Hang Lung Mathematics Awards
Vol. 10 (2023), pp. 99–148

ON THE PARAMETRIZATION OF EGYPTIAN FRACTIONS

A RESEARCH REPORT SUBMITTED TO THE SCIENTIFIC COMMITTEE OF THE HANG LUNG MATHEMATICS AWARDS

TEAM NUMBER
2347707

TEAM MEMBER
TAM HEI TUNG

TEACHER
MR. FONG YUK LUN

SCHOOL
HKUGA COLLEGE

AUGUST 2023

Abstract. This study explores Egyptian fractions, focusing on parametrization to construct a unified approach to open problems in this field. The paper introduces a symmetric parametrization for Egyptian fraction equations, demonstrating its effectiveness through three applications. It also investigates conjectures related to the shortest length of Egyptian expansion and the Generalized Erdös-Straus conjecture, and explores connections with semiperfect numbers. The research leverages Geometry to transform Egyptian equations into a parametrized system, offering a novel perspective on tackling open problems with and within the field of Egyptian fractions.

Contents

1. Introduction

Egyptian fraction is one of the oldest branches of Number theory, involving the representation of nonzero fractions as a sum of distinct unit fractions. For more historical background, please refer to [4]. Despite centuries of study, these problems persistently present challenges.

Definition 1.1. *A non-zero fraction* $\frac{m}{n}$ *is said to have* l *as its* ***shortest length of Egyptian expansion*** *if* $\frac{m}{n}$ *can be expressed as a sum of a* ***minimum*** *of* l *distinct unit fractions.*

Conjecture 1 (Generalized Erdös Conjecture)**.** *For every integer* $a \geq 1$*, there exists a solution to the Diophantine equation* $\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ *for* $n > n_0$*, where* n_0 *is a constant depending on* a*.*

The original conjecture is a special case, where $a = 4$ and $n_0(4) = 1$.

Conjecture 2. *Do there exist finite or infinitely many nonzero fractions with a given shortest length of Egyptian expansion?*

The conjecture's successful proof historically hinges on examining various modular identities on n, first enumerated by Mordell [7]. However, no definitive results indicate a particular set of fractions sharing the same shortest Egyptian expansion length.

Transitioning from these open problems, we navigate to an intriguing intersection of semiperfect numbers and Egyptian fractions. This connection, though well-known, is explored in [3].

Prompted by a philosophical question—If Egyptian fractions are a form of Diophantine equations, can we construct a **unified** approach to these open problems?—This paper presents a significant achievement: a natural, symmetric parametrization for Egyptian fraction equations. The derivation of this parametrization will be discussed, and its value showcased through applications in three separate Egyptian fractions fields, marking remarkable progress

2. Egyptian parametrization formula

2.1. **Motivations of Egyptian parametrization.** Most Diophantine equations do not share trivial patterns in solutions from their original forms. For example, the famous Pythagorean triple $a^2 + b^2 = c^2$. But Euclid's formula reveals the following identity:

$$(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$$

and indeed the following parametrization

$$a = x^2 - y^2, \quad b = 2xy, \quad c = x^2 + y^2 \tag{1}$$

covers all primitive Pythagorean triples. Since x, y are restricted to integers only, a, b, c can be seen as the results of substituting different combinations of free integer variables (x, y). Therefore (1) is called Euclid's formula. Also, notice that the parametrized pair (x, y) is **symmetric**, meaning that exchanging (x, y) into (y, x) in (1) makes no difference. One of the famous methods to discover (1) is using coordinate geometry and considering rational points on a unit circle.

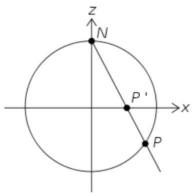

FIGURE 1. Geometric model of finding rational points on a unit circle, source: from Wikipedia

The derivation of (1) is skipped because Pythagorean triples are not studied objects of this paper. Nevertheless, We are interested in searching for parametrization of the general Diophantine **Egyptian equation**

$$\frac{m}{n} = \frac{1}{x_1} + ... + \frac{1}{x_l} \tag{2}$$

Despite pure interest, we have strong reasons why parametrization is preferred. When $l = 3$, (2) is equivalent to solve for the Diophantine equation

$$mx_1x_2x_3 = n(x_1x_2 + x_1x_3 + x_2x_3) \tag{3}$$

Any variables x_1, x_2, x_3 exist in 3 terms in (3). If we choose x_1 to be the subject,

$$x_1 = \frac{nx_2x_3}{mx_2x_3 - nx_2 - nx_3}$$

The three-term denominator leads to a difficult understanding of x_1. Hence, the objectives for Egyptian parameterization are:

(1) To have **fewer** parameterized variables than the original in (2).
(2) To **decrease** the solving complexity of the parametrized equation(s) compared to the original.
(3) To ensure the parameterized variables are **symmetric**, meaning the exchange of parametrized variables' values doesn't affect solutions to (2).

The subsequent subsections will reveal that, while the first objective may not be fully achievable, the third is certainly attainable. The accomplishment of the second objective, believed to be viable, is demonstrated in Section 3.

2.2. **Case Studies on Two-Term and Three-Term Egyptian expansions.** In this subsection, we aim to identify a suitable parametrization of Egyptian fraction expansion through a case study approach using Geometry. Unlike previous researchers, we propose that preprocessing and transforming the Diophantine equation (2) of different lengths l into a parametrized system can offer advantages during the solving process.

Geometric models representing unit fractions as sums of unit fractions have previously been established in an HLMA paper [2]. To aid understanding and maintain clarity, these models will be conveniently represented in Proposition 2.1 and 2.3.

Proposition 2.1. *In Figure 1, $AC//BD//FE$. Let $AC = x,\ BD = y,\ FE = n$. Then*

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{y} \tag{4}$$

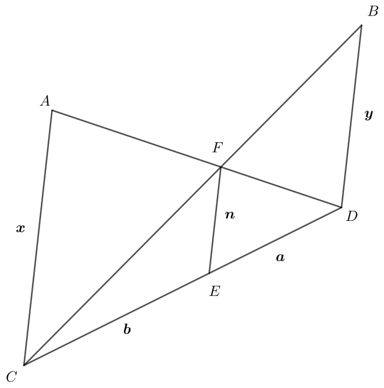

FIGURE 2. Geometric model of 4

Proof. By $AC//BD//FE$, $\triangle CEF \sim \triangle CDB$ and $\triangle DEF \sim \triangle DCA$. Therefore,

$$\frac{n}{x} = \frac{a}{a+b}, \quad \frac{n}{y} = \frac{b}{a+b}$$

Notice

$$\frac{n}{x} + \frac{n}{y} = \frac{a}{a+b} + \frac{b}{a+b} = 1 \quad \Longleftrightarrow \quad \frac{1}{n} = \frac{1}{x} + \frac{1}{y}$$

□

The above geometric model has not suggested directions of parametrization yet. However, if we focus on variables a and b in Figure 2.2 only, a beautiful and symmetric parametrization of two terms Egyptian fraction expansion can be obtained.

Proposition 2.2. *In Figure 2.2,*

(i) $\frac{1}{ab} = \frac{1}{b(a+b)} + \frac{1}{a(a+b)}$

(ii) $ab,\ (a+b)a,\ (a+b)b \in \mathbb{N}$ *iff* $a^2,\ b^2,\ ab \in \mathbb{N}$

Proof. (i) Recall

$$\frac{a}{a+b} = \frac{n}{x} \quad , \quad \frac{b}{a+b} = \frac{n}{y}$$

Then $n : x : y = ab : b(a+b) : a(a+b)$. Combining with Proposition 2.1, that $\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$, we have

$$\frac{1}{ab} = \frac{1}{a(a+b)} + \frac{1}{b(a+b)}$$

The parametrization when $n = ab, x = a(a+b), y = b(a+b)$ is symmetric because the values of a and b can interchange with each other.

(ii) The proof is obvious by $(a+b)a - ab = a^2$, $(a+b)b - ab = b^2$.

□

Remark 1. *Proposition 2.2(i) revealed the* ***Egyptian parametrization*** *on* (4). *However, we need algebraic skills, i.e. Proposition 2.2(ii) to identify the natures of* $a,\ b$ *to solve* (4).

Example 2.1. *The following demonstrates how to find all solutions to the Diophantine equation*

$$\frac{1}{12} = \frac{1}{x} + \frac{1}{y}$$

Pick $n = ab = 12$. By Proposition 2.2(ii),

$$x = a(a+b) \text{ and } y = b(a+b) \text{ with } a^2, b^2 \in \mathbb{N}$$

Notice that $x,\ y$ are allowed to exchange their values. Without loss of generality, assume $a \le b$.

$$a^2, b^2 \in \mathbb{N} \Longleftrightarrow \begin{cases} a = 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{6}, 3 \\ b = 12, 6\sqrt{2}, 4\sqrt{3}, 6, 2\sqrt{6}, 4 \end{cases}$$

Then the following are all solutions.

x	y	a	b
13	156	1	12
14	84	$\sqrt{2}$	$6\sqrt{2}$
15	60	$\sqrt{3}$	$4\sqrt{3}$
16	48	2	6
18	36	$\sqrt{6}$	$2\sqrt{6}$
21	28	3	4

Notice that if we pick $a, b \in \mathbb{N}$ with $\gcd(a,b) > 1$, $\gcd(ab, a(a+b), b(a+b)) > 1$ and cause the solution to be degenerated. For example, if $a = 2$ and $b = 6$,

$$\frac{1}{12} = \frac{1}{16} + \frac{1}{48} \iff \frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

meaning that the solution $(16, 48)$ solving (4) when $n = 12$ is an enlargement of $(4, 12)$ solving (4) when $n = 3$. To a certain extent, we may regard $(16, 48)$ as a "meaningless" solution.

Now we start the study on three-term Egyptian expansion.

Proposition 2.3. *Figure 3 depicts 3 excircles of $\triangle ABC$ with radii r_1, r_2, r_3 and corresponding centres K, M, N respectively. Let n be the radius of the inscribed circle of $\triangle ABC$. Then*

$$\frac{1}{n} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

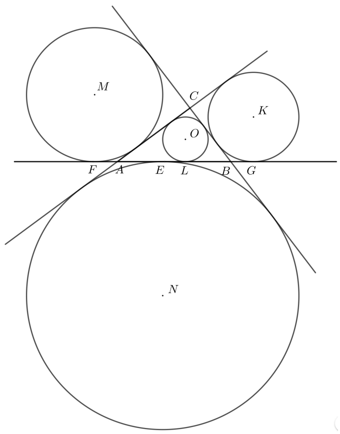

FIGURE 3. Geometric model of 3 fractions

Proof. Consider the quadrilateral of $MABC$.

$$\text{Area of } \triangle ABC = \text{ Area of quadrilateral } MABC - \text{Area of } \triangle MAC$$

By definition of tangent,

$$\begin{aligned}\text{Area of quadrilateral } MABC &= \text{Area of } \triangle MAB + \text{Area of } \triangle MBC \\ &= \frac{ABr_2}{2} + \frac{BCr_2}{2}\end{aligned}$$

Also, we have

$$\text{Area of } \triangle ABC = \frac{ABn}{2} + \frac{BCn}{2} + \frac{ACn}{2}$$

Therefore,

$$\frac{(AB + BC + AC)n}{2} = \frac{(AB + BC - AC)r_1}{2}$$

$$r_2 = \frac{(AB+BC+AC)n}{AB+BC-AC}$$

By a similar argument, we will obtain

$$r_1 = \frac{(AB+BC+AC)n}{AB+AC-BC} \quad \text{and} \quad r_3 = \frac{(AB+BC+AC)n}{BC+AC-AB}$$

And we have

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$$
$$= \frac{AB+AC-BC}{(AB+BC+AC)n} + \frac{AB+BC-AC}{(AB+BC+AC)n} + \frac{BC+AC-AB}{(AB+BC+AC)n} = \frac{1}{n}$$

□

In the context of a three-term Egyptian expansion, achieving a symmetric parametrization akin to Proposition 2.2 is not straightforward due to the numerous possible length and area-related quantities present in Figure 3. To allow "geometry to reveal the truth," we propose to further adapt the circle-excircles model on the coordinate plane using linear transformations, including translation and rotation. This adjustment aims to minimize the number of free variables derived from the coordinates of points, enabling us to redefine length-related quantities in the original model using these coordinates. The following strategy is potentially the most effective for this placement.

Refer to Figure 4 for visualization. We fix the center of the inscribed circle at the origin and align AB parallel to the x-axis. With points A and B given, the intersection point C is determined by the extended tangents. In this placement, while the x-coordinates of A and B are free variables, the y-coordinates of A and B are identical and equal to $-n$. Hence, we only need two free-moving variables, which are the x-coordinates of A and B.

Proposition 2.4. *Figure 4 is formed by Figure 3 with the center of the inscribed circle having radius n at the origin.* $A(x_1, -n)$ *and* $B(x_2, -n)$ *are on the line* $y = -n.$ *, where* $x_1 < -n,\ x_2 > n$*. Then*

$$r_3 = -\frac{x_1 x_2}{n} \tag{5}$$

Proof. In Figure 4, the slope of L_{AO} is $-\frac{n}{x_1}$ and the slope of L_{BO} is $-\frac{n}{x_2}$.

By properties of incentre and excircles of triangles, FN bisects $\angle CAF$, and OA bisects $\angle CAB$. Also $\angle CAF + \angle CAB = 180°$.

Therefore, $NA \perp OA$ and similarly, $NB \perp OB$.

Consequently, the slope of L_{AN} and L_{BN} are $\frac{x_1}{n}$ and $\frac{x_2}{n}$ respectively.

Therefore, equations of L_{AN} and L_{BN} are as follows:

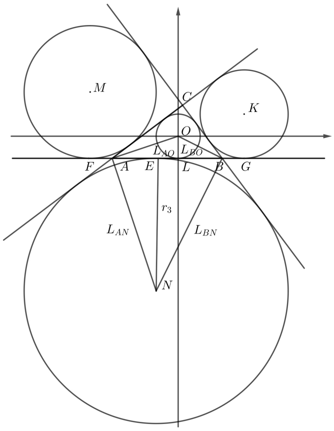

FIGURE 4. Geometric model of 3 fractions on coordinate plane

$$L_{AN} : \frac{y+n}{x-x_1} = \frac{x_1}{n} \qquad\qquad L_{BN} : \frac{y+n}{x-x_2} = \frac{x_2}{n}$$

$$y = \frac{x_1(x-x_1)}{n} - n \qquad\qquad y = \frac{x_2(x-x_2)}{n} - n$$

N is the intersecting point of L_{AN} and L_{BN}.

$$\frac{x_1(x-x_1)}{n} - n = \frac{x_2(x-x_2)}{n} - n$$
$$x = x_1 + x_2$$

Substituting $x = x_1 + x_2$ into L_{AN}, we have

$$y = \frac{x_1[(x_1+x_2) - x_1]}{n} - n$$
$$y = \frac{x_1x_2}{n} - n$$

Therefore, the coordinates of N can be determined as $\left(x_1 + x_2, \frac{x_1x_2}{n} - n\right)$.

Recall that N is the center of the excircle below $y = -n$. The radius of the incircle is $OL = n$.

The radius of this excircle is exactly $-\left(\frac{x_1x_2}{n} - n\right) - n = -\frac{x_1x_2}{n}$. □

Now the readers should notice that we did not express r_1 and r_2 in terms of x_1 and x_2. The reason is that although we can further express r_1 and r_2, the formula has lost its beauty from symmetry. Indeed the formulas for r_1 and r_2 can also be beautiful after further revealing length relations in $\triangle ABC$.

Proposition 2.5. *In Figure 5, D, H, L are the points on the inscribed circle that touch $\triangle ABC$ and n is the radius of the inscribed circle. Denote $CH = CD = a$, $LB = BH = b$, $LA = AD = c$. Then* *(i)* $n^2 = \dfrac{abc}{a+b+c}$,

(ii) $r_1 = \dfrac{ab}{n}$, $r_2 = \dfrac{ac}{n}$, $r_3 = \dfrac{bc}{n}$.

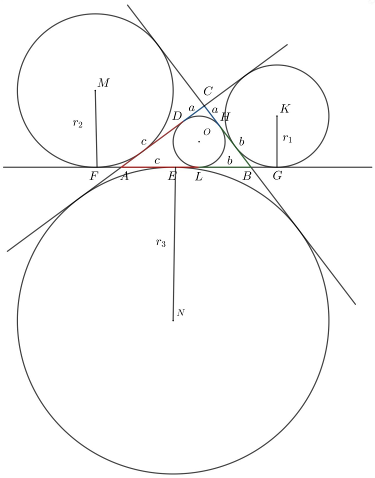

FIGURE 5. Geometry model in Proposition 2.2 with a, b, c

Proof. (i)

$$\begin{aligned}\text{Area of } \triangle ABC &= \text{Area of } \triangle BOC + \text{Area of } \triangle AOC + \text{Area of } \triangle AOB \\ &= \frac{1}{2}n(a+b) + \frac{1}{2}n(a+c) + \frac{1}{2}n(b+c) \\ &= n(a+b+c)\end{aligned}$$

Also by Heron's formula, we have

$$\begin{aligned}&\text{Area of} \triangle ABC \\ &= \sqrt{s(s-BC)(s-AC)(s-AB)} \\ &\qquad \text{where } s = \frac{AB+BC+AC}{2} = a+b+c \\ &= \sqrt{s(s-(a+b))(s-(a+c))(s-(b+c))} \\ &= \sqrt{(a+b+c)c \times b \times a}\end{aligned}$$

Therefore, $n(a+b+c) = \sqrt{(a+b+c)abc}$

By squaring both sides of the above equation, we obtain $n^2 = \dfrac{abc}{a+b+c}$.

(ii) From Proposition 2.3, we know that the radius of the excircle below y-axis, denoted as r_3, is given by $-\dfrac{x_1 x_2}{n}$.

By substituting $-x_1 = c, x_2 = b$, we can find that the radius of r_3 is $\dfrac{bc}{n}$. Now rotate $\triangle ABC$ about the origin such that BC lies on the horizontal line $y = -n$. As a result, the center and the radius of the excircle below the horizontal axis are replaced by the original K and r_1 respectively.

Since $HC = a, BH = b$, the radius for r_1 can be expressed as $\dfrac{ab}{n}$.

