Hang Lung Mathematics Awards
Vol. 2 (2006), pp. 197–218

TWO INTERESTING MATHEMATICAL GAMES

TEAM MEMBERS
Kin-Ying Chan, Man-Chung Cheung, Kwok-Chung Li,
Chi-Fai Ng, Ming-Leung Sin[1]

SCHOOL
Shatin Tsung Tsin Secondary School

Abstract. In this project, we try to find out the patterns of Sudoku and the best strategy of a game in game theory.
For Sudoku part, we added three symmetries and used permutation to deal with the problem. We found that there are only 16 possible permutation patterns. Some properties of the combinations of these 16 patterns are also included in the report.
The game in game theory is game about 3 players choose a number that is greater than or equal to 3 respectively. The player who chooses the smallest and not repeated number is the winner. We successfully found the best strategy.

1. Introduction

In the 2005 summer holiday, our group members discovered a funny game in the newspaper.

The game we found in newspaper is called 'Sudoku', which is a puzzle game invented by Euler. Since a Sudoku has a unique solution by given just a few numbers, we think it is amazing and interesting. We are keen on finding the reason of this phenomenon, how can the given numbers control the whole Sudoku? While we were finding some information about Sudoku, we found that many people think that Sudoku is a game absolutely not related to mathematics. However, we don't agree with those ideas. We believe that any things that relate to logic must relate to mathematics and Sudoku is such a good example. At that time, we found that it was very valuable for us to study.

[1]This work is done under the supervision of the authors' teacher, Ms. Pik-Yee Lam.

In a 9×9 puzzle, 14-plus numbers are generally given and players need to fill the puzzle with numbers 1 to 9. The numbers should not be repeated in each column, row and 3×3 box. Sudoku is very popular nowadays as it is challenging. Everyone does Sudoku in the train, bus, and school, home. Sudoku spreads very fast as the newspapers post a game each day. Also, large number of websites about Sudoku can let people know Sudoku more easily. They will find questions of Sudoku on the web and feed their passion of doing Sudoku. Actually, we seldom play Sudoku because we just want to find out the logic inside it, not want to play it for fun.

Most people think that it is a game, which depends logical thinking only. However, we do not agree that there is no mathematics inside the game. In fact, logic is a part of mathematics. The way we solve the problem is to minimize the number of boxes, from 9×9 to 4×4. We find that there are only 16 patterns and from them, we know the least numbers need to be given. We want to find the minimum numbers need to be provided in a 9×9 Sudoku by this method. However, we find that it is not as easy as we thought. The situation becomes more complicated. So we began to study something beside Sudoku. A normal Sudoku has three conditions that are each column, each row and each 3×3 box should contain 1 to 9. Now we add three more to minimize the possible number of solutions. This brings us the concept of permutation. We discover that there are special permutations from the first box to the second box.

Beside Sudoku, another game we studied is told by one of our friends and the rules are as follows: each player decides a number and it must be an integer. They will tell the numbers at the same time. The one who gives the least value but not a repeated number will win. Since two persons playing the game is meaningless and the game needed at least three persons. We choose this game because it is worth studying. Also, the game is about game theory and it raises our interest to study it. Game theory is common in the world. It is used in economic, making decisions, etc. It is weird that the best decision is not the way we can get the best result because it is not only probability, but also psychology. Since everyone wants to get the best result and others will not affect his action.

What we want to find is the relationship between the number of players and the most suitable number to give. First we start our work from three persons. We set up equations to solve the problem. Let's consider the situation of three players. Suppose the probability of giving one is p_1, and the probability of winning is that others don't give one, that is $(1-p_1)^2$. Similarly, we find the probabilities of giving two and three, which are $(p_1^2+p_3^2)$ and $(p_1^2+p_2^2)$ respectively. According to the game theory, the probabilities of victory by giving one, and that of giving two and that of giving three should be equal. So that we can solve how many one, how many two and how many three we should give to have the greatest winning chance. We have found the result of the situation of three players. However, as the number of players is increasing, the equations become more complicated and more difficult to solve. So we hope we can solve it by computer.

Both games are new and worth studying. We believe we can find some interesting results.

2. Sudoku

Sudoku is a game invented by Euler. We now concern 9×9 Sudoku. Let us introduce the rules.

