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A STUDY ON POLYHEDRON WITH ALL TRIANGULAR FACES: NINE-POINT CIRCLE CO-SPHERE

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Abstract. Previous articles have discussed about the properties of orthocentric tetrahedrons: nine-point circles on each face cospherical and the 3D Euler line. This paper aims at finding the sufficient and necessary conditions for the nine-point circles to be cospherical in the triangular polyhedrons. First, we discussed the conditions for the nine-point circles to be cospherical in a tetrahedron, in a hexahedron and in an octahedron. Next, we found that the 3D orthocenter H_C, the center of the 24-point sphere (48-point sphere) N_C and the 3D circumcenter O_C of a tetrahedron (an octahedron), if they exist, must be collinear and the ratio of the distance between them is $H_CN_C : N_CO_C = 1 : 1$. After studying the properties of triangular polyhedrons, we have found that the existence of the 3D orthocenter and the 3D circumcenter is the necessary condition for the nine-point circles to be cospherical.

1. Introduction

In this paper, we continue the study and the proof in "*the 24-point sphere*" ⟨⟨ *24* 點球面 ⟩⟩ (Liu & Lin, 2007) regarding the properties of the *orthocentric tetrahedron*. We try to use different ways to prove the sufficient and necessary conditions for the nine-point circles to be cospherical. Liu & Lin have also made some assumptions (without proof) about the 36-point sphere and the 48-points sphere in a hexahedron and in an octahedron respectively. In this paper, we develop those concepts and prove the sufficient and necessary conditions for the nine-point circles to be cospherical in a hexahedron and an octahedron.

We use vectors to prove several simple properties of *orthocentric tetrahedron* and to explain why the 36-point sphere will never exists. In the study of the octahedron, we use vectors as well as the 3-dimensional coordinate geometry to obtain the sufficient and necessary conditions for the nine-point circles to be cospherical. "Vector" serves as an important mathematical tool in our studies.

In the second part, we use the 3-dimensional coordinate geometry to analyse the properties regarding the *3D Euler line* in a tetrahedron and in an octahedron. It is amazing to discover that the *3D centroid* G_C and the center of the 24-point sphere N_C are coincided in an *orthocentric tetrahedron.* After our studies about the *orthocentric tetrahedron*, we have found that the *3D orthocenter* H_C, the center of the 24-point sphere (48-point sphere) N_C and the *3D circumcenter* O_C in a tetrahedron (an octahedron), if they exist, must be collinear and ratio of the distance between them is $H_CN_C : N_CO_C = 1 : 1$. The relation between the three centers inspires us in studying the properties of the $6N$-point sphere in *traiangular polyhedron.*

In the third part, we use the properties found in the second part to prove some necessary conditions for the nine-point circles of *triangular polyhedron* to be cospherical. In the future, we hope that the sufficient and necessary conditions for the existence of the $6N$-point sphere will be further discussed and developed.

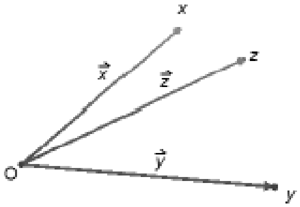
2. Part Zero: Basic Knowledge

1. **Properties of the Euler line**
   1.1. The orthocenter(H), the centroid(G), the circumcenter(O) of any triangles are collinear.
   1.2. $HG : GO = 2 : 1$.
   1.3. The center of the nine-point circle(N) also lies on the Euler line, where $HN : NO = 1 : 1$ and $HN : NG : GO = 3 : 1 : 2$.
2. **Properties of nine-point Circle**
   2.1. In a triangle, the three mid-points on each side, the three foots of each altitude and the three mid-points of the line segments joining the vertices and the orthocenter are concyclic.
3. **Properties of the 3D orthocenter, the 3D centroid and the 3D circumcenter of a tetrahedron**
   3.1. Definition: In this research paper, we define, in a tetrahedron,
      3.1.1 The *3D median* to be the line joining a vertex to the centroid of the opposite face.
      3.1.2 The *3D altitude* to be the line passing through the orthocenter of a face and is perpendicular to that face.
      3.1.3 The *3D normal* (or the *3D perpendicular bisector*) to be the line passing through the circumcenter of a face and perpendicular to that face.
   3.2. The four *3D medians* are concurrent. We define the intersection point to be the *3D centroid.*
   3.3. The distance from a vertex to the *3D centroid* on a *3D median* is three fourth of the length of that *3D median.*

3.4. The six perpendicular plain bisectors of the edges are concurrent. We define the intersection point to be the *3D circumcenter*.
3.5. We define a tetrahedron to be an *orthocentric tetrahedron* if every edge of the tetrahedron is perpendicular to the opposite edge (that is, if we let $\vec{a}$ be a vector parallel to an edge and $\vec{b}$ be a vector parallel to the opposite edge, then $\vec{a} \cdot \vec{b} = 0$).
3.6. The four *3D altitudes* of an *orthocentric tetrahedron* are concurrent. We define the intersection point to be the *3D orthocenter*.
3.7. In an *orthocentric tetrahedron*, the *3D altitude* will pass through the vertex opposite to the face as well, i.e. the *3D altitude* will be the height of the tetrahedron as well.

4. **Properties of an *orthocentric tetrahedron***
4.1. The *orthocentric tetrahedron* OXYZ has the following properties.

$$\begin{cases} \vec{x} \cdot \vec{y} = \vec{z} \cdot \vec{x} \\ \vec{z} \cdot \vec{y} = \vec{z} \cdot \vec{x} \\ \vec{y} \cdot \vec{z} = \vec{y} \cdot \vec{x} \end{cases}$$



In other words,
the projection of $\vec{y}$ on $\vec{x}$ = the projection of $\vec{z}$ on $\vec{x}$,
the projection of $\vec{x}$ on $\vec{z}$ = the projection of $\vec{y}$ on $\vec{z}$,
the projection of $\vec{z}$ on $\vec{y}$ = the projection of $\vec{x}$ on $\vec{y}$,

4.2. The *3D Euler line* exists in an *orthocentric tetrahedron.*
4.3. The other properties of the *3D Euler line* will be proved in this research paper.

5. **Triangular polyhedron**
5.1. We define a polyhedron with all triangular faces to be a *triangular polyhedron.*

3. Part One: The sufficient and necessary conditions for the nine-point circles on each face to be cospherical in a tetrahedron, a hexahedron and an octahedron

If a triangular polyhedron with N faces contains N cospherical nine-point circles, the sphere are said to be a '$6N$-point sphere'.

Proof. Assume that all the nine-point circles on each face of a polyhedron with N triangle faces are cospherical. Let P be the polyhedron and S be the sphere. Note that

1. S intersects each edge at two points (the foot of the perpendicular and the mid-point of the edge) and S intersects each face at three other points (the mid-points of the orthocenter and the vertices), and
2. P has N faces and $\frac{3N}{2}$ edges.

Therefore, there are $\left[3(N)+2\left(\frac{3N}{2}\right)\right]$ cospherical points on the surface of S and we define S to be a "$6N$-point sphere". □

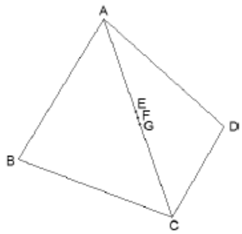
3.1. The sufficient and necessary conditions for the existence of the 24-point circle

Theorem 1. *If the nine-point circles on each face of a tetrahedron are cospherical, then the tetrahedron is an orthocentric tetrahedron.*

Lemma 2. *In a triangular polyhedron, if the foots of the perpendicular from the vertices of two adjacent faces are coincided, we call the two faces having "coincided foot". If the nine-point circles on each face of a polyhedron are cospherical, then any two adjacent faces on the polyhedron must have "coincided foot".*

Proof. Assume the nine-point circle on each face of tetrahedron $ABCD$ are co-spherical.

Let
E be the foot of the perpendicular of B to AC in ΔABC,
G be the foot of the perpendicular of D to AC in ΔACD,
F be the mid-point of AC.



Note that there are at most two intersection points for a sphere and a straight line. Hence, at least two of the points E, F or (and) G must be coincided.

Assume that E and F are coincided, the nine-point circle of ΔABC will intersect AC at one point E only, where the nine-point circle of ΔACD will intersects AC at two points E and G. Hence, the two nine-point circles are not cospherical. Contradiction occur. Therefore, E and F are not coincided. Similarly, F and G are not coincided as well.

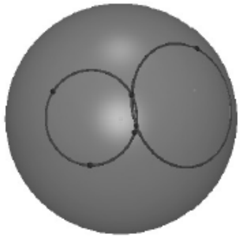
Hence, E and G are coincided. The two adjacent face ΔABC and ΔACD have the "coincided foot". □

Hence, the edge BD is perpendicular to the edge AC ($\overrightarrow{BD}\cdot\overrightarrow{AD}=0$). In other words, the tetrahedron is an *orthocentric tetrahedron* and the *3D orthocenter* exists.

Theorem 3. *If a tetrahedron is an orthocentric tetrahedron, then the nine-point circles on each face of the tetrahedron are cospherical.*

Lemma 4. *If any two intersecting circles are not coplanar, the two circles must be cospherical. The two non-coplanar intersecting circles define a sphere.*

Proof. We can cut two intersecting circles with any ratio on a sphere. In the other words, two intersecting circles with any ratio are *cospherical* if they are not co-planar. The intersection point of the normals from the centers of the two circles is the center of the sphere. □



Lemma 5. *If any four points are not co-planar, the four points must be cospherical. The four points define a sphere.*

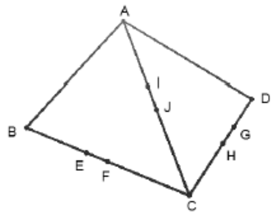
Proof. In the 3D coordinate system, the equation of a sphere with center (x_c, y_c, z_c) is:

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2.$$

There are four unknowns x_c, y_c, z_c and r in the equation, hence, we need four points to get an unique solution for the four unknowns. □

Assume that the tetrahedron $ABCD$ is an *orthocentric tetrahedron.*

Let E, I be the foots of the perpendiculars of ΔABC,
G, I be the foots of the perpendiculars of ΔAPC,
J, F, H be the mid-point of AC, BC, CD respectively.



