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EQUIDECOMPOSITION PROBLEM

TEAM MEMBER
Cheuk-Ting Li[1]

SCHOOL
La Salle College

Abstract. The equidecomposition problem is to divide a shape into pieces, and then use the pieces to form another shape. In this project, we are going to investigate the conditions under which a given shape can be broken down and combined into another specified shape. The classical problem on the equidecomposability of polygons has already been solved by mathematicians. We start by presenting the proof of the classical problem, which is the keystone of this research. Then the problem is generalized to weighted shapes, shape with curves, etc. Some interesting new results are obtained.

1. Introduction

Like paper cutting, the equidecomposition problem is on dissecting a shape into pieces, moving and rotating the pieces, and then combining them into a new shape. If it is possible to form a shape from another shape by the above operation, we say that the two shapes are equidecomposable. It is well-known that every two polygons with equal area are equidecomposable, but there are still many plane figures that mathematicians know little about their equidecomposability. The aim of the project is to generalize the problem, and to study some interesting variants. In the following sections, we are going to find out the conditions under which a pair of polygons, rectilinear shapes and weighted shapes is equidecomposable. We will also try to add some new conditions, such as forbidding rotation, limiting rotation and allowing scaling.

[1]This work is done under the supervision of the author's teacher, Ms. Mee-Lin Luk

Section 2 gives the proof of the classical problem. The idea of the proof is not original. However, it is included since it is used in the succeeding sections.

Section 3 shows a generalization of the original problem. Its proof is very similar to that in Section 2. This section is included so as to facilitate the presentation in the sections afterwards.

Shapes with weight are studied in Section 4. The equidecomposability of these shapes are investigated to provide an easy and elegant approach to the original problem and its variants.

We forbid rotations of pieces in Section 5, and study the conditions that make two shapes equidecomposable. The proof relies on the results in the first three sections. This section is a crucial part of the project.

In Section 6, the equidecomposability between shapes with curves as boundary is studied. It is difficult to consider all shapes with curves. So we add a little restriction.

In Section 7, more variants of the problem are discussed, namely equidecomposability when some points are fixed, when only one scaling is allowed, when only one rotation is allowed, and when scaling instead of rotation is allowed. In Sections 7.3 and 7.4, some surprising discoveries are presented.

Equidecomposition problem is fascinating as a challenging logic puzzle, yet it is probable that the results of the problem can be applied in practical usage, like packing goods and cutting paper. My interest on this problem was aroused by a book which mentions the classical problem, and then I started to generalize the problem and add more constraints on it. This project includes most of my original works. I acknowledge my teacher and my father for their comments on the presentation of the final report.

2. The Original Problem

Shapes are represented by a set of points in the 2D space. A shape may not be connected. Also it may have holes. In this project, only shapes with finite area and perimeter are considered. Since shapes are sets, we can take union and intersection of shapes. $A \cup B$ represents the set of points which are in shape A or shape B. $A \cup B$ represents the set of points which are in shape A and shape B. Allow an abuse of notataion: $(A = B)$ means that the symmetric difference of the two sets has zero area (measure zero).

We can translate and rotate shapes. A translation of shape A with vector v is denoted by $T_v(A)$. A rotation of shape A with centre O and angle θ? is denoted as $R_{O,\theta}(A)$. Note that the position of the shapes has no importance in most parts of the report, but it matters in the above operations.

Note that the term "disjoint" in this project means that the area of the intersection of the two shapes considered equals 0. If we say more than 2 shapes are disjoint, it means that each pair of shapes is disjoint.

Dissecting and Combining

To dissect a shape is to divide the shape into a finite number of shapes. That means fractals are not considered. Mathematically, we call a set of disjoint shapes A_1, A_2, ..., An a dissection of shape A if there exists vectors $v_1, v_2, ..., v_n$, points $O_1, O_2, ..., O_n$ and real numbers $\theta_1, \theta_2, \ldots, \theta_n$ such that $\bigcup_{i=1}^{n} T_{v_i} \circ R_{O_i,\theta_i}(A_i) = A$, where "$\circ$" means composition of transformations (rotation is applied first). To combine shapes is to put the shapes together to form one shape. To say that shape A is a combination of shapes $A_1, A_2, \ldots, A_n$ is equivalent to say that $A_1, A_2, \ldots, A_n$ is a dissection of shape A.

Define **equidecomposability** as follows: If shape A can be dissected into a collection of shapes that can be combined to form shape B, then shape A and shape B are equidecomposable. In other words, A and B are equidecomposable if and only if there exists disjoint. If shape A can be dissected into a collection of shapes that can be combined to form shape B, then shape A and shape B are equidecomposable. In other words, A and B are equidecomposable if and only if there exists disjoint shapes $A_1, A_2, \ldots, A_n$, vectors $v_1, v_2, \ldots, v_n$, points $O_1, O_2, \ldots, O_n$ and real numbers $\theta_1, \theta_2, \ldots, \theta_n$ such that

$$\bigcup_{i=1}^{n} A_i = A, \ \bigcup_{i=1}^{n} T_{v_i} \circ R_{O_i,\theta_i}(A_i) = B$$

and $T_{v_i} \circ R_{O_i,\theta_i}(A_i)$ are disjoint. We represent the above relation as $A \sim B$. Clearly $A \sim B$ is equivalent to $B \sim A$.

It is well known that every two polygons with the same area are equidecomposable. This result is called Wallace-Bolyai-Gerwien Theorem. A proof written in my own words is presented below.

Theorem 1. *Every two polygons with the same area are equidecomposable.*

Proof. To prove the above theorem, we need the following lemmas:

Lemma 2. *Equidecomposability is transitive, i.e. $A \sim B$ and $B \sim C$ implies $A \sim C$.*

Proof. The basic idea of the proof is to put the cutting lines when shape B is dissected to form shape A and those when shape B is dissected to form shape C together to cut more fine pieces that can be used to form shapes A, B and C. See appendix A for the detailed proof. □

Lemma 3. *Two parallelograms with the same area are equidecomposable.*

Proof. First we prove the following lemma:

Lemma 4. *Two parallelograms with the same area and base length are equidecomposable.*

Proof. Since the two parallelograms have the same area and base length, their heights on the equal base are equal as well. Apply rotation and translation to the parallelograms to make their bases overlaps each other. Let the point sets representing the two parallelograms after the transformations be A and B. Let the vector with the same length and parallel to the base be v. Vector v can be in any one of the two possible directions.

Define shape C_i such that $C_i = A \cap T_{iv}(B)\ \forall\ i \in \mathbb{Z}$ (set of integers) (iv is the product of a real number and a vector). Let $D = \{i \in \mathbb{Z} : C_i \neq \varnothing\}$. Since shape A is bounded, D is a finite set. $T_{iv}(B)$ are disjoint due to the nature of parallelogram (see Figure 1). Since A and B share the same height, every point in A must be in one of the C's. Therefore $\{C_i : i \in D\}$ is a dissection of A.

When translation T_{-iv} is applied on $Ci(= A \cap T_{iv}(B))\ \forall\ i \in \mathbb{Z}$, $T_{-iv}(C_i)$ $(= B \cap T_{-iv}(A))$ is obtained. Similarly $\{B \cap T_{-iv}(A) : i \in D\}$ is a dissection of B. Therefore two parallelograms with same area and base length are equidecomposable. □

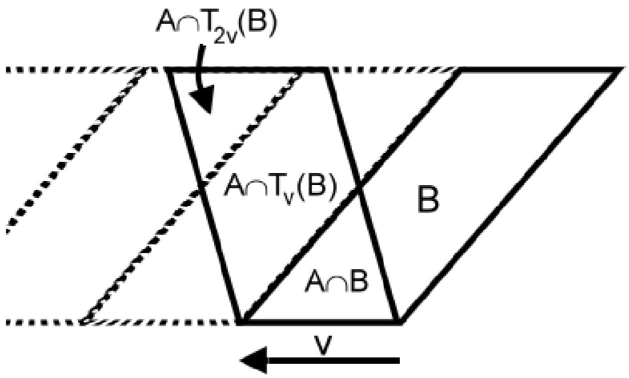

FIGURE 1

We return to the general statement. For two parallelograms A and B with the same area, let a_1, a_2, a_1, a_2 be the lengths of the sides of A and b_1, b_2, b_1, b_2 be the lengths of the sides of B. Without loss of generality, assume a_2 is the smallest among a_1, a_2, b_1, b_2. Consider a parallelogram with side lengths b_1, a_1, b_1, a_1. Its area can take any value in the interval $[0, a_1 b_1]$. Since $a_1 b_1 \geqslant a_1 a_2 \geqslant$ area of parallelogram A, a parallelogram with side lengths b_1, a_1, b_1, a_1 can have area equal to that of A. Let the parallelogram obtained be C.

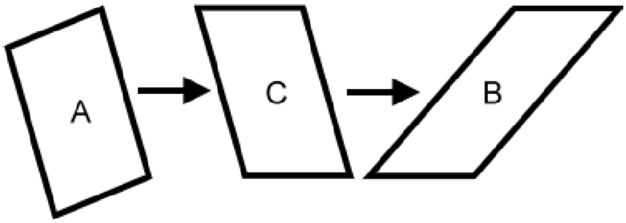

FIGURE 2

By Lemma 4, $A \sim C$. By Lemma 4 again, $C \sim B$. By Lemma 2, $A \sim B$. $\square$

Lemma 5. *A polygon can be dissected into finite number of triangles.*

Proof. We want to prove the correctness of proposition $P(n)$ which is defined below for all positive integer n.

$P(n)$: "An n-sided polygon can be dissected into finite number of triangles".

When $n = 1$ or 2, the polygon does not have area, and thus no triangles are needed. $P(1), P(2)$ are true.
When $n = 3$, if the polygon is a degenerated triangle, no triangles are needed. If the polygon is a non-degenerated triangle, it can be dissected into one triangle (that is itself). Therefore $P(3)$ is true.
Assume $P(n)$ is true for $n = 1, \ldots, k$. When $n = k + 1$, let A, B and C be three adjacent vertices such that the interior $\angle ABC < 180°$. Since the angle sum of n-sided polygon is $(n - 2)180°$, the average of the interior angles is less than $180°$. Such A, B and C must exist. For each vertex P of the polygon that is included in ΔABC (other than A and B, but may be C), note the angle $\angle PAB$. Let D be the vertex such that $\angle DAB$ is the minimum among all the angles. If two or more vertices tie, select the one closer to A.

Case 1: $D \neq C$ (See Figure 3)

Cut the polygon into three parts along line segments AD and DB. One of the parts is ΔADB (no other vertices are in ΔADB since $\angle DAB$ is the minimum angle), while the other two are polygons with number of vertices less than $k + 1$. By the induction hypothesis, the two polygons can be dissected into triangles. Therefore the $(k+1)$-sided polygon can also be dissected into triangles.