By using a similar argument, we can deduce that the radius of r_2 is $\dfrac{ac}{n}$. □

Therefore, combining the Proposition 2.5(i) and 2.5(ii), we have our final formula:

$$n^2 = \frac{abc}{a+b+c}, \quad x = \frac{ab}{n}, \quad y = \frac{ac}{n} \text{ and } \quad z = \frac{bc}{n}.$$

Combining with Proposition 2.3,

$$\frac{1}{\dfrac{abc}{n(a+b+c)}} = \frac{1}{\dfrac{ab}{n}} + \frac{1}{\dfrac{ac}{n}} + \frac{1}{\dfrac{bc}{n}} \tag{6}$$

with $n^2 = \dfrac{abc}{a+b+c}$ is a symmetric parametrization of three terms Egyptian fractions expansion.

Drawing parallels to Proposition 2.2(ii), we aim to adopt an algebraic approach to ascertain the characteristics of variables $a,\ b,\ c$ in the aforementioned parametrization. Intriguingly, while a, b, c might not necessarily be integers, they are required to appear in surd form with common irrational parts. Moreover, we can establish a bijection between any integer triple (x, y, z) that satisfies the Egyptian expansion equation $\dfrac{m}{n} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$ and integer triples that are closely associated with the above parametrization. These methodologies are elaborated in the following sections. For ease of reference, we introduce the following set of notations.

Definition 2.1. *Given $m, n \in \mathbb{N}$.*

(i) $SolE_3(m) = \{\, (x, y, z) \in \mathbb{N}^3 \mid \dfrac{m}{n} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \,\}$

(ii) $SolI_3(m) = \{\, (a^*, b^*, c^*) \in \mathbb{R}^{+3} \mid n^2 = \dfrac{ma^*b^*c^*}{a^* + b^* + c^*},\ n|a^*b^*,\ n|a^*c^*,\ n|b^*c^* \,\}$

(iii) $SolP_3(m) = \{\, (a, b, c) \in \mathbb{N}^3 \mid n^2 = \dfrac{m\sigma abc}{a+b+c},\ n|ab\sigma,\ n|ac\sigma,\ n|bc\sigma\ \exists \sigma \in \mathbb{N} \,\}$

We are going to establish the bijective relations between these sets in the following.

Proposition 2.6. *Let $E^{-1} : SolI_3(m) \longrightarrow SolE_3(m)$ be a mapping such that $E^{-1}(a^*, b^*, c^*) = \left(\dfrac{a^*b^*}{n}, \dfrac{a^*c^*}{n}, \dfrac{b^*c^*}{n}\right)$ for a fixed $n \in \mathbb{N}$. Then E^{-1} is bijective.*

Proof. We first need to prove that E^{-1} is well-defined. Take $(a^*, b^*, c^*) \in SolI_3(m)$.

The check is directed by setting $x = \dfrac{a^*b^*}{n}, y = \dfrac{a^*c^*}{n}, z = \dfrac{b^*c^*}{n}$.

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{n(a^* + b^* + c^*)}{a^*b^*c^*} = n\left(\frac{m}{n^2}\right) = \frac{m}{n}$$

by the definition of $(a^*, b^*, c^*) \in SolI_3(m)$.

$$\therefore \quad (x, y, z) = \left(\frac{a^*b^*}{n}, \frac{a^*c^*}{n}, \frac{b^*c^*}{n}\right) \in SolE_3(m)$$

(1) To prove the surjectivity of E, observe

$$E^{-1}\left(\sqrt{\frac{nxy}{z}}, \sqrt{\frac{nxz}{y}}, \sqrt{\frac{nyz}{x}}\right) = (x, y, z) \; \forall \; x, y, z \in \mathbb{R}^{+3}$$

Now we need to check that $\forall (x, y, z) \in SolE_3(m)$, $\left(\sqrt{\frac{nxy}{z}}, \sqrt{\frac{nxz}{y}}, \sqrt{\frac{nyz}{x}}\right) \in SolI_3(m)$.

The first condition of $SolE_3(m)$ requires a direct expansion on

$$\begin{aligned} \frac{ma^*b^*c^*}{a^* + b^* + c^*} &= \frac{m\sqrt{\frac{nxy}{z}}\sqrt{\frac{nxz}{y}}\sqrt{\frac{nyz}{x}}}{\sqrt{\frac{nxy}{z}} + \sqrt{\frac{nxz}{y}} + \sqrt{\frac{nyz}{x}}} \\ &= \frac{m\sqrt{\frac{nxyz}{z^2}}\sqrt{\frac{nxyz}{y^2}}\sqrt{\frac{nxyz}{x^2}}}{\sqrt{\frac{nxyz}{z^2}} + \sqrt{\frac{nxyz}{y^2}} + \sqrt{\frac{nxyz}{x^2}}} \\ &= \frac{mn}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} = n^2 \quad \text{by using } \frac{m}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \end{aligned}$$

The divisibility conditions in $SolE_3(m)$ are valid by observing that any product of two terms from $\sqrt{\frac{nxy}{z}}, \sqrt{\frac{nxz}{y}}, \sqrt{\frac{nyz}{x}}$ are integers (by cancellation) and are multiples of n.

Therefore, for all $(x, y, z) \in SolE_3(m)$. By substitution. there exists a^*, b^*, c^* such that

$$(E^{-1}) \qquad E^{-1}(a^*, b^*, c^*) = \left(\frac{a^*b^*}{n}, \frac{a^*c^*}{n}, \frac{b^*c^*}{n}\right) = (x, y, z)$$

Hence, E^{-1} is surjective.

(2) To prove injectivity of E^{-1}, suppose

$$\begin{cases} \frac{a_1^*b_1^*}{n} = \frac{a_2^*b_2^*}{n} \\ \frac{a_1^*c_1^*}{n} = \frac{a_2^*c_2^*}{n} \\ \frac{b_1^*c_1^*}{n} = \frac{b_2^*c_2^*}{n} \end{cases}$$

for some $(a_1^*, b_1^*, c_1^*), (a_2^*, b_2^*, c_2^*) \in SolI_3(m)$. By rearranging terms, we have $\frac{a_1^*}{a_2^*} = \frac{b_2^*}{b_1^*} = \frac{c_1^*}{c_2^*} = \frac{a_2^*}{a_1^*}$, implying that $a_1^{*2} = a_2^{*2}$. Therefore, $a_1^* = a_2^*$ by $a_1^*, a_2^* \in \mathbb{R}^+$. Then, obviously $b_1^* = b_2^*$ and $c_1^* = c_2^*$. Hence E^{-1} is injective.

(1) and (2) implies that E^{-1} is bijective. □

Remark 2. *Hence* $E : SolE_3(m) \longrightarrow SolI_3(m)$ *is also bijective and*

$$E(x, y, z) = \left(\sqrt{\frac{nxy}{z}}, \sqrt{\frac{nxz}{y}}, \sqrt{\frac{nyz}{x}}\right) \tag{E}$$

Lemma 2.1. *If* $\frac{m}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ *for some* $x, y, z \in \mathbb{N}$, *then* $\frac{nxy}{z}, \frac{nxz}{y}, \frac{nyz}{x} \in \mathbb{N}$.

Proof. Consider $mxyz = n(xy + xz + yz)$, by rearranging terms, we have

$$\frac{nxy}{z} = mxy - nx - ny \in \mathbb{N}$$

Similarly, $\frac{nxz}{y}, \frac{nyz}{x} \in \mathbb{N}$. □

Proposition 2.7. *It is given that* $(a^*, b^*, c^*) = E(x, y, z)$ *for some* $(x, y, z) \in \mathbb{N}^3$ *and* $\frac{m}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. *Then* $a^* = a\sqrt{\sigma}, b^* = b\sqrt{\sigma}, c^* = c\sqrt{\sigma}$ *for some* $a, b, c, \sigma \in \mathbb{N}$ *and* σ *is square-free.*

Proof. After mapping (E) on (x, y, z), $a^* = \sqrt{\frac{nxy}{z}}, b^* = \sqrt{\frac{nxz}{y}}$ and $c^* = \sqrt{\frac{nyz}{x}}$.

By Lemma 2.1, $\frac{nxy}{z}, \frac{nxz}{y}, \frac{nyz}{x} \in \mathbb{N}$ and let $\sqrt{\frac{nxy}{z}} = a\sqrt{\sigma_1}, \sqrt{\frac{nxz}{y}} = b\sqrt{\sigma_2}$, $\sqrt{\frac{nyz}{x}} = c\sqrt{\sigma_3}$ such that $\sigma_1, \sigma_2, \sigma_3$ are square-free. We want to show $\sigma_1 = \sigma_2 = \sigma_3$. Consider

$$\frac{\sqrt{\frac{nxy}{z}}}{\sqrt{\frac{nxz}{y}}} = \frac{a\sqrt{\sigma_1}}{b\sqrt{\sigma_2}}$$

$$\frac{\sqrt{\sigma_1}}{\sqrt{\sigma_2}} = \frac{by}{az}$$

Since $y, z, a, b \in \mathbb{N}, \sigma_1, \sigma_2$ are square-free,

$$\frac{\sqrt{\sigma_1}}{\sqrt{\sigma_2}} = \frac{by}{az} \in \mathbb{Q} \quad \text{iff} \quad \frac{\sqrt{\sigma_1}}{\sqrt{\sigma_2}} = 1 \text{ iff } \sigma_1 = \sigma_2$$

$$\therefore \sigma_1 = \sigma_2$$

Similarly, we have $\sigma_2 = \sigma_3$, $\sigma_1 = \sigma_3$, therefore $\sigma_1 = \sigma_2 = \sigma_3$. □

Theorem 2.1 (Three-term Egyptian parametrization formula)**.** *Fixed* $n \in \mathbb{N}$. *Denote*

$\sqrt{\frac{nxy}{z}} = a\sqrt{\sigma}, \sqrt{\frac{nxz}{y}} = b\sqrt{\sigma}, \sqrt{\frac{nyz}{x}} = c\sqrt{\sigma}$ *where* $a, b, c, \sigma \in \mathbb{N}, \sigma$ *is square-free.*

Define another bijective mapping $T : SolI_3(m) \longrightarrow SolP_3(m)$ *such that*

$$T(a\sqrt{\sigma}, b\sqrt{\sigma}, c\sqrt{\sigma}) = (a, b, c)$$

Then $T \circ E : SolE_3(m) \longrightarrow SolP_3(m)$ *is also bijective, and is given by*

$$(EGY_3^+) \qquad T \circ E(x, y, z) = \left(\sqrt{\frac{nxy}{\sigma z}}, \sqrt{\frac{nxz}{\sigma y}}, \sqrt{\frac{nyz}{\sigma x}}\right) = (a, b, c)$$

The inverse bijective mapping $E^{-1} \circ T^{-1} : SolP_3(m) \longrightarrow SolI_3(m)$ *is given by*

$$(EGY_3^-) \qquad E^{-1} \circ T^{-1}(a, b, c) = \left(\frac{ab\sigma}{n}, \frac{ac\sigma}{n}, \frac{bc\sigma}{n}\right) = (x, y, z)$$

Proof. The map T is well-defined because $a, b, c, \sigma \in \mathbb{N}$ are well-defined by Proposition 2.7 and injective by that σ is square-free. T is also bijective by observing

$$n^2 = \frac{m(a\sqrt{\sigma})(b\sqrt{\sigma})(c\sqrt{\sigma})}{(a\sqrt{\sigma}) + (b\sqrt{\sigma}) + (c\sqrt{\sigma})} = \frac{m\sigma abc}{a + b + c}$$

$(a\sqrt{\sigma}, b\sqrt{\sigma}, c\sqrt{\sigma})$ and (a, b, c) satisfy divisibility conditions of $SolI_3(m)$ and $SolP_3(m)$ respectively given that $(a\sqrt{\sigma}, b\sqrt{\sigma}, c\sqrt{\sigma}) \in SolI_3(m)$. Hence T is bijective.

Also, E is bijective by Proposition 2.6.

Combining the above, $T \circ E$ is bijective, an hence the inverse mapping $E^{-1} \circ T^{-1}$. Interested readers may work on the explicit form of T^{-1}. □

Theorem 2.1 tells us that solving $\frac{m}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is equivalent to solving the Diophantine system for unknowns (a, b, c) in

$$(7) \qquad \begin{cases} n^2 = \dfrac{m\sigma abc}{a + b + c} \\ x = \dfrac{\sigma ab}{n}, \ y = \dfrac{\sigma ac}{n}, \ z = \dfrac{\sigma bc}{n} \end{cases} \qquad \text{for some } \sigma \in \mathbb{N}$$

x, y, z are arbitrary until a, b, c are decided. Then (x, y, z) are exactly the triples solving the m-E_3 equation on n. Efforts will be made to solve such a system in Section 3.

Example 2.2. *Correspondence between* (a, b, c) *and* (x, y, z)

We will use an example to show the correspondence between (a, b, c) and (x, y, z). We will fix $m = 1, n = 7$, and find solutions of $\frac{1}{7} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ by computer program. One set of solutions that we obtained is $x = 8, y = 57, z = 3192$. Using the inverse bijective mapping of $E^{-1} \circ T^{-1}$ as mentioned in Theorem 2.1,

$$a = \sqrt{\frac{nxy}{\sigma z}}, \ b = \sqrt{\frac{nxz}{\sigma y}}, \ c = \sqrt{\frac{nyz}{\sigma x}}$$

By substituting x, y, z into the formulas, we will obtain $a = 1, b = 56, c = 399$. σ is the unique square-free integer that ensures a, b, c to be integers. In this case, $\sigma = 1$. Below is a list of (a, b, c, σ) after a part of the solutions is transformed using the same mapping.

σ	x	y	z	a	b	c
1	8	57	3192	1	56	399
2	8	58	1624	1	28	203
3	9	33	693	1	21	77
5	10	25	350	1	14	35
6	12	18	252	1	14	21

Curious readers may see that the variable a maintains 1 and c is always a multiple of 7. These observations eventually contributed to solving (7) in Section 3.

2.3. **General Egyptian parametrization on $\frac{m}{n} = \frac{1}{x_1} + ... + \frac{1}{x_l}$.** Continuing from the last section, we aspire to leverage the insights gained from the successful two and three fractions Egyptian expansions to establish a robust, symmetric parametrization for $\frac{m}{n} = \frac{1}{x_1} + ... + \frac{1}{x_l}$. However, upon investigating a l-dimensional model for a general Egyptian equation, we encountered two phenomena:

(1) Such a geometric model does exist, and the relationships between the inscribed sphere of a l-dimensional simplex and its $(l + 1)$ exspheres are detailed in [10].
(2) Contrarily, we **cannot** derive a parametrization from [10].

The reason why a higher-dimensional model is unamenable to our parametrization goal is far from trivial, leading to a standstill in progress for several months. Eventually, we attempted to hypothesize a parametrization formula for $l = 4$ and we found that by setting

$$a^2 = \frac{nxyz}{w},\ b^2 = \frac{nxyw}{z},\ c^2 = \frac{nxzw}{y},\ d^2 = \frac{nyzw}{x}$$

From a, b, c, d formulae, we change the subjects to x, y, z and w to obtain

$$x^2 = \frac{abc}{nd},\ y^2 = \frac{abd}{nc},\ z^2 = \frac{acd}{nb},\ w^2 = \frac{bcd}{na}$$

We substitute x, y, z, w into the Egyptian fraction equation in 4 terms as follows:

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}$$

$$n^3 = \frac{abcd}{(a+b+c+d)^2} \quad \cdots\cdots (*)$$

Such parametrization exhibits symmetric properties among x, y, z, w and we believe (*) should be the parametrization that we are looking for, despite that (*) is found in algebraic testing and observations. Now we look at the geometric model as shown in the figure below.

The properties in the figure can be linked with a, b, c, and d. These properties may include various aspects like side lengths, areas, or coordinates. However, upon

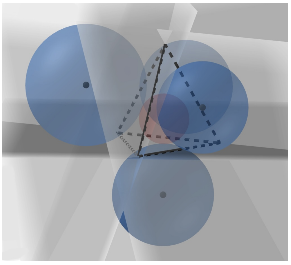

FIGURE 6. Geometric model of $\frac{1}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$, n is the radius of the inscribed sphere and r are radii of exshperes

examining (*), we note that the dimension of n is 3, while the dimension on the right-hand side is $4 - 2 = 2$. Due to this mismatch in dimensions, we posit that integrating a, b, c, and d into the geometric model could be exceptionally complex. Curious readers can utilize the radii information from [10] to explore geometric paths towards unveiling Egyptian parametrization.

Subsequently, after a series of algebraic attempts, we will infer the general form of $\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_l}$ and formally substantiate that our conjecture on (*) indeed presents a correct parametrization.

Definition 2.2. *Given $m, n \in \mathbb{N}$*

(i) $SolE_l(m) = \left\{ (x_1, x_2, ..., x_l) \in \mathbb{N}^l \mid \frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_l} \right\}$

(ii) $SolI_l(m)$

$$= \left\{ (\alpha_1^*, \alpha_2^*, ..., \alpha_l^*) \in \mathbb{R}^{+^l} \mid n^{l-1} = \frac{m^{l-2} \prod_{i=1}^{l} \alpha_i^*}{\left(\sum_{i=1}^{l} \alpha_i^* \right)^{l-2}}, \quad \sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n \alpha_j^{*\,l-2}}} \in \mathbb{N}, j = 1, 2, ..., l \right\}$$

(iii) $SolP_l(m)$

$$= \left\{ (\alpha_1, \alpha_2, ..., \alpha_l) \in \mathbb{N}^l \mid n^{l-1} = \frac{m^{l-2} \sigma \prod_{i=1}^{l} \alpha_i}{\left(\sum_{i=1}^{l} \alpha_i \right)^{l-2}}, \quad \sqrt[l-2]{\frac{\sigma \prod_{i=1}^{l} \alpha_i}{n \alpha_j^{l-2}}} \in \mathbb{N}, j = 1, 2, ..., l \right\},$$

for some $\sigma \in \mathbb{N}$

We will establish the bijective relations between these sets in the following.

Proposition 2.8. *Let $E^{-1} : SolI_l(m) \to SolE_l(m)$ be a mapping such that*

$$E^{-1}(\alpha_1^*, \alpha_2^*, ..., \alpha_l^*) = \left(\sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n \alpha_l^{*\,l-2}}}, \sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n \alpha_{l-1}^{*\ \,l-2}}}, ..., \sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n \alpha_1^{*\,l-2}}} \right)$$ *for a fixed*

$n \in \mathbb{N}$. Then E^{-1} is bijective.