Let the nine-point of ΔABC, ΔADC and ΔBCD be C_α, C_β and C_γ respectively.

According the lemma 4, any two non-coplanar circles must be cospherical, we let

$S_{(\alpha,\beta)}$ be the sphere defined by C_α and C_β
$S_{(\beta,\gamma)}$ be the sphere defined by C_β and C_γ, and
$S_{(\gamma,\alpha)}$ be the sphere defined by C_γ and C_α.

Note that I, J, F and H lie on the surface of $S_{(\alpha,\beta)}$, $S_{(\beta,\gamma)}$ and $S_{(\gamma,\alpha)}$. Since I, J, F, H are no coplanar, according to lemma 5, $S_{(\alpha,\beta)}$, $S_{(\beta,\gamma)}$ and $S_{(\gamma,\alpha)}$ are the same sphere. Therefore, C_α, C_β and C_γ are cospherical. Similarly, we can prove that the nine-point circles on any three faces of tetrahedron are cospherical. Hence, all nine-point circles on the faces of the tetrahedron are cospherical.

Combining theorem 1 and 3:

24-point sphere exists if and only if the tetrahedron is an orthocentric tetrahedron.

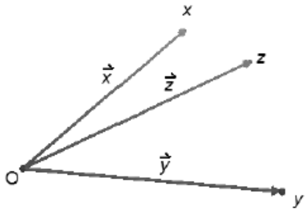
3.2. The conditions for the existence of the 36-point circle

Theorem 6. *The 36-point sphere does not exist.*

Lemma 7. *If three adjacent faces of a tetrahedron have "coincided foots", then any two adjacent faces of the tetrahedron must have "coincided foots" as well.*

Proof. Assume that ΔOXY, ΔOYZ and ΔOZX have "*coincided foots*", we have:

$$|\overrightarrow{OX}|\cos\angle XOY = |\overrightarrow{OZ}|\cos\angle ZOY$$
$$|\overrightarrow{OX}||\overrightarrow{OY}|\cos\angle XOY = |\overrightarrow{OZ}||\overrightarrow{OY}|\cos\angle ZOY$$
$$\vec{x}\cdot\vec{y} = \vec{z}\cdot\vec{y}$$



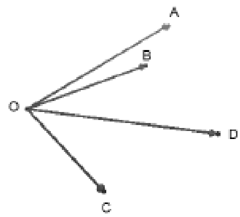
Similarly, $\vec{y}\cdot\vec{x} = \vec{z}\cdot\vec{x}$ and $\vec{x}\cdot\vec{z} = \vec{y}\cdot\vec{z}$.

Hence $\begin{cases} \vec{x}\cdot\vec{y} = \vec{z}\cdot\vec{x} \\ \vec{z}\cdot\vec{y} = \vec{z}\cdot\vec{x} \\ \vec{y}\cdot\vec{z} = \vec{y}\cdot\vec{x} \end{cases} \Rightarrow \begin{cases} (\vec{x}-\vec{z})\cdot(\vec{y}-\vec{z}) = -\vec{z}\cdot(\vec{y}-\vec{z}) \\ (\vec{z}-\vec{x})\cdot(\vec{y}-\vec{x}) = -\vec{x}\cdot(\vec{y}-\vec{x}) \\ (\vec{y}-\vec{x})\cdot(\vec{z}-\vec{x}) = -\vec{x}\cdot(\vec{z}-\vec{x}) \end{cases}$.

In other words, the foot of the perpendicular from the vertex of the forth face are also coincided with the foot of the perpendicular form the vertices of the other three faces. ☐

Hence, the tetrahedron $OXYZ$ is an *orthocenric tetrahedron.*

Now, we consider the hexahedron $OABCD$.
Assume that the 36-point sphere exists in the hexahedron $OABCD$.
According to the lemma 2, then the foots of the perpendicular of ΔOAB, ΔOAD and ΔBAD are coincided. Hence, according the lemma 7, the tetrahedron $OABD$ is an *orthocentric tetrahedron.* Similarly, the tetrahedron $COBD$ is an *orthocentric tetrahedron.*



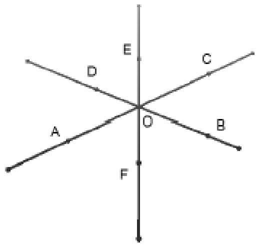
However, we have proved that the center of the 24-point sphere is the *3D centroid* of the tetrahedron (Part 4, Theorem 13), which lies inside the tetrahedron. In other words, the centers of the two 24-point spheres of the tetrahedron $OABD$ and the tetrahedron $COBD$ are not coincided. Contradiction occurs as it is impossible for the 36-point sphere having two center. Hence, the 36-point sphere does not exist.

3.3. The sufficient and necessary conditions for the existence the 48-point sphere

Theorem 8. *If the 48-point sphere exists, then the octahedron $EABCDF$ fulfills the following conditions:*

I The three diagonals of the octahedron are perpendicular to each other (i.e. $AC \perp BD \perp EF$) and concurrent, and

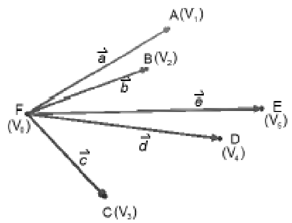
II $(OA)(OC) = (OB)(OD) = (OE)(OF)$, where O is the intersection of AC, BD and EF.



Lemma 9. *If the 48-point sphere exists in the octahedron $EABCEF$, then the vertices A, B, C, D, vertices F, B, E, D and vertices F, A, E, C must be coplanar.*

Proof.
Assume the 48-point sphere exists in the octahedron $EABCEF$.
Consider the four edges $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ connenting to V_0.
According to lemma 2,

$$S_0 \begin{cases} \vec{a}\cdot\vec{b} = \vec{c}\cdot\vec{b} \\ \vec{b}\cdot\vec{c} = \vec{d}\cdot\vec{c} \\ \vec{c}\cdot\vec{d} = \vec{a}\cdot\vec{d} \\ \vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{d} \end{cases}$$



Similarly, consider the four edges $-\vec{a}, \vec{d}-\vec{a}, \vec{e}-\vec{a}, \vec{b}-\vec{a}$ connecting to V_1:

$$S_1 \begin{cases} -\vec{a}\cdot(\vec{d}-\vec{a}) = (\vec{e}-\vec{a})\cdot(\vec{d}-\vec{a}) \\ (\vec{d}-\vec{a})\cdot(\vec{e}-\vec{a}) = (\vec{b}-\vec{a})\cdot(\vec{e}-\vec{a}) \\ (\vec{e}-\vec{a})\cdot(\vec{b}-\vec{a}) = -\vec{a}\cdot(\vec{b}-\vec{a}) \\ (\vec{b}-\vec{a})\cdot(-\vec{a}) = (\vec{d}-\vec{a})\cdot(-\vec{a}) \end{cases} \Rightarrow \begin{cases} \vec{e}\cdot\vec{d} = \vec{e}\cdot\vec{a} \\ \vec{d}\cdot\vec{e} - \vec{d}\cdot\vec{a} = \vec{b}\cdot\vec{e} - \vec{e}\cdot\vec{a} \\ \vec{e}\cdot\vec{b} = \vec{e}\cdot\vec{a} \\ \vec{b}\cdot\vec{a} = \vec{d}\cdot\vec{a} \end{cases} \Rightarrow \begin{cases} \vec{e}\cdot\vec{d} = \vec{e}\cdot\vec{a} \\ \vec{d}\cdot\vec{e} = \vec{b}\cdot\vec{e} \\ \vec{e}\cdot\vec{b} = \vec{e}\cdot\vec{a} \\ \vec{b}\cdot\vec{a} = \vec{d}\cdot\vec{a} \end{cases}$$

Now, consider the four edges $-\vec{b}$, $\vec{a}-\vec{b}$, $\vec{e}-\vec{b}$, $\vec{c}-\vec{b}$ connecting to V_2,
the four edges $-\vec{c}$, $\vec{b}-\vec{c}$, $\vec{e}-\vec{c}$, $\vec{d}-\vec{c}$ connecting to V_3,
the four edges $-\vec{d}$, $\vec{c}-\vec{d}$, $\vec{e}-\vec{d}$, $\vec{a}-\vec{d}$ connecting to V_4,
the four edges $\vec{a}-\vec{e}$, $\vec{d}-\vec{e}$, $\vec{c}-\vec{e}$, $\vec{b}-\vec{e}$ connecting to V_5,

we have $S_2\begin{cases}\vec{e}\cdot\vec{a}=\vec{e}\cdot\vec{b}\\ \vec{a}\cdot\vec{e}=\vec{e}\cdot\vec{c}\\ \vec{e}\cdot\vec{c}=\vec{e}\cdot\vec{b}\\ \vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{b}\end{cases}$, $S_3\begin{cases}\vec{e}\cdot\vec{b}=\vec{e}\cdot\vec{c}\\ \vec{e}\cdot\vec{b}=\vec{d}\cdot\vec{e}\\ \vec{e}\cdot\vec{d}=\vec{e}\cdot\vec{c}\\ \vec{d}\cdot\vec{c}=\vec{b}\cdot\vec{c}\end{cases}$, $S_4\begin{cases}\vec{e}\cdot\vec{c}=\vec{e}\cdot\vec{d}\\ \vec{e}\cdot\vec{c}=\vec{a}\cdot\vec{e}\\ \vec{e}\cdot\vec{a}=\vec{e}\cdot\vec{d}\\ \vec{a}\cdot\vec{d}=\vec{c}\cdot\vec{d}\end{cases}$, and

