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HOW TO KEEP WATER COLD – A STUDY ABOUT THE WET CONTACT SURFACE AREA IN CYLINDER

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Abstract. The question investigated in this essay is: Given the dimensions of a cylindrical container and the volume of the water contained in it, which position would give the minimum wet contact area? In part 1, we will discuss about which position of the container, horizontal or vertical, will give a smaller wet contact surface area when the volume of water varies. In part 2, we will still discuss the wet contact area in the cylinder, but considering the volume of water as a constant and allowing the cylinder to incline with a variable angle α. We will try to find out the value of α such that the total wet surface area is minimum.

1. Introduction

Actually, doing this research was not initially for Hang Lung Mathematics Award, but for a winter holiday assignment from the Mathematics Enrichment Class in my secondary school two years before writing this paper:

"Think of an interesting mathematics question, then try to solve it."

This reminded me of a problem which was long in my mind.

One day, I looked at a bottle of water on my desk. I laid it down and put it back to upright, repeatedly. Then I observed that the contact area between the bottle and water might be different when the bottle is placed horizontally and vertically. I would like to find out which position of the bottle, placed horizontally or vertically, can give the minimum wet contact area for different volume of water. I found this problem interesting because

[1] This work is done under the supervision of the author's teacher, Mr. Kwok-Tai Cathay Wong.

it may have some significance in the storage of some viscous liquids such as crude oil or chemicals. Or, at least I would know how to reduce the wet surface area and keep my water cold for a longer time.

After I described the problem to my teacher in the Enrichment Class, he told me that I have to study differentiation to solve this kind of problem. Later when I studied differentiation and solved the problem, I felt the happiness of using textbook knowledge to solve daily-life problems. I discovered an interesting result which was out of my expectation. I expected that, as the volume of water decreases, there will be only one switching point to change the position with smaller wet surface area, but finally I found that there can be either one or three switching points.

After I have solved the above problem (which is part 1 in this report), my mathematics teacher suggested me to investigate some generalisation of the original problem, such as a different shape of the container. Finally I decided to generalise the inclined angle of the container, so that the container is not restricted to only two positions. I found the problem in part two is more difficult, as there are more dependent variables which are interrelated, but they cannot be expressed explicitly in terms of other variables. Eventually I got another unexpected result: When the water level is lower than half the height of the cylinder, you should incline the cylinder for some angle to minimize the wet contact area!

In this report, when an equation apparently cannot be solved analytically, the Solver programme in Microsoft Excel will be used. However, I tried to use mathematical analysis to tackle the question whenever possible.

2. Definitions

As there are many variables and functions introduced throughout the report, a list of their definitions and notations is given here. The page number in the bracket indicates the first page the notation appears.

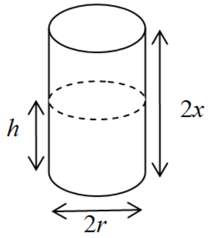
r	the radius of the cylinder (p.4)
$2x$	the height of the cylinder (p.4)
h	the water level when the cylinder is placed vertically (p.4)
V	the volume of the water (p.4)
A_1	the wet contact area when the cylinder is placed vertically (p.4)

A_2	the wet contact area when the cylinder is placed horizontally (p.4)
θ	the angle at center formed by the water segment on the base when the cylinder is placed horizontally, which is a variable (p.5)
$K(\theta)$	the wet contact area on the base when the cylinder is placed horizontally (p.5)
$f(\theta)$	$A_1 - A_2$ (p.6)
a	the first θ-intercept of the curve $y = f(\theta)$ (p.10)
b	the third θ-intercept of the curve $y = f(\theta)$ (p.10)
α	the inclined angle made with the ground when the cylinder is placed obliquely (p.12)
A_3	the wet contact area when the water covers the base of the cylinder totally and does not touch the lid (p.13)
A_4	the wet contact area when the water does not touch the lid and does not totally cover the base (p.13)
A_5	the wet contact area when the water touches both the lid and the base (p.13)
l	the shortest height of the water when the cylinder is placed obliquely (p.14)
p	the longest height minus the shortest height of the water when the cylinder is placed obliquely (p.14)
β	the angle at center formed by the water segment on the base when the cylinder is placed obliquely (p.14)
h'	the height of the cone formed by the water when the cylinder is placed obliquely (p.14)
γ	the angle at centre of the base formed by water when the water just touches the lid (p.15)
d	the depth of water on the base when the cylinder is placed obliquely (p.15)
$g(\beta)$	$\beta - \sin\beta$ (p.17)
ϕ	the angle at centre formed by the water segment when the cylinder is placed horizontally, which is a constant (p.17)
λ	the angle at centre formed by the water segment on the lid when the cylinder is placed obliquely (p.18)

3. Part 1 - Vertical or Horizontal?

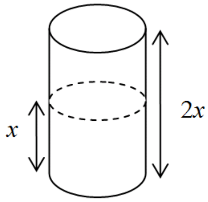
The problem we will investigate in this part is: Given the dimensions of a cylinder and the volume of water in the cylinder, which position, horizontal or vertical, would give a smaller total wet contact surface area? How will the result change if the volume of water varies?

Let the radius of the cylinder be r ($r > 0$), the height of the cylinder be $2x$ ($x > 0$), and the water level when the cylinder is in vertical position be h ($h \geqslant 0$). Then the volume of water V, which equals to $\pi r^2 h$, directly varies as h.



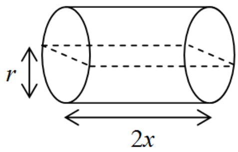
We can easily observe that if the cylinder is half full of water, the wet contact areas in the two positions are equal. They should be both equal to half of the curved surface area plus the area of one base of the cylinder.

The following is the proof.



When the cylinder is placed vertically,

$$\text{contact area}(A_1) = r^2 + 2r\pi x.$$



When the cylinder is placed horizontally,

$$\begin{aligned}\text{contact area}(A_2) &= 2 \times \frac{1}{2}r^2\pi + \frac{1}{2} \times 2r\pi \times 2x \\ &= r^2\pi + 2r\pi x.\end{aligned}$$

Therefore, $A_1 = A_2$.

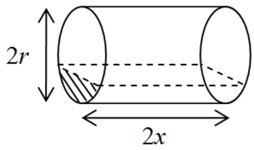
Now we consider the general case of $0 \leqslant h \leqslant 2x$.

It is obvious that when h is just slightly greater than 0, the horizontal position will give a smaller wet contact area, because the water touches the whole base in the vertical position, but just touch a very small area in the horizontal position. However, when h is just slightly less than $2x$, the vertical position will give a smaller wet contact area, because the water almost touches all the surface in the horizontal position, but the water will not touch the top surface of the cylinder in the vertical position. Therefore, we would like to investigate in what range of the value of h, the vertical (or the horizontal) position would give a smaller wet contact area.

When the cylinder is placed vertically,

$$\text{contact area}(A_1) = \begin{cases} \pi r^2 + 2r\pi h & \text{when } 0 < h < 2x, \\ 0 & \text{when } h = 0, \\ 2\pi r^2 + 2r\pi h & \text{when } h = 2x, \end{cases}$$

$$\text{volume of water}(V) = r^2\pi h.$$



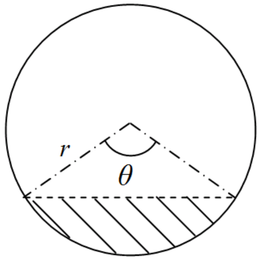


Figure of the base

When the cylinder is placed horizontally, let the shaded area, which is the wet contact area on the base, be $K(\theta)$. Then $K(\theta) = \frac{r^2\theta}{2} - \frac{r^2 \sin\theta}{2}$, where θ is the angle at center and $0 \leqslant \theta \leqslant 2\pi$.

$$\begin{aligned} \text{Contact area}(A_2) &= 2K(\theta) + r\theta \times 2x \\ &= 2K(\theta) + 2rx\theta \\ &= 2\left[\frac{r^2\theta}{2} - \frac{r^2\sin\theta}{2}\right] + 2rx\theta \\ &= r^2\theta - r^2\sin\theta + 2rx\theta. \end{aligned}$$

$$\begin{aligned} \text{Volume of water}(V) &= K(\theta) \times 2x \\ &= \left[\frac{r^2\theta}{2} - \frac{r^2\sin\theta}{2}\right] \times 2x \\ &= (r^2\theta - r^2\sin\theta)x. \end{aligned}$$

As r and x are fixed, $A_1 = r^2\pi + 2r\pi h$ is a function of h whereas $A_2 = r^2\theta - r^2\sin\theta + 2rx\theta$ is a function of θ. As V increases from 0 to $\pi r^2 h$, h increases from 0 to $2x$, and θ also increases from 0 to $2x$. Therefore, we will try to write h in terms of θ, so that A_1 and A_2 both can be expressed as functions of θ. Then we can compare the magnitudes of A_1 and A_2 for different values of θ.

