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CONSTRUCTION OF TANGENTS TO CIRCLES IN POINCARÉ MODEL

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Abstract. In this project, we study Poincaré disk model of hyperbolic geometry and compare it with Euclidean geometry we have learnt in school. We investigate some basic properties of the model and derive some theorems comparable to those in Euclidean geometry.
The main objective of our work is to construct four common (non-Euclidean) tangents to two circles with Euclidean compass and Euclidean straightedge, as well as two other construction problems, in Poincaré disk model. With non-Euclidean transformations, we can transform a point to anywhere inside the Poincaré disk, with lengths and angles preserved. So we first focus on performing the transformation by compass and straightedge, and then solve the problems with a centre of the circle placed at the centre of the disk. Finally we can transform the picture back to the given position by the inverse function.

1. Introduction

In secondary school, we learn Euclidean Geometry, which is based on *Elements* by Euclid. It is the geometry in our daily life, and people used to think it as the geometry of our world. But is it the only geometry?

As the fifth postulate is much more complicated than the other four, it is hard for many mathematicians to accept. Many people tried to deduce the fifth postulate from the other four postulates, but they did not succeed.

After centuries of trials, people developed some new ideas. Russian mathematician Nikolai Ivanovich Lobachevsky (1792-1856) assumed that the fifth postulate was not true and replaced it by the following statement: "*Given*

[1]This work is done under the supervision of the authors' teacher, Mr. Chun-Yu Kwong.

any line L and a point P not on L, there are infinitely many lines through P that do not meet L." He then successfully developed a new geometry, the **hyperbolic geometry** (also called **Lobachevskian geometry**).

When we started to do our project, we tried to investigate whether the theorems in geometry we have learnt in school are true in non-Euclidean geometry. Lacking time and background knowledge, we chose to work on Poincaré disk model of hyperbolic geometry first, instead of proving or disproving those theorems in general situations.

In the course of our work, we used Excel to calculate the Cartesian equations of hyperbolic lines and circles. This helped us find easily that many theorems about circles are not valid in Poincaré model. We were also interested in the existence of Euler line and nine-point circle, but found that both do not exist.

Our interest then shifted to construction problems. We learnt methods to construct hyperbolic lines (d-lines) and circles using Euclidean compass and straightedge, from "Compass and Straightedge in the Poincaré Disk" written by Chaim Goodman-Strauss. Bearing in our minds that in Poincaré model, circles were Euclidean circles while lines were circular arcs, we thought that the construction problems of tangents to circles in Poincaré model should be interesting.

We have solved three construction problems by Euclidean compass and straightedge in our project, namely,

1. construction of the tangent to a circle at a point,
2. construction of the tangents to a circle from an external point,
3. construction of the four common tangents to two circles.

In the process, we tried to imitate those methods used in Euclidean geometry to construct tangents to circles. But the methods we use in Euclidean geometry require the fact that the angle in a semi-circle is a right angle, which is not true in non-Euclidean case. Finally, we developed the method of construction in a totally different way.

We tried all construction methods described in this report using Sketchpad, and most of the figures in the report were drawn with this software.

2. Hyperbolic Geometry and Poincaré Disk Model

2.1. From Euclidean to non-Euclidean Postulates

In Euclid's *Elements*, propositions in geometry are deduced from the following five **Euclidean postulates**[1]:

Postulate 1. *A straight line segment can be drawn joining any two points.*

Postulate 2. *Any straight line segment can be extended indefinitely in a straight line.*

Postulate 3. *Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.*

Postulate 4. *All right angles are congruent.*

Postulate 5. (Parallel Postulate) *If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough,*

or equivalently,

given any straight line and a point not on it, there exists one and only one straight line which passes through that point and never intersects the first line, no matter how far they are extended.

The fifth postulate is lengthy and not as trivial as the other four. Many mathematicians felt uncomfortable with it and tried to deduce it from the other postulates, or reduce it into a simpler statement. In 1829, **Nikolai Ivanovich Lobachevsky** (1792-1856) published a book describing a consistent geometry with the parallel postulate replaced by the **Non-Euclidean Parallel Postulate**[2]:

"*Given any straight line L and a point P not on L, there are at least two straight lines which pass through P and do not meet L.*"

2.2. Möbius Transformations

Definition 2.1. [3] *The **extended complex plane** is the union of the Euclidean plane and one extra point, the point at infinity.*

Definition 2.2. *A **generalized circle** in the extended complex plane is a set that is either a circle or an extended line. ($l \cup \{\infty\}$ is an extended line if l is a line.)*[4]

Definition 2.3. [4] *The **cross ratio** of four complex numbers z_1, z_2, z_3 and z_4 is defined as* $[z_1, z_2; z_3, z_4] = \dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$.

Definition 2.4. [4] *In the extended complex plane, a **Möbius transformation** is a bijective function that preserves the cross ratio of any four points in the plane.*

In the following, we will show that a Möbius transformation preserves

1. generalized circles,
2. the angle between two arcs, and
3. inversion.

General Form of Möbius transformations

Theorem 2.5. [4] *If f is a Möbius transformation in the extended complex plane, then there exist complex constants a, b, c and d such that* $f(z) \equiv \dfrac{az + d}{cz + d}$.

Proof. Let $w = f(z)$ be a Möbius transformation, $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$. By the fact that f is surjective, there are three points z_1, z_2 and z_3 such that $w_1 = f(z_1)$, $w_2 = f(z_2)$ and $w_3 = f(z_3)$. By the fact that f is injective, z_1, z_2 and z_3 are three different points. Since a Möbius transformation preserves cross ratio, for any $z \in \mathbb{C} \cup \{\infty\}$ ($z \neq z_1, z_2, z_3$), we have $[z_1, z_2; z_3, z] = [0, 1; \infty, w]$.

If z_1, z_2, z_3 are all not equal to ∞, then $\dfrac{(z_1 - z_3)(z_2 - z)}{(z_2 - z_3)(z_1 - z)} = \dfrac{1 - w}{-w}$ and hence $w = \dfrac{z_1 - z}{(A-1)z + z_1 - Az_2}$, where $A = \dfrac{z_1 - z_3}{z_2 - z_3}$.

If $z_3 = \infty$, then $\dfrac{z_2 - z}{z_1 - z} = \dfrac{1-w}{-w}$ and hence $w = \dfrac{z_1 - z}{z_1 - z_2}$.

Similarly, it is easy to deduce that $w = \dfrac{z_2 - z_3}{z - z_3}$ when $z_1 = \infty$ and $w = \dfrac{z - z_1}{z - z_3}$ when $z_2 = \infty$.

Therefore, in any condition, the general form of Möbius transformation is $w = \dfrac{az+b}{cz+d}$ where a, b, c, d are complex constants. □

Remark We can verify that a function of the form $f(z) \equiv \dfrac{az+b}{cz+d}$ is a Möbius transformation.

Preservation of Generalized Circles

Theorem 2.6. [4] *A Möbius transformation maps a generalized circle in the extended complex plane to a generalized circle.*

Proof. Let $w = f(z)$ be a Möbius transformation. Let C be a generalized circle in the extended complex plane, and $f(C) = \{z \in \mathbb{C} \cup \infty \mid z = f(u), u \in C\}$ be the image of C. We are going to prove that $f(C)$ is also a generalized circle in the extended complex plane.

It is easy to prove that four distinct points are concyclic if and only if their cross-ratio is real. (If one of the points is at infinity, then they are concyclic means that the three other points are collinear.)

For three arbitrary points z_1, z_2, z_3 on C, let $w_i = f(z_i)$, $i = 1, 2, 3$. For any point z on C other than z_1, z_2, z_3 as f is a Möbius transformation, we have $[z_1, z_2; z_3, z] = [w_1, w_2; w_3, f(z)]$. Since z_1, z_2, z_3 and z are concyclic, $[z_1, z_2; z_3, z]$ is real. Thus $[w_1, w_2; w_3, f(z)]$ is also real and hence w_1, w_2, w_3 and $f(z)$ are also concyclic. Let $\tilde{C}$ be the generalized circle passing through w_1, w_2, w_3. Then $z \in C$ implies that $f(z) \in \tilde{C}$, and hence $f(C) \subset \tilde{C}$.

Consider a variable point w $(w \neq w_1, w_2, w_3)$ on $\tilde{C}$, as f^{-1} is also a Möbius transformation, we also have

$$\begin{aligned}[w_1, w_2; w_3, w] &= [f^{-1}(w_1), f^{-1}(w_2); f^{-1}(w_3), f^{-1}(w)] \\ &= [z_1, z_2; z_3, f^{-1}(w)].\end{aligned}$$

So $w \in \tilde{C}$ implies that $f^{-1}(w) \in C$, and hence $f^{-1}(\tilde{C}) \subset C$. Thus $\tilde{C} \subset f(C)$. Hence $f(C) = \tilde{C}$ and the result follows. □

Preservation of Angle between Two Circular Arcs

Theorem 2.7. [4] *Let C_1 and C_2 be arcs of two intersecting generalized circles with a unique common end point and this common end point is not at infinity. If f is a Möbius transformation, then the angle between C_1 and C_2 is equal to that between $f(C_1)$ and $f(C_2)$.*

Proof. Let z_1 be the common end point of C_1 and C_2 and $z_1' = f(z_1)$. Let z_2 be the other intersection point of the circle containing C_1 and the circle containing C_2. The angle between the two arcs is $\arg[z_1, z_2; \zeta_1, \zeta_2]$.

Let $C_j' = f(C_j)$, $\zeta_j' = f(\zeta_j)$, $j = 1, 2$; $z_2' = f(z_2)$. Obviously, z_1' and z_2' are the intersection points of images of the circles containing C_1 and C_2. ζ_1' and ζ_2' are other terminal points of C_1' and C_2' is $\arg[z_1', z_2'; \zeta_1', \zeta_2']$.

Since f preserves the cross-ratio of four points,

$$\arg[z_1, z_2; \zeta_1, \zeta_2] = \arg[z_1', z_2'; \zeta_1', \zeta_2']$$

and hence it preserves angles. □

Introduction of Inversion

Definition 2.8. *Let C be a circle, with centre z_0 and radius r. For a point $z \neq z_0$, and a half-line l starting from z_0 and passing through z, define the inverse point z^* of z with respect to C as the point on l satisfying $|z - z_0| \cdot |z^* - z_0| = r^2$. Besides, we define the inverse point of z_0 as ∞, and vice versa.*
If C is an extended line $l \cup \{\infty\}$, then z^* is the inverse point of z with respect to C if l is the perpendicular bisector of the line segment joining z and z^*.

Theorem 2.9. [4] *If C is an extended line, the proof is simple. let C be a circle in the extended complex plane. Let z^* be the inverse point of z with* *respect to C, and $z^* \neq z$. Then any generalized circle passing through z and z^* is orthogonal to C.*

Proof. Let C be a generalized circle in the extended complex plane, with centre z_0 and radius r.

Case 1 $\quad z \neq z_0$ and $z^* \neq z_0$
Let C_1 be a generalized circle passing through z and z^* with centre z_1, and intersecting C at ζ_1 and ζ_2.

Then $|\zeta_1 - z_0|^2 = r^2 = |z - z_0| \cdot |z^* - z_0|$. By the Intersecting Chords Theorem (Appendix A), the line $z_0\zeta_1$ touches C_1, thus $z_0\zeta_1 \perp \zeta_1 z_1$. But $z_0\zeta_1$ is the radius of C, so $\zeta_1 z_1$ is also a tangent to C, thus C is orthogonal to C_1 at ζ_1. Similarly, C is also orthogonal to C_1 at ζ_2.

Case 2 $\quad z = z_0$ or $z^* = z_0$
Since $\{z, z^*\} = \{z_0, \infty\}$, a generalized circle passing through z and z^* is a straight line passing through the centre of C, obviously, it is orthogonal to C. Hence the result follows. $\square$

Theorem 2.10. [4] *Let C be a generalized circle in the extended complex plane. Let z^* be the inverse point of z with respect to C, and $z^* \neq z$. Then any generalized circle passing through z and orthogonal to C must pass through z^*.*

The proof is similar to that of Theorem 2.9.

Theorem 2.11. [4] *Let C be a generalized circle in the extended complex plane with equation $\alpha z\bar{z} + \bar{\beta}z + \beta\bar{z} + \gamma = 0$, where α, $\gamma \in \mathbb{R}$ and $\alpha\gamma < |\beta|^2$. If z^* is the inverse point of with respect to C, then we have $\alpha z^*\bar{z} + \bar{\beta}z^* + \beta\bar{z} + \gamma = 0$.*

Proof. Let z^* be the inverse point with respect to $C = \{z \in \mathbb{C} : |z - z_0| = r\}$. Then we have $|z - z_0| \cdot |z^* - z_0| = r^2$.

