarity transformation) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron and T be a similarity transformation of $\mathbb{R}^3$. Then,*

(a) $\{T(\mathbf{v}_0), T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are affinely independent,
(b) $T(\Delta) = [T(V_0), T(V_1), T(V_2), T(V_3)]$,
(c) $T(E_{i,j}) = [T(V_i), T(V_j)]$, where $i \neq j$, and
(d) $T(F_i) = [T(V_j), T(V_k), T(V_l)]$, where $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

Proof. Under similarity transformation T of the form (64), we have

$$T(\mathbf{v}_i) - T(\mathbf{v}_0) = tA(\mathbf{v}_i - \mathbf{v}_0) \quad \text{for } i = 1, 2, 3$$

Suppose there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\begin{aligned}
\alpha_1(T(\mathbf{v}_1) - T(\mathbf{v}_0)) + \alpha_2(T(\mathbf{v}_2) - T(\mathbf{v}_0)) + \alpha_3(T(\mathbf{v}_3) - T(\mathbf{v}_0)) &= 0 \\
tA(\alpha_1(\mathbf{v}_1 - \mathbf{v}_0) + \alpha_2(\mathbf{v}_2 - \mathbf{v}_0) + \alpha_3(\mathbf{v}_3 - \mathbf{v}_0)) &= 0 \\
\alpha_1(\mathbf{v}_1 - \mathbf{v}_0) + \alpha_2(\mathbf{v}_2 - \mathbf{v}_0) + \alpha_3(\mathbf{v}_3 - \mathbf{v}_0) &= A^{-1}0 = 0 \\
&\quad (A \text{ is invertible})
\end{aligned}$$

Since $\{\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \mathbf{v}_3 - \mathbf{v}_0\}$ are linearly independent, we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, the transformed vertices $\{T(\mathbf{v}_0), T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are affinely independent too, hence proving (a).

As mentioned in Section 1.3, any point u in the tetrahedron can be represented as a convex combination as in (1), so we have

$$\begin{aligned} T(u) &= tA(\lambda_0\mathbf{v}_0 + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3) + b \\ &= \lambda_0(tA\mathbf{v}_0 + b) + \lambda_1(tA\mathbf{v}_1 + b) + \lambda_2(tA\mathbf{v}_2 + b) + \lambda_3(tA\mathbf{v}_3 + b) \\ &= \lambda_0 T(\mathbf{v}_0) + \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2) + \lambda_3 T(\mathbf{v}_3) \end{aligned}$$

This shows that as $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ run through the condition in (1) so that u runs through $\Delta, T(u)$ will run through every point the tetrahedron

$$[T(\mathbf{v}_0), T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)]$$

in a one-to-one correspondence manner. More precisely, similarity transformations preserve convex combination. Hence, (b), (c) and (d) follow. □

We now verify that centroid, circumcentre, incentre, excentre, Monge point, quasi-orthocentre, antimedial circumcentre, orthic inexcentre and χ_{25} are tetrahedron centres:

Proposition 34. *(A source of tetrahedron centres) Centroid, circumcentre, incentre, Monge point, quasi-orthocentre, antimedial circumcentre, orthic inexcentre and* χ_{25} *are tetrahedron centres.*

Excentres form 'a group of' tetrahedron centres. Here, 'a group of' tetrahedron centres is defined as a function $\chi : \mathbb{S}^3 \to \mathscr{P}_4(\mathbb{R}^3)$*, where* $\mathscr{P}_4(\mathbb{R}^3)$ *denotes the set of all the* 4*-element subsets of* $\mathbb{R}^3$*, such that it is equivariant under similarity transformations of* $\mathbb{R}^3$*.*

Proof. That is to verify that $\mathcal{G}, \mathcal{O}, \mathcal{I}, \{I_0, I_1, I_2, I_3\}$, $\mathcal{M}$, $\mathcal{Q}_k$, $\mathcal{J}$ and $\mathcal{K}$ are equivariant under similarity transformations of $\mathbb{R}^3$. To this end, let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron, and let T be any similarity transformation of the form (64). Recall from Lemma 33 that $T(\Delta) = [T(V_0), T(V_1), T(V_2), T(V_3)]$, $T(E_{i,j}) = [T(V_i), T(V_j)]$ for $i \neq j$, and $T(F_i) = [T(V_j), T(V_k), T(V_l)]$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$. We shall show that

$$T(\mathbf{x}(\Delta)) = \mathbf{x}(T(\Delta)) \quad \text{for } \mathcal{G}, \mathcal{O}, \mathcal{I}, \{I_0, I_1, I_2, I_3\},\ \mathcal{M},\ \mathcal{Q}_k,\ \mathcal{J}\ ,\ \mathcal{K}$$

Centroid:

$$\begin{aligned} T(g(\Delta)) &= tA\left(\frac{1}{4}(\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)\right) + b \quad \text{(by (2))} \\ &= \frac{1}{4}(tA\mathbf{v}_0 + b) + \frac{1}{4}(tA\mathbf{v}_1 + b) + \frac{1}{4}(tA\mathbf{v}_2 + b) + \frac{1}{4}(tA\mathbf{v}_3 + b) \\ &= \frac{1}{4}T(\mathbf{v}_0) + \frac{1}{4}T(\mathbf{v}_1) + \frac{1}{4}T(\mathbf{v}_2) + \frac{1}{4}T(\mathbf{v}_3) \quad \text{(by (2))} \\ &= g(T(\Delta)) \end{aligned} \tag{69}$$