Since the volume of water is unchanged,

$$\begin{aligned}\pi r^2 h &= (r^2\theta - r^2\sin\theta)x \\ \pi h &= (\theta - \sin\theta)x \\ h &= \frac{(\theta - \sin\theta)x}{\pi}\end{aligned}$$

Putting $h = \dfrac{(\theta - \sin\theta)x}{\pi}$ into $A_1 = r^2\pi + 2r\pi h$, we have

$$A_1 = \pi r^2 + 2r\pi\left[\frac{(\theta - \sin\theta)x}{\pi}\right].$$

Therefore,

$$A_1 = \pi r^2 + 2rx\theta - 2rx\sin\theta.$$

To compare the values of A_1 and A_2, we define a function $f(\theta) = A_1 - A_2$, where $0 < \theta < 2\pi$.

$$\begin{aligned}f(\theta) &= A_1 - A_2 \\ &= (r^2\pi + 2rx\theta - 2rx\sin\theta) - (r^2\theta - r^2\sin\theta + 2rx\theta) \\ &= r^2\pi - 2rx\sin\theta - r^2\theta + r^2\sin\theta.\end{aligned}$$

If $f(\theta) > 0$, then $A_1 > A_2$, i.e. placing horizontally would give a smaller wet area.
If $f(\theta) < 0$, then $A_1 < A_2$, i.e. placing vertically would give a smaller wet area.

First, we consider some trivial cases. Obviously, when $\theta = 0$ (there is no water) and 2π (the container is full of water), $f(\theta) = 0$. Also, from the result shown in the beginning of this part, when the water is half full, i.e. $\theta = \pi$, we have $f(\theta) = 0$.

Now, consider when h is just slightly greater than 0, i.e. θ is just slightly greater than 0, A_1 is about πr^2 but A_2 is about 0, hence $f(\theta) \approx \pi r^2$. Actually, it is easy to show that $\lim_{\theta \to 0+} f(\theta) = \pi r^2$. Similarly, it is easy to show that $\lim_{\theta \to 2\pi-} f(\theta) = -\pi r^2$. Therefore, as θ increases from 0 to 2π, $f(\theta)$ varies from πr^2, via 0 when $\theta = \pi$, to $-\pi r^2$.

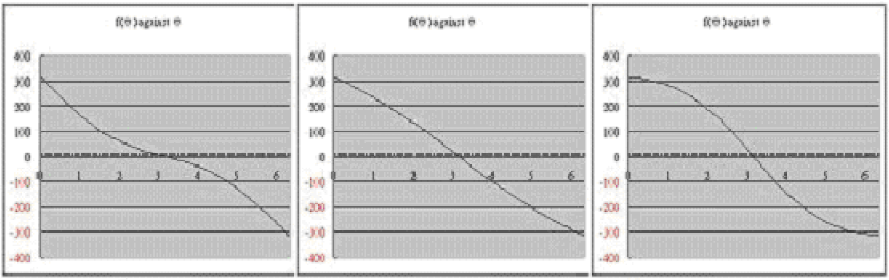
As $f(\theta)$ is continuous and differentiable in the range $(0, 2\pi)$, if we can find all the roots of the equation $f(\theta) = 0$, we can find the range of θ for which $f(\theta)$ is greater than zero, and hence our problem will be solved.

Setting $f(\theta) = 0$, we have

$$\begin{aligned} r^2\pi - 2rx\sin\theta - r^2\theta + r^2\sin\theta &= 0 \\ r\pi - 2x\sin\theta - r\theta + r\sin\theta &= 0 \end{aligned}$$

We cannot solve this kind of equation analytically, so we have to use computer programme (such as Solver in Microsoft Excel) or numerical method to find the roots of this equation when the numerical values of r and x are known.

However, we can still investigate some properties about $f(\theta)$. As $\lim_{\theta \to 0+} f(\theta)$, $f(\pi)$, and $\lim_{\theta \to 2\pi-} f(\theta)$ are equal to πr^2, 0 and $-\pi r^2$ respectively, if $f(\theta)$ is a decreasing function in the range $(0, 2\pi)$, then $\theta = \pi$ must be the only root for the equation $f(\theta) = 0$. This result can be illustrated by the following graphs(The graphs are drawn by the Excel Programme attached.):



So we now investigate whether the curve $y = f(\theta)$ has any stationary point. If there is no stationary point, then $f(\theta)$ must be a decreasing function.

Consider $f'(\theta)$.

$$\begin{aligned} f'(\theta) &= \frac{d}{d\theta}(r^2\pi - 2rx\sin\theta - r^2\theta + r^2\sin\theta) \\ &= -2rx\cos\theta - r^2 + r^2\cos\theta \end{aligned}$$

Setting $f'(\theta) = 0$, we have

$$\begin{aligned} -2rx\cos\theta - r^2 + r^2\cos\theta &= 0 \\ -2x\cos\theta - r + \cos\theta &= 0 \\ (r-2x)\cos\theta &= r \end{aligned}$$

If $r - 2x = 0$(i.e. $r = 2x$), as $r > 0$, $f'(\theta) = 0$ has no solution. (1)

Otherwise,

$$\cos\theta = \frac{r}{r-2x}.$$

Since $-1 \leqslant \cos\theta \leqslant 1$, when $\frac{r}{r-2x} < -1$ or $\frac{r}{r-2x} > 1$, $f'(\theta) = 0$ would have no solution, i.e. no stationary point for the graph $y = f(\theta)$. Therefore, we will discuss the following three cases of the value of $\frac{r}{r-2x}$.

Case 1: $\frac{r}{r-2x} < -1$ or $\frac{r}{r-2x} > 1$

Case 2: $\frac{r}{r-2x} = -1$ or $\frac{r}{r-2x} = 1$

Case 3: $-1 \leqslant \frac{r}{r-2x} \leqslant 1$

Case 1:

When $\frac{r}{r-2x} < -1$,

$$\begin{aligned} \frac{r}{r-2x} + 1 &< 0 \\ \frac{r+r-2x}{r-2x} &< 0 \\ \frac{2r-2x}{r-2x} &< 0 \\ \frac{r-x}{r-2x} &< 0 \end{aligned}$$

Since $r - 2x < r - x$,

$$\begin{aligned} r - 2x < 0 &\text{ and } r - x > 0 \\ r < 2x &\text{ and } r > x \end{aligned}$$

Hence $x < r < 2x$. (2)

When $\dfrac{r}{r-2x} > 1$,

$$\begin{aligned}
\frac{r}{r-2x} - 1 &> 0 \\
\frac{r-r+2x}{r-2x} &> 0 \\
\frac{2x}{r-2x} &> 0 \\
r - x &> 0 \quad \text{(since } x > 0\text{)}
\end{aligned}$$

Hence $r > 2x$. (3)

Therefore, when $x < r > 2x$ or $r > 2x$, $\cos\theta = \dfrac{r}{r-2x}$ has no solution.

Case 2:

When $\cos\theta = 1$ or -1, there will be one root as θ is between 0 to 2π, i.e. the curve $y = f(\theta)$ only has one stationary point. Because $\lim\limits_{\theta\to 0+} f(\theta)$, $f(\pi)$ and $\lim\limits_{\theta\to 2\pi-} f(\theta)$ are equal to πr^2, 0 and $-\pi r^2$ respectively, if there is only 1 stationary point, this stationary point must be a point of inflexion. That means $f(\theta)$ is still a decreasing function when $\dfrac{r}{r-2x} = -1$ or $\dfrac{r}{r-2x} = 1$.

When $\dfrac{r}{r-2x} = -1$,

$$\begin{aligned}
r &= 2x - r \\
r &= x
\end{aligned} \tag{4}$$

When $\dfrac{r}{r-2x} = 1$,

$$\begin{aligned}
r &= r - 2x \\
2x &= 0 \quad \text{(which is rejected because } x > 0\text{)}
\end{aligned}$$

Combining (1), (2), (3) and (4), when $r \geqslant x$, $f(\theta)$ is a decreasing function in $(0, 2\pi)$, i.e. when the container has the size of $r \geqslant x$, placing horizontally would give a smaller wet contact area when the volume of water is smaller than half of the container capacity, and placing vertically would give a smaller wet contact area when the volume of water is lager than half of the container capacity.