Case 2: $D = C$ (See Figure 3)

Cut the polygon into two parts along line segment AC. One of the part is ΔACB, while the other is a polygon with number of vertices k. By the induction hypothesis, the k-sided polygon can be dissected into triangles. Therefore the $k + 1$ sided polygon can also be dissected into triangles.

Therefore $P(k + 1)$ is also true. By mathematical induction, $P(n)$ is true $\forall\ n \in \mathbb{N}$. $\square$

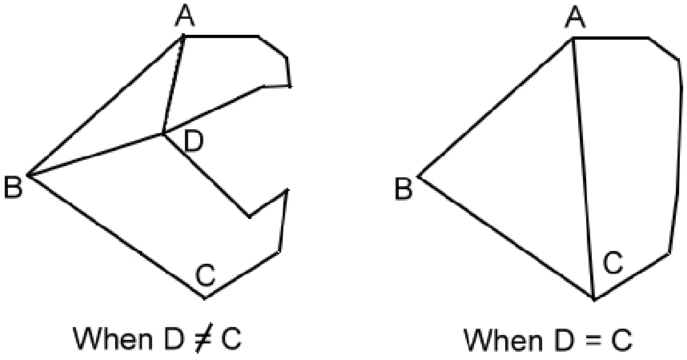

FIGURE 3

After proving the above lemmata, we come to the crucial part of the proof of Theorem 1. Let the two polygons be A and B. Let the equal area be S. By Lemma 5, A and B can be dissected into a finite number of triangles. Let $\{A_i\}$ be a dissection of A consisting of triangles. Let $\{B_i\}$ be a dissection of B consisting of triangles. Any triangle can be dissected and combined into a rectangle (see Figure 4).

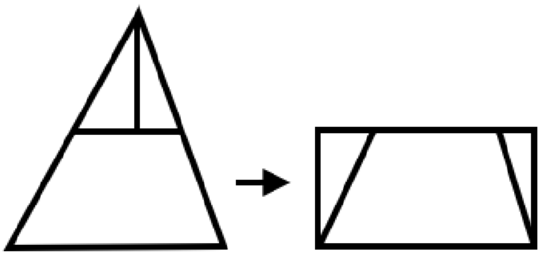

FIGURE 4

Consider a rectangle with side length m and n. By Lemma 3, it is equidecomposable with a rectangle with side length $\sqrt{S}$ and $\frac{mn}{\sqrt{S}}$.

For each A_i , dissect and combine it into a rectangle with one side equals $\sqrt{S}$. Then combine all the rectangles together by putting the sides with length $\sqrt{S}$ together. Since the area of A is S, a square with side length S must be obtained. Shape A is equidecomposable with the square. Similarly

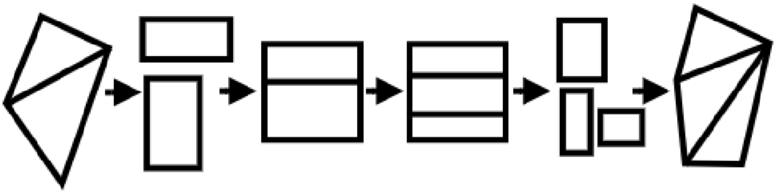

FIGURE 5

shape B is also equidecomposable with the square. By Lemma 2, $A \sim B$. $\square$

3. Equidecomposition of rectilinear shapes

In Section 2, we have seen that every two polygons with the same area are equidecomposable. Then we are going to generalize the statement a bit to shapes that may not be connected and may have holes. Rectilinear shapes are shapes with straight line segments as boundary, which may consist of several connected components and holes.

Theorem 6. *Every two rectilinear shapes with the same area are equidecomposable.*

Proof. From the proof of Theorem 1, we can see that every two shapes with the same area that can be dissected into triangles are equidecomposable. By Lemma 5, polygons can be dissected into triangles. Therefore we need only to dissect the rectilinear shapes into polygons, then apply Lemma 5 to dissect them into triangles, then the same proof in Theorem 1 can be applied.

For each connected components of the rectilinear shape (let it be A), let the hole in A with the smallest distance to the outer boundary of A be B. Let P and P' be two points on the boundary of the hole B and the outer boundary of A (i.e. not the boundary around the holes) respectively such that the length of PP' is the minimal distance between the outer boundary of A and the boundary of hole B. Since B is the hole closest to the outer boundary of A, line segment PP' does intersect any other points on the boundary of A or its holes. Cut A along PP' as in Figure 6.

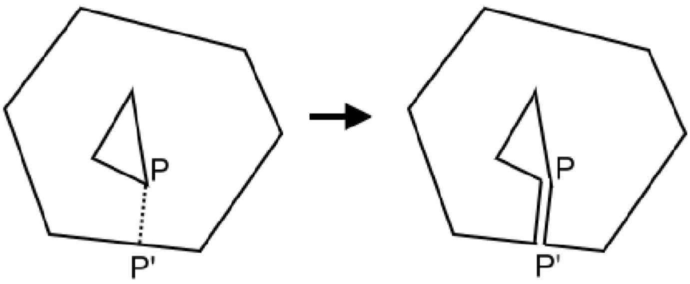

FIGURE 6

After the cutting, B is no longer a hole of A. Repeat the process until no holes can be found on A. Do the same on each connected component. Then a set of polygons is obtained. By Lemma 5, they can be dissected into triangles. The rest of the proof is the same as that of Theorem 1. □

4. Equidecomposition of shapes with weight

A *weighted shape* is a shape with a weight assigned to each point in it. Weights are integers that can be either positive or negative. When two weighted shapes overlap, their weights add up in the overlapped area.

A weighted shape A can be represented by a function $A : \mathbb{R}^2 \to \mathbb{Z}$ where $A(P)$ gives the weight at point P. If a point P is not in the weighted shape, $A(P) = 0$. Since it is a function, we can perform addition and subtraction on them.

A weighted shape A is said to be of constant weight if $A(P)$ is either c or 0 for all point P, where c is a constant integer.

Define a *constant region* of a weighted shape to be a connected region of constant weight, and there does not exist a larger connected region containing it satisfying the first condition. In this project, we consider only weighted shapes with finite number of constant regions. Also each constant region has finite perimeter.

Define an *edge* of a weighted shape to be a line segment such that the difference between weights on two sides of the line segment at every point on the line segment is a non-zero constant, and there does not exist a longer

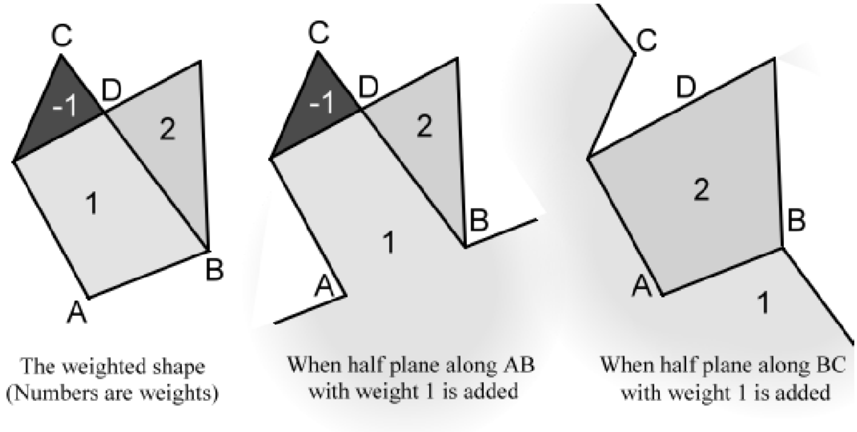

FIGURE 7

line segment containing it satisfying the first condition. In other words (cf. **Reviewer's Comment** 1), it is a line segment such that

1. every point on it is on the boundary of a constant region, and
2. there exists a half plane along the line segment and with constant weight W such that when it is added to the weighted shape, there are only finitely many points on the line segment that are also on the boundary of a constant region, and
3. there does not exist a longer line segment containing it that satisfies the first two conditions.

Figure 7 shows two examples of edges. (AB and BC are edges, but AD, BD and CD are not.)

The weighted shape is called rectilinear if every point on the boundary of the constant regions of it is on an edge, i.e. there are no curves on the boundary of its constant regions.

To dissect a weighted shape A is to find a set of weighted shapes with sum equaling A.

Define *unit shape* to be a weight shape with constant weight 1 or -1. Any shape can be dissected into unit shapes. For example, a shape with weight 3 can be dissected into 3 unit shapes.

Define the area of a weighted shape as the sum of the product of the area and the weight of each constant region in the shape.

$$\text{Area} = \int A(v)\,dv$$

We say that two weighted shape is equidecomposable if there exist weighted shapes $A_1, A_2, \ldots, A_n$, vectors $v_1, v_2, \ldots, v_n$, points $O_1, O_2, \ldots, O_n$ and real numbers $\theta_1, \theta_2, \ldots, \theta_n$ such that $\sum_{i=1}^{n} A_i(P) = A(P)$, $\sum_{i=1}^{n} T_{v_i} \circ R_{O_i,\theta_i}\ (A_i(P)) = B(P)\ \forall\ P \in \mathbb{R}^2$. (For simplicity we write $T_v \circ R_{O,\theta}(A(P))$ instead of $A(R_{O,-\theta} \circ T_{-v}(P))$, which gives shape A after the transformations $T_v \circ R_{O,\theta}$. Transformation $T_v \circ R_{O,\theta}$ is considered as an operator here.)

Note that non-weighted shapes considered in Sections 2 and 3 are special cases of weighted shapes. Non-weighted shapes can be treated as weighted shapes with constant weight 1. However the condition of equidecomposability of weighted shapes seems much different from the original one, since there may exist shapes with negative weight in the dissection. The theorem below shows that the two conditions are actually equivalent on non-weighted shapes.

Theorem 7. *Two non-weighted shapes are equidecomposable if and only if they are equidecomposable when they are treated as a weighted shape with constant weight 1.*

Proof. The only if part is obvious, since you cannot dissect shape A into weighted shapes and combine them into B if you cannot do so with non-weighted shapes.