As z, z^*, z_0 are collinear, there exists $k > 0$ such that $z - z_0 = k(z^* - z_0)$. Then

$$\begin{aligned} |z - z_0| \cdot |z^* - z_0| &= k|z^* - z_0|^2 \\ &= k(\overline{z^* - z_0})(z^* - z_0) \\ &= [\overline{k(z^* - z_0)}](z^* - z_0) \\ &= \overline{(z - z_0)}(z^* - z_0). \end{aligned}$$

So we have $(\bar{z}-\bar{z_0})(z^*-z_0)=r^2$, i.e. $z^*\bar{z}-\bar{z_0}z^*-z_0\bar{z}+|z_0|^2-r^2=0$.

Since the equation of C can be written as $z\bar{z}-\bar{z_0}z-z_0\bar{z}+|z_0|^2-r^2=0$, the result follows in this case.

Let L be the straight line $\bar{\beta}z+\beta\bar{z}+\gamma=0$, where $\gamma\in\mathbb{R}$ and $\beta\neq 0$. Let u be a variable point on L. If z^* is the inverse point z with respect to L, then we have $|u-z|=|u-z^*|$.

Hence

$$\begin{aligned}(u-z)(\bar{u}-\bar{z})&=(u-z^*)(\bar{u}-\bar{z^*})\\ u\bar{u}-u\bar{z}-\bar{u}z+z\bar{z}&=u\bar{u}-u\bar{z^*}-\bar{u}z^*+z^*\bar{z^*}\\ (\bar{z^*}-\bar{z})u+(z^*-z)\bar{u}+z\bar{z}-z^*\bar{z^*}&=0\end{aligned}$$

Comparing the coefficients with $k(\bar{\beta}u+\beta\bar{u}+\gamma)=0$ where $k\neq 0$, we get

$$k\bar{\beta}=\bar{z^*}-\bar{z} \tag{1}$$

$$k\beta=z^*-z \tag{2}$$

$$k\gamma=z\bar{z}-z^*\bar{z^*} \tag{3}$$

From (2), we have $\overline{k\beta}=\bar{z^*}-\bar{z}$. By (1), $k\bar{\beta}=\overline{k\beta}$ and hence $k\in\mathbb{R}\backslash\{0\}$. Then

$$\begin{aligned}&\bar{\beta}z^*+\beta\bar{z}+\gamma\\ =&\frac{1}{k}(\bar{z^*}-\bar{z})z^*+\frac{1}{k}(z^*-z)\bar{z}+\frac{1}{k}(z\bar{z}-z^*\bar{z^*})\\ =&0.\end{aligned}$$

Hence z^* satisfies $\alpha z^*\bar{z}+\bar{\beta}z^*+\beta\bar{z}+\gamma=0$, where C is a circle if $\alpha\neq 0$ and is a straight line if $\alpha=0$. □

By the above theorem, if the equation of C is $\alpha z\bar{z}+\bar{\beta}z+\beta\bar{z}+\gamma=0$, then the inverse transformation with respect to C can be written as

$$z^*=-\frac{\beta\bar{z}+\gamma}{\alpha\bar{z}+\bar{\beta}}\quad(\alpha,\gamma\in\mathbb{R},\alpha\gamma<|\beta|^2).$$

If we let $f(z)=-\dfrac{\bar{\beta}z+\gamma}{\alpha z+\beta}$, then $z^*=\overline{f(z)}$.

Therefore, **an inverse transformation is the conjugate of a Möbius transformation**.

An immediate consequence is that an inverse transformation maps the cross ratio of any 4 points into its conjugate, as $[z_1^*,z_2^*;z_3^*,z_4^*]=[\bar{z_1},\bar{z_2};\bar{z_3},$

$\bar{z}_4] = \overline{[z_1, z_2; z_3, z_4]}$.

Preservation of Inversion

Theorem 2.12. [4] *Let $w = f(z)$ be a Möbius transformation and C be a generalized circle in $\mathbb{C} \cup \infty$. If z_0^* is the inverse point of z_0 with respect to C, then $w_0^* = f(z_0^*)$ is the inverse point of $w_0 = f(z_0)$ with respect to $f(C)$.*

Proof. Let C_1 be a generalized circle passing through z_0 and z_0^*. Then C_1 is orthogonal to C. By properties of Möbius transformations, $f(C)$ and $f(C_1)$ are orthogonal circles. Also, $f(C_1)$ passes through w_0 and w_0^*.

Suppose that there is another generalized circle C_2 passing through z_0 and z_0^*. Then $f(C_2)$ is also a circle orthogonal to $f(C)$ and passing through w_0 and w_0^*. By Theorem 2.10, any generalized circle which passes through w_0 and is orthogonal to $f(C)$ must passes through the inverse point of w_0, so the inverse point of w_0 with respect to $f(C)$ must lie on $f(C_1)$ and $f(C_2)$, and this point must be w_0^*. □

2.3. Poincaré Model

Notation 2.13. *We denote the **Poincaré unit disk** by $D = \{z \in \mathbb{C} : |z| < 1\}$ and its circumference by $U = \{z \in \mathbb{C} : |z| = 1\}$*

Definition 2.14. [3] *A **d-line** is that part of a generalized circle which meets U at right angles and which lies in D.*

A d-line passing through two given points always exists and is unique (Construction 4.5)

Definition 2.15. [3] *Two d-lines that do not meet in D are **parallel** if the generalized Euclidean circles of which they are parts meet at a point on U.*

Definition 2.16. [3] *Two d-lines that do not meet in D are **ultra-parallel** if the generalized Euclidean circles of which they are parts do not meet on U.*

Definition 2.17. [3] *The **angle between two d-lines** passing through a point A in D is the Euclidean angle between their tangents at A.*

Definition 2.18. [4] *For any two points (z_1, z_2), there exists a unique d-line l passing through them. Let ζ_1 and ζ_2 be the terminal points of l in* *which ζ_1 and ζ_2 are on U. (ζ_1 is nearer to z_1 and ζ_2 is nearer to z_2). The* ***non-Euclidean/hyperbolic distance*** *between this two points is defined by*

$$d(z_1, z_2) = \begin{cases} \log[z_1, z_2; \zeta_1, \zeta_2]^{-1} = \log \dfrac{|z_2 - \zeta_1||z_1 - \zeta_2|}{|z_1 - \zeta_1||z_2 - \zeta_2|} & \text{if } z_1 \neq z_2 \\ 0 & \text{if } z_1 = z_2 \end{cases}.$$

It is easy to verify that $d(z_1, z_2)$ is well-defined.

Note If z_1, z_2 interchange, $d(z_2, z_1) = \log \dfrac{|z_1 - \zeta_2||z_2 - \zeta_1|}{|z_2 - \zeta_2||z_1 - \zeta_1|} = d(z_1, z_2)$.

When $|z_1| \to 1$, $z_1 \to \zeta_1$ and hence $d(z_1, z_2) \to \infty$ (similar for $|z_2| \to 1$).
When $z_1 = 0$ and $z_2 = r$ $(0 < r < 1)$, $d(0, r) = \log \left| \dfrac{(r+1)(0-1)}{(0+1)(r-1)} \right| = \log \dfrac{1+r}{1-r}$.

Theorem 2.19. *Let z_1 and z_2 be two points on a d-line l with end-points n_1 and n_2, where n_1 is closer to z_1 and n_2 is closer to z_2. If z_3 is a point on l between z_1 and z_2, then $d(z_1, z_3) + d(z_3, z_2) = d(z_1, z_2)$.*

Proof. By definition,

$$d(z_1, z_3) = \log \frac{|z_3 - n_1||z_1 - n_2|}{|z_1 - n_1||z_3 - n_2|}$$

and

$$d(z_3, z_2) = \log \frac{|z_2 - n_1||z_3 - n_2|}{|z_3 - n_1||z_2 - n_2|}.$$

Therefore,

$$\begin{aligned} d(z_1, z_3) + d(z_3, z_2) &= \log \frac{|z_3 - n_1||z_1 - n_2|}{|z_1 - n_1||z_3 - n_2|} + \log \frac{|z_2 - n_1||z_3 - n_2|}{|z_3 - n_1||z_2 - n_2|} \\ &= \log \frac{|z_2 - n_1||z_3 - n_2|}{|z_3 - n_1||z_2 - n_2|} \frac{|z_2 - n_1||z_3 - n_2|}{|z_3 - n_1||z_2 - n_2|} \\ &= \log \frac{|z_2 - n_1||z_1 - n_2|}{|z_1 - n_1||z_2 - n_2|} \\ &= d(z_1, z_2). \end{aligned}$$

$\square$

Definition 2.20. *A* ***circle*** *is the locus of all the points which have the same hyperbolic distance (radius) from a fixed point (centre).*

2.4. Non-Euclidean Transformations

Definition 2.21. [4] *Let f be a bijective function on D. If f preserves the hyperbolic distance between two points, i.e. $d(z_1, z_2) = d(f(z_1), f(z_2))$ for all z_1, $z_2 \in D$, then f is called a* ***non-Euclidean transformation****.*

Theorem 2.22. [4] *If f is a Möbius transformation such that $f(D) = D$, then f is a non-Euclidean transformation.*

Proof. As f preserves circles and angles between arcs, it transforms d-lines to d-lines. Since the hyperbolic distance between two points depends on the cross ratio and f preserves cross ratios, f preserves hyperbolic distances. So it is a non-Euclidean transformation. $\square$

Corollary 2.23. [4] *Let f be a Möbius transformation of the form $f(z) = e^{i\theta}\dfrac{z - z_0}{1 - \bar{z}_0 z}$, where z_0 is a point inside D and θ is a real constant. Then f is a non-Euclidean transformation.*

Proof. If $f(z) = e^{i\theta}\dfrac{z - z_0}{1 - \bar{z}_0 z}$,

$$\begin{aligned} |f(z)|^2 &= \frac{(z - z_0)(\bar{z} - \bar{z}_0)}{(1 - \bar{z}_0 z)(1 - z_0 \bar{z})} \\ &= \frac{|z|^2 - z_0\bar{z} - z\bar{z}_0 + |z_0|^2}{1 - z_0\bar{z} - z\bar{z}_0 + |z_0|^2|z|^2} \\ &= 1 \quad \text{ if } |z| = 1. \end{aligned}$$

Hence $f(U) = U$. As $f(z_0) = 0$, we have $f(D) = D$. So f is a non-Euclidean transformation. $\square$

Note If $f(z) = e^{i\theta}\dfrac{z - z_0}{1 - \bar{z}_0 z}$ and $g(z) = \dfrac{z - z_1}{1 - \bar{z}_1 z}$, then $g^{-1} \circ f$ is a non-Euclidean transformation that transforms z_0 to z_1.

Corollary 2.24. [4] *Given any two points u and v in the Poincaré disk, the hyperbolic distance between the points is $d(u, v) = \log\left(1 + \dfrac{|u - v|}{|1 - \bar{u}v|}\right) - \log\left(1 - \dfrac{|u - v|}{|1 - \bar{u}v|}\right)$.*

Proof. Let f be the function $f(z) = e^{i\theta}\dfrac{z - u}{1 - \bar{u}z}$, where θ is a real constant such that $f(v)$ is a positive real number. Note that $f(u) = 0$.

Let $r = f(v)$. Then $r = |f(v)| = \dfrac{|u-v|}{|1-\bar{u}v|}$. Here f is a non-Euclidean transformation and hence

$$\begin{aligned} d(u,v) &= d(f(u), f(v)) \\ &= d(0,r) \\ &= \log \frac{1+r}{1-r} \\ &= \log\left(1 + \frac{|u-v|}{|1-\bar{u}v|}\right) - \log\left(1 - \frac{|u-v|}{|1-\bar{u}v|}\right). \end{aligned}$$

□

Theorem 2.25. [4] *A hyperbolic circle in Poincaré model is a Euclidean circle.*

Proof. Let C be a hyperbolic circle with hyperbolic centre z_0 and hyperbolic radius r. Let f be the Möbius transformation $f(z) = \dfrac{z-z_0}{1-\bar{z}_0 z}$. Then f is a non-Euclidean transformation. Therefore, for any $z \in \mathbb{C}$,

$$\begin{aligned} d(z, z_0) &= r \\ d(f(z), f(z_0)) &= d(f(z), 0) = r \\ r &= \log \frac{1+|f(z)|}{1-|f(z)|} \\ |f(z)| &= \frac{e^r - 1}{e^r + 1} \end{aligned}$$

So $f(C)$ is a Euclidean circle and $C = f^{-1}(f(C))$ is also a Euclidean circle.

□

Theorem 2.26. [4] *The inversion with respect to a circle which is orthogonal to U is a non-Euclidean transformation.*

Proof. Let C be a generalized circle which is orthogonal to the unit circle U. Since an inversion is the conjugate of a Möbius transformation (Theorem 2.11), it also preserves angles, and transforms a generalized circle to a generalized circle.

Let f be the inversion with respect to C. As C is orthogonal to U, the inverse of an arbitrary point on U with respect to C also lies on U. So $f(U) = U$ and thus $f(D) = D$. Since an inversion preserves circles and angles between arcs, f transforms d-lines to d-lines.