$$S_5\begin{cases}(\vec{a}-\vec{e})\cdot(\vec{d}-\vec{e})=(\vec{c}-\vec{e})\cdot(\vec{d}-\vec{e})\\ (\vec{d}-\vec{e})\cdot(\vec{c}-\vec{e})=(\vec{b}-\vec{e})\cdot(\vec{c}-\vec{e})\\ (\vec{c}-\vec{e})\cdot(\vec{b}-\vec{e})=(\vec{a}-\vec{e})\cdot(\vec{b}-\vec{e})\\ (\vec{b}-\vec{e})\cdot(\vec{a}-\vec{e})=(\vec{d}-\vec{e})\cdot(\vec{a}-\vec{e})\end{cases}\Rightarrow\begin{cases}\vec{a}\cdot\vec{d}-\vec{a}\cdot\vec{e}=\vec{c}\cdot\vec{d}-\vec{c}\cdot\vec{e}\\ \vec{d}\cdot\vec{c}-\vec{d}\cdot\vec{e}=\vec{b}\cdot\vec{c}-\vec{b}\cdot\vec{e}\\ \vec{c}\cdot\vec{b}-\vec{c}\cdot\vec{e}=\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{e}\\ \vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{e}=\vec{d}\cdot\vec{a}-\vec{d}\cdot\vec{e}\end{cases}$$

Combining S_0, S_1, S_2, S_3, S_4 and S_5, we have $S_T\begin{cases}\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{b}\\ \vec{b}\cdot\vec{c}=\vec{d}\cdot\vec{c}\\ \vec{c}\cdot\vec{d}=\vec{a}\cdot\vec{d}\\ {}^*\vec{b}\cdot\vec{e}=\vec{e}\cdot\vec{a}\\ {}^*\vec{d}\cdot\vec{e}=\vec{e}\cdot\vec{b}\\ {}^*\vec{a}\cdot\vec{e}=\vec{e}\cdot\vec{c}\end{cases}$.

From the equation with *, we know A, B, C, D are coplanar.
Next, consider the vertices F, B, E and D. Let

$$-\vec{a}\text{ be }\vec{i},$$
$$\vec{d}-\vec{a}\text{ be }\vec{k},$$
$$\vec{e}-\vec{a}\text{ be }\vec{l},$$
$$\vec{b}-\vec{a}\text{ be }\vec{m},\text{ and}$$
$$\vec{c}-\vec{a}\text{ be }\vec{n},$$

Consider the vectors connecting V_1 and other vertices

$$\begin{cases}\vec{k}\cdot\vec{n}=\vec{n}\cdot\vec{i}\\ \vec{m}\cdot\vec{n}=\vec{n}\cdot\vec{k}\\ \vec{i}\cdot\vec{n}=\vec{n}\cdot\vec{l}\end{cases}\Leftrightarrow\begin{cases}(\vec{c}-\vec{a})\cdot(\vec{b}-\vec{a})=(\vec{c}-\vec{a})\cdot(-\vec{a})\\ (\vec{b}-\vec{a})\cdot(\vec{c}-\vec{a})=(\vec{d}-\vec{a})\cdot(\vec{c}-\vec{a})\\ (-\vec{a})\cdot(\vec{c}-\vec{a})=(\vec{c}-\vec{a})\cdot(\vec{e}-\vec{a})\end{cases}$$

$$\Leftrightarrow\begin{cases}\vec{b}\cdot\vec{c}=\vec{a}\cdot\vec{b}\\ \vec{b}\cdot\vec{c}-\vec{b}\cdot\vec{a}=\vec{c}\cdot\vec{d}-\vec{d}\cdot\vec{a}\\ \vec{e}\cdot\vec{c}=\vec{e}\cdot\vec{a}\end{cases}\Leftrightarrow\begin{cases}\vec{b}\cdot\vec{c}=\vec{a}\cdot\vec{b}\\ 0=0\\ \vec{e}\cdot\vec{c}=\vec{e}\cdot\vec{a}\end{cases}$$

In other words, F, B, E and D are coplanar.
Next, consider the vertices F, A, E and C. Let

$$-\vec{d}\text{ be }\vec{u},$$
$$\vec{c}-\vec{d}\text{ be }\vec{v},$$
$$\vec{e}-\vec{d}\text{ be }\vec{w},$$
$$\vec{a}-\vec{d}\text{ be }\vec{x},\text{ and}$$
$$\vec{b}-\vec{d}\text{ be }\vec{y}.$$

Consider the vectors connecting V_4 and other vertices.

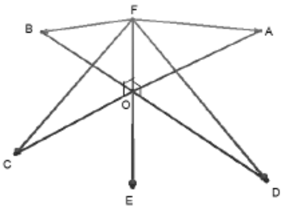
$$\begin{cases} \vec{v}\cdot\vec{y}=\vec{y}\cdot\vec{u} \\ \vec{x}\cdot\vec{y}=\vec{y}\cdot\vec{v} \\ \vec{v}\cdot\vec{y}=\vec{y}\cdot\vec{w} \end{cases} \Leftrightarrow \begin{cases} (\vec{b}-\vec{d})\cdot(\vec{a}-\vec{d})=(\vec{b}-\vec{d})\cdot(-\vec{d}) \\ (\vec{b}-\vec{d})\cdot(\vec{a}-\vec{d})=(\vec{c}-\vec{d})\cdot(\vec{b}-\vec{d}) \\ (-\vec{d})\cdot(\vec{b}-\vec{d})=(\vec{b}-\vec{d})\cdot(\vec{e}-\vec{d}) \end{cases}$$

$$\Leftrightarrow \begin{cases} \vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{d} \\ \vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{d}=\vec{c}\cdot\vec{b}-\vec{c}\cdot\vec{d} \\ \vec{e}\cdot\vec{b}=\vec{e}\cdot\vec{d} \end{cases} \Leftrightarrow \begin{cases} \vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{d} \\ 0=0 \\ \vec{e}\cdot\vec{b}=\vec{e}\cdot\vec{d} \end{cases}$$

Hence, F, A, E and C are coplanar. □

Lemma 10. *If the 48-point sphere exists in the octahedron $EABCEF$, then the diagonals AC, BD and EF must be perpendicular to each other.*

Proof.



Note that

$(\vec{a}-\vec{c})\cdot(\vec{b}-\vec{d}) = (\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{d})-(\vec{c}\cdot\vec{b}-\vec{c}\cdot\vec{d}) = 0,$

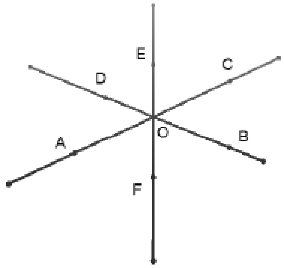
Hence, AC is perpendicular to BD.

Similarly, we know the diagonals AC, BD, EF are perpendicular to each other. □

Lemma 11. *If the 48-point sphere exists in the octahedron $EABCEF$, then $(OA)(OC) = (OB)(OD) = (OE)(OF)$, where O is the intersection point of the diagonals AC, BD and EF.*

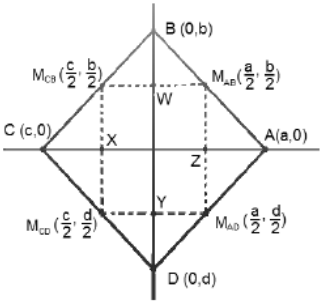
Proof. Now we introduce the 3D coordinate system to the octahedron $EABCDF$. Since the diagonals are perpendicular to each other, we transform the octahedron $EABCDF$ such that AC lies on the x-axis, BD lies on the y-axis and EF lies on the z-axis.

Let
$A = (a,0,0)$, $B = (0,b,0)$, $C = (c,0,0)$,
$D = (0,d,0)$, $E = (0,0,e)$, $C = (0,0,f)$, and
$M_{\alpha\beta}$ be the mid-point of α and β,
where α, $\beta \in \{A,B,C,D,E,F\}$.



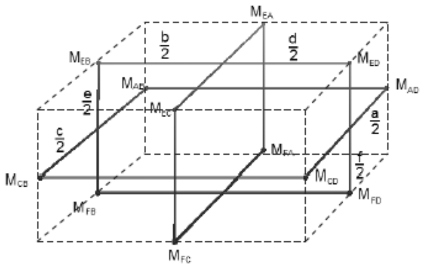
Consider the plane $ABCD$.

Note that $\Delta CBA \sim \Delta M_{CB}BM_{AB}$ and $\Delta CDA \sim \Delta M_{CD}DM_{AD}$.
Hence, $M_{AB}M_{CB}//AC//M_{AD}M_{CD}$.
Similarly, $M_{CB}M_{CD}//BD//M_{AB}M_{AD}$.
In other words, $M_{AB}M_{CB}M_{CD}M_{AD}$ is a parallelogram.



Moreover, since M_{AB}, M_{CB}, M_{CD} and M_{AD} are co-circular, and $\angle M_{CB}M_{AB}M_{AD} = \angle M_{AD}M_{CD}M_{CB}$ (property of parallelogram), $M_{AB}M_{CB}M_{CD}M_{AD}$ is a rectangle.
Similarly, $M_{EB}M_{FB}M_{FD}M_{ED}$ and $M_{EA}M_{EC}M_{FC}M_{FA}$ are rectangles also.

Hence, the 12 mid-points of the edges of the octahedron $EABCDF$,
i.e. $M_{EA}, M_{EB}, M_{EC}, M_{ED}, M_{AB}, M_{CB}$, $M_{CD}, M_{AD}, M_{FA}, M_{FB}, M_{FC}$ and M_{FD} lies on the 12 edges of a cuboid.