For the if part, let the two shapes with constant weight be A and B. Let $A_1, A_2, \ldots, A_m$ be unit shapes with weight 1, $A_{m+1}, A_{m+2}, \ldots, A_n$ be unit shapes with weight $-1, v_1, v_2, \ldots, v_n$ be vectors, $O_1, O_2, \ldots, O_n$ be points and $\theta_1, \theta_2, \ldots, \theta_n$ be real numbers such that $\sum_{i=1}^{n} A_i(P) = A(P)$, $\sum_{i=1}^{n} T_{v_i} \circ R_{O_i,\theta_i}(A_i(P)) = B(P)\ \forall\ P \in \mathbb{R}^2$. Note that we can find unit shapes $A_1, A_2, \ldots, A_n$ satisfying the requirement since every weighted shape can be dissected into unit shapes. Select a unit shape A_i among $A_{m+1}, A_{m+2}, \ldots, A_n$ with weight -1. Since every point in shape A have 0 or 1 weight, for each point in A_i, there must be another shape with weight 1 in $A_1, A_2, \ldots, A_m$ that also contain that point to cancel the negative weight. Therefore it is possible to find disjoint shapes $A'_1, A'_2, \ldots, A'_m$ (may have empty sets) such that $A'_j \subset A_j$ for $j = 1, \ldots, m$ and $\bigcup_{j=1}^{m} A'_j = A_i$ when the shapes are considered as point sets with no weight. Similarly find disjoint $B'_1, B'_2, \ldots, B'_m$ such that $B'_j \subset T_{v_j} \circ R_{O_j,\theta_j}(A_j)$ for $j = 1, \ldots, m$ and $\bigcup_{j=1}^{m} B'_j = T_{v_i} \circ R_{O_i,\theta_i}(A_i)$.

$$\bigcup_{j=1}^{m} A'_j = A_i \sim T_{v_i} \circ R_{O_i,\theta_i}(A_i) = \bigcup_{j=1}^{m} B'_j$$

Therefore $\{A'_j\}$ as a whole is equidecomposable with $\{B'_j\}$ as a whole when they are treated as non-weighted shapes. $\{T_{v_j} \circ R_{O_j,\theta_j}(A'_j)\}$ as a whole is also equidecomposable with $\{B'_j\}$ as a whole when they are treated as non-weighted shapes. Take out A_i and $\{A'_j\}$ from the dissection. Each piece A_j becomes $A_j \backslash A'_j$. Since we remove A_i with weight -1 and $\{A'_j\}$ with weight 1, the impact of the removal on the sum cancels out. The sum of all dissection pieces before transformation is still A.

After the transformation, each of the dissection pieces $A_j \backslash A'_j$ becomes $T_{v_j} \circ R_{O_j,\theta_j}(A_j) \backslash T_{v_j} \circ R_{O_j,\theta_j}(A'_j)$. Since we can dissect the shapes in $\{B'_j\}$ and transform the pieces to form shapes $\{T_{v_j} \circ R_{O_j,\theta_j}(A'_j)\}$, we can use the shapes $\{B'_j\}$ to compensate the removal of $\{A'_j\}$ after the transformation. When B'_j is taken out from $T_{v_j} \circ R_{O_j,\theta_j}(A_j)$, the weight in the region $\bigcup_{j=1}^{m} B'_j$ will be decreased by 1, which cancels out the removal of $T_{v_i} \circ R_{O_i,\theta_i}(A_i)$, the piece A_i with weight -1 after transformation. Therefore we can dissect shape A into pieces to form shape B without using the piece A_i, thus using one less piece with weight -1. Repeat the procedure on each A_i with weight -1, then all the pieces with weight -1 are eliminated. It is possible to dissect shape A into pieces to form shape B without using pieces with weight -1. $\square$

Theorem 8. *Every two weighted rectilinear (cf.* ***Reviewer's Comment*** *2) shapes with the same area are equidecomposable.*

Proof.

Lemma 9. $A \sim B$ *and* $C \sim D \;\Rightarrow\; A + C \sim B + D.$

Proof. The lemma is rather obvious. Put the dissection used to form B from A together with the dissection used to form D from C together, we can get the dissection that can be used to form $B+D$ from $A+C$. Proving this lemma is easy because unlike non-weighted shapes, weighted shapes can overlap each other freely. □

(**Remarks**: Put $-B$ into both C and D, the result $A \sim B \Leftrightarrow A - B \sim \varnothing$ is obtained, where $\varnothing$ is the empty set, i.e. the shape with weight equals 0 on every point. This result is useful in the sections afterward.)

Lemma 10. *Equidecomposability is transitive on weighted shapes.*

Proof. Suppose $A \sim B$ and $B \sim C$. Then $-C \sim -B$. Apply Lemma 9. $A - C \sim \varnothing$. $A - C + C\ \varnothing + C$. $A \sim C$. □

(**Remarks**: From Theorem 7, this lemma is actually equivalent to Lemma 2. The proof of this lemma is included to show the operations of weighted shapes and how are them similar to arithmetic operations.)

We return to the proof of the theorem. Let the two weighted shapes considered be A and B. Assume A can be dissected into unit shapes $A_1, A_2, \ldots, A_m$ with weight 1 and $A_{m+1}, ..., A_n$ with weight -1. Assume B can be dissected into unit shapes $B_1, B_2, \ldots, B_{m'}$ with weight 1 and $B_{m'+1}, \ldots, B_{n'}$ with weight -1. Let C be a sufficiently large square with weight 1 that can contain all the pieces $A_{m+1}, \ldots, A_n, B_{m'+1}, \ldots, B_{n'}$ with weight -1 without any two of the pieces overlapping each other. Translate and rotate pieces $A_1, A_2, \ldots, A_m$ such that any two of them do not overlap and none of them overlap the square C, and then translate and rotate $A_{m+1}, \ldots, A_n$ such that any two of them do not overlap and all of them are contained in C. Let the shape obtained after the transformations be A'. The negative weight is eliminated by the square, so the weight of A' is 0 or 1 for each point on it. A' can be treated as non-weighted rectilinear shape. Since only translations and rotations of pieces are used to transform $A + C$ to A', $A + C \sim A'$. Similarly define $B'.B + C \sim B'$. Since the area of A' and B' are equal, by Theorem 6, $A' \sim B'$. By Lemma 10, $A + C \sim B + C$. By Lemma 9, $A \sim B$. □

5. Equidecomposition without Rotation

In previous sections, we have seen that many shapes with the same area are equidecomposable. This may not be true if rotation is taken out from the transformation procedure, that is, only translation is allowed. In this section, we are going to investigate the conditions under which two shapes are equidecomposable without rotation.

We call disjoint shapes $A_1, A_2, \ldots, A_n$ a rotationless dissection of shape A if there exists vectors $v_1, v_2, \ldots, v_n$ such that $\bigcup_{i=1}^{n} T_{v_i}(A_i) = A$. We say that shape A is a rotationless combination of shapes $A_1, A_2, \ldots, A_n$ if $A_1, A_2, \ldots, A_n$ is a rotationless dissection of shape A.

Define rotationless equidecomposability as follows:

If shape A can be rotationlessly dissected into a collection of shapes that can be rotationlessly combined into shape B, then shape A and shape B are rotationlessly equidecomposable.

In this section, we are going to find out the conditions under which two polygons are rotationlessly equidecomposable.

Define the direction of an edge of a non-weighted rectilinear shape to be the direction of movement (expressed as the angle with the horizontal line) when a point is moving along that edge such that the interior of the rectilinear shape is on the left of the moving path.

Direction of an edge can be defined similarly on weighted shapes as the direction of movement (expressed as the angle with the horizontal line) when a point is moving along that edge such that weight of the weighted shape on the left of the moving path is larger than that on the right of the moving path.

Consider a right-angled isosceles triangle and a square with the same area in the below orientation:

By Theorem 1, they are equidecomposable. But the same result cannot be obtained when rotation is forbidden. Consider the hypotenuse of the triangle. Its direction is $135°$, but no edges with the same direction can be found in the square. However, when we cut the triangle or the square along a line in order to produce an edge with direction $135°$, an edge with direction $315°$

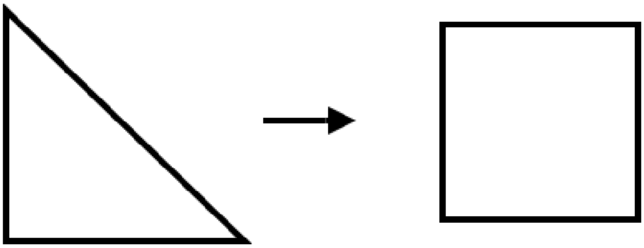

FIGURE 8

of equal length will be produced on the other side of the cutting line. We can see that in the dissection of the triangle, the total length of the edges with direction $135°$ is always longer than the total length of the edges with direction $315°$, while the two lengths must be the same in the dissection of the square. This makes the two shapes not rotationlessly equidecomposable.

Consider a square and a square that is rotated by $45°$. It seems that they are not equidecomposable without using rotation.

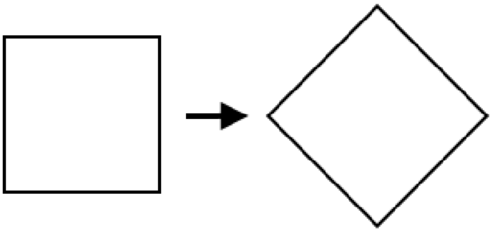

FIGURE 9

But actually they are

Even more surprisingly, one can actually perform rotation on parallelograms by dissecting and combining without using rotation. See Lemma 13 for a general statement.

In a square (or parallelogram), the opposite edge have opposite direction (i.e. difference is $180°$), which can be eliminated by putting them together. This makes it possible to dissect a parallelogram to form another parallelogram with totally different directions of edges.

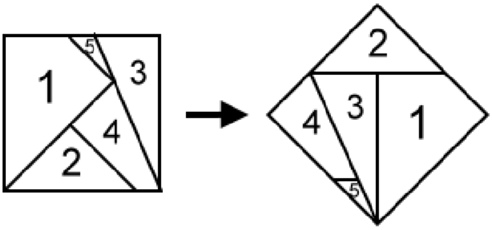

FIGURE 10

Define direction distribution function of shape A as $F_A : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} F_A(\theta) = {} & (\text{total length of edges in shape } A \text{ with direction } \theta) \\ & - (\text{total length of edges in shape } A \text{ with direction } \theta + 180^\circ) \end{aligned}$$

Note that $F_A(\theta)$ is periodic with period 360°.

It can be defined similarly on a weighted shape B:

$$\begin{aligned} F_B(\theta) = {} & (\text{total power of edges in shape } B \text{ with direction } \theta) \\ & - (\text{total power of edges in shape } B \text{ with direction } \theta + 180^\circ) \end{aligned}$$

where the power of an edge is the product of its length and the absolute difference of weights on two sides of it. Power is always positive.