Circumcentre: Recall from (3) that

$$||\mathbf{o}(\Delta)-\mathbf{v}_0|| = ||\mathbf{o}(\Delta)-\mathbf{v}_i|| = ||\mathbf{o}(\Delta)-\mathbf{v}_2|| = ||\mathbf{o}(\Delta)-\mathbf{v}_3||$$

But by (65),

$$||T(\mathbf{o}(\Delta))-T(\mathbf{v}_i)||^2 = t^2||\mathbf{o}(\Delta)-\mathbf{v}_i||^2$$

so

$$\begin{aligned} ||T(\mathbf{o}(\Delta))-T(\mathbf{v}_0)|| &= ||T(\mathbf{o}(\Delta))-T(\mathbf{v}_1)|| \\ &= ||T(\mathbf{o}(\Delta))-T(\mathbf{v}_2)|| \\ &= ||T(\mathbf{o}(\Delta))-T(\mathbf{v}_3)|| \\ &= tR \end{aligned} \tag{70}$$

where R is the circumradius of Δ. Therefore,

$$T(\mathbf{o}(\Delta)) = \mathbf{o}(T(\Delta)) \tag{71}$$

Incentre: Recall from (9) that

$$\mathbf{n}_0\cdot(\mathbf{i}(\Delta)-\mathbf{p}_0) = \mathbf{n}_1\cdot(\mathbf{i}(\Delta)-\mathbf{p}_1) = \mathbf{n}_2\cdot(\mathbf{i}(\Delta)-\mathbf{p}_2) = \mathbf{n}_3\cdot(\mathbf{i}(\Delta)-\mathbf{p}_3),$$

where $\mathbf{n}_i$ is the inward normal vector of the face F_i and $\mathbf{p}_i$ is a point F_i. Then,

$$\tilde{\mathbf{n}}_i := \operatorname{sgn}(t)A\mathbf{n}_i$$

will become the inward normal vector of the face $T(F_i)$. It is because $||\tilde{\mathbf{n}}_i||^2 = ||\operatorname{sgn}(t)A\mathbf{n}_i||^2 = (A\mathbf{n}_i)^T(A\mathbf{n}_i) = \mathbf{n}_i^T A^T A\mathbf{n}_i = \mathbf{n}_i^T\mathbf{n}_i = ||\mathbf{n}_i||^2 = 1$, and by (66),

$$\tilde{\mathbf{n}}_i\cdot(T(\mathbf{v}_j)-T(\mathbf{v}_k)) = \operatorname{sgn}(t)t\mathbf{n}_i\cdot(\mathbf{v}_j-\mathbf{v}_k) = |t|\mathbf{n}_i\cdot\mathbf{e}_{j,k},$$

so that $\tilde{\mathbf{n}}_i\cdot(T(\mathbf{v}_j)-T(\mathbf{v}_k)) = 0$ for $j,k\neq i$ and $\tilde{n}_i\cdot(T(\mathbf{v}_i)-T(\mathbf{v}_j)) > 0$ for $j\neq i$. By (66) again, we will also have

$$\tilde{\mathbf{n}}_i\cdot(T(\mathbf{i}(\Delta))-T(\mathbf{p}_i)) = |t|\mathbf{n}_i\cdot(\mathbf{i}(\Delta)-\mathbf{p}_i),$$

so

$$\begin{aligned} \tilde{\mathbf{n}}_0\cdot(T(\mathbf{i}(\Delta)-T(\mathbf{p}_0))) &= \tilde{\mathbf{n}}_1\cdot(T(\mathbf{i}(\Delta))-T(\mathbf{p}_1)) \\ &= \tilde{\mathbf{n}}_2\cdot(T(\mathbf{i}(\Delta))-T(\mathbf{p}_2)) \\ &= \tilde{\mathbf{n}}_3\cdot(T(\mathbf{i}(\Delta))-T(\mathbf{p}_3)) \end{aligned}$$

Therefore,

$$T(\mathbf{i}(\Delta)) = \mathbf{i}(T(\Delta))$$

Excentre: Let $\{i,j,k,l\} = \{0,1,2,3\}$. Recall from (13) that

$$\mathbf{n}_i'\cdot(\mathbf{i}_i(\Delta)-\mathbf{p}_i) = \mathbf{n}_j\cdot(\mathbf{i}_i(\Delta)-\mathbf{p}_j) = \mathbf{n}_k\cdot(\mathbf{i}_i(\Delta)-\mathbf{p}_k) = \mathbf{n}_l\cdot(i_i(\Delta)-p_l)$$

Defining $\tilde{\mathbf{n}}_i' = -\tilde{\mathbf{n}}_i$, by (66),

$$\begin{aligned} &\tilde{\mathbf{n}}_i'\cdot(T(\mathbf{i}_i(\Delta))-T(\mathbf{p}_i)) = |t|\mathbf{n}_i'\cdot(\mathbf{i}_i(\Delta)-\mathbf{p}_i) \quad \text{and} \\ &\tilde{\mathbf{n}}_j'\cdot(T(\mathbf{i}_i(\Delta))-T(\mathbf{p}_j)) = |t|\mathbf{n}_j\cdot(\mathbf{i}_i(\Delta)-\mathbf{p}_j) \quad \text{for } j\neq i, \end{aligned}$$

so

$$\begin{aligned}\tilde{\mathbf{n}}_i' \cdot (T(\mathbf{i}(\Delta) - T(\mathbf{p}_i))) &= \tilde{\mathbf{n}}_j' \cdot (T(\mathbf{i}(\Delta)) - T(\mathbf{p}_j)) \\ &= \tilde{\mathbf{n}}_k' \cdot (T(\mathbf{i}(\Delta)) - T(\mathbf{p}_k)) \\ &= \tilde{\mathbf{n}}_l' \cdot (T(\mathbf{i}(\Delta)) - T(\mathbf{p}_l))\end{aligned}$$