1. Each row should contain 1 to 9, e.g.

1	2	3	4	5	6	7	8	9

2. Each column should contain 1 to 9, e.g.

1								
2								
3								
4								
5								
6								
7								
8								
9								

3. Each 3×3 grids should contain 1 to 9, e.g.

1	2	3						
4	5	6						
7	8	9						

Since the number of patterns is too large so that we cannot write down the entire pattern for observing in order to find out the minimum numbers that needed to be given for having a unique solution. Hence, we add more symmetry to control the number of possible solutions. After the long period of trial and error, we decided to add three symmetries.

1. Each corresponding box in every 3×3 box must contain 1 to 9, e.g.

1			2			4		
7			9			3		
5			6			8		

2. Each small row of 3×3 box in a big column of the game puzzle should contain 1 to 9, e.g.

1	2	6						
7	3	5						
9	4	8						

3. Each small column of 3×3 box in a big row of the game puzzle should contain 1 to 9, e.g.

1			2			9		
3			4			5		
6			8			7		

Including these three new symmetries, there are all together six rules and these minimize the number of possible patterns, which is 256. That is not just good news for us but we also discover some patterns that are related to permutation.

We look at one example first.

1	6	8	2	4	9	3	5	7
5	7	3	6	8	1	4	9	2
9	2	4	7	3	5	8	1	6
4	9	2	5	7	3	6	8	1
8	1	6	9	2	4	7	3	5
3	5	7	1	6	8	2	4	9
7	3	5	8	1	6	9	2	4
2	4	9	3	5	7	1	6	8
6	8	1	2	9	4	5	7	3

In this example, six rules are applied in it. Here let us introduce the method to represent permutation. We record the position as follows:

1	2	3						
4	5	6						
7	8	9						

Then we record the change of position by the following method. To introduce this clearly, we use the following example:

a	b	c	e	d	f			
d	e	f	g	h	i			
g	h	i	b	a	c			

Since a changes its position from the 1^{st} small box (in the first big box) to the 8^{th} small box (in the second big box), h changes its position from the 8^{th} small box (in the first big box) to the 5^{th} small box (in the second big box) and e changes its position from the 5^{th} small box (in the first big box) to the 1^{st} small box (in the second big box). We give the notation $(1, 8, 5)$ to record this change in position.

Let's come back to the example and focus on the first big row:

1	6	8	2	4	9	3	5	7
5	7	3	6	8	1	4	9	2
9	2	4	7	3	5	8	1	6

Since the 1^{st} small box goes to the 6^{th} small box, the 6th small box goes to the 8^{th} small box and the 8^{th} small box goes to the 1^{st} small box, so the notation is $(1, 6, 8)$. Similarly, we get $(2, 4, 9)$ and $(3, 5, 7)$. Hence, the permutation is $(1, 6, 8)(2, 4, 9)(3, 5, 7)$.

Now, we are going to find out the permutation from the 2^{nd} box to the 3^{rd} box. Using the above method, we get $(1, 6, 8)(2, 4, 9)(3, 5, 7)$, it is the same as the permutation from the 1^{st} box to the 2^{nd} box. So we would ask: must the permutation from the 2^{nd} box to the 3^{rd} box be the same as the permutation from the 1^{st} box to the 2^{nd} box when six rules are applied? This would never happen in a normal Sudoku.

2.1. Permutation

Here are the six rules of Sudoku:

Rule 1 There are number 1 to number 9 in each row.
Rule 2 There are number 1 to number 9 in each column.
Rule 3 There are number 1 to number 9 in each big box.
Rule 4 In the same big row, the same number cannot exist in the same small column.
Rule 5 In the same big column, the same number cannot exist in the same small row.
Rule 6 The same number cannot exist in the same position of each big box in the whole Sudoku.

First of all, let us introduce the notation of the following parts. We use $(abc)(def)(ghi)$ to represent a permutation pattern. Since $(123) = (231) =$

(312), we will choose the one in which the first number is the smallest in this bracket. Moreover, the position of brackets will not affect the result, so we will put the bracket which the first number is the smallest in the front.