Let z_1 and z_2 be two arbitrary points inside D such that $z_1 \neq z_2$. Construct a circle C' through z_1 and z_2 that is orthogonal to U, and intersecting U at ζ_1 and ζ_2 , with ζ_1 nearer to z_1 while ζ_2 nearer to z_2. We have

$$[z_1, z_2; \zeta_1, \zeta_2] = \overline{[f(z_1), f(z_2); f(\zeta_1), f(\zeta_2)]}.$$

Since the four points are concyclic, the cross ratios above must be real numbers. Therefore,

$$[z_1, z_2; \zeta_1, \zeta_2] = [f(z_1), f(z_2); f(\zeta_1), f(\zeta_2)].$$

From the definition of hyperbolic distance in Poincaré model, we have

$$d(z_1, z_2) = d(f(z_1), f(z_2)).$$

So f preserves the hyperbolic distance and is a non-Euclidean transformation. □

Theorem 2.27. *Given two arbitrary points which are the inverse of each other with respect to a d-line, the d-line is the hyperbolic perpendicular bisector of the d-line (segment) joining the points.*

Proof. Let A be a point and γ be a d-line in the Poincaré disk such that A is not on γ. Let A' be the inverse of A with respect to γ, then the d-line joining A and A' is orthogonal to γ(Theorem 2.9), i.e. $AA' \perp \gamma$.

Let B be the intersection of AA' and γ. Since B is a point on γ, it is the inverse of itself with respect to γ. By Theorem 2.26, the inversion with respect to γ is a non-Euclidean transformation. As this inversion transforms the segment AB to the segment $A'B$, the two segments are equal in (hyperbolic) length. Hence, γ is the hyperbolic perpendicular bisector of AA'. □

3. More on Poincaré Model

3.1. Hyperbolic Triangles

Theorem 3.1. [4] *The sum of the angles in a triangle must be less than π.*

Proof. Let w_1, w_2, w_3 be any three points in the unit disk, $\triangle w_1 w_2 w_3$ be the triangle described by the d-lines joining them. By a suitable non-Euclidean transformation, we can transform the triangle, with angles and lengths of sides preserved, so that w_1 coincides with the origin. Without loss of generality, suppose that $w_1 = 0$. Then the sides with end point w_1 lie on radii of the Poincaré disk. Denote the interior angles of the hyperbolic triangle at w_1, w_2, w_3 by α ,β ,γ respectively.

Construct also the Euclidean triangle with vertices w_1, w_2, w_3. Denote its interior angles at w_1, w_2, w_3 by α' ,β' ,γ' respectively.

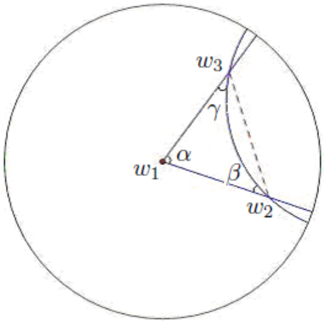


FIGURE 1

As shown in Figure 1, $\alpha = \alpha'$, $\beta < \beta'$, $\gamma < \gamma'$. Since $\alpha' + \beta' + \gamma' = \pi$, we have $\alpha + \beta + \gamma < \pi$. $\square$

Take any three points ζ_1, ζ_2, ζ_3 on the circumference of the unit disk, mark the vectors from the origin to these points and take any three points z_1, z_2, z_3 on the vectors respectively. Let α, β ,γ be the interior angles of $\triangle z_1 z_2 z_3$.

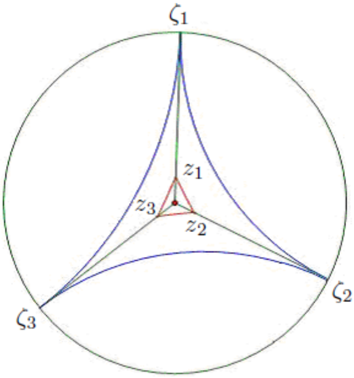


FIGURE 2

When z_1, z_2, z_3 tend to ζ_1, ζ_2, ζ_3 along their vectors, the angles α, β, γ tend to 0.

When z_1, z_2, z_3 tend to the origin along their vectors, the d-lines joining z_1,

z_2, z_3 tend to Euclidean straight lines, and the angle sum $\alpha+\beta+\gamma$ tend to π.

Theorem 3.1 implies that there is no rectangle in Poincaré model. A rectangle can be divided into two triangles by its diagonal, and hence the sum of its interior angles should be less than $\pi + \pi = 2\pi$. However, a rectangle should have four right angles, with a sum of $4 \times \frac{\pi}{2} = 2\pi$.

Note In Euclidean geometry, there are 5 rules for the congruence of triangles, namely, SAS, SSS, ASA, AAS and RHS. All these 5 rules are valid in Poincaré model. Besides, we can prove that when two hyperbolic triangles have equal corresponding angles, they are congruent to each other (AAA).

The proof of these six rules of congruence can be found in Appendix B.

3.2. Properties of Parallel Lines

Theorem 3.2. [4] *A straight line cuts another two straight lines with the corresponding angles equal if and only if this straight line passes through the mid-point of a common perpendicular line within the region bounded by those two straight lines.*

Proof. Let l and l' be any two straight lines with a common perpendicular line h, l'' be another straight line which cuts l and l' at z_1 and z_1' respectively and passes through the mid-point of the common perpendicular line (Figure 2.3).

$$\begin{aligned}
\angle z_1 z_0 z_2 &= \angle z_1' z_0 z_2' \text{ (vertically opposite angles)} \\
z_0 z_2 &= z_0 z_2' \text{ (given)} \\
\angle z_1 z_2 z_0 &= \angle z_1' z_2' z_0' \text{ (given)} \\
\text{Therefore, } \triangle z_0 z_1 z_2 &\cong \triangle z_0 z_1' z_2'. \text{ (ASA)} \\
\text{Therefore, } \angle z_0 z_1 z_2 &\cong \angle z_0 z_1' z_2'. \ (\triangle z_0 z_1 z_2 \cong \triangle z_0 z_1' z_2')
\end{aligned}$$

So the corresponding angles are equal as vertically opposite angles are equal.

Let l and l' be any two straight lines, l'' be another straight line which cuts l and l' at z_1 and z_1' respectively and makes the corresponding angles equal. Let z_0 be the mid-point of $z_1 z_1'$. Draw a line h which passes through z_0 and is perpendicular to l(Figure 3).

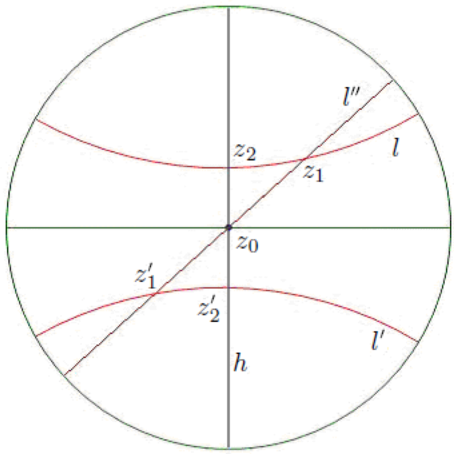


FIGURE 3

Since vertically opposite angles are equal, we have

$$\begin{aligned}
\angle z_0 z_1 z_2 &= \angle z_0 z_1' z_2' \\
z_0 z_1 &= z_0 z_1' \text{ (given)} \\
\angle z_1 z_0 z_2 &= \angle z_1' z_0 z_2' \text{ (vetically opposite angles)}
\end{aligned}$$

Therefore, $\triangle z_0 z_1 z_2 \cong \triangle z_0 z_1' z_2'$. (ASA)

Therefore, $\angle z_1 z_2 z_0 \cong \angle z_1' z_2' z_0$ and $z_2 z_0 = z_2' z_0$. ($\triangle z_0 z_1 z_2 \cong \triangle z_0 z_1' z_2'$)

Therefore, h is also perpendicular to l' and z_0 is the mid-point of within h the region bounded by l and l'. $\square$

This theorem holds not only in hyperbolic geometry, but also in Euclidean geometry and spherical geometry, with the number of common perpendicular lines differing only.

In the following, we will discuss more basic properties of parallel lines in the Poincaré model (hyperbolic geometry). The main results are:

1. The existence of common perpendicular line of two d-lines implies that the 2 d- lines are ultra-parallel, and vice versa.
2. Any two d-lines have at most one common perpendicular line.

Consequently, from our theorems and these results we have:
"If a line cuts another two straight lines with corresponding angles equal, the two straight lines are ultra-parallel, which means one would not meet the

other one", which is a tool in proving AAA rule of congruence of triangles (Appendix B) in Poincaré model.

Theorem 3.3. [4] *Let l be a d-line and w be a point that does not lie on l. If l' is a d-line parallel to l and passing through w, then the angle ω between l' and the perpendicular from w to l is given by $\sin\omega = \dfrac{1}{\cosh d}$, where d is the perpendicular non-Euclidean distance between w and l.*

Proof. Draw a d-line h from w perpendicular to l, which cuts l at z_0. Then d is the non-Euclidean distance between w and z_0. First consider the case when l is part of the real axis, $z_0 = 0$, $w = \rho i$ where ρ is a positive real number, and l' meets l at 1. Then h is part of the imaginary axis.

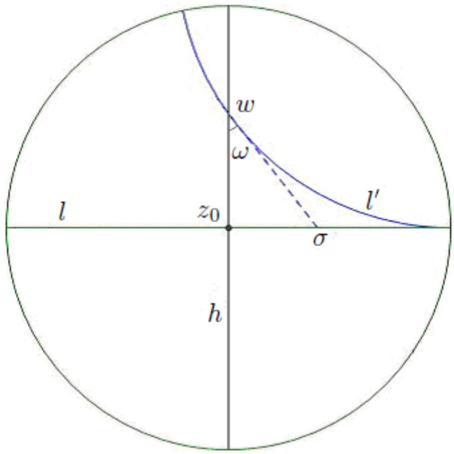


FIGURE 4

We have $d = \log\left(\dfrac{0-i}{\rho i - i}\cdot\dfrac{\rho i + i}{0+i}\right) = \log\left(\dfrac{1+\rho}{1-\rho}\right)$. Thus $\rho = \dfrac{e^d-1}{e^d+1}$.

Draw the Euclidean tangent of l' at w, which cuts the real-axis at σ. We have

$$\sigma = \rho\tan\omega,\ |\rho i - \sigma| = \frac{\rho}{\cos\omega}.$$

Noting that l' meets the real axis at 1, we have $|\rho i - \sigma| = 1 - \sigma$. Hence

$$\rho\tan\omega + \frac{\rho}{\cos\omega} = \sigma + (1-\sigma) = 1$$
$$\rho + \rho\sin\omega = \cos\omega$$

Squaring both sides, we have $(\rho^2+1)\sin^2\omega + 2\rho^2\sin\omega + (\rho^2-1) = 0$.

Solving for $\sin\omega$, we have $\sin\omega = \dfrac{1-\rho^2}{1+\rho^2}$.

Since $\rho = \dfrac{e^d-1}{e^d+1}$,

$$\sin\omega = \frac{2}{e^d+e^{-d}} = \frac{1}{\cosh d}.$$

For the general case, we can transform l to a part of the real axis and w to a point on the positive imaginary axis by a non-Euclidean transformation, and then use the above argument to prove the theorem. □

Remark The angle ω above is called the **angle of parallelism** for perpendicular distance d. It depends only on d and it is a continuous, strictly decreasing function of d which takes all values in the interval $(0, \frac{\pi}{2})$.

Theorem 3.4. [4] *The non-Euclidean distance from a point on a d-line to another d-line which is parallel to the first one tends to 0 when this point tends to the point where the two d-lines meet at* $U = \{z \in \mathbb{C} : |z| = 1\}$*; and tends to infinity when this point tends to another end of the line.*

Proof. Without loss of generality, assume l is the interval $(-1, 1)$ on the real axis.

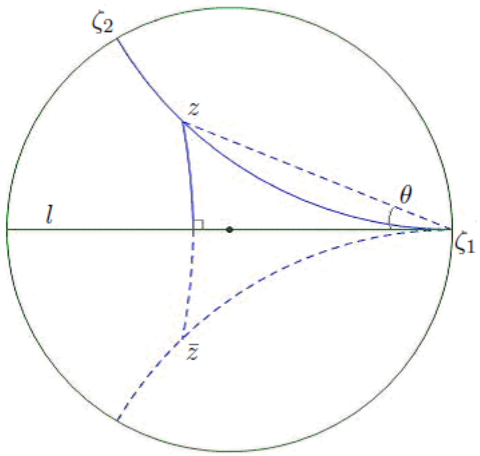


FIGURE 5

Let l' be a d-line which is parallel to l and meets U at $\zeta_1 = 1$ and ζ_2.

Let z be a point on l' and denote the non-Euclidean distance from z to l by $d(z,l)$. By symmetry, we have $d(z,l) = d(\bar{z},l)$. By Theorem 2.19, $d(z,l) + d(\bar{z},l) = d(z,\bar{z})$.