However, we may not conclude that $T(\mathbf{i}_i(\Delta)) = \mathbf{i}_i(T(\Delta))$, but only

$$T(\mathbf{i}_i(\Delta)) = \mathbf{i}_{\sigma(\mathbf{i})}(T(\Delta)),$$

where σ is a permutation of $\{0,1,2,3\}$, as $T(\Delta) = [T(V_0), T(V_1), T(V_2), T(V_3)] = [T(V_{\sigma(0)}), T(V_{\sigma(1)}), T(V_{\sigma(2)}), T(V_{\sigma(3)})]$ in general. Nevertheless, we must have

$$T(\{\mathbf{i}_0(\Delta), \mathbf{i}_1(\Delta), \mathbf{i}_2(\Delta), \mathbf{i}_3(\Delta)\}) = \{\mathbf{i}_0(T(\Delta)), \mathbf{i}_1(T(\Delta)), \mathbf{i}_2(T(\Delta)), \mathbf{i}_3(T(\Delta))\}$$

Monge point:

$$\begin{aligned}T(\mathbf{m}(\Delta)) &= tA(2\mathbf{g}(\Delta) - \mathbf{o}(\Delta)) + b \quad \text{(by (19))} \\ &= 2(tA\mathbf{g}(\Delta) + b) - (tA\mathbf{o}(\Delta) + b) \\ &= 2T(\mathbf{g}(\Delta)) - T(\mathbf{o}(\Delta)) \\ &= 2\mathbf{g}(T(\Delta)) - \mathbf{o}(T(\Delta)) \quad \text{(by (69) and (71))} \\ &= \mathbf{m}(T(\Delta)) \quad \text{(by (19))}\end{aligned}$$

k-quasi-orthocentre:

$$\begin{aligned}T(\mathbf{q}_k(\Delta)) &= tA\left(\frac{4}{3+k}\mathbf{g}(\Delta) - \frac{1-k}{3+k}\mathbf{o}(\Delta)\right) + \mathbf{b} \quad \text{(by (26))} \\ &= \frac{4}{3+k}(tA\mathbf{g}(\Delta) + \mathbf{b}) - \frac{1-k}{3+k}(tA\mathbf{o}(\Delta) + \mathbf{b}) \\ &= \frac{4}{3+k}(tA\mathbf{g}(\Delta) + \mathbf{b}) - \frac{1-k}{3+k}T(\mathbf{o}(\Delta)) \\ &= \frac{4}{3+k}(tA\mathbf{g}(\Delta) + \mathbf{b}) - \frac{1-k}{3+k}\mathbf{o}(T(\Delta)) \quad \text{(by (69) and (71))} \\ &= \mathbf{q}_k(T(\Delta)) \quad \text{(by (26))}\end{aligned}$$

Antimedial circumcentre:

$$\begin{aligned}T(\mathbf{j}(\Delta)) &= tA(4\mathbf{g}(\Delta) - 3\mathbf{o}(\Delta)) + \mathbf{b} \quad \text{(by (38))} \\ &= 4(tA\mathbf{g}(\Delta) + \mathbf{b}) - 3(tA\mathbf{o}(\Delta) + \mathbf{b}) \\ &= 4T(\mathbf{g}(\Delta)) - 3T(\mathbf{o}(\Delta)) \\ &= 4\mathbf{g}(T(\Delta)) - 3\mathbf{o}(T(\Delta)) \quad \text{(by (69) and (71))} \\ &= \mathbf{j}(T(\Delta)) \quad \text{(by (38))}\end{aligned}$$

Orthic inexcentre: Similar to the proof about circumcentre above, we have

$$T(\mathbf{o}(F_i)) = \mathbf{o}(T(F_i)), \tag{72}$$

where this o refers to circumcentre of triangles, and the circumradius of the face $T(F_i)$ of $T(\Delta)$