The proof below shows that every two weighted rectilinear shapes with the same area are rotationlessly equidecomposable if and only if they have the same direction distribution function. We consider weighted shapes instead of non-weighted shapes since weighted shapes can perform addition and subtraction without considering whether the pieces are overlapping. After the problem on weighted shapes is proved, the problem on non-weighted shapes is also proved by the idea of Theorem 7.

Theorem 11. *Two weighted rectilinear shapes with the same area are rotationlessly equidecomposable if and only if they have the same direction distribution function.*

Proof. For the only if part, treat the dissection of a shape as a sequence of steps that break a shape into two along a cutting line (straight, bent or curved). The curved part of the cutting line can be approximated by line segments. The length of the line segments tends to 0 as the approximation becomes better. Therefore curves have no effect on the direction distribution function. Each line segment on the cutting line creates two line segments on the boundary of the divided shapes, each with opposite direction with the other. Therefore their impact on the direction distribution function cancels out. From the above argument we can see that the direction distribution function remains the same after dissection. Similar arguments apply to the combining procedure, which is the reverse of dissecting. If two rectilinear shapes have different direction distribution functions, it is not possible for one shape to dissect and combine into another. For the if part, we need the following lemmata:

Lemma 12. *Rotationless equidecomposability is transitive.*

Proof. Similar proof of Lemma 2 can be applied.

□

Lemma 13. *Two parallelograms of constant weight with same area and weight are rotationlessly equidecomposable.*

Proof. The proof of Lemma 4 works on two parallelograms with equal and parallel bases even without rotation. (Parallel bases means that rotation is not needed in the first step of the proof.) After the two parallelograms are translated such that they share a common vertex, let the parallelogram be $ABCD$ and $AEFG$ (see Figure 11).

Case 1: $DG//BC$

$DG//BC \Leftrightarrow DG//DA \Leftrightarrow G, A, D$ collinear $\Leftrightarrow GA//BC$.
Translate parallelogram $AEFG$ to $A'E'F'A$ such that point G coincides with A after the translation. G is the point after the common vertex A when the vertices of the parallelogram is ordered anti-clockwisely. After the translation, the new point after A is F'.
Since $F'A//FG$, FG is not parallel to GA since $S_{AEFG} \neq 0$. Since $GA//BC$, $F'A$ is not parallel to BC.
When we consider parallelogram $A'E'F'A$ instead of $AEFG$, case 1 will not be encountered. So we need only to consider the case where DG is not parallel to BC.

Case 2: Otherwise
Let I be the intersection of BC and DG (or their extensions). Construct H such that $\overrightarrow{HI}=\overrightarrow{AD}$, and J such that $\overrightarrow{HJ}=\overrightarrow{AG}$. Parallelograms $ABCD$ and $AHID$ shares the same base and height. Thus they are rotationlessly equidecomposable by Lemma 4. Parallelograms $AHID$ and $AHJG$ share the same base and height. Therefore they are rotationlessly equidecomposable. Parallelograms $AHJG$ and $AEFG$ share the same base, and $S_{AHJG} = S_{AHID} = S_{ABCD} = S_{AEFG}$ ($SAHJG$ is the area of quadrilateral $AHJG$). Therefore they are rotationlessly equidecomposable. By Lemma 12, parallelogram $ABCD$ and $AEFG$ are rotationlessly equidecomposable. □

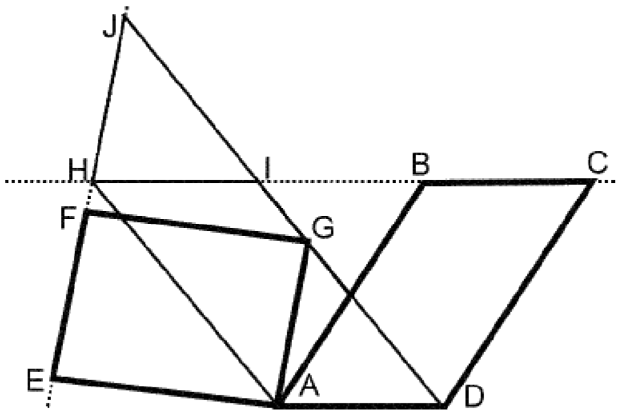

FIGURE 11

Now we return to the proof of the theorem. Let the two weighted rectilinear shapes be A and B. By Lemma 9, to prove $A \sim B$ is equivalent to prove $A - B \sim \varphi$. Let $C = A - B$.

Define a step as follow:

1. Find a pair of non-horizontal, non-vertical (i.e. direction is not a multiple of 90°) 19 edges with opposite direction (i.e. the difference in direction is 180°) in C. Such pair must exist in shape C not consisting of horizontal or vertical edges only, because if we pick an edge in C that is non-horizontal and non-vertical, an edge with opposite direction in C must exist since C has a direction distribution function that is always 0.

2. If the two edges do not have the same length, divide the longer edge into two segments, one with length equals that of the shorter edge, and consider that divided segment and the shorter edge only. The two line segments have equal lengths.
3. Consider the shorter edge in step 2. (Select any one of them if they are of equal length.) Construct a right-angled triangle with it as hypotenuse and the other two sides either horizontal or vertical. Let the triangle be D. The weight of D equals the difference in weight on two sides of the edge, such that when D is added to C, the shorter edge is no longer an edge. Subtract D' that is the translated copy of D with the other line segment as hypotenuse from C (see Figure 12). A new shape C' which has the same direction distribution function as C is obtained after the addition and subtraction. By Lemma 9, since $D \sim D', C \sim \varnothing \Leftrightarrow C + D \sim \varnothing + D' \Leftrightarrow C + D - D' \sim \varnothing \Leftrightarrow C' \sim \varnothing$. Therefore we need only to consider the new shape C' and continue the process.

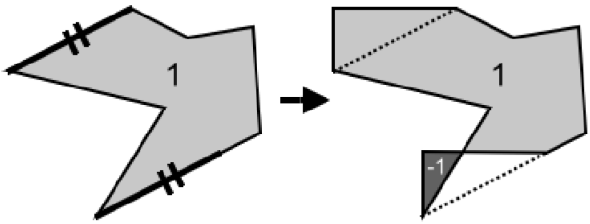

FIGURE 12

Since the shorter edge of the pair is removed in each step, each step decreases the total number of non-horizontal, non-vertical edges by at least one. Since the number of edges of C is finite, after a finite number of steps, only horizontal or vertical edges are left.

For each edge, dissect the shape along the extension of the edge. Since all edges are horizontal or vertical, a set of rectangles is obtained (see Figure 13).

Dissect each rectangle with weight not equal to 1 or -1 into multiple copies of rectangles with weight equals to 1 or -1 with the same widths and heights. Only rectangles with weight 1 or -1 is left. By Lemma 4.2, we can dissect each rectangle and combine the pieces into a rectangle consisting of horizontal or vertical edges with width 1. Combine all these rectangles with weight

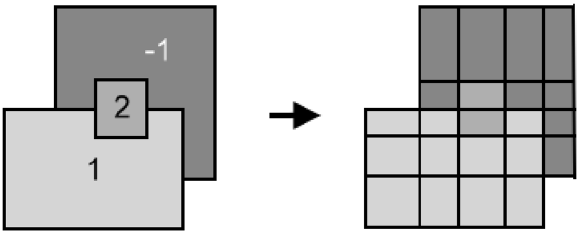

FIGURE 13

1 into a large rectangle with width 1 and height x. Combine all rectangles with weight -1 into a large rectangle with width 1. Since $C = A - B$ have zero area, the height of the large rectangle with weight -1 is also x. Combine the two large rectangles together and an empty set is obtained. Thus $A - B \sim \varnothing$ and $A \sim B$. $\square$

6. Equidecomposition on shapes with curves

In the preceding sections, only rectilinear shapes are considered. The problem becomes more complicated when curves can appear on the boundary of the shapes.

For example, is a square equidecomposable with a circle with the same area?

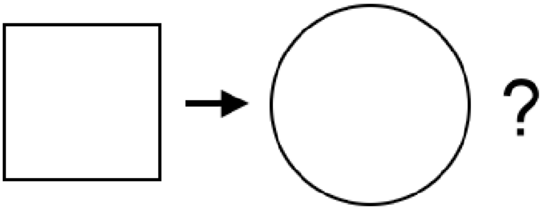

FIGURE 14

To dissect a square to form a circle, an edge with an arc (or arcs) must exist in the dissection. But when an arc is cut in a square, another arc with different side as the interior of the shape will be produced, which needs another arc to compensate. Therefore we can see that the two shapes are not equidecomposable. The logic here is similar to that in Section 5. (Actually they are equidecomposable when non-measurable sets are allowed and when the Axiom of Choice is used. See the discussion in Section 8.)

It seems that the general problem is unsolvable when all shapes are allowed. But it can be solved when a restriction is added to the shapes.

Let a weighted shape with curves be A. Define a component of shape A as the union of a constant region of A with finite area and the constant regions surrounded by it. A component has no holes. The weight of a component is the weight of the constant region if it is not surrounded by a single constant region, the weight of the constant region minus the weight of the surrounding region if it is surrounded by one constant region. This is to make that the sum of all the components gives the original shape. Figure 15 shows the components of a shape.

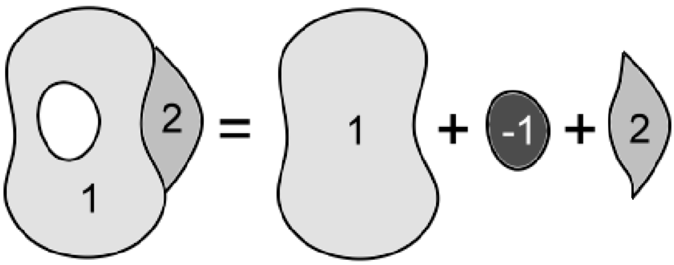

FIGURE 15

Define "unit component" to be a component with weight 1 or -1. Any component can be dissected into unit components. Let C_i be the i-th unit component of shape A (the order of the components has no significance).

$$\sum_{i=1}^{n} C_i = A.$$

Describe a unit component C by its "direction function" $f(x)$ that gives the complex number with modulus 1 representing the direction of the boundary at a point P which is x unit-length from a starting point B along the boundary (following its direction) of shape C. Note that if the weight of C is 1, P runs anti-clockwisely as x increases. If the weight is -1, P runs clockwisely as x increases. In this section, a point is represented by a complex number.

Direction of a curve at a point P is the tangential direction at P, and the interior of the shape is on the left when looked along the direction, or the weight on the left is larger than that on the right if the shape is weighted.