Hence

$$\begin{aligned}
d(z,l) &= \frac{1}{2}d(z,\bar{z}) \\
&= \frac{1}{2}\log\left(1+\frac{|z-\bar{z}|}{|1-z^2|}\right) - \frac{1}{2}\log\left(1-\frac{|z-\bar{z}|}{|1-z^2|}\right) \\
&= \frac{1}{2}\log\left(1+\frac{2|y|}{|1-z||1+z|}\right) - \frac{1}{2}\log\left(1-\frac{2|y|}{|1-z||1+z|}\right) \\
&= \frac{1}{2}\log\left(1+\frac{2\sin\theta}{|1+z|}\right) - \frac{1}{2}\log\left(1-\frac{2\sin\theta}{|1+z|}\right),
\end{aligned}$$

where y is the imaginary part of z and θ is the angle between the real axis and the Euclidean straight line joining z and ζ_1.

When $z \in l'$ tends to ζ_1, $\theta \to 0$. So $d(z,l) \to \frac{1}{2}\log(1) - \frac{1}{2}\log(1) = 0$.

When $z \in l'$ tends to $\zeta_2 \neq \pm 1$, $\sin\theta \to \frac{|\zeta_2+1|}{2}$, hence $\frac{2\sin\theta}{|1+z|} \to \frac{2|\zeta_2+1|}{2|1+\zeta_2|} = 1$ and therefore $d(z,l) \to \infty$. □

Theorem 3.5. *The non-Euclidean distance from a point on a d-line to another d-line which is ultra-parallel to the first one tends to infinity when this point tends to either end of the line.*

The proof is similar to that of Theorem 3.4.

Theorem 3.6. [4] *Let l be a d-line and z_0 be a point that does not lie on l. The non-Euclidean distance between z_0 and a point on l is the least when it is the perpendicular distance from z_0 to l.*

Proof. Let l be a d-line, z_0 be a point that does not lie on l, h be the d-line passing through z_0 and perpendicular to l. Suppose that l meets h at z_0'.

Without loss of generality, let l be the line that ends at 1 and -1, and z_0 be ρi, where $\rho > 0$. Then the perpendicular line h from z_0 to l lies on the imaginary axis and $z_0' = 0$.

We have $d(z_0,l) = d(z_0,z_0') = \frac{1+\rho}{1-\rho}$.

For any point z on l, we have $\bar{z} = z$ and hence

$$d(z_0, z) = \log \frac{1 + |\rho i - z|/|1 - z\rho i|}{1 - |\rho i - z|/|1 - z\rho i|}.$$

For all $z \in \mathbb{R}$, $\dfrac{|\rho i - z|}{|1 - z\rho i|} = \dfrac{\sqrt{\rho^2 + z^2}}{\sqrt{1 + z^2\rho^2}} \geqslant \dfrac{\sqrt{\rho^2 + z^2\rho^4}}{\sqrt{1 + z^2\rho^2}} = \rho.$

As $\dfrac{1+t}{1-t} = \dfrac{2}{1-t} - 1$ is increasing on $(0, 1)$,

$$\log \frac{1+\rho}{1-\rho} \leqslant \log \frac{1 + |\rho i - z|/|1 - z\rho i|}{1 - |\rho i - z|/|1 - z\rho i|}.$$

Hence $d(z_0, l) \leqslant d(z_0, z)$ for all $z \in I$.

The equality holds if and only if z is the foot of the perpendicular.

Therefore, the perpendicular distance $d(z_0, l)$ of z_0 to l is the shortest distance from z_0 to any point on l. □

Corollary 3.7. *For any right-angled triangle in Poincaré model, the hypotenuse is the longest side among its three sides.*

Theorem 3.8. [4] *Two straight lines are ultra-parallel if and only if they have a common perpendicular line.*

Proof. Let l and l' have a common perpendicular line h, which cuts l and l' at z_0 and z_0' respectively. Let ω be the angle of parallelism of z_0' with respect to l, then $0 < \omega < \dfrac{\pi}{2}$(Theorem 3.3). Therefore, the angle made by l' and h is within ω to $\pi - \omega$, that means l' is ultra-parallel with l.

Let l and l' be two d-lines which are ultra-parallel.

When a point $z' \in l'$ moves along l' and tends to the boundary of the Poincaré disk at either end of l', $d(z', l)$ tends to infinity (Theorem 3.5). Since $d(z', l)$ is a continuous function of z', it has a minimum at some point $z_0' \in l'$. Let z_0 be the foot of perpendicular from z_0' to l. i.e.

$$d(z_0', l) \leqslant d(z', l) \quad \text{for all } z' \in l'.$$

Since $d(z', l) \leqslant d(z', z)$ for all $z \in l$ and for all $z' \in l'$, we have

$$d(z_0', l) \leqslant d(z', z) \quad \text{for all } z \in l \text{ and for all } z' \in l'.$$

In particular,

$$d(z_0', l) \leqslant d(z', z_0) \quad \text{for all } z' \in l'.$$

Since $d(z_0, l') = d(z_0, z')$ for some $z' \in l'$, we have

$$d(z_0', z_0) = d(z_0', l) \leqslant d(z_0, l').$$

But we also have $d(z_0', z_0) \geqslant d(z_0, l')$, therefore, $d(z_0, l') = d(z_0, z_0')$ and z_0' is the foot of perpendicular from z_0 to l'. Hence the d-line joining z_0 and z_0' is a common perpendicular line to l and l'. □

From the non-existence of rectangle, there is one and only one common perpendicular line to two d-lines.

By Theorems 3.2 and 3.8, if a line cuts another two straight lines with corresponding angles equal, then the two straight lines are ultra-parallel.

3.3. Tangents to Circles

Theorem 3.9. *Let C be a circle with hyperbolic centre at z_0. If a d-line l is a tangent to C at w, then $l \perp z_0 w$.*

Proof. Assume that l is not perpendicular to $z_0 w$, then there exists a point w' on l such that $l \perp z_0 w'$. Since w' is outside the circle, $z_0 w'$ is longer then $z_0 w$. But as we know that the hypotenuse is the longest side of a right-angled triangle, $z_0 w$ should be longer than $z_0 w'$. This leads to a contradiction. So l is perpendicular to $z_0 w$. □

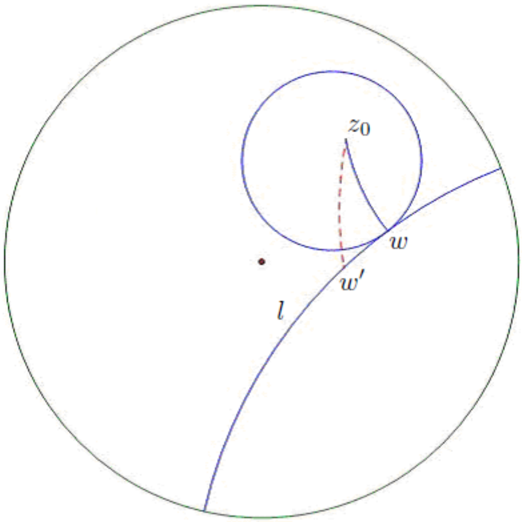


FIGURE 6

Corollary 3.10. *If we define the angle between a hyperbolic circle and a d-line as the angle between their Euclidean tangents at the intersection, then from Theorem 3.9, a hyperbolic radius of a circle is always perpendicular to the circle as the circle and its tangent at the point of contact have a common Euclidean tangent.*

Theorem 3.11. *If two tangents l and l' are drawn from an external point z to a circle with centre z_0, then*

(i) *the lengths of the tangents are equal;*
(ii) *the two tangents subtend equal angles at the centre;*
(iii) *the line joining the centre of the circle and the external point bisects the angle included by the tangents.*

Proof. Let w, w' be the points where the tangents meet the circle. Consider the hyperbolic triangles $\triangle z_0wz$ and $\triangle z_0w'z$.

$$\angle z_0wz = \angle z_0w'z = \frac{\pi}{2} \quad \text{(Theorem 3.9)}$$
$$z_0w = z_0w' \quad \text{(radii of the circle)}$$
$$z_0z = z_0z \quad \text{(common side)}$$
$$\text{Hence } \triangle z_0wz \cong \triangle z_0w'z. \quad \text{(RHS)}$$

Therefore,

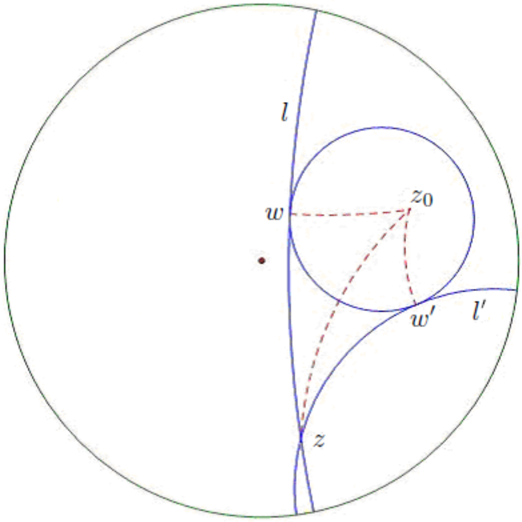


FIGURE 7

(i) $zw = zw'$
(ii) $\angle zz_0w = \angle zz_0w'$
(iii) $\angle z_0zw = \angle z_0zw'$ □

4. Constructions in Poincaré Disk

4.1. Elementary Constructions

Construction 4.1. *Construct a circle passing through three non-collinear points.*

1. Three points A, B and C are given.
2. Draw line segments AB and BC.
3. Construct the perpendicular bisectors of AB and BC.
4. The point of intersection of the two perpendicular bisectors is the centre of the circle.

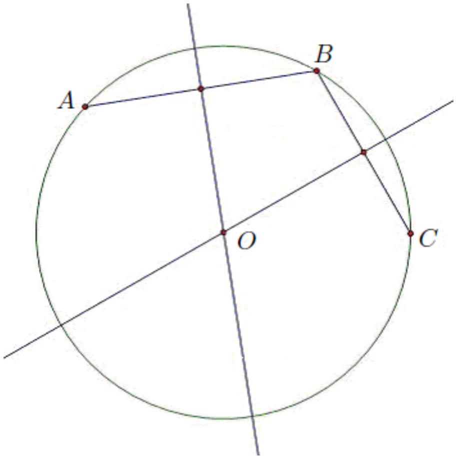


FIGURE 8

Construction 4.2.a. [5] *Construct the inverse point of a given point with respect to a circle C with centre O, where the given point lies outside the circle.*

1. A circle C centred at O and a point B, which is outside C, are given.
2. Draw circle C' with OB as a diameter.
3. C and C' intersect at points P and Q.
4. Join PQ to meet OB at a point A, which is the inverse of B.

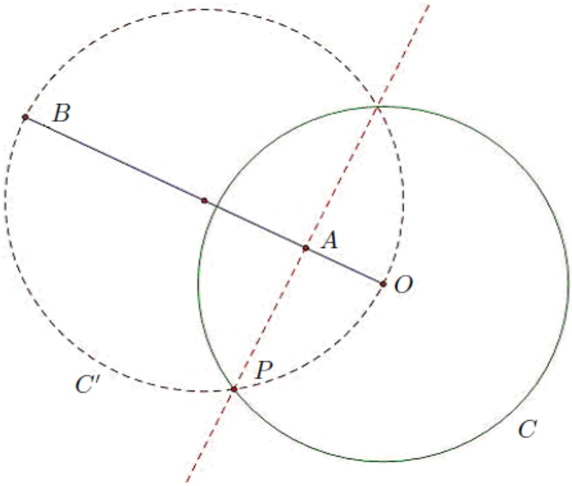


FIGURE 9

Explanation of Construction 4.2.a

$$\triangle OPB \sim \triangle OAP$$
$$\text{So, } \frac{OP}{OB} = \frac{OA}{OP}$$
$$\text{i.e. } OA \cdot OB = OP^2$$

By definition, A is the inverse of B.

Construction 4.2.b. [5] *Construct the inverse point of a given point with respect to a circle C with centre O, where the given point lies inside the circle.*

1. A circle C centred at O and a point A, which is outside C, are given.
2. Draw a line passing through O and A.
3. Through A draw a line perpendicular to OA, intersecting C at two points. Name one of them as P.
4. Draw a perpendicular line to OP at P.
5. The lines drawn in Step 2 and Step 4 will intersect a point B, which is the inverse of A.

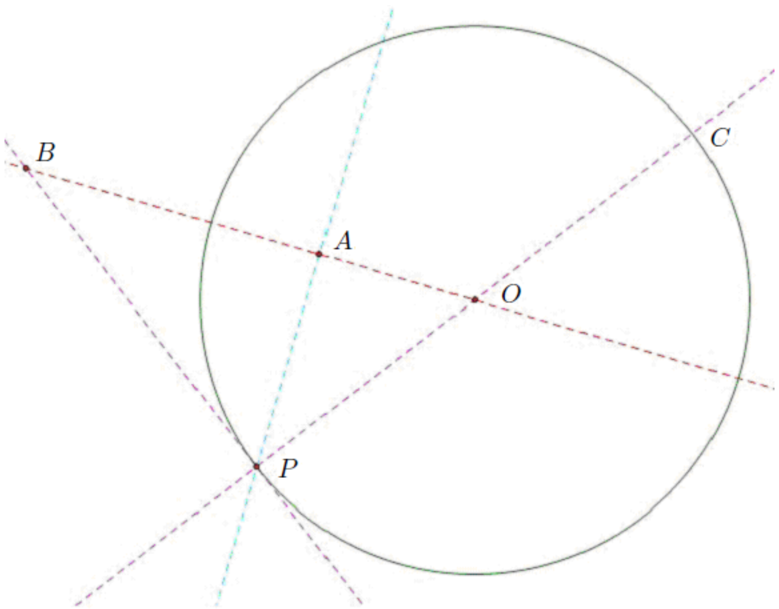


FIGURE 10

Explanation of Construction 4.2.b

$$\triangle OPB \sim \triangle OAP$$

$$\text{So, } \frac{OP}{OB} = \frac{OA}{OP}$$

$$\text{i.e. } OA \cdot OB = OP^2$$

By definition, B is the inverse of A.