Proof. We first need to prove that E^{-1} is well-defined. Take $(\alpha_1^*, \alpha_2^*, ..., \alpha_l^*) \in SolI_l(m)$. The check is direct by setting $x_k = \sqrt[l-2]{\frac{\prod_{i=1}^{l}\alpha_i^*}{n{\alpha_j^*}^{l-2}}}$, where $j = l-k+1$.

$$
\begin{aligned}
\sum_{k=1}^{l}\frac{1}{x_k} &= \sqrt[l-2]{\frac{n{\alpha_l^*}^{l-2}}{\prod_{i=1}^{l}\alpha_i^*}} + \sqrt[l-2]{\frac{n{\alpha_{l-1}^*}^{l-2}}{\prod_{i=1}^{l}\alpha_i^*}} + ... + \sqrt[l-2]{\frac{n{\alpha_1^*}^{l-2}}{\prod_{i=1}^{l}\alpha_i^*}} \\
&= \alpha_l^*\sqrt[l-2]{\frac{n}{\prod_{i=1}^{l}\alpha_i^*}} + \alpha_{l-1}^*\sqrt[l-2]{\frac{n}{\prod_{i=1}^{l}\alpha_i^*}} + ... + \alpha_1^*\sqrt[l-2]{\frac{n}{\prod_{i=1}^{l}\alpha_i^*}} \\
&= \sum_{i=1}^{l}\alpha_i^*\sqrt[l-2]{\frac{n}{\prod_{i=1}^{l}\alpha_i^*}} = \sqrt[l-2]{n}\times\frac{\sum_{i=1}^{l}\alpha_i^*}{\sqrt[l-2]{\prod_{i=1}^{l}\alpha_i^*}} \\
&= \sqrt[l-2]{n}\times\sqrt[l-2]{\frac{m^{l-2}}{n^{l-1}}} = m\sqrt[l-2]{\frac{n}{n^{l-1}}} = \frac{m}{\sqrt[l-2]{n^{l-2}}} = \frac{m}{n}
\end{aligned}
\tag{8}
$$

by the definition of $(\alpha_1^*, \alpha_2^*, ..., \alpha_l^*) \in SolI_l(m)$.

$$
\therefore \quad (x_1, x_2, ..., x_k) = \left(\sqrt[l-2]{\frac{\prod_{i=1}^{l}\alpha_i^*}{n{\alpha_l^*}^{l-2}}}, \sqrt[l-2]{\frac{\prod_{i=1}^{l}\alpha_i^*}{n{\alpha_{l-1}^*}^{l-2}}}, ..., \sqrt[l-2]{\frac{\prod_{i=1}^{l}\alpha_i^*}{n{\alpha_1^*}^{l-2}}}\right) \in SolE_l(m)
$$

(1) To prove the surjectivity of E, observe

$$
E^{-1}\left(\sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_l^2}}, \sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_{l-1}^2}}, ..., \sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_1^2}}\right) = (x_1, x_2, ..., x_l)\ \forall\ x_1, x_2, ..., x_l \in {\mathbb{R}^+}^l
$$

Now we need to check that $\forall\ (x_1, x_2, ..., x_l) \in SolE_l(m)$,

$$
\left(\sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_l^2}}, \sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_{l-1}^2}}, ..., \sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_1^2}}\right) \in SolI_l(m).
$$

The first condition of $SolE_l(m)$ requires a direct expansion on

$$\frac{m^{l-2}\prod_{i=1}^{l}\alpha_i^*}{\left(\sum_{i=1}^{l}\alpha_i^*\right)^{l-2}} = \frac{m^{l-2}\prod_{j=1}^{l}\sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_j^2}}}{\left(\sum_{j=1}^{l}\sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_j^2}}\right)^{l-2}} = \frac{m^{l-2}\sqrt{\frac{\left(n\prod_{i=1}^{l}x_i\right)^{l}}{\left(\prod_{i=1}^{l}x_i\right)^{2}}}}{\left(\sqrt{n\prod_{i=1}^{l}x_i}\right)^{l-2}\left(\sum_{i=1}^{l}\frac{1}{x_i}\right)^{l-2}}$$

$$= \frac{m^{l-2}\sqrt{n^l\left(\prod_{i=1}^{l}x_i\right)^{l-2}}}{\left(\sqrt{n\prod_{i=1}^{l}x_i}\right)^{l-2}\left(\sum_{i=1}^{l}\frac{1}{x_i}\right)^{l-2}}$$

$$= \frac{m^{l-2}\sqrt{n^2}}{\left(\sum_{i=1}^{l}\frac{1}{x_i}\right)^{l-2}} = \frac{m^{l-2}n}{\left(\sum_{i=1}^{l}\frac{1}{x_i}\right)^{l-2}} = n^{l-1} \text{ by using } \frac{m}{n} = \sum_{i=1}^{l}\frac{1}{x_i}$$

The remaining conditions in $SolE_l(m)$ are valid by letting $\alpha_{l-j+1}^* = \sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_j^2}}$, for $j = 1, 2, ..., l$. Then,

$$\sqrt[l-2]{\frac{\prod_{i=1}^{l}\alpha_i^*}{n\alpha_{l-j+1}^{*\,l-2}}} = \sqrt[l-2]{\frac{\prod_{j=1}^{l}\sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_{l-j+1}^2}}}{n(\alpha_{l-j+1}^*)^{l-2}}} = \sqrt[l=2]{\frac{\prod_{j=1}^{l}\sqrt{\frac{n\prod_{i=1}^{l}x_i}{x_{l-j+1}^2}}}{n\sqrt{\left(\frac{n\prod_{i=1}^{l}x_i}{x_j^2}\right)^{l-2}}}}$$

$$= \sqrt[l-2]{\frac{\sqrt{\frac{n^l\left(\prod_{i=1}^{l}x_i\right)^{l}}{\left(\prod_{j=1}^{l}x_j\right)^{2}}}}{\sqrt{\frac{n^2\cdot n^{l-2}\left(\prod_{i=1}^{l}x_i\right)^{l-2}}{(x_j^2)^{l-2}}}}} = \sqrt[l-2]{\sqrt{\frac{n^l\left(\prod_{i=1}^{l}x_i\right)^{l-2}}{\frac{n^l\left(\prod_{i=1}^{l}x_i\right)^{l-2}}{(x_j^2)^{l-2}}}}} = x_j$$

for $=x_1, x_2, ..., x_l \in {\mathbb{R}^+}^l$

Therefore, for all $(x_1, x_2, ..., x_l) \in SolE_l(m)$. By substitution, there exists $\alpha_1^*, \alpha_2^*, ..., \alpha_l^*$ such that

(E^{-1})

$$E^{-1}(\alpha_1^*, \alpha_2^*, ..., \alpha_l^*) = \left(\sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n{\alpha_l^*}^{l-2}}}, \sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n{\alpha_{l-1}^*}^{l-2}}}, ..., \sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n{\alpha_1^*}^{l-2}}} \right) = (x_1, x_2, ..., x_l)$$

Hence, E^{-1} is surjective.

(2) To prove injectivity of E^{-1}, suppose

$$\begin{cases} \sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n{\alpha_l^*}^{l-2}}} = \sqrt[l-2]{\frac{\prod_{i=1}^{l} {\alpha_i^*}'}{n{{\alpha^*}'_l}^{l-2}}} & \cdots\cdots \quad (I_1) \\ \sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n{\alpha_{l-1}^*}^{l-2}}} = \sqrt[l-2]{\frac{\prod_{i=1}^{l} {\alpha_i^*}'}{n{{\alpha^*}'_{l-1}}^{l-2}}} & \cdots\cdots \quad (I_2) \\ \quad\vdots \qquad\qquad \vdots & \\ \sqrt[l-2]{\frac{\prod_{i=1}^{l} \alpha_i^*}{n{\alpha_1^*}^{l-2}}} = \sqrt[l-2]{\frac{\prod_{i=1}^{l} {\alpha_i^*}'}{n{{\alpha^*}'_1}^{l-2}}} & \cdots\cdots \quad (I_l) \end{cases}$$

for some $(\alpha_1^*, \alpha_2^*, ..., \alpha_l^*)$, $({\alpha_1^*}', {\alpha_2^*}', ..., {\alpha_l^*}') \in SolI_l(m)$. By multiplying (I_1), (I_2), ..., and (I_l) that the denominators in every equation are the same, we have

$$\frac{\prod_{i=1}^{l} \alpha_i^*}{n{\alpha_l^*}^{l-2}} \cdot \frac{\prod_{i=1}^{l} \alpha_i^*}{n{\alpha_{l-1}^*}^{l-2}} \cdots \frac{\prod_{i=1}^{l} \alpha_i^*}{n{\alpha_1^*}^{l-2}} = \frac{\prod_{i=1}^{l} {\alpha_i^*}'}{n{{\alpha^*}'_l}^{l-2}} \cdot \frac{\prod_{i=1}^{l} {\alpha_i^*}'}{n{{\alpha^*}'_{l-1}}^{l-2}} \cdots \frac{\prod_{i=1}^{l} {\alpha_i^*}'}{n{{\alpha^*}'_1}^{l-2}}$$

$$\frac{\left(\prod_{i=1}^{l} \alpha_i^*\right)^l}{\left(\prod_{i=1}^{l} \alpha_i^*\right)^{l-2}} = \frac{\left(\prod_{i=1}^{l} {\alpha_i^*}'\right)^l}{\left(\prod_{i=1}^{l} {\alpha_i^*}'\right)^{l-2}}$$

$$\prod_{i=1}^{l} \alpha_i^* = \prod_{i=1}^{l} {\alpha_i^*}'$$

Substitute the above product in (I_1) and further simplifying, we have $\alpha_l^* = {\alpha_l^*}'$ by $\alpha_l^*, {\alpha_l^*}' \in \mathbb{R}^+$. Iterate from (I_1),(I_2), ..., to (I_l), obviously $\alpha_i^* = {\alpha_i^*}'$ where $i = 1, 2, 3, ..., l$. Hence E^{-1} is injective.

(1) and (2) implies E^{-1} is bijective. □

Remark 3. *Hence* $E : SolE_l(m) \longrightarrow SolI_l(m)$ *is also bijective and*

$$(E) \qquad E(x_1, x_2, ..., x_l) = \left(\sqrt{\frac{n \prod_{i=1}^{l} x_i}{x_l^2}}, \sqrt{\frac{n \prod_{i=1}^{l} x_i}{x_{l-1}^2}}, ..., \sqrt{\frac{n \prod_{i=1}^{l} x_i}{x_1^2}} \right)$$

Lemma 2.2. *If* $\frac{m}{n} = \sum_{i=1}^{l} \frac{1}{x_i}$ *for some* $x_i \in \mathbb{N}$*, then* $\frac{n \prod_{i=1}^{l} x_i}{x_j^2} \in \mathbb{N}$*, for some* $j = 1, 2, ..., l$*.*

Proof. Consider $m \prod_{i=1}^{l} x_i = n \sum_{j=1}^{l} \frac{\prod_{i=1}^{l} x_i}{x_j}$, by rearrange the terms, we have

$$\frac{n \prod_{i=1}^{l} x_i}{x_j} = m \prod_{i=1}^{l} x_i - n \sum_{\substack{k=1 \\ k \neq j}}^{l} \frac{\prod_{i=1}^{l} x_i}{x_k}$$

Dividing both side by x_j, $\frac{n \prod_{i=1}^{l} x_i}{x_j^2} = m \prod_{\substack{i=1 \\ i \neq j}}^{l} x_i - n \sum_{\substack{k=1 \\ k \neq j}}^{l} \frac{\prod_{\substack{i=1 \\ i \neq j}}^{l} x_i}{x_k} \in \mathbb{N}$

□

Proposition 2.9. *It is given that* $(\alpha_1^*, \alpha_2^*, ..., \alpha_l^*) = E(x_1, x_2, ..., x_l)$ *for some* $(x_1, x_2, ..., x_l) \in \mathbb{N}^l$ *and* $\frac{m}{n} = \sum_{i=1}^{l} \frac{1}{x_i}$*. Then,* $\alpha_i^* = \alpha_i \sqrt{\sigma}$ *for some* $\alpha_i, i \in \mathbb{N}$ *and* σ *is square-free.*

Proof. After mapping (E) on $(x_1, x_2, ..., x_l)$, $\alpha_i^* = \sqrt{\frac{n \prod_{i=1}^{l} x_i}{x_j^2}}$, for $j = 1, 2, ..., l$. By Lemma 2.2, $\frac{n \prod_{i=1}^{l} x_i}{x_j^2} \in \mathbb{N}$ and let $\sqrt{\frac{n \prod_{i=1}^{l} x_i}{x_j^2}} = \alpha_j \sqrt{\sigma_j}$ such that σ_j are square-free for $j = 1, 2, ..., l$. We want to show $\sigma_i = \sigma_j$. Consider

$$\frac{\sqrt{\frac{n\prod_{i=1}^{l} x_i}{x_l^2}}}{\sqrt{\frac{n\prod_{i=1}^{l} x_i}{x_{l-1}^2}}} = \frac{\alpha_1\sqrt{\sigma_1}}{\alpha_2\sqrt{\sigma_2}}$$

$$\therefore \quad \frac{\sqrt{\sigma_1}}{\sqrt{\sigma_2}} = \frac{\alpha_2 x_{l-1}}{\alpha_1 x_l}$$

Since $\alpha_1, \alpha_2, x_{l-1}, x_l \in \mathbb{N}, \sigma_1, \sigma_2$ are square-free

$$\frac{\sqrt{\sigma_1}}{\sqrt{\sigma_2}} = \frac{\alpha_2 x_{l-1}}{\alpha_1 x_l} \in \mathbb{Q} \text{ iff } \frac{\sqrt{\sigma_1}}{\sqrt{\sigma_2}} = 1 \text{ iff } \sigma_1 = \sigma_2$$

$$\therefore \sigma_1 = \sigma_2$$

Similarly, we have $\sigma_i = \sigma_j$. □

Theorem 2.2 (General-term Egyptian parametrization formula)**.** *Fixed $n \in \mathbb{N}$. Denote*

$$\sqrt{\frac{n\prod_{i=1}^{l} x_i}{x_j^2}} = \alpha_{l-j+1}\sqrt{\sigma} \text{ where } \alpha_i,\ \sigma \in \mathbb{N},\ \sigma \text{ is square-free, } j = 1, 2, ..., l.$$

Define another bijective mapping $T : SolI_l m \to SolP_l(m)$ such that

$$T(\alpha_1\sqrt{\sigma}, \alpha_2\sqrt{\sigma}, ..., \alpha_l\sqrt{\sigma}) = (\alpha_1, \alpha_2, ..., \alpha_l).$$

Then $T \circ E : SolE_l(m) \to SolP_l(m)$ is also bijective and is given by
(EGY_l^+)

$$T \circ E(x_1, x_2, ..., x_l) = \left(\sqrt{\frac{n\prod_{i=1}^{l} x_i}{\sigma x_l^2}}, \sqrt{\frac{n\prod_{i=1}^{l} x_i}{\sigma x_{l-1}^2}}, ..., \sqrt{\frac{n\prod_{i=1}^{l} x_i}{\sigma x_1^2}}\right) = (\alpha_1, \alpha_2, ..., \alpha_l)$$

The inverse of bijective mapping $E^{-1} \circ T^{-1} : SolP_l(m) \to SolI_l(m)$ is given by

$(EGY_l^-) \quad E^{-1} \circ T^{-1}(\alpha_1, \alpha_2, ..., \alpha_l)$

$$= \left(\sqrt[l-2]{\frac{\sigma\prod_{i=1}^{l} \alpha_i}{n\alpha_l^{l-2}}}, \sqrt[l-2]{\frac{\sigma\prod_{i=1}^{l} \alpha_i}{n\alpha_{l-1}^{l-2}}}, ..., \sqrt[l-2]{\frac{\sigma\prod_{i=1}^{l} \alpha_i}{n\alpha_1^{l-2}}}\right) = (x_1, x_2, ..., x_l)$$

Proof. The map T is well-defined because $\alpha_i \in \mathbb{N}$ are well-defined by Proposition 2.9 and injective by that σ is square-free. T is also bijective by observing

$$n^{l-1} = \frac{m^{l-2}\prod_{i=1}^{l}\alpha_i\sqrt{\sigma}}{\left(\sum_{i=1}^{l}\alpha_i\sqrt{\sigma}\right)^{l-2}} = \frac{m^{l-2}\sigma\prod_{i=1}^{l}\alpha_i}{\left(\sum_{i=1}^{l}\alpha_i\right)^{l-2}}$$

$\alpha_i\sqrt{\sigma}$ and α_i where $i = 1, 2, 3, ..., l$ satisfy divisibility conditions of $SolI_l(m)$ and $SolP_l(m)$ respectively. Given that $(\alpha_1\sqrt{\sigma}, \alpha_2\sqrt{\sigma}, ..., \alpha_l\sqrt{\sigma}) \in SolI_l(m)$. Hence, T is bijective.

Also, E is bijective by Proposition 2.8.

Combining the above, $T \circ E$ is bijective, and hence the inverse mapping $E^{-1} \circ T^{-1}$. □

We finally verify the formula discovered at the beginning of this subsection is correct. Applying $l = 4$ on Theorem 2.2,

$$\begin{cases} n^3 = \dfrac{m^2\sigma abcd}{(a+b+c+d)^2} \\ x = \sqrt{\dfrac{\sigma abc}{nd}},\ y = \sqrt{\dfrac{\sigma abd}{nc}},\ z = \sqrt{\dfrac{\sigma acd}{nb}},\ w = \sqrt{\dfrac{\sigma bcd}{na}} \end{cases} \tag{9}$$

for some $\sigma \in \mathbb{N}$ while variables $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are replaced by a, b, c, d. When a, b, c, d become subjects,

$$a = \sqrt{\frac{nxyz}{\sigma w}},\ b = \sqrt{\frac{nxyw}{\sigma z}},\ c = \sqrt{\frac{nxzw}{\sigma y}},\ d = \sqrt{\frac{nyzw}{\sigma x}} \tag{10}$$

3. Egyptian parametrization and the shortest length of Egyptian expansion

This section unveils new findings concerning the shortest length of Egyptian fraction expansions. To elucidate the intricacies involved in identifying and validating the shortest expansion lengths for specific fractions, we underscore two pivotal points:

(1) The restriction against repeated denominator selections in the expansion $\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_l}$, as stated in [5] P.154, increases complexity to the task of pinpointing fractions with the minimal l length of Egyptian expansion. If denominators are allowed to repeat, fractions with arbitrary l as its shortest length of Egyptian expansion can be answered by $l = \sum_{i=1}^{l} 1$.

(2) While the **upper bound** for the shortest expansion length can be easily determined, say, using a greedy algorithm, the **lower bound** presents a challenge as it necessitates proving the absence of Egyptian equation solutions shorter than this lower bound.