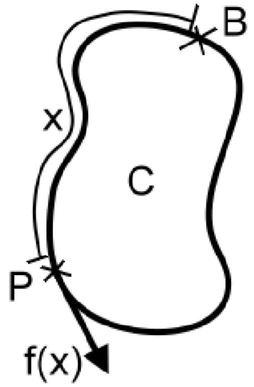

FIGURE 16

The direction function of a weighted shape is obtained by combining the direction functions of its unit components in the following way. When we take the sum of two unit components C_1 and C_2, we combine their direction functions $f_1(x)$ and $f_2(x)$ into one $f(x)$ by connecting their starting points B_1 and B_2. Let the perimeter of C_1 and C_2 be p_1 and p_2 respectively.

$$f(x) = \begin{cases} f_1(x) & \text{when } x \in [0, p_1) \\ \frac{B_2 - B_1}{|\,B_2 - B_1\,|} & \text{when } x \in [p_1, p_1 + |\,B_1 - B_2\,|) \\ f_2(x - p_1 - |\,B_1 - B_2\,|) & \text{when } x \in [p_1 + |\,B_1 - B_2\,|, \\ & \qquad p_1 + p_2 + |\,B_1 - B_2\,|) \\ \frac{B_1 - B_2}{|\,B_1 - B_2\,|} & \text{when } x \in [p_1 + p_2 + |\,B_1 - B_2\,|, \\ & \qquad p_1 + p_2 + 2|\,B_1 - B_2\,|) \end{cases}$$

The perimeter of the combined $C_1 + C_2$ is $p_1 + p_2 + 2|\,B_1 - B_2\,|$, and its starting point is B_1.

By a sequence of operations, we can get the direction function of the original shape A, which can describe shape A. Given the direction function $f(x)$, the starting point B, for any point P, its weight can be given by the following formula:

$$\text{Weight} = \sum_{i=1}^{n} \operatorname{sgn}(\operatorname{Im}(f(x_i))),$$

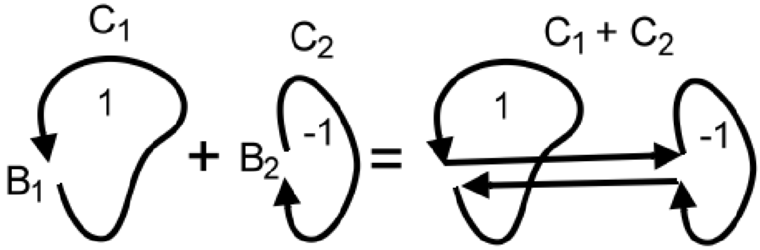

FIGURE 17

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases},$$

$\operatorname{Im}(x)$ is the imaginary part of complex number x, $\{x_i\}$ is the set of solutions of

$$\arg\left(\int_0^x f(y)\,dy + B - P\right) = 0.$$

We then add a restriction to the direction function such that the equidecomposability between two shapes with direction functions satisfying the restriction can be determined.

First introduce the term "analytic function". Analytic function is a function that can be expressed as $f(x) = \sum_{i=0}^{\infty} a_i x^i$ for an infinite sequence a_i. Most of the commonly used functions are analytic, for example, $x^2, \sin x, e^x$ are analytic. Functions that involve cases are usually not analytic, for example, $\operatorname{sgn}(x)$ is not analytic.

A property of analytic function is that if two analytic functions $f_1(x)$ and $f_2(x)$ are equal $\forall\ x \in (a, b)$ where $a < b$, then $f_1(x) = f_2(x)\ \forall\ x \in (-\infty, \infty)$. This suggests that the function value in a small interval is enough to characterize the whole function.

Define "analytic decomposition" of function $f(x)$ with domain $[0, p)$ to be a sequence of analytic functions $f_0(x), f_1(x), f_2(x), \ldots, f_n(x)$ and a sequence $a_1, a_2, \ldots, a_n$ such that $f(x) = f_i(x) \forall\ x \in [a_i, a_{i+1})$ $(a_0 = 0, a_{n+1} = p)$. If such finite sequence exists, $f(x)$ is called "analytic decomposable".

In this section, we consider only shapes with direction functions that are analytic decomposable. For example, the direction function of a square is

$$f_{\text{square}}(x) = \begin{cases} 1 & \text{when } x \in [0, \frac{p}{4}) \\ i & \text{when } x \in [\frac{p}{4}, \frac{p}{2}) \\ -1 & \text{when } x \in [\frac{p}{2}, \frac{3p}{4}) \\ -i & \text{when } x \in [\frac{3p}{4}, p) \end{cases}$$

which can be analytically decomposed into 4 functions. The direction function of a circle is $f_{\text{circle}}(x) = e^{2i\pi x/p}$, which is analytic, and thus analytic decomposable.

Define a "common interval" of a function $f(x)$ on another function $g(x)$ to be an interval (a, b) such that there exist θ and α such that $g(x) = e^{i\theta} f(x + \alpha)\ \forall\ x \in (a, b)$ (see remark at the end of the section), and there are no intervals that properly contain (a, b) satisfying the first requirement. We say $f(x)$ is similar to $g(x)$ if there exist θ and α such that $g(x) \equiv f(x + \alpha)$ (i.e. one can rotate $f(x + \alpha)$ to form $g(x)$).

Define the "direction distribution operator" of a shape with analytic decomposable direction function $f(x)$ be an operator $h : F \to F$ (a function that maps a function to a function) which takes an analytic function as its argument and gives a real-valued function. $h(g)$ gives a real-valued function $g'(y)$ that equals the number of common intervals of $f(x)$ on $g(x)$ that contains y, minus the number of common intervals of $f(-x)$ on $g(x)$ that contains y. In other words, $g'(y)$ is the difference between the frequency of a segment of $g(x)$ that contains y on the boundary of the shape in forward direction, and the frequency of a segment of $g(x)$ that contains y on the boundary of the shape in backward direction.

Let the direction distribution operators of shapes A and B be $h_A(g)$ and $h_B(g)$ respectively. There are some properties on the direction distribution operator:

Property 14. *The direction distribution operator of nA (nA is the sum of n copies of A) is $nh_A(g)$.*

Property 15. *The direction distribution operator of $A+B$ is $h_A(g)+h_B(g)$.*

Note that rotation is allowed in this section.

Theorem 16. *Two weighted shapes with analytic decomposable direction functions with the same area are equidecomposable if and only if they have the same direction distribution operator.*

Proof. For the only if part, the same logic in the proof of Theorem 11 can be applied. Treat the dissection of a shape as a sequence of steps that breaks a shape into two along a cutting line (straight or bent or curved). Keep track of the direction distribution operator of the shapes as a whole. Each analytic part on the cutting line creates two analytic curves on the boundary of the divided shapes, each with different side of it to be the interior of the shape (or with different side of it to have greater weight in the weighted shape case). Therefore their impacts on the direction distribution operator cancel out. From the above argument we can see that the direction distribution operator remains the same after dissection. Similar arguments apply on the combining procedure, which is the reverse of dissecting.

If two rectilinear shapes have different direction distribution operators, it is not possible to dissect one shape and combine the pieces into the other shape.

For the if part, let the two weighted shapes with analytic decomposable direction functions be A and B. Let $C = A - B$. $A \sim B \Leftrightarrow A - B \sim \varnothing \Leftrightarrow C \sim \varnothing$. By the method of combining two direction functions mentioned above, we can see that the direction function of C is also analytic decomposable. By property 14 of direction distribution operator, the direction distribution operator of $-B$ is the negative of the direction distribution of B. By property 15 of direction distribution operator, the direction distribution operator of C equals the sum of the direction distribution operators of A and $-B$. Since A and B have the same direction distribution operator, the sum of the direction distribution operators of A and $-B$ is zero. Therefore the direction distribution operator of C gives a constant function which is always 0 for all input $g(x)$.

Define a step as follows:

1. Find a pair of non-straight-line analytic curves segments $g_1(x)$ and $g_2(x)$ on the direction function $f(x)$ of C such that $g_1(x)$ is similar to $g_2(-x)$. Such pair must exist in shape C with boundary of constant regions not consisting of straight line segments only, because if you pick an analytic curve $g_1(x)$ in $f(x)$, an analytic curve with opposite

direction must exist since C has a direction distribution operator that gives a constant function which is always 0 for the input $g_1(x)$.

2. Connect the end-points of the curve $g_1(x)$ by a line segment. Let the region bounded by $g_1(x)$ and the line be D. The weight of D equals the difference in weight on two sides of the curve $g_1(x)$, such that when D is added to C, the curve is replaced by the line in the direction function of C. Subtract D' that is the translated copy of D which is also the region bounded by $g_2(x)$ and the line connecting its end-points. (See Figure 18) A new shape C' which has the same direction distribution function as C is obtained after the addition and subtraction. By Lemma 9, since $D \sim D'$,

$$C \sim \varnothing \;\Leftrightarrow\; C + D \sim \varnothing + D' \;\Leftrightarrow\; C + D - D' \sim \varnothing \;\Leftrightarrow\; C' \sim \varnothing.$$

Therefore we need only consider the new shape C' and continue the process.

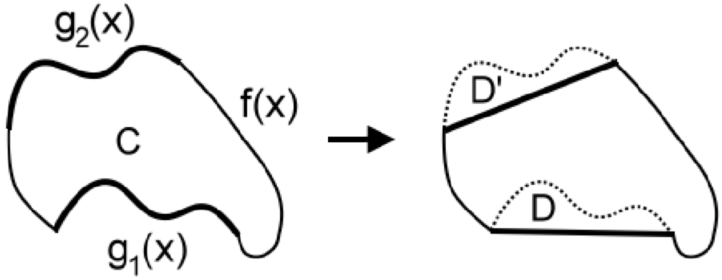

FIGURE 18

Each step can eliminate at least one non-straight-line analytic curve segment in the direction function of C. After a series of steps, only straight edges are left. Since each step does not affect the area of C, the area of the final version of C is still 0. Apply Theorem 8 on C. The result follows. $\square$

(**Remark**: Similar results can be obtained if rotation is not allowed. Change the line $g(x) \equiv e^{i\theta} f(x + \alpha)$ into $g(x) \equiv f(x + \alpha)$ in this case.)

7. More variants

7.1. When some points are fixed

The equidecomposability problem is like paper cutting. One can cut a piece of paper of a rectilinear shape to pieces and use the pieces to form another rectilinear shape with the same area. But what will happen if we put some marks on the paper and restrict the original and final position of the marks? When we dissect the shapes into pieces and transform the pieces, each mark will undergo the same transformation as the piece on which it lies.

Mathematically, given points $P_1, P_2, \ldots, P_n$ on rectilinear shape A, and $P'_1, P'_2, \ldots, P'_n$ on rectilinear shape B with the same area as A, the problem is to dissect shape A together with the points on it into pieces, then translate and rotate the pieces together with the points on it to form shape B with the final position of P_i equals P'_i.