$$R_i(T(\Delta)) = tR_i \tag{73}$$

where R_i is the circumradius of F_i. Then,

$$\begin{aligned}
T(k(\Delta)) &= tA(\mathbf{o}_0(\Delta) + \mathbf{o}_1(\Delta) + \mathbf{o}_2(\Delta) + \mathbf{o}_3(\Delta) - 3\mathbf{o}(\Delta)) + b \quad \text{(by (61))} \\
&= tA(\mathbf{o}(F_0) + \mathbf{o}(F_1) + \mathbf{o}(F_2) + \mathbf{o}(F_3) - 3\mathbf{o}(\Delta)) + b \\
&= (tA\mathbf{o}(F_0) + b) + (tA\mathbf{o}(F_1) + b) + (tA\mathbf{o}(F_2) + b) + (tA\mathbf{o}(F_3) + b) \\
&\qquad - 3(tA\mathbf{o}(\Delta) + b) \\
&= T(\mathbf{o}(F_0)) + T(\mathbf{o}(F_1)) + T(\mathbf{o}(F_2)) + T(\mathbf{o}(F_3)) - 3T(\mathbf{o}(\Delta)) \\
&= \mathbf{o}(T(F_0)) + \mathbf{o}(T(F_1)) + \mathbf{o}(T(F_2)) + \mathbf{o}(T(F_3)) - 3\mathbf{o}(T(\Delta)) \\
&\qquad \text{(by (71) and (72))} \\
&= \mathbf{o}_0(T(\Delta)) + \mathbf{o}_1(T(\Delta)) + \mathbf{o}_2(T(\Delta)) + \mathbf{o}_3(T(\Delta)) - 3\mathbf{o}(T(\Delta)) \\
&= k(T(\Delta)) \quad \text{(by (61))}
\end{aligned} \tag{74}$$

χ_{25}:

$$\begin{aligned}
&T(\mathbf{x}_{25}(\Delta)) \\
&= tA\Bigg(\frac{R^2}{R_0^2+R_1^2+R_2^2+R_3^2-2R^2}k(\Delta) \\
&\qquad + \frac{R_0^2+R_1^2+R_2^2+R_3^2-3R^2}{R_0^2+R_1^2+R_2^2+R_3^2-2R^2}o(\Delta)\Bigg) + b \quad \text{(by (63))} \\
&= \frac{R^2}{R_0^2+R_1^2+R_2^2+R_3^2-2R^2}(tAk(\Delta)+b) + \\
&\qquad \frac{R_0^2+R_1^2+R_2^2+R_3^2-3R^2}{R_0^2+R_1^2+R_2^2+R_3^2-2R^2}(tA\mathbf{o}(\Delta)+b) \\
&= \frac{R^2}{R_0^2+R_1^2+R_2^2+R_3^2-2R^2}T(k(\Delta)) + \frac{R_0^2+R_1^2+R_2^2+R_3^2-3R^2}{R_0^2+R_1^2+R_2^2+R_3^2-2R^2}T(\mathbf{o}(\Delta)) \\
&= \frac{(tR)^2}{(tR_0)^2+(tR_1)^2+(tR_2)^2+(tR_3)^2-2(tR)^2}k(T(\Delta)) \\
&\qquad + \frac{(tR_0)^2+(tR_1)^2+(tR_2)^2+(tR_3)^2-3(tR)^2}{(tR_0)^2+(tR_1)^2+(tR_2)^2+(tR_3)^2-2(tR)^2}\mathbf{o}(T(\Delta)) \\
&\quad \text{(by (71) and (74))} \\
&= \chi_{25}(T(\Delta)) \quad \text{(by (63), (70) and (73))}
\end{aligned}$$

Hence, centroid, circumcentre, incentre, excentre, Monge point, quasi-orthocentre, antimedial circumcentre, orthic inexcentre and χ_{25} are tetrahedron centres. □

Proposition 34 motivates us to prove two general ways to construct tetrahedron centres from others:

Proposition 35. *(Affine combination of tetrahedron centres) An affine combination of tetrahedron centres is also a tetrahedron centre.*

Proof. Let $\chi/\mathbf{x}_1, \ldots, \chi_m/\mathbf{x}_m$ be some tetrahedron centres, and let $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$ such that $\sum_{i=1}^m \gamma_i = 1$. Consider the affine combination $\chi/\mathbf{x}$ defined by

$$\mathbf{x}(\Delta) := \sum_{i=1}^{m} \gamma_i \mathbf{x}_i(\Delta) \quad \text{for } \Delta \in \mathbb{S}^3$$

Under a similarity transformation T as described in (64),

$$\begin{aligned}
T(\mathbf{x}(\Delta)) &= tA \sum_{i=1}^{m} \gamma_i \mathbf{x}_i(\Delta) + b \\
&= \sum_{i=1}^{m} \gamma_i tA\mathbf{x}_i(\Delta) + \sum_{i=1}^{m} \gamma_i b \\
&= \sum_{i=1}^{m} \gamma_i (tA\mathbf{x}_i(\Delta) + b) \\
&= \sum_{i=1}^{m} \gamma_i T(\mathbf{x}_i(\Delta)) \\
&= \sum_{i=1}^{m} \gamma_i \mathbf{x}_i(T(\Delta)) \\
&= \mathbf{x}(T(\Delta)),
\end{aligned}$$

and hence χ is equivariant under similarity transformations. □

While Proposition 35 may be quite intuitive, the next will be less obvious.

Proposition 36. *(Combination of facial centres) If $\chi/\mathbf{x}$ is a triangle centre and $\mathcal{Y}/\mathbf{y}$ is a tetrahedron centre, then $\mathcal{Z}/\mathbf{z}$ defined as*

$$\mathbf{z}(\Delta) := \mathbf{y}(\Delta) + \sum_i (\mathbf{x}(F_i) - \mathbf{y}(\Delta)) \quad \textit{for } \Delta \in \mathbb{S}^3$$

is also a tetrahedron centre.