In recent years, no substantial results have been uncovered that allow manipulation of the shortest expansion length for a particular category of fractions ([5] P.155).

However, we have discerned specific fraction classes with verifiable shortest Egyptian expansion lengths of **3, 4, and 5**, employing Egyptian parametrization methodologies. Parametrization is crucial as it facilitates **exhaustion** and proofs of nonexistence for solutions to the Egyptian equation $\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_l}$ under certain conditions.

The proofs presented in this section are primarily **constructive**. The forthcoming theorem, though predicated on a straightforward trick, is pivotal for ensuing arguments. The definition of $SolE_l(m)$ is provided in Definition 2.2.

Theorem 3.1. *Given* $n \in \mathbb{N}$ *and* $\gcd(n, m) = 1$. *If*

(i) $SolE_l(m) \neq \emptyset$ *on* n *and*
(ii) $SolE_l(m+1) = \emptyset$ *on* n *for some* $l \geq 2$,

then the shortest length of Egyptian expansion of $\frac{m+1}{n}$ *is* $l+1$.

Proof. From (i), There exist $x_1, x_2, ..., x_l \in \mathbb{N}$ such that $\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_l}$
$\iff \frac{m+1}{n} = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_l} + \frac{1}{n}$ implying that $SolE_{l+1}(m+1) \neq \emptyset$.

Therefore, the shortest length of Egyptian expansion of $\frac{m+1}{n} \leq l+1$.

From (ii), as $\frac{m+1}{n}$ can't be expressed in $\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_l}$ for all $x_1, x_2, ..., x_l \in \mathbb{N}$.

Hence, the shortest length of expansion for $\frac{m+1}{n} > l$.

Therefore, the shortest length of expansion for $\frac{m+1}{n}$ is $l+1$. □

In general, the size of $SolE_l(m)$ on prime n is usually smaller and easier to exhaust compared to composite n. Therefore, in the following arguments, readers can expect that the special set of fractions has have prime as their denominator.

We will give the special set of fractions achieving 3, 4, and 5 as their shortest lengths of Egyptian expansion respectively in the following theorems.

Theorem 3.2. *For odd prime* $n \geq 5$, $\frac{F+1}{n}$ *has a shortest Egyptian Expansion length 3 if* $F | n+1$.

Proof. Let $n+1 = FT$, F, $T \in \mathbb{N}$. From Proposition 2.2, by setting $n = ab, a \leq b$ where $a^2, b^2 \in \mathbb{N}$. All solutions to the two-term Egyptian equation $\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$ are generated from $\begin{cases} a = 1, \sqrt{n} \\ b = n, \sqrt{n} \end{cases}$ which solves the equation in the form of

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} \quad \text{and} \quad \frac{1}{n} = \frac{1}{2n} + \frac{1}{2n}$$

Notice

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} \iff \frac{F}{n} = \frac{1}{T} + \frac{1}{nT}$$

Hence, $SolE_2(F) \neq \emptyset$ on n.

On the other hand, by $\gcd(F,\, F+1) = 1$, we have $F+1 \nmid n+1$ or $F+1 \nmid 2n$. Therefore, $\dfrac{F+1}{n}$ can't be expressed in the sum of 2 unit fractions $\dfrac{1}{x} + \dfrac{1}{y}$ for all $x, y \in \mathbb{N}$. Hence, $SolE_2(F+1) = \emptyset$ on n.

By Theorem 3.1, $\dfrac{F+1}{n}$ has 3 as the shortest length of Egyptian expansion. ☐

Corollary 3.1. *There are* ***infinitely many*** *fractions having 3 as the shortest length of Egyptian expansion.*

Proof. There are infinitely many primes n, and hence infinitely many F satisfying Theorem 3.2. ☐

For the case of having 4 as the shortest length of Egyptian expansion, we need preparations such that a very narrow choice of the smallest denominator of the Egyptian expansion at the price of restricting the values of m.

Lemma 3.1. *Fixed a prime $n \geq 5$. It is given that $\dfrac{5n}{6} < m < n$ and $SolE_3(m) \neq \emptyset$ on n. Then for any $(x, y, z) \in SolE_3(m)$ on n, $x = 2$.*

Proof. Notice that $\dfrac{m}{n} > \dfrac{\frac{5n}{6}}{n} = \dfrac{5}{6} > \dfrac{1}{4} + \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{3}{4} > \dfrac{1}{4} + \dfrac{1}{y} + \dfrac{1}{z} \quad \forall\, z \geq y \geq 4$

Therefore $x \leq 3$.

When $x = 3$, $\quad \dfrac{m}{n} > \dfrac{5}{6} = \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{4} > \dfrac{1}{3} + \dfrac{1}{y} + \dfrac{1}{z} \quad \forall\, y, z \in \mathbb{N}$ where $z \geq y \geq 4$, implying the only possible choice of y is 3.

If $y = 3$, then $\dfrac{m}{n} = \dfrac{2}{3} + \dfrac{1}{z} = \dfrac{2z+3}{3z}$, where $z|3$.

When $z = 1$, $\dfrac{m}{n} = \dfrac{5}{3}$ where $m = \dfrac{5n}{3} > n$. When $z = 3$, $\dfrac{m}{n} = 1$ where $m = n$.

The above two cases are contradicting to the bound $\dfrac{5n}{6} < m < n$. Therefore, $x = 2$. ☐

Now we are ready to reveal such a special of fraction having 4 as the shortest length of Egyptian expansion.

Theorem 3.3. *Given $n = 6t+1$ is a prime where $t \neq 1$. Then the shortest length of Egyptian expansion for*

$$\frac{5t+2}{6t+1} \quad \textit{is} \quad 4.$$

Proof. Let $m = 5t+1$, where $m \in \mathbb{N}$. Notice that

$$1 > \frac{m}{n} = \frac{5t+1}{6t+1} > \frac{5t}{6t} = \frac{5}{6}$$

By Lemma 3.1, for any $(x, y, z) \in SolE_3(m)$ with $x \leq y \leq z$, $x = 2$. Similarly, $(x, y, z) \in SolE_3(m+1)$ with $x \leq y \leq z$ also implies $x = 2$.

Applying Theorem 2.1, $(x, y, z) \in SolE_3(m) \iff \exists\, (a, b, c) \in \mathbb{N}^3$ such that

$$\text{(11)} \qquad \begin{cases} n^2 = \dfrac{m\sigma abc}{a+b+c} \\ x = \dfrac{\sigma ab}{n}, \quad y = \dfrac{\sigma ac}{n} \quad z = \dfrac{\sigma bc}{n} \end{cases} \text{, for some } \sigma \in \mathbb{N}$$

If $x \leq y \leq z$, then $a \leq b \leq c$ in which $x = 2 \iff ab\sigma = 2n$.

From (11), we have $n^2 = \dfrac{2nmc}{a+b+c}$ and $n = \dfrac{2mc}{a+b+c}$. As $\gcd(m,n) = 1$, then $n|2c$, $n|c$. Denote $c' = \dfrac{c}{n}$.

Notice that $n|\sigma ab$, σac, σbc regardless of the combinations of a, b, σ. Therefore, the necessary and sufficient conditions for (11) to be solvable is

$$n = \frac{2mnc'}{a+b+nc'} \iff c' = \frac{a+b}{2m-n} = \frac{a+b}{2(5t+1)-(6t+1)} = \frac{a+b}{4t+1}$$

The only possible choices of b are n and $2n$ respectively. When $b = 2n = 12t+2$, $c' = 3$, implying $SolE_3(m) \neq \emptyset$.

When m is replaced by $m+1$, the deduction procedure above is the same and $c' = \dfrac{a+b}{2(m+1)-n} = \dfrac{a+b}{4t+3}$. When $b = n$, $1 < \dfrac{a+b}{4t+3} < 2$. When $b = 2n$, $2 < \dfrac{a+b}{4t+3} < 3$.

Hence $c' \notin \mathbb{N}$, contracting to the existence of (x, y, z).

Therefore, $SolE_3(m+1) = \emptyset$ on n.

By Theorem 3.1, $\dfrac{m+1}{n} = \dfrac{5t+2}{6t+1}$ has 4 as the shortest length of Egyptian expansion. □

Readers can write down the explicit form of such expansion as an exercise.

Corollary 3.2. *There are* ***infinitely many*** *fractions having 4 as the shortest length of Egyptian expansion.*

Proof. By Dirichlet's Theorem of primes in arithmetic progression, there are infinitely many primes in the form of $6t+1$, and therefore there are infinitely such n. □

Readers may be curious about did Egyptian parametrization suggests the numerator of the fraction in Theorem 3.3 to be $5t+2$. Unfortunately, we chose $5t+2$ mainly based on the experiment data and the restriction range of m.

The last progress on the shortest length of Egyptian expansion is 5 as the shortest length. Below is a quick preparation for the values of a part of denominators.

Lemma 3.2. *Fix a prime $n \geq 7$. It is given that $M > \dfrac{4n}{3}$ and $SolE_4(M) \neq \emptyset$. Then for any $(x, y, z, w) \in SolE_4(m)$ on n with $x \leq y \leq z \leq w, x \leq 2$.*

Proof. Notice that $\dfrac{M}{n} > \dfrac{4}{3} = \dfrac{1}{3} + \dfrac{1}{3} + \dfrac{1}{3} + \dfrac{1}{3} > \dfrac{1}{3} + \dfrac{1}{y} + \dfrac{1}{z} + \dfrac{1}{w}$ $\forall\, w \geq z \geq y \geq 4$. Therefore, $x \leq 2$. □

Now we are ready for the last result of this section.

Theorem 3.4. *Given $n = 6t+1$ where $t > 1$. Then, the shortest length of Egyptian expansion for*

$$\frac{11t+3}{6t+1} \quad \textit{is } 5.$$

Proof. Let $m = 5t+1$, $M = 11t+2$, where $m,\ M \in \mathbb{N}$. Notice that $\dfrac{M}{n} = \dfrac{11t+2}{6t+1} > \dfrac{8t}{6t} = \dfrac{4}{3}$.

By Lemma 3.2, for any $(x, y, z, w) \in SolE_4(M)$ with $x \leq y \leq z \leq w,\ x \leq 2$. Similarly, $(x, y, z, w) \in SolE_4(M+1)$ with $x \leq y \leq z \leq w$ also implies $x \leq 2$.

Below we will prove the shortest length of Egptyian expansion of $\dfrac{M+1}{n}$ and tackle $\dfrac{M}{n}$ in the last paragraph. Suppose $SolE_4(M+1) \neq \emptyset$ on n.

When $x = 1$, $\dfrac{M+1}{n} = \dfrac{11t+3}{6t+1} = 1 + \dfrac{5t+2}{6t+1} = 1 + \dfrac{1}{y} + \dfrac{1}{z} + \dfrac{1}{w}\ \forall\ w \geq z \geq y \geq 2$.

As proved in Theorem 3.3, the shortest length of Egyptian expansion for $\dfrac{5t+2}{6t+1}$ is 4, which contradicts to $\dfrac{5t+2}{6t+1} = \dfrac{1}{y} + \dfrac{1}{z} + \dfrac{1}{w}$ for some y, z, w. Therefore, $x = 2$.

Now suppose $x = 2$. On the other hand, observe

$$\frac{M}{n} > \frac{4}{3} > \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \geq \frac{1}{2} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} \quad \forall\ w \geq z \geq y \geq 4$$

Therefore, $y \leq 3$.

When $x = y = 2$, $\dfrac{M+1}{n} = 1 + \dfrac{5t+2}{6t+1} = \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{z} + \dfrac{1}{w} = 1 + \dfrac{1}{z} + \dfrac{1}{w}$. Same as above, $\dfrac{5t+2}{6t+1} = \dfrac{1}{z} + \dfrac{1}{w}$ contradicts to the shortest length of $\dfrac{5t+2}{6t+1}$ being 4. Therefore, $y \neq 2$. As $y \leq 3$ and $2 = x \leq y$, $y = 3$.

By applying Theorem 2.2, $(x, y, z, w) \in SolE_4(M+1) \iff \exists (a, b, c, d) \in \mathbb{N}^4$ such that

(12)

$$\begin{cases} n^3 = \dfrac{(M+1)^2 \sigma abcd}{(a+b+c+d)^2} \\ x = \sqrt{\dfrac{\sigma abc}{nd}}, \quad y = \sqrt{\dfrac{\sigma abd}{nc}} \quad z = \sqrt{\dfrac{\sigma acd}{nb}}, \quad w = \sqrt{\dfrac{\sigma bcd}{na}} \qquad \text{for some } \sigma \in \mathbb{N} \end{cases}$$

If $x \leq y \leq z \leq w$, then $a \leq b \leq c \leq d$, and hence $x = 2,\ y = 3$ from the above argument. Multiplying x and y gives $ab\sigma = 6n$. By substituting $\dfrac{ab\sigma}{n} = 6$ into x, we have $\dfrac{c}{d} = \dfrac{2}{3}$. Let $c = 2k, d = 3k$. From (12), we have $n^3 = \dfrac{(M+1)^2 \sigma abcd}{(a+b+c+d)^2}$. By substituting a, b, c, d, σ,

$$\begin{aligned} (M+1)^2 \cdot 6n \cdot 6k^2 &= n^3 (a+b+5k)^2 \\ \iff 6(M+1)k &= 5kn + (a+b)n \\ \iff k &= \frac{(a+b)n}{6(M+1) - 5n} \\ \iff k &= \frac{(a+b)(6t+1)}{6(11t+3) - 5(6t+1)} \\ \iff k &= \frac{(a+b)(6t+1)}{36t+13} \end{aligned}$$

Notice $\gcd(36t+13, 6t+1) = \gcd(7, 6t+1) = 1$ by $t > 1$. Therefore, $k \in \mathbb{N}$ if and only if $36t+13$ divides $a+b$. The maximum value of $a+b$ attains at $a = 1, b = 6n$. Implying $1 + 6(6t+1) < 36t+13$. Hence $k \notin \mathbb{N}$ and $c, d \notin \mathbb{N}$.

$\therefore \quad SolE_4(M+1) = \emptyset$

Now consider $\frac{M}{n}$. Observe that

$$\frac{M}{n} = \frac{n+m}{n} = 1 + \frac{5t+1}{6t+1}$$

By the construction process in Theorem 3.3, we have $6t+1 \in SolE_3(5t+1)$. Therefore, $\frac{M}{n} = 1 + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}$ for some $y, z, w \in \ltimes$ and $SolE_4(M) \neq \emptyset$ on n.

By Theorem 3.1, $\frac{M+1}{n} = \frac{11t+3}{6t+1}$ has 5 as its shortest length of Egyptian expansion. □

Corollary 3.3. *There are* ***infinitely many*** *fractions having 5 as the shortest length of Egyptian expansion.*

Proof. Exactly the same proof as Corollary 3.2. □

Readers may observe that the fractions proposed in Theorem 3.4 differ from those in Theorem 3.3 by only 1. We cannot indefinitely increase the fraction by adding 1 and propose fractions with increasingly longer shortest lengths of Egyptian expansion due to historical constraints. Moreover, a meaningful expansion typically showcases diverse denominators. While incrementing an existing fraction by 1 may seem straightforward, the process of **exhausting** and subsequently **eliminating** the potential existence of other solutions to Egyptian fraction equations of a certain length is not as self-evident as one might initially presume.

We conclude this section by highlighting that Egyptian parametrization serves as an **instrument** to verify whether the set of fractions proposed possesses a consistent shortest length of Egyptian expansion. Readers are urged to investigate and formulate theories to estimate, with relative accuracy, the shortest length of Egyptian expansion. In this regard, Egyptian parametrization proves formidable in providing verifications.

4. Egyptian Parametrization and the Generalized Erdös-Straus Conjecture

This section is dedicated to harnessing the power of Egyptian parametrization as delineated in Theorem 2.1, to shed new light on the Generalized Erdös-Straus Conjecture. This conjecture, notable for its association with Egyptian fractions, remains one of the most captivating puzzles in the field.

4.1. Terminologies and Foundational Results for Parameterized k-E_3 Triple.

In an effort to simplify the study target and given the conjecture's requirement to hold universally for all natural numbers n, we will confine our examination to prime numbers n. It is a logical deduction that if the conjecture associated with Erdös-Straus holds for all prime numbers n, it would consequently hold for all natural numbers.

To ensure swift retrieval and application of equations in our ongoing analysis, we will reiterate the definitions provided in Section 2.2, which will persistently be in use throughout the rest of the paper.

Definition 4.1. *Given that* $n \in \mathbb{N}$.

(i) ***m-E$_l$* equation on *n*** *refers to the Diophantine equation*

$$\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_l},$$

where $l \geq 3$, $\gcd(n, x_1, x_2, \ldots, x_l) = 1$ *and* $\gcd(m, n) = 1$. *m refers to the numerator and l refers to the length of Egyptian expansion. Without loss of generality,* $x_1 \leq x_2 \leq \ldots \leq x_l$. ***m-E$_3$* equation on *n*** *is studied in this section.*

(ii) Let $k = m\sigma$, $(a, b, c) \in \mathbb{N}^3$ *is called an k-**parametrized** triple **on n** if a, b, c satisfies the system of the Diophantine equations and divisibilities*

$$\begin{cases} n^2 = \dfrac{kabc}{a+b+c} \\ n|abk, \quad n|ack \quad, n|bck \end{cases}$$

If $a < b < c$, *we further call* (a, b, c) *an **ascending parametrized** triple on n.*

Depending on whether n divides k, there are two variations of the system.

The **standard type** of triple refers to the case when $\gcd(n, k) = 1$, and the system remains unchanged.

The **degenerated type** of triple refers to the case when $\gcd(k, n) > 1$. Because n is a prime, n divides k. That means the divisibility conditions automatically hold, and the system is reduced to the Diophantine equation $n = \dfrac{k'abc}{a+b+c}$ only with $k' = \dfrac{k}{n}$.

We will see in Theorem 4.1 that these two systems indeed possibly produce different modular identities.

Besides, the identities

$$\frac{1}{n} = \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n}, \quad \frac{2}{n} = \frac{1}{n} + \frac{1}{2n} + \frac{1}{2n}, \quad \frac{3}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{n}$$

are true for every natural number n and hence we are only interested in the case when $k \geq 4$. Differing from Section 3, in the consideration of Diophantine equations only, denominators of m-E_3 equations can be equal. The first proposition verifies on when the ascending triple is well-defined using strict inequalities.

Proposition 4.1. *Given a prime n and an integer* $k \geq 4$ *where* $\gcd(k, n) = 1$. *Then the system*

$$\begin{cases} n^2 = \dfrac{kabc}{a+b+c} \\ n|abk, \quad n|ack \quad, n|bck \end{cases} \tag{13}$$

(i) does not have solutions when $a = b = c$ *or* $a = b < c$.

