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ISOAREAL AND ISOPERIMETRIC DEFORMATION OF CURVES

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Abstract. In the report, we want to answer the following question: How to deform a curve such that the rate of change of perimeter is minimum while the area and the total kinetic energy are fixed? This means that the perimeter shrinks fastest when $\frac{dP}{dt} > 0$, increases most slowly when $\frac{dP}{dt} > 0$ First we work on isosceles triangle as a trial. Then we study smooth simple closed curve and obtain the following results:

1. The radial velocity of each point of the curve in polar coordinates. (3.1.6)
2. The magnitude of the velocity at each point of the curve along the normal direction is equal to standard score of the curvature at that point (3.2.2).
3. Application of the results on Isoperimetric inequality (3.3).
4. The velocity for the dual isoperimetric problem (4).

1. Introduction

Deformation is a change of an object in its size or shape. Only from this word, you may not know what our project is really about. Apparently, you may think that it is a study of Isoperimetric inequality [2]. But actually, we are doing a research about a similar problem which is totally a new idea. As we have searched for any problem similar to our project in the internet, by Yahoo and even Google. However, maybe our searching skill is bad, we found nothing like this!

Before starting to study this project, we loved to play war type computer games and were interested by the army array. We thought that the best

[1] This work is done under the supervision of the authors' teacher, Mr. Wing-Kay Chang

strategy is to increase the perimeter so the army can have a higher chance attacking the enemy. The first idea exists in our mind is to find out how the army should respond so as to increase the contact surface with the enemy. It is obvious that there will not be a final state for the army array. However, how can we modify such a vague thinking to a precise mathematic problem?

On one raining day, we were shocked by a slug near the window. To expel the slug, we were inspired by its reaction. It tried to turn its body to a ball shape to minimize its surface area. We then abandoned maximizing its perimeter, trying the opposite direction. Finally, we linked up our thinking with what we were seeing, creating the present research!

In nature, there are many cases that related to minimizing the perimeter or surface area in the fast rate, for instance, animals want to protect themselves by shrinking its body or plants want to prevent loss of water by reducing its surface area. So, it is meaningful for us to study it.

Isoperimetric inequality states that among all closed curves, circle has the minimum perimeter with an enclosed fixed area. We do not pursue to prove or doubt it, but were attracted by its variable process of deformation. If a curve wants to change to this final state as fast as possible, what process will it choose? In other words, how can it minimize its perimeter at the fastest rate? That's our project's aim.

Beside Isoperimetric inequality, we also got the idea from another classic problem that is Brachistochrone problem. We are also going to find the least time for a shape to minimize its perimeter with fixed area. (cf. **Reviewer's Comments** 1) The difference between our problem and the Brachistochrone problem is that Brachistochrone curve describes the process of a point to another point; our problem is to describe the deformation process of a curve.

In chapter 1, we study isosceles triangle as it is the simplest polygon. Using simple algebra and calculus, we calculate how the triangle should be deformed so that the rate of change of perimeter is maximum. We also find the locus of the vertices by solving a suitable differential equation.

In chapter 2, we use two methods, Euler Lagrange equation[3] and inner product[5], to find the velocity of the points of a smooth simple closed curve under isoareal deformation so that the rate of change of perimeter is maximum. Then we relate the velocity at each point along the normal direction with the curvature at that point. We show that the velocity is equal to the standard score of the curvature. Then we derive the analog formulae for isoperimetric deformation.

2. Triangle

In this part, we are going to study the deformation process of triangle. Since triangle is the simplest polygon, it allows us to have a taste of how to study our project. In the following parts, we are

Minimizing the rate of change of perimeter with the area of the triangle fixed.

2.1. Isosceles Triangle

We start the study by considering isosceles triangle as its symmetric property helps us simplify the calculation. We need to think of one side of the triangle only. It is because the movement of the two bottom vertices should be the same (cf. **Reviewer's Comments** 2), otherwise if we turn over the isosceles triangle, the movement will be different in spite of the same isosceles triangle.

Consider a symmetric triangle where the three points are $A(0,a)$, $B(b,0)$, $C(-b,0)$.

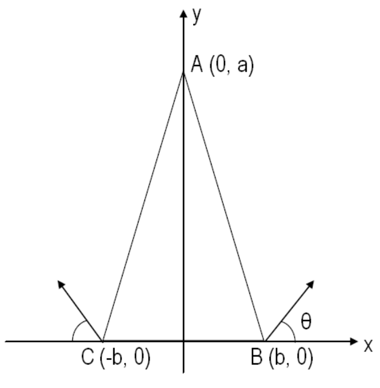


FIGURE 1

Let
A be the area of the triangle at time t.
P be the perimeter of the triangle at time t.
θ be the angle made between the horizontal and the instantaneous direction of the movement of the bottom vertices where $\theta \in [0, 2\pi]$.

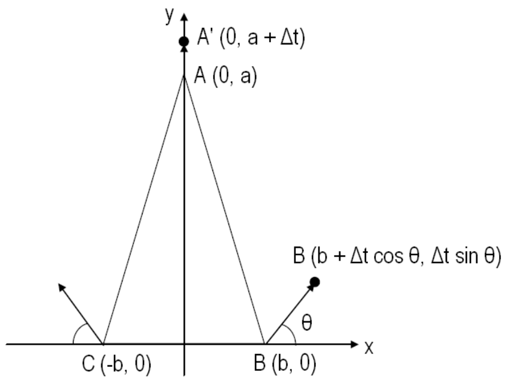


FIGURE 2

We assume that the vertices move with constant velocity 1 unit/second.

At time $t = 0$, the area $A_0 = \frac{1}{2}a(2b) = ab$.

At time $t = 0$, the perimeter $P_0 = 2b + 2\sqrt{a^2 + b^2}$.

Because of the symmetry property of isosceles triangle, the top vertex $A(0, a)$ is also forced to move either up or down, so we study the problem in two cases.

Case 1: The point $A(0, a)$ moves upwards

After time interval Δt, the area

$$A_{\Delta t} = \frac{1}{2}(a + \Delta t - \Delta t \sin\theta)[2(b + \Delta t \cos\theta)]$$

Hence

$$\begin{aligned}
\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} &= \lim_{\Delta t \to 0} \frac{A_{\Delta t} - A_0}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(a + \Delta t - \Delta t \sin\theta)(b + \Delta t \cos\theta) - ab}{\Delta t} \\
&= b(1 - \sin\theta) + a\cos\theta
\end{aligned}$$

As the area of the triangle is fixed,

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} &= 0 \\ \frac{a}{b} &= \frac{\sin\theta - 1}{\cos\theta} = \frac{\sin\frac{\theta}{2} - \cos\frac{\theta}{2}}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \qquad (1) \\ \frac{a}{b} &= \tan\left(\frac{a}{2} - \frac{\pi}{4}\right) \qquad (2) \end{aligned}$$

Refer to 1. If $a > \sqrt{3}b$, then

$$\tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right) > \sqrt{3} = \tan\frac{\pi}{3}$$

$$\frac{7\pi}{6} < \theta < \frac{3\pi}{2}$$

If $a < \sqrt{3}b$, then

$$\tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right) < \sqrt{3} = \tan\frac{\pi}{3}$$

$$\frac{\pi}{2} < \theta < \frac{7\pi}{6}$$

The above results mean that the two bottom vertices, B and C, will move differently, depending on what the original triangle is.

If it is thinner than an equilateral triangle, i.e. $a > \sqrt{3}b$

$$\frac{7\pi}{6} < \theta < \frac{3\pi}{2}$$

If it is fatter than an equilateral triangle, i.e. $a < \sqrt{3}b$

$$\frac{\pi}{2} < \theta < \frac{7\pi}{6}$$

Also, from (2), we can see that the vertices B and C has only one way to move for particular values of a and b.

Our objective is to minimize the change of perimeter under constant area. Therefore, we should also consider the rate of change of perimeter.

After time interval Δt, the perimeter

$$P_{\Delta t} = 2(b + \Delta t\cos\theta) + 2\sqrt{(a + \Delta t - \Delta t\sin\theta)^2 + (b + \Delta t\cos\theta)^2}$$

Hence

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\Delta P}{\Delta t} &= \lim_{\Delta t\to 0}\frac{P_{\Delta t}-P_0}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\begin{array}{l}2(b+\Delta t\cos\theta)\\ \quad +2\sqrt{(a+\Delta t-\Delta t\sin\theta)^2+(b+\Delta t\cos\theta)^2}\\ \quad -2b-2\sqrt{a^2+b^2}\end{array}}{\Delta t}\\ &= 2\left(\cos\theta+\frac{a(1-\sin\theta)+b\cos\theta}{\sqrt{a^2+b^2}}\right)\end{aligned}$$

Substitute (1) in it,

$$\lim_{\Delta t\to 0}\frac{\Delta P}{\Delta t} = 2\cos\theta\left(1+\frac{b^2-a^2}{b\sqrt{a^2+b^2}}\right) \tag{3}$$

If the triangle is thinner than an equilateral triangle, i.e. $a > \sqrt{3}b$,

$$1+\frac{b^2-a^2}{b\sqrt{a^2+b^2}} < 0$$

$$\frac{7\pi}{6} < \theta < \frac{3\pi}{2}$$

$$-\frac{\sqrt{3}}{2} < \cos\theta < 0$$

Refer to (3),

$$\lim_{\Delta t\to 0}\frac{\Delta P}{\Delta t} > 0$$

If the triangle is fatter than an equilateral triangle, i.e. $a < \sqrt{3}b$,

$$1+\frac{b^2-a^2}{b\sqrt{a^2+b^2}} > 0$$

$$\frac{\pi}{2} < \theta < \frac{7\pi}{6}$$

$$-\frac{\sqrt{3}}{2} < \cos\theta < 0$$

Refer to (3),

$$\lim_{\Delta t\to 0}\frac{\Delta P}{\Delta t} < 0$$

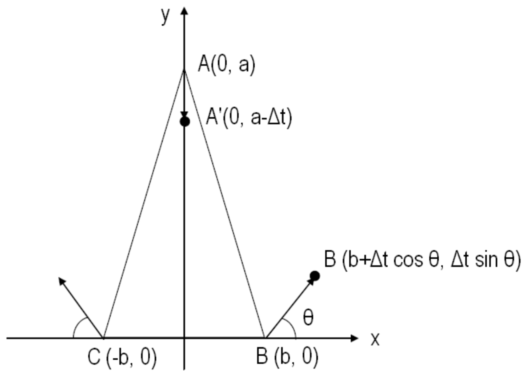


FIGURE 3

Above we find that if the vertex A moves upwards, the rate of change of perimeter will be positive for triangle thinner than an equilateral triangle but negative for triangle fatter than an equilateral triangle.

Case 2: The case of the point $A(0, a)$ moves downwards

After time interval Δt, the area

$$A_{\Delta t} = (a - \Delta t - \Delta t \sin\theta)(b + \Delta t \cos\theta)$$

Hence

$$\begin{aligned}\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} &= \lim_{\Delta t \to 0} \frac{(a - \Delta t - \Delta t \sin\theta)(b + \Delta t \cos\theta) - ab}{\Delta t} \\ &= a\cos\theta - b(1 + \sin\theta)\end{aligned}$$

As the area of the triangle is fixed,

$$\begin{aligned}\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} &= 0 \\ \frac{a}{b} &= \frac{\sin\theta + 1}{\cos\theta} = \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}} \qquad (4) \\ \frac{a}{b} &= \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \qquad (5)\end{aligned}$$

Refer to (5). If $a > \sqrt{3}b$, then

$$\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) > \sqrt{3} = \tan\frac{\pi}{3}$$

$$\frac{\pi}{6} < \theta < \frac{\pi}{2}$$

If $a < \sqrt{3}b$, then

$$\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) < \sqrt{3} = \tan\frac{\pi}{3}$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{6}$$

The above results mean that the two bottom vertices, B and C, will move differently, depending on what the original triangle is.