Construction 4.3. [5] *Given two orthogonal circles, construct the inverse of a point, which lies on one of the circles, with respect to the other circle.*

1. Two orthogonal circles C and C', with centres O and O' respectively, are given. Point A lies on C'.
2. Draw the line OA.
3. The line will intersect C' again at B, and this is the inverse of A with respect to C. (Theorem 2.10)

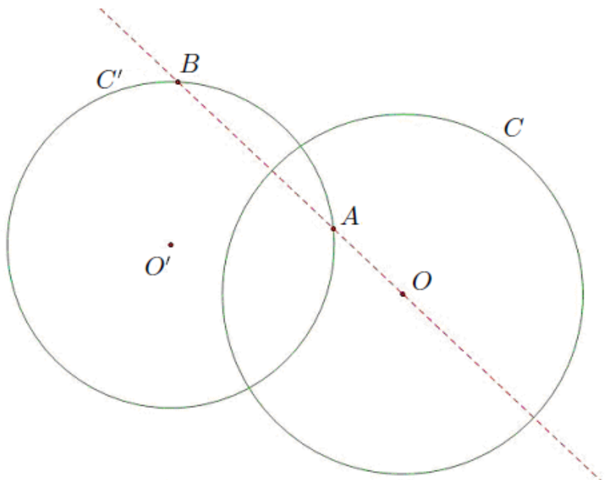


FIGURE 11

Construction 4.4. *Given a circle C and a point A outside C, construct a circle C'' which has the centre at A and is orthogonal to circle C.*

1. A circle C with centre O and a point A are given.
2. Draw a circle C'' with AO as a diameter.
3. Let P be a point of intersection of C'' and C.
4. Draw a circle C'' with centre A and radius AP. This circle is orthogonal to C.

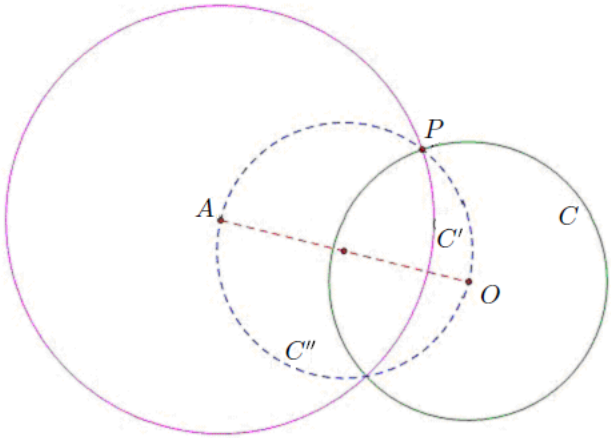


FIGURE 12

4.2. Hyperbolic Straightedge and Compass Constructions

Construction 4.5. *Construct a d-line passing through two given points inside the Poincaré disk.*

1. Points A and B in the Poincaré disk are given.
2. Construct the inverse A' of A with respect to U by Construction 4.2.b.
3. Draw a circle C' passing through A', A and B by Construction 4.1.
4. The portion of C' inside the Poincaré disk is the d-line.

Note The centre of C' is called the **pole** of the d-line.

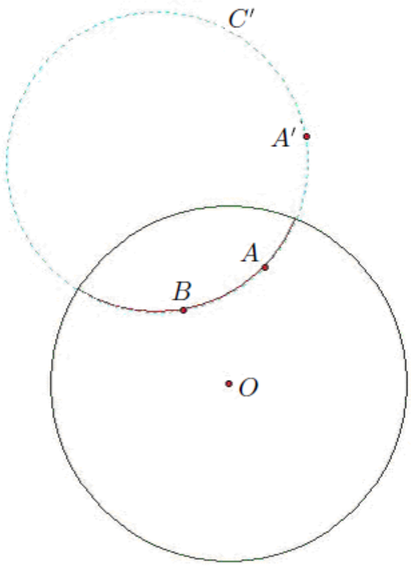


FIGURE 13

Construction 4.6. [5] *Construct a d-line AB when point A, point B and their polars are given.*
N.B. The polar of a point A in D is the Euclidean perpendicular bisector of A and its inverse, which is also the locus of the poles of d-lines that pass through A. (Theorems 2.9 and 2.10)

1. Let P be the intersection point of the polars of A and B.
2. Draw a circle with centre P and radius PA.

3. The portion of the circle in Step 2 inside D is the d-line passing through A and B.

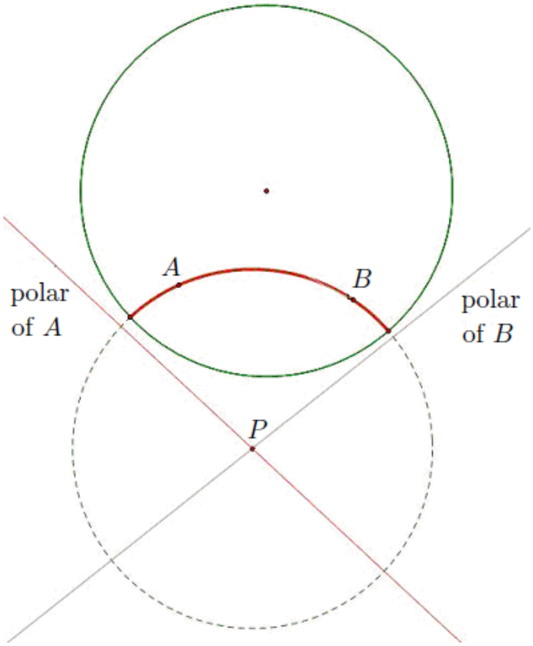


FIGURE 14

Explanation of Construction 4.6
The polar of a point inside the Poincaré disk is the locus of the poles of d-lines passing that point. So the intersection of the polars of A and B is the pole of the d-line passing through both A and B.

Construction 4.7. [5] *Construct a hyperbolic circle with hyperbolic centre A and passing through B, where B is not the centre of the Poincaré disk.*

1. Points A and B in Poincaré disk are given.
2. Construct the d-line passing through A and B.
3. Draw the Euclidean tangent at B to the d-line, intersecting Euclidean line OA at a point E.
4. Draw a Euclidean circle with E as centre and EB as radius. It is a hyperbolic circle centred at A.

Explanation of Construction 4.7
By Theorem 2.25, a hyperbolic circle is also a Euclidean circle. In Euclidean geometry, we know that the radius is perpendicular to the circumference of a

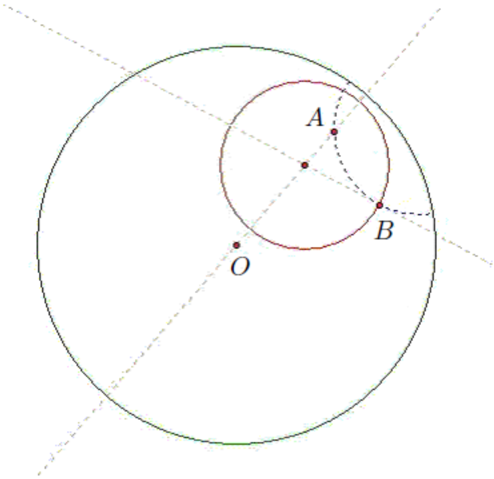


FIGURE 15

circle. By Corollary 3.10, the hyperbolic radius AB is also perpendicular to the circumference of the hyperbolic circle. Therefore, when we construct the Euclidean tangent of the d-line AB at B, it will pass through the Euclidean centre of the circle.

Construction 4.8. *Construct a hyperbolic circle with given hyperbolic centre and passing through the centre of the Poincaré disk.*

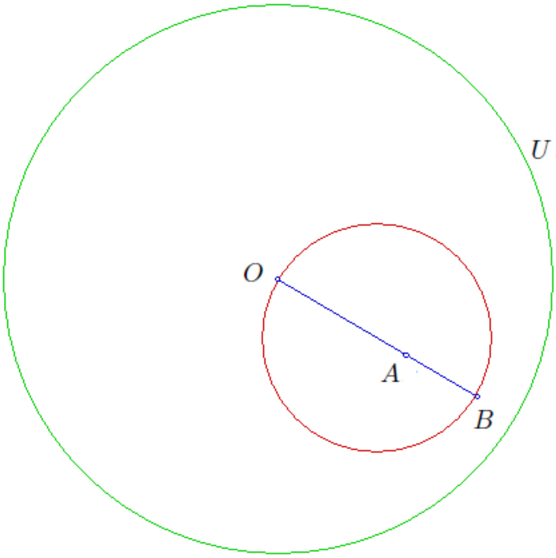


FIGURE 16

Let A be the hyperbolic centre of the required circle and OB be a diameter of the circle. If the Euclidean length of OA and OB are r and s respectively, then $\log \dfrac{1+s}{1-s} = 2\log \dfrac{1+r}{1-r}$ and hence $s = \dfrac{2r}{1+r^2}$.

Construct a Euclidean circle with centre O and radius s. Its intersection point with OA produced B . The required circle is the Euclidean circle with diameter OB.

Note We can construct Euclidean lengths which are the products or the quotients of other given lengths by constructing similar triangles[6].

4.3. Elementary Constructions in the Poincaré Model

Construction 4.9.a. [5] *Construct the perpendicular bisector of a segment AB (d-line).*

1. Points A and B inside Poincaré disk are given.
2. Construct the inverses of A and B with respect to U.

3. Draw a Euclidean line through AB and a Euclidean line through the inverses of A and B.
4. The lines intersect at a point E, which is the pole of the d-line (perpendicular bisector) we want.
5. Using Construction 4.4, construct a circle with Euclidean centre at E that is orthogonal to U. The hyperbolic perpendicular bisector of AB is a part of this circle.

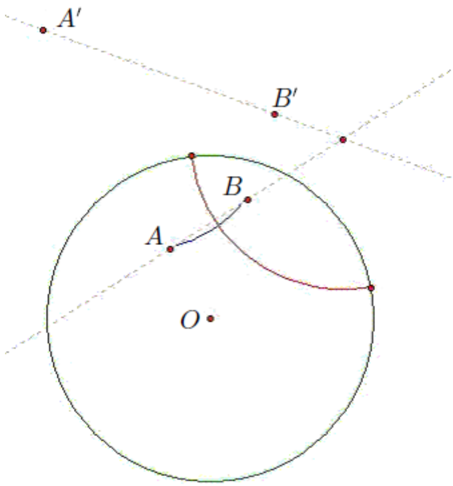


FIGURE 17

Explanation of Construction 4.9.a

Let γ be the perpendicular bisector of AB. By Theorem 2.27, B is the inverse of A with respect to γ. With respect to γ, the inverses of A and B with respect to U are inverses of each other. So if we draw two Euclidean lines, one passing through A and B and one passing through the inverses of A and B with respect to U, the lines intersect at a point E, which is the pole of γ.

Construction 4.9.b. [5] *Construct the perpendicular bisector of a segment AB (d-line).*

1. Points A and B inside Poincaré disk are given.
2. Use Construction 4.7 to construct two hyperbolic circles, one with centre A and passing through B, one with centre B and passing through A.
3. The two hyperbolic circles will intersect at two points.

4. Use Construction 4.5 to construct a d-line passing through the intersection points found in step 3. This d-line is the hyperbolic perpendicular bisector of AB.

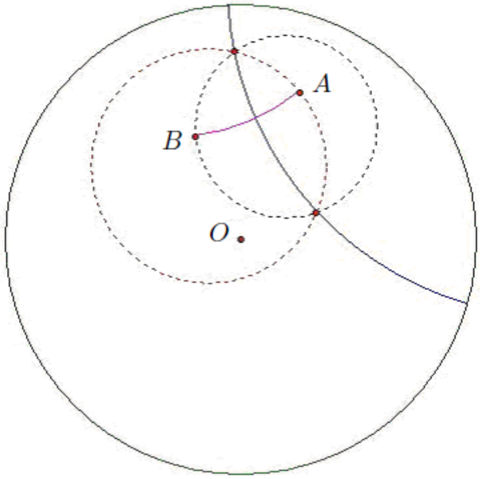


FIGURE 18

Construction 4.10. [5] *Given a triangle ABC and a segment $A'B'$ such that $AB = A'B'$, construct a triangle $A'B'C'$ which is congruent to $\triangle ABC$.*

The construction is the same as in Euclidean geometry.