(ii) has solution when $a < b = c$ if and only if $2n + 1 = kb^{'2}$ or $n + 1 = kb^{'2}$for some $b^{'} \in \mathbb{N}$.

Proof. (i) If $a = b = c$, then

$$k = 3\left(\frac{n}{a}\right)^2 \iff a = 1 \text{ } or \text{ } n$$

$\gcd(n, k) = 1 \iff a = n \Rightarrow k = 3 < 4$, contradicting to $k \geq 4$.

If $a = b < c$, then $n^2 = \dfrac{ka^2c}{2a + c}$. From the divisibility condition of the system,

$$n|a^2k \text{ } and \text{ } \gcd(k, n) = 1 \Rightarrow n|a$$

Let $a = a'n, a' \in \mathbb{N}$. Then

$$n^2 = \frac{ka^2c}{2a + c} \iff 2a'n = c(ka'^2 - 1) > a'n(ka'^2 - 1) \Rightarrow ka'^2 < 3 \Rightarrow k < 3 < 4$$

Therefore, when $a = b = c$ or $a = b < c$ or $a < b = c$, the necessary condition system have solutions is $k < 4$, and hence no solutions when $k \geq 4$.

(ii) If $a < b = c$, then $n \mid bck$, $\gcd(k, n) = 1$ imply $n \mid b$. Denote $b' = \dfrac{b}{n}$. Then (13) becomes

$$\begin{aligned}
n^2 = \frac{kab^2}{a + 2b} &\iff a + 2b^{'}n = kab^{'2} \quad \text{implies } a \mid 2b^{'}n \Rightarrow a \mid 2b^{'} \\
&\iff kb^{'2} - \left(\frac{2b^{'}}{a}\right) n = 1 \\
&\Longrightarrow \gcd(b^{'}, \frac{2b^{'}}{a}) = 1 \\
&\Longrightarrow a = b^{'} \text{ } or \text{ } a = 2b^{'}
\end{aligned}$$

Then $2n + 1 = kb^{'2}$ or $n + 1 = kb^{'2}$. In the reverse direction, a direct checking on setting

$$b = c = n\sqrt{\frac{2n + 1}{k}}, a = \frac{b}{n} \text{ and } b = c = n\sqrt{\frac{n + 1}{k}}, a = \frac{b}{n}$$

satisfy (13).

□

A proposition is essential to reveal the relationship between the m-E_3 equation and ascending parametrized triples.

Proposition 4.2. *m-E_3 equation on n has a solution if and only if (i) there exists a pair of ascending parametrized $m\sigma$-E_3 triple on n for some $\sigma \geq 1$, or (ii) $m \mid n+1$ or $m \mid 2n + 1$.*

Proof. By Theorem 2.1, there exists a bijection between $SolE_3(m)$ and $SolP_3(m)$ (readers can trace back to Definition 2.1) which solves the system

$$\begin{cases} n^2 = \dfrac{m\sigma abc}{a+b+c} \\ x = \dfrac{m\sigma ab}{n}, \quad y = \dfrac{m\sigma ac}{n} \quad , z = \dfrac{m\sigma bc}{n} \end{cases}$$

for some square-free $\sigma \in \mathbb{N}$. Also, $x \le y \le z$ if and only if $a \le b \le c$. Therefore, (a, b, c) is parametrized $m\sigma$-E_3 triple on n. From Proposition 4.1, (a, b, c) can be an ascending parametrized $m\sigma$-E_3 triple on n which is the case(i), or case (ii) where $a < b = c$.

In case (ii), Proposition 4.1 tells us that

$$2n + 1 = m\sigma b'^2 \text{ or } n + 1 = m\sigma b'^2 \tag{14}$$

Notice that σ, b'^2 are arbitrary except that σ is square-free and b'^2 is a square. Then $\sigma b'^2$ is an arbitrary integer and (14) holds if and only if $m \mid n + 1$ or $m \mid 2n + 1$. The reverse direction holds by applying reverse mapping in Theorem 2.1. □

From now on, we can shift our focus to solving m-E_3 equation to find valid ascending parametrized $m\sigma$-E_3 triples on n by introducing a new free variable σ. Proposition 4.3, 4.4 build foundations towards our major results in Section 4.2.

Proposition 4.3. *It is given that (a, b, c) is an ascending parametrized triple on a prime n of standard type, i.e.* $\gcd(n, k) = 1$ *and* $k \ge 4$. *Then the following holds.*

(i) $ab < \dfrac{3n^2}{k}$
(ii) Exactly two of the variables from (a, b, c) is a multiple of n, and n^2 divides none of a, b, c.
(iii) $\gcd(a, n) = 1$.
(iv) $b = b'n, c = c'n$ *for some* $b', c' \in \mathbb{N}$ *with* $\gcd(n, b') = 1$ *and* $\gcd(n, c') = 1$.

Proof. (i) By definition 4.1, we have

$$\frac{1}{n} = \frac{1}{\frac{abk}{n}} + \frac{1}{\frac{ack}{n}} + \frac{1}{\frac{bck}{n}} < \frac{3}{\frac{abk}{n}}$$

where $a < b < c$ implies $ab < \dfrac{3n^2}{k}$.

(ii) Again from definition 4.1, $\gcd(k, n) = 1$ and the following holds.

$$\begin{cases} n^2 = \dfrac{kabc}{a+b+c} \\ n|abk, \quad n|ack \quad , n|bck \end{cases}$$

By applying Euclid's Lemma on prime n, n divides at least two variables from a, b, c.

Suppose n divides both a, b, and c. Then

$$\frac{1}{n} = \frac{1}{\frac{abk}{n}} + \frac{1}{\frac{ack}{n}} + \frac{1}{\frac{bck}{n}} \iff k = \frac{1}{\frac{ab}{n^2}} + \frac{1}{\frac{ac}{n^2}} + \frac{1}{\frac{bc}{n^2}} \le 3 < 4$$

contradicting to $k \geq 4$.

$\therefore$ Exactly two of the variables from (a, b, c) are multiples of n, and the immediate consequence is that $\gcd(n, a, b, c) = 1$.

Also, notice that

$$n^2 = \frac{kabc}{a+b+c} \Longleftrightarrow n^2(a+b+c) = kabc$$

implying the maximum powers of n dividing $kabc$ is 2.

Combining with the proved result that exactly two of the variables from (a, b, c) is a multiple of n, then none of(a, b, c) is a multiple of n^2.

(iii) Assume on the contrary that $\gcd(a, n) > 1$.

By that n is a prime, denote $a = na^{'}$ for some $a^{'} \in \mathbb{N}$.

By proposition 4.3 (i), $ab < \frac{3n^2}{k}$. Combine with $b > a = a'n$,

$$n^2 {a'}^2 = a^2 < ab < \frac{3n^2}{k} \Rightarrow a'^2 < \frac{3}{k} \leq \frac{3}{4} < 1$$

$\therefore a' \notin \mathbb{N}$ and contradiction occurs. Therefore, $\gcd(a, n) = 1$.

(iv) An exact consequence from Proposition 4.3(ii) and 4.3(iii).

□

Proposition 4.4. *Given that n is a prime and a positive integer $m < n$. Denote $b = b'n,\ c = c'n$. Then (a, b, c) is an ascending parametrized triple on a prime n of standard type if and only if $b',\ c' \in \mathbb{N}$, $b' < c'$ and $kab' - n | a + nb'$, or equivalently, $kab' - n | kb'^2 + 1$.*

Proof. $(\Longleftarrow)$: Suppose $kab' - n | a + nb'$ for some $b' \in \mathbb{N}$.

Denote $nc' = \frac{(a + nb')n}{kab' - n}$ where $c' \in \mathbb{N}$. Equivalently,

$$n^2 = \frac{ka(b'n)(c'n)}{a + nb' + nc'}$$

Obviously, $n | ab'n,\ n | ac'n,\ n | (b'n)(c'n)$.

By definition $(a, nb', nc') = (a, b, c)$ is an ascending parametrized triple on n.

$(\Longrightarrow)$: Suppose (a, b, c) is an ascending parametrized triple on a prime n.

By Proposition 4.3(iv), $b',\ c' \in \mathbb{N}$.

Consider $n^2 = \frac{kabc}{a+b+c} = \frac{kn^2 a'b'c'}{a + b'n + c'n}$. By making the subject of c',

$$c' = \frac{(a + nb')n}{kab' - n}$$

$\gcd(kab' - n, n) = \gcd(kab', n) = 1$ because n divides none of a, b',k.

Again by Euclid's Lemma, $kab' - n | a + nb'$.

The equivalence is established by considering

$$\begin{aligned} a + nb' &\equiv 0 \pmod{kab' - n} \\ \Longleftrightarrow a + (kab')b' &\equiv 0 \pmod{kab' - n} \\ \Longleftrightarrow a(kb'^2 + 1) &\equiv 0 \pmod{kab' - n} \end{aligned}$$

$\therefore kab' - n | a + nb' \iff kab' - n | kb'^2 + 1$.

□

4.2. **Congruence generation theorem.** We are now prepared to delineate the precise range of modular identities such that n has a solution to the m-E_3 equation, provided n satisfies one of these modular identities.

Theorem 4.1. *(Congruence generation theorem)*

m-E_3 equation, where $m \geq 4$, on a prime n has at least one solution if and only if n satisfies one of the following:

(i) $\exists F, b', \sigma \in \mathbb{N}$ such that

$$\begin{cases} \sigma m b' \mid Fn+1 \\ F \mid \sigma m b'^2 + 1 \end{cases} \tag{15}$$

if the corresponding $m\sigma$-E_3triple is of ***ascending and of standard type****.*

(ii)

$$n \equiv -ab'^{-1} \pmod{m\sigma ab' - 1} \tag{16}$$

for some $a, \sigma', b' \in \mathbb{N}$ if the corresponding $m\sigma$-E_3 triple is of ***degenerated type****.*

(iii) $m \mid n+1$ or $m \mid 2n+1$ ($m\sigma$-E_3 triple is ***not ascending****).*

Proof. By Proposition 4.2, m-E_3 equation on n has at least one solution if and only if there exists at least one pair of ascending parametrized $m\sigma$-E_3 triple (a, b, c) on n for some $\sigma \geq 1$. Then we split σ into two cases:

(i) $\gcd(n, m\sigma) = 1$, i.e. $m\sigma$-E_3 triple is ascending and of standard type

m-E_3 equation on n has at least one solution

$\Longleftrightarrow$ there exists at least one pair of ascending parametrized $m\sigma$-E_3 triple (a, b, c) with $\gcd(n, m\sigma) = 1$

$\Longleftrightarrow m\sigma ab' - n | m\sigma b'^2 + 1$ where $b' = \dfrac{b}{n} \in \mathbb{N}$ by Proposition 4.4

$\Longleftrightarrow \dfrac{m\sigma b'^2 + 1}{m\sigma ab' - n}$ is a positive integer for some m, σ, a, b$' \in \mathbb{N}$.

($\Longleftarrow$): Suppose $\dfrac{m\sigma b'^2 + 1}{m\sigma ab' - n} = F \in \mathbb{N}$.

By rearranging terms, $Fn + 1 = \sigma F m a b' - \sigma m b'^2$.

$\Longleftrightarrow \sigma m b'(Fa - b') = Fn + 1$

$\Longrightarrow \sigma m b' | Fn + 1$

A second rearrangement on F gives $F(\sigma m a b' - n) = \sigma m b'^2 + 1$.

$\Longrightarrow F | \sigma m b'^2 + 1$.

Combine two conditions,

$$\begin{cases} \sigma m b' \mid Fn+1 \\ F \mid \sigma m b'^2 + 1 \end{cases}$$

($\Longrightarrow$) : Suppose (15) holds for some F, σ,b$' \in \mathbb{N}$

Let $\sigma m b'^2 + 1 = dF$ for some $d \in \mathbb{N}$.

In terms of congruence, $d \equiv F^{-1} (mod\ \sigma m b')$. (Well-defined for F^{-1} by $\gcd(d, \sigma m b^{'}) = 1$)

Consider the first divisibility. $\sigma m b' | Fn + 1 \iff n \equiv -F^{-1} \equiv -d \pmod{\sigma m b'}$.

Therefore, we can let $d = a\sigma mb' - n$ for some $a \in \mathbb{N}$.
$\iff \sigma mb'^2 + 1 = dF = F(a\sigma mb' - n)$
$\iff F = \dfrac{m\sigma b'^2 + 1}{m\sigma ab' - n}$ and $m, \sigma, a, b' \in \mathbb{N}$.
Therefore, $\dfrac{m\sigma b'^2 + 1}{m\sigma ab' - n}$ is a positive integer for some $m, \sigma, a, b' \in \mathbb{N}$ if and only (15) holds for some $F, b', \sigma \in \mathbb{N}$.

(ii) $n|m\sigma$, i.e. $m\sigma$-E_3 triple is of degenerated type
Then m-E_3 equation on n has at least one solution if and only if $n = \dfrac{m\sigma' abc}{a+b+c}$ for some $(a, b, c) \in \mathbb{N}^3$, equivalently,

$$a + b + c = \frac{m\sigma' abc}{n} \tag{17}$$

Notice $\gcd(m\sigma', n) = 1$ and without loss of generality, assume $n|b$.
Let $b = nb'$, $b' \in \mathbb{N}$. (17)holds

$$\iff a + b'n + c = m\sigma' ab'c$$
$$\iff c = \frac{a + nb'}{m\sigma' ab' - 1}$$
$$\iff a + nb' \equiv 0 \pmod{m\sigma' ab' - 1}$$

By $\gcd(b', m\sigma' ab' - 1) = \gcd(b', -1) = 1$ and hence $b'^{-1} \pmod{m\sigma' ab' - 1}$ is well-defined. $\therefore n \equiv -ab'^{-1} \pmod{m\sigma' ab' - 1}$

(iii) $m\sigma$-E_3 triple is **not ascending**
From Proposition 4.2(ii), $m \mid n+1$ or $m \mid 2n+1$.

Combining cases (i), (ii), and (iii), the theorem follows. □

Remark 4. *(i) For composite n, Theorem 4.1 becomes* ***sufficient*** *conditions for n to satisfy m-E_3 equation. Reversely, if any natural number $n > 2$ does not have solutions on m-E_3 equation, then it is* ***necessary*** *for n to escape all congruences produced by Theorem 4.1.*
(ii) case (iii) is two special cases of (i) by setting $b' = \sigma = 1, F = 1$ and $b' = \sigma = 1, F = 2$ respectively. These choices easily satisfy (15). Therefore, case(iii) will be absorbed in case(i) in later discussions.

Readers now can understand the meaning of congruence generation in the following example.

Example 4.1. *(i) Regarding to standard type triples, pick $\sigma = 2, m = 4, b' = 1, F = 3$. Then*

$$\begin{cases} 8|3n+1 \\ 3|8\ \ +1 \end{cases}$$

is true if and only if $3n + 1 \equiv 0 \pmod 8$. In other words, whenever $n \equiv 5 (mod\ 8)$, n has a solution on 4-E_3 equation, and equivalently satisfies Erdös-Straus Conjecture.

(ii) *Regarding to dengenerated type of triples, pick* $a = 1, b' = 2, \sigma = 1, m = 4$. *Then* $n \equiv -1 \times 2^{-1} \equiv 3 \pmod 7$ *is another congruence condition for* n *to satisfies Erdös-Straus Conjecture.*

To compile a comprehensive list of congruence conditions, it is essential to include both standard and degenerate types of parametrized triples. A complete cataloging was previously accomplished in [8], albeit, it necessitated the employment of seven conditions. Remarkably, our parametrization methodology led us to discern that merely two conditions are required.

At the present juncture, we can only pinpoint F, b', σ under limited conditions, such as when constraining the value of $\sigma m b'$. Nonetheless, the interplay between F, b', σ, as well as the method to assure all potential combinations that comply with Theorem 4.1 (i), remains an open question. A comprehensive list of congruencies, arranged in ascending order up to mod 20, is provided in Figure 4.2 (the basic assumption being that n is odd, i.e., $n \equiv 0 \pmod 2$). The values of F, b', σ exhibit an as yet unidentified pattern.

```
n con to  0  mod  2
n con to  2  mod  3
n con to  3  mod  4
n con to  3  mod  7
n con to  5  mod  7
n con to  6  mod  7
n con to  5  mod  8
n con to  7  mod  11
n con to  8  mod  11
n con to  10  mod  11
n con to  7  mod  15
n con to  13  mod  15
n con to  14  mod  19
n con to  15  mod  19
n con to  18  mod  19
n con to  13  mod  20
n con to  17  mod  20
```

FIGURE 7. a complete list of congruencies in ascending order up to mod 20

An immediate consequence of Theorem 4.1 is that we can easily tell that some of the residue classes of $n \pmod m$ are the easiest to guarantee the existence of m-E_3 equation on n.

Corollary 4.1. *If* $n \equiv -F^{-1} \pmod m$ *for some* F *which is a factor of* $m+1$, *then* m-E_3 *equation has a solution on* n.

Proof. Pick $b' = \sigma = 1$. $F^{-1} \pmod m$ is well-defined because $\gcd(F, m) \leq \gcd(m + 1, m) = 1$ where F is a factor of $m + 1$.

By theorem 4.1, we just need to check $\begin{cases} m | Fn + 1 \\ F | m \; + 1 \end{cases}$.

$n \equiv -F^{-1} \pmod m$ is the condition of the first divisibility and F is a factor of $m + 1$ is the condition of the second divisibility. □

For the case of the original Erdös-Straus Conjecture, $m = 4$ and therefore primes n where

$n \equiv -5^{-1} \equiv -1^{-1} \equiv 3$ $(mod\ 4)$ always are the easiest elimination target when we are looking for counterexamples of n. Hence usually, we would assume that $n \equiv 1$ $(mod\ 4)$.

A fundamental inquiry relating to these congruence conditions is whether the modulus of these congruences will extend to infinity and whether each new congruence will encompass some unique n. For degenerated types of parametrized triples (case (ii) in Theorem 4.1), we don't require a specific theorem to prove the existence of infinite congruence conditions. This is because the variables σ' and c' are free, and it's evident that there are infinitely many distinct congruence conditions by iterating σ' and c'. For standard type triples, it's not immediately clear that there are infinite valid combinations stemming from σ' and F. Therefore, we require the following construction. In the sections that follow, we represent the concept of a residue class as $[r]_m = \{a \in \mathbb{N} \mid a \equiv r \pmod{m}\}$.