Case 3:

When $-1 < \dfrac{r}{r-2x} < 1$, i.e. $0 < r < x$, there will be 2 roots for the equation $\cos\theta = \dfrac{r}{r-2x}$ for $0 < \theta < 2\pi$, which implies $y = f(\theta)$ has 2 stationary points. By the nature of cosine function, one of these two roots must be smaller than π, and the other must be greater than π. That means, there is one stationary point between 0 and π (exclusive), and the other stationary point between π and 2π (exclusive). As $\lim_{\theta\to 0+} f(\theta)$, $f(\pi)$, and $\lim_{\theta\to 2\pi-} f(\theta)$ are equal to πr^2, 0 and $-\pi r^2$ respectively, if these two stationary points are not both points of inflexion, then the stationary point between 0 and π must lie below the θ-axis, and the stationary point between π and 2π must lie above the θ-axis.

To test whether the two stationary points are points of inflexion, we consider $f''(\theta)$.

$$\begin{aligned} f'(\theta) &= -2rx\cos\theta - r^2 + r^2\cos\theta \\ f''(\theta) &= 2rx\sin\theta - r^2\sin\theta \\ &= r\sin\theta(2x - r) \end{aligned}$$

Set $f''(\theta) = 0$. As $r > 0$ and $2x - r > 0$, we have $\sin\theta = 0$. Hence $\theta = \pi$ for $0 < \theta < 2\pi$.

But this stationary point must not be at $\theta = \pi$, therefore these two stationary points cannot be points of inflexion.

Therefore, when $0 < r < x$, there are two turning points, one is a local minimum point between 0 and π and it lies below the θ-axis, the other one is a local maximum point between π and 2π and it lies above the θ-axis. It implies that there must be three θ-intercepts of the curve $y = f(\theta)$. Let the three θ-intercepts be a, π and b.

To further confirm the existence of these θ-intercepts, we can consider $f'(\pi)$.

$$\begin{aligned} f'(\theta) &= -2rx\cos\theta - r^2 + r^2\cos\theta \\ f'(\pi) &= 2rx - 2r^2 \\ &= 2r(x - r) \end{aligned}$$

As $0 < r < x$, $x - r > 0$, hence $f'(\pi) > 0$, which implies that $f(\theta)$ is increasing at $\theta = \pi$. So the turning point between 0 and π must be below the θ-axis, and the turning point between π and 2π must be above the θ-axis.

As there are three θ-intercepts of the curve $y = f(\theta)$, we need to find a and b, which are the roots of the equation $f(\theta) = 0$, to know the range of θ for which $f(\theta) > 0$. Although we cannot solve the equation analytically, can we find some relationship between a and b?

As a is a root of the equation $f(\theta) = 0$ and $f(\theta) = r^2\pi - 2rx\sin\theta - r^2\theta + r^2\sin\theta$, we have $f(a) = r^2\pi - 2rx\sin a - r^2a + r^2\sin a = 0$. Then

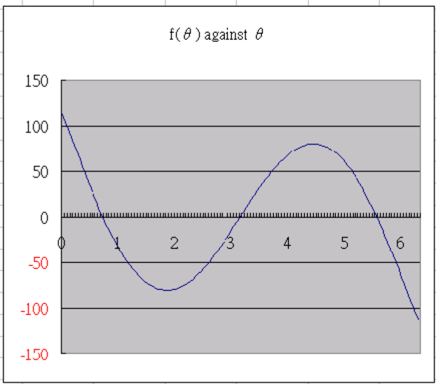
$$\begin{aligned} f(2\pi - a) &= r^2\pi - 2rx\sin(2\pi - a) - r^2(2\pi - a) + r^2\sin(2\pi - a) \\ &= r^2\pi + 2rx\sin a - 2r^2\pi + r^2a - r^2\sin a \\ &= -r^2\pi + 2rx\sin a + r^2a - r^2\sin a \\ &= -f(a) \\ &= 0. \end{aligned}$$

Therefore, if a is a root of the equation $f(\theta) = 0$, $2\pi - a$ is also a root of the equation. Hence $b = 2\pi - a$. From this result, we only need to use numerical method to find either a or b, and the other one can be easily calculated.

Note that as $f(\theta) = -f(2\pi - \theta)$, the whole curve of $y = f(\theta)$ is symmetric about the point $(\pi, 0)$.

In order to find a by computer, I have written an Excel file which uses Solver to find an estimate of a.

	A	B	C	D	E	F	G	H	I	J	K	L
1	θ	A1	A2	f(θ)	r=	6		θ	A1	A2	f(θ)	
2	0	113.0973	0	113.097	x=	14	Answer:	0.721025	123.3235	123.3235	0.000	Run Solver
3	0.05	113.1008	8.40075	104.700								
4	0.1	113.1253	16.806	96.319								
5	0.15	113.1917	25.22023	87.972								
6	0.2	113.3209	33.6479	79.673								
7	0.25	113.5335	42.09346	71.440								
8	0.3	113.8499	50.56127	63.289								
9	0.35	114.2905	59.05568	55.235								
10	0.4	114.8751	67.58094	47.294								
11	0.45	115.6231	76.14124	39.482								
12	0.5	116.5538	84.74068	31.813								
13	0.55	117.6859	93.38326	24.303								
14	0.6	119.0374	102.0729	16.965								
15	0.65	120.626	110.8133	9.813								
16	0.7	122.4688	119.6082	2.861								
17	0.75	124.582	128.461	-3.879								
18	0.8	126.9815	137.3752	-10.394								
19	0.85	129.6822	146.3539	-16.672								
20	0.9	132.6984	155.4002	-22.702								
21	0.95	136.0435	164.517	-28.474								
22	1	139.7302	173.707	-33.977								



The picture is captured from the ‘part 1’ sheet of the Excel file attached. When the values of r and x are input in the orange boxes(F1 and F2), the curve of $f(\theta)$ against θ will be plotted. If the approximate answer is input in the yellow box and then press the 'Run Solver' button, a more accurate estimate will appear in the yellow box(H2). (The Solver add-in must be installed first). This programme can solve the problem numerically, and it can let us verify the properties we have proved.

We can now summarise the findings in this part. For containers with $0 < r < x$, the results are as follows:

Range/Value of θ	Sign/Value of $f(\theta)$	Position which will give a smaller wet contact surface area
$0 < \theta < a$	Positive	Horizontal
$\theta = a$	Zero	Both
$a < \theta < \pi$	Negative	Vertical
$\theta = \pi$	Zero	Both
$\pi < \theta < 2\pi - a$	Positive	Horizontal
$\theta = 2\pi - a$	Zero	Both
$2\pi - a < \theta < 2\pi$	Negative	Vertical

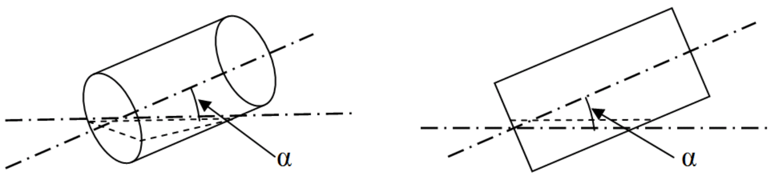
Here a denotes the root of the equation $f(\theta) = 0$ in the range $0 < a < \pi$, and in general it needs to be found numerically.

For containers with $r \geqslant x$, the results are as follows:

Range/Value of θ	Sign/Value of $f(\theta)$	Position which will give a smaller wet contact surface area
$0 < \theta < \pi$	Positive	Horizontal
$\theta = \pi$	Zero	Both
$\pi < \theta < 2\pi$	Negative	Vertical

4. Part 2 - Generalisation in Inclination Angle

Besides placing the cylinder vertically and horizontally, the cylinder can be placed obliquely to make an inclined angle (α) with the ground.



So the question in this part is, for different volume of water, what is the minimum contact area when the inclined angle α varies?

When the cylinder is turned from vertical position to horizontal position, α decreases from $\frac{\pi}{2}$ to 0.