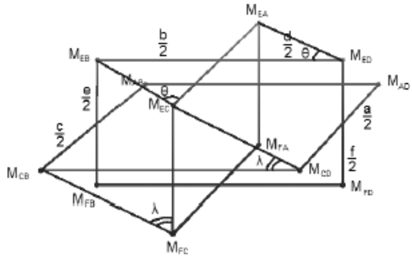


Note that M_{EB}, M_{EA}, M_{EC} and M_{ED} are con-cyclic and $M_{EB}M_{ED} \perp M_{EA}M_{EC}$, therefore

$$\tan\angle M_{EA}M_{ED}M_{EB} = \tan\angle M_{EB}M_{EC}M_{EA}.$$

$$\text{i.e. } \tan\theta = \frac{\frac{b}{2}}{\frac{c}{2}} = \frac{\frac{a}{2}}{\frac{d}{2}}$$

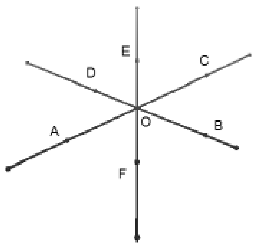
$$\text{Similarly, } \tan\lambda = \frac{\frac{e}{2}}{\frac{d}{2}} = \frac{\frac{b}{2}}{\frac{f}{2}}.$$



Therefore we have $ac = bd = ef$. $\square$

Theorem 12. *The 48-point sphere exists if the octahedron $EABCDF$ fulfills the following conditions:*

I The three diagonals of the octahedron are perpendicular to each other (i.e. $AC \perp BD \perp EF$) and concurrent, and
II $(OA)(OC) = (OB)(OD) = (OE)(OF)$, where O is the intersection of AC, BD and EF.



The coordinates of the 12 mid-points of the 12 edges are:

$$M_{EA}=\left(\frac{a}{2},0,\frac{e}{2}\right),\ M_{EC}=\left(\frac{c}{2},0,\frac{e}{2}\right),\ M_{EB}=\left(0,\frac{b}{2},\frac{e}{2}\right),\ M_{ED}=\left(0,\frac{d}{2},\frac{e}{2}\right)$$
$$M_{AD}=\left(\frac{a}{2},\frac{d}{2},0\right),\ M_{AB}=\left(\frac{a}{2},\frac{b}{2},0\right),\ M_{CD}=\left(\frac{c}{2},\frac{d}{2},0\right),\ M_{BC}=\left(\frac{c}{2},\frac{b}{2},0\right)$$
$$M_{FA}=\left(\frac{a}{2},0,\frac{f}{2}\right),\ M_{FC}=\left(\frac{a}{2},0,\frac{f}{2}\right),\ M_{FB}=\left(0,\frac{b}{2},\frac{f}{2}\right),\ M_{FD}=\left(0,\frac{d}{2},\frac{f}{2}\right)$$

Since $ac=bd=ef$, there exist a point $G\left(\frac{a+c}{4},\frac{b+d}{4},\frac{e+f}{4}\right)$ such that G is equidistant with all the 12 mid-points above. Let r be the distance between G and the mid-points,

$$r^2=\frac{a^2+b^2+c^2+d^2+e^2+f^2-2x}{4^2}\quad \text{where } x=ac=bd=ef.$$

Hence, G is the center and r is the radius of the 48-point sphere. In other words, the 12 mid-points are cospherical.

In addition, the three mid-points of the triangle on each face define the nine-point circle, i.e. the 8 nine-point circles are cospherical. In other words, the 48 points of the 8 nine-point circles are cospherical.

Combining theorem 8 and 12:

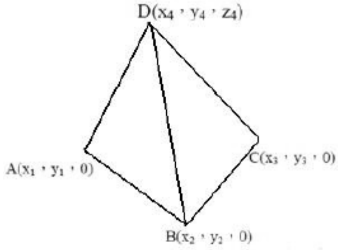
> The 48-point sphere exists if and only if the octahedron $EABCDF$ fulfills the following conditions:
> I. The three diagonals of the octahedron are perpendicular to each other and concurrent, and
> II. $(OA)(OC)=(OB)(OD)=(OE)(OF)$, where O is the intersection of AC, BD and EF.

4. Part Two: Analysis of the tetrahedron and octahedron

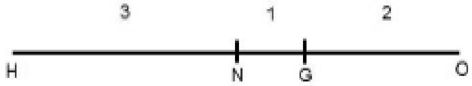
4.1. Analysis of the tetrahedron

Theorem 13. *In an orthocentric tetrahedron, the center of the 24-point sphere and the 3D centroid coincided.*

Let the four vertices of an *orthocentric tetrahedron* be $A(x_1, y_1, 0)$, $B(x_2, y_2, 0)$, $C(x_3, y_3, 0)$ and $D(x_4, y_4, z_4)$. In an *orthocentric tetrahedron*, the *3D altitude* will be the height of the tetrahedron which passes through the orthocenter of the opposite face, therefore, the coordinates of the orthocenter H of ΔABC is $(x_4, y_4, 0)$.



Consider the properties of Euler line and nine-point circle, the ratio of the distance between the orthocenter H, the center of nine-point circle N, the centroid G and the circumcenter O of ΔABC are $HN : NG : GO = 3 : 1 : 2$.



Using the section formula, the coordinates of the center of the nine-point circle of ΔABC is

$$\left(\frac{3\left(\frac{x_1+x_2+x_3}{3}\right)+x_4}{4}, \frac{3\left(\frac{y_1+y_2+y_3}{3}\right)+y_4}{4}, 0\right) = \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, 0\right)$$

Let the *3D centroid* of tetrahedron be G_C, the coordinates of G_C is:

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_4}{4}\right)$$

In other words, N is the projection of G_C to ΔABC. Hence, G_C is equidistant from any points on the nine-point circle of ΔABC.

Now, we consider the coordinates of the mid-points of the three vertices and the orthocenter $H(x_4, y_4, 0)$ of ΔABC:

$$M_{HA}\left(\frac{x_1+x_4}{2}, \frac{y_1+y_4}{2}, 0\right),\ M_{HB}\left(\frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}, 0\right),\ M_{DC}\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}, \frac{z_4}{2}\right)$$

These three points are equidistant from G_C as they lie on the nine-point circle of ΔABC.

Considering the coordinates the mid-points of DA, DB and DC:

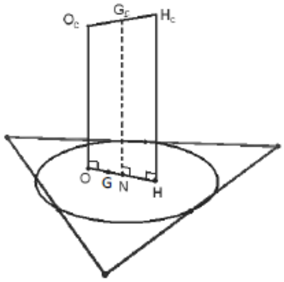
$$M_{DA}\left(\frac{x_1+x_4}{2}, \frac{y_1+y_4}{2}, \frac{z_4}{2}\right),\ M_{DB}\left(\frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}, \frac{z_4}{2}\right),\ M_{DC}\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}, \frac{z_4}{2}\right)$$

These three points are also equidistant from G_C. In other words, these above six points are all equidistant from G_C and they all lie on the surface of the 24-point sphere.

According to lemma 5, these six points define the sphere, hence, the *3D centroid* G_C is the center of the 24-point sphere.

Theorem 14. *The center of the 24-point sphere G_C is the mid-point of 3D orthocenter H_C and the 3D circumcenter O_C of the tetrahedron.*

Since the *3D Euler line* exists in an *orthocentric tetrahedron*, the *3D orthocenter* H_C, the *3D centroid* G_C and the *3D circumcenter* O_C are collinear.



Note that the projection of O_C to ΔABC is the circumcenter O of ΔABC, the projection of H_C to ΔABC is the orthocenter H.

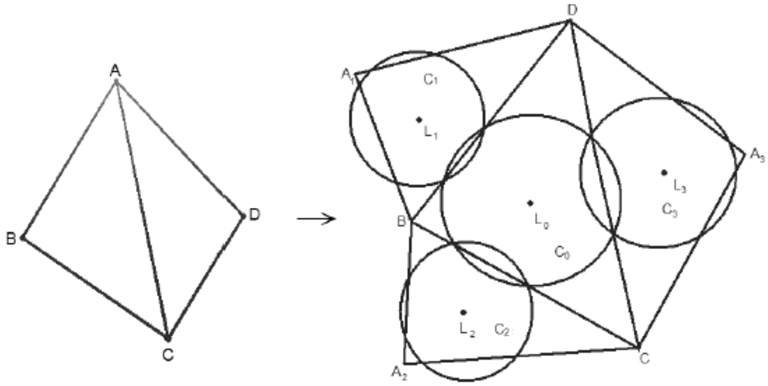
According to theorem 13, the center of the nine-point circle N is the projection of G_C to ΔABC. Hence, using the intercept theorem,

$$H_C G_C : G_C O_C = HN : NO = 1 : 1$$

Hence, the center of the 24-point spher N_C is the mid-point of *3D orthocenter* H_C and the *3D circumcenter* O_C of the tetrahedron.

4.2. Another proof for the existence of the 24-point sphere in an *orthocentric tetrahedron*

Assume that the tetrahedron $ABCD$ is an *orthocentric tetrahedron.*



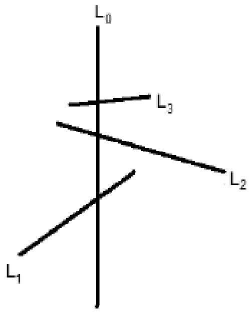
Let $A_1BA_2CA_3D$ to be the net of tetrahedron $ABCD$,

C_1, C_2, C_3 and C_0 be the center of the nine-point circles of $\Delta A_1BD, \Delta A_2BC, \Delta A_3CD$ and ΔBDC respectively, and

L_1, L_2, L_3 and L_0 be the normal passing through the center of C_1, C_2, C_3 and C_4 respectively.