Proof. Under a similarity transformation T as described in (64),

$$\begin{aligned} T(\mathbf{z}(\Delta)) &= tA\left(\mathbf{y}(\Delta) + \sum_i (\mathbf{x}(F_i) - \mathbf{y}(\Delta))\right) + \mathbf{b} \\ &= tA\mathbf{y}(\Delta) + \mathbf{b} + \sum_i (tA\mathbf{x}(F_i) - tA\mathbf{y}(\Delta)) \\ &= T(\mathbf{y}(\Delta)) + \sum_i (T(\mathbf{x}(F_i)) - T(\mathbf{y}(\Delta))) \\ &= \mathbf{y}(T(\Delta)) + \sum_i (\mathbf{x}(T(F_i)) - \mathbf{y}(T(\Delta))) \\ &= \mathbf{z}(T(\Delta)), \end{aligned}$$

and hence $\mathcal{Z}$ is equivariant under similarity transformations. In the last step, Lemma 33(d) is used so that $T(\Delta) = [T(F_0), T(F_1), T(F_2), T(F_3)]$. □

Figure 48 illustrates Proposition 36: the point $\mathcal{Z}(\Delta)$ is obtained by translating the point $\mathcal{Y}(\Delta)$ by the resultant vector (solid arrow) of the other four vectors (dotted arrows).

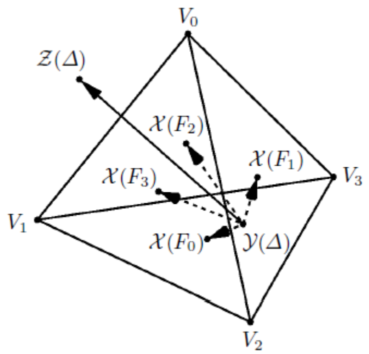

FIGURE 48. Proposition 36

5.2. Constructing Tetrahedron Centres Using Barycentric Functions

Each point in the plane containing a triangle in the space can be expressed uniquely in barycentric coordinates with respect to the vertices of the triangle, so it is desirable to generalize this idea to the space and subsequently express the tetrahedron centres in barycentric coordinates with respect to the vertices of the tetrahedron.

Definition 37. *(Barycentric function) Let $\mathbb{V}$ be the set of all affinely independent quadraples of vectors in $\mathbb{R}^3$. Then, $\lambda : \mathbb{V} \to \mathbb{R}$ is a barycentric function if:*

(i) *Invariance under similarity transformations:*

$$\lambda(T(\mathbf{x}_0), T(\mathbf{x}_1), T(\mathbf{x}_2), T(\mathbf{x}_3)) = \lambda(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

for $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in V$*, for any similarity transformation* T *of* $\mathbb{R}^3$*.*

(ii) *Symmetry in the second, third and fourth variables:*

$$\lambda(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \lambda(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \quad \textit{for} \quad (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in V,$$

where $\{i, j, k\} = \{1, 2, 3\}$*.*

(iii) *Normalization:*

$$\begin{aligned} &\lambda(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + \lambda(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_0) + \lambda(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_0, \mathbf{x}_1) + \lambda(\mathbf{x}_3, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) \\ &= 1 \quad \textit{for } (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in V \end{aligned}$$

Because of the symmetry (ii), we can abbreviate $\lambda(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ as $\lambda(\mathbf{x}_0)$, so that (iii) can be rewritten as

$$\lambda(\mathbf{x}_0) + \lambda(\mathbf{x}_1) + \lambda(\mathbf{x}_2) + \lambda(\mathbf{x}_3) = 1$$

when the context is clear. It is then routine to check that barycentric functions can generate tetrahedron centres:

Proposition 38. *(Barycentric functions generate tetrahedron centres) If* λ *is a barycentric function, then* $\chi/\mathbf{x}$ *defined as*

$$\mathbf{x}(\Delta) := \sum_i \lambda(\mathbf{v}_i)\mathbf{v}_i \quad \textit{for } \Delta = [V_0, V_1, V_2, V_3] \in \mathbb{S}^3$$

is a tetrahedron centre.

Proof. Under a similarity transformation T as described in (64),

$$\begin{aligned} T(\mathbf{x}(\Delta)) &= tA \sum_i \lambda(\mathbf{v}_i)\mathbf{v}_i + \mathbf{b} \\ &= \sum_i \lambda(\mathbf{v}_i) tA\mathbf{v}_i + \sum_i \lambda(\mathbf{v}_i)\mathbf{b} \\ &= \sum_i \lambda(\mathbf{v}_i)(tA\mathbf{v}_i + \mathbf{b}) \\ &= \sum_i \lambda(T(\mathbf{v}_i))T(\mathbf{v}_i) \\ &= \mathbf{x}(T(\Delta)), \end{aligned}$$

and hence χ is equivariant under similarity transformations. In the last step, Lemma 33(b) is used. $\square$

Then $(\lambda(\mathbf{v}_0), \lambda(\mathbf{v}_1), \lambda(\mathbf{v}_2), \lambda(\mathbf{v}_3))$ form the *barycentric coordinates* of the tetrahedron centre χ, and λ will be called the *barycentric function* of the tetrahedron centre χ.

In fact, barycentric functions generate all tetrahedron centres.

Theorem 39. *(Barycentric functions generate all tetrahedron centres) If $\chi/\mathbf{x}$ is a tetrahedron centre of the form*

$$\mathbf{x}(\Delta) := \sum_i \lambda_i(\Delta)\mathbf{v}_i \quad \text{for } \Delta = [V_0, V_1, V_2, V_3] \in \mathbb{S}^3,$$

where $\lambda_0, \lambda_1, \lambda_2, \lambda_3 : \mathbb{S}^3 \to \mathbb{R}$ are functions such that

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1, \tag{75}$$

then,

$$\lambda_i(\Delta) = \lambda(\mathbf{v}_i)$$

where $\lambda : \mathbb{V} \to \mathbb{R}$ is a barycentric function.