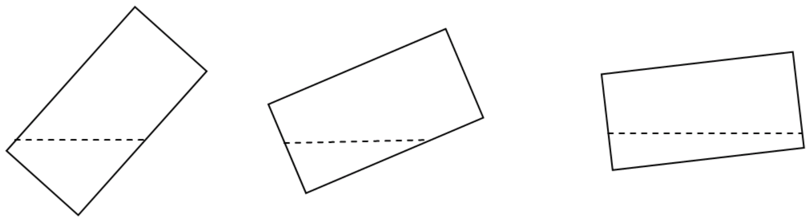
We will first consider the case of $h < x$, then we will use the result to deduce the solution for the case of $h > x$.

Before we consider the case of $h < x$, first we discuss about the trivial case of $h = x$. When $h = x$, the volume of water is half of the capacity of the cylinder. In this case, no matter how α varies, the shape of water is congruent to the shape of empty space in the cylinder. Hence they must have the same area of contact. Therefore, when $h = x$, the wet contact surface area is always equal to half of the total surface area of the cylinder, no matter how α varies.

Now we discuss the question for the two cases, i.e., when $h < x$ and when $h > x$.

Case 1: $h < x$

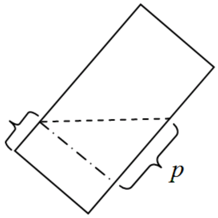
When $h < x$, the process will be as follows: First, when α begins to decrease from π, the water covers the base of the cylinder totally and does not touch the lid. When α become small enough, the water does not cover the base of the cylinder completely, but the water does not touch the lid yet. When α become even smaller, the water touches both the base and the lid. Finally, $\alpha = 0$ and the cylinder is placed horizontally.



A_3 A_4 A_5

Let the wet contact area be A_3, A_4 and A_5 respectively in the figures shown above. We will try to find the minimum value among A_1, A_3, A_4, A_5 and A_2 as α varies.

First we try to calculate A_3. Let l and p be defined as shown in the figure below.



The curved wet contact surface area $= 2r\pi l + \frac{1}{2} \times 2r\pi p$. Therefore,

$$\begin{aligned} A_3 &= r^2\pi + 2r\pi l + \frac{1}{2} \times 2r\pi p \\ &= r^2\pi + 2r\pi l + r\pi p. \end{aligned}$$

Considering the volume of water (V), we have

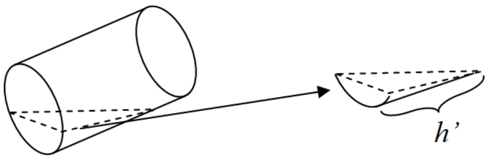
$$r^2\pi l + \frac{1}{2}r^2\pi p = r^2\pi h$$
$$l = h - \frac{1}{2}p$$

Therefore,

$$\begin{aligned} A_3 &= r^2\pi + 2r\pi l + r\pi p \\ &= r^2\pi + 2r\pi(h - \frac{1}{2}p) + r\pi p \\ &= r^2\pi + 2r\pi h \\ &= A_1. \end{aligned}$$

This implies that, as long as the water covers the base completely, the wet contact surface area is equal to that in vertical position.

Now consider A_4(A revised version of this part by the author is on page 25, cf. **Reviewer's Comments** 1).



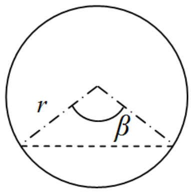


Figure of the base

The left figure show the cone formed by the water, while the right figure shows the base of the cone. h' and β are defined as shown above. Note that β will decrease from 2π to γ, where γ is the angle at centre of the base formed by the water when the water just touches the lid.

$$\text{The wet contact area on the base } = \left[\frac{r^2\beta}{2} - \frac{r^2\sin\beta}{2}\right].$$

Note that when $\beta > \pi$, the wet surface area should be the area of sector plus the area of triangle. However, since $\sin(2\pi - \beta) = -\sin\beta$, the above expression still holds when $\beta > \pi$.

The wet curved surface area is the same as the area of a triganle with base equal to the wet arc on the base and with height equal to h'. Therefore, the wet curved surface area $= \dfrac{r\beta h'}{2}$.

Combining the results,

$$\begin{aligned} A_4 &= \left(\frac{r^2\beta}{2} - \frac{r^2\sin\beta}{2}\right) + \frac{r\beta h'}{2} \\ &= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta h'. \end{aligned}$$

To relate β and h' to other known constants or the variable α, we consider the following figure.

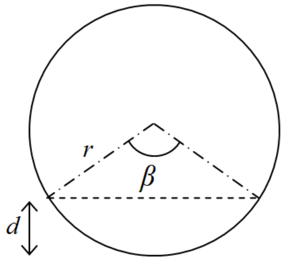


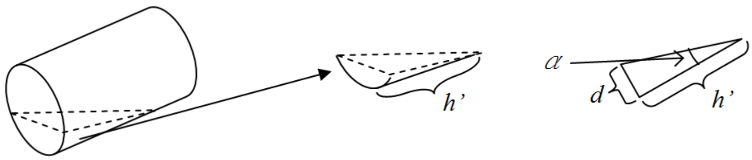
Figure of the base

Let the depth of water on the base be d.

$$\begin{aligned} d &= r - r\cos\frac{\beta}{2} \\ &= r(1 - \cos\frac{\beta}{2}) \end{aligned}$$

Note that the above expression is still correct when $\beta > \pi$.

Consider the triangle formed by the water:



$$h' = \frac{d}{\tan\alpha} = \frac{r(1-\cos\frac{\beta}{2})}{\tan\alpha}$$

The shape of the water is half of a cylinder with base as the shape formed on the base of the cylinder and with height h'. The volume of water,

$$\begin{aligned} V &= \frac{1}{2} \times \text{base area } \times \text{height} \\ &= \frac{1}{2}\left[\frac{1}{2}r^2(\beta - \sin\beta)h'\right] \\ &= \frac{r^3}{4}(\beta - \sin\beta)\frac{1-\cos\frac{\beta}{2}}{\tan\alpha}. \end{aligned}$$

By substituting $V = \pi r^2 h$, we have

$$\tan\alpha = \frac{r(\beta - \sin\beta)(1-\cos\frac{\beta}{2})}{4\pi h}.$$

When α is given, it is difficult to calculate the value of β, but it is easy to do the opposite. Therefore, we will switch the variable β to make our question a bit easier to analyse.

So we write A_4 in terms of β.

$$\begin{aligned} A_4 &= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta h' \\ &= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta\frac{r(1-\cos\frac{\beta}{2})}{\tan\alpha} \\ &= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta\frac{4\pi h}{\beta - \sin\beta} \end{aligned}$$

A_4 is defined only when the water does not touch the lid. Therefore, A_4 is defined only when

$$h' \leqslant 2x$$
$$\frac{4\pi h}{\beta - \sin\beta} \leqslant 2x$$
$$\beta - \sin\beta \geqslant \frac{2\pi h}{x}$$

Let $g(\beta) = \beta - \sin\beta$. Since β decreases from 2π to ϕ, where ϕ is the angle at centre of the base when the cylinder is placed horizontally and $g'(\beta) = 1 - \cos\beta \geqslant 0$ for all values of β, $g(\beta)$ is non-increasing from $g(2\pi) = 2\pi$ to $g(\phi)$. Considering the volume of water when the cylinder is placed horizontally, we have

$$\pi r^2 h = \frac{r^2}{2}(\phi - \sin\phi) \cdot 2x$$
$$\phi - \sin\phi = \frac{\pi h}{x}$$

Therefore, $g(\phi) = \dfrac{\pi h}{x}$.

Since $h < x$, $g(2\pi) > \dfrac{2\pi h}{x} > g(\phi)$, and the equation $\beta - \sin\beta = \dfrac{2\pi h}{x}$ must have a root between ϕ and 2π. Therefore, we can find the range of β for which $\beta - \sin\beta \geqslant \dfrac{2\pi h}{x}$.