Definition 15. *In a 3-dimensional space, we denote the situation "L_n and L_m has a point of intersection" by $(L_n \circ L_m)$.*

Note that the line segment joining the centers of two adjacent circles is perpendicular to chord formed by joining the two intersection points of that two circles. Hence, $(L_1 \circ L_0), (L_2 \circ L_0), (L_3 \circ L_0), (L_1 \circ L_2), (L_2 \circ L_3)$ and $(L_3 \circ L_1)$. Suppose L_1, L_2 and L_3 do not intersect at the same point at L_0. Then L_1, L_2 and L_3 will never intersect with each other, however, since $L_1 \circ L_2$, contradict occurs. Hence, at least two of the normals L_1, L_2 and L_3 are concurrent with L_0.



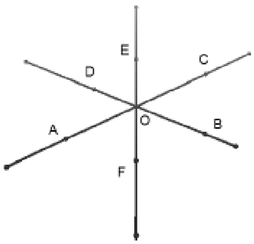
Suppose $L_1 \circ L_2 \circ L_0$, but they do not intersect with L_3, then L_3 and L_1 will never intersect with each other, however, since $L_3 \circ L_1$, contradict occurs. Therefore, $L_1 \circ L_2 \circ L_0 \circ L_3$. And, thus, C_1, C_2, C_3 and C_0 are co-spherical and the 24-point sphere exists

4.3. The analysis of the octahedron

Theorem 16. *The 3D circumcenter of an octahedron exists if the 48-point sphere exists.*

The *3D circumcenter* of is the intersection of planes which perpendicularly bisect each edge. Assume that the 48-point sphere exists in the octahedron $EABCDF$.

Considering the $x-y$ plane.
Let L_{AB} and L_{BC} be the perpendicular bisectors of AB and BC respectively,



$$m_{L_{AB}} = -\frac{b}{a},\ m_{L_{BC}} = -\frac{b}{c}$$

$$L_{AB} : b\Big(y - \frac{b}{2}\Big) = a\Big(x - \frac{a}{2}\Big) \Rightarrow by - \frac{b^2}{2} = ax - \frac{a^2}{2}.$$

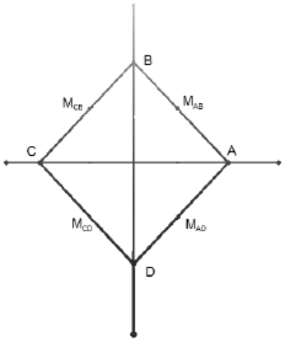
$$L_{BC} : b\Big(y - \frac{b}{2}\Big) = c\Big(x - \frac{a}{2}\Big) \Rightarrow by - \frac{b^2}{2} = cx - \frac{c^2}{2}.$$

Let the intersection of L_{AB} and L_{BC} be $P(u, v)$, we have

$$au - \tfrac{a^2}{2} = cu - \tfrac{c^2}{2} \qquad bv - \tfrac{b^2}{2} = \tfrac{a(a+c)}{2} - \tfrac{a^2}{2}$$

$$(a-c)u = \tfrac{a^2-c^2}{2} \quad \text{and} \qquad v = \tfrac{ac+b^2}{2b}$$

$$u = \tfrac{a+c}{2} \qquad v = \tfrac{bd+b^2}{2b} = \tfrac{b+d}{2}$$



Hence, $P = \Big(\dfrac{a+c}{2}, \dfrac{b+d}{2}\Big)$.

Similarly, P is the intersection of L_{AB}, L_{BC}, L_{CD} and L_{DA}.
Therefore, the '*3D perpendicular bisector*' on x-y plane is: $\left(\dfrac{a+c}{2}, \dfrac{b+d}{2}, t\right), t \in \mathbb{R}$.
Similarly, the '*3D perpendicular bisector*' on x-z plane is: $\left(\dfrac{a+c}{2}, s, \dfrac{e+f}{2}\right), s \in \mathbb{R}$,
while the '*3D perpendicular bisector*' on $y-z$ plane is: $\left(w, \dfrac{b+d}{2}, \dfrac{e+f}{2}\right), w \in \mathbb{R}$.
Hence, the *3D circumventer* is: $\left(\dfrac{a+c}{2}, \dfrac{b+d}{2}, \dfrac{e+f}{2}\right)$.

By considering the coordinates of the six vertices of the octahedron: $A(a,0,0), B(0,b,0), C(c,0,0), D(0,d,0), E(0,0,e)$ and $F(0,0,f)$, the radius R of the circumsphere of the octahedron is:

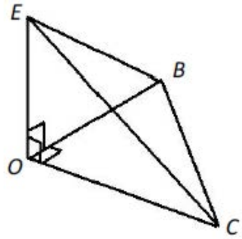
$$R^2 = \frac{a^2+b^2+c^2+d^2+e^2+f^2+2x}{2^2}, \text{ where } x = ac = bd = ef.$$

Theorem 17. *The 3D orthocenter of an octahedron exists if the 48-point sphere exists.*

In a *orthocentric tetrahedron*, the *3D altitude* to be the line passing through the orthocenter of a face and is perpendicular to that face, which is also the height of the tetrahedron, however, there is not easy to define the 'height' of a octahedron. Hence, we define the *3D altitude* of an octahedron as follow:

Definition 18. *A '3D altitude' of a triangular polyhedron is normal to a face a t the orthocenter.*

Suppose the 48-point sphere exists in an octahedron, the three diagonals of the octahedron are perpendicular to each other and concurrent, therefore, the octahedron can be split into eight right-angled tetrahedrons. One of the right-angled tetrahedron is shown in the figure on the right.



Since O is the *3D orthocenter* of the tetrahedron $OBCE$, the normal of the orthocenter of ΔBCE must pass through the point O. Similarly, the normal of the orthocenter of all other faces will pass through the point O. Now, from the definition of the *3D altitude* of polyhedron, $O(0,0,0)$ will be the *3D orthocenter* of the octahedron.

Now, according to theorem 12, 16 and 17, the coordinates of the *3D circumcenter* O_C, the center of the 48-point sphere N_C, and the *3D orthocenter* H_C are $\left(\frac{a+c}{2}, \frac{b+d}{2}, \frac{e+f}{2}\right), \left(\frac{a+c}{4}, \frac{b+d}{4}, \frac{e+f}{4}\right), (0,0,0)$ respectively. Hence

The 3D orthocenter H_C, the center of 48-point sphere N_C and 3D circumcenter O_C are collinear and $H_CN_C : N_CO_C = 1:1$. We call this line "the 3D Euler line".

According to theorem 13,

The center of the 24-point sphere is the 3D centroid of the tetrahedron.

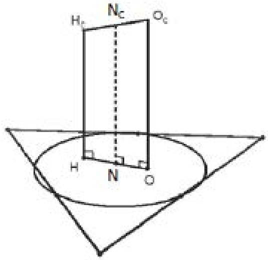
Therefore we guess, there will be a relation between the *3D centroid* of a polyhedron the center of the $6N$-point sphere. Under a certain condition, the *3D centroid* and the center of the $6N$-point sphere will coincide.

5. Part Three:Analysis of the triangular polyhedron

5.1. The condition of $6N$-point sphere of a polyhedron

Theorem 19. *If the 3D orthocenter and 3D circumcenter of a triangular polyhedron exist, then the center of the 6N-point sphere must exist.*

Suppose the *3D orthocenter* H_C and the *3D circumcenter* O_C exist in a *triangular polyhedron.* H_C is the intersection of the *3D altitudes*, while O_C is the intersection of the *3D normals* (or the *3D perpendicular bisectors*). Therefore, the orthocenter and circumcenter on each face can be viewed is the projection of H_C and O_C on that face respectively.



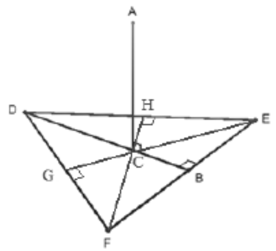
Note that, on any faces of a *triangular polyhedron*, the center of a nine-point circle N on is the mid-point of orthocenter H and circumcenter O. Using the intercept theorem, the normals passing through the centre of the nine-point circles on each face must intersect at N_C and $H_CN_C : N_CO_C = 1 : 1$, where N_C is the center of the $6N$-point sphere.

Hence, the center of sphere of $6N$-point sphere exists and it must be the mid-point of the *3D orthocenter* and the *3D circumcenter.*

Theorem 20. *If the 3D orthocenter and the 3D circumcenter exist in a triangular polyhedron, then the polyhedron can be divided into N orthocentric tetrahedrons.*

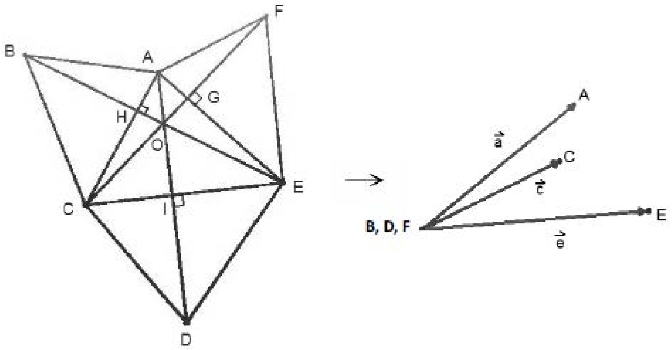
Lemma 21. *In the tetrahedron $ADEF$, if the 3D altitude is the height of a tetrahedron, i.e. the 3D altitude at C passes through A as well, then $AB \perp EE, AG \perp DF$ and $AH \perp DE$.*

Proof.
Since $EF \perp DB$, AB will be the shortest distance from A to EF. Hence $AB \perp EF$.
Similarly, $AG \perp DF, AH \perp DE$.