Similarly, from (5), we can see that the vertices B and C has only one way to move for particular values of a and b.

If it is thinner than an equilateral triangle, i.e. $a > \sqrt{3}b$

$$\frac{\pi}{6} < \theta < \frac{\pi}{2}$$

If it is fatter than an equilateral triangle, i.e. $a < \sqrt{3}b$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{6}$$

Our objective is to minimize the change of perimeter under constant area. Therefore, we should also consider the rate of change of perimeter.

After time interval Δt, the perimeter

$$P_{\Delta t} = 2(b + t\cos\theta) + 2\sqrt{(a - t - t\sin\theta)^2 + (b + t\cos\theta)^2}$$

Hence

$$\begin{aligned}
\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} &= \lim_{\Delta t \to 0} \frac{P_{\Delta t} - P_0}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\begin{array}{l} 2(b + \Delta t\cos\theta) \\ \quad +2\sqrt{(a - \Delta t - \Delta t\sin\theta)^2 + (b + \Delta t\cos\theta)^2} \\ \quad -2b - 2\sqrt{a^2 + b^2} \end{array}}{\Delta t} \\
&= 2\left(\cos\theta + \frac{-a(1 + \sin\theta) + b\cos\theta}{\sqrt{a^2 + b^2}}\right)
\end{aligned}$$

Substitute (4) in it,

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = 2\cos\theta \left(1 + \frac{b^2 - a^2}{b\sqrt{a^2+b^2}}\right) \tag{6}$$

If it is thinner than an equilateral triangle, i.e. $a > \sqrt{3}b$

$$1 + \frac{b^2 - a^2}{b\sqrt{a^2+b^2}} < 0$$

$$\frac{\pi}{6} < \theta < \frac{\pi}{2} \Rightarrow 0 < \cos\theta < \frac{\sqrt{3}}{2}$$

Refer to (6),

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} < 0$$

If it is fatter than an equilateral triangle, i.e. $a < \sqrt{3}b$

$$1 + \frac{b^2 - a^2}{b\sqrt{a^2+b^2}} > 0$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{6} \Rightarrow 0 < \cos\theta < \frac{\sqrt{3}}{2}$$

Refer to (6),

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} > 0$$

Above we find that if the vertex A moves downwards, the rate of change of perimeter will be positive for triangle fatter than an equilateral triangle but negative for triangle thinner than an equilateral triangle.

Our aim is to find out how to minimize the rate of change of the perimeter of an isosceles triangle and in above part, we have obtained all the information we needed and below is a short summary:

	Triangle thinner than an equilateral triangle	Triangle fatter than an equilateral triangle
The point $A(0, a)$ moves upwards	$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} > 0$	$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} < 0$
The point $A(0, a)$ moves downwards	$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} < 0$	$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} > 0$

Also, the direction of the movement of the vertices B and C is unique for certain values of a and b.

That means for each type of triangle, i.e. thinner or fatter than an isosceles triangle, it has only two ways to change its shape if it has to keep its area fixed. One way makes the rate of change of the perimeter positive and another makes it negative. Therefore, if we choose the way making the rate of change of the perimeter negative, we have already minimized it.

So, to minimize the rate of change of the perimeter,

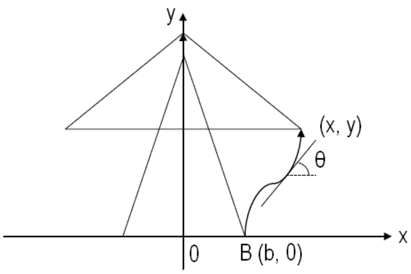
Triangle **thinner** than an equilateral triangle	The point $A(0,a)$ moves **downwards**	$\frac{a}{b} = \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$
Triangle **fatter** than an equilateral triangle	The point $A(0,a)$ moves **upwards**	$\frac{a}{b} = \tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$

where the meaning of θ is shown on Figure 1.

In the previous part, we just consider the change of the triangle simultaneously. How about the deformation process? We will now explore the process.

As the vertex A only moves up or down, we are more interested in the path of the bottom vertex. Besides, the paths of vertices B and C should be the same and just symmetric to the y-axis, so below we only consider the path of B.

Suppose the point $(b, 0)$ moves to the point (x, y) along a path.



Case 1: The point $A(0,a)$ moves upwards

By definition,

$$\frac{dy}{dx} = \tan\theta$$

By (1),

$$\frac{\text{height}}{\text{half of base}} = \frac{\sin\theta - 1}{\cos\theta}$$
$$\frac{\frac{ab}{x}}{x} = \frac{\sin\theta - 1}{\cos\theta}$$

From which we obtain

$$\frac{ab}{x^2} = \tan\theta - \sec\theta$$
$$\frac{ab}{x^2} = \frac{dy}{dx} - \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$$
$$\frac{dy}{dx} = \frac{ab}{2x^2} - \frac{x^2}{2ab}$$

Hence

$$y = \frac{ab}{2}\int \frac{dx}{x^2} - \frac{1}{2ab}\int x^2\,dx$$
$$= -\frac{ab}{2x} - \frac{x^3}{6ab} + C.$$

When $x = b$, $y = 0$,

$$0 = -\frac{ab}{2b} - \frac{b^3}{6ab} + C$$
$$C = \frac{a}{2} + \frac{b^2}{6a}$$

The required path is

$$y = -\frac{ab}{2x} - \frac{x^3}{6ab} + \frac{a}{2} + \frac{b^2}{6a}$$

Case 2: If the point $(0, a)$ moves downwards

$$\begin{cases} \dfrac{dy}{dx} = \tan\theta \\ \dfrac{\frac{ab}{x}}{x} = \dfrac{\sin\theta + 1}{\cos\theta} \end{cases}$$

From which we obtain

$$\frac{ab}{x^2} = \tan\theta + \sec\theta$$

$$\frac{ab}{x^2} = \frac{dy}{dx} + \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$$

$$\frac{dy}{dx} = \frac{ab}{2x^2} - \frac{x^2}{2ab}$$

Hence

$$\begin{aligned} y &= \frac{ab}{2}\int \frac{dx}{x^2} - \frac{1}{2ab}\int x^2\,dx \\ &= -\frac{ab}{2x} - \frac{x^3}{6ab} + C. \end{aligned}$$

When $x = b$, $y = 0$,

$$0 = -\frac{ab}{2b} - \frac{b^3}{6ab} + C$$

$$C = \frac{a}{2} + \frac{b^2}{6a}$$

The required path is

$$y = -\frac{ab}{2x} - \frac{x^3}{6ab} + \frac{a}{2} + \frac{b^2}{6a}$$

We can see that the two paths are of the same equation:

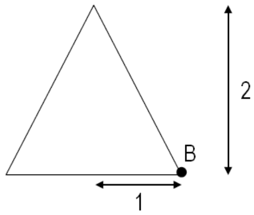
$$y = -\frac{ab}{2x} - \frac{x^3}{6ab} + \frac{a}{2} + \frac{b^2}{6a} \tag{7}$$

That means the triangle has only one path to change its shape if its area is kept fixed. To achieve the minimization of the rate of change of the perimeter, different portions of the path will be adopted.

Let's consider the following examples to explain the deformation process.

2.1.1. Example

Consider a triangle with height 2 and base 2 (thinner than an equilateral triangle).



Hence $a = 2$, $b = 1$. By (7), the path of the vertex B is given by

$$y = -\frac{1}{x} - \frac{x^3}{12} + \frac{13}{12}$$

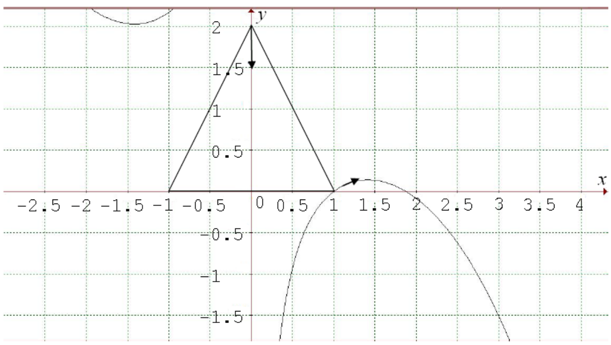


FIGURE 4

Since we are considering the right bottom vertex B, x will not be negative, we need to consider the right hand side of the graph only.

If we want to minimize

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t}$$

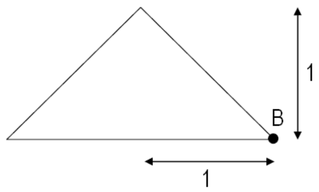
the point $A(0, a)$ moves downwards. As shown in Figure 4, the arrow shows the direction of the path of B because

$$\frac{\pi}{6} < \theta < \frac{\pi}{2}$$

Note that while the deformation process, the triangle crosses the equilibrium state: an equilateral triangle, because equilateral triangle is the triangle with the minimum perimeter among all triangles with the same area. So, when we are minimizing the rate of change of the perimeter, the perimeter is decreasing in the fastest way and the triangle must cross the equilibrium state.

2.1.2. Example

Consider a triangle with height 1 and base 2 (fatter than an equilateral triangle).



Hence $a = 1$, $b = 1$. By (7), the path of the vertex B is given by

$$y = -\frac{1}{2x} - \frac{x^3}{12} + \frac{2}{3}$$

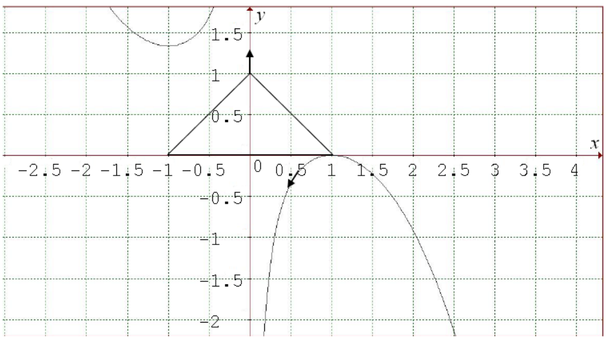


FIGURE 5

Since we are considering the right bottom vertex B, x will not be negative, we need to consider the right hand side of the graph only.

If we want to minimize

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t}$$

the point $A(0, a)$ moves upwards. As shown in Figure 5, the arrow shows the direction of the path of B because

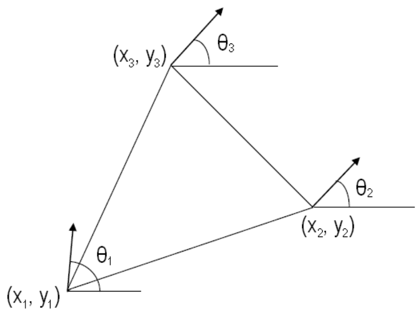
$$\frac{\pi}{2} < \theta < \frac{7\pi}{6}$$

Similarly in the deformation process, the triangle crosses the equilibrium state: an equilateral triangle, when we are minimizing the rate of change of the perimeter, the perimeter is decreasing in the fastest way and the triangle must cross the equilibrium state.