Now, we are going to explain how to time two permutations together. We time two permutations by the following method. Look at this example: (123)(456)(145)(236)

We will start from the last bracket. 2 goes to 3, 3 goes to 1, so 2 goes to 1. 1 goes to 4, 4 goes to 5, then 1 goes to 5. 5 goes to 1 and 1 goes to 2. After this process, we get (215). Similarly, we get (346). Therefore (123)(456)(145)(236) = (215)(346).

If $\sigma = (159)$, Then $\sigma^2 = (159)(159) = (195)$. Now we can find out the possible patterns.

For $a = 1$,

b	
1	X - because $a = 1$
2	X - against rule 5
3	X - against rule 5
4	X - against rule 4
5	
6	
7	X - against rule 4
8	
9	

Therefore, $(1bc)$, $b \neq c$, b, $c = 5$, 6, 8, 9.

Case 1: $b = 5$

c	
6	X - against rule 5
8	X - against rule 4
9	

The case $b = 9$ shares the same combination where only $c = 5$ is acceptable.

Case 2: $b = 6$

c	
5	X - against rule 5
8	
9	X - against rule 4

The case $b = 8$ shares the same combination where only $c = 6$ is acceptable.

Therefore, there are only 4 cases: (159), (195), (168) and (186) in the first bracket. Since (159) and (195), (168) and (186) share the same cases, only (159) and (168) will be considered in the following parts.

For $d = 2$,

e	
1	X - against rule 5
2	X - because$d = 2$
3	X - against rule 5
4	
5	X - against rule 4
6	X - cannot be used if first bracket is (168)
7	
8	X - against rule 4
9	X - cannot be used if first bracket is (159)

Therefore, $(2ef)$, $e \neq f$, e, $f = 4$, 6, 7, 9.

Case 1: $e = 4$

f	
6	X - against rule 5
7	X - against rule 4
9	X if the first bracket is (159)

The case $e = 7$ shares the same combination and $f = 4$.

Case 2: $e = 6$

f	
4	X - against rule 5
7	X if the first bracket is (168)
9	X - against rule 4

The case $e = 7$ shares the same combination where only $f = 4$. Similarly, we get the result of case 2.2.

Conclusion:

There are 16 combinations in the first big column. They are grouped in 2 big groups (159) and (168). (Some combinations are ignored.) In each group, there are 8 combinations due to 3 brackets which have two combinations each.

To make the conclusion simpler, other but same combinations are ignored. Then we get two permutation expressions(ascending order of numbers): (159)(267)(348) and (168)(249)(357)

Finally, we found 16 patterns:

1. (159)(267)(348)
2. (159)(267)(384)
3. (159)(276)(348)
4. (159)(276)(384)
5. (195)(267)(348)
6. (195)(267)(384)
7. (195)(276)(348)
8. (195)(276)(384)
9. (168)(249)(357)
10. (168)(249)(375)
11. (168)(294)(357)
12. (168)(294)(375)
13. (186)(249)(357)
14. (186)(249)(375)
15. (186)(294)(357)
16. (186)(294)(375)

In fact, there are two different starting of permutation. One starts with (159) and one starts with (168).We denote them as σ type and τ type.

Each pattern is combined of three brackets. For σ type, the brackets are chosen from each column of the following table:

(159)	(267)	(348)
(195)	(276)	(384)

For τ type, the brackets are chosen from each column of the following table:

(168)	(249)	(357)
(186)	(294)	(375)

The following parts will use $\sigma_1\sigma_2\sigma_3$ and $\tau_1\tau_2\tau_3$ to represent a permutation pattern. For example, $\sigma_1 = (168)$, $\sigma_2 = (249)$ and $\sigma_3 = (357)$. If we use

$\sigma_1\sigma_2\sigma_3$ to be the permutation of square $(1,1)$ to square $(1,2)$, what is the permutation of square $(1,2)$ to square $(1,3)$? The answer is $\sigma_1\sigma_2\sigma_3$. Why?

We can find the reason step by step.

First, for each bracket σ_n^2 cannot be used because a certain number will go back to the original position in box 1. To explain it clearly, suppose we use $\sigma_1\sigma_2\sigma_3$ to be the permutation of square $(1,1)$ to square $(1,2)$ and $\sigma_1^2\sigma_2\sigma_3$ to be the permutation of square $(1,2)$ to square $(1,3)$.