□

Lemma 22. *If the net $ABCDEF$ of a tetrahedron fulfills the following condition: (I)$AE \perp CF$, (II) $AC \perp BE$ and (III) $CE \perp AD$, the 3D orthocenter of a tetrahedron exists*



Proof. Considering the projection of $\overrightarrow{AB}$ on $\overrightarrow{AC}$ and $\overrightarrow{AE}$ on $\overrightarrow{AC}$,

$$\vec{a} \cdot (\vec{a} - \vec{c}) = (\vec{a} \cdot \vec{e}) \cdot (\vec{a} - \vec{c}) \Rightarrow \vec{a} \cdot \vec{e} = \vec{a} \cdot \vec{c}$$

Similarly,

$$\begin{cases} \vec{a} \cdot (\vec{a} - \vec{c}) = (\vec{a} - \vec{e}) \cdot (\vec{a} - \vec{c}) \\ \vec{e} \cdot (\vec{e} - \vec{c}) = (\vec{e} - \vec{a}) \cdot (\vec{e} - \vec{c}) \\ \vec{e} \cdot (\vec{e} - \vec{a}) = (\vec{e} - \vec{c}) \cdot (\vec{e} - \vec{a}) \end{cases} \Rightarrow \begin{cases} \vec{e} \cdot \vec{a} = \vec{e} \cdot \vec{a} \\ \vec{a} \cdot \vec{e} = \vec{a} \cdot \vec{c} \\ \vec{c} \cdot \vec{e} = \vec{c} \cdot \vec{a} \end{cases}$$

Hence, the *3D orthocenter* of the tetrahedron exists. □

If the *3D orthocenter* of a N-face triangular polyhedron exists, using the 3 vertices of any face and *3D orthocenter*, we can divide the polyhedron into N tetrahedrons. Considering the tetrahedrons formed, the *3D altitudes* will also be the height of the tetrahedrons as well, hence, according to lemma 21 and lemma 22, all the tetrahedrons formed are *orthocentric tetrahedrons*.

6. Conclusion

In the above study, we have shown that:

1. The 24-point sphere exists if and only if the tetrahedron is an *orthocentric tetrahedron.*
2. The 36-point sphere does not exist.
3. The 48-point sphere exists if and only if the octahedron $EABCDF$ fulfills the following conditions:
   I. *The three diagonals of the octahedron are perpendicular to each other and concurrent, and*
   II. $(OA)(OC) = (OB)(OD) = (OE)(OF)$, *where* O *is the intersection of* AC, BD *and* EF.
4. In an *orthocentric tetrahedron*, the center of the 24-point sphere is the *3D centroid* of the tetrahedron.
5. The *3D orthocenter* H_C, the center of the 24-point sphere N_C, and the *3D circumcenter* O_C of an *orthocentric tetrahedron* are collinear. In addition, $H_CN_C : N_CO_C = 1 : 1$.
6. The *3D circumcenter* of an octahedron exists if the 48-point sphere exists.
7. If the 48-point sphere exists in an octahedron, the *3D orthocenter* H_C, the center of 48-point sphere N_C and *3D circumcenter* O_C are collinear and $H_CN_C : N_CO_C = 1 : 1$. We call this line "the line of centers of sphere".
8. If the *3D orthocenter* and *3D circumcenter* of a *triangular polyhedron* exist, then the center of the 6N-point sphere exists.
9. If the *3D orthocenter* and the *3D circumcenter* exist in a *triangular polyhedron*, all the vertices of the polyhedron are cospherical and the polyhedron can be divided into N *orthocentric tetrahedrons.*

7. Discussion

According to "*the 24-point sphere*" ⟨⟨ *24* 點球面 ⟩⟩ (Liu & Lin, 2007), Liu & Lin suggested that the $6N$-point sphere exists only if the numbers of the faces connecting to every vertex is the same. Someone also suggested that if the *3D orthocenter exists* in a *triangular polyhedron*, the adjacent vertices for any vertex are coplanar and any two adjacent faces must have "*coincided foot*". Their suggestions are to be proved.

Yet, we are still looking for the sufficient and necessary condition for the existence of the *3D orthocenter*, the *3D circumcenter* and the $6N$-point sphere in a *triangular polyhedron.* In addition, it will be challenging to find out the relation between the *3D orthocenter*, the *3D circumcenter*, the *3D centroid* and the center of the $6N$-point sphere, if they exist, in a *triangular polyhedron.* We are interested in the questions like: Are the centers collinear? What will be the ratio of the distance between the centers? We hope that, in the future, the properties of the $6N$-point sphere will be further discussed and developed.

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[2] 熊斌, 田廷彥, 國際數學奧林匹克研究, 世紀出版, 2008.
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Reviewer's Comments

First of all, I congratulate the students and the supervising teacher for their achievements. I wish I had a mathematics teacher like that in my high school years. I was told not to focus too much on the correctness of the proofs, rather I should comment on the readability and read-friendliness of the article. Hence I take the liberty of suggesting a re-arrangement of the starting part of the essay.

I found that the toughest part is "Part Zero: Basic Knowledge". Not enough details were given to motivate the subsequent investigations. The reader had to look up a lot of different preparation material before he/she can start reading the subsequent parts. I looked up the various facts and proofs as usual via Google, and finally pieced everything together. Here is what I found:

There are many centers for a triangle, amongst them: centroid, circumcenter, orthocenter. The first amazing theorem is the Euler line saying that all these centers are collinear, for ANY triangle. Of course, for an equilateral triangle, they all coincide.
http://mathforum.org/pcmi/hstp/sum2011/afternoon/TetraEulerLine.pdf is a good source of information. It does not give you all the details, and proof. The first question one can ask is: Does any tetrahedron have an Euler line, joining the centroid, circumcenter, orthocenter of the tetrahedron, given that tetrahedron is a natural generalization of the triangle in 3D space.

Before we can answer this question, we have to generalize the notion of centroid, circumcenter, orthocenter to a tetrahedron. Related to that of course, we have to generalize the notion of median, perpendicular bisectors, altitude to a tetrahedron.

It is natural to expect the medians for a tetrahedron to be concurrent, because we expect the center of gravity to exist for all objects. Similarly we expect the perpendicular bisectors of a tetrahedron to be concurrent, because we know that 4 non-coplanar points determine a sphere. For the altitudes, we do not expect them to be concurrent, unless there are some restrictions on the shape of the tetrahedron. Just imagine you have an altitude on one face, just moving the other vertex slightly would ruin the concurrency. The extra condition we need turns out to be "orthocentric". Then the Euler line in an orthocentric tetrahedron.

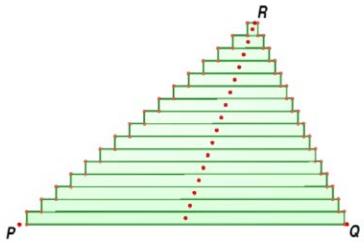
Going into another direction, on each face of the tetrahedron, there is a nine-point circle, which passes through the three mid-points on each side, the three foots of each altitude and the three mid-points of the line segments joining the vertices and the orthocenter of that face. There are four faces in a tetrahedron. Hence there are four such nine-point circles associated with each tetrahedron. It is natural to ask whether these four nine-point circles are co-spherical, ie lie on the same sphere. This is one of the questions studied in this essay.

This sphere is called the 24-point sphere. It helps if a simple explanation is given as to why it is "24-point".[1]

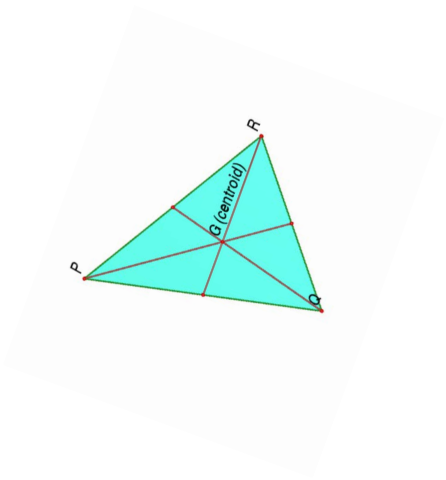
For your consideration, below is how I would present the beginning part of the essay:

The Centroid of a triangle:

The following pictures have been taken from
http://mathforum.org/pcmi/hstp/sum2011/afternoon/TetraEulerLine.pdf



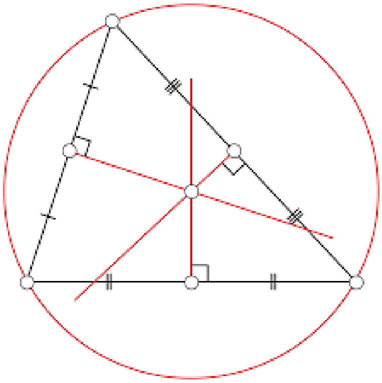
The locus of balance points of strips parallel to a side of the triangle is the median.



The three medians are concurrent at a point called the centroid, which is the center of gravity for the triangle.

(It depends on how much detail you want to go into, you may want to add a few more elementary facts on centroid, or point people to some suitable websites.)

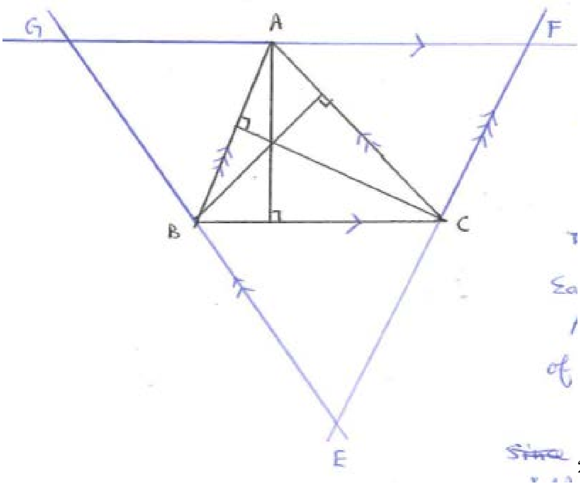
[1] Every face of a tetrahedron has a nine-pint circle. There are 4 faces. $4 \times 9 = 36$ points There are 6 edges. On each edge the midpoint, foot of altitude get double counted. Getting rid of the double counting, we get $36 - 6 \times 2 = 24$ distinct points.