2.2. Irregular Triangle

After we have solved the case of isosceles triangles, we proceed to solve a more difficult case: irregular triangle.

We consider an irregular triangle where the three vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3).



Let
A be the area of the triangle at time t.
P be the perimeter of the triangle at time t.
$\theta_1, \theta_2, \theta_3$ be the angles made between the horizontal and the instantaneous direction of the movement of the three vertices where $\theta_1, \theta_2, \theta_3 \in [0, 2\pi]$.

We assume that the vertices move with constant velocity 1 unit/second.

At time $t=0$, the area $A_0 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

After time interval Δt, the area

$$\begin{aligned} A_{\Delta t} &= \frac{1}{2}\begin{vmatrix} x_1+\Delta t\cos\theta_1 & y_1+\Delta t\sin\theta_1 & 1 \\ x_2+\Delta t\cos\theta_2 & y_2+\Delta t\sin\theta_2 & 1 \\ x_3+\Delta t\cos\theta_3 & y_3+\Delta t\sin\theta_3 & 1 \end{vmatrix} \\ &= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \frac{\Delta t}{2}\begin{vmatrix} x_1 & \sin\theta_1 & 1 \\ x_2 & \sin\theta_2 & 1 \\ x_3 & \sin\theta_3 & 1 \end{vmatrix} + \frac{\Delta t}{2}\begin{vmatrix} \cos\theta_1 & y_1 & 1 \\ \cos\theta_2 & y_2 & 1 \\ \cos\theta_3 & y_3 & 1 \end{vmatrix} \\ &\quad + \frac{\Delta t^2}{2}\begin{vmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_3 & \sin\theta_3 & 1 \end{vmatrix} \end{aligned}$$

As the area of the triangle is fixed,

$$\lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = 0$$

$$\frac{1}{2}\begin{vmatrix} x_1 & \sin\theta_1 & 1 \\ x_2 & \sin\theta_2 & 1 \\ x_3 & \sin\theta_3 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} \cos\theta_1 & y_1 & 1 \\ \cos\theta_2 & y_2 & 1 \\ \cos\theta_3 & y_3 & 1 \end{vmatrix} = 0$$

At time $t=0$, the perimeter

$$\begin{aligned} P_0 = &\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} + \sqrt{(x_2-x_3)^2+(y_2-y_3)^2} \\ &+ \sqrt{(x_3-x_1)^2+(y_3-y_1)^2} \end{aligned}$$

After time interval Δt, the perimeter

$$\begin{aligned} P_{\Delta t} = &\sqrt{(x_1+\Delta t\cos\theta_1-x_2-\Delta t\cos\theta_2)^2+(y_1+\Delta t\sin\theta_1-y_2-\Delta t\sin\theta_2)^2} \\ &+\sqrt{(x_2+\Delta t\cos\theta_2-x_3-\Delta t\cos\theta_3)^2+(y_2+\Delta t\sin\theta_2-y_3-\Delta t\sin\theta_3)^2} \\ &+\sqrt{(x_3+\Delta t\cos\theta_3-x_1-\Delta t\cos\theta_1)^2+(y_3+\Delta t\sin\theta_3-y_1-\Delta t\sin\theta_1)^2} \end{aligned}$$

Hence

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\Delta P}{\Delta t} = &\frac{(\cos\theta_1-\cos\theta_2)(x_1-x_2)+(\sin\theta_1-\sin\theta_2)(y_1-y_2)}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}\\ &+\frac{(\cos\theta_2-\cos\theta_3)(x_2-x_3)+(\sin\theta_2-\sin\theta_3)(y_2-y_3)}{\sqrt{(x_2-x_3)^2+(y_2-y_3)^2}}\\ &+\frac{(\cos\theta_3-\cos\theta_1)(x_3-x_1)+(\sin\theta_3-\sin\theta_1)(y_3-y_1)}{\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}}\end{aligned}$$

By Lagrange Multiplier [4], we construct

$$H(\theta_1,\theta_2,\theta_3) = f(\theta_1,\theta_2,\theta_3) - \lambda g(\theta_1,\theta_2,\theta_3)$$

where

$$\begin{aligned}f(\theta_1,\theta_2,\theta_3) = &\frac{(\cos\theta_1-\cos\theta_2)(x_1-x_2)+(\sin\theta_1-\sin\theta_2)(y_1-y_2)}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}\\ &+\frac{(\cos\theta_2-\cos\theta_3)(x_2-x_3)+(\sin\theta_2-\sin\theta_3)(y_2-y_3)}{\sqrt{(x_2-x_3)^2+(y_2-y_3)^2}}\\ &+\frac{(\cos\theta_3-\cos\theta_1)(x_3-x_1)+(\sin\theta_3-\sin\theta_1)(y_3-y_1)}{\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}}\end{aligned}$$

$$g(\theta_1,\theta_2,\theta_3) = \begin{vmatrix} x_1 & \sin\theta_1 & 1\\ x_2 & \sin\theta_2 & 1\\ x_3 & \sin\theta_3 & 1\end{vmatrix} + \begin{vmatrix} \cos\theta_1 & y_1 & 1\\ \cos\theta_2 & y_2 & 1\\ \cos\theta_3 & y_3 & 1\end{vmatrix}$$

and set

$$\frac{\partial H}{\partial\theta_1} = \frac{\partial H}{\partial\theta_2} = \frac{\partial H}{\partial\theta_3} = \frac{\partial H}{\partial\lambda} = 0$$

However, we failed to solve such complicated equation explicitly, even by computer program (*Mathematica 6.0*).

We have also tried the symmetric quadrilateral by the similar method. However, as it is too complicated to solve, we have put it in the appendix.

3. Closed Curve

Now, we come to the core part of our report, which is closed curve. As our aim is to investigate the fastest change of some shapes, so we shall consider curves in polar coordinates system. Here the curves we studied are smooth.

Our objective here is

Minimizing the rate of change of perimeter with the area of the triangle fixed.

3.1. Optimization of the rate of change of perimeter of a curve with fixed area

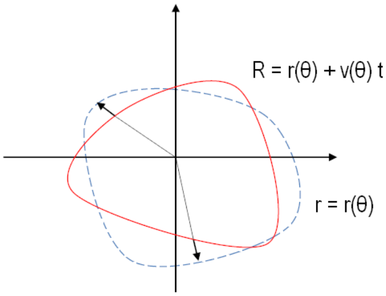
Let the equation of the dashed curve be

$$r = r(\theta)$$

and that of the solid curve be

$$R = R(\theta) = r(\theta) + v(\theta)t$$

where $v(\theta)$ is the radial velocity of each point of the curve at time t.



Let
A be the area bounded by the curve at time t
L be the length of the curve at time t.

Hence

$$\Delta A = \frac{1}{2}\int_0^{2\pi} (R^2 - r^2)\, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} (2rv\,\Delta t + v^2\Delta t^2)\, d\theta$$

$$\lim_{\Delta t\to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0} \frac{\frac{1}{2}\int_0^{2\pi} (2rv\Delta t + v^2\Delta t^2)\, d\theta}{\Delta t}$$
$$= \int_0^{2\pi} rv\, d\theta$$

$$\begin{aligned}
\Delta L &= \int_0^{2\pi} (\sqrt{R^2 + R'^2} - \sqrt{r^2 + r'^2}) d\theta \\
&= \int_0^{2\pi} (\sqrt{(r + v\Delta t)^2 + (r' + v'\Delta t)^2} - \sqrt{r^2 + r'^2}) d\theta \\
&= \int_0^{2\pi} \frac{2rv\Delta t + v^2\Delta t^2 + 2r'v'\Delta t + v'^2\Delta t^2}{\sqrt{(r + v\Delta t)^2 + (r' + v'\Delta t)^2} + \sqrt{r^2 + r'^2}} d\theta
\end{aligned}$$

$$\begin{aligned}
\lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t} &= \lim_{\Delta t \to 0} \frac{\int_0^{2\pi} \frac{2rv\Delta t + v^2\Delta t^2 + 2r'v'\Delta t + v'^2\Delta t^2}{\sqrt{(r+v\Delta t)^2 + (r'+v'\Delta t)^2} + \sqrt{r^2 + r'^2}} d\theta}{\Delta t} \\
&= \lim_{\Delta t \to 0} \int_0^{2\pi} \frac{2rv + v^2\Delta t + 2r'v' + v'^2\Delta t}{\sqrt{(r + v\Delta t)^2 + (r' + v'\Delta t)^2} + \sqrt{r^2 + r'^2}} d\theta \\
&= \int_0^{2\pi} \frac{rv + r'v'}{\sqrt{r^2 + r'^2}} d\theta
\end{aligned}$$

As the area is fixed,

$$\begin{aligned}
\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} &= 0 \\
\int_0^{2\pi} rv \, d\theta &= 0 \qquad (8) \\
& \qquad (9)
\end{aligned}$$

and we are going to minimize

$$\lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t} = \int_0^{2\pi} \frac{rv + r'v'}{\sqrt{r^2 + r'^2}} d\theta \qquad (10)$$

and we achieve the optimization by two methods:

Method 1: Euler Lagrange Equation

Euler Lagrange equation is an important formula of the calculus of variations [1, 3]. It provides a way to solve for functions which extremize a given functional. Therefore, it first comes to our mind when we handle this problem.

Set

$$J[v(\theta)] = \int_0^{2\pi} \left(\frac{rv + r'v'}{\sqrt{r^2 + r'^2}} + \lambda rv \right) d\theta = \int_0^{2\pi} F\, d\theta$$

where

$$F = F(\theta, v, v') = \frac{rv + r'v'}{\sqrt{r^2 + r'^2}} + \lambda rv$$

To optimize $J[v(\theta)]$, $v(\theta)$ should satisfy the Euler Lagrange equation,

$$F_v - \frac{d}{d\theta} F_{v'} = 0$$

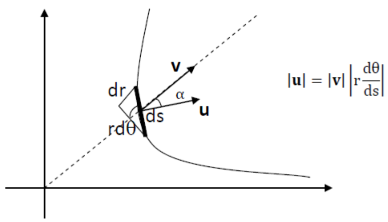
$$\frac{r}{\sqrt{r^2 + r'^2}} + \lambda r - \frac{d}{d\theta} \frac{r'}{\sqrt{r^2 + r'^2}} = 0$$

The above equation is independent of $\mathbf{v}$. (cf. **Reviewer's Comments** 3) It is because the question cannot define the function $\mathbf{v}$ well. If the question is set like this, $\mathbf{v}$ can oscillate with high frequency such that after it is added to the original curve, the area is still fixed but the rate of change of the length of the curve can be as large as possible. So we have to introduce the total kinetic energy of the curve as another constraint here.

We cannot use $\mathbf{v}$ (cf. **Reviewer's Comments** 3) directly to describe the kinetic energy, otherwise the density of the curve will not be changed uniformly and makes the question become more complicated.

On the other hand, we can consider the actual velocity $\mathbf{u}$ of the string at each point which is perpendicular to the string so that the density of the string will remain uniform.

Here
$v = |\mathbf{v}|$ is the speed of the string along the radial direction,
$u = |\mathbf{u}|$ is the speed of the string perpendicular to the string.