Proposition 4.5 (Infinity of meaningful congruence conditions on the standard type triples). *Given a fixed $m \geq 4$ and prime $n \in [r]_m$ such that $n \not\equiv -f^{-1}$ $(\text{mod } m)$ for any f which is a factor of $m + 1$. Then there exists infinite subsets $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ and $\{r_0, r_1, r_2, \ldots\}$ of $\mathbb{N}$ such that*

(i) whenever $n \in [r_i]_{m\sigma_i}$ for some $i \geq 0$, m-E_3 equation has at least one solution on n,
(ii) $[r_x]_{m\sigma_x} \not\subseteq [r_y]_{m\sigma_y}$ for any distinct $x, y \geq 0$.

In other words, there exist infinitely many congruence conditions for n to have a solution on m-E_3 equation and every congruence condition covers some unique prime n.

Proof. The proof is based on construction using Theorem 4.1(i) only. The key is to strategically choose σ_i and F to fulfill the necessary and sufficient condition (15).

Notice that if $[-f^{-1}]_m$, where f which is a factor of $m+1$, forms a reduced residue class mod m, then from Corollary 4.1 m-E_3 equation has a solution on every prime n. Therefore, we assume $[r]_m$ such that $r \not\equiv -f^{-1}$ $(mod\ m)$ exists.

From Dirichlet's theorem of arithmetic progression, there exist infinitely many prime F such that $-F^{-1} \not\equiv -f^{-1}$ $(mod\ m)$ for any f which is a factor of $m+1$.

Consider the linear Diophantine equation $m\sigma - Fk = 1$. $\gcd(m, F) = 1$ implies that the equation has infinitely many pairs of solutions (σ, k). Among all possible σ, there are infinitely many of them being primes by using Dirichlet's theorem on arithmetic progression again on σ. Now we are ready for the construction.

Construction procedure of $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ and $\{r_0, r_1, r_2, \ldots\}$

(1) Take $b' = 1$.
(2) Pick prime σ_0 such that $F | m\sigma_0 + 1$ where F is a prime and $-F^{-1} \not\equiv -f^{-1}$ $(mod\ m)$. (For existence, please see the argument above)
(3) Define σ_i to be the i^{th} prime after σ_0 in the arithmetic progression $\sigma_0, \sigma_0 + F,\ \sigma_0 + 2F, \ldots$.
(4) r_i is the residue where $r_i \equiv -F^{-1}$ $(mod\ \sigma_i m)$, $i \geq 0$.

In short, b', F are fixed, but σ varies to form different congruence conditions.

Verification

(i) From step 2, $F|m\sigma_0 + 1$ implies $F|m\sigma_i + 1 \ \forall i \geq 1$ because $F|m(\sigma_i - \sigma_0)$.
Also, $\forall i \geq 1$, $\gcd(m\sigma_i, m\sigma_i + 1) = 1$.
$\Rightarrow \forall i \geq 1$, $\gcd(m\sigma_i, F) = 1$
$\Rightarrow -F^{-1}(mod\ \sigma_i m)$ is well-defined and $[r_i]_{\sigma_i m} = [-F^{-1}]_{\sigma_i m}$ forms a residue class.
Whenever $n \in [r_j]_{m\sigma_j}$ for some $j \geq 0$, $n \equiv r_j \equiv -F^{-1} \pmod{\sigma_j m} \iff Fn + 1 \equiv 0 \pmod{\sigma_j m}$.
$F|m\sigma_j + 1$ by construction.
By Theorem 4.1, m-E_3 equation has at least one solution on n.

(ii) Notice $n \in [r]_m$ by definition.
Suppose $[r_x]_{m\sigma_x}, [r_y]_{m\sigma_y}$ are distinct residue classes satisfying the construction in (i).
σ_x, σ_y are distinct primes. By Chinese remainder theorem, $[r_x]_{m\sigma_x} \cap [r_y]_{m\sigma_y}$ is a unique residue class mod $m\sigma_x\sigma_y$ and $[r_x]_{m\sigma_x} \cap [r_y]_{m\sigma_y} \subsetneq [r_x]_{m\sigma_x}$ and $[r_y]_{m\sigma_y}$.
Therefore, $[r_x]_{m\sigma_x} \not\subseteq [r_y]_{m\sigma_y}$.

□

Example 4.2. *Consider $m = 4$ and $n \equiv 1\ (mod\ 4)$. Pick $b' = 1$ and $\sigma_0 = 2, F = 3$. Such a choice is valid because $3 \mid 4 \times 2 + 1$. Below are the first 4 congruences from the infinite sequence generated by such selection:*
$n \equiv -3^{-1} \equiv 5 \pmod 8$, $n \equiv -3^{-1} \equiv 13 \pmod{20}$,
$n \equiv -3^{-1} \equiv 29 \pmod{44}$, $n \equiv -3^{-1} \equiv 45 \pmod{68}$, ...

If n satisfies any of these congruences, then n has a solution on the 4-E_3 equation.
The modulus is found by picking primes along the arithmetic progression $4 \times 2, 4(2 + 3), 4(2 + 2 \times 3), \ldots$

However, readers can see that many congruencies are missing by referring to Figure 4.2 if we follow the path of congruence generation using such narrow choice of b', σ_0 and F.

On the other hand, one may be interested in common features of all congruencies. Surprisingly, Mordell has proved that for $m = 4$, all the modular identities generated do not contain any square. Here we show a second version of such proof using 4.1 and quadratic reciprocity to show an alignment with existing results in the field of Egyptian fractions.

Proposition 4.6 (properties of congruence conditions)**.** *Consider two types of congruence conditions generated by Theorem 4.1.*

(1) $-F^{-1} \pmod{4\sigma b'}$ is a quadratic nonresidue for any σ, F, b' which satisfies Theorem 4.1(i).
(2) $-ab'^{-1} \pmod{4\sigma ab' - 1}$ is a quadratic nonresidue for any $\sigma, a, b' \in \mathbb{N}$ which satisfies Theorem 4.1(ii).

Simple consequences from the above are **square numbers do not** satisfy any congruence conditions generated by Theorem 4.1 when $m = 4$.

Proof. Consider $[-F^{-1}]_{4\sigma b'}$ satisfying Theorem 4.1(i).

Recall that $\sigma, b', F \in \mathbb{N}$ satisfy $\begin{cases} 4\sigma b' | Fn+1 \\ F | 4\sigma b'^2 \ +1 \end{cases}$.

If $n \equiv -F^{-1} \ \equiv 3(mod\ 4)$ for some F, then n is a nonresidue mod 4 and therefore nonresidue mod $4\sigma b'$. Hence we proceed by assuming $n \equiv -F^{-1} \equiv 1\ (mod\ 4)$ and $F \equiv 3\ (mod\ 4)$.

Let $b' = 2^\alpha b^*$ and $\sigma = 2^\beta \sigma^*$ such that $b^*$and σ^* are odd numbers, where $\alpha, \beta \geq 0$.

Suppose $[-F^{-1}]_{4\sigma b'}$ is a quadratic nonresidue mod $4\sigma b'$.

Then $Fn+1 \equiv 0 \pmod{2^{\alpha+\beta+2}}$ and $[-F^{-1}]_{4\sigma b'}$ is also a quadratic nonresidue mod σ^*.

$\implies \left(\frac{-F^{-1}}{\sigma^*}\right) = \left(\frac{-F}{\sigma^*}\right) = -1$ in Jacobi Symbol.

We want to show contradiction occurs on $\left(\frac{-F}{\sigma^*}\right) = 1$.

In the second division, $\sigma(2b')^2 + 1 \equiv 0 \pmod{F} \iff (2b')^2 \equiv -\sigma^{-1} \pmod{F}$

$\implies \left(\frac{-\sigma^{-1}}{F}\right) = \left(\frac{-\sigma}{F}\right) = \left(\frac{-2^\beta\sigma^*}{F}\right) = 1 \implies \left(\frac{\sigma^*}{F}\right) = \left(\frac{-2^\beta}{F}\right) = (-1)^{\frac{F-1}{2}}\left(\frac{2^\beta}{F}\right)$

By quadratic reciprocity,

$$\left(\frac{\sigma^*}{F}\right)\left(\frac{F}{\sigma^*}\right) = (-1)^{\frac{(F-1)}{2}\frac{(\sigma^*-1)}{2}} \iff \left(\frac{F}{\sigma^*}\right) = \left(\frac{\sigma^*}{F}\right)(-1)^{\frac{(F-1)}{2}\frac{(\sigma^*-1)}{2}}$$

Then

$$\begin{aligned}
\left(\frac{-F}{\sigma^*}\right) &= (-1)^{\frac{\sigma^*-1}{2}}\left(\frac{F}{\sigma^*}\right) \\
&= (\frac{\sigma^*}{F})(-1)^{\frac{\sigma^*-1}{2}}(-1)^{\frac{(F-1)}{2}\frac{(\sigma^*-1)}{2}} \\
&= (-1)^{\frac{\sigma^*-1}{2}}(-1)^{\frac{(F-1)}{2}\frac{(\sigma^*-1)}{2}}(-1)^{\frac{F-1}{2}}\left(\frac{2^\beta}{F}\right) \\
&= (-1)^{\frac{(F+1)}{2}\frac{(\sigma^*+1)}{2}-1}\left(\frac{2^\beta}{F}\right) \\
&= \begin{cases} (-1) \text{ if } \beta \equiv 0 \pmod{2} \\ (-1)(-1)^{\frac{F^2-1}{8}} \text{ if } \beta \equiv 1 \pmod{2} \end{cases} \quad \text{by that } F \equiv 3 \pmod{4}
\end{aligned}$$

Consider $Fn+1 \equiv 0(mod\ 2^{\alpha+\beta+2})$. When $\beta \equiv 1 \pmod{2}$, $2^{\alpha+\beta+2} \geq 8$.

When $F \equiv 7\ (mod\ 8)$, $(-1)(-1)^{\frac{F^2-1}{8}} = -1$.

When $F \equiv 3\ (mod\ 8)$, $Fn+1 \equiv 0 \pmod{8}$ gives $n \equiv -F^{-1} \equiv 5\ (mod\ 8)$, and 5 is a nonresidue mod 8. Hence $-F^{-1}$ is a nonresidue mod $4\sigma b'$. Combining above, $\left(\frac{-F}{\sigma^*}\right) = -1$ which contradicts to $\left(\frac{-F}{\sigma^*}\right) = 1$.

$[-F^{-1}]_{4\sigma b'}$ is quadratic nonresidue mod $4\sigma b'$ for any $\sigma, b', F \in \mathbb{N}$.

Consider $[-ab'^{-1}]_{4'-1}$ which satisfies Theorem 4.1(ii).

Let $ab' = 2^q a_0 b_0$ such that a_0, b_0 are odd.

Then $\left(\frac{-ab'^{-1}}{4\sigma ab'-1}\right) = \left(\frac{-ab'}{4\sigma ab'-1}\right)$

$= \left(\frac{2^q}{2^{q+2}\sigma a_0 b_0 - 1}\right)\left(\frac{a_0 b_0}{2^{q+2}\sigma_0 b_0 - 1}\right)\left(\frac{-1}{2^{q+2}\sigma_0 b_0 - 1}\right)$

in Jacobi symbol.

$\left(\frac{-1}{2^{q+2}\sigma' a_0 b_0 - 1}\right) = -1$ because $2^{q+2}\sigma' a_0 b_0 - 1 \equiv 3 \ (mod\ 4)$

By quadratic reciprocity,

$$\left(\frac{a_0 b_0}{2^{q+2}\sigma a_0 b_0 - 1}\right)\left(\frac{2^{q+2}\sigma a_0 b_0 - 1}{a_0 b_0}\right) = (-1)^{\frac{a_0 b_0 - 1}{2}\frac{2^{q+2}\sigma a_0 b_0 - 2}{2}} = (-1)^{\frac{a_0 b_0 - 1}{2}}$$

$$\iff \left(\frac{a_0 b_0}{2^{q+2}\sigma' a_0 b_0 - 1}\right)\left(\frac{-1}{a_0 b_0}\right) = \left(\frac{-1}{a_0 b_0}\right)$$

$$\iff \left(\frac{a_0 b_0}{2^{q+2}\sigma' a_0 b_0 - 1}\right) = 1$$

$$\left(\frac{2^q}{2^{q+2}\sigma' a_0 b_0 - 1}\right) = \begin{cases} 1 \ when \ q \equiv 0 \ (mod\ 2) \\ \left(\frac{2}{2^{q+2}\sigma' a_0 b_0 - 1}\right) \ when \ q \equiv 1 \ (mod\ 2) \end{cases}$$

Notice $2^{q+2}\sigma' a_0 b_0 - 1 \equiv 7 \ (mod\ 8)$ when $q \geq 1 \Longrightarrow \left(\frac{2}{2^{q+2}\sigma' a_0 b_0 - 1}\right) = 1.$

Combine the above results, we have $\left(\frac{-ab'^{-1}}{4\sigma' ab'^{-1}}\right) = (1)(1)(-1) = -1.$

$\therefore [-ab'^{-1}]_{4\sigma ab'-1}$ is a quadratic nonresidue mod $4\sigma' ab' - 1$ for any $\sigma, a, b' \in \mathbb{N}$. ☐

Notice that all escaping numbers are suspected to be composite numbers, and hence these numbers are not counted as counterexamples to Erdös-Straus Conjecture . For computational approaches and the searching algorithm for solutions, please see Appendix.

4.3. **Natural density and Prime density of potential counterexamples to generalized Erdös-Straus Conjecture.** In this subsection, our objective is to derive results concerning the density of potential counterexamples to the generalized Erdös-Straus Conjecture. It's noteworthy that our examination is centered on prime n, which makes the use of natural density unsuitable for gauging the frequency of potential counterexamples. Instead of this, our evaluation of potential counterexamples of prime numbers must be about the overall count of prime numbers within specified limits. Employing advanced results from analytic number theory, we introduce Theorem 4.6 as a conclusive finding, asserting that regardless of the m value, the prime density of potential counterexamples to the m-E_3 equation is zero.

Below are commonly used density-related notations in Number theory.

Definition 4.2. *Given a subset a of natural numbers. $|S|$ refers to the cardinality of a finite set S.*

(i) The ***Natural density*** *$D_{\mathbb{N}}(A)$ is defined as*

$$\lim_{n\to\infty} \frac{|A \cap \{1, 2, ..., n\}|}{n}$$

*(ii) The **upper density** $U_{\mathbb{N}}(A)$ is defined as*

$$\lim_{n\to\infty}\sup\frac{|A\cap\{1,2,...,n\}|}{n}$$

*(iii) The **prime density** $D_{\mathbb{P}}(A)$ is defined as*

$$\lim_{n\to\infty}\frac{|A\cap\{1,2,...,n\}|}{\pi(n)} = \lim_{n\to\infty}\frac{(\ln n)|A\cap\{1,2,...,n\}|}{n}$$

where $\pi(n)$ stands for the number of prime numbers less than or equal to n. The equality comes from the famous Prime number theorem.

Obviously, $D_{\mathbb{N}}(A)\le U_{\mathbb{N}}(A)$ and $0\le D_{\mathbb{N}}(A),\ U_{\mathbb{N}}(A),\ D_{\mathbb{P}}(B)\le 1$ for all $A\subseteq\mathbb{N}$ and subset B of prime numbers. Besides, Dirichlet density is considered a better replacement for Prime density. However, our referenced theorems did not use Dirichlet density explicitly. Therefore, to complete Theorem 4.6, we finish the proof using prime density.

Below are theorems referenced from an additive and analytical number theory.

Theorem 4.2 (Szemerédi's theorem)**.** *Let a be a subset of $\mathbb{N}$ where $U_{\mathbb{N}}(A)>0$. Then for every integer $k\ge 1$, $\exists a,r\in\mathbb{N}$ such that $a,\ a+r,\ a+2r,\ \dots,\ a+(k-1)r$ forms a arithmetic progression of length k.*

Proof. See [1]. □

Theorem 4.3 (Prime number theorem for arithmetic progression)**.** *Let $\pi(x,q)$ denote the number of all primes p no greater than x, congruent to a mod q, for $a,q\in\mathbb{N}$ such that $\gcd(a,q)=1$. Then $\pi(x,q)\sim\frac{1}{\phi(q)}\frac{x}{lnx}$ where $\phi(q)$ is the Euler totient function.*

Proof. See [9]. □

The following Corollary secures the density behavior of primes in multiple residue classes.

Corollary 4.2. *Denote $\pi(x,q,\{a_1,a_2,\dots,a_k\})$ to be the number of all primes no greater than x and congruent to a_i (mod q) for some $i\in\{1,2,\dots,k\}$, where $\{a_1,a_2,\dots,a_k\}$ is a subset of the reduced residue system mod q.*
Then $\pi(x,q,\{a_1,a_2,\dots,a_k\})\sim\frac{k}{\phi(q)}\frac{x}{lnx}$.

Theorem 4.4 (Merten's formula for arithmetic progression)**.** *Define*

$$P(x;q,a)=\prod_{p\le x,p\equiv a\ modq}\left(1-\frac{1}{p}\right)$$

Then

$$P(x;q,a)=\frac{C(q,a)}{(\ln x)^{\frac{1}{\phi(q)}}}+\mathcal{O}\left(\frac{1}{(\ln x)^{\frac{1}{\phi(q)}+1}}\right) \tag{18}$$

where $C(q,a)$ *is a constant such that* $C(q,a)^{\phi(q)} = e^{-\gamma}\prod_p \left(1-\frac{1}{p}\right)^{\alpha(p;q,a)}$ *where* $\alpha(p;q,a) = \phi(q)-1$ *if* $p \equiv a \pmod q$ *and* $\alpha(p;q,a) = -1$ *otherwise.*

Proof. See [6]. □

Because of having $\ln x$ in the denominator of the asymptotic formula of $P(x;q,a)$, we can conclude the following.

Corollary 4.3.

$$\lim_{x\to\infty} P(x;q,a) = 0.$$

Now we are ready to prove results regarding natural density and prime density respectively. To keep the expression short, we use the following notation.

Denote the set of counterexamples of m-E_3 equation as follows (Eventually escaping set can be empty).

Definition 4.3. *Denote the following sets:*

(i) $DE_3(m)^c = \{\, n \in \mathbb{N} \mid \frac{m}{n} \neq \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ \forall x,y,z \in \mathbb{N}\,\}$

(ii) $DP_3(m)^c = \left\{ prime\ p \middle| \frac{m}{p} \neq \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ \forall x,y,z \in \mathbb{N}\right\}$

The letter "D" stands for the consideration of the denominator n. Clearly, $D\mathbb{P}_3(m)^c \subset DE_3(m)^c\ \forall m \in \mathbb{N}$ and $DE_3(1)^c = DE_3(2)^c = DE_3(3)^c = \phi$.