The **circumcenter** has to be on the perpendicular bisector of each of its sides of a triangle. The perpendicular bisectors of a triangle are concurrent.

(again, depending on how much detail you want to go into, you may want to add a few more elementary facts and proofs, or point people to some suitable websites.)

The **orthocenter** is the point where the **altitudes (height)** of a triangle meet. There are several proofs that the three altitudes of a triangle are concurrent. Or one can see the orthocenter of one triangle as the circumcenter of another related triangle, as follows:



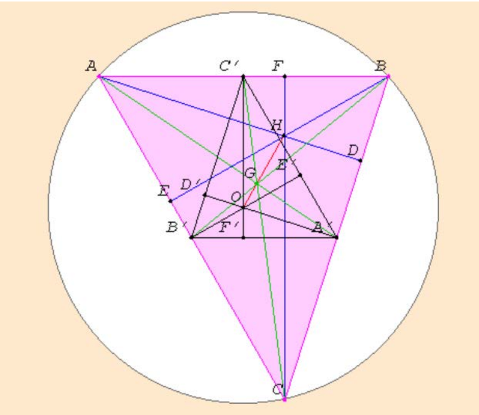
[2]

Given ΔABC, we construct ΔEFG by drawing lines through vertices which are parallel to their opposite sides. In the above diagram, GAF (resp. GBE, ECF) is the straight line through A (B, C), parallel to BC (resp. AC, BA). Since $GACB$, $FABC$ are parallelograms, easy to see that $GA = AF$, and the altitude of ΔABC

[2]I don't have any drawing software to use. So I draw the picture, and scan it.

through A is the perpendicular bisector of GF in ΔEFG. Since perpendicular bisectors of a triangle are concurrent, so must be the altitudes.

http://aleph0.clarku.edu/∼djoyce/java/Geometry/eulerline.html:

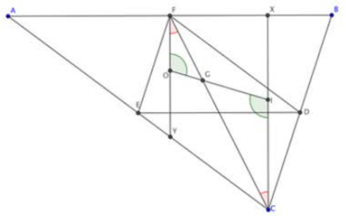


Focus your attention on the centroid G. For each point, like A on one side of it, there is another, like A' on the other side of it but half as far away. On one side is B, the other B'; on one side C, the other C'. In fact, this correspondence sends the whole triangle ABC to the smaller, but similar, triangle $A'B'C'$, called the *medial triangle*. The sides of the medial triangle $A'B'C'$ are parallel and half the length of the sides of the original triangle ABC.

You can see from the figure that this correspondence sends the altitudes of the original triangle, which are AD, BE, and CF, to the altitudes of the medial triangle, which are $A'D'$, $B'E'$, and $C'F'$. Since the altitudes of the original triangle meet at the orthocenter H of the original triangle, the altitudes of the medial triangle will meet at its orthocenter H' which you can see in the figure is labelled O. Behold! This orthocenter O of the medial triangle is the circumcenter of the original triangle! Thus, this correspondence sends H to O, that is, H and O are on the opposite sides of the centroid G, and O is half as far away from G as H is.

The above gives you an intuitive reason why the centroid(G), circumcenter(O) and orthocentre(H) of a triangle should be collinear, why $HG : GO = 2 : 1$. HGO is called the Euler line.

The following diagram from http://ninepointcircle.weebly.com/history.html may also help:



Euler's line
is for any non equilateral triangle. It passes through many points, including the orthocenter, the circumcenter, the centroid, and the center of the nine point circle of the triangle.

On an interesting note, the following article by Ed Sandifer[3] gives an account on how Euler discovered the line known as Euler line today:

http://eulerarchive.maa.org/hedi/HEDI-2009-01.pdf

Now some information about the **nine-point circle**:

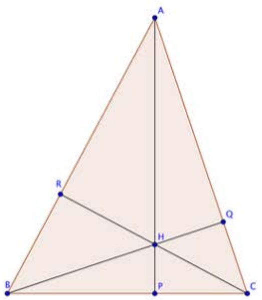
http://ninepointcircle.weebly.com/history.html :

The Nine Point Circle

Home History Definitions Mathematical Background Proofs Interactive tools Other resources

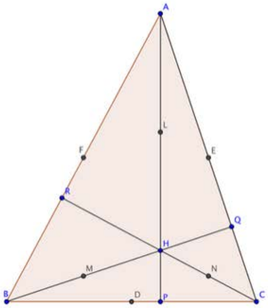
The theorem we are going to prove is the existence of the nine point circle, which is a circle created using nine important points of a triangle. Those nine points are the midpoint of each side, the feet of each altitude, and the midpoints of the segments connecting the orthocenter with each vertex.

Proof 1:

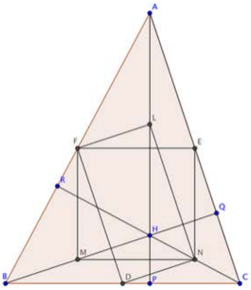


This shows the triangle with points ABC as the vertices, R, Q, and P as the feet of the altitudes BA, AC, and CA respectively and the orthocenter, H. We will also add midpoints to the legs of the triangle.

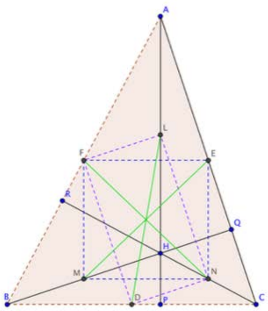
[3]Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT.



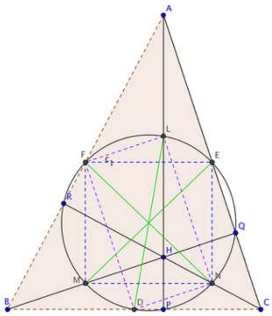
For segments BC, AC and AB, the midpoints are D, E and F respectively. We will also create midpoints of the segments connecting the orthocenter and the vertices. For the segments HA, HB and HC, the midpoints are L, M and N respectively.



Next we will draw the segments for FENM: FE, EN, NM and NF and the segments for FLND: FL, LN, ND and DF.



Due to the theorem stating that the line joining two midpoints of a side of triangle is parallel to the third side (See Mathematical Background page), using triangle ABC, FE is parallel to BC. Using triangle BHC, MN parallel BC and by the transitivity of parallel lines, FE is parallel to MN.
By triangle CHA, EN is parallel to AH and by triangle BHA, FM is parallel to AH. By transitivity, FM is parallel to EN.
FEMN is a rectangle. To get this fact we use the perpindicular bisectors. AP is perpendicular to BC and since BC is parallel to FE, AP is perpendicular to FE. By the same logic FE is perpendicular MN. Since FM is parallel to EN and is parallel to AH, FM is perpendicular to BC. By the definition of a rectangle, FEMN is one.
FLND is a rectangle using the same logic as above.



DL, EM, and FN are diameters of the same circle. This proof is on the Mathematical Background Page.

DP is perpendicular to LP because AH is perpendicular BC. So P is on the circle (See Mathematical Background: Perpendicular on the circle). BQ is perpendicular to EQ and CR is perpendicular FR. So points R, Q, and P are on the circle.

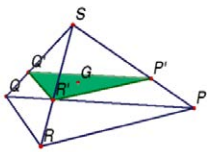
Now we come to the first question about whether there exists Euler line in a tetrahedron (a tetrahedron being a 3D triangle).

First of all, how do we generalise the notion of centroid, circumcenter, orthocentre of a tetrahedron?

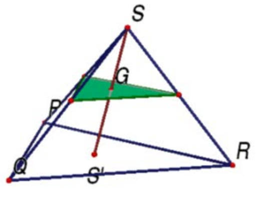
Centroid of a tetrahedron:

Pictures taken from **tjones@alpinedistrict.org**

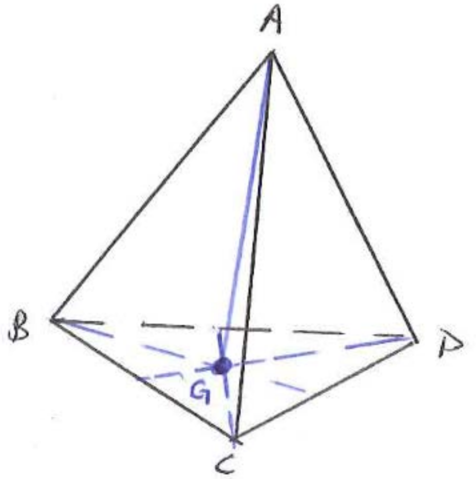
We could slice a tetrahedron into thin slices parallel to a *face*. Each slice would be a triangle that balances at its *centroid*.



This segment in the tetrahedron is the analog of the *median* in a triangle. It is the locus of the *centroids* of each triangular slice, and a skewer piercing the tetrahedron along this segment will balance it.



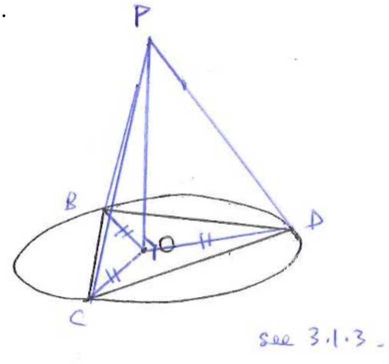
The median is the line joining a vertex to the centroid of the opposite face of the tetrahedron.