To solve our main problem, we need to find the minimum value of

$$A_4 = \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta\frac{4\pi h}{\beta - \sin\beta}$$

as β varies from 2π to γ, where $\gamma - \sin\gamma = \dfrac{2\pi h}{x}$. If we differentiate A_4 with respect to β, we have, after simplification,

$$\frac{dA_4}{d\beta} = \frac{1}{2}r^2(1 - \cos\beta) + 2\pi rh\frac{\beta\cos\beta - \sin\beta}{(\beta - \sin\beta)^2}.$$

However, the above expression is so complicated that we cannot solve the equation $\dfrac{dA_4}{d\beta} = 0$ analytically. But we can deduce that, since r^2, $1 - \cos\beta$, r, h and $(\beta - \sin\beta)^2$ are all non-negative numbers, if $\beta\cos\beta - \sin\beta > 0$, then $\dfrac{dA_4}{d\beta} > 0$. When β is in the range of $\dfrac{3\pi}{2} < \beta < 2\pi$, $\cos\beta > 0$, $\cos\beta > 0$

and $-\sin\beta > 0$, hence $\dfrac{dA_4}{d\beta} > 0$, i.e. when β starts decreasing from 2π, the total wet surface area must be decreasing.

To find the turning point numerically, a programme is written in the Excel file attached in the worksheet "Part 2 (A4)". The input and output of the programme are very similar to the "Part 1" worksheet.

Finally, we consider A_5(A revised version of this part by the author is on page 25, cf. **Reviewer's Comments** 1).

When α further decreases so that $\beta - \sin\beta \leqslant \dfrac{2\pi h}{x}$, the water will touch the lid. The wet contact surface area will then be equal to the sum of the wet area on the base, the wet area on the lid, and the wet curved surface area.

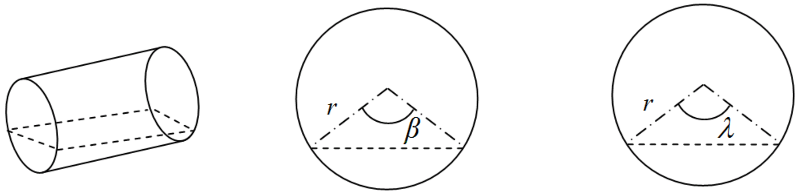


Figure of the base Figure of the lid

$$\text{The wet area on the base} = \frac{1}{2}r^2(\beta - \sin\beta)$$

$$\text{The wet area on the lid} = \frac{1}{2}r^2(\lambda - \sin\lambda),$$

where λ is the angle at centre formed by the water segment on the lid. Note that $0 \leqslant \lambda \leqslant \phi$.

The wet curved surface area is actually a trapezium, with height equals to $2x$, and two parallel sides equal to $r\beta$ and $r\lambda$. Therefore, the wet curved surface area $= (r\beta + r\lambda) \times (2x) \times \dfrac{1}{2} = rx(\beta + \lambda)$. Hence

$$A_5 = \frac{1}{2}r^2(\beta - sin\beta) + \frac{1}{2}r^2(\lambda - sin\lambda) + rx(\beta + \lambda).$$

In order to find the minimum value of A_5 as β varies, we will try to express λ in terms of β.

Now consider the volume of water. We can cut the shape of water into two parts - a cylinder with base equal to the shape of water on the lid, and half of a cylinder with base equal to a section of circle cut by the water depth of base and water depth of lid, as shown in the following figure.

Then

$$\begin{aligned}\text{the volume of water} &= \frac{1}{2}r^2(\lambda - \sin\lambda)\cdot 2x \\ &\quad + \frac{1}{2}\cdot\frac{1}{2}r^2[(\beta - \sin\beta) - (\lambda - \sin\lambda)]\cdot 2x \\ &= xr^2\{(\lambda - \sin\lambda) + \frac{1}{2}[(\beta - \sin\beta) - (\lambda - \sin\lambda)]\} \\ &= \frac{xr^2}{2}[(\beta - \sin\beta) + (\lambda - \sin\lambda)].\end{aligned}$$

Therefore,

$$\begin{aligned}\pi r^2 h &= \frac{xr^2}{2}[(\beta - \sin\beta) + (\lambda - \sin\lambda)] \\ \lambda - \sin\lambda &= \frac{2\pi h}{x} - (\beta - \sin\beta)\end{aligned}$$

Therefore,

$$\begin{aligned}A_5 &= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r^2(\lambda - \sin\lambda) + rx(\beta + \lambda) \\ &= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r^2[\frac{2\pi h}{x} - (\beta - \sin\beta)] + rx(\beta + \lambda) \\ &= \frac{r^2\pi h}{x} + rx(\beta + \lambda).\end{aligned}$$

The above expression implies that, as α varies, the sum of the area of wet surface on the base and the area of wet surface on the lid is a constant $\frac{r^2\pi h}{x}$. Therefore, we only need to find the minimum value of the wet curved surface area $rx(\beta + \lambda)$, i.e. to find the minimum value of $(\beta + \lambda)$. However, from the expression $\lambda - \sin\lambda = \frac{2\pi h}{x} - (\beta - \sin\beta)$, we cannot express λ in terms of β explicitly, hence we cannot write A_5 in terms of β.

Although we cannot write A_5 in terms of β only, we can use other method to investigate when A_5 will be minimum. Now let the area of the wet segment on the base or the lid be $K(\cdot)$, then the wet segment on the lid is

$$K(\lambda) = \frac{r^2}{2} g(\lambda)$$
$$= \frac{r^2}{2}(\lambda - \sin\lambda).$$

As $h < x$, the water depth on the lid would never exceed r, i.e. $0 \leqslant \lambda < \pi$, for cylinders of any dimensions.

The wet segment on the base is

$$K(\beta) = \frac{r^2}{2} g(\beta)$$
$$= \frac{r^2}{2}(\beta - \sin\beta),$$

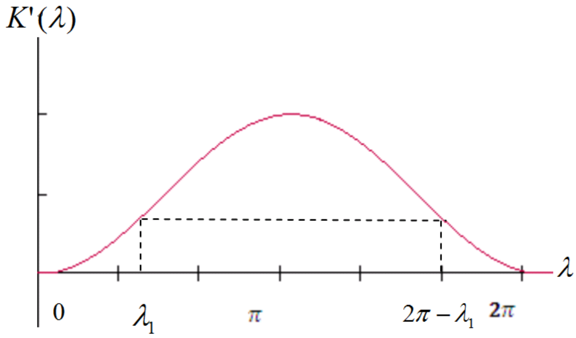
where $0 < \beta < 2\pi$ for cylinders of any dimensions.

Note that $K(\beta) + K(\lambda) = \dfrac{r^2\pi h}{x}$, which is a constant.

Now consider

$$K'(\lambda) = \frac{r^2}{2}(1 - \cos\lambda).$$

The graph of $K'(\lambda)$ against λ is as follows:



Let λ take a value λ_1, where $0 \leqslant \lambda_1 < \pi$. Then there will be a corresponding value of β, and let it be β_1, where $\lambda_1 < \beta_1 < 2\pi - \lambda_1$. (The requirement of $\beta_1 < 2\pi - \lambda_1$, though obvious, can be mathematically proved by considering the trivial case when $\lambda_1 = 0$ and $\beta_1 < 2\pi$. The reasoning is the same as the following proof, but in this trivial case the condition of $\beta_1 < 2\pi - \lambda_1$ is not required.)

Noting that the graph is reflectional symmetric along $\phi = \pi$ and opens downward, we have $0 < K'(\lambda_1) < K'(\beta_1)$ as $\lambda_1 < \beta_1 < 2\pi - \lambda_1$, i.e. no matter what the dimensions of the cylinder are, the increasing rate of change of lid area $K(\lambda_1)$ is lower than the increasing rate of change of the base area $K(\beta_1)$. However, as $0 < K'(\lambda_1) < K'(\beta_1)$, if λ_1 increases to λ_2, in order to keep $K(\lambda) + K(\beta)$ as a constant, the increase from λ_1 to λ_2 will be greater than the decrease from β_1 to β_2, i.e. $\lambda_1 + \beta_1 < \lambda_2 + \beta_2$. Therefore, as λ increases, $\lambda + \beta$ will increase, i.e. the value of A_5 is minimum when $\lambda = 0$. After the water touches the lid, the smaller the inclination angle, the larger the total wet surface area.

Conclusion for the case of $h < x$

When the water still covers the base completely, the total wet surface area remains unchanged.

When the water starts not covering the base completely, the total wet surface area starts decreasing. The minimum total wet surface area can be attained at a time when the water is not covering the base completely but not yet touches the lid.

After the water touches the lid, the total wet surface area starts increasing until the cylinder is placed horizontally.

To calculate the exact inclination angle needed for the minimum wet surface area, we need to use some numerical methods, such as Solver in the Excel files attached.