Since $\mathbf{u}$ is a component of $\mathbf{v}$,

$$|\mathbf{u}| = |\mathbf{v}| \cos \alpha$$

Note that

$$\cos\alpha = r\frac{d\theta}{ds}$$

so

$$|\mathbf{u}| = |\mathbf{v}|\cos\alpha = |\mathbf{v}|\left(r\frac{d\theta}{ds}\right) \tag{11}$$

Together with $\frac{ds}{d\theta} = \sqrt{r^2 + r'^2}$, the kinetic energy of the string should be

$$\int_0^{2\pi} \rho|\mathbf{u}|^2\, ds = \int_0^{2\pi} \rho\left(r\frac{d\theta}{ds}\right)^2 v^2\, ds = \int_0^{2\pi} \rho\frac{r^2}{\sqrt{r^2+r'^2}}v^2\, d\theta$$

where

$$\text{density } \rho = \frac{\text{mass}}{\text{length}} = \frac{M}{\int_0^{2\pi}\sqrt{r^2+r'^2}\, d\theta}.$$

Therefore

$$\frac{1}{2}\int_0^{2\pi} \rho\frac{r^2}{\sqrt{r^2+r'^2}}v^2\, d\theta = C$$

$$\frac{M}{\int_0^{2\pi}\sqrt{r^2+r'^2}\, d\theta}\int_0^{2\pi} \frac{r^2}{\sqrt{r^2+r'^2}}v^2\, d\theta = 2C$$

$$M\int_0^{2\pi} \frac{r^2}{\sqrt{r^2+r'^2}}v^2\, d\theta = 2C\int_0^{2\pi}\sqrt{r^2+r'^2}\, d\theta$$

For simplicity, we choose $M = 2C = 1$,

$$\int_0^{2\pi}\left(\frac{r^2}{\sqrt{r^2+r'^2}}v^2 - \sqrt{r^2+r'^2}\right)\, d\theta = 0 \tag{12}$$

Again set

$$\begin{aligned} J[v(\theta)] &= \int_0^{2\pi}\left[\frac{rv + r'v'}{\sqrt{r^2+r'^2}} + \lambda rv + \lambda'\left(\frac{r^2}{\sqrt{r^2+r'^2}}v^2 - \sqrt{r^2+r'^2}\right)\right] d\theta \\ &= \int_0^{2\pi} F\, d\theta \end{aligned}$$

where

$$F = F(\theta, v, v') = \frac{rv + r'v'}{\sqrt{r^2+r'^2}} + \lambda rv + \lambda'\left(\frac{r^2}{\sqrt{r^2+r'^2}}v^2 - \sqrt{r^2+r'^2}\right)$$

By the Euler Lagrange Equation,

$$\begin{aligned} F_v - \frac{d}{d\theta}F_{v'} &= 0 \\ \frac{r}{\sqrt{r^2+r'^2}} + \lambda r + 2\lambda'\frac{r^2}{\sqrt{r^2+r'^2}}v - \frac{d}{d\theta}\frac{r'}{\sqrt{r^2+r'^2}} &= 0 \\ \frac{r}{\sqrt{r^2+r'^2}} + \lambda r + 2\lambda'\frac{r^2}{\sqrt{r^2+r'^2}}v - \frac{r^2r''-rr'^2}{(r^2+r'^2)^{3/2}} &= 0 \qquad (13) \end{aligned}$$

Hence

$$\begin{aligned} v &= \frac{1}{2\lambda'\frac{r^2}{\sqrt{r^2+r'^2}}}\left[\frac{r^2r''-rr'^2}{(r^2+r'^2)^{3/2}} - \frac{r}{\sqrt{r^2+r'^2}} - \lambda r\right] \\ v &= \frac{1}{2\lambda'}\left[\frac{r^2r''-rr'^2}{r(r^2+r'^2)} - \frac{1}{r} - \frac{\lambda\sqrt{r^2+r'^2}}{r}\right] \qquad (14) \end{aligned}$$

Multiply (13) both sides by $\frac{\sqrt{r^2+r'^2}}{r}$,

$$1 + \lambda\sqrt{r^2+r'^2} + 2\lambda' rv - \frac{rr''-r'^2}{r^2+r'^2} = 0$$

Integrate both sides with respect to θ,

$$\begin{aligned} \int_0^{2\pi}\left[1 + \lambda\sqrt{r^2+r'^2} + 2\lambda' rv - \frac{rr''-r'^2}{r^2+r'^2}\right]d\theta &= 0 \\ \lambda\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta - \int_0^{2\pi}\left[\frac{rr''-r'^2}{r^2+r'^2} - 1\right]d\theta &= 0 \\ \lambda &= \frac{\int_0^{2\pi}\left[\frac{rr''-r'^2}{r^2+r'^2} - 1\right]d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta} \qquad (15) \end{aligned}$$

Besides, from the constraint of total KE (12),

$$\int_0^{2\pi}\left[\frac{r^2}{\sqrt{r^2+r'^2}}v^2 - \sqrt{r^2+r'^2}\right]d\theta = 0.$$

We find that

$$\lambda' = \sqrt{\frac{\int_0^{2\pi}\frac{1}{4}\left[\frac{rr''-r'^2}{r(r^2+r'^2)} - \frac{\lambda\sqrt{r^2+r'^2}}{r}\right]^2\frac{r^2}{\sqrt{r^2+r'^2}}\,d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}} \qquad (16)$$

We can see that by Euler Lagrange Equation, we can find the function v such that it optimizes the rate of change of the perimeter with the area bounded by the curve is still fixed.

In order to show our result more clearly, below is an example illustrate how the function **v** works on the curve.

3.1.1. Example

The following is the equation of an ellipse

$$r = \frac{\sqrt{2}}{\sqrt{2\sin^2\theta + \cos^2\theta}}$$

By using computer program (*Mathematica 6.0*), we calculate the function **v**.

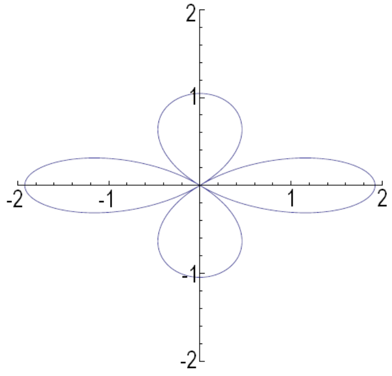
$$\lambda = \frac{\int_0^{2\pi}\left(\frac{rr''-r'^2}{r^2+r'^2} - 1\right)d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}d\theta} = -0.822364$$

$$\lambda' = \sqrt{\frac{\int_0^{2\pi}\frac{1}{4M}\left(\frac{rr''-r'^2}{r(r^2+r'^2)} - \frac{1}{r} - \frac{\lambda\sqrt{r^2+r'^2}}{r}\right)^2\frac{r^2}{\sqrt{r^2+r'^2}}d\theta}{\int_0^{2\pi}C\sqrt{r^2+r'^2}d\theta}}$$
$$= 0.154052$$

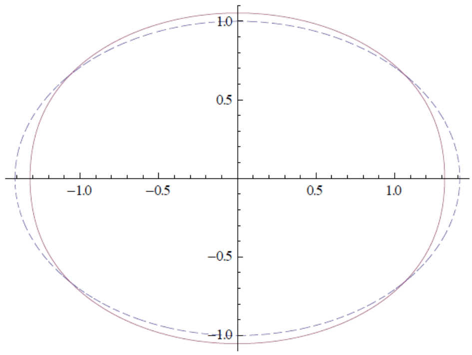
where

$$\begin{aligned}
v = 3.245671196522956\Bigg(& -\frac{\sqrt{\cos^2\theta + 2\sin^2\theta}}{\sqrt{2}} \\
& + 0.5814990719760088\sqrt{\cos^2\theta + 2\sin^2\theta} \\
& \times \sqrt{\frac{2\cos^2\theta\sin^2\theta}{(\cos^2\theta+2\sin^2\theta)^3} + \frac{2}{\cos^2\theta+2\sin^2\theta}} \\
& + \frac{\sqrt{\cos^2\theta+2\sin^2\theta}}{\sqrt{2}\left(\frac{2\cos^2\theta\sin^2\theta}{(\cos^2\theta+2\sin^2\theta)^3} + \frac{2}{\cos^2\theta+2\sin^2\theta}\right)} \\
& \times \Bigg(-\frac{2\cos^2\theta\sin^2\theta}{(\cos^2\theta+2\sin^2\theta)^3} \\
& + \frac{2\left(\frac{3\cos^2\theta\sin^2\theta}{(\cos^2\theta+2\sin^2\theta)^{5/2}} - \frac{\cos^2\theta}{(\cos^2\theta+2\sin^2\theta)^{3/2}} + \frac{\sin^2\theta}{(\cos^2\theta+2\sin^2\theta)^{3/2}}\right)}{\sqrt{\cos^2\theta+2\sin^2\theta}}\Bigg)\Bigg)
\end{aligned}$$

Here is the curve of $\mathbf{v}$:



and here are the curves $r = r$(dashed one) and $R = r + (0.05)v$(solid one):



After we have found the function $\mathbf{v}$, it is natural to ask: is the solution a maximum or a minimum?

In calculus of variation, to check whether the solution is a maximum or a minimum, it is necessary to check whether then solution satisfies Jacobi Condition and Legendre Condition [1] or not. However it is difficult and tedious to do so. We try another approach instead: inner product.

Method 2: Inner Product

The reason why we use inner product to solve the problem is that the result we obtained by the method of calculus of variation is too complicated to check whether it is a maximum or a minimum by ordinary method. So we think of using inner product as integrals involved can be converted into inner products.

Inner product has the following properties:

1. $\langle \mathbf{u}, \mathbf{u} \rangle \geqslant 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$
2. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
3. $\langle a\mathbf{u}_1 + b\mathbf{u}_2, \mathbf{v} \rangle = a\langle \mathbf{u}_1, \mathbf{v} \rangle + b\langle \mathbf{u}_2, \mathbf{v} \rangle$

From (2), the rate of change of the length of the curve is

$$\begin{aligned}\lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t} &= \int_0^{2\pi} \frac{rv + r'v'}{\sqrt{r^2 + r'^2}}\, d\theta \\ &= \int_0^{2\pi} \frac{rv}{\sqrt{r^2 + r'^2}}\, d\theta + \left[\frac{r'v}{\sqrt{r^2 + r'^2}}\right]_0^{2\pi} - \int_0^{2\pi} v \left(\frac{r'}{\sqrt{r^2 + r'^2}}\right)' d\theta\end{aligned}$$

Since $\mathbf{r}$ is a simple closed curve and smooth on $[0, 2\pi]$, $r(0) = r(2\pi)$, $r'(0) = r'(2\pi)$, $v(0) = v(2\pi)$

$$\left[\frac{r'v}{\sqrt{r^2 + r'^2}}\right]_0^{2\pi} = 0$$

$$\lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t} = \int_0^{2\pi} \frac{rv}{\sqrt{r^2 + r'^2}}\, d\theta - \int_0^{2\pi} v \left(\frac{r'}{\sqrt{r^2 + r'^2}}\right)' d\theta$$

Our aim is to minimize the above integral under the conditions (8) and (12):

1. $\displaystyle\int_0^{2\pi} rv\, d\theta = 0$
2. $\displaystyle\int_0^{2\pi} \left(\frac{r^2}{\sqrt{r^2 + r'^2}} v^2 - \sqrt{r^2 + r'^2}\right) d\theta = 0$

$$\int_0^{2\pi} v^2 \left(\frac{r^2}{\sqrt{r^2 + r'^2}}\right) d\theta = \int_0^{2\pi} \sqrt{r^2 + r'^2}\, d\theta = L.$$

We define

$$\langle a, b \rangle = \int_0^{2\pi} ab \left(\frac{r^2}{\sqrt{r^2 + r'^2}}\right) d\theta$$

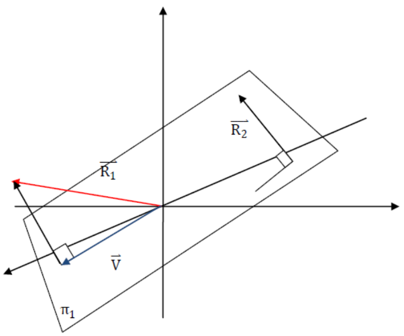
Let $R_1 = \left[\frac{r}{\sqrt{r^2 + r'^2}} - \left(\frac{r'}{\sqrt{r^2 + r'^2}}\right)'\right] \left(\frac{\sqrt{r^2 + r'^2}}{r^2}\right)$ and $R_2 = r \left(\frac{\sqrt{r^2 + r'^2}}{r^2}\right)$.