1. $\triangle ABC$ and points A' and B' are given, with $AB = A'B'$.
2. Construct the perpendicular bisector γ_1 of A and A'.
3. Locate the inverse B'' of B with respect to γ_1.
4. Construct the line γ_2 passing through A' and the mid-point of $B'B''$. Since $A'B' = A'B''$, γ_2 is the perpendicular bisector of $B'B''$.
5. Construct the inverse C'' of C with respect to γ_1 and then the inverse of C'' with respect to γ_2. This is the point C' we want.

Note that $AC = A'C'' = A'C'$, $BC = B''C'' = B'C'$, $AB = A'B'' = A'B'$. So $\triangle ABC \cong \triangle A'B'C'$.

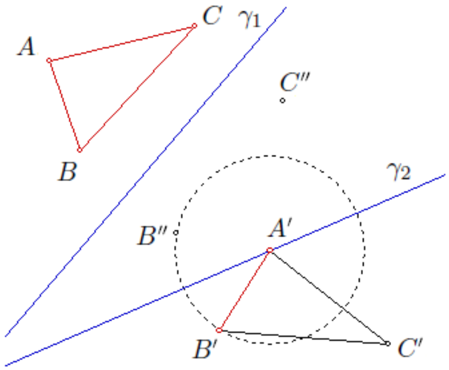


FIGURE 19

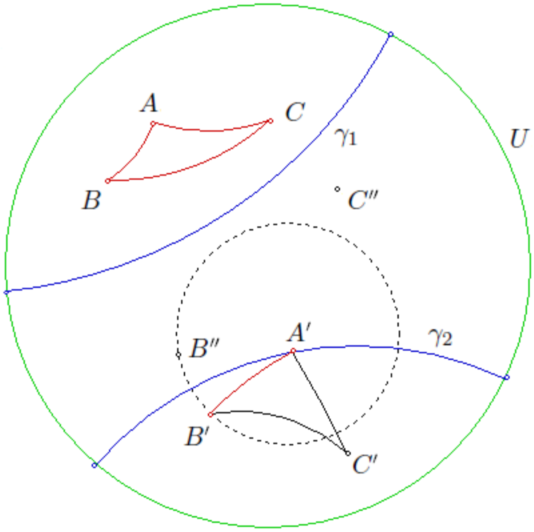


FIGURE 20

The above transformation is independent of the position of C. So for any point C in the Poincaré disk, we have $\triangle ABC \cong \triangle A'B'C'$, which means that the transformation is length-preserving. Hence it is a non-Euclidean transformation.

Construction 4.11. [5] *Construct a hyperbolic circle with hyperbolic centre A and radius equal to a hyperbolic length of BC.*

1. A, B and C inside Poincaré disk are given.
2. Construct the perpendicular bisector γ of the segment BA by Construction 4.9.
3. Construct the inverse P of C with respect to γ.
4. Draw the circle by using Construction 4.7, with hyperbolic centre A and hyperbolic radius AP.

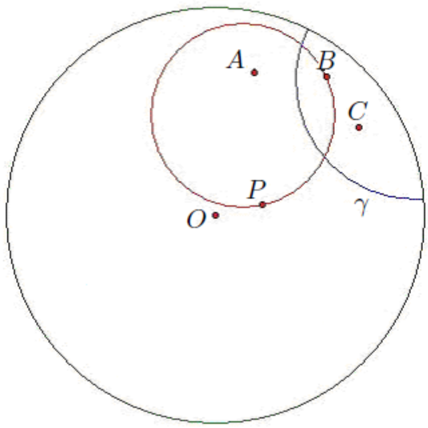


Figure 21

5. Construction of non-Euclidean Tangents to Circles

In this section, we are going to do three construction problems, namely,

1. to construct a tangent to a circle at a point;
2. to construct tangents from an external point to a circle;
3. to construct common tangents of two given circles using the methods discussed in Section 4.

5.1. Tangent at a Point

Given a circle $C_1 \subset D$, with non-Euclidean centre B and a point A on C_1, construct a tangent to C_1 at A. (Figure 22)

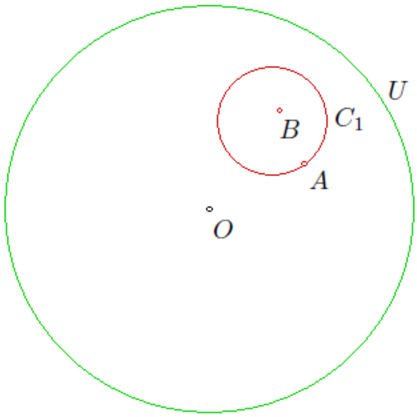


FIGURE 22

1. Construct a circle centred at O with non-Euclidean radius AB (Construction 4.11) and on it take an arbitrary point B'. We have $OB' = AB$.
2. Construct a non-Euclidean circle C_1' centred at B' and passing through O. (Figure 23)

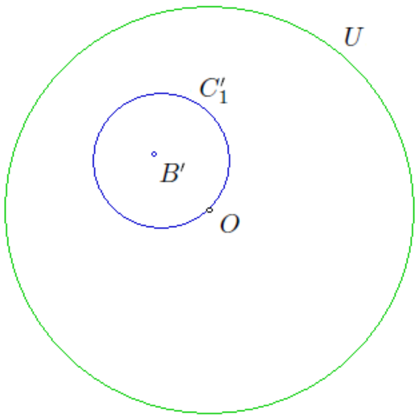


FIGURE 23

3. Construct a straight line L' which is perpendicular to OB' at O. It is the tangent to C_1' at O. (Figure 24)

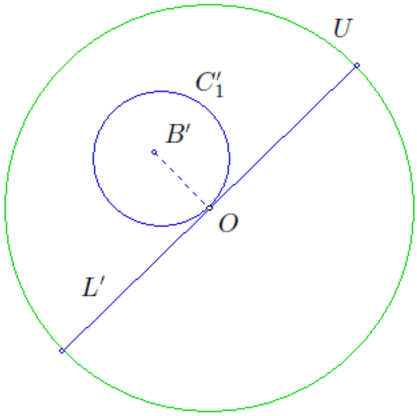


FIGURE 24

4. Arbitrarily take a point E' on L'. Use Construction 4.10 to locate a point E such that $\triangle BAE \cong \triangle B'OE'$. The d-line joining A and E is the non-Euclidean tangent to C_1 at A. (Figure 25)

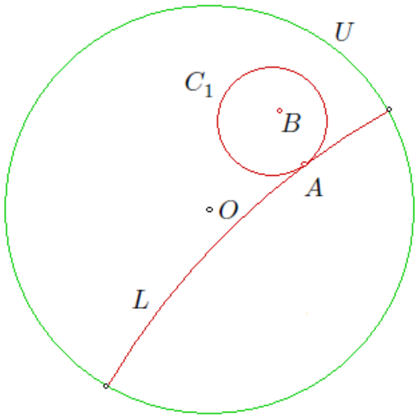


FIGURE 25

5.2. Tangents from an External Point

Given a circle D, with hyperbolic centre B, and an external point $A \in D$, construct the two tangents from A to C_1. (Figure 26)

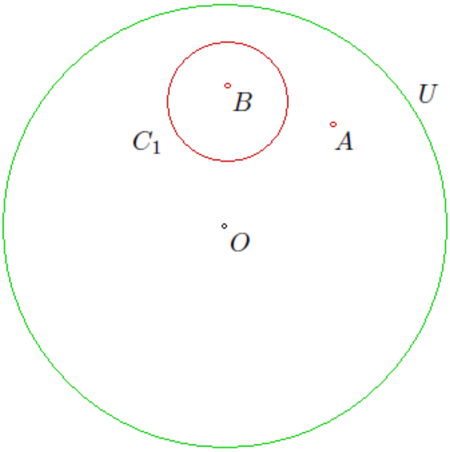


FIGURE 26

1. Locate a point B' such that $OB' = AB$. This can be done by using Step 1 in 5.1. Use construction 4.11 to draw a circle C_1' with centre at B' and radius equal to that of C_1. (Figure 27)

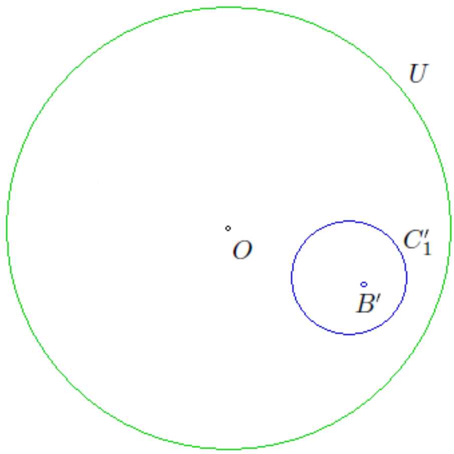


FIGURE 27

2. Construct the tangents from O to C_1', which are indeed Euclidean tangents, by the following steps:
   (a) Locate the Euclidean centre X of C_1'.
   (b) Construct a circle with diameter OX to meet C_1' at E' and F'.
   (c) OE' and OF' are the tangents. (Figure 28)

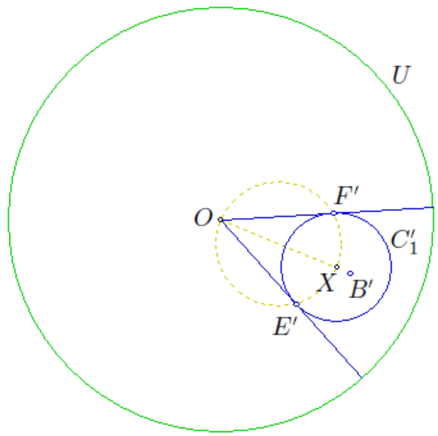


FIGURE 28

3. By Construction 4.10, construct $\triangle ABE$ congruent to $\triangle OB'E'$. Then the d-line passing through A and E is a tangent to C_1. We can draw another tangent AF similarly. (Figure 29)

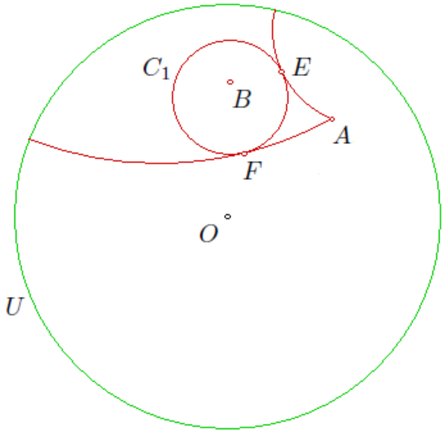


FIGURE 29

This construction can be carried out through another way, by using a theorem which is very important to our project:

Theorem 5.1. *Let U be a unit circle centred at O, and C be another circle with centre also at O and radius r, where $r < 1$. If there is a circle orthogonal to U and touching C externally, then the radius of this circle is* $\dfrac{1-r^2}{2r}$.

Proof. Let R be the radius of the circle which is orthogonal to U and touches C externally.

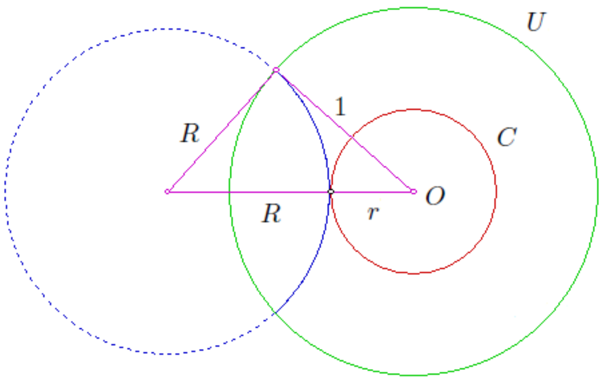


FIGURE 30

As shown in Figure 30, we have $R^2+1^2 = (R+r)^2$ (Pythagorean Theorem). Simplifying this equation, we have $R = \dfrac{1-r^2}{2r}$. □

Given a Euclidean length r, we can construct a line segment of length $R = \dfrac{1-r^2}{2r}$ using compass and straightedge. Thus Theorem 5.1 provides a convenient way to construct tangents.

To construct tangents from an external point to a circle, we can use the following method:

1. Instead of transforming A to O, we transform the picture so that B goes to O. Suppose that A goes to A'' and the circle C_1 goes to C_1''. (Figure 31)

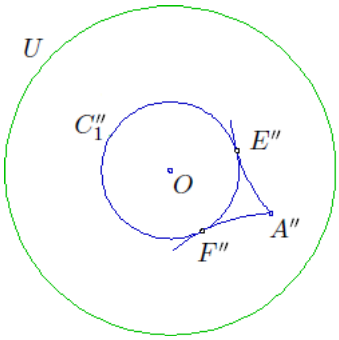


FIGURE 31

2. Let the Euclidean radius of C_1 be r. Construct a segment of length $R = \dfrac{1-r^2}{2r}$. Construct a circle centred at O with radius $R+r$ and a circle centred at A'' with radius R. The intersection of the circles are the poles of the two tangents to C_1'' from A''.
3. Transform the whole picture back to the original position.