Case 2: $h > x$

We apply the result of the $h < x$ case to the $h > x$ case. As

$$\text{capacity of the cylinder} = \text{volume of water} + \text{volume of empty space}$$

and the first two terms are constant, the volume of empty space is a constant. Also,

$$\begin{aligned}\text{total surface area of the cylinder} = {} & \text{total wet surface area} \\ & + \text{total “untouched” surface area.}\end{aligned}$$

Since the total surface area of the cylinder is unchanged, to find the minimum total wet surface area, we can find the maximum total untouched surface area. The total untouched surface area with water level h is exactly the same as the total wet surface area with water level $2x - h$. Since $x < h < 2x$, we have $0 < 2x - h < x$ and hence we can use the result in case 1, but this time we need to find the maximum 'wet surface area'. According to our result in case 1, when the inclination angle changes, the variation of the total untouched surface area is as follows:

When the water still covers the base completely, the total untouched surface area remains unchanged.

When the water starts not covering the base completely, the total untouched surface area starts decreasing. It may or may not bounce back to increase before the water touches the lid.

After the water touches the lid, the total untouched surface area starts increasing until the cylinder is placed horizontally.

Therefore, the maximum total untouched area occurs either in the vertical position (or any A_3 position) or the horizontal position. According to the result in part 1, if $r \geqslant x$, then vertical position always gives the smallest total wet surface area; if $r < x$, then we can simply calculate the area in both horizontal and vertical positions, and compare which position would give a smaller total wet surface area, then that position would be the position giving the smallest total wet surface area among all inclination angles.

We can conclude the result of this part in the following table:

Water level h	The position where the wet contact area is smallest.
$h < x$	Some position where the water is not totally covering the base, and not yet touches the lid (A_4 position). The inclination angle α can be calculated by numerical methods.
$h = x$	Any position will give the same wet contact area.
$h > x$	If $r \geqslant x$, vertical position. If $r < x$, either vertical position or horizontal position, need to use direct checking to find the position giving smallest wet contact area.

5. Conclusion

In both parts, we have analysed how the wet surface area changes when the variables vary. We have found a way to find out the solution for all kinds of cylinder. Not only having found the corresponding values of the independent variables, we have examined the properties of the functions as far as I can. The results are clearly presented in the end of each part. In the process I found different interesting properties (such as the constant sum of base areas in A_5) and used various mathematical tools (such as differentiation, numerical methods, and investigating property of functions). I am very happy that I can solve my problem with the knowledge learnt in mathematics lessons.

6. Acknowledgement

I would like to take this opportunity to show the gratitude to my teacher advisor, Mr. Wong Kwok Tai, Cathay, for his supervision and guidance during the completion of the project. I am also thankful for the valuable comments given by another Mathematics teacher, Mr. Chu Lap Foo, Samuel, in my secondary school.

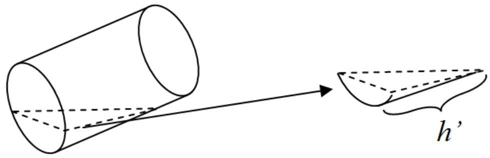
Reviewer's Comments

1. In part 2, in the case of A_4, the shape of the water is not a cone(p.14) nor half of a cylinder(p.16) and the wet curved surface area is not a triangle(p. 15). Similarly, in the case of A_5, the shape of the water surface is not a trapezium(p.18) and the shape of the water is not the combination of two cylinders(p.18). Therefore, the subsequent calculations are incorrect. A revised version of these two parts by the author is on the next page.

2. The consideration given by the author in part 2 is "over-simplified", and may be worth further exploration.

The discussion of A_4 and A_5 revised by the author

Now consider A_4.



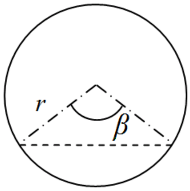


Figure of the base

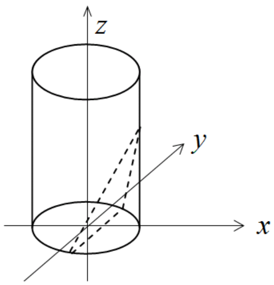
The left figure show the shape of the water, while the right figure shows the base of the cone. h' and β are defined as shown above. Note that β will decrease from 2π to γ, where γ is the angle at centre of the base formed by the water when the water just touches the lid.

$$\text{The wet contact area on the base } = \left[\frac{r^2\beta}{2} - \frac{r^2 \sin\beta}{2}\right].$$

Note that when $\beta > \pi$, the wet surface area should be the area of sector plus the area of triangle. However, since $\sin(2\pi - \beta) = -\sin\beta$, the above expression still holds when $\beta > \pi$.

The curve formed on the water surface is a part of an ellipse. Therefore, the wet curved surface area is lined by a part of an elliptical curve. By integration, the value of this area is calculated as follows:

The inclined cylindrical container is placed in a 3-dimensional space:



$$r\cos\frac{\beta}{2} \le x \le r$$

$$0 \le z \le (\cot\alpha)(x - r\cos\frac{\beta}{2})$$

$$-\sqrt{r^2 - x^2} \le y \le \sqrt{r^2 - x^2}$$

The wet curved area

$$
\begin{aligned}
&= 2 \times \int_{r\cos\frac{\beta}{2}}^{r} \int_{0}^{\cot\alpha(x - r\cos\frac{\beta}{2})} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dzdx \\
&= 2 \times \int_{r\cos\frac{\beta}{2}}^{r} \int_{0}^{\cot\alpha(x - r\cos\frac{\beta}{2})} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dzdx \\
&= 2 \times \int_{r\cos\frac{\beta}{2}}^{r} \frac{r\cot\alpha(x - r\cos\frac{\beta}{2})}{\sqrt{r^2 - x^2}}\, dzdx \\
&= 2r\cot\alpha \left[-\sqrt{r^2 - x^2} - r\cos\frac{\beta}{2}\sin^{-1}\frac{x}{r}\right]_{r\cos\frac{\beta}{2}}^{r} \\
&= r^2\cot\alpha(2\sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2}).
\end{aligned}
$$

The volume of water V

$$
\begin{aligned}
&= \int_{r\cos\frac{\beta}{2}}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{0}^{\cot\alpha(x - r\cos\frac{\beta}{2})} dzdydx \\
&= \int_{r\cos\frac{\beta}{2}}^{r} 2\cot\alpha(x - r\cos\frac{\beta}{2})\sqrt{r^2 - x^2}\, dzdydx \\
&= 2\cot\alpha \left[-\frac{1}{3}(r^2 - x^2)^{\frac{3}{2}} - r\cos\frac{\beta}{2} \times \frac{r^2}{2}\left(\sin^{-1}\frac{x}{r} + \frac{x\sqrt{r^2 - x^2}}{r^2}\right)\right]_{r\cos\frac{\beta}{2}}^{r} \\
&= 2\cot\alpha \left[-\frac{1}{2}r^3\cos\frac{\beta}{2}\left(\frac{\pi}{2}\right) + \frac{1}{3}r^3\sin^3\frac{\beta}{2} \right. \\
&\quad \left. + \frac{1}{2}r^3\cos\frac{\beta}{2}\left(\frac{\pi}{2} - \frac{\beta}{2} + \cos\frac{\beta}{2}\sin\frac{\beta}{2}\right)\right] \\
&= r^3\cot\alpha\left(\frac{2}{3}\sin^3\frac{\beta}{2} - \frac{1}{2}\beta\cos\frac{\beta}{2} + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\right).
\end{aligned}
$$

By substituting $V = \pi rh$, we have

$$
\tan\alpha = \frac{r}{\pi h}\left(\frac{2}{3}\sin^3\frac{\beta}{2} - \frac{1}{2}\beta\cos\frac{\beta}{2} + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\right).
$$

Therefore,

$$\text{the wet curved surface area} = r^2 \cot\alpha \left(2\sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2}\right)$$
$$= \frac{\pi r h \left(2\sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2}\right)}{\frac{2}{3}\sin^3\frac{\beta}{2} - \frac{1}{2}\beta\cos\frac{\beta}{2} + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}}.$$

So

$$A_4 = \frac{1}{2}r^2(\beta - \sin\beta) + \frac{\pi r h \left(2\sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2}\right)}{\frac{2}{3}\sin^3\frac{\beta}{2} - \frac{1}{2}\beta\cos\frac{\beta}{2} + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}}.$$

Though the wet curved surface area can now be written as a function of a single variable β only, this complicated expression makes it almost impossible to use analytical method to have any further exploration about the relationship between A_4 and β. Therefore, in the following discussion we will use some simple estimations in wet curved surface area and volume, so that we can further investigate the change of A_4 when β varies.