Consider a pair of points P_i and P'_i, let the distance of P_i to the closest point to P_i among $P_1, P_2, \ldots, P_n$ be R, and the distance of P'_i to the closest point to P'_i among $P'_1, P'_2, \ldots, P'_n$ be R'. Let C be a square with centre P_i and side length $\min\{R, R'\}$. Let T_v be the translation that maps P_i to P'_i. Consider the intersection $D = A \cap T_{-v}(B) \cap C$, it is contained in A. Also it does not contain any points in $P_1, P_2, \ldots, P_n$ other than P_i. When shape D is translated by vector v, it is contained in B and does not contain any points in $P'_1, P'_2, \ldots, P'_n$ other than P'_i . Therefore we can cut shape D from shape A and translate it by v to shape B, then we need only to consider the shapes $A \backslash D$ and $B \backslash T_v(D)$ together with the $n-1$ pairs of points.

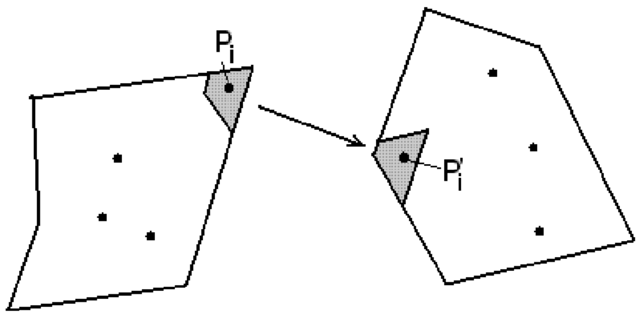

FIGURE 19

Repeat the procedure on each pair of points, then two rectilinear shapes without marked points are left. Apply Theorem 6 on the two shapes. Therefore it is possible to dissect shape A and use the pieces to form shape B under the restriction.

7.2. When the two shapes are of different area

In previous sections, the two shapes considered must have equal area. If the two shapes have different area, it is clearly impossible for the shapes to be equidecomposable. However, when scaling of dissection pieces is allowed, the problem becomes too trivial since one can just scale each piece by the ratio between the two shapes. Therefore we allow scaling on one of the pieces only. We are going to find out whether any two shapes are equidecomposable under this new condition.

For simplicity, only non-weighted polygons are considered. Rotation is allowed.

Theorem 17. *Every two polygons are equidecomposable when only one scaling is allowed.*

For the two shapes considered, let the one with greater area be A and the one with smaller area be B. Let the smallest circle that contains A be C_1 and its radius be R_1. Let the biggest circle that is contained in B be C_2 and its radius be R_2. Let S_A (S_B) be the area of A (B respectively). Then (cf. **Reviewer's Comment** 3)

$$\pi R_1^2 > S_A > S_B > \pi R_2^2$$

$$\begin{aligned}
\frac{R_1^2}{R_2^2} &> \frac{S_A}{S_B} \\
S_B R_1^2 &> S_A R_2^2 \\
S_A R_1^2 - S_B R_1^2 &< S_A R_1^2 - S_A R_2^2 \\
\frac{R_1^2(S_A - S_B)}{R_1^2 - R_2^2} &< S_A \\
S_A &> \frac{S_A - S_B}{1 - \frac{R_2^2}{R_1^2}}
\end{aligned}$$

Therefore it is possible to cut a polygon X with area $\frac{S_A - S_B}{1-\frac{R_2^2}{R_1^2}}$ from shape A. Consider the scaling that maps C_1 to C_2. Let the image of X be X'. See Figure 20.

$$S_{X'} = \frac{R_2^2}{R_1^2}\left(\frac{S_A - S_B}{1 - \frac{R_2^2}{R_1^2}}\right) = \frac{S_A - S_B}{\frac{R_1^2}{R_2^2} - 1}$$

$$\begin{aligned}
S_A - S_X &= S_A - \frac{S_A - S_B}{1 - \frac{R_2^2}{R_1^2}} = \frac{S_A - \frac{S_A R_2^2}{R_1^2} - S_A + S_B}{1 - \frac{R_2^2}{R_1^2}} \\
&= \frac{S_B - \frac{S_A R_2^2}{R_1^2}}{1 - \frac{R_2^2}{R_1^2}} = \frac{S_B - \frac{S_B R_2^2}{R_1^2} - \frac{S_A R_2^2}{R_1^2} + \frac{S_B R_2^2}{R_1^2}}{1 - \frac{R_2^2}{R_1^2}} \\
&= S_B - \frac{\frac{(S_A - S_B)R_2^2}{R_1^2}}{1 - \frac{R_2^2}{R_1^2}} = S_B - \frac{S_A - S_B}{\frac{R_1^2}{R_2^2} - 1} \\
&= S_B - S_{X'}
\end{aligned}$$

Apply Theorem 6 to shape $A \backslash X$ and shape $B \backslash X'$. Shape $A \backslash X$ can be dissected to pieces and combined into shape $B \backslash X'$. Then piece X in A is scaled to X' in B. ($A \backslash X$ contains the points that are in A but not in X.) Therefore any pair of non-weighted polygons is equidecomposable when one scaling is allowed.

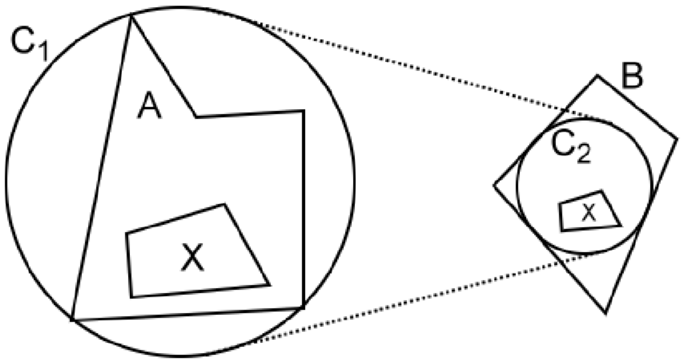

FIGURE 20

7.3. When only one rotation is allowed

The original equidecomposition problem allows unlimited usage of rotation in the dissection pieces. What will happen if we add a restriction that only one of the pieces can perform rotation and translation while the others can only perform translation? The piece that is rotated must be connected. We also add a restriction that there is no hole on the piece, i.e. it is a polygon. The proof below shows that it is actually unnecessary to have unlimited number of rotations.

Note that we consider non-weighted shapes here.

Theorem 18. *Every two rectilinear shapes with the same area are equidecomposable when only one rotation is allowed.*

Proof. To prove the theorem is equivalent to prove that for any pair of rectilinear shapes A and B, one can find a shape D contained in A such that $A \backslash D$ is rotationlessly equidecomposable with $B \backslash D'$, where D' is a translated and rotated copy of D.

Lemma 19. *For any rectilinear shape A without holes that can be bounded by a circle with radius less than R, there exist a polygon with the same direction distribution function as A and can also be bounded by a circle with radius less than R.*

Proof. The only difference between rectilinear shapes without holes and polygons is that rectilinear shapes without holes can contain disjoint polygons. The problem can be solved if we find a way to connect the polygons. Among the component polygons in A, select two polygons with the smallest distance. Let the polygons be B and C. Let point P_B on the boundary of B and P_C on the boundary of C be two points such that their distance is the minimal distance between B and C. Among the intersection points of the ray $P_C P_B$ with polygon B, let P'_B be the one that is not P_B and is closest to P_B, and also the line segment $P_B P'_B$ is contained in polygon B. Define P'_C similarly. If such P'_B (or P'_C) does not exist (line $P_B P_C$ cut B or C at one point only), add a square with extremely small side length to shape B (or shape C) on the edge connected to P_B (or P_C) such that the square does not touch any other parts of A other than the edge (see Figure 21), and the addition of the square does not make the smallest circle containing A have radius greater than or equals R (this is possible since the smallest circle containing A have radius less than R, not equal to R). After the addition of square, such P'_B and P'_C must exist.

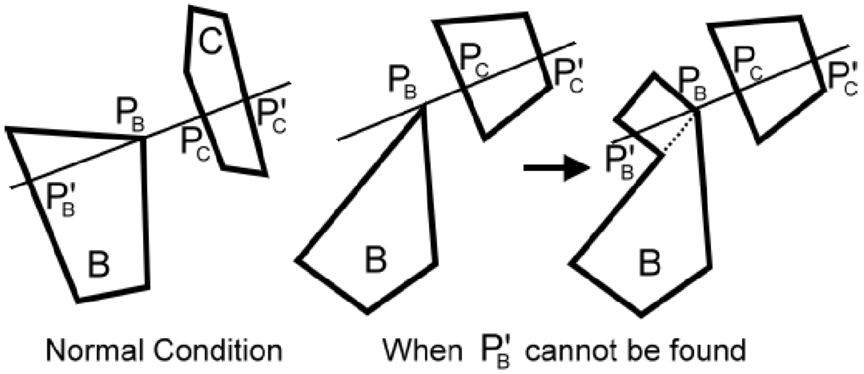

Figure 21

Since P_BP_C is the closest distance among pairs of polygons, the line segment P_BP_C does not cut any other points. Therefore line segment $P'_BP'_C$ cuts exactly 4 points of the boundary shape A. Dissect shape A along line segment $P'_BP'_C$ (not the extension of it). Then each of the polygons B and C will be divided into two polygons. This does not affect the direction distribution function. Let B be divided into B' and B'', and C be divided into C' and C'', where B' and C' are on the same side of $P'_BP'_C$. Translate B', C' along vector v that is perpendicular to $P'_BP'_C$ with extremely small magnitude such that the translation does not make B' and C' intersect any other parts of A, and also it does not make the smallest circle containing A have radius greater than or equals R (this is possible since the smallest circle containing A has radius less than R, not equal to R). Let the translated copy of P'_B and P'_C be P''_B and P''_C respectively. Add the rectangle $P'_BP'_CP''_CP''_B$ to A. See Figure 22.

Since there are no intersection between line segment P_BP_C with A other than the two end points, the rectangle does not intersect any other parts of shape A. The addition of rectangle does not affect the direction distribution function. After the addition, B', B'', C' and C'' becomes one connected component. Repeat the process until only one polygon is left. It is the polygon needed. $\square$

Lemma 20. *For any positive real number R and any rectilinear shape A, there exists a polygon with the same direction distribution function as A such*

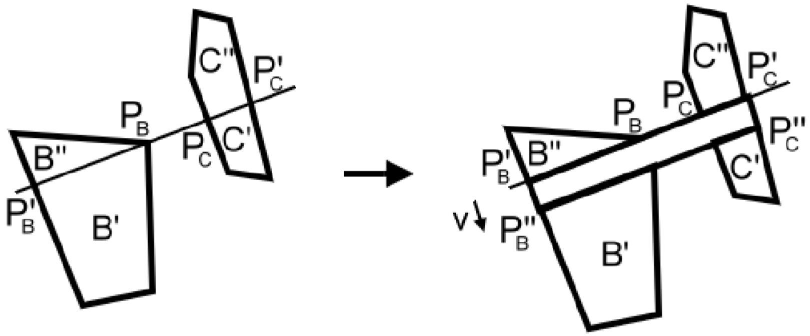

FIGURE 22

that it is bounded by a circle with radius less than R.