Proof. Writing

$$\begin{aligned}
\lambda_0(\Delta) &= \lambda_0(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), \lambda_1(\Delta) = \lambda_1(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3),\\
\lambda_2(\Delta) &= \lambda_2(\mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3), \lambda_3(\Delta) = \lambda_3(\mathbf{v}_3, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)
\end{aligned}$$

and $\mathbf{x}(\Delta) = \mathbf{x}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ as Δ is determined by its vertices V_0, V_1, V_2, V_3, we have

$$\begin{aligned}
\mathbf{x}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \sum_i \lambda_i(\Delta)\mathbf{v}_i\\
&= \lambda_0(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_0 + \lambda_1(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_1\\
&\qquad + \lambda_2(\mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3)\mathbf{v}_2 + \lambda_3(\mathbf{v}_3, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)\mathbf{v}_3\\
&\qquad \text{for } (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in V
\end{aligned} \tag{76}$$

Since Δ is independent of the order of $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we can swap $\mathbf{v}_0$ and $\mathbf{v}_1$ to obtain

$$\begin{aligned}
\mathbf{x}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \mathbf{x}(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3)\\
&= \lambda_0(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_1 + \lambda_1(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_0\\
&\qquad + \lambda_2(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_3)\mathbf{v}_2 + \lambda_3(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2)\mathbf{v}_3
\end{aligned} \tag{77}$$

Comparing the coefficients of $\mathbf{v}_0$ and $\mathbf{v}_1$ with those in (76), we have

$$\lambda_0(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \lambda_1(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \quad \text{and} \quad \lambda_1(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3) = \lambda_0(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3),$$

both of which suggest that $\lambda_0 = \lambda_1$, as the above relations hold for all $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in V$. Similarly, by swapping $\mathbf{v}_1$ and $\mathbf{v}_2$ and by swapping $\mathbf{v}_2$ and $\mathbf{v}_3$, we will see that $\lambda_1 = \lambda_2$ and $\lambda_2 = \lambda_3$ respectively. Hence,

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda$$

for some function $\lambda : \mathbb{V} \to \mathbb{R}$, and we have to verify that λ is a barycentric function.

In terms of λ, (76) and (77) can be written as

$$\begin{aligned}\mathbf{x}(\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) &= \lambda(\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)\mathbf{v}_0+\lambda(\mathbf{v}_1,\mathbf{v}_0,\mathbf{v}_2,\mathbf{v}_3)\mathbf{v}_1\\ &\quad+\lambda(\mathbf{v}_2,\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_3)\mathbf{v}_2+\lambda(\mathbf{v}_3,\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2)\mathbf{v}_3 \quad\text{and} \qquad (78)\\ \mathbf{x}(\mathbf{v}_1,\mathbf{v}_0,\mathbf{v}_2,\mathbf{v}_3) &= \lambda(\mathbf{v}_1,\mathbf{v}_0,\mathbf{v}_2,\mathbf{v}_3)\mathbf{v}_1+\lambda(\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)\mathbf{v}_0\\ &\quad+\lambda(\mathbf{v}_2,\mathbf{v}_1,\mathbf{v}_0,\mathbf{v}_3)\mathbf{v}_2+\lambda(\mathbf{v}_3,\mathbf{v}_1,\mathbf{v}_0,\mathbf{v}_2)\mathbf{v}_3 \quad\text{and} \qquad (79)\end{aligned}$$

respectively. Comparing the coefficients of $\mathbf{v}_2$ and $\mathbf{v}_3$ in (78) and (79), we see that λ is symmetric in the second and third variables. Now, swapping $\mathbf{v}_2$ and $\mathbf{v}_3$ in (78), we have

$$\begin{aligned}\mathbf{x}(\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_2) &= \lambda(\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_2)\mathbf{v}_0+\lambda(\mathbf{v}_1,\mathbf{v}_0,\mathbf{v}_3,\mathbf{v}_2)\mathbf{v}_1\\ &\quad+\lambda(\mathbf{v}_3,\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2)\mathbf{v}_3+\lambda(\mathbf{v}_2,\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_3)\mathbf{v}_2\end{aligned}$$

Comparing the coefficients of $\mathbf{v}_0$ and $\mathbf{v}_1$ with those in (78), we see that λ is symmetric in the third and fourth variables as well. Hence λ is symmetric in the second, third and fourth variables, and the symmetry condition (ii) for a barycentric function is satisfied.

With the symmetry just proved, write $\lambda(\mathbf{v}_i,\mathbf{v}_j,\mathbf{v}_k,\mathbf{v}_l)$ as $\lambda(\mathbf{v}_i)$ for $\{i,j,k,l\} = \{0,1,2,3\}$. Then, the given condition (75) can be written as

$$\lambda(\mathbf{v}_0)+\lambda(\mathbf{v}_1)+\lambda(\mathbf{v}_2)+\lambda(\mathbf{v}_3)=1 \qquad (80)$$

matching the normalization condition (iii) for a barycentric function.