The wet curved surface area is estimated as the area of a triangle with base equal to the wet arc on the base and with height equal to h'. Therefore, the wet curved surface area $\approx \dfrac{r\beta h'}{2}$.

Therefore,

$$\text{the estimated value of } A_4 = \left(\frac{r^2\beta}{2} - \frac{r^2\sin\beta}{2}\right) + \frac{r\beta h'}{2}$$
$$= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta h'.$$

To relate β and h' to other known constants or the variable α, we consider the following figure.

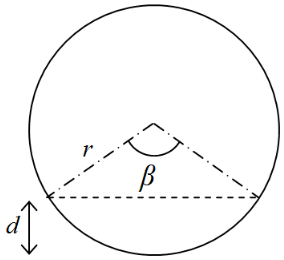


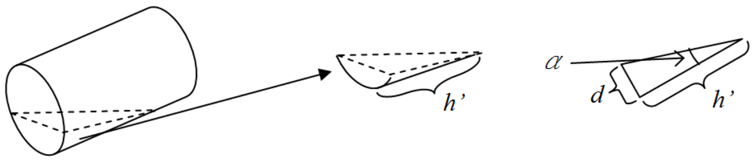
Figure of the base

Let the depth of water on the base be d.

$$\begin{aligned} d &= r - r\cos\frac{\beta}{2} \\ &= r(1-\cos\frac{\beta}{2}) \end{aligned}$$

Note that the above expression is still correct when $\beta > \pi$.

Consider the triangle formed by the water:



$$\begin{aligned} h' &= \frac{d}{\tan\alpha} \\ &= \frac{r(1-\cos\frac{\beta}{2})}{\tan\alpha} \end{aligned}$$

The volume of water is estimated as half of a cylinder with base as the shape formed on the base of the cylinder and with height h'. The volume of water,

$$\begin{aligned} V &\approx \frac{1}{2} \times \text{base area } \times \text{height} \\ &= \frac{1}{2}\left[\frac{1}{2}r^2(\beta - \sin\beta)h'\right] \\ &= \frac{r^3}{4}(\beta-\sin\beta)\frac{1-\cos\frac{\beta}{2}}{\tan\alpha}. \end{aligned}$$

By substituting $V = \pi r^2 h$, we have

$$\tan\alpha \approx \frac{r(\beta-\sin\beta)(1-\cos\frac{\beta}{2})}{4\pi h}.$$

When α is given, it is difficult to calculate the value of β, but it is easy to do the opposite. Therefore, we will switch the variable β to make our question a bit easier to analyse.

So we write A_4 in terms of β.

$$\begin{aligned}
A_4 &\approx \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta h' \\
&= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta\frac{r(1-\cos\frac{\beta}{2})}{\tan\alpha} \\
&= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta\frac{4\pi h}{\beta - \sin\beta}
\end{aligned}$$

A_4 is defined only when the water does not touch the lid. Therefore, A_4 is defined only when

$$\begin{aligned}
h' &\leqslant 2x \\
\frac{4\pi h}{\beta - \sin\beta} &\leqslant 2x \\
\beta - \sin\beta &\geqslant \frac{2\pi h}{x}
\end{aligned}$$

Let $g(\beta) = \beta - \sin\beta$. Since β decreases from 2π to ϕ, where ϕ is the angle at centre of the base when the cylinder is placed horizontally and $g'(\beta) = 1 - \cos\beta \geqslant 0$ for all values of β, $g(\beta)$ is non-increasing from $g(2\pi) = 2\pi$ to $g(\phi)$. Considering the volume of water when the cylinder is placed horizontally, we have

$$\begin{aligned}
\pi r^2 h &= \frac{r^2}{2}(\phi - \sin\phi)\cdot 2x \\
\phi - \sin\phi &= \frac{\pi h}{x}
\end{aligned}$$

Therefore, $g(\phi) = \dfrac{\pi h}{x}$.

Since $h < x$, $g(2\pi) > \dfrac{2\pi h}{x} > g(\phi)$, and the equation $\beta - \sin\beta = \dfrac{2\pi h}{x}$ must have a root between ϕ and 2π. Therefore, we can find the range of β for which $\beta - \sin\beta \geqslant \dfrac{2\pi h}{x}$.

To solve our main problem, we need to find the minimum value of

$$A_4 = \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r\beta\frac{4\pi h}{\beta - \sin\beta}$$

as β varies from 2π to γ, where $\gamma - \sin\gamma = \dfrac{2\pi h}{x}$. If we differentiate A_4 with respect to β, we have, after simplification,

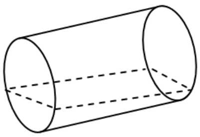
$$\frac{dA_4}{d\beta} = \frac{1}{2}r^2(1-\cos\beta) + 2\pi rh\frac{\beta\cos\beta - \sin\beta}{(\beta - \sin\beta)^2}.$$

However, the above expression is so complicated that we cannot solve the equation $\dfrac{dA_4}{d\beta} = 0$ analytically. But we can deduce that, since r^2, $1-\cos\beta$, r, h and $(\beta - \sin\beta)^2$ are all non-negative numbers, if $\beta\cos\beta - \sin\beta > 0$, then $\dfrac{dA_4}{d\beta} > 0$. When β is in the range of $\dfrac{3\pi}{2} < \beta < 2\pi$, $\cos\beta > 0$, $\cos\beta > 0$ and $-\sin\beta > 0$, hence $\dfrac{dA_4}{d\beta} > 0$, i.e. when β starts decreasing from 2π, the total wet surface area must be decreasing.

To find the turning point numerically, a programme is written in the Excel file attached in the worksheet "Part 2 (A4)". The input and output of the programme are very similar to the "Part 1" worksheet.

Finally, we consider A_5.

When α further decreases so that $\beta - \sin\beta \leqslant \dfrac{2\pi h}{x}$, the water will touch the lid. The wet contact surface area will then be equal to the sum of the wet area on the base, the wet area on the lid, and the wet curved surface area.



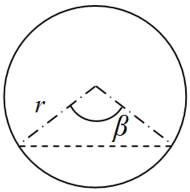


Figure of the base

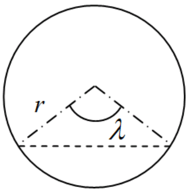


Figure of the lid

$$\text{The wet area on the base} = \frac{1}{2}r^2(\beta - \sin\beta)$$

$$\text{The wet area on the lid} = \frac{1}{2}r^2(\lambda - \sin\lambda),$$

where λ is the angle at centre formed by the water segment on the lid. Note that $0 \leqslant \lambda \leqslant \phi$.

The wet curved surface area is formed by a part of elliptical curve and two straight lines. By integration, the value of this area is calculated as follows:

The wet curved area

$$= 2 \times \int_{r\cos\frac{\beta}{2}}^{r} \int_{0}^{\cot\alpha(x - r\cos\frac{\beta}{2})} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dz dx$$

$$- 2 \times \int_{r\cos\frac{\lambda}{2}}^{r} \int_{0}^{\cot\alpha(x - r\cos\frac{\lambda}{2})} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dz dx$$

$$= r^2 \cot\alpha \left(2\sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2} + \lambda\cos\frac{\lambda}{2} - 2\sin\frac{\lambda}{2}\right).$$

The volume of water

$$= \int_{r\cos\frac{\beta}{2}}^{r} \int_{-\sqrt{r^2 - x^2}}^{\sqrt{r^2 - x^2}} \int_{0}^{\cot\alpha(x - r\cos\frac{\beta}{2})} dz dy dx$$

$$- \int_{r\cos\frac{\lambda}{2}}^{r} \int_{-\sqrt{r^2 - x^2}}^{\sqrt{r^2 - x^2}} \int_{0}^{\cot\alpha(x - r\cos\frac{\lambda}{2})} dz dy dx$$

$$= r^3 \cot\alpha \left[\left(\frac{2}{3}\sin^3\frac{\beta}{2} - \frac{1}{2}\beta\cos\frac{\beta}{2} + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\right) - \left(\frac{2}{3}\sin^3\frac{\lambda}{2} - \frac{1}{2}\lambda\cos\frac{\lambda}{2} + \cos^2\frac{\lambda}{2}\sin\frac{\lambda}{2}\right)\right].$$