In the following theorems 4.5 and 4.6, for the sake of simplicity, we only use congruence conditions generated by Theorem 4.1(ii), i.e. those generated by the standard type triples.

Theorem 4.5. $D_{\mathbb{N}}(DE_3(m)^c) = 0$ *for all integers* $m \geq 4$.

Proof. Let C_m be the collection of all congruences generated by $\begin{cases} \sigma m b' | Fn+1 \\ F | \sigma m b' \ +1 \end{cases}$.

From Remark 4, no matter n is prime or composite, a necessary condition for $n \in DE_3(m)^c$ is that n does not satisfy any congruence in C_m.

Suppose for an arbitrary large k, there exists $\{a, a+r, a+2r, .., a+(k-1)r\} \subset DE_3(m)^c$.

From Proposition 4.5, by the infinite possible choices of σ, we can always pick $[-F^{-1}]_{m\sigma} \in C_m$ such that $m\sigma < k$ and $\gcd(a, m\sigma) = 1$. Then

$$a, a+r, a+2r, .., a+(k-1)r$$

forms a reduced residue system mod $m\sigma$ and $\exists\ i \in \{0,1,2,\ldots,k-1\}$ such that $a + ir \equiv -F^{-1} (mod\ m\sigma)$, contradicting to $a+ir$ does not satisfy any congruence in C_m.

$\therefore$ $DE_3(m)^c$ does not contain arithmetic progressions of length k.

From Szemerédi's Theorem and its double negation,

$D_{\mathbb{N}}(DE_3(m)^c) \leq U_{\mathbb{N}}(DE_3(m)^c) = 0 \implies D_{\mathbb{N}}(DE_3(m)^c) = 0.$ □

The proof of Theorem 4.5 is rather short because natural density is easier to handle than prime density.

The next Lemma and theorem deals with our concerned prime density behaviors of potential counterexamples of primes n on m-E_3 equation.

Lemma 4.1. *Let $S_1, S_2, \dots, S_k$ be a finite number of disjoint subsets of prime numbers such that $D_{\mathbb{P}}(S_i) = 0$ for $i = 1, 2, \dots, k$. Then $D_{\mathbb{P}}(S_1 \cup S_2 \cup \dots S_k) = 0$.*

Proof. By definition of prime density,

$$\begin{aligned} &D_{\mathbb{P}}(S_1 \cup S_2 \cup \dots S_k) \\ &= \lim_{n\to\infty} \frac{|S_1 \cup S_2 \cup \dots S_k \cap \{1,2,\dots,n\}|}{\pi(n)} \\ &= \lim_{n\to\infty} \frac{|S_1 \cap \{1,2,\dots,n\}|}{\pi(n)} + \frac{|S_2 \cap \{1,2,\dots,n\}|}{\pi(n)} + \dots \frac{|S_k \cap \{1,2,\dots,n\}|}{\pi(n)} \end{aligned}$$

Notice that k is finite and

$$D_{\mathbb{P}}(S_i) = \lim_{n\to\infty} \frac{|S_i \cap \{1,2,\dots,n\}|}{\pi(n)} = 0$$

for $i = 1, 2, \dots, k$.

$\therefore D_{\mathbb{P}}(S_1 \cup S_2 \cup \dots S_k) = 0 + 0 + \dots + 0 = 0.$ □

Theorem 4.6. $D_{\mathbb{P}}(DP_3(m)^c) = 0 \quad \forall m \in \mathbb{N}.$

Proof. The density argument comes from an explicit construction using proposition 4.5 and split prime n according to residue classes mod m. Notice we only pick congruencies from standard-type parametrized triple. Readers may reconstruct the proof and improve the asymptotic density by using congruencies generated from both standard-type and degenerated-type parametrized triple.

From Corollary 4.1, if If $n \equiv -F^{-1}$ $(mod\ m)$ for some F which is a factor of $m+1$, then m-E_3 equation has a solution on n.

Hence $DP_3(m)^c = \phi$ for $n \equiv -F^{-1}$ $(mod\ m)$, $F \mid m+1$.

We then consider the remaining classes, namely $[r_1]_m, [r_2]_m, \dots, [r_t]_m$, where $t < m$.

Using Proposition 4.5, construct t separate infinite sequences of congruences on separate cases of n belonging to t different residue classes. For a clear visualization, please see below:

$$(CS_1) \qquad \left(n \equiv r_1 \pmod m \textbf{ and } \begin{cases} n & \equiv -F_1^{-1}(mod\ b_{1,1}) \textbf{ or} \\ n & \equiv -F_1^{-1}(mod\ b_{1,2}) \textbf{ or} \\ & \vdots \end{cases} \right)$$

$$(CS_2) \qquad \left(n \equiv r_2 \pmod m \textbf{ and } \begin{cases} n & \equiv -F_2^{-1}(mod\ b_{2,1}) \textbf{ or} \\ n & \equiv -F_2^{-1}(mod\ b_{2,2}) \textbf{ or} \\ & \vdots \end{cases} \right)$$

$$\vdots$$

$$(CS_t) \qquad \left(n \equiv r_t \pmod{m} \textbf{ and } \begin{cases} n & \equiv -F_t^{-1}(mod\ b_{t,1}) \textbf{ or} \\ n & \equiv -F_t^{-1}(mod\ b_{t,2}) \textbf{ or} \\ & \vdots \end{cases} \right)$$

where $b_{i,j}, F_i$ are primes and$b_{i,j} \equiv b_{i,1}(mod\ F_i)\ \forall i = 1, 2 \ldots, t$ and $j \geq 1$.

If n satisfies any one of the compound conditions of congruencies above, for example,

$$n \equiv r_1 \pmod{m} \quad \text{and} \quad n \equiv -F_1^{-1} \pmod{b_{1,2}}$$

then from Theorem 4.1(i), m-E_3 equation has a solution on n. Now we define the complement of (CS) and difine each sequence:

$$(ES_1) \qquad \left(n \equiv r_1 \pmod{m} \textbf{ and } \begin{cases} n & \not\equiv 0, -F_t^{-1}(mod\ b_{t,1}) \textbf{ and} \\ n & \not\equiv 0, -F_t^{-1}(mod\ b_{t,2}) \textbf{ and} \\ & \vdots \end{cases} \right)$$

$$(ES_2) \qquad \left(n \equiv r_2 \pmod{m} \textbf{ and } \begin{cases} n & \not\equiv 0, -F_2^{-1}(mod\ b_{2,1}) \textbf{ and} \\ n & \not\equiv 0, -F_2^{-1}(mod\ b_{2,2}) \textbf{ and} \\ & \vdots \end{cases} \right)$$

$$\vdots$$

$$(ES_t) \qquad \left(n \equiv r_t \pmod{m} \textbf{ and } \begin{cases} n & \not\equiv 0, -F_t^{-1}(mod\ b_{t,1}) \textbf{ and} \\ n & \not\equiv 0, -F_t^{-1}(mod\ b_{t,2}) \textbf{ and} \\ & \vdots \end{cases} \right)$$

Notice that 0 is added in all (ES) because n is a prime. For any n, n satisfies either (CS_q) or (ES_q) for some $q \geq 0$.

If n satisfies (ES_i) for some $i \in \{1, 2, \ldots, t\}$, as (ES) are the complement of (CS), n is a potential prime counter-example to m-E_3 equation.

Let $S_i = \{prime\ n \mid n \equiv r_i(mod\ m)\ and\ n\ satisfies\ (ES_i)\}$.

We now verify that $D_{\mathbb{P}}(S_i) = 0$ for $i = 1,\ 2\ ,\ \ldots,\ t$.

Consider the first k congruences in (ES_i). Because $b_{i,1}, b_{i,2}, \ldots, b_{i,k}$ are primes and by Chinese remainder theorem, n can congruent to $(b_{i,1} - 2)(b_{i,2} - 2) \ldots (b_{i,k} - 2)$ residue classes mod $mb_{i,1}b_{i,2}, \ldots b_{i,k}$.

The number of primes less than or equal to x that are **not** congruent to $0, -F_i^{-1}$ (mod $b_{i,1}, b_{i,2}, \ldots, b_{i,k}$)

$$
\begin{aligned}
&= \pi(x, mb_{i,1}b_{i,2}\dots b_{i,k}, \{r \mid \ r \not\equiv 0, -F_i^{-1} \pmod{b_{i,j}} \ \forall j \in \{1,2,\dots,k\}\}) \\
&\sim \frac{(b_{i,1}-2)(b_{i,2}-2)\dots(b_{i,k}-2)}{\phi(mb_{i,1}b_{i,2}\dots b_{i,k})}\frac{x}{lnx} \text{ by Corollary 4.2} \\
&= \frac{1}{m}\frac{(b_{i,1}-2)(b_{i,2}-2)\dots(b_{i,k}-2)}{(b_{i,1}-1)(b_{i,2}-1)\dots(b_{i,k}-1)}\frac{x}{lnx} \text{ by } b_{i,1}, b_{i,2}, \dots, b_{i,k} \text{ are primes} \\
&= \frac{1}{m}\left(1-\frac{1}{b_{i,1}-1}\right)\left(1-\frac{1}{b_{i,2}-1}\right)\dots\left(1-\frac{1}{b_{i,k}-1}\right)\frac{x}{lnx} \\
&\leq \frac{1}{m}\left(1-\frac{1}{b_{i,1}}\right)\left(1-\frac{1}{b_{i,2}}\right)\dots\left(1-\frac{1}{b_{i,k}}\right)\frac{x}{lnx}
\end{aligned}
$$

$$
\begin{aligned}
&D_{\mathbb{P}}(S_i) \\
&= \lim_{x\to\infty}\lim_{k\to\infty}\frac{\pi(x, mb_{i,1}b_{i,2}\dots b_{i,k}, \{r|r \not\equiv 0, -F_t^{-1} \pmod{b_{i,j}} \ \forall j \in \{1,2,\dots,k\}\})}{\pi(x)} \\
&\leq \lim_{x\to\infty}\lim_{k\to\infty}\frac{\frac{1}{m}\left(1-\frac{1}{b_{i,1}}\right)\left(1-\frac{1}{b_{i,2}}\right)\dots\left(1-\frac{1}{b_{i,k}}\right)\frac{x}{lnx}}{\frac{x}{lnx}} \\
&= \frac{1}{m}\lim_{x\to\infty}\lim_{k\to\infty}\left(1-\frac{1}{b_{i,1}}\right)\left(1-\frac{1}{b_{i,2}}\right)\dots\left(1-\frac{1}{b_{i,k}}\right) \\
&= \frac{1}{m}\lim_{x\to\infty}\prod\left(1-\frac{1}{p}\right) \text{where } p \leq x \text{ are primes and } b_{i,j} \equiv b_{i,1} \pmod{F} \ \forall j \geq 1 \\
&= \frac{1}{m}\lim_{x\to\infty} P(x, F, b_{i,1}) \\
&= 0 \quad \text{by Corollary 4.3}
\end{aligned}
$$

Notice that

$$D_{\mathbb{P}}(DP_3(m)^c) \leq D_{\mathbb{P}}(S_1 \cup S_2 \cup \dots S_t)$$

because counterexamples m-E_3 equation must satisfy one of the (ES).

Also, t is finite for a given m. By Lemma 4.1, $D_{\mathbb{P}}(S_1 \cup S_2 \cup \dots S_t) = 0$.

$\therefore \quad D_{\mathbb{P}}(DP_3(m)^c) = 0 \quad \forall m \geq 4$ □

Readers with Analysis background should know that density-based (18) provides poor estimation because many congruencies other than the selected ones are missing. The progress for prime density approaching 0 is reflected by the slow growth of $\ln x$.

5. Egyptian parametrization and Semiperfect numbers

In this section, a technique aimed to narrow the choices and number of prime factors of a coprime semi-perfect number, which is a broader version of a primitive semiperfect number, is introduced using the parametrization of Egptyain fractions. It is very well known that perfect numbers and semi-perfect numbers have associated representations in Egyptian fractions.

Below are the terminologies to be used in this section.

Definition 5.1. *Given $S \in \mathbb{N}$.*

*(i) S is **semiperfect** if it is equal to the sum of some of its distinct proper factors.*

*(ii) S is **primitive semiperfect** if it is not a multiple of a semiperfect number other than itself.*

*(iii) S is **coprime semiperfect in n factors** if it can be expressed as a sum of distinct proper factors $\alpha_1, \alpha_2, ..., \alpha_n$ of S with $\gcd(\alpha_1, ..., \alpha_n) = 1$.*

For example, $1, 2, 3, 4, 6$, and 12 are factors of 12. We see that $1 + 2 + 3 + 6 = 12$. Therefore, 12 is a coprime semi-perfect number.

Remark 5. *Given that S is semiperfect, **primitive semiperfect** implies **coprime semiperfect** because of the double negation of the definition of coprime semiperfect numbers. If S is semiperfect but **not** coprime semiperfect in n factors for any $n \in \mathbb{N}$, then whenever*

$$S = \alpha_1 + \alpha_2 + ... + \alpha_n, \quad \gcd(\alpha_1, \alpha_2, ..., \alpha_n) = g > 1 \Longleftrightarrow \frac{S}{g} = \frac{\alpha_1}{g} + \frac{\alpha_2}{g} + ... + \frac{\alpha_n}{g}$$

*Then S is the multiple of $\frac{S}{g}$, which is also a semiperfect number. Therefore, S is **not** primitive semiperfect.*

However, another direction may not be true. Observe $24 = 1 + 3 + 8 + 12$ which is a coprime semiperfect number in 4 factors. But 24 is a multiple of 12, where 12 is a semiperfect number as shown. That means 24 is **not** primitive semiperfect. Hence our studying target, coprime semiperfect, has a slightly bigger size compared to primitive semiperfect numbers.

The classical relationship between Egyptian fraction expansion and semi-perfect numbers can be seen in the following.

Proposition 5.1. *S is semiperfect if and only if there exists factors $x_1, x_2, ..., x_n$ of S such that*

$$\sum_{i=1}^{n} \frac{1}{x_i} = 1.$$

Proof. Let $S = \sum_{i=1}^{n} \alpha_i$ where $\alpha_i \in \mathbb{N}$ and $\alpha_i | S$. Dividing both sides by S, we have

$$\frac{\sum_{i=1}^{n} \alpha_i}{S} = \sum_{i=1}^{n} \frac{1}{\frac{S}{\alpha_i}} = 1 \text{ where } \frac{S}{\alpha_1}, \frac{S}{\alpha_2}, ..., \frac{S}{\alpha_n} \text{ are also factors of } S$$

For the reverse direction, just let $x_1 = \frac{S}{\alpha_1}, x_2 = \frac{S}{\alpha_2}, ..., x_n = \frac{S}{\alpha_n}$. Then $\alpha_1 + \alpha_2 + ... + \alpha_n = S$. □

Now we are ready to introduce the Egyptian parametrization technique. As this topic is an application of the parametrization and the formula of Egyptian parametrization requires much deeper research when the length of the expansion increases, we only use the technique in an Egyptian expansion of 4 unit fractions.

Theorem 5.1. *It is given that S is a coprime semiperfect number in 4 factors and it can be expressed in a sum of four distinct proper factors $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ with* $\gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 1$. *Then $S = k\sqrt{\alpha_1\alpha_2\alpha_3\alpha_4\sigma}$ for some $\sigma \in \mathbb{N}$ and $k \in \mathbb{N}$.*

Proof. Without loss of generality, assume $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$.

We first transform the sum of factors into an Egyptian expansion problem

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 1 \text{ with } x_i = \frac{S}{\alpha_{5-i}}, \ i = 1, 2, 3, 4 \tag{19}$$

From $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$, $x_1 < x_2 < x_3 < x_4$.

Then by Theorem 2.2, for (x_1, x_2, x_3, x_4) there exists a corresponding ascending 1σ-E_4 triple $(a, b, c, d) \in \mathbb{N}^4$, i.e. $a < b < c < d$, such that

$$\begin{cases} 1^3 = \dfrac{1^2\sigma abcd}{(a+b+c+d)^2} \iff a+b+c+d = \sqrt{\sigma abcd} \\ x_1 = \sqrt{\dfrac{\sigma abc}{d}}, \ x_2 = \sqrt{\dfrac{\sigma acd}{b}}, \ x_3 = \sqrt{\dfrac{\sigma abd}{c}}, \ x_4 = \sqrt{\dfrac{\sigma bcd}{a}} \end{cases} \tag{20}$$

for some **square-free** $\sigma \in \mathbb{N}$. As demonstrated in Theorem 2.2, a, b, c, d can also be expressed in terms of x_1, x_2, x_3, x_4 by

$$a = \sqrt{\frac{x_1x_2x_3}{x_4\sigma}}, \ b = \sqrt{\frac{x_1x_2x_4}{x_3\sigma}}, \ c = \sqrt{\frac{x_1x_2x_4}{x_2\sigma}}, \ d = \sqrt{\frac{x_2x_3x_4}{x_1\sigma}} \tag{21}$$

Then

$$\begin{aligned} a : b : c : d &= \sqrt{\frac{x_1x_2x_3}{x_4\sigma}} : \sqrt{\frac{x_1x_2x_4}{x_3\sigma}} : \sqrt{\frac{x_1x_2x_4}{x_3\sigma}} : \sqrt{\frac{x_2x_3x_4}{x_1\sigma}} \\ &= \sqrt{\frac{x_1x_2x_3x_4}{x_4^2\sigma}} : \sqrt{\frac{x_1x_2x_3x_4}{x_3^2\sigma}} : \sqrt{\frac{x_1x_2x_3x_4}{x_2^2\sigma}} : \sqrt{\frac{x_1x_2x_3x_4}{x_1^2\sigma}} \\ &= \frac{1}{x_4} : \frac{1}{x_3} : \frac{1}{x_2} : \frac{1}{x_1} \\ &= \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 \end{aligned}$$

By the assumption of coprime semiperfect number, $\gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 1$ and therefore $\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4$ is a simplified ratio. Let $a = k\alpha_1, \ b = k\alpha_2, \ c = k\alpha_3, \ d = k\alpha_4$. Combining with the first equation from (20) and definition of S,

$$\begin{aligned} kS = k\alpha_1 + k\alpha_2 + k\alpha_3 + k\alpha_4 &= k^2\sqrt{\alpha_1\alpha_2\alpha_3\alpha_4\sigma} \\ \iff S &= k\sqrt{\alpha_1\alpha_2\alpha_3\alpha_4\sigma} \end{aligned}$$

for some $\sigma \in \mathbb{N}$ and $k \in \mathbb{N}$. □

A superior advantage of the argument of Theorem 5.1 using Egptyian parametrization is that we do not rely on any information from S itself and factors of S except that the factor combinations have to be coprime only. Besides, we can limit S further by a necessary condition $k\sqrt{\alpha_1\alpha_2\alpha_3\alpha_4\sigma}$ which can be a very strong restriction as readers can see in the following argument. Hence, this argument has big generalization potential.