Hence

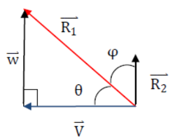
$$\langle v, R_1\rangle = \int_0^{2\pi} v\left[\frac{r}{\sqrt{r^2+r'^2}} - \left(\frac{r'}{\sqrt{r^2+r'^2}}\right)'\right]\left(\frac{\sqrt{r^2+r'^2}}{r^2}\right)\left(\frac{r^2}{\sqrt{r^2+r'^2}}\right)d\theta$$
$$= \int_0^{2\pi} v\left[\frac{r}{\sqrt{r^2+r'^2}} - \left(\frac{r'}{\sqrt{r^2+r'^2}}\right)'\right]d\theta$$
$$\langle v, R_2\rangle = \int_0^{2\pi} vr\left(\frac{\sqrt{r^2+r'^2}}{r^2}\right)\left(\frac{r^2}{\sqrt{r^2+r'^2}}\right)d\theta$$
$$= \int_0^{2\pi} rv\,d\theta = 0$$
$$\langle v, v\rangle = \int_0^{2\pi} v^2\left(\frac{r^2}{\sqrt{r^2+r'^2}}\right)d\theta = L$$

Since $\langle v, R_2\rangle = \int_0^{2\pi} rv\,dx = 0$, $\mathbf{v}$ can be any vector lying on the plane π_1 which is perpendicular to $\mathbf{R}_2$.

Suppose $\mathbf{V}$ lies on the plane π_1, and $\mathbf{R}_1$ is another vector represented by the arrow in the second quadrant, and $\mathbf{v} = n\mathbf{V}$.



To minimize $\int_0^{2\pi} \frac{rv + r'v'}{\sqrt{r^2+r'^2}}\,d\theta = \int_0^{2\pi} v\left[\frac{r}{\sqrt{r^2+r'^2}} - \left(\frac{r'}{\sqrt{r^2+r'^2}}\right)'\right]d\theta = \langle v, R_1\rangle$, we first maximize this by choosing the projection of $\mathbf{R}_1$ on the plane π_1 , i.e. the bottom arrow,



Since $\mathbf{V} = \mathbf{R}_1 - \mathbf{w}$ and $\mathbf{w}$ is parallel to $\mathbf{R}_2$, we have

$$\begin{aligned}\mathbf{w} &= |\,\mathbf{w}\,|\frac{\mathbf{R}_2}{|\,\mathbf{R}_2\,|} = (\,|\,\mathbf{R}_1\,|\sin\theta\,)\frac{\mathbf{R}_2}{|\,\mathbf{R}_2\,|} \\ &= \frac{|\,\mathbf{R}_1\,||\,\mathbf{R}_2\,|\cos\varphi}{|\,\mathbf{R}_2\,|^2}\mathbf{R}_2 = \frac{\langle \mathbf{R}_1, \mathbf{R}_2\rangle}{\langle \mathbf{R}_2, \mathbf{R}_2\rangle}\mathbf{R}_2\end{aligned}$$

So

$$\begin{aligned}\mathbf{V} &= \mathbf{R}_1 - \mathbf{w} \\ \mathbf{V} &= \mathbf{R}_1 - \frac{\langle \mathbf{R}_1, \mathbf{R}_2\rangle}{\langle \mathbf{R}_2, \mathbf{R}_2\rangle}\mathbf{R}_2\end{aligned}$$

To find $\mathbf{v}$ such that $\langle\, v, R_1\rangle$ is minimized and $\langle v, v\rangle = L$, we must have $\mathbf{v} = -k\mathbf{V}$ where k is a positive constant.

$$\begin{aligned}v = -k &\left\{\left[\frac{r}{\sqrt{r^2+r'^2}} - \left(\frac{r}{\sqrt{r^2+r'^2}}\right)'\right]\left(\frac{\sqrt{r^2+r'^2}}{r^2}\right)\right. \\ &- \frac{\int_0^{2\pi}\left[\frac{r}{\sqrt{r^2+r'^2}} - \left(\frac{r'}{\sqrt{r^2+r'^2}}\right)'\right]\left(\frac{\sqrt{r^2+r'^2}}{r^2}\right) r \left(\frac{\sqrt{r^2+r'^2}}{r^2}\right)\left(\frac{r^2}{\sqrt{r^2+r'^2}}\right) d\theta}{\int_0^{2\pi} r^2\left(\frac{\sqrt{r^2+r'^2}}{r^2}\right)^2\left(\frac{r^2}{\sqrt{r^2+r'^2}}\right) d\theta} \\ &\qquad \left. r\left(\frac{\sqrt{r^2+r'^2}}{r^2}\right)\right\} \\ = k &\left[\frac{rr''-r'^2}{r(r^2+r'^2)} - \frac{\int_0^{2\pi}\left[\frac{r^2r''-rr'^2}{r(r^2+r'^2)} - 1\right] d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\, d\theta}\left(\frac{\sqrt{r^2+r'^2}}{r}\right)\right]\end{aligned}$$

Since

$$\begin{aligned}
\langle v, v\rangle &= L \\
\int_0^{2\pi} v^2 \left(\frac{r^2}{\sqrt{r^2+r'^2}}\right) d\theta &= L \\
\int_0^{2\pi} k^2 \left[\frac{rr''-r'^2}{r(r^2+r'^2)} \right. \\
\left. -\frac{\int_0^{2\pi}\left[\frac{r^2r''-rr'^2}{r(r^2+r'^2)}-1\right]d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}\left(\frac{\sqrt{r^2+r'^2}}{r}\right)\right]^2 \\
\left(\frac{r^2}{\sqrt{r^2+r'^2}}\right) d\theta &= \int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta
\end{aligned}$$

we can solve that

$$k = \sqrt{\frac{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}{\int_0^{2\pi}\left[\frac{rr''-r'^2}{r(r^2+r'^2)}-\frac{\int_0^{2\pi}\left[\frac{r^2r''-rr'^2}{r(r^2+r'^2)}-1\right]d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}\left(\frac{\sqrt{r^2+r'^2}}{r}\right)\right]^2\left(\frac{r^2}{\sqrt{r^2+r'^2}}\right)d\theta}}$$

When we compare this with (14), (15) and (16),

$$v = \frac{1}{2\lambda'}\left[\frac{rr''-r'^2}{r(r^2+r'^2)} - \frac{\lambda\sqrt{r^2+r'^2}}{r}\right]$$

where

$$\lambda = \frac{\int_0^{2\pi}\left[\frac{rr''-r'^2}{r^2+r'^2}-1\right]d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}$$

$$\lambda' = \sqrt{\frac{\int_0^{2\pi}\frac{1}{4}\left[\frac{rr''-r'^2}{r(r^2+r'^2)}-\frac{\lambda\sqrt{r^2+r'^2}}{r}\right]^2\frac{r^2}{\sqrt{r^2+r'^2}}\,d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}}$$

we can see that

$$\begin{aligned}
\frac{1}{2\lambda'} &= \sqrt{\frac{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}{\int_0^{2\pi}\left[\frac{rr''-r'^2}{r(r^2+r'^2)}-\frac{\lambda\sqrt{r^2+r'^2}}{r}\right]^2\frac{r^2}{\sqrt{r^2+r'^2}}\,d\theta}} \\
&= k
\end{aligned}$$

Hence

$$v = k\left[\frac{rr''-r'^2}{r(r^2+r'^2)} - \frac{\int_0^{2\pi}\left[\frac{rr''-r'^2}{r^2+r'^2}-1\right]d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}\left(\frac{\sqrt{r^2+r'^2}}{r}\right)\right]$$
$$= \frac{1}{2\lambda'}\left[\frac{rr''-r'^2}{r(r^2+r'^2)} - \frac{\lambda\sqrt{r^2+r'^2}}{r}\right]$$

The two different approaches give the same result. Since the function v solved by inner product method minimizes

$$\langle v, R_1\rangle = \int_0^{2\pi} v\left[\frac{r}{\sqrt{r^2+r'^2}} - \left(\frac{r'}{\sqrt{r^2+r'^2}}\right)'\right]d\theta$$

i.e.

$$\lim_{\Delta t\to 0}\frac{\Delta L}{\Delta t} = \int_0^{2\pi}\frac{rv+r'v'}{\sqrt{r^2+r'^2}}\,d\theta = \int_0^{2\pi} v\left[\frac{r}{\sqrt{r^2+r'^2}} - \left(\frac{r'}{\sqrt{r^2+r'^2}}\right)'\right]d\theta$$

is minimized by the function v.

3.2. Curvature

We can see that the expressions in (14), (15) and (16) are so tedious for us to derive the deformation of the curve. Is it possible for us to predict how the curve deforms without complicated calculation?

We think that the solution should be coordinates free, that is no matter how the coordinates are put, the curve should have the same change such that the rate of change of the perimeter is the minimum while its area is kept fixed. The change should be the same in spite of the presentation of the curve.

So we think of some coordinates independent quantities, one of them is the curvature of the curve. Since curvature depends on the shape of the curve, not the position of the curve, i.e. the equation of the curve, it may be possible for us to express the deformation in terms of it. We also believe that the change of the curve will depend on the other quantities which are independent of the coordinates, e.g. the length of the curve, the area bounded by the curve, etc. Also, the function v in (14) depends on the coordinates system as it defines the radial velocity of each point. If the curve is put in different direction, v will change.

So we use u, which is the velocity of each point along the normal direction to the curve and is independent of the coordinates system.

The curvature of closed curve is defined as

$$\kappa = \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}}$$

The length of the closed curve is defined as

$$L = \int_0^{2\pi} \sqrt{r^2 + r'^2}\, d\theta$$

From (11) together with (14), the function u is

$$\begin{aligned} u &= v\left(r\frac{d\theta}{ds}\right) \\ &= \frac{1}{2\lambda'}\left[\frac{rr'' - r'^2}{r(r^2 + r'^2)} - \frac{\lambda\sqrt{r^2 + r'^2}}{r}\right]\left(\frac{r}{\sqrt{r^2 + r'^2}}\right) \\ &= \frac{1}{2\lambda'}\left[\frac{rr'' - r'^2}{(r^2 + r'^2)^{3/2}} - \frac{1}{\sqrt{r^2 + r'^2}} - \lambda\right] \\ &= \frac{1}{2\lambda'}\left[-\frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} - \lambda\right] \end{aligned}$$

Hence $u = \frac{1}{2\lambda'}(-\kappa - \lambda)$.