Then the permutation of square $(1,1)$ to square $(1,3) = (\sigma_1\sigma_2\sigma_3)(\sigma_1^2\sigma_2\sigma_3) = \sigma_1^3\sigma_2^2\sigma_3^2$.

Since $\sigma_1^3 = \sigma_1^3$, the permutation from square $(1,1)$ to square $(1,3) = \sigma_1\sigma_2^2\sigma_3^2$. Notice that the permutation from square $(1,1)$ to square $(1,2) = \sigma_1\sigma_2\sigma_3$.

Then the changing of position of number according to σ_1 will be equal in the second box and the third box. The numbers will appear in the same position in the second big box and the third big box. Similarly, in the permutation of square $(1,2)$ to square $(1,3)$, σ_2^2 and σ_3^2 are not permitted. Therefore, the only answer is from square $(1,2)$ to square $(1,3)$ is $\sigma_1\sigma_2\sigma_3$.

Second, τ type cannot be used because some numbers that are the same will be in the same row. (Note that the directions of σ type are always from top-left to down-right and from down-right to top-left. The directions of τ type are always from top-right to down-left and from down-left to top-right.)

With these two reasons, the only possible permutation of square $(1,2)$ to square $(1,3)$ is $\sigma_1\sigma_2\sigma_3$.

Now we concern the permutation from $(1,1)$ square to square $(2,1)$. If the permutation pattern of square $(1,1)$ to square $(1,2)$ is of type σ, what is the permutation pattern of square $(1,1)$ to $(2,1)$ square? The permutation pattern should be of type τ because if the pattern is of type σ, it will be against rule 6 of the game in square $(2,1)$ or $(3,1)$.

To show the case clearly, we use σ type to represent the permutation from the left to the right and τ type to represent the permutation from the top the bottom in the following proofs.

Among all the permutations, there are two types of permutation. One comes from switching rows and columns and another does not.

For example,

1	6	8	2	4	9	3	5	7
5	7	3	6	8	1	4	9	2
9	2	4	7	3	5	8	1	6

the permutation of this one is $(1,6,8)(2,4,9)(3,5,7)$. This is a permutation coming from switching rows and columns. Notice the first small row in the first box, it is switched to the second small row in the second box. The second row is switched to the third row in the second row and the third row is switched to the first row in the second box. Similarly, the columns are switched like this. the first is switched to the third, the second switched to the first and the third is switched to the second. Look at this one,

1	2	3	9	6	4			
4	5	6	8	1	7			
7	8	9	2	3	5			

This one does not come from switching rows and columns. Originally, 1,2 and 3 are on the same row, after this permutation, 2 and 3 are on the same row but 1 is not. However, this method to determine which come from switching rows and columns and which are not is too slow. Now, let us introduce a method to determine which one comes from switching rows and columns. For example, $(1,6,8)(2,4,9)(3,5,7)$. It means the first place goes to the sixth place. The second one goes to the fourth place and the third one goes to the fifth place. Since 6, 4, 5 are in the same row, 1, 2, 3 go to the same row. Similarly, the second row will move to the third row according to this rule.

Among the 16 permutations

$$
\begin{array}{l}
(159)(267)(348)\\
(159)(267)(384)\\
(159)(276)(348)\\
(159)(276)(384)\\
(195)(267)(348)\\
(195)(267)(384)\\
(195)(276)(348)\\
(195)(276)(384)\\
(168)(249)(357)\\
(168)(249)(375)\\
(168)(294)(357)\\
(168)(294)(375)\\
(186)(249)(357)\\
(186)(249)(375)\\
(186)(294)(357)\\
(186)(294)(375),
\end{array}
$$

only (159)(267)(348), (195)(276)(384), (168)(249)(357) and (186)(294)(375) are permutations coming from switching rows and columns.

Now, we are going to study if σ and τ are given, what will be permutation from square $(1,1)$ to square $(2,2)$? There are three cases:

1. Both σ and τ (σ is the permutation pattern of square(1, 1) to square (1,2) and τ is the permutation pattern of square (1, 1) to square (2, 1)) come from permutation of switching rows and columns,
2. One comes from switching rows and columns, another does not,
3. Both do not come from permutations of switching rows and columns.