Proof. Consider adding a square hole on shape A. It adds two pairs of edges to shape A, while the impact of the opposite sides cancels out. Therefore adding a square hole will not affect the direction distribution function.

Put shape A on a square grid. For each square that is contained in shape A, make it a hole on shape A. Only the parts of shape A that do not completely occupy a square is left. The shape left have the same direction distribution function as A. We call a square that is only partially contained in A as "incomplete square", and the intersection points of the boundary of shape A and the sides of any square as "contact points". Clearly there are at least 2 contact points on the sides of an incomplete square, and each contact point is on the sides of at most 4 incomplete squares (we count the points on the vertex of a square 4 times). Let the number of incomplete squares be X, and the number of contact points be Y. We can deduce that $X \leqslant 2Y$.

For each edge in shape A, when the number of horizontal (or vertical) division lines of the square grid per unit length (let it be n) increase linearly to infinity, the number of intersections of the edges with the horizontal and vertical division lines (i.e. contact points on the edge) increases at linear speed. Y increases at linear speed, so X increases at no faster speed than linear speed. Therefore there must exist n such that $\lfloor nR \rfloor^2 \geqslant X$. We can put all the X incomplete squares (together with the part of A inside it) in a big square with $\lfloor nR \rfloor^2$ small squares. The side length of the big square

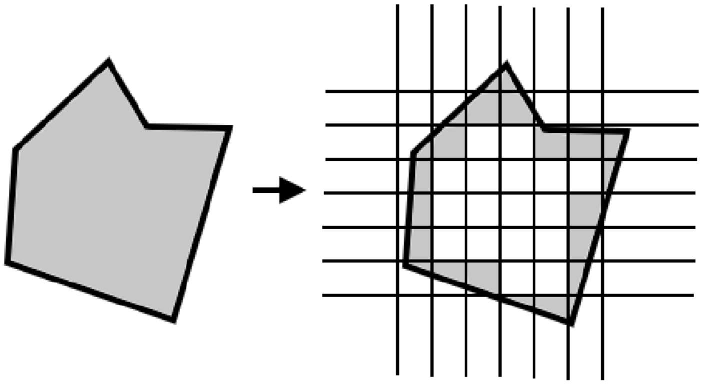

FIGURE 23

is $\frac{1}{n}\lfloor nR \rfloor \leqslant R$. It can be contained in a larger square with side length R, which can be contained in a circle with radius less than R.

It seems that the lemma is proved, but it is not, since the resultant shape is a collection of incomplete squares that may not be connected. If we rotate this piece as a whole, actually many disjoint pieces are rotated together. But Lemma 19 cannot be applied since there may be holes. Therefore we need the following trick.

Instead of finding n such that $\lfloor nR \rfloor^2 \geqslant X$, we find n such that $\lfloor nR \rfloor^2 \geqslant 4X$. If there is a hole of A contained in an incomplete square, further increase n such that the area of a square is smaller than the area of the smallest hole in A, thus making each part of A inside an incomplete square has no holes. Then we put the incomplete squares in a way that no two squares may touch, thus making it impossible to have holes. See Figure 24.

The shape with all the incomplete squares aligned as above (let the shape be B) has no holes. Since $\lfloor nR \rfloor^2 \geqslant 4X$, and $\frac{1}{4}$ of the small squares are used in the above alignment, the shape B can be put in a square with side length R (let it be C), which can be contained in a circle with radius less than R. The direction function of B equals that of A. Apply Lemma 19 on shape B. The resultant polygon is the shape needed. $\square$

We return to the proof of the theorem. Let A and B be the two rectilinear shapes considered. Let the radius of the largest circle that can be contained in both shape A and shape B be R. Let the direction distribution functions

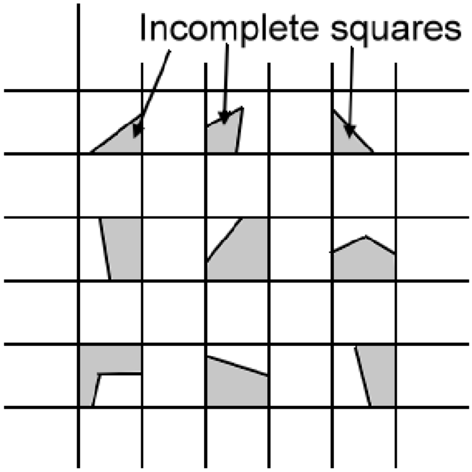

FIGURE 24

of A and B be $F_A(\theta)$ and $F_B(\theta)$ respectively. If we consider shape B rotated by 180° together with shape A as one shape, the new shape has direction distribution function $F_A(\theta) - F_B(\theta)$. If we scale the new shape to half of its size, its direction function is $\frac{F_A(\theta)-F_B(\theta)}{2}$. Apply Lemma 19 on this shape with R, we can get a shape C which can be contained in a circle with radius R (therefore can be contained in A or B) with direction distribution function

$$F_A(\theta) - \frac{F_A(\theta) - F_B(\theta)}{2} = \frac{F_A(\theta) + F_B(\theta)}{2}.$$

Let C' be shape C rotated 180°. Shape $B \backslash C'$ has direction distribution function

$$F_B(\theta) - \left(-\frac{F_A(\theta) - F_B(\theta)}{2}\right) = \frac{F_A(\theta) + F_B(\theta)}{2}.$$

$A \backslash C$ shares the same direction distribution function as $B \backslash C'$. By Theorem 11, they are rotationlessly equidecomposable. □

(**Remark**: It is surprising that such a harsh restriction can be added to the original equidecomposition problem. The number of rotation is restricted to 1, also its angle can be restricted to 180°.)

7.4. Scaling but no rotation

In this part, we allow scaling instead of rotation in the dissection pieces. Note that a rotation with angle 180° is a scaling with scaling factor -1. If scaling with negative scaling factor is allowed, Theorem 18 can be applied here to prove that every two rectilinear figures are equidecomposable under the new condition. Therefore we allow only scaling with positive scaling factor. Scaling with factor 0 which makes the shape disappears is not allowed. Scaling with factor 1 is allowed.

Also we would like to have the minimal number of scaling. The piece scaled must be a polygon. When the two shapes considered have equal area and are not rotationlessly equidecomposable, it is impossible to dissect one shape into pieces, scale only one of the pieces, then combine the pieces to form another shape. Therefore two or more scaling must be used if the two shapes considered have equal area and are not rotationlessly equidecomposable. However, it may be possible to use one scaling only when the two shapes have different area.

Note that non-weighted polygons are considered in this section.

Theorem 21. *Every two polygons with different area are equidecomposable when no rotations are allowed and only one scaling is allowed.*

Proof. Define "attach" a polygon B to an edge XY of a polygon A by a square with side length α to be the following operation:

1. Translate B to a position near the edge XY such that the minimal distance between B and XY is less than α, and B has no intersection with A.
2. Cut polygon B along a straight line that is perpendicular to edge XY into two polygons B' and B''.
3. Translate B'' along vector v parallel to XY with magnitude α in a direction away from B'.
4. Add a square with side length α touching side XY, polygon B' and B'' as shown in Figure 25.

   After attaching, a single polygon with area equals $S_A + S_B + \alpha^2$ and direction distribution function equals the sum of the direction distribution functions of A and B is obtained.

   We return to the proof of the theorem. Let the one with larger area of the two shapes considered be A and the other be B. Let

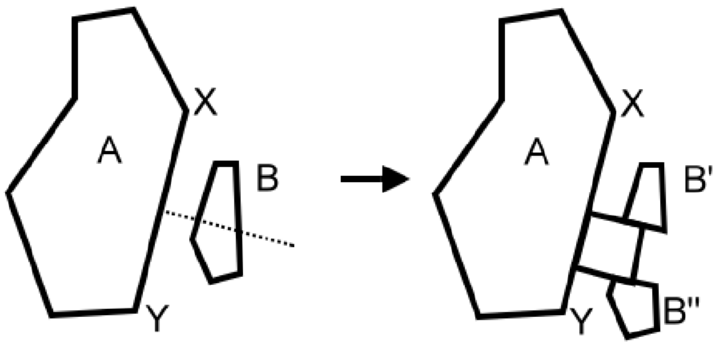

FIGURE 25

the smallest circle that contains A be C_1 and its radius be R_1. Let the biggest circle that is contained in B be C_2 and its radius be R_2. Similar to Section 7.2, if we can find a polygon in A with area $\frac{S_A - S_B}{1-\frac{R_2^2}{R_1^2}}$ with suitable direction distribution function, then we can scale it by $\frac{R_2}{R_1}$ and translate it to shape B.

Divide shape A by a line segment XY into two polygons A' and A'', where X and Y lies on the boundary of A, such that the area of A' equals $\frac{S_A - S_B}{1-\frac{R_2^2}{R_1^2}} - \varepsilon$, where ε is a very small positive number. XY is now an edge of A' and A''. Let the direction distribution function of A' be $F_{A'}(\theta)$. Apply Lemma 20 on A' rotated 180° to get a polygon D with direction distribution function $-F_{A'}(\theta)$ that can be contained in a small circle with area less than $\frac{\varepsilon}{2}$.

Consider the shape with shape A rotated 180° and shape B put together. It has direction distribution function $F_B(\theta) - F_A(\theta)$. Scale it by $\frac{R_1}{R_1 - R_2}$ to obtain a shape with direction distribution function $\frac{R_1(F_A(\theta) - F_B(\theta))}{R_1 - R_2}$. Apply Lemma 20 on the shape with a sufficiently small R to get a polygon E with direction distribution function $\frac{R_1(F_A(\theta) - F_B(\theta))}{R_1 - R_2}$ that can be contained in a small circle with area less than $\frac{\varepsilon}{2}$.