AG is the locus of centroids of triangular slices parallel to ΔBCD. Since we expect the center of gravity to be a point, therefore we expect the four 3D medians of a tetrahedron to be concurrent. (A proof is not provided here. There are many proofs available; for example, using coordinate geometry.) At least this fact is believable.

Circumcenter of a tetrahedron:

circumcentre .

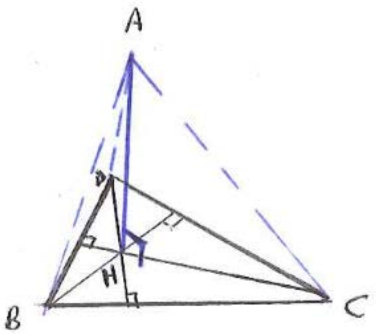


PO is a perpendicular bisector of ΔBCD. Any point on the perpendicular bisector is equidistant from B, C, D.

See Section 3.1.3 of the Essay. The 3D normal (or the 3D perpendicular bisector) to be the line passing through the circumcenter of a face and perpendicular to that face. Using congruent triangles, it is easy to see that any point on the perpendicular bisector is equi-distant from the vertices of the base triangle B, C, D.

Any 3 points not on a straight line determines a circle. Any 4 points not on the same plane determines a sphere.(this was mentioned in lemma 5.) So we expect the four perpendicular bisectors on the four faces of the tetrahedron to be concurrent. The common intersection point is called the circumcenter of the tetrahedron, which is the center of the sphere circumscribing the tetrahedron.

The Orthocenter of a tetrahedron:
Orthocentre.
The 3D altitude is the line passing through the orthocentre of a face, perpendicular to the face.



$\overrightarrow{DA} = \overrightarrow{DH} + \overrightarrow{HA} \qquad \overrightarrow{DA} \cdot \overrightarrow{BC} = \overrightarrow{DH} \cdot \overrightarrow{BC} + \overrightarrow{HA} \cdot \overrightarrow{BC} = 0 + 0 = 0$

3.1.2 The 3D altitude to be the line passing through the orthocenter of a face and is perpendicular to that face.

From the picture, it is easy to see that

$$\overrightarrow{DA} = \overrightarrow{DH} + \overrightarrow{HA}, \overrightarrow{DA} \cdot \overrightarrow{BC} = \overrightarrow{DH} \cdot \overrightarrow{BC} + \overrightarrow{HA} \cdot \overrightarrow{BC} = 0 + 0 = 0. \qquad (**)$$

3.5 We define a tetrahedron to be an **orthocentric** tetrahedron if every edge of the tetrahedron is perpendicular to the opposite edge (that is, if we let $\vec{a}$ be a vector parallel to an edge and $\vec{b}$ be a vector parallel to the opposite edge, then $\vec{a} \cdot \vec{b} = 0$.

So the argument in (**) shows that if all altitudes are actually heights (ie. passing through the opposite vertex), then the tetrahedron must be orthocentric.

[I suppose the converse is also true. Can the authors provide a proof of this?]

This motivates the introduction of orthocentric tetrahedron very naturally.

It would be nice if the proof of "The 3D Euler line exists in an orthocentric tetrahedron" is also included in the paper. (This fact is mentioned in 4.2 of Part Zero of the Paper.)

From here on, I think the subsequent Part One, Part Two,... would be much easier to read.

The following gives a few more suggestions on some specific parts of the Paper.

Part Zero Section 4.1

From definition of orthocentric tetrahedron, one gets $\vec{x}\cdot(\vec{z}-\vec{y})=0$, $\vec{y}\cdot(\vec{x}-\vec{z})=0$, $\vec{z}\cdot(\vec{y}-\vec{x})=0$, which imply $\vec{x}\cdot\vec{z}=\vec{x}\cdot\vec{y}$, $\vec{y}\cdot\vec{x}=\vec{y}\cdot\vec{z}$, $\vec{z}\cdot\vec{y}=\vec{z}\cdot\vec{x}$.

Part One:

If a triangular polyhedron with N faces contains N co-spherical nine-point circles, the sphere are said to be a '$6N$-point sphere'.

I would provide a proof like this:

One face has 3 sides of a triangle[4]. N faces have $3N$ sides of triangles. 2 sides form 1 edge. There are altogether $3N/2$ edges.

There are 9 points of the nine-point circle per face, $9N$ points for N faces. 2 points per edge are double counted. Without double counting, there are $9N-2\times\frac{3N}{2}=6N$ distinct points.

Part One, bottom of page 4:

"Hence, the edge BD is perpendicular to the edge $AD(\overrightarrow{BD}\cdot\overrightarrow{AC}=0))$. In other words, the tetrahedron is an orthocentric tetrahedron and the 3D orthocenter exists."

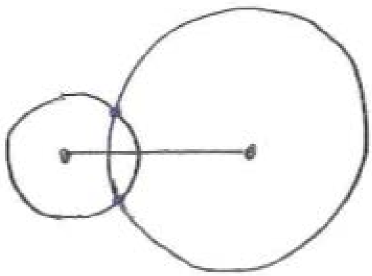
I would present the proof as follows:

$\overrightarrow{BD}=\overrightarrow{BE}+\overrightarrow{ED}=\overrightarrow{BE}+\overrightarrow{GD}$, because by Lemma 2, $E=G$.

$\overrightarrow{BD}\cdot\overrightarrow{AC}=(\overrightarrow{BE}+\overrightarrow{GD})\cdot\overrightarrow{AC}=\overrightarrow{BE}\cdot\overrightarrow{AC}+\overrightarrow{GD}\cdot\overrightarrow{AC}=0+0=0.$

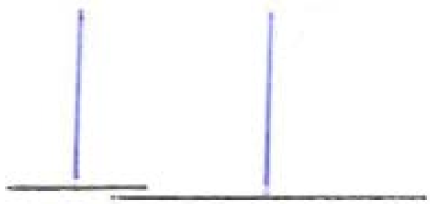
Lemma 4

Top view: if the two circle are intersecting and co-planar:



[4]This is where we used the fact that the polyhedron is formed by only triangular faces.

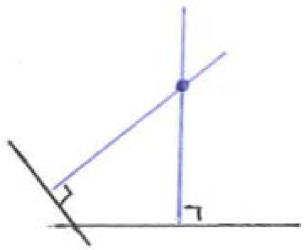
Side view: if the two circle are intersecting and co-planar:



The blue line on the left is perpendicular through the centre of the circle. Any point on this perpendicular is equidistance from the circumference of the circle.

Any point on the perpendicular through the center of the planar circle is equidistant from any point on that circle.

Imagine that the smaller circle is tilited away from the plane of the bigger circle, hinged at the two intersecting points. Then we have the following side view:

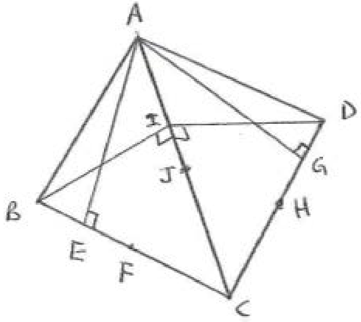


The blue point represents the centre of the sphere containing the two circles.

Then the intersecting point of the two perpendiculars through the two centers is the center of the sphere containing the two circles. (That is the proof of Lemma 4.) I think the important point is to let the reader "see" what is going on.

Assume that the tetrahedron $ABCD$ is an orthocentric tetrahedron.

Let E, I be the foots of the perpendiculars of ΔABC.
Let G, I be the foots of the perpendiculars of ΔADC.
J, F, H be the mid points of AC, BC, CD respectively.



The foots I in the above statements coincide because of the orthocentric assumption.
Assume only that $BI \perp AC$.
By orthocentricity of the tetrahedron,

$$0 = \overrightarrow{BD} \cdot \overrightarrow{AC} = (\overrightarrow{BC} + \overrightarrow{ID}) \cdot \overrightarrow{AC} = \overrightarrow{BI} \cdot \overrightarrow{AC} + \overrightarrow{ID} \cdot \overrightarrow{AC} = \overrightarrow{ID} \cdot \overrightarrow{AC}.$$

$\Rightarrow I$ is the foot of height DI on ΔADC.

Let the nine-point circle of ΔABC, ΔADC and ΔBCD be C_α, C_β and C_γ respectively.

According to Lemma 4, any two non-planar circles must be cospherical, we let

$S_{(\alpha,\beta)}$ be the sphere defined by C_α and C_β,
$S_{(\beta,\gamma)}$ be the sphere defined by C_β and C_γ,
$S_{(\gamma,\alpha)}$ be the sphere defined by C_γ and C_α.

Note that

$$\left.\begin{array}{ll} I, J, F & \text{lie on } S_\alpha \\ I, J, H & \text{lie on } S_\beta \end{array}\right\} \Rightarrow I, J, F, H \text{ lie on } S_{(\alpha,\beta)}.$$

Note that

$$\left.\begin{array}{ll} I, J, H & \text{lie on } S_\beta \\ F, H & \text{lie on } S_\gamma \end{array}\right\} \Rightarrow I, J, F, H \text{ lie on } S_{(\beta,\gamma)}.$$

Note that

$$\left.\begin{array}{ll} F, H & \text{lie on } S_\gamma \\ I, J, F & \text{lie on } S_\alpha \end{array}\right\} \Rightarrow I, J, F, H \text{ lie on } S_{(\gamma,\alpha)}.$$

Since I, J, F, H are not coplanar, according to lemma 5, $S_{(\alpha,\beta)}$, $S_{(\beta,\gamma)}$, and $S_{(\gamma,\alpha)}$ are the same sphere.
Therefore C_α, C_β and C_γ are co-spherical.

. . . $\square$