Case 1

Lemma 1. *If both σ and τ come from permutation of switching rows and columns, then $\sigma\tau = \tau\sigma$ is the unique solution for the permutation from square $(1,1)$ to square $(2,2)$.*

Proof. When $\sigma = (1,9,5)(2,7,6)(3,8,4)$ and $\tau = (1,6,8)(2,4,9)(3,5,7)$, then

1	2	3	5	6	4			
4	5	6	8	9	7			
7	8	9	2	3	1			
8	9	7	3	1	2			
2	3	1	6	4	5			
5	6	4	9	7	8			

It is easy to find out that the permutation from square $(2,1)$ to square $(2,2)$ is

$$(1,9,5)(2,7,6)(3,8,4) = \sigma$$

and the permutation from square $(1,2)$ to square $(2,2)$ is

$$(1,6,8)(2,4,9)(3,5,7) = \tau.$$

Therefore, in this case, the permutation from square $(1,1)$ to square $(2,2)$ is $\sigma\tau = \tau\sigma$.

When $\sigma = (1,9,5)(2,7,6)(3,8,4)$ and $\tau = (1,8,6)(2,9,4)(3,7,5)$, then

1	2	3	5	6	4			
4	5	6	8	9	7			
7	8	9	2	3	1			
6	4	5	7	8	9			
9	8	7	1	2	3			
3	1	2	4	5	6			

It is easy to find out that the permutation from square $(2,1)$ to square $(2,2)$ is

$$(1,9,5)(2,7,6)(3,8,4) = \sigma$$

and the permutation from square $(1,2)$ to square $(2,2)$ is

$$(1,8,6)(2,9,4)(3,7,5) = \tau.$$

Therefore, in this case, the permutation from square $(1,1)$ to square $(2,2)$ is $\sigma\tau = \tau\sigma$. $\square$

Case 2

Lemma 2. *If σ comes from switching rows and columns, τ does not. Then $\sigma\tau$ is the unique solution for the permutation from square* $(1,1)$ *to square* $(2,2)$.

Proof. In the following, σ will be considered as coming from permutation and τ will not. Obviously, the permutation pattern of square $(1,1)$ to square $(2,2)$ should be $\sigma\tau$ or $\tau\sigma$ but, in fact, only one of them is right and we are now going to present our reasons.

σ	τ
(159)(267)(348)	(168)(249)(357)
(195)(276)(384)	(186)(294)(375)

For convenience, we use some symbols to represent the brackets.

$$\sigma_1 = (159) \quad \tau_1 = (168)$$
$$\sigma_2 = (267) \quad \tau_2 = (249)$$
$$\sigma_3 = (348) \quad \tau_3 = (357)$$

Since τ is not from switching rows and columns, there must be one or two brackets in which τ is square, e.g. $\tau_1\tau_2\tau_3^2$. Besides, there are no repeating numbers in the three brackets so we can always put τ_n (which exist two times in τ) at the end of τ. Therefore, the permutations $\sigma\tau$ and $\tau\sigma$ are different. And by observation $\sigma\tau$ is the permutation of square $(1,1)$ to square $(2,2)$.

For example, $\sigma = (159)(267)(348)$ and $\tau = (168)(249)(375)$

1	2	3	9	7	8			
4	5	6	3	1	2			
7	8	9	6	4	5			
8	9	5	4	3	6			
2	7	1	5	8	9			
3	6	4	1	2	7			

Notice that the permutation from square $(2,1)$ to square $(2,2)$ is

$$(159)(267)(348) = \sigma.$$

Therefore, the permutation from square $(1,1)$ to square $(2,2)$ is $\sigma\tau$.

However, the permutation from square $(1,2)$ to square $(2,2)$ is

$$(168)(294)(357) \neq \tau.$$

Therefore, the permutation from square $(1,1)$ to square $(2,2)$ is not $\tau\sigma$. $\square$

Case 3

Lemma 3. *If both σ and τ do not come from permutations of switching rows and columns, then there will be no solution for square* $(2,2)$.