Attach each of D and E to edge XY of A' by a square with side length $\sqrt{\frac{\varepsilon - S_D - S_E}{2}}$ to get polygon G which can be contained in A (it is possible since ε can be arbitrarily small). The area of G is

$$\frac{S_A - S_B}{1 - \frac{R_2^2}{R_1^2}} - \varepsilon + S_D + S_E + \left(\sqrt{\frac{\varepsilon - S_D - S_E}{2}}\right)^2 = \frac{S_A - S_B}{1 - \frac{R_2^2}{R_1^2}}$$

The direction distribution function of G is

$$F_{A'}(\theta) - F_{A'}(\theta) + \frac{R_1(F_A(\theta) - F_B(\theta))}{R_1 - R_2} = \frac{R_1(F_A(\theta) - F_B(\theta))}{R_1 - R_2}.$$

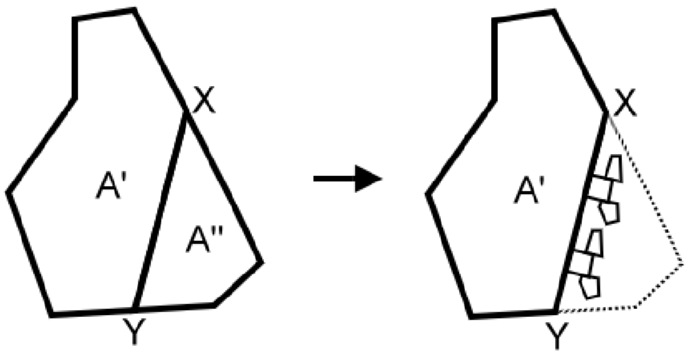

FIGURE 26

Consider the scaling that maps C_1 to C_2. Let the image of G be G'. Similar to Section 7.2, the area of $A\backslash G$ equals the area of $B\backslash G'$. The direction distribution function of $A\backslash G$ is

$$F_A(\theta) - \frac{R_1(F_A(\theta) - F_B(\theta))}{R_1 - R_2} = \frac{R_1 F_B(\theta) - R_2 F_A(\theta)}{R_1 - R_2},$$

while the direction distribution function of $B\backslash G'$ is

$$F_B(\theta) - \frac{R_2}{R_1}\left(\frac{R_1(F_A(\theta) - F_B(\theta))}{R_1 - R_2}\right) = \frac{R_1 F_B(\theta) - R_2 F_A(\theta)}{R_1 - R_2}$$

which equals that of $A\backslash G$. By Theorem 11, $A\backslash G$ is rotationlessly equidecomposable with $B\backslash G'$. Only one scaling is used to transform G to G'. □

Theorem 22. *Every two polygons are equidecomposable when no rotations are allowed and only two scalings are allowed.*

Proof. Let the two shapes considered be A and B. Dissect A into A' and A'', B into B' and B'' such that the area of A' is not equal to that of B', and the area of A'' is not equal to that of B''. Apply Theorem 21 to shape A' and B', and then to A'' and B''. □

Therefore the minimum number of scaling is

0 when the two polygons share the same area and direction distribution function;
1 when the two polygons does not share the same area; and
2 when the two polygons share the same area and have different direction distribution functions.

8. Conclusion and Discussion

In the above sections, we have proved the equidecomposability of rectilinear shapes, weighted shapes and shapes with curves as boundary. We have also found out the conditions for two rectilinear shapes to be equidecomposable when rotation is forbidden. Some variants of the problem are also investigated. It is remarkable that one can dissect a rectilinear shape and use the pieces to form another rectilinear shape with the same area with one rotation only.

It is natural to extend the statement from 2D shape to 3D polyhedron. Elegant though the proof of the 2D case is, the conditions for two 3D figures to be equidecomposable is still unknown. The general problem on equidecomposability of polyhedron is the famous Hilbert's Third Problem, which was disproved using a counter example.

At the beginning of the project, we limit the choice of the shape and the dissection to shapes with finite area and perimeter. If this restriction is removed, strange things may be observed. Some of the shapes in the dissection may be fractals with infinite perimeter. The proved Tarski's Circle-squaring Problem states that one can dissect a circle into finitely many pieces and combine them into a square with equal area even without rotation, but Axiom of Choice is used in the proof. This suggests that if non-measurable sets are allowed, it is almost impossible to obtain any results without involving the basic definitions of the mathematics system.

Appendix A. Proof of the transitivity of equidecomposability of polygon

Suppose shape A can be dissected into disjoint shapes $A_1, A_2, \ldots, A_n$ such that $\bigcup_{i=1}^{n} A_i = A$, $\bigcup_{i=1}^{n} T_{v_i} \circ R_{O_i,\theta_i}(A_i) = B$ and $T_{v_i} \circ R_{O_i,\theta_i}(A_i)$ are disjoint for some vectors $v_1, v_2, \ldots, v_n$, points $O_1, O_2, \ldots, O_n$ and real numbers $\theta_1, \theta_2, \ldots, \theta_n$. Suppose shape B can be dissected into disjoint shapes $B_1, B_2, \ldots, B_n$ such that $\bigcup_{i=1}^{n} B_i = B$, $\bigcup_{i=1}^{n} T_{w_i} \circ R_{P_i,\phi_i}(B_i) = C$ and $T_{w_i} \circ R_{P_i,\phi_i}(B_i)$ are disjoint for some vectors $w_1, w_2, \ldots, w_n$, points $P_1, P_2, \ldots, P_n$ and real numbers $\phi_1, \phi_2, \ldots, \phi_n$. (Note that the "$n$" for shape A and B are the same. If fewer than n pieces are needed in the dissection of A or B, the rest of the pieces are empty sets.)

Consider the point sets $A_i \cap [R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)]$ for $i, j = 1, \ldots, n$

$$
\begin{aligned}
&\bigcup_{i,j\in[1,n]} A_i \cap [R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)] \\
&= \bigcup_{i=1}^{n} [A_i \cap \bigcup_{j=1}^{n} R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)] = \bigcup_{i=1}^{n} [A_i \cap R_{O_i,-\theta_i} \circ T_{-v_i}(B)] \\
&= \bigcup_{i=1}^{n} R_{O_i,-\theta_i} \circ T_{-v_i}[T_{v_i} \circ R_{O_i,\theta_i}(A_i) \cap B] \\
&= \bigcup_{i=1}^{n} R_{O_i,-\theta_i} \circ T_{-v_i}[T_{v_i} \circ R_{O_i,\theta_i}(A_i)] \quad \text{(since } T_{v_i} \circ R_{O_i,\theta_i}(A_i) \text{ is a part} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{of shape } B) \\
&= \bigcup_{i=1}^{n} A_i = A
\end{aligned}
$$

Consider the intersection of any 2 point sets in $A_i \cap [R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)]$

$$
\begin{aligned}
&\{A_i \cap [R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)]\} \cap \{A_{i'} \cap [R_{O_{i'},-\theta_{i'}} \circ T_{-v_{i'}}(B_{j'})]\} \\
&= A_i \cap A_{i'} \cap [R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)] \cap [R_{O_{i'},-\theta_{i'}} \circ T_{-v_{i'}}(B_{j'})]
\end{aligned}
$$

Case 1: $i \neq i'$

$A_i \cap A_{i'} = \varnothing$. The intersection is an empty set

Case 2: $i = i$ **and** $j \neq j'$

$$\begin{aligned} &A_i \cap A_{i'} \cap [R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)] \cap [R_{O_{i'},-\theta_{i'}} \circ T_{-v_{i'}}(B_{j'})] \\ &= A_i \cap R_{O_i,-\theta_i} \circ T_{-v_i}(B_j \cap B_{j'}) = \varnothing \end{aligned}$$

since $B_j \cap B_{j'} = \varnothing$. Therefore the intersection is empty set. $A_i \cap [R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)]$ are disjoint point sets for $i, j \in [1, n]$. Therefore $\{A_i \cap [R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)]\}$ is a dissection of shape A.

When transformation $T_{w_j} \circ R_{P_j,\phi_j} \circ T_{v_i} \circ R_{O_i,\theta_i}$ is applied on $A_i \cap [R_{O_i,-\theta_i} \circ T_{-v_i}(B_j)]$ for $i, j \in [1, n]$, point sets $T_{w_j} \circ R_{P_j,\phi_j} \circ T_{v_i} \circ R_{O_i,\theta_i}(A_i) \cap T_{w_j} \circ R_{P_j,\phi_j}(B_j)$ are obtained. Note that the composition of multiple translations and rotations can still be written as the composition of one translation and one rotation.

$$\begin{aligned} &\bigcup_{i,j \in [1,n]} T_{w_j} \circ R_{P_j,\phi_j} \circ T_{v_i} \circ R_{O_i,\theta_i}(A_i) \cap T_{w_j} \circ R_{P_j,\phi_j}(B_j) \\ &= \bigcup_{j=1}^{n} \left\{ \left[\bigcup_{i=1}^{n} T_{w_j} \circ R_{P_j,\phi_j} \circ T_{v_i} \circ R_{O_i,\theta_i}(A_i) \right] \cap T_{w_j} \circ R_{P_j,\phi_j}(B_j) \right\} \\ &= \bigcup_{j=1}^{n} \left[T_{w_j} \circ R_{P_j,\phi_j}(B) \cap T_{w_j} \circ R_{P_j,\phi_j}(B_j) \right] \\ &= \bigcup_{j=1}^{n} T_{w_j} \circ R_{P_j,\phi_j}(B_j) = C \end{aligned}$$

From $A \sim B$ and $B \sim C$, we can see that the area of A, B and C are the same. Shape A can be dissected into shapes $\{A_i \cap [R_{O_i-\theta_i} \circ T_{-v_i}(B_j)]\}$ that can cover shape C under certain transformations that preserves area. If there are overlapping areas of the shapes after the transformations, the area of C will be smaller than that of A, which leads to a contradiction. Therefore there are no overlapping areas in shapes $T_{w_j} \circ R_{P_j,\phi_j} \circ T_{v_i} \circ R_{O_i,\theta_i}(A_i) \cap T_{w_j} \circ R_{P_j,\phi_j}(B_j)$. They are disjoint.

Therefore $\{T_{w_j} \circ R_{P_j,\phi_j} \circ T_{v_i} \circ R_{O_i,\theta_i}(A_i) \cap T_{w_j} \circ R_{P_j,\phi_j}(B_j)\}$ is a dissection of C. Therefore $A \sim C$.

REFERENCES

[1] *Wolfram Mathworld* [mathworld.wolfram.com/Dissection.html]

[2] *Equidecomposability History* [http://www.geocities.com/cnowlen/Cathy/Math5200/equihist.html]

Reviewer's Comments

1. On page 8, paragraph 5, line 4, why is the requirement before the phrase"In other words" logically equivalent to that after it, i.e. the conditions 1, 2 and 3?

2. For the statement in Theorem 8, how is "rectilinear" involved in the proof?

3. On page 23, last paragraph, line 4, it is better to add the phrase "Let S_A (S_B) be the area of A (B respectively). Then…"