Now we proceed to reveal the limited choices of factors and hence list all coprime semiperfect numbers in 4 factors. Theorem 5.1 established a necessary condition

$$S = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = k\sqrt{\sigma\alpha_1\alpha_2\alpha_3\alpha_4} \tag{22}$$

for choosing factors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ from a potentially coprime perfect number S.

Suppose

$$\sqrt{\sigma\alpha_1\alpha_2\alpha_3\alpha_4} = p_1p_2...p_n \Longleftrightarrow \sigma\alpha_1\alpha_2\alpha_3\alpha_4 = p_1^2p_2^2...p_n^2 \tag{23}$$

where $p_1, p_2, ..., p_n$ are primes (**not** necessarily distinct) and essentially prime factors of S. The following proposition showcases the power of (23).

Proposition 5.2. *In (23), if* $\gcd(p_i, k) = 1$*, then* p_i *divides* ***two*** *variables from* $\sigma, \alpha_1, \alpha_2, \alpha_3, \alpha_4$*, where* $i = 1, 2, ..., n$.

Proof. Recall and observe the prime powers in the later 4 equations in (20).

If $p_i \mid \sigma$, by that σ is square-free, p_i must divide one more variable from $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

If $p_i^2 \mid \alpha_j$ and does not divide other variables for some $j \in \{1, 2, 3, 4\}$, by $a = k\alpha_1,\ b = k\alpha_2,\ c = k\alpha_3,\ d = k\alpha_4$ and (20),

$$x_{5-j} = \frac{k\sqrt{\dfrac{\sigma\prod\limits_{t\neq j}\alpha_t}{\alpha_j}}}{p_i} \Longrightarrow p_i \mid k$$

contradicting to $\gcd(p_i, k) = 1$. Therefore, the Proposition holds. □

Proposition 5.2 tells us that p_i has to be separately divided σ or different α. For intuitive imagination, we suggest solving the restriction $\sqrt{\sigma\alpha_1\alpha_2\alpha_3\alpha_4}$ as playing an interesting math game of inputting primes into the following arithmetic structure

$$S = k\sqrt{\sigma\alpha_1\alpha_2\alpha_3\alpha_4} = kp_1p_2...p_n = \overset{(\alpha_1)}{\square} + \overset{(\alpha_2)}{\square} + \overset{(\alpha_3)}{\square} + \overset{(\alpha_4)}{\square} \tag{box game}$$

Be noted that the values of the boxes at the final stage of the game are potential choices of α.

Procedure for Box game(S, k, σ)

(1) Pick factors $k, \sigma \in \mathbb{N}$ from S. We suggest starting from $k, \sigma = 1$ and iterating k, σ in ascending order.
(2) From (23), $\sigma\alpha_1\alpha_2\alpha_3\alpha_4 = p_1^2p_2^2...p_n^2$. As the 4 boxes represent the "storage" of the prime factors of 4 αs, then we start to distribute factors $p_1^2, p_2^2, ..., p_n^2$ into the boxes one by one according to the rules set up by Proposition 5.2.
(3) After all prime factors are input in the boxes, notice that these factors are "excess" because σ (if $\sigma > 1$) exists on the left-hand side of (23). Now take out the factors where their product is equal to σ.
(4) Check the final sum of the boxes, i.e. $\alpha_1+\alpha_2+\alpha_3+\alpha_4$. If $\gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 1$ and equality in (box game) still holds, then we can say that a coprime

semiperfect number S is discovered. If not, repeat the process with alternative σ or k.

$$p_i^2 \textit{ already filled in, } \sigma \textit{ factors deleted} \rightarrow \left(\square + \square + \square + \square\right) = S$$

To pass through (box game), a major concern is that if S is arbitrarily chosen, there are possibly unlimited and unbounded choices for $p_1, p_2, ..., p_m$. Meanwhile, the following lemma posted a strong boundary on the largest possible prime dividing a coprime semiperfect number S into 4 factors.

Firstly, observe that S must be an even number by the sum of 4 factors must be even.

For convenient discussion of the set of factor sum of numbers, denote

$$F_4(S) = \{\alpha_1+\alpha_2+\alpha_3+\alpha_4 \mid \text{distinct factors } \alpha_1, \alpha_2, \alpha_3, \alpha_4 \text{ of } S,\ \gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 1\}$$

Lemma 5.1. *Given S is a coprime semiperfect number in 4 factors.*

(i) S contains some odd factors.
(ii) Let p be a prime factor of S and $p^s \mid S$. Then $p^s \leq 9$.

Proof. (i) Assume $S = 2^n$ for some $n \geq 4$. If $n \leq 3$, then S contains less than 4 factors.

$$\begin{aligned}
\max\{F_4(2^n)\} &= \sum_{j=n-4}^{n-1} 2^j \\
&= 2^{n-2}\left(2 + 1 + \frac{1}{2} + \frac{1}{4}\right) \\
&= 3.75 \times 2^{n-2} \\
&< 4 \times 2^{n-2} = 2^n
\end{aligned}$$

Therefore 2^n cannot be a semi-perfect number in a sum of 4 proper factors.

(ii) By (i), S must contain some odd factors, and therefore the statement is well-defined.

We now play (box game) on S for arbitrary $k \mid S$. Observe that if we set $\sigma > 1$, eventually fewer prime factors are being distributed to boxes, which leads to a smaller sum of $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ at the final stage. Hence, from the perspective of maximizing $F_4(S)$, we should set $\sigma = 1$ to play (box game).

Without loss of generality, assume $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ at the last stage of the box placement. Then the optimal strategy to maximize $F_4(S)$ is to naturally insert the maximum number of prime factors such that the value of the fourth box $= \dfrac{S}{\text{smallest prime factor of } S} = \dfrac{S}{2}$. For the third box, since every prime factor of S has one more chance to be inserted in one more box other than the fourth box, the maximum value of the third box can be as closed as the fourth box but not exceeding it, i.e. $\dfrac{S}{3}$, if 3 is a factor of S. With this setup, the following inequality is established for

$p \geq 5$.

$$\max\{F_4(S)\} \leq \frac{S}{2p^s} + \frac{S}{p^s} + \frac{S}{3} + \frac{S}{2} \tag{24}$$

The maximum value of the first and second boxes have p^s in the denominator because we insert every p into the third and fourth boxes to maximize $F_4(S)$. If not, the maximum power of p dividing S is less than S, implying

$$\frac{S}{p} + \frac{S}{p} + \frac{S}{p} + \frac{S}{p} < \frac{4S}{p} < \frac{5S}{6} < \frac{S}{2p^s} + \frac{S}{p^s} + \frac{S}{3} + \frac{S}{2}$$

The above tells us that **not** inserting all the factor ps into the 4th box leads to a smaller value of $F_4(S)$ when $p \geq 5$. With further simplification,

$$\begin{aligned} \max\{F_4(S)\} \leq \frac{S}{2p^s} + \frac{S}{p^s} + \frac{S}{3} + \frac{S}{2} &< S \\ \iff \frac{1}{2p^s} + \frac{1}{p^s} + \frac{1}{3} + \frac{1}{2} &< 1 \\ \iff \frac{3}{2p^s} + \frac{1}{3} + \frac{1}{2} &< 1 \\ \iff \frac{1}{p^s} &< \frac{1}{9} \\ \iff p^s &> 9 \end{aligned}$$

When $p^s > 9$, $\max\{F_4(S)\} < S$. Therefore, for $\max\{F_4(S)\} \geq S$, $p^s \leq 9$.

When $p = 2$ or 3, if $s \leq 2$, then the Proposition holds. Suppose $s \geq 3$. Notice that the inequality

$$pa + \frac{b}{p} < a + b \iff a < \frac{b}{p} \tag{25}$$

holds for all $p, a, b \in \mathbb{R}^+$. If p is taken out from the third or fourth box of the placement

$$\boxed{\frac{S}{2p^s}} + \boxed{\frac{S}{p^s}} + \boxed{\frac{S}{3}} + \boxed{\frac{S}{2}}$$

and being inserted in the first or second box, let $a = \frac{S}{p^s}$ and $b = \frac{S}{3}$ in (25). By $s > 2$,

$$\frac{S}{p^s} < \frac{S}{3p}$$

Applying (25), it is directly to see the sum of two changed boxes decreases, and hence the final sum of all boxes of the exchanged placement decreases. Thus (24) is still the maximum possible sum of 4 coprime factors of S and $p^s \leq 9$.

□

Once the bound using Lemma 5.1(ii), the possible choices of S are then finite and all coprime semiperfect numbers with 4 factors can be found using brute force checking by computer programs.

Theorem 5.2. *If S is a coprime semiperfect number in 4 factors, then*

$$S = 2^{q_1}3^{q_2}5^{q_3}7^{q_4}$$

for some $0 \leq q_1 \leq 3,\ 0 \leq q_2 \leq 2,\ 0 \leq q_3, q_4 \leq 1.$

Proof. A direct consequence from Lemma 5.1 (ii). $\square$

However, it is still necessary for us to demonstrate once how (box game) suggests the correct placement of factors such that S is a coprime semiperfect number.

Proposition 5.3. $30 = 2 \times 5 \times 7$ *is a coprime semiperfect number with 4 factors.*

Proof. The statement is easy to verify by directly checking on factor combinations of 30 and 12. The purpose of the statement is to show (box game) procedure on a numerical example. Again we following the arrangement of the placement $\square < \square < \square < \square$.

We start with the number 30 and pick $\sigma = k = 1$. We claim that the following is the only possible initial placement of some prime factors of 30.

$$\overset{(4)}{\boxed{2}} + \overset{(3)}{\boxed{3}} + \overset{(2)}{\boxed{2 \times 5}} + \overset{(1)}{\boxed{3 \times 5}}$$

Reason for (1): If (1) doesn't contain the factor 5, $(1) \leq 2 \times 3 = 6$. By the ascending arrangement of boxes, the maximum approximation of S is $4 \times 6 = 24 < 2 \times 3 \times 5$. If $(1) = 2 \times 5$, maximum value of $(2) = 2 \times 3$. Then the maximum approximation of the sum of the first 3 boxes $= 3 \times (2) = 18$. Then $S \leq 28 < 30$. Therefore $(1) = 3 \times 5$.

Reason for (2): As $(2) \neq (1) \neq 3 \times 5$, if (2) doesn't contain the factor 5, $(2) \leq 2 \times 3 = 6$. As we used factor 3 twice in (1) and (2), and 2 once in (2). Either $2 \times 5 = 10$ in (3), which contradicts the ascending arrangement of boxes, or factor 2 is in (4), and 5 is in (3). Then $2 + 5 + 2 \times 3 + 3 \times 5 = 28 < 30$. If $(2) = 5$, the sum of all boxes $< 5 \times 3 + 15 = 30$. Therefore $(2) = 2 \times 5$.

Reason for (3) and (4): As the remaining factors are 2 and 3 to be insert one more time into the boxes. Either $(3) = 2 \times 3$, which contradicts to the ascending arrangement of boxes, or $(3) = 3$ and $(4) = 2$.

Clearly, from the observation that when $(4) = 2, (3) = 3, (2) = 10, (1) = 15, (4) + (3) + (2) + (1) = 30$. Therefore 30 is a coprime semiperfect number. $\square$

Lastly, with trials on the (box game) argument with all potential S, we have the ending result of this section. After checking either by using a computer program or procedure following Proposition 5.3, there exist only 6 coprime semiperfect numbers with 4 factors as follows.

$$2 \times 3^2 = 18 = 1 + 2 + 2 \times 3 + 3 \times 3$$

$$2 \times 3 \times 5 = 30 = 2 + 3 + 2 \times 5 + 3 \times 5$$

$$2 \times 3 \times 7 = 42 = 1 + 2 \times 3 + 2 \times 7 + 3 \times 7$$

$$2^2 \times 3 = 12 = 1 + 2 + 3 + 2 \times 3$$

$$2^2 \times 5 = 20 = 1 + 2 \times 2 + 5 + 2 \times 5$$

$$2^3 \times 3 = 24 = 1 + 3 + 2 \times 2 \times 2 + 2 \times 2 \times 3$$

Only 20 is primitive among 6 possible values of S. The above Egyptian parametrization approach has further investigation potential in longer expansion length, i.e. finding coprime semiperfect numbers in factors more than 4. In particular, when the expansion length is odd, we can search for **odd coprime** semiperfect numbers and possibly provide insights into the odd perfect number conjecture.

6. Conclusion

In short, this paper aims to reinvigorate the study of Egyptian fractions by transitioning from traditional methods to a novel system. We introduce a symmetric parametrization for Egyptian fraction equations, designed to streamline the solution process and ensure symmetry among the solutions. Our approach addresses the conjecture of the shortest length of the Egyptian expansion up to length 5, and we propose the potential to expand this parametrization to arbitrary cases. Furthermore, our parametrization offers fresh perspectives on the Generalized Erdös-Straus Conjecture. We are confident that our methodology presents promising opportunities for tackling the formidable Odd Perfect Number Conjecture.

References

[1] E.Szemeredi. “On sets of integers containing no k elements in arithmetic progression”. In: *Acta Arithmetica* (1975). DOI: `http://matwbn.icm.edu.pl/ksiazki/aa/aa27/aa27132.pdf`.

[2] Alan Fong. “On the investigation of the Erdos-Straus Conjecture”. In: *Hang Lung Mathematics Awards* 6 (2014).

[3] C.N. Friedman. “Sums of Divisors and Egyptian Fractions”. In: *Journal of Number Theory* 44.3 (1993), pp. 328–339. ISSN: 0022-314X. DOI: `https://doi.org/10.1006/jnth.1993.1057`. URL: `https://www.sciencedirect.com/science/article/pii/S0022314X83710577`.

[4] Ronald L. Graham. “Paul Erdős and Egyptian Fractions”. In: *Erdős Centennial*. Ed. by László Lovász, Imre Z. Ruzsa, and Vera T. Sós. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 289–309. ISBN: 978-3-642-39286-3. DOI: `10.1007/978-3-642-39286-3_9`. URL: `https://doi.org/10.1007/978-3-642-39286-3_9`.

[5] Paul Hoffman. *The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth*. Hyperion, 1998.

[6] Alessandro Languasco and Alessandro Zaccagnini. “A note on Mertens’ formula for arithmetic progressions”. In: *Journal of Number Theory - J NUMBER THEOR* 127 (Nov. 2007), pp. 37–46. DOI: `10.1016/j.jnt.2006.12.015`.

[7] Louis J. Mordell. *Diophantine Equations*. Academic Press, 1967, pp. 287–290.

[8] Serge E. Salez. *The Erdős-Straus conjecture New modular equations and checking up to* $N = 10^{17}$. 2014. arXiv: `1406.6307 [math.NT]`.

[9] Ivan Soprunov. “A SHORT PROOF OF THE PRIME NUMBER THEOREM FOR ARITHMETIC PROGRESSION”. In: (Jan. 2010).

[10] Alexis Toda. “Radii of the Inscribed and Escribed Spheres of a Simplex”. In: *International Journal of Geometry* 3 (Oct. 2014), pp. 5–13.

Appendix

The code is stored in this link:

https://drive.google.com/file/d/1PP1LGYgUaH81pZavkig0j-ohfl9lYqh_/view?usp=sharing

For additional information on the checking algorithm, please see:

https://drive.google.com/file/d/1OgbnQa6AHhzLKJEbSFHT--xBK7f8rrxa/view?usp=sharing

Below is the explanation of different parameters.

```
# Function 1: Verifying wthether 1 up to (checking_range) has solutions on m-E3 eqation
# parameters setting
m=5
enhancer_limit=2000
lcm_bound=3*10**8
congruence_limit=2200*m
checking_range=1*10**12

# Verification command. There are some progress bars showing congruences generating process
count,e_list,final_mod,escape_residue=ultimate_check(m,checking_range)
```

```
# Function 2: Check solutions of m-E3 equation on a single n
# parameters setting
m=13
congruence_limit=5000*m
# Preparations of congruence conditions up to the range that you assigned
t1m_list,t1r_list,t1a_collect_list,c,sig,categ = ntype_con_generator_final(m,1,congruence_limit,2)
t2m_list,t2r_list,t2b_collect_list,F,sig,categt = n_con_generator_final(m,1,congruence_limit,add_prime_con)

# Check command. Once congruence conditions are generated, different solutions from n can be found without generat
single_escape_check(2521,t1m_list,t1r_list,t2m_list,t2r_list)
```

```
#Function 3: Obtain guess from value of n_0(m)

#parameters setting
n_0_lower=9
n_0_upper=20
enhancer_limit_mul=800
lcm_bound=2*10**8
congruence_limit_mul=900
checking_range=1*10**10
suggest_gap=10**8

# Search command.
n_0_search(n_0_lower,n_0_upper,enhancer_limit_mul,lcm_bound,congruence_limit_mul,checking_range,suggest_gap)
```

Figure 8. Functions 1, 2, and 3

Readers are welcome to obtain experimental data using our program and make further progress on the Generalized Erdös-Straus Conjecture.

REVIEWERS' COMMENTS

The author considered the generalized Erdös-Straus Conjecture, which asks whether for any $a \in \mathbb{Z}_{\geq 1}$, there is a solution to the Diophantine equation

$$\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

for $n > n_0$ where n_0 is an integer depending on a.

The conjecture is proved to be true for $n \leq 10^{14}$ using scientific computations, but is still open in general.

The main results of the article are:

- Section 2: a symmetric solution to Egyptian expansion, and the proof of equivalence to asymmetric solutions. Geometric interpretation for three-term and form-term expansions.
- Section 3: some results on the shortest length expansion, focused some cases with length 3 ,4, 5.
- Section 4: discussion on the generalization of Erdos-Straus conjecture, and using prime density to produce some possibly counter-examples.
- Section 5: application of Egyptian parametrization to semi perfect numbers.

Overall, the author chose an interested topic in number theory, which has a long history. He/she addressed several questions on this topic and made progress. Although the main statements are in number theory, the approaches also include various techniques, such as geometry and computer programming. The report is nicely written with both details and sketch of ideas.