From (15) and (16)

$$\begin{aligned} \lambda &= \frac{\int_0^{2\pi}\left[\frac{rr'' - r'^2}{r^2 + r'^2} - 1\right] d\theta}{\int_0^{2\pi}\sqrt{r^2 + r'^2}\, d\theta} = \frac{\int_0^{2\pi}\left[-\frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}}\right]\sqrt{r^2 + r'^2}\, d\theta}{\int_0^{2\pi}\sqrt{r^2 + r'^2}\, d\theta} \\ &= -\frac{\int_0^L \kappa\, ds}{L} \end{aligned}$$

$$\lambda' = \sqrt{\frac{\int_0^{2\pi} \frac{1}{4}\left[\frac{rr''-r'^2}{r(r^2+r'^2)} - \frac{\lambda\sqrt{r^2+r'^2}}{r}\right]^2 \frac{r^2}{\sqrt{r^2+r'^2}}\,d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}}$$

$$= \sqrt{\frac{\int_0^{2\pi} \frac{1}{4}\left[\frac{rr''-r'^2}{(r^2+r'^2)^{3/2}} - \frac{1}{\sqrt{r^2+r'^2}} - \lambda\right]^2 \sqrt{r^2+r'^2}\,d\theta}{\int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta}}$$

$$= \sqrt{\frac{\int_0^L \frac{1}{4}(-\kappa-\lambda)^2\,ds}{L}}$$

we can also see that the above quantities are independent of the coordinates, but dependent of the unchanged quantities L and κ.

Therefore,

	Closed curve
u	$\frac{1}{2\lambda'}(-\kappa-\lambda)$
λ	$\frac{\int_0^L -\kappa\,ds}{L}$
λ'	$\sqrt{\frac{\int_0^L \frac{1}{4}(-\kappa-\lambda)^2\,ds}{L}}$

By referring to Appendix 5.2, the expressions of u, λ and λ' for open curve and closed curve are the same. It is reasonable because the above expressions are independent of the coordinates, as well as the coordinates system.

In fact, they have a deeper meaning, note that

$$-\lambda = \frac{\int_0^L \kappa\,ds}{L} = \frac{2\pi}{L}$$

which can be interpreted as the average curvature of the curve.

So,

$$u = \frac{1}{2\lambda'}(-\kappa-\lambda) = \frac{1}{2\lambda'}(-\kappa+\bar{\kappa}) \tag{17}$$

where $\bar{\kappa} = -\lambda$ defines the average curvature.

Also,

$$\lambda' = \sqrt{\frac{\int_0^L \frac{1}{4}(-\kappa - \lambda)^2\, ds}{L}} = \sqrt{\frac{\frac{1}{4}\int_0^L (\kappa^2 + 2\lambda\kappa\lambda^2)\, ds}{L}}$$

$$= \sqrt{\frac{\frac{1}{4}\left(\int_0^L \kappa^2\, ds - \frac{8\pi^2}{L} + \frac{4\pi^2}{L}\right)}{L}} = \frac{1}{2}\sqrt{\frac{\int_0^L \kappa^2\, ds}{L} - \left(\frac{2\pi}{L}\right)^2} = \frac{1}{2}\sigma$$

where σ is the standard deviation of curvature. Therefore, the maximum decreasing rate of perimeter is obtained when the velocity along the normal direction,

$$u = -\frac{\kappa - \bar{\kappa}}{\sigma} \tag{18}$$

where the right hand side is simply the standard score of $-\kappa$.

We can deduce how the curve changes by the above expressions. The following example illustrates how we can predict the deformation process of the curve.

3.2.1. Example

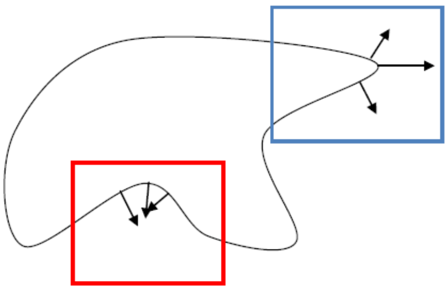


FIGURE 6

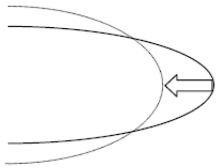
Figure 6 shows a smooth simple closed curve where the arrows show the normal to the curve at some points.

The right-boxed portion of the curve has a larger curvature than its average curvature as the normal of the curve changes sharply.

From (17),

$$u = \frac{1}{2\lambda'}(-\kappa + \bar{\kappa}) < 0$$

so the portion will move like this:

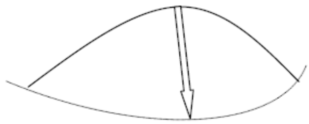


The left-boxed portion of the curve has a smaller curvature than the average curvature because the normal turns clock-wisely.

From (17),

$$u = \frac{1}{2\lambda'}(-\kappa + \bar{\kappa}) > 0$$

so the portion will move like this:



3.3. Isoperimetric Inequality

The isoperimetric problem can be stated as follows: Among all closed curves in the plane of fixed perimeter, which curve maximizes the area of its enclosed region? This question can be shown to be equivalent to the following problem: Among all closed curves in the plane enclosing a fixed area, which curve minimizes the perimeter? Circle has been proved to be the answer to the question, that means circle has the minimum perimeter among all closed curves enclosing a fixed area.

How about in our deforming process, what will be the curve if it continuously minimizes its rate of change of perimeter? We guess the shape will finally deform to a circle as fast as possible, that is when it reaches a circle, it will have no further change.

From our result, the function u can minimize the perimeter of a given closed curve as fast as possible at every moment. But the curve must be continuously changing because the KE of the particles is constant. But we believe the curve will pass through an equilibrium state in the process and then the perimeter increases again. So what is the equilibrium state? When the curve reaches the equilibrium state, its perimeter cannot be minimized anymore and has no change at that particular moment.

From (17),

$$u = \frac{1}{2\lambda'}(-\kappa + \bar{\kappa})$$

When $u = 0$,

$$\frac{1}{2\lambda'}(-\kappa + \bar{\kappa}) = 0$$

$$\kappa = \bar{\kappa} = \frac{\int_0^L \kappa \, ds}{L} = \frac{2\pi}{L}$$

The curvature equals a constant if and only if the curve is a circle.

Therefore we have proved that the equilibrium state of the curve is a circle under this process but it does not imply that the curve will reach an equilibrium state. Hence if there exists a shape with minimum perimeter, then it must be a circle.

4. Dual Problem - Isoperimetric Deformation

By similar technique, we can solve the dual problem, that is maximizing the rate of change of the area with the perimeter and the total kinetic energy keeping fixed. It is the same as what we do before - minimizing the rate of change of the perimeter with the area and the total kinetic energy keeping fixed. As we believe that it will be easier for us to find the deformation process from the results obtained, we do this problem too. (cf. **Reviewer's Comments** 4)

We find that

$$u = \frac{1}{2\lambda'}(\lambda\kappa + 1)$$

where

$$\lambda = -\frac{2\pi}{\int_0^L \kappa^2 \, ds}$$

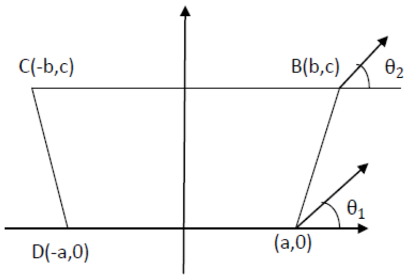
$$\lambda' = \sqrt{\frac{\int_0^L \frac{1}{4}(\lambda\kappa + 1)^2 \, ds}{L}}$$

5. Appendix

5.1. Symmetric Quadrilateral

Since we cannot solve the irregular triangle, we want to have a breakthrough if we can consider the case of a regular quadrilateral.

We consider a symmetric quadrilateral where the four vertices are $A(a,0)$, $B(b,c)$, $C(-b,c)$, $D(-a,0)$.



Let
A be the area of the quadrilateral at time t.
P be the perimeter of the quadrilateral at time t.
θ be the angle made between the horizontal and the instantaneous direction of the movement of the bottom vertices where $\theta \in [0, 2\pi]$.

We assume that the vertices move with constant velocity 1 unit/second.

At time $t = 0$, the area $A_0 = (a+b)c$.

After time interval Δt, the area

$$A_{\Delta t} = (a + \Delta t \cos\theta_1 + b + \Delta t \cos\theta_2)(c + \Delta t \sin\theta_2 - \Delta t \sin\theta_1)$$

As the area of the quadrilateral is fixed,

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = 0$$
$$c(\cos\theta_1 + \cos\theta_2) + (a+b)(\sin\theta_2 - \sin\theta_1) = 0$$

At time $t = 0$, the perimeter $P_0 = 2a + 2b + 2\sqrt{(b-a)^2 + c^2}$.

After time interval Δt, the perimeter

$$\begin{aligned} P_{\Delta t} &= 2(a + \Delta t \cos\theta_1) + 2(b + \Delta t \cos\theta_2) \\ &\quad + 2\sqrt{(b + \Delta t\cos\theta_2 - a - \Delta t\cos\theta_1)^2 + (c + \Delta t \sin\theta_2 - \Delta t\sin\theta_1)^2} \end{aligned}$$

Hence

$$\lim_{\Delta t\to 0} \frac{\Delta P}{\Delta t} = 2(\cos\theta_1 + \cos\theta_2) + \frac{2[(b-a)(\cos\theta_2 - \cos\theta_1) + c(\sin\theta_2 - \sin\theta_1)]}{\sqrt{(b-a)^2 + c^2}}$$

By Lagrange Multiplier, we construct

$$\begin{aligned} H(\theta_1, \theta_2) &= 2(\cos\theta_1 + \cos\theta_2) + \frac{2[(b-a)(\cos\theta_2 - \cos\theta_1) + c(\sin\theta_2 - \sin\theta_1)]}{\sqrt{(b-a)^2 + c^2}} \\ &\quad - \lambda[c(\cos\theta_1 + \cos\theta_2) + (a+b)(\sin\theta_2 - \sin\theta_1)] \end{aligned}$$

and set

$$\begin{aligned} \frac{\partial H}{\partial\theta_1} &= -2\sin\theta_1 + 2\left[\frac{(b-a)\sin\theta_1 + c(-\cos\theta_1)}{\sqrt{(b-a)^2 + c^2}}\right] \\ &\qquad - \lambda[(a+b)(-\cos\theta_1) + c(1 - \sin\theta_1)] \\ \frac{\partial H}{\partial\theta_2} &= -\lambda[(a+b)\cos\theta_2 - c\sin\theta_2] \\ &\qquad + 2\left[\frac{\cos\theta_2 - [-a + b + \sqrt{(a-b)^2 + c^2}]\sin\theta_2}{\sqrt{(a-b)^2 + c^2}}\right] \\ \frac{\partial H}{\partial\lambda} &= -c\cos\theta_1 - c\cos\theta_2 + (a+b)(\sin\theta_1 - \sin\theta_2) \end{aligned}$$

Here is the same situation like irregular triangle. Although, it seems to be easier than the case of irregular triangle, the equations are still too complicated to be solved.