Proof. Remember there are totally 4 permutations of switching rows and columns? They are $(159)(267)(348)$, $(195)(276)(384)$, $(168)(249)(357)$ and $(186)(294)(375)$. We classify them into two types.

$$\begin{array}{ll} \sigma a. & (159)(267)(348) \\ \sigma b. & (195)(276)(384) \\ \tau a. & (168)(249)(357) \\ \tau b. & (186)(294)(375) \end{array}$$

Type a presents the first number comes from the first row, the **second** number comes from the **second** row and the **third** number comes from the **third** row.

Type b presents the first number comes from the first row, the **second** number comes from the **third** row and the **third** number comes from the **second** row.

The following proof can just cover the case when both σ_n and τ_n come from the same type. For example, $\sigma : aab$ and $\tau : aab$.

Case 3.1 To prove this statement, we introduce this notation to present 9 positions in a box,

a_1	a_2	a_3
b_1	b_2	b_3
c_1	c_2	c_3

In this notation, the same character represents the same row and the same number represents the same column.

For permutation, each bracket should consist of every character and number, e.g. (a_1, b_2, c_3), otherwise, the same number will appear in the same box, row or column.

For permutation of switching rows and columns, the same position in every bracket should come from the same row or column, like $(a_1, b_2, c_3)(a_2, b_3, c_1)$ (a_3, b_1, c_2).

To prove if both σ and τ do not come from switching rows and columns, suppose $\sigma = (a_1, b_2, c_3)(a_2, b_3, c_1)(a_3, c_2, b_1)$ and $\tau = (a_1, b_3, c_2)(a_2, b_1, c_3)$ (a_3, c_1, b_2), then the permutation from square $(1, 1)$ to square $(2, 2)$ is

$$\begin{aligned}\sigma\tau &= (a_1, b_2, c_3)(a_2, b_3, c_1)(a_3, c_2, b_1)(a_1, b_3, c_2)(a_2, b_1, c_3)(a_3, c_1, b_2)\\ &= (a_1, c_1, c_3, b_3, b_1)(a_2, a_3)(b_2, c_2).\end{aligned}$$

However, box a_2 goes to box b_3 when it is transformed from square $(1, 1)$ to square $(1, 2)$ but box a_2 goes to box a_3 when it is transformed from square $(1, 1)$ to square $(2, 2)$. Since a_3 and b_3 are in the same column, this is against rule 4, $\sigma\tau$ is rejected.

How about $\tau\sigma$?

$$\begin{aligned}\tau\sigma &= (a_1, b_3, c_2)(a_2, b_1, c_3)(a_3, c_1, b_2)(a_1, b_2, c_3)(a_2, b_3, c_1)(a_3, c_2, b_1)\\ &= (a_1, a_3)(a_2, c_2, c_3, b_3, b_2)(b_1, c_1)\end{aligned}$$

However, box a_1 goes to box b_3 when it is transformed from square $(1, 1)$ to square $(2, 1)$ but box a_1 goes to box a_3 when it is transformed from square $(1, 1)$ to square $(2, 2)$. Since a_3 and b_3 are in the same column, this is against rule 4, $\tau\sigma$ is rejected.

Case 3.2 To prove if σ and τ are different type, we may consider the case σ : ab and τ : ba because if the result is positive, no matter what bracket 3 is, it proves when they do not come from the same type, there is no solution for square (2,2).

Suppose $\sigma = (159)(276)\sigma_3$ and $\tau = (186)(249)\tau_3$. Then

1	2	3	9	6				
4	5	6		1	7			
7	8	9	2		5			
6	9				1			
2		8						
	1	4						

There is no free space to fill in 2. Against rule 2, 2 cannot be filled in the first column in square $(2, 2)$. Against rule 1, 2 cannot be filled in the second row in square $(2, 2)$. Against rule 5, 2 cannot be filled in the third row in square $(2, 2)$. Against rule 6, 2 cannot be filled in the second small box in square $(2, 2)$.

Therefore, for $\sigma = (159)(276)\sigma_3$ and $\tau = (186)(249)\tau_3$, there is no solution for square $(2, 2)$.

Since the position of brackets can be exchanged, whenever two or three brackets are different, there is no solution for square $(2, 2)$.

The last case is one bracket is different, e.g. $\sigma : aab$ and $\tau : bab$.