Let T be any similarity transformation T as described in (64). Since (80) holds for all affinely independent $\{\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, we will also have

$$\lambda(T(\mathbf{v}_0))+\lambda(T(\mathbf{v}_1))+\lambda(T(\mathbf{v}_2))+\lambda(T(\mathbf{v}_3))=1$$

as $\{T(\mathbf{v}_0),T(\mathbf{v}_1),T(\mathbf{v}_2),T(\mathbf{v}_3)\}$ are also affinely independent according to Lemma 33(a). Then,

$$\begin{aligned}T(\mathbf{x}(\Delta)) &= \mathbf{x}(T(\Delta))\\ tA\sum_i \lambda(\mathbf{v}_i)\mathbf{v}_i+b &= \sum_i \lambda(T(\mathbf{v}_i))(tA\mathbf{v}_i+b)\\ &= tA\sum_i \lambda(T(\mathbf{v}_i))\mathbf{v}_i+\sum_i \lambda(T(\mathbf{v}_i))b\\ tA\sum_i \lambda(\mathbf{v}_i)\mathbf{v}_i &= tA\sum_i \lambda(T(\mathbf{v}_i))\mathbf{v}_i\\ \sum_i \lambda(\mathbf{v}_i)\mathbf{v}_i &= \sum_i \lambda(T(\mathbf{v}_i))\mathbf{v}_i\end{aligned}$$

as $t\neq 0$ and A is invertible. By comparing the coefficients of $\mathbf{v}_0$ again, we have

$$\lambda(\mathbf{v}_0)=\lambda(T(\mathbf{v}_0))$$

and the invariance condition (i) for a barycentric function is satisfied. $\square$

6. Summary

In this paper, we have accomplished the aims and objectives stated in Section 1.2.

In Section 2, we have presented new characterizations of the classical triangle centres, namely centroid, circumcentre, incentre, excentre and orthocentre, and have proved their properties that carry over to tetrahedra.

In Section 3, we have generalized Monge point of tetrahedra to a family of tetrahedron centres lying on the Euler lines. As a consequence, Monge point and twelve-point centre of tetrahedra have been shown to share the common geometric feature of being the points of concurrence of special lines derived from their triangle counterparts.

In Section 4, we have constructed new generalizations of orthocentre of triangles to tetrahedra, namely antimedial circumcentre and orthic inexcentre. We did not merely define some new points, but have actually, and most importantly, found the geometric and algebraic properties that carry over to tetrahedra through these generalizations. More precisely, the homothethy between a triangle and its antimedial triangle, as well as that between its tangential and orthic triangles are preserved. The collinear of orthocentre and circumcentre with centroid or χ_{25}, as well as the vector representations of orthocentre are preserved.

In Section 5, we have built a framework to study tetrahedron centres in general. While our definition of tetrahedron centre is geometric in nature, we have also found its algebraic representation in terms of barycentric function.

During this research, we have also observed signs of feasibility to extend all our work to higher-dimensional simplices. It is because our analytic approach requires only basic linear algebra, and we have intentionally avoided the use of cross product throughout. We can even expect synthetic proofs in higher-dimensions similar to those presented in Lemma 25. But due to our limited knowledge of higher-dimensional simplices, hyperplanes and hyperspheres, we were unready for such ambition at the current stage. We hope that after acquiring the necessary knowledge, we will revisit this problem and explore more tetrahedron centres in the future.

REFERENCES

[1] S. Abu-Saymeh and M. Hajja, *On the Fermat–Torricelli points of tetrahedra and of higher dimensional simplexes*, Math. Mag. 70 (1997), no. 5, 372-378.

[2] M. Buba-Brzozowa, *Analogues of the nine-point circle for orthocentric n-simplexes*, J. Geom. 81 (2004), no. 1-2, 21-29.

[3] M. Buba-Brzozowa, *The Monge point and the $3(n+1)$ point sphere of an n-simplex*, J. Geom. Graph. 9 (2005), no. 1, 31-36.

[4] M. Buba-Brzozowa and K. Witczyński, Some properties of orthocentric simplexes, Demonstr. Math. 37 (2004), no. 1, 191-196.