5.3. Common Tangents to Two Circles

Given two circles $C_1, C_2 \subset D$, with hyperbolic centres A, B respectively, construct 4 common tangents of C_1 and C_2. (Figure 32)

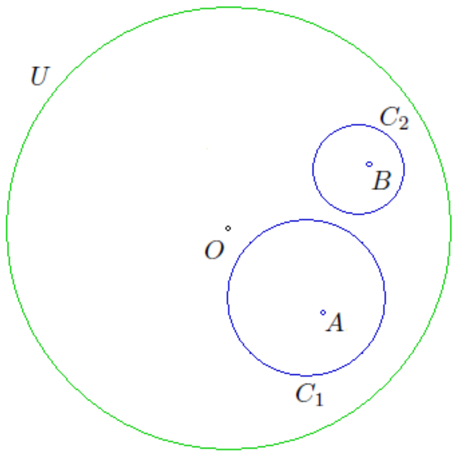


FIGURE 32

1. Using Construction 4.11, we can translate one hyperbolic centre, say A, to O and with hyperbolic distance OB' equal to that of AB. (where O, B' are the images of A, B respectively.) Also, we can construct hyperbolic circles C_1', C_2' with centres at O, B' and hyperbolic radii the same as C_1, C_2 respectively. (Figure 33)

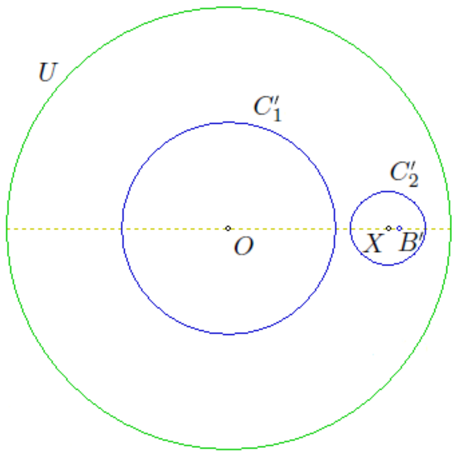


FIGURE 33

2. Construct the line OB'. Let the Euclidean centre of C_2' be X. Let the Euclidean radii of C_1', C_2' be r_1, r_2 respectively. By Theorem 5.1, the Euclidean radii R of the 4 common tangents are equal, which is determined only by r_1.

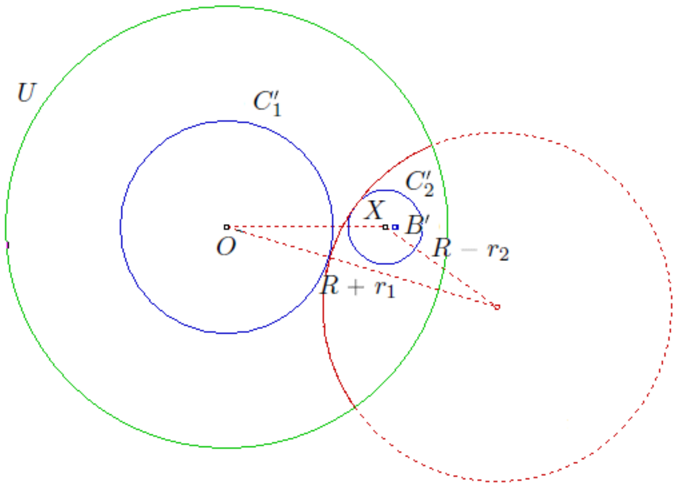


FIGURE 34

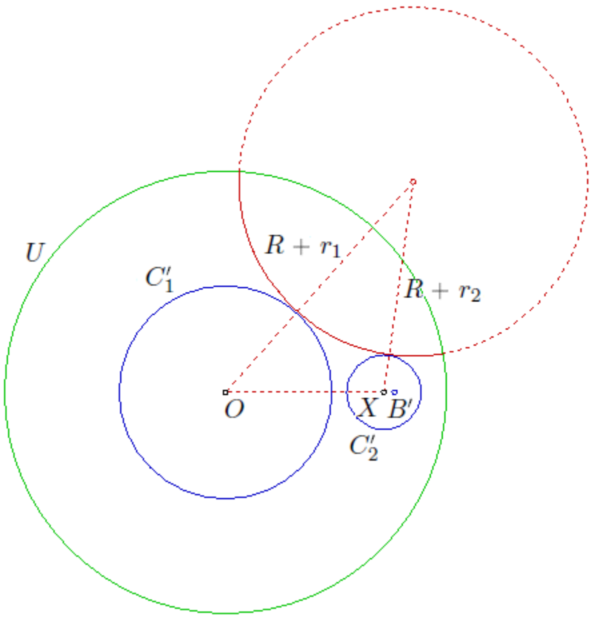


FIGURE 35

3. Construct Euclidean straight lines between poles of the 4 common tangents and O, X as in Figure 36. By sum and difference of radii, the Euclidean distances are $R + r_1$, $R + r_2$, $R - r_2$. Then the poles of the 4 common tangents can be located using a Euclidean compass.

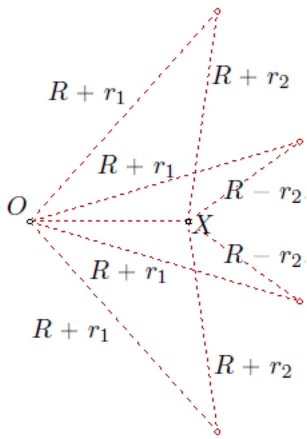


FIGURE 36

4. As we have r_1, we can work out R and thus we can construct the common tangents with radii R from the centres located. (Figure 37).

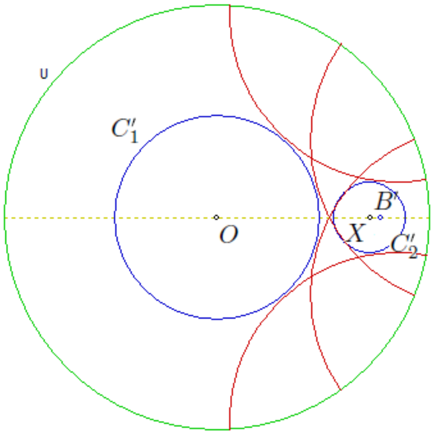


FIGURE 37

5. Arbitrarily locate two points on each common tangent. Then we can translate back C_1', C_2' to C_1, C_2 using Construction 4.10, and the two points on each common tangents are translated under the same transformation. Finally, we construct 4 d-lines passing through those corresponding points; these are the 4 common tangents to C_1, C_2. (Figure 38)

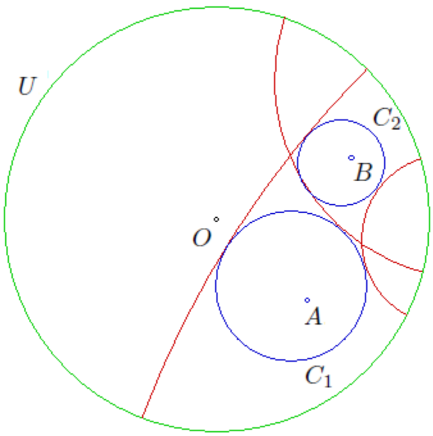


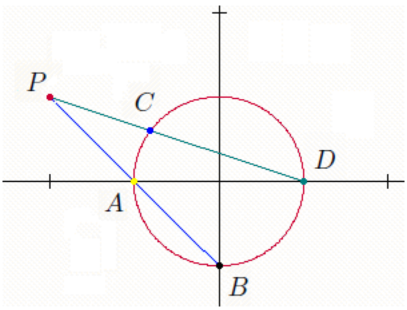
FIGURE 38

6. Conclusion

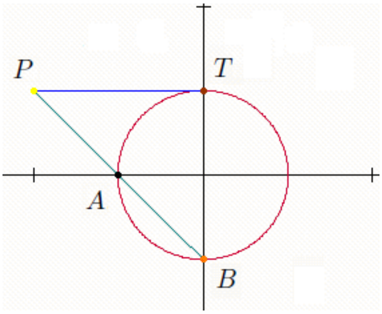
When we were considering the construction of tangents, we faced some difficulties. We tried to use the well-known methods in Euclidean geometry, but found at once that they are inadequate as the diameter does not subtend a right angle on the circumference in non-Euclidean geometry. All our attempts failed until we realized that the Euclidean radius of a tangent, to a circle centred at the origin, depends only on the radius of the circle (Theorem 5.1).

The construction methods are not only proved but also carried out by ourselves. We have done all the constructions with Sketchpad. We have improved some of our construction methods by doing this. Sometimes we thought a method was correct, but when we tried to do the construction, we found that something was missing. For example, if we want to translate a point to the origin, we have to construct a circle passing through the origin. But the usual construction of non-Euclidean circles did not work in this situation. So we worked to find a method to do that, which appeared in Construction 4.8 of this report.

Appendix A. Intersecting Chords Theorem



For any circle, if from a point P which is not on the circle, two lines are drawn to meet the circle at A, B and C, D respectively, then $PA \cdot PB = PC \cdot PD$.



In particular, if C and D coincide at T ,i.e., PT is a tangent to that circle, then $PA \cdot PB = PT^2$.

The converse is also true.

Appendix B. Congruence of Triangles

In this appendix, we are going to investigate the conditions of congruence of triangles in Poincaré Model. The proofs are actually similar to those in Euclidean geometry.

In Euclid's *Elements*, the congruence theorems for triangles were put on proposition I.4(SAS), I.8(SSS) and I.26(ASA), and the concept of superposition was used. But such a concept is not clear. So we decide to follow David Hilbert's axioms in "*The Foundations of Geometry*", and start the work from an axiom of congruence of triangles.

First of all, there are only $2^3 = 8$ possible combinations for congruence of triangles, which are: SSS, AAA, SSA, AAS, SAS, ASA, SAA, ASS. But SSA, AAS are equivalent to ASS, SAA respectively.

Here, we use Hilbert's Axiom of congruence and a consequence of the axioms of congruence[7].

"*If, in two triangles ABC and $A'B'C'$ the congruence $AB \cong A'B'$, $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$ hold, then the congruence $\angle ABC \cong \angle A'B'C'$ and $\angle ACB \cong \angle A'C'B'$ also hold.*"

"*If, for two triangles ABC and $A'B'C'$, the congruence $AB \cong A'B'$, $AC \cong A'C'$ and $\angle A \cong \angle A'$ hold, then the two triangles are congruent to each other.*"

Therefore we accept SAS as a condition of congruence of triangles.

We are going to investigate the other five combinations.

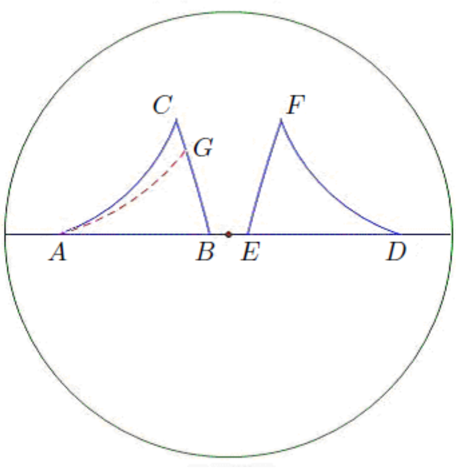
Some of the proofs require Proposition 7 of Euclid's *Elements* Book I (Proposition I.7):

"*Let ABC be a triangle and D be a point on the same side of AB as C. If $AC = AD$ and $BC = BD$, then C and D are the same point.*"

We will prove this later.

B.1. The ASA Rule

Proof. Let $\angle CAB = \angle FDE$, $AB = DE$, $\angle CBA = \angle FED$. Assume that $\triangle ABC$ is not congruent to $\triangle DEF$. Then $BC \neq EF$, otherwise $\triangle ABC \cong \triangle DEF$(SAS). Without loss of generality, we may assume that $BC > EF$. Let G be a point on BC such that $BG = EF$.



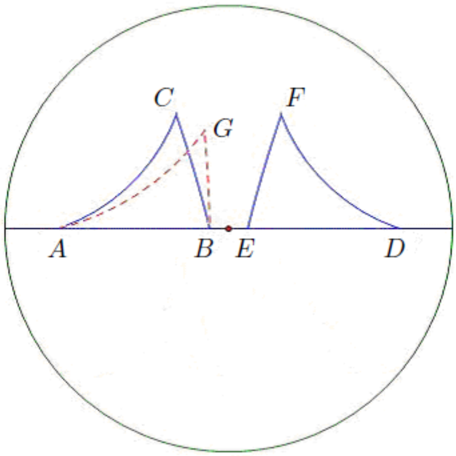
$$\begin{aligned} \triangle ABG &\cong \triangle DEF && \text{(SAS)} \\ \angle GAB &= \angle FDE && (\triangle ABG \cong \triangle DEF) \\ \angle CAB &= \angle FDE && \text{(given)} \end{aligned}$$

Hence $\angle GAB = \angle CAB$.