By substituting $V = \pi r^2 h$, we have

$$\tan\alpha = \frac{r}{\pi h}\left[\left(\frac{2}{3}\sin^3\frac{\beta}{2} - \frac{1}{2}\beta\cos\frac{\beta}{2} + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\right) - \left(\frac{2}{3}\sin^3\frac{\lambda}{2} - \frac{1}{2}\lambda\cos\frac{\lambda}{2} + \cos^2\frac{\lambda}{2}\sin\frac{\lambda}{2}\right)\right].$$

Therefore,

$$
\begin{aligned}
&\text{the wet curved surface area}\\
&= r^2 \cot\alpha \left(2\sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2} + \lambda\cos\frac{\lambda}{2} - 2\sin\frac{\lambda}{2}\right)\\
&= \pi r h \left(2\sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2} + \lambda\cos\frac{\lambda}{2} - 2\sin\frac{\lambda}{2}\right)\\
&\quad \left[\left(\frac{2}{3}\sin^3\frac{\beta}{2} - \frac{1}{2}\beta\cos\frac{\beta}{2} + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\right)\right.\\
&\quad \left. - \left(\frac{2}{3}\sin^3\frac{\lambda}{2} - \frac{1}{2}\lambda\cos\frac{\lambda}{2} + \cos^2\frac{\lambda}{2}\sin\frac{\lambda}{2}\right)\right]^{-1}.
\end{aligned}
$$

So,

$$
\begin{aligned}
&A_5\\
&= \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r^2(\lambda - \sin\lambda)\\
&\quad + \pi r h \left(2\sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2} + \lambda\cos\frac{\lambda}{2} - 2\sin\frac{\lambda}{2}\right)\\
&\quad \left[\left(\frac{2}{3}\sin^3\frac{\beta}{2} - \frac{1}{2}\beta\cos\frac{\beta}{2} + \cos^2\frac{\beta}{2}\sin\frac{\beta}{2}\right)\right.\\
&\quad \left. - \left(\frac{2}{3}\sin^3\frac{\lambda}{2} - \frac{1}{2}\lambda\cos\frac{\lambda}{2} + \cos^2\frac{\lambda}{2}\sin\frac{\lambda}{2}\right)\right]^{-1}.
\end{aligned}
$$

Similar to the case of A_4, the above expression of wet curved surface area makes it almost impossible to use analytical method to have any further exploration about the relationship of A_5 and β. Therefore, in the following discussion we will use some simple estimations in wet curved surface area and volume, so that we can further investigate the change of A_5 when β varies.

The wet curved surface area is estimated as a trapezium, with height equals to $2x$, and two parallel sides equal to $r\beta$ and $r\lambda$. Therefore, the wet curved surface area $\approx (r\beta + r\lambda) \times (2x) \times \frac{1}{2} = rx(\beta + \lambda)$. Hence

$$A_5 \approx \frac{1}{2}r^2(\beta - \sin\beta) + \frac{1}{2}r^2(\lambda - \sin\lambda) + rx(\beta + \lambda).$$

In order to find the minimum value of A_5 as β varies, we will try to express λ in terms of β.

To make the question simple, we estimate the volume of water by cutting the shape of water into two parts – a cylinder with base equal to the shape of water on the lid, and half of a cylinder with base equal to a section of circle cut by the water depth of base and water depth of lid, as shown in the following figure.

Then

$$\begin{aligned}\text{the volume of water} &\approx \frac{1}{2}r^2(\lambda - \sin\lambda)\cdot 2x \\ &\quad + \frac{1}{2}\cdot\frac{1}{2}r^2[(\beta-\sin\beta)-(\lambda-\sin\lambda)]\cdot 2x \\ &= xr^2\{(\lambda-\sin\lambda)+\frac{1}{2}[(\beta-\sin\beta)-(\lambda-\sin\lambda)]\} \\ &= \frac{xr^2}{2}[(\beta-\sin\beta)+(\lambda-\sin\lambda)].\end{aligned}$$

Therefore,

$$\begin{aligned}\pi r^2 h &\approx \frac{xr^2}{2}[(\beta-\sin\beta)+(\lambda-\sin\lambda)] \\ \lambda-\sin\lambda &\approx \frac{2\pi h}{x}-(\beta-\sin\beta)\end{aligned}$$

Therefore,

$$\begin{aligned}A_5 &\approx \frac{1}{2}r^2(\beta-\sin\beta)+\frac{1}{2}r^2(\lambda-\sin\lambda)+rx(\beta+\lambda) \\ &= \frac{1}{2}r^2(\beta-\sin\beta)+\frac{1}{2}r^2[\frac{2\pi h}{x}-(\beta-\sin\beta)]+rx(\beta+\lambda) \\ &= \frac{r^2\pi h}{x}+rx(\beta+\lambda).\end{aligned}$$

The above expression implies that, as α varies, the sum of the area of wet surface on the base and the area of wet surface on the lid is a constant $\frac{r^2\pi h}{x}$. Therefore, we only need to find the minimum value of the wet curved surface area $rx(\beta+\lambda)$, i.e. to find the minimum value of $(\beta+\lambda)$. However, from the expression $\lambda-\sin\lambda=\frac{2\pi h}{x}-(\beta-\sin\beta)$, we cannot express λ in terms of β explicitly, hence we cannot write A_5 in terms of β.

Although we cannot write A_5 in terms of β only, we can use other method to investigate when A_5 will be minimum. Now let the area of the wet segment on the base or the lid be $K(\cdot)$, then the wet segment on the lid is

$$\begin{aligned} K(\lambda) &= \frac{r^2}{2} g(\lambda) \\ &= \frac{r^2}{2}(\lambda - \sin\lambda). \end{aligned}$$

As $h < x$, the water depth on the lid would never exceed r, i.e. $0 \leqslant \lambda < \pi$, for cylinders of any dimensions.

The wet segment on the base is

$$\begin{aligned} K(\beta) &= \frac{r^2}{2} g(\beta) \\ &= \frac{r^2}{2}(\beta - \sin\beta), \end{aligned}$$

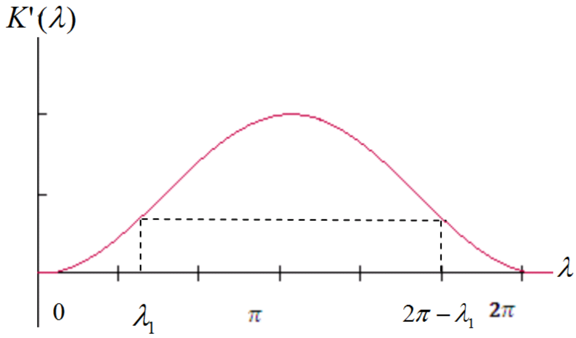
where $0 < \beta < 2\pi$ for cylinders of any dimensions.

Note that $K(\beta) + K(\lambda) = \dfrac{r^2\pi h}{x}$, which is a constant.

Now consider

$$K'(\lambda) = \frac{r^2}{2}(1 - \cos\lambda).$$

The graph of $K'(\lambda)$ against λ is as follows:



Let λ take a value λ_1, where $0 \leqslant \lambda_1 < \pi$. Then there will be a corresponding value of β, and let it be β_1, where $\lambda_1 < \beta_1 < 2\pi - \lambda_1$. (The requirement of $\beta_1 < 2\pi - \lambda_1$, though obvious, can be mathematically proved by considering the trivial case when $\lambda_1 = 0$ and $\beta_1 < 2\pi$. The reasoning is the same as the following proof, but in this trivial case the condition of $\beta_1 < 2\pi - \lambda_1$ is not required.)

Noting that the graph is reflectional symmetric along $\phi = \pi$ and opens downward, we have $0 < K'(\lambda_1) < K'(\beta_1)$ as $\lambda_1 < \beta_1 < 2\pi - \lambda_1$, i.e. no matter what the dimensions of the cylinder are, the increasing rate of change of lid area $K(\lambda_1)$ is lower than the increasing rate of change of the base area $K(\beta_1)$. However, as $0 < K'(\lambda_1) < K'(\beta_1)$, if λ_1 increases to λ_2, in order to keep $K(\lambda) + K(\beta)$ as a constant, the increase from λ_1 to λ_2 will be greater than the decrease from β_1 to β_2, i.e. $\lambda_1 + \beta_1 < \lambda_2 + \beta_2$. Therefore, as λ increases, $\lambda + \beta$ will increase, i.e. the value of A_5 is minimum when $\lambda = 0$. After the water touches the lid, the smaller the inclination angle, the larger the total wet surface area.