5.2. Open Curve

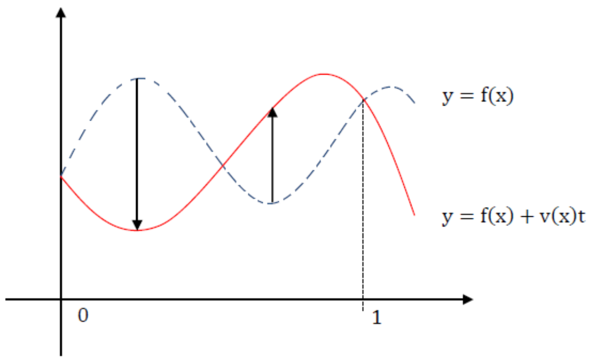
Before we try the closed curve, we have tried the open curve as a trial. Let the equation of the dashed curve be

$$y = f(x)$$

and that of the solid curve be

$$y = f(x) + v(x)t$$

where $v(x)$ is the vertical velocity of each point of the curve at time t.



Let
A be the area bounded by the curve and the x-axis at time t
L be the length of the curve at time t.

At time $t = 0$, the area bounded by the curve and the x-axis

$$A_0 = \int_0^1 f\,dx$$

At time $t = 0$, the length of the curve

$$L_0 = \int_0^1 \sqrt{1 + f'^2}\,dx$$

After time interval Δt, the area bounded by the curve and x-axis

$$A_{\Delta t} = \int_0^1 (f + v\Delta t)\,dx$$

After time interval Δt, the length of the curve

$$L_{\Delta t} = \int_0^1 \sqrt{1 + (f' + v'\Delta t)^2}\,dx$$

Hence

$$\Delta A = A_{\Delta t} - A_0 = \int_0^1 (f + v\Delta t)\,dx - \int_0^1 f\,dx = \Delta t \int_0^1 v\,dx$$

$$\lim_{\Delta t\to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta t \int_0^1 v\,dx}{\Delta t} = \int_0^1 v\,dx$$

$$\begin{aligned}\Delta L &= L_{\Delta t} - L_0 \\ &= \int_0^1 \sqrt{1 + (f' + v'\Delta t)^2}\,dx - \int_0^1 \sqrt{1 + f'^2}\,dx \\ &= \int_0^1 (\sqrt{1 + (f' + v'\Delta t)^2} - \sqrt{1 + f'^2})\,dx\end{aligned}$$

$$\begin{aligned}\lim_{\Delta t\to 0} \frac{\Delta L}{\Delta t} &= \lim_{\Delta t\to 0} \frac{\int_0^1 (\sqrt{1 + (f' + v'\Delta t)^2} - \sqrt{1 + f'^2})\,dx}{\Delta t} \\ &= \lim_{\Delta t\to 0} \int_0^1 \frac{1 + (f' + v'\Delta t)^2 - 1 - f'^2}{\Delta t(\sqrt{1 + (f' + v'\Delta t)^2} - \sqrt{1 + f'^2})}\,dx \\ &= \int_0^1 \frac{f'v'}{\sqrt{1 + f'^2}}\,dx\end{aligned}$$

As the area is fixed,

$$\begin{aligned}\lim_{\Delta t\to 0} \frac{\Delta A}{\Delta t} &= 0 \\ \int_0^1 v\,dx &= 0 \qquad (19)\end{aligned}$$

and we are going to minimize

$$\lim_{\Delta t\to 0} \frac{\Delta L}{\Delta t} = \int_0^1 \frac{f'v'}{\sqrt{1 + f'^2}}\,dx \qquad (20)$$

Similarly, we achieve the optimization by two methods:

Method 1: Euler-Lagrange Equation

Set

$$J[v(x)] = \int_0^1 \left(\frac{f'v'}{\sqrt{1+f'^2}} + \lambda v \right) dx = \int_0^1 F\, dx$$

where

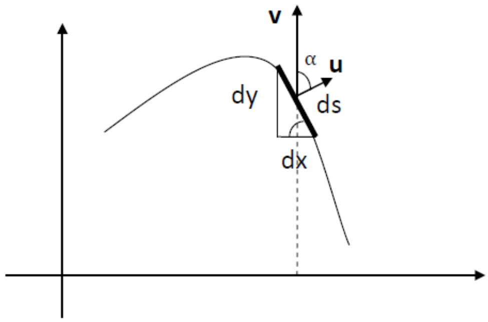
$$F = F(x, v, v') = \frac{f'v'}{\sqrt{1+f'^2}} + \lambda v.$$

Again by the Euler-Lagrange Equation,

$$F_v - \frac{d}{dx} F_{v'} = 0$$
$$\lambda - \frac{d}{dx} \frac{f'}{\sqrt{1+f'^2}} = 0$$

which is independent of v.

Like what we have done in section 3, we add the total kinetic energy of all element on the curve as the constraint.



Since $\mathbf{u}$ is a component of $\mathbf{v}$,

$$|\mathbf{u}| = |\mathbf{v}| \cos\alpha$$

Note that

$$\cos\alpha = \frac{dx}{ds}$$

so

$$|\mathbf{u}| = |\mathbf{v}| \cos\alpha = |\mathbf{v}| \left(\frac{dx}{ds} \right) \tag{21}$$

Together the kinetic energy of the string should be

$$\int_0^1 \rho|\mathbf{u}|^2\, ds = \int_0^1 \rho \left(\frac{dx}{ds}\right)^2 v^2\, ds = \int_0^1 \rho \frac{1}{\sqrt{1+f'^2}} v^2\, dx$$

Let ρ be the density of the curve, hence

$$\rho = \frac{\text{mass}}{\text{length}} = \frac{M}{\int_0^1 \sqrt{1+f'^2}\, dx}.$$

If the total kinetic energy is constant, we have

$$\begin{aligned}
\frac{1}{2}\int_0^1 \rho \frac{1}{\sqrt{1+f'^2}} v^2\, dx &= C \\
\frac{M}{\int_0^1 \sqrt{1+f'^2}\, dx} \int_0^1 \frac{1}{\sqrt{1+f'^2}} v^2\, dx &= 2C \\
M \int_0^1 \frac{1}{\sqrt{1+f'^2}} v^2\, dx &= 2C \int_0^1 \sqrt{1+f'^2}\, dx
\end{aligned}$$

For simplicity, we choose $M = 2C = 1$,

$$\int_0^1 \left(\frac{1}{\sqrt{1+f'^2}} v^2 - \sqrt{1+f'^2} \right) dx = 0 \tag{22}$$

Set

$$\begin{aligned}
J[v(x)] &= \int_0^1 \left[\frac{f'v'}{\sqrt{1+f'^2}} + \lambda v + \lambda' \left(\frac{1}{\sqrt{1+f'^2}} v^2 - \sqrt{1+f'^2} \right) \right] dx \\
&= \int_0^1 F\, dx
\end{aligned}$$

where

$$F = F(x, v, v') = \frac{f'v'}{\sqrt{1+f'^2}} + \lambda v + \lambda' \left(\frac{1}{\sqrt{1+f'^2}} v^2 - \sqrt{1+f'^2} \right)$$

By the Euler Lagrange Equation,

$$F_v - \frac{d}{dx}F_{v'} = 0$$

$$\lambda + 2\lambda' \frac{1}{\sqrt{1+f'^2}} v - \frac{d}{dx}\frac{f'}{\sqrt{1+f'^2}} = 0$$

$$\lambda + 2\lambda' \frac{1}{\sqrt{1+f'^2}} v - \frac{f''}{(1+f'^2)^{3/2}} = 0 \tag{23}$$

Hence

$$\begin{aligned} v &= \frac{1}{2\lambda' \frac{1}{\sqrt{1+f'^2}}}\left[\frac{f''}{(1+f'^2)^{3/2}} - \lambda\right] \\ v &= \frac{1}{2\lambda'}\left(\frac{f''}{1+f'^2} - \lambda\sqrt{1+f'^2}\right) \end{aligned} \tag{24}$$

Multiply (23) both sides by $\sqrt{1+f'^2}$ and integrate both sides with respect to x, we have

$$\begin{aligned} \int_0^1 \left[\lambda\sqrt{1+f'^2} + 2\lambda' v - \frac{f''}{1+f'^2}\right] dx &= 0 \\ \lambda \int_0^1 \sqrt{1+f'^2}\, dx - \int_0^1 \frac{f''}{1+f'^2}\, dx &= 0 \\ \lambda &= \frac{\int_0^1 \frac{f''}{1+f'^2}\, dx}{\int_0^1 \sqrt{1+f'^2}\, dx} \end{aligned} \tag{25}$$

Besides, from the constraint of total KE (22),

$$\int_0^1 \left(\frac{1}{\sqrt{1+f'^2}} v^2 - \sqrt{1+f'^2}\right) dx = 0.$$

We find that

$$\lambda' = \sqrt{\frac{\int_0^1 \frac{1}{4}\left(\frac{f''}{1+f'^2} - \lambda\sqrt{1+f'^2}\right)^2 \frac{1}{\sqrt{1+f'^2}}\, dx}{\int_0^1 \sqrt{1+f'^2}\, dx}} \tag{26}$$

Method 2: Inner Product

From (20), the rate of change of the length of the curve is

$$\lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t} = \int_0^1 \frac{f'v'}{\sqrt{1+f'^2}}\, dx = -\int_0^1 v \left(\frac{r'}{\sqrt{1+f'^2}} \right)' dx$$

Our aim is to minimize the above integral under the conditions (19) and (22):

1. $\int_0^1 v\, dx = 0$
2. $\int_0^1 \left(\frac{1}{\sqrt{1+f'^2}} v^2 - \sqrt{1+f'^2} \right) dx = 0$

$$\int_0^1 v^2 \left(\frac{1}{\sqrt{1+f'^2}} \right) dx = \int_0^1 \sqrt{1+f'^2}\, dx = L.$$

We define

$$\langle a, b \rangle = \int_0^1 ab \left(\frac{1}{\sqrt{1+f'^2}} \right) dx$$

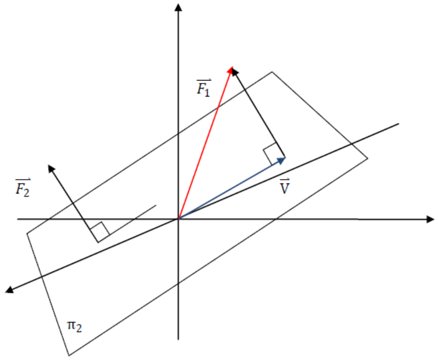
Let $F_1 = \left(\frac{f'}{\sqrt{1+f'^2}} \right)' \sqrt{1+f'^2}$ and $F_2 = \sqrt{1+f'^2}$.

Hence

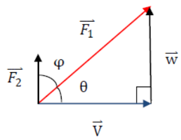
$$\begin{aligned}
\langle v, F_1 \rangle &= \int_0^1 v \left(\frac{f'}{\sqrt{1+f'^2}} \right)' \sqrt{1+f'^2} \left(\frac{1}{\sqrt{1+f'^2}} \right) dx \\
&= \int_0^1 v \left(\frac{f'}{\sqrt{1+f'^2}} \right)' dx \\
\langle v, F_2 \rangle &= \int_0^1 v \sqrt{1+f'^2} \left(\frac{1}{\sqrt{1+f'^2}} \right) dx \\
&= \int_0^1 v\, dx = 0 \\
\langle v, v \rangle &= \int_0^1 v^2 \left(\frac{1}{\sqrt{1+f'^2}} \right) dx = L
\end{aligned}$$

Since $\langle v, F_2 \rangle = \int_0^1 v\,dx = 0$, $\mathbf{v}$ can be any vector lying on the plane π_2 which is perpendicular to $\mathbf{F}_2$.