Suppose $\sigma : aa$ and $\tau : ba$. σ_3 will not be a because $\sigma = aaa$ and $\tau = ba$ have been mentioned in Case 2. The only possible pattern for σ is aab. For τ_3, it will not be a because $\sigma : a\sigma_2 b$ and $\tau : b\tau_2 a$ have been mentioned in case 3.1. The only possible pattern for τ is bab.

Let $\sigma = (159)(267)(384)$ and $\tau = (186)(249)(375)$. Then

1	2	3	9	7	4			
4	5	6	8	1	2			
7	8	9	6	3	5			
6	9	5	7	8	1			
2	7	8	5	4	3			
3	1	4		2	6			

There is no free space to fill in 9. Against rule 2, 9 cannot be filled in the first column in square $(2, 2)$.

Therefore, for $\sigma = (159)(267)(384)$ and $\tau = (186)(249)(375)$, there is no solution for square $(2, 2)$.

Since the position of brackets can be exchanged, for all permutation which have one different bracket, there is no solution for square $(2, 2)$.

To conclude, there is no solution when both σ and τ do not come from permutations of switching rows and columns. □

3. A Game in Game Theory

This is a game in game theory. It is a long name therefore we always use the short form, GGT, in the following discussion.

Here are some information of GGT:

Number of players	Indeterminate
Tools needed	Fingers, papers, computer, etc. Anything that can show numbers(The choice of players).
Time required to play one set of game	No fixed time. Players can either use one round as the whole set or tens, hundreds or even thousands rounds as a set. The time of a round is about 3-5 seconds.

It is the time for our royal referees to know how the game process. The steps are given to you now:

1. There should be at least 2 players join together to play the game.
2. Each player should decide a number as his/her choice. The number should be kept secretly (not saying out) before all players are ready.
3. All players show their numbers at the same time.
4. The winner exists (sometimes not) according to the following rule:
   I. The winner should not choose the same number with any other players. (The players who choose the same number with others are known as losers. His/Her number will be considered as meaningless)
   II. The winner should have the smallest "meaningful" number. (Those do not have the smallest number will also be considered as losers.)

Now, we are going to show some examples to you: There are 6 players. For convenience, we use P_n to represent the player n.

Round 1

$$P_1 : 1 \quad P_2 : 2 \quad P_3 : 3 \quad P_4 : 4 \quad P_5 : 5 \quad P_6 : 6$$

P_1 is the winner.

Round 2

$$P_1 : 1 \quad P_2 : 1 \quad P_3 : 3 \quad P_4 : 4 \quad P_5 : 5 \quad P_6 : 6$$

P_3 is the winner.

Round 3

$$P_1 : 1 \quad P_2 : 1 \quad P_3 : 2 \quad P_4 : 2 \quad P_5 : 5 \quad P_6 : 6$$

P_5 is the winner.

Round 4

$$P_1 : 1 \quad P_2 : 1 \quad P_3 : 1 \quad P_4 : 4 \quad P_5 : 5 \quad P_6 : 6$$

P_4 is the winner.

Round 5

$$P_1 : 1 \quad P_2 : 1 \quad P_3 : 2 \quad P_4 : 4 \quad P_5 : 4 \quad P_6 : 6$$

P_3 is the winner.

Round 6

$$P_1 : 1 \quad P_2 : 1 \quad P_3 : 1 \quad P_4 : 1 \quad P_5 : 2 \quad P_6 : 2$$

No winner.

Let's put an end to the examples.

The following are what we get in our study.

In the case with 3 players, we have the result as follow:

Assume the other two players will have the same strategy. Let P_1, P_2 and P_3 be the percentages of other players who make the choices of 1, 2 or 3 respectively. We must choose 1, 2 or 3 in each round so that $P_1+P_2+P_3 = 1$.

If I choose 1, the probability that I win is $(1 - P_1)^2$. It means that if the other two players don't choose 1, I will win.

If I choose 2, the probability that I win is $(P_1^2 + P_3^2)$. It means that if the two players choose 1 or 3 together, I will win.

If I choose 3, the probability that I win is $(P_1^2 + P_2^2)$. It means that if the two players choose 1 or 2 together, I will win.

If all the players want to win a large amount of round together, we will have the same strategy. The percentages of me to choose 1, 2 or 3 will be P_1, P_2 and P_3 respectively.