Therefore, AG and AC are the same line. Since AC and BC intersect at both C and G, C and G are the same point. Then $BC = BG = EF$, which contradicts our assumption. So $\triangle ABC \cong \triangle DEF$. □

B.2. The SSS Rule

Proof. Let $AB = DE$, $BC = EF$, $AC = DF$. Assume that $\triangle ABC$ is not congruent to $\triangle DEF$. Then $\angle ABC \neq \angle DEF$, otherwise $\triangle ABC \cong \triangle DEF$(SAS). Without loss of generality, we may assume that $\angle ABC < \angle DEF$. Construct a point G such that $\angle ABG = \angle DEF$ and $BG = EF$.



$$\begin{aligned} \triangle ABG \cong \triangle DEF &\quad \text{(SAS)} \\ AG = DF &\quad (\triangle ABG \cong \triangle DEF) \\ AC = DF &\quad \text{(given)} \end{aligned}$$

Hence $AG = AC$.

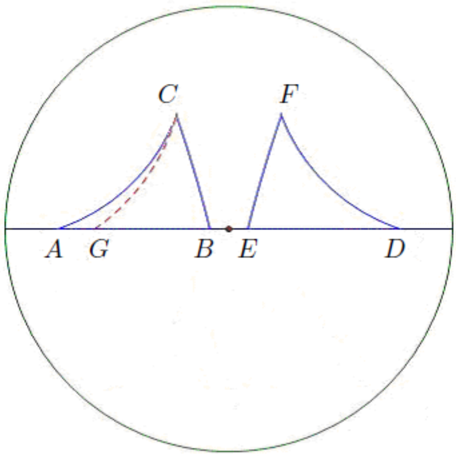
$$\begin{aligned} BG = EF &\quad (\triangle ABG \cong \triangle DEF) \\ BC = EF &\quad \text{(given)} \end{aligned}$$

Hence $BC = BG$.

Thus $AC = AG$, $BC = BG$, and both C and G are on the same side of the base AB. From Euclid's Elements, Proposition I.7, C and G are the same point; BC and BG are the same line. Therefore, $\angle ABG = \angle ABC = \angle DEF$, which contradicts our assumption. So $\triangle ABC \cong \triangle DEF$. $\square$

B.3. The AAS Rule

Proof. Let $\angle CAB = \angle FDE$, $\angle ABC = \angle DEF$, $BC = EF$. Assume that $\triangle ABC$ is not congruent to $\triangle DEF$. Then $AB \neq DE$, otherwise $\triangle ABC \cong \triangle DEF$(SAS). Without loss of generality, we may assume that $AB > DE$. Let G be the point on AB such that $GB = DE$.



$$\begin{aligned} &\triangle GBC \cong DEF \quad \text{(SAS)} \\ &\angle CGB = \angle FDE \quad (\triangle GBC \cong \triangle DEF) \\ &\angle CAB = \angle FDE \quad \text{(given)} \end{aligned}$$

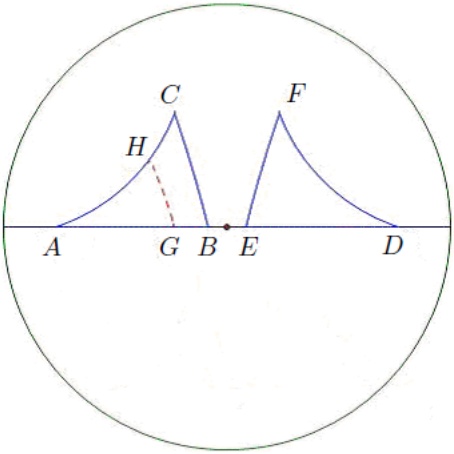
Hence $\angle CAB = \angle CGB$.

$$\begin{aligned} \angle CAG + \angle CGA &= \angle CAB + \angle CGA \\ &= \angle CAB + (\pi - \angle CGB) \quad (\angle \text{s on a straight line}) \\ &= \angle CAB + (\pi - \angle CAB) \\ &= \pi \end{aligned}$$

Since AC and GC intersect at C, the angle sum of $\triangle AGC \geqslant \pi$, which is impossible in hyperbolic geometry. Hence A and G are the same point. Therefore, $AB = GB = DE$, which contradicts our assumption. So $\triangle ABC \cong \triangle DEF$. $\square$

B.4. The AAA Rule

Proof. Let $\angle CAB = \angle FDE$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$. Assume that $\triangle ABC$ is not congruent to $\triangle DEF$. Then $AB \neq DE$, otherwise $\triangle ABC \cong \triangle DEF$(ASA). Without loss of generality, we may assume that $AB > DE$. Let G be the point on AB such that $AG = DE$. Let H be a point such that $\angle AGH = \angle DEF$ and $\angle GAH = \angle EDF$.



Since $\angle AGH = \angle DEF$ and $\angle ABC = \angle DEF$, $\angle ABC = \angle AGH$ and hence BC and GH are ultra-parallel (and do not intersect). Thus H lies on the line segment AC, and $BCHG$ will form a quadrilateral.

$$\begin{aligned} &\triangle AGH \cong \triangle DEF \quad \text{(ASA)} \\ &\angle AHG = \angle DFE \quad (\triangle AGH \cong \triangle DEF) \\ &\angle ACB = \angle DFE \end{aligned}$$

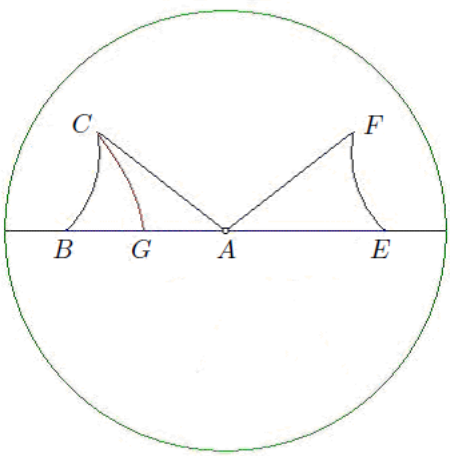
Hence $\angle ACB = \angle AHG$.

$$\begin{aligned} &\text{The angle sum of } BCHG \\ &= \angle BGH + \angle ABC + \angle GHC + \angle ACB \\ &= (\pi - \angle AGH) + \angle ABC + (\pi - \angle AHG) + \angle ACB \\ &= (\pi - \angle ABC) + \angle ABC + (\pi - \angle ACB) + \angle ACB \\ &= 2\pi \end{aligned}$$

Divide $BCHG$ into two triangles, then at least one of them has the angle sum $\geqslant \pi$, which is impossible in hyperbolic geometry. Thus $\triangle ABC \cong \triangle DEF$. ∎

B.5. SSA and Congruence

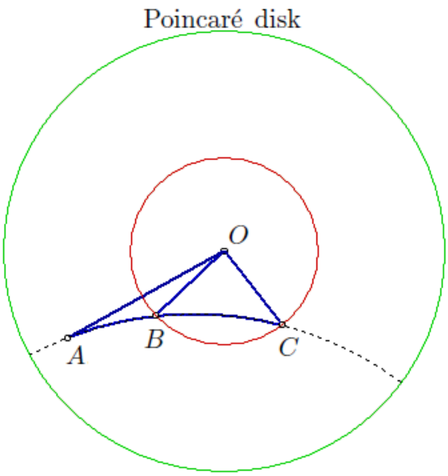
Proof. Let $BC = EF$, $AC = AF$, $\angle BAC = \angle EAF$. Assume that $\triangle ABC$ is not congruent to $\triangle AEF$. Then $AB \neq AE$, otherwise $\triangle ABC \cong \triangle AEF$(SAS). Without loss of generality, we may assume that $AB > AE$. Let G be the point on AB such that $AG = AE$.



$$\begin{aligned} \triangle AGC &\cong \triangle AEF \quad \text{(SAS)} \\ GC &= EF \quad (\triangle AGC \cong \triangle AEF) \\ BC &= EF \quad \text{(given)} \end{aligned}$$

Hence $BC = GC$.

We can construct a pair of triangles which satisfy the SSA condition but are not congruent. Let the centre of the Poincaré disk be O. Construct a circle centred at O to get two points B and C which are equidistant from O and B, O, C are not collinear. There exists a d-line passing through B and C. On this d-line, choose an arbitrary point A which is not between B and C. Then $\triangle OAB$ and $\triangle OAC$ satisfy the SSA condition but are not congruent.



Therefore, SSA does not imply the congruence of two triangles. □

B.6. Proof of Euclid's Proposition I.7

In the proof of SSS congruence, we have used Proposition I.7 of Euclid's Elements. Commentators over the centuries pointed out that Euclid's proof of this proposition was incomplete[8].

Theorem B.1. *The angles at the base of an isosceles triangle are equal*

Proof. Let $\triangle ABC$ be an isosceles triangle which $AB = BC$. Consider $\triangle ABC$ and $\triangle CBA$.

$$AB = BC$$
$$\angle ABC = \angle CBA$$
$$BC = AB$$

Hence $\triangle ABC \cong \triangle CBA$(SAS). $\angle BAC = \angle BCA(\triangle ABC \cong \triangle CBA)$. Therefore, the base angles of an isosceles triangle are equal. □

Theorem B.2. *In an isosceles triangle, the angle bisector of the angle opposite to the base is the perpendicular bisector of the base.*

Proof. Let $\triangle ABC$ be an isosceles triangle which $AB = BC$. Draw the angle bisector of $\angle ABC$ to meet the base AC at D. Consider $\triangle ABD$ and

$\triangle CBD$.

$$AB = BC$$
$$\angle ABD = \angle CBD$$
$$BD = BD$$

Hence $\triangle ABD \cong \triangle CBD$(SAS).

$$AD = CD \quad (\triangle ABD \cong \triangle CBD)$$
$$\angle ADB = \angle CDB \quad (\triangle ABD \cong \triangle CBD)$$
$$\angle ADB + \angle CDB = \pi \quad (\angle\text{s on a staight line})$$

Hence $\angle ADB = \angle CDB = \dfrac{\pi}{2}$.

Therefore, the bisector of the angle opposite to the base of an isosceles triangle is the perpendicular bisector of the base.

Since the line joining two points is unique, we can say that in an isosceles triangle, the line joining the mid-point of the base and its opposite vertex is the bisector of the angle at the vertex and the perpendicular bisector of the base. □

Proof of Proposition I.7. Let ABC be a triangle and D be a point on the same side of AB as C such that $AC = AD$ and $BC = BD$. Suppose that C and D are not the same point.

Join CD. Since AC equals AD and BC equals BD, both $\triangle ACD$ and $\triangle BCD$ are isosceles triangles with the base CD.

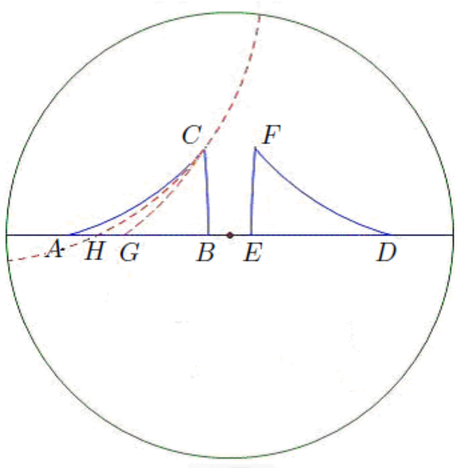
Let M be the midpoint of CD, then AM and BM are both perpendicular bisectors of CD and hence they are the same line. Thus A, B and M are collinear, which is impossible as C and D lie on the same side of AB.

So C and D are the same point. □

In Euclidean Geometry, if the hypotenuse and one other side of a right-angled are equal to the hypotenuse and one side of another right-angled triangle, then the two triangles are congruent (RHS). RHS is a special case of SSA when the angle has a measurement of $\dfrac{\pi}{2}$. The RHS rule is also valid in hyperbolic geometry and it is valuable as it contains the concept of measurement. So we put this part as an extension.

B.7. The RHS Rule

Proof. Let $\angle ABC = \angle DEF = \frac{\pi}{2}$, $BC = EF$, $AC = DF$. Assume that $\triangle ABC$ is not congruent to $\triangle DEF$. Then $AB \neq DE$, otherwise $\triangle ABC \cong \triangle DEF$(SAS). Without loss of generality, we may assume that $AB > DE$. Let G be a point on AB such that $GB = DE$.



$$\begin{aligned} \triangle GBC &\cong \triangle DEF \quad \text{(SAS)} \\ GC &= DF \quad \triangle GBC \cong \triangle DEF \\ AC &= DF \quad \text{(given)} \end{aligned}$$

Hence $AC = GC$. Therefore, $\triangle AGC$ is an isosceles triangle.

Take H as the mid-point of the base AG. Then the line joining C and H will be the perpendicular bisector of AG, and $\angle AHC = \frac{\pi}{2}$. Now we have $\angle ABC = \angle AHC = \frac{\pi}{2}$, and the lines HC and BC are ultra-parallel. But they meet at the point C, and this leads to a contradiction. Therefore, $\triangle ABC \cong \triangle DEF$. $\square$

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Reviewer's Comments

The reviewer has only comments on the wordings, which have been amended in this paper.