Suppose $\mathbf{V}$ lies on the plane π_2, and $\mathbf{F}_1$ is another vector represented by the arrow in the second quadrant, and $\mathbf{v} = n\mathbf{V}$.



To minimize $\displaystyle\int_0^1 \frac{f'v'}{\sqrt{1+f'^2}}\,dx = -\int_0^1 v\left(\frac{f'}{\sqrt{1+f'^2}}\right)' dx$, i.e. to maximize $\langle v, F_1 \rangle$, we choose the projection of $\mathbf{F}_1$, i.e. the bottom arrow, to be $\mathbf{V}$.



Since $\mathbf{V} = \mathbf{F}_1 - \mathbf{w}$ and $\mathbf{w}$ is parallel to $\mathbf{F}_2$, we have

$$\begin{aligned} \mathbf{w} &= |\,\mathbf{w}\,|\frac{\mathbf{F}_2}{|\,\mathbf{F}_2\,|} = (\,|\,\mathbf{F}_1\,|\sin\theta\,)\frac{\mathbf{F}_2}{|\,\mathbf{F}_2\,|} \\ &= \frac{|\,\mathbf{F}_1\,||\,\mathbf{F}_2\,|\cos\varphi}{|\,\mathbf{F}_2\,|^2}\mathbf{F}_2 = \frac{\langle \mathbf{F}_1, \mathbf{F}_2 \rangle}{\langle \mathbf{F}_2, \mathbf{F}_2 \rangle}\mathbf{F}_2 \end{aligned}$$

So

$$\mathbf{V} = \mathbf{F}_1 - \mathbf{w}$$

$$\mathbf{V} = \mathbf{F}_1 - \frac{\langle \mathbf{F}_1, \mathbf{F}_2 \rangle}{\langle \mathbf{F}_2, \mathbf{F}_2 \rangle}\mathbf{F}_2$$

and

$$
\begin{aligned}
v = k \Bigg[& \left(\frac{f'}{\sqrt{1+f'^2}} \right)' \sqrt{1+f'^2} \\
& - \frac{\int_0^1 \left(\frac{f'}{\sqrt{1+f'^2}} \right)' \sqrt{1+f'^2}\sqrt{1+f'^2} \left(\frac{1}{\sqrt{1+f'^2}} \right) dx}{\int_0^1 (1+f'^2) \left(\frac{1}{\sqrt{1+f'^2}} \right) dx} \sqrt{1+f'^2} \Bigg] \\
= k & \left(\frac{f''}{\sqrt{1+f'^2}} - \frac{\int_0^1 \frac{f''}{1+f'^2}\, dx}{\int_0^1 \sqrt{1+f'^2}\, dx} \sqrt{1+f'^2} \right)
\end{aligned}
$$

Since

$$
\begin{aligned}
\langle v, v \rangle &= L \\
\int_0^1 v^2 \left(\frac{1}{\sqrt{1+f'^2}} \right) dx &= L \\
\int_0^1 k^2 \left(\frac{f''}{\sqrt{1+f'^2}} - \frac{\int_0^1 \frac{f''}{1+f'^2}\, dx}{\int_0^1 \sqrt{1+f'^2}\, dx} \sqrt{1+f'^2} \right)^2 & \\
\left(\frac{1}{\sqrt{1+f'^2}} \right) dx &= \int_0^1 \sqrt{1+f'^2}\, dx
\end{aligned}
$$

we can solve that

$$
k = \sqrt{\frac{\int_0^1 \sqrt{1+f'^2}\, dx}{\int_0^1 \left(\frac{f''}{\sqrt{1+f'^2}} - \frac{\int_0^1 \frac{f''}{1+f'^2}\, dx}{\int_0^1 \sqrt{1+f'^2}\, dx} \sqrt{1+f'^2} \right)^2 \left(\frac{1}{\sqrt{1+f'^2}} \right) dx}}
$$

When we compare this with (24), (25) and (26),

$$
v = \frac{1}{2\lambda'} \left(\frac{f''}{1+f'^2} - \lambda \sqrt{1+f'^2} \right)
$$

where

$$\lambda = \frac{\int_0^1 \frac{f''}{1+f'^2}\,dx}{\int_0^1 \sqrt{1+f'^2}\,dx}$$

$$\lambda' = \sqrt{\frac{\int_0^1 \frac{1}{4}\left(\frac{f''}{1+f'^2} - \lambda\sqrt{1+f'^2}\right)^2 \frac{1}{\sqrt{1+f'^2}}\,dx}{\int_0^1 \sqrt{1+f'^2}\,dx}}.$$

We can see that

$$\begin{aligned}\frac{1}{2\lambda'} &= \sqrt{\frac{\int_0^1 \sqrt{1+f'^2}\,dx}{\int_0^1 \left(\frac{f''}{1+f'^2} - \lambda\sqrt{1+f'^2}\right)^2 \frac{1}{\sqrt{1+f'^2}}\,dx}} \\ &= k\end{aligned}$$

Hence

$$\begin{aligned}v &= k\left(\frac{f''}{1+f'^2} - \frac{\int_0^1 \frac{f''}{1+f'^2}dx}{\int_0^1 \sqrt{1+f'^2}\,dx}\sqrt{1+f'^2}\right) \\ &= \frac{1}{2\lambda'}\left(\frac{f''}{1+f'^2} - \lambda\sqrt{1+f'^2}\right)\end{aligned}$$

Since the function v solved by inner product method maximizes

$$\langle v, F_1\rangle = \int_0^1 v\left(\frac{f'}{\sqrt{1+f'^2}}\right)'\,dx$$

i.e.

$$\lim_{\Delta t\to 0}\frac{\Delta L}{\Delta t} = \int_0^1 \frac{f'v'}{\sqrt{1+f'^2}}\,dx = -\int_0^1 v\left(\frac{f'}{\sqrt{1+f'^2}}\right)'\,dx$$

is minimized by the function v.

For curvature, we define the curvature of open curve as

$$\kappa = -\frac{f''}{(1+f'^2)^{3/2}}$$

The length of the open curve is defined as

$$L = \int_0^1 \sqrt{1 + f'^2}\, dx$$

The function u is

$$\begin{aligned} u &= v\left(r\frac{dx}{ds}\right) \\ &= \frac{1}{2\lambda'}\left(\frac{f''}{1+f'^2} - \lambda\sqrt{1+f'^2}\right)\left(\frac{1}{\sqrt{1+f'^2}}\right) \\ &= \frac{1}{2\lambda'}\left(\frac{f''}{(1+f'^2)^{3/2}} - \lambda\right) \end{aligned}$$

Hence $u = \dfrac{1}{2\lambda'}(-\kappa - \lambda)$.
where

$$\begin{aligned} \lambda &= \frac{\int_0^1 \frac{f''}{1+f'^2}\, dx}{\int_0^1 \sqrt{1+f'^2}\, dx} = \frac{\int_0^1 \left(-\frac{f''}{(1+f'^2)^{3/2}}\right)\sqrt{1+f'^2}\, dx}{\int_0^1 \sqrt{1+f'^2}\, dx} \\ &= -\frac{\int_0^L \kappa\, ds}{L} \\ \lambda' &= \sqrt{\frac{\int_0^1 \frac{1}{4}\left(\frac{f''}{1+f'^2} - \lambda\sqrt{1+f'^2}\right)^2 \frac{1}{\sqrt{1+f'^2}}\, dx}{\int_0^1 \sqrt{1+f'^2}\, dx}} \\ &= \sqrt{\frac{\int_0^1 \frac{1}{4}\left(\frac{f''}{(1+f'^2)^{3/2}} - \lambda\right)^2 \sqrt{1+f'^2}\, dx}{\int_0^1 \sqrt{1+f'^2}\, dx}} \\ &= \sqrt{\frac{\int_0^L \frac{1}{4}(-\kappa-\lambda)^2\, ds}{L}} \end{aligned}$$

We can see that the above quantities are independent of the coordinates, but dependent of the unchanged quantities L and κ.

6. Conclusion

Thank you for reading our report! Of course, there are some we have done and some we can have further study on this topic. To end up our report, we will have a brief conclusion and prospect after we have such result of this topic.

In fact, we didn't have a wide range of knowledge of mathematics. By reading reference books, searching for the information in the internet, consulting teachers, we had the researching mind gradually. Because of this project, we have learnt some we didn't know before, including Euler Lagrange equation, Lagrange multiplier and inner product. Here, we want to thank all the teachers who have helped us in this project.

To summarize, we have the following results in our report. The deformation and path of the points of an isosceles triangle is the first result we got. Also, we have found the velocity function of the points in every moment that a smooth simple closed curve can minimize its rate of change of perimeter with a fixed area in the fastest rate by 2 different methods. Finally, let's state the main result of this report. The maximum rate of change of perimeter of a smooth simple closed curve under isoareal deformation with normalized kinetic energy is attained when the magnitude of the velocity at each point of the curve along the normal direction is equal to the standard score of the curvature at that point. This result can also be used to prove the isoperimetric inequality.

We know we have just solved a minor part of this area of mathematics problem; there are still a lot of areas you and me can explore.

First, we can have study on the change of the momentum of each point on the curve. The aim is that we can see whether the centre of mass of the shape moves throughout the process. If it moves, how it moves.

Second, we can find out the rate of change of curvature of the curve. Throughout the deformation, the shape of the curve is changing, so the curvature is also changing. By studying how the curvature changes with respect to time, we may find out the total time for the process as we have the initial curvature and final curvature.

No matter you love our work or not, we are very delighted to have research on this mathematical topic. As what we said before, we were not pursue to prove or doubt the "final state", but were attracted by the process. We were enjoyable enough when doing this research. I hope that you will have any inspiration after reading this report as we also love the thinking process of tackling the mathematical problems.

In the near future, we will continue to have research on it as it is only the end of our report but not the end of our project!

Acknowledgement: Mr. Chung-Wa Ho, Mr. Kwun-Wing Wong.

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[2] http://en.wikipedia.org/wiki/Isoperimetric_inequality.
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Reviewer's Comments

Main Comment: On first reading of the report, the reviewer was very confused about whether the paper aimed at maximizing or minimizing $\frac{dP}{dt}$. The reviewer then think the aim is to minimize $\frac{dP}{dt}$ while keeping the area constant. When $\frac{dP}{dt} < 0$, minimizing it means the shape is reducing its perimeter at the fastest rate. When $\frac{dP}{dt} > 0$, minimizing it means to increase its perimeter at the slowest rate. Since the area is kept constant and the kinetic energy has to be kept constant as well, therefore the shape does not stop changing even though it may have reached the minimum perimeter state. In this sense, the problem this report is trying to solve differs from the usual minimum perimeter problem.

Of the few formulae and calculations that the reviewer has checked, no error was found. The following suggestions are only meant to improve the readability of the paper.

1. It was said that "We are also going to find the least time for a shape to minimize its perimeter with fixed area ..." Such a result cannot be found in the report.

2. Triangle: "It is because the movement of the two bottom vertices should be the same, ..." Reviewer's comment: In this section the assumption of symmetry is crucial. More explanation is needed to convince others that the isosceles triangle has to move in a symmetrical manner.

3. The two paragraphs "The above equation is independent of v ..." & "We cannot use v directly ..." are not clear. A bit more explanation may help the readers.

4. Dual Problem. In this section, it looks like a lot of steps were skipped. More details may help the readers.

Final Words: Congratulations on Supervisor and the authors of this paper. Good to see mathematics education is moving forward in Hong Kong.