Then the percentage for me to win is

$$P = P_1(1-P_1)^2 + P_2(P_1^2+P_3^2) + P_3(P_1^2+P_2^2).$$

To maximize this probability, it is obvious that $P_2 = P_3$.

Since $P_1+P_2+P_3 = 1$, $P_1 = 1-2P_2$.

$$\begin{aligned}
&P_1(1-P_1)^2 + P_2(P_1^2+P_3^2) + P_3(P_1^2+P_2^2)\\
=\ &P_1(1-P_1)^2 + P_2(P_1^2+P_2^2) + P_2(P_1^2+P_2^2)\\
=\ &P_1(1-P_1)2 + 2P_2(P_1^2+P_2^2)\\
=\ &(1-2P_2)(1-1+2P_2)^2 + 2P_2((1-2P_2)^2+P_2^2)\\
=\ &4P_2^2(1-2P_2) + 2P_2(1-4P_2+4P_2^2+P_2^2)\\
=\ &4P_2^2 - 8P_2^3 + 2P_2 - 8P_2^2 + 10P_2^3\\
=\ &2P_2 - 4P_2^2 + 2P_2^3
\end{aligned}$$

Differentiating with respect to P_2, the result is $2-8P_2+6P_2^2$. When this equals zero, the chance for us to win is the highest. Therefore,

$$\begin{aligned}
2-8P_2+6P_2^2 &= 0\\
3P_2^2-4P_2+1 &= 0\\
(3P_2-1)(P_2-1) &= 0\\
3P_2-1 = 0 \text{ or } P_2-1 &= 0 \text{ (rejected)}\\
P_2 &= \frac{1}{3}\\
P_3 &= P_2\\
&= \frac{1}{3}
\end{aligned}$$

You may think that all the people will use this strategy. However, according to the theory of Nash, the players will not choose this strategy. It is because this strategy can not prevent other player to choose a better plan to oppose your plan. If we want to prevent other players to change a better plan to oppose your plan, you need to make sure that the probabilities of victory of choosing 1, 2 or 3 are the same.

It means that $(1-P_1)^2 = (P_1^2+P_3^2) = (P_1^2+P_2^2)$.

Since $(P_1^2+P_3^2) = (P_1^2+P_2^2)$, $P_2 = P_3$.

Since $P_1+P_2+P_3 = 1$, $2P_2 = 1-P_1$.

Since $(1-P_1)^2 = P_1^2+P_2^2$,

$$\begin{aligned}
1-2P_1+P_1^2 &= P_1^2+(1-P_1)^2/2^2 \\
1-2P_1 &= (1-2P_1+P_1^2)/4 \\
4-8P_1 &= 1-2P_1+P_1^2 \\
P_1^2+6P_1-3 &= 0
\end{aligned}$$

Therefore, $P_1 = \dfrac{-6+\sqrt{6^2-4\times 1\times(-3)}}{2\times 1} = \dfrac{-6+\sqrt{48}}{2} = -3+2\sqrt{3}$.

Since $2P_2 = 1-P_1$,

$$\begin{aligned}
2P_2 &= 1+3-2\sqrt{3} \\
P_2 &= 2-\sqrt{3} \\
P_3 &= 2-\sqrt{3}
\end{aligned}$$

The results tells us that we should choose 1 more frequently than 2 and 3.

P_1^3, P_2^3 and P_3^3 are the sum of the percentage of the choices of all the players 1, 2 or 3 together. It means that $P_1^3+P_2^3+P_3^3$ is the probability that all players lose.

So that the percentage of each people to win the game is

$$\frac{1-(P_1^3+P_2^3+P_3^3)}{3} = 0.287187078.$$

REFERENCES

[1] 梁煉，數學建模，華南理工大學出版社，頁264 [Liang Lian, *Mathematical Modeling*, South China University of Technology Press, 264.]
[2] 楊啓帆，方道元，數學建模，浙江大學出版社，頁315 [Yang Qifan, Fan Daoyuan, *Mathematical Modeling*, Zhejiang University Press, 315.]
[3] *Stanford Encyclopedia of Philosophy*, http://plato.stanford.edu/entries/game-theory.

Reviewer's Comments

The reviewer has only comments on the wordings, which have been amended in this paper.