[5] N. A. Court, *The biratio of the altitudes of a tetrahedron*, Duke Math. J. 13 (1946), no. 3, 383-386.
[6] R. A. Crabbs, *Gaspard Monge and the Monge point of the tetrahedron*, Math. Mag. 76 (2003), no. 3, 193-203.
[7] A. L. Edmonds, *The geometry of an equifacetal simplex*, Mathematika 52 (2005), no. 1-2, 31-45.
[8] A. L. Edmonds, *The center conjecture for equifacetal simplices*, Adv. Geom. 9 (2009), no. 4, 563-576.
[9] M. Hajja, *The Gergonne and Nagel centers of an n-dimensional simplex*, J. Geom. 83 (2005), no. 1-2, 46-56.
[10] M. Hajja and H. Martini, *Concurrency of the altitudes of a triangle*, Math. Semesterber. 60 (2013), no. 2, 249-260.
[11] M. Hajja and H. Martini, *Orthocentric simplices as the true generalizations of triangles*, Math. Intelligencer 35 (2013), no. 2, 16-28.
[12] M. Hajja and P. Walker, *The inspherical Gergonne center of a tetrahedron*, J. Geom. Graph. 8 (2004), no. 1, 23-32.
[13] H. Havlicek and G. Weiss, *Altitudes of a tetrahedron and traceless quadratic forms*, Amer. Math. Monthly 110 (2003), no. 8, 679-693.
[14] C. Kimberling, *Central points and central lines in the plane of a triangle*, Math. Mag. 67 (1994), no.3, 163-187.
[15] C. Kimberling, Encyclopedia of Triangle Centers, http://faculty.evansville.edu/ck6/encyclopedia/
[16] A. O. Konnully, *Orthocentre of a simplex*, J. Lond. Math. Soc. 39 (1964), no. 1, 685-691.
[17] S. R. Mandan, *Altitudes of a simplex in n-space*, J. Aust. Math. Soc. 2 (1962), no. 4, 403-424.
[18] S. Outudee, *Various kinds of centres of simplices*, M.Sc. thesis, Chulalongkorn University, 1999. http://cuir.car.chula.ac.th/handle/123456789/3638/
[19] J. Sadek, *Isogonal conjugates in an n-simplex*, Forum Geom. 17 (2017), 425-432.
[20] J. Sadek, M. Bani-Yaghoub and N. H. Rhee, *Isogonal conjugates in a tetrahedron*, Forum Geom. 16 (2016), 43-50.
[21] A. A. Toda, *Radii of the inscribed and escribed spheres of a simplex*, Int. J. Geom. 3 (2014), no. 2, 5-13.
[22] P. Yiu, *The uses of homogeneous barycentric coordinates in plane Euclidean geometry*, Internat. J. Math. Ed. Sci. Tech. 31 (2000), no.4, 569-578.

Reviewer's Comments

The paper under review studies generalizations of various classical centres of triangles (centroid, circumcentre, incentre, excentre and orthocentre) to those of tetrahedra. The authors starts with the generalizations of first four centres, which are straightforward, by describing them as points of concurrence of certain straight lines (note that the corresponding centres of a triangle are also defined to be points of concurrence of some straight lines associated to the triangle). When it comes to orthocentres, no generalizations as straightforward as those of the other centres are available, as the three heights of a generic tetrahedron are not concurrent. It is for this reason that the authors devote a major part of the paper to studying a number of notions of orthocentres of a tetrahedron.

1. The Monge point and quasi-orthocentres, which form a family of points lying on the Euler line, the line joining the centroid and circumcentre of a tetrahedron, parametrized by the ratio of division. This generalizes the classical result that the orthocentre of a triangle lies on the line which joins the circumcentre and the centroid of the triangle.
2. Antimedial circumcentre, which is defined to be the circumcentre of the tetrahedron whose four sides are tangent to the circumsphere of the original tetrahedron. This definition is inspired by the result that the orthocentre of a triangle is the circumcentre of the triangle whose incircle is the circumcircle of the original triangle. This centre is represented as the sum of the position vectors of the four vertices if the origin is taken to be the circumcentre (Theorem 22).
3. Orthic inexcentre, defined as the incentre or excentre of a certain tetrahedron which is homothetic to the tetrahedron whose in-sphere is the circumsphere of the original tetrahedron. The orthic inexcentre is represented as the sum of the position vectors of the circumcentres of the four triangular sides of the tetrahedron (Theorem 29), and lies on the line joining the circumcentre and the centre of homothety (Theorem 28).

Finally, the authors define general tetrahedron centres intrinsically as a map from the set of tetrahedron to $\mathbb{R}^3$ which is equivariant with respect to rigid motions. Then they characterize general tetrahedron centres as linear combinations of the position vectors of vertices, where the coefficients are barycentric functions (Theorem 39). Thus the various tetraheron centres studied before are special examples of general tetrahedron centres under this characterization.

The paper is well-written and organized, with ample illustrations and motivations for generalizations to tetrahedron centres well-explained. However, it appears that the mathematics involved in this paper is elementary and well-known (vectors and some high-school geometry). I would like to see more results of centres of higher dimensional simplices (e.g. barycentric description of centres) rather than multiple proofs of the same elementary result (e.g. Lemma 25). The following are some specific comments on the papers.

1. p.4, last two lines: the sentence should read '...we will use a unified approach to generalize the classical triangle centres to those of tetrahedra by generalizing the vertices of a triangle to those of a tetrahedron and generalizing the edges...to the faces of a tetrahedron'.
2. p.26, the first paragraph: it should be placed after Definition 23 as otherwise tangent planes, which are mentioned there, have not been defined.
3. p.27, line -3, first sentence: it should read 'Beware of the terminologies...by angle sizes...'.
4. p.28, Lemma 25: it is unnecessary to give three different proofs to a lemma, which is not one of the main results in the paper.
5. p.38, Definition 32: the notation $\mathbb{S}^2$ is usually reserved for spheres. It is better to use another notation for the set of all triangles.
6. p.38, line 12: 'lies' should read 'lie'.
7. p.38, line 16: 'equivariances' should read 'equivariance'. Then 'require' should read 'requires'.
8. p.38, line 19: '...are also a triangle...' should read '...be also a triangle...'.
9. p.39, Proposition 34: avoid using the terminology 'a group of tetrahedron centres' as a group in mathematics means a certain algebraic object. Use 'a collection of tetrahedron centres' instead.
10. p.39, line of proof of Proposition 34: add $\mathcal{X}_{25}$ after $\mathcal{K}$.
11. p.43, Definition 37: 'quadraples' should read 'quadruples'.
12. p.45: it would be better if they could spell out what the barycentric functions are for special examples of tetrahedron centres after the proof of